University of Southampton

Faculty of Social Sciences
School of Mathematical Sciences

Social finance: a model of social influence on financial markets

DOI: ...

by

Giuseppe Di Nardo
MPhys
ORCiD: 0000-0003-4259-3767

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Abstract

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The purpose of this research project is to investigate the possible destabilizing effects generated by social interactions between investors in financial markets, with particular regard to excess volatility in price fluctuations and financial bubbles. More specifically, I build a model describing a financial market populated by different categories of investors, distinguished by their level of sophistication, which measures investors' net worth and experience in financial markets and having heterogeneous beliefs in estimating the value of the traded asset. I represent the interaction among traders through a coupled map, which describes the evolution and dissemination of the perceived value of the asset over time. Its esteem will depend on the trading strategy adopted by these investors, having the choice between being fundamentalists, that is, believing the asset will tend to its fundamental value, contrarian fundamentalists, believing the asset will move away from its fundamental value, or followers, imitating one of the two previous types of investors with a delay. My goal is to perform an analysis of this model and understand how the structure of this system and the set of investors affected by social biases influence those critical points beyond which these undesired phenomena appear.

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Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. None of this work has been published before submission

Signed:.. Date:..................

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To all the people who do their best to follow their dreams

Chapter 1

Introduction

Over recent years, various scientific communities have emerged with the common objective of providing a mathematical explanation for the many regular patterns present in the time series of a stock price in financial markets, whose potential cause might be associated with investors' behaviour and their corresponding interactions. In particular, economists, mathematicians, physicists and others have contributed to behavioural modelling in financial markets within two important research fields, namely econophysics and financial economics[1]. Some of the models developed by these authors are called agent-based models (ABMs), which, in general, are very computationally-oriented, although in this context they also include models that are analytically tractable[2]. Econophysics is a branch of physics led by physicists who use ideas, models and techniques from statistical physics in order to describe socio-economic phenomena (see Kutner et al. (2019) for a recent overview). Here, financial agent-based models are built by exploiting the properties of critical systems found in nature but used in an economic context, such as percolation theory applied to a random graph (Cont and Bouchaud (2000)) or the Ising paradigm on complex networks (Iori (1999)), whose nodes represent the investors and the links, for example, their social circle. Agent-based models in financial economics, instead, is a relatively new topic, which was born of the need to move from the theoretical limitations imposed by traditional economics (representative agent with rational expectations and perfect rationality) to a more realistic vision of the economy (boundedly rational agents with heterogeneous expectations) (see Farmer and Foley (2009), Farmer et al. (2012), Battiston et al. (2016) for a discussion on this matter, and Hommes and LeBaron (2018) for a recent overview). Following Lux and Alfarano (2016), and Franke and Westerhoff (2009), it is possible to classify the models from the financial economics

[1] Jovanovic and Schinckus (2017) (two financial economists) investigate the interface between these two fields, highlighting their differences and similarities.

[2] These models were once called, and still are by some authors, Agent-Based Computational Economics (ACE) models, but dropping the word 'computational' allows the inclusion of models that do not require computational tools for analysis such as the Heterogeneous Agents Models (HAMs), see Hommes and LeBaron (2018).

branch into two categories based on the approach adopted to modelling the switching mechanisms for investors between different strategies present in a financial market: models using the transition probability approach (TPA) and those using the discrete choice approach (DCA).

Transition probability approach (TPA)

Weidlich and Haag (2012) were the first authors (back in 1983) to adopt the TPA in a social context, while Kirman (1991a) (see also Kirman (1993)) was the first to apply it in a financial context, followed immediately by Lux (1995) incorporating it into an asset pricing model. The main feature of this approach is that of integrating herding and contagion phenomena into economic models by letting investors interact *directly* with each other. The general framework of these models is based on a financial market populated by two types of investors (fundamentalists/chartists or buyers/sellers), who switch between one strategy and another according to a transition probability (defined for a single agent) which depends on, for example, the opinion index associated with a specific strategy (one typically uses a 'noisy' voter model to describe the opinion formation among investors). In order, then, to describe the evolution in time of the number of investors adopting a specific strategy, one resorts to a master equation, which is typically hard to solve analytically, forcing the model-maker to employ a mean-field approximation. After that, one substitutes the result into a market-clearing equation which dictates the evolution of the price of a stock according to a specific rule(see Lux and Marchesi (1999) and Lux and Marchesi (2000a)). At this point, one either proceeds with an analytical study of a differential equation, even though it is almost impossible given the high non-linearity of the resulting system, or focuses on statistical analysis in order to find possible correspondences with the so-called *stylised facts*, that is regular patterns found over different time series across several stocks in financial markets (see Cont (2001) for a discussion on this matter). Unfortunately, two pieces of work cast doubt on many models based on this paradigm by testing their robustness with respect to an enlargement of the system size (the number of investors) and its consequences for their possible application on complex networks. The first work, by Egenter et al. (1999), showed that an increase in the number N of investors would inevitably make all the regular statistical properties of the fluctuations in financial time series (stylised facts) disappear, turning all into normal distributions (this phenomenon is called the *N-dependence* problem). Subsequently, Alfarano and Milaković (2009) analysed the impact of different network structures (regular lattice, random graph, small-world, scale-free networks) on the statistical equilibrium outcome of this type of model, showing that none of these structures overcome the N-dependence problem, except the random graph, which is the most unrealistic social structure in this context. The same authors proposed that a possible solution could be the introduction of particular hierarchical structures (see

Alfarano and Milaković (2009), and Alfarano et al. (2013)), and more recently Artime et al. (2019) introduced two different mechanisms to the noisy voter model on complex networks (non-linear interactions among investors and reluctance to change their opinions) that make it robust and well-defined in the limit of a large number of agents.

Discrete choice approach (DCA)

The DCA shares many similarities with TPA, although the former has gained way more attention over the past years (Franke and Westerhoff (2009)). This approach to modelling the population dynamics was proposed by Brock and Hommes (1997), who adapted the random utility framework for empirical analysis of discrete choice problems (Manski and McFadden (1981)) to a demand-supply cobweb type (Brock and Hommes (1997)) and an asset pricing model (Brock and Hommes (1998)). Unlike TPA, the parameters related to the fitness of a particular strategy directly determine the evolution of the fraction of the investors adopting the strategy (so not the transition probabilities as in TPA), which allows the hurdle of solving the master equation to be bypassed. Moreover, this approach can be easily extended to more than two strategies (while with a TPA this is less obvious) and does not face the N-dependence problem since these models deal with fractions of investors. In fact, one can assume the presence of an infinite number of investors in the market, which gives the possibility of analysing the model via systems of difference equations derived for the infinite population limit. Brock and Hommes introduced a concept known as *Adaptive Belief Systems* (ABS), a standard asset pricing model variation based on discounted values originating from mean-variance maximization. A model adapted for scenarios involving heterogeneous beliefs, where agents possessing limited rationality choose their forecasting or investment strategies based on recent, comparative performance.

DCA Roots This approach to modelling financial markets draws on a rich lineage of theoretical constructs emphasising the role of heterogeneous agent models (HAMs) in explaining complex market dynamics. Central to this theoretical evolution was the work of Frankel and Froot (1986), who introduced a dynamic model where the foreign exchange market was populated by investors adopting different strategies. Their revolutionary model incorporated time-varying weights assigned to these forecasting strategies based on ongoing performance evaluations, a method later reflected in the ABS (which adopts the DCA). Frankel and Froot observed (thanks to some evidence from survey data for exchange rate expectations) that different investor types rely on distinct information and trading rules, shaping the market's overall behaviour. So, they developed a model by including three different groups of investors: fundamentalists, chartists and portfolio managers. Fundamentalists model their trading decisions based on fundamental analysis, while chartists make trading decisions based on historical price movements using tools like autoregressive time series models. Portfolio managers form their expectations as a weighted average of predictions from fundamentalists and chartists, adjusting these weights through a Bayesian updating process depending on which group has recently provided more accurate forecasts. Further contributing to the conceptual groundwork for DCA, Day and Huang (1990) introduced one of the first discrete-time models to exhibit complex, chaotic behaviours in asset prices that mirrored real-world market phenomena such as bull runs and crashes. This model, alongside contributions from Chiarella (1992) and Lux (1998), highlighted how simple behavioural rules, when applied to interactions among different types of agents, could lead to endogenous price fluctuations. The literature on HAMs broadened the understanding of financial markets by integrating the concept of noise traders—investors whose decision-making deviates from the rational actor model. This group includes chartists and naive traders, as discussed in works by authors like De Long et al. (1990a), De Long et al. (1990b) and Gennotte and Leland (1990). These models explore how misperceptions of expected returns or the application of simple feedback rules can significantly influence market dynamics. So, DCA originates from various economic theories that highlight the significance of different behaviours among agents and the interactions between their beliefs and market results. It captures the complex behaviours in financial markets by focusing on how agents switch between different types based on performance, a central ingredient in the ABS introduced in the works of Brock and Hommes (Brock and Hommes (1997), Brock and Hommes (1998)), showing how the combination of fundamental and technical strategies influences asset price movements.

Later works Building on Brock and Hommes' theories, subsequent research has further developed the ABS framework across various market segments. De Grauwe and Grimaldi (2006) applied it to exchange rate modelling, introducing transaction costs that affect market behaviour significantly. Their model delineated a threshold around the fundamental equilibrium where fundamentalists become inactive due to high costs, leading to distinct trading patterns and explaining intermittent market volatility. Ap Gwilym (2010) extended this to equity markets, specifically testing against the FTSE All-Share Index. He highlighted that the model's heuristic rules, based on past rule profitability rather than strict rational expectations, could effectively explain the market data, emphasizing its evolutionary rationality. Furthermore, Frijns et al. (2010) adapted the ABS to the options market, developing a model that considers differing beliefs about future volatility among fundamentalists and chartists. This model, tested on the DAX30 index options, showed superior pricing performance over standard GARCH-type models, highlighting the active role of different trader types and the flexibility of the ABS approach in capturing complex market dynamics. These advancements collectively enhanced the understanding of financial markets from the perspective of heterogeneous agent models.

Extension to social interactions The class of models using this approach also has its drawbacks. In particular, they are not really meant to account for herding or contagion phenomena, since the agents do not interact directly or locally with each other but globally through the price and the performance of the adopted strategy. Very few authors have attempted to extend these types of models by including social phenomena. As pointed out by Dieci and He (2018) in one of their suggestions about future research regarding these models, little space has been given to social interactions and social networks. Anufriev et al. (2018) make similar observations, especially regarding the incorporation of networks within models following either TPA or DCA, noting that only two works (at the time they published their review) are present in the literature that accomplished this feat (see Alfarano and Milaković (2009) and Panchenko et al. (2013)). They stressed that more work in this direction would be desirable since clustering and hubs, as well as the endogenous formation of information networks, might have consequences for the financial system. Luckily, in recent years, the integration of social networks into the DCA has seen promising developments, mainly through the efforts of researchers like Gong and Diao (2023) and Hatcher and Hellmann (2022). Gong and Diao (2023) use the SIS model (diffusion models originating in epidemiology) to analyze the spread of chartist and fundamentalist strategies among investors, demonstrating how these strategies can continuously influence market behaviour akin to the spread of a common cold. Their approach emphasizes the role of transient strategies within financial markets, showing how they can cyclically affect investor behaviour and market outcomes. On the other hand, Hatcher and Hellmann (2022) incorporate continuous belief dynamics from opinion dynamics literature, allowing for an exploration of how individual beliefs evolve within a social network. Their model illustrates that depending on the network structure, investors' beliefs can converge to a consensus or remain diverse, significantly affecting market volatility and pricing. Both these studies underscore the importance of network effects in financial markets, revealing how investors' performance and network position can dramatically shape market dynamics.

Social influence on financial markets and research questions

Based on financial economists' studies about investor behaviour[3], it is clear that *social biases* influence investors in destabilizing the financial markets by creating bubbles and anomalous volatility in price fluctuations, thereby endangering the single investor's activity, a listed company, and even the entire stock market. Several exogenous sources (social proximity, social media, mass media, financial recommendations, internet forums, financial TV channels) trigger them and affect the stock market. We intend to investigate and understand how the structure of this system and the set of investors affected by social biases influence those critical points beyond which these

[3] See Nofsinger (2017) and Ackert and Deaves (2009) for a review and references therein

undesired phenomena appear. More specifically, the research questions that we will try to address are: in which way does social interaction influence investors' perceptions of the value of an asset? Can it give rise to abnormal volatilities by spreading investors' beliefs away from each other? Or does it cluster them towards a common value, potentially leading to bubbles or bursts? In this thesis, we attempt to answer these questions by building dynamic models of the asset market, belonging to the second class described above, in which a financial market is populated by investors whose social interaction is described through a coupled map, which dictates the evolution and dissemination of the perceived value of the asset over time. The price esteem will depend on the trading strategy adopted by these investors, having the choice between being *pure fundamentalists*, that is, believing the asset will tend to its fundamental value, *contrarian fundamentalists*, believing the asset will move away from its fundamental value, or *followers*, imitating one of the two previous types of investors with a delay. This thesis is divided into four further chapters: in Chapter 2, we will present the general framework of a DCA model, reviewing all past works which took into account social interactions, and will discuss the findings of a series of works regarding data analysis of a real stock market; in Chapter 3, we will develop our model of social influence on financial markets based on the conclusions we made in the previous chapter and perform an analytical analysis when the market is populated by pure fundamentalists only; in Chapter 4, we will consider the case when the market is populated by pure fundamentalists and followers; finally, in Chapter 5, we will analyse the case when the market is populated by three groups of investors: pure fundamentalists, contrarian fundamentalists, and followers.

Chapter 2

Literature review

In the first part of this chapter, we will describe the general framework of a model adopting the discrete choice approach (DCA), assuming that the reader has no knowledge of asset pricing, discrete choice or dynamical systems theory (we will be following BH's seminal work Brock and Hommes (1998), together with Hommes (2018), and Chiarella et al. (2009)), and we will review the state of the literature on the extensions to the setting described that take social interactions into account. In the second part, we will discuss the findings of a series of works regarding data analysis of investors' trading activity in the Finnish stock market. In the last part, we will draw some conclusions based on the discussion addressed in the previous sections, which will support us in the modelling approach we will adopt in the next chapter.

2.1 Modelling of a financial market

2.1.1 DCA general framework

2.1.1.1 Portfolio optimization

We assume investors have the option to invest in two assets[1], bearing two different levels of risk (or degree of price volatility): a risk-free and a risky asset. The risk-free asset is perfectly elastically supplied (that is, whatever the demand/supply is for that asset, there will not be any increase/decrease in its price) and pays a fixed rate of return $r > 0$ (investing in this asset would guarantee a safe profit over a certain

[1] The fact that there are two different assets provides a benchmark regarding the optimization of an investor's portfolio (collection of all investments held by an investor). This optimization allows determining a theoretically appropriate required rate of return (gain or loss of an investment over a specified period of time) for an asset that investors want to add to their portfolio (Cochrane (2009)).

period)[2]. So, if investors purchased only this asset investing all their wealth W_t at time t, then W_{t+1} is given by

$$W_{t+1} = W_t + rW_t = RW_t, \tag{2.1}$$

where $R = 1 + r$. The risky asset, of which we are interested in studying the evolution of its price p_t per share (for example, the unit of a company's capital owned by an investor, thus called 'shareholder'), pays a stochastic dividend y_t (an investor gets paid by owning a percentage of a company), which is mostly related to a company's profit distribution to its shareholders. If traders used all their wealth W_t to purchase the risky asset at time t, the wealth at time $t+1$ would become

$$W_{t+1} = W_t + \left(\frac{p_{t+1} + y_{t+1} - p_t}{p_t}\right) W_t = W_t + r^{nf}_{t+1} W_t = R^{nf}_{t+1} W_t \tag{2.2}$$

where r^{nf}_{t+1} is the rate of return of the risky asset ("nf" stands for not risk-free), with $R^{nf}_{t+1} = (p_{t+1} + y_{t+1})/p_t$. From equation (2.2), one deduces that $W_{t+1} > W_t$ if $p_{t+1} + y_{t+1} > p_t$ (investors' wealth increases having purchased the asset when its price at time t was lower than the sum of the price and dividend at time $t+1$) and $W_{t+1} < W_t$ in the opposite case. So here, unlike the risk-free asset case, investors can potentially lose money. Using equations (2.1) and (2.2), we can write the general case when investors optimize their portfolio by investing a proportion π_t of their wealth at time t in the risky asset and $1 - \pi_t$ in the risk-free one, so that W_{t+1} is given by

$$W_{t+1} = RW_t(1 - \pi_t) + R^{nf}_{t+1} W_t \pi_t = RW_t + (R^{nf}_{t+1} - R) W_t \pi_t \tag{2.3}$$

We now want to write this last equation expressing the proportion π_t in terms of the number of shares z_t purchased at time t, better known as the *demand function*, with their relationship given by

$$\pi_t = \frac{z_t p_t}{W_t} \tag{2.4}$$

The wealth dynamics (2.3) becomes, thus

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t) z_t \tag{2.5}$$

where we skipped some straightforward algebraic steps expressing all in terms of the price and dividend. The term $R_{t+1} = p_{t+1} + y_{t+1} - Rp_t$ in equation 2.5 represents the *excess return* for the risky asset over the risk-free one. A positive excess return at time $t+1$ is equivalent to saying that purchasing the risky asset at time t would have been more profitable than opting for the risk-free one, and the opposite is true for the

[2] An example of a risk-free asset is a bond.

negative case. In fact, the excess return R_{t+1} multiplied by the demand z_t (the second term in (2.5) is called the realized profit and will be used as a performance measure for the strategy adopted by the investors. At this point, given the wealth dynamics (2.5), one has to specify a utility function $U(W_{t+1})$ that measures investors' satisfaction or preference over the allocation of their wealth at time $t+1$ to both assets and choose the correct value for the demand z_t in order to get the highest value of that function. To derive the expression for $U(W_{t+1})$, we require that for $W_{t+1} = 0$, investors do not feel satisfied at all, meaning that $U(W_{t+1} = 0) = 0$, while for larger payoffs their satisfaction increases approaching eventually a maximum value, for which we have $U(W_{t+1}) = 1$ (completely satisfied). Furthermore, we also suppose that investors are risk-averse, that is, they get satisfied more or less quickly depending on how risky the investment would be, in their opinion. If they are very risk-averse, then they are content with little ($U(W_{t+1}) \rightarrow 1$ rapidly), on the contrary, if they are little risk-averse ($U(W_{t+1}) \rightarrow 1$ slowly). A function (among others) that satisfies these conditions is given by

$$U(W_{t+1}) = 1 - e^{-aW_{t+1}} \tag{2.6}$$

where a represents the investor's risk aversion, averaged over all investors. Equation (2.6) is generally referred to as the *constant absolute risk aversion* (CARA) utility function (we will soon explain why it is called "absolute")(Figure 2.1 shows the function for different values a)[3]. So investors' aim is to choose z_t that maximizes

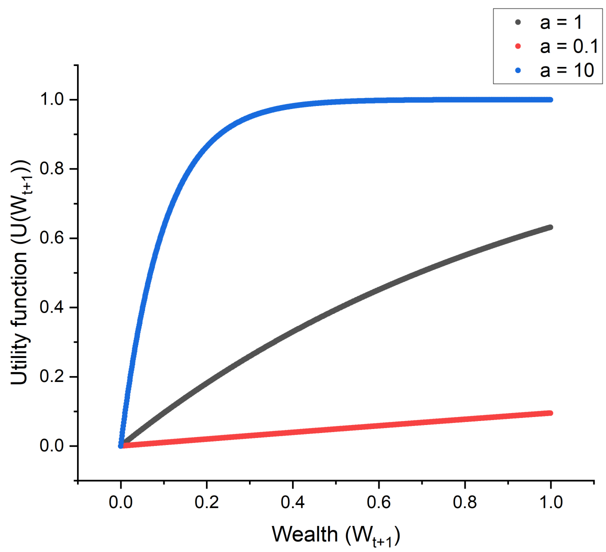


FIGURE 2.1: The utility function $U(W_{t+1})$ for $a = 0.1$, $a = 1$, and $a = 10$

$U(W_{t+1})$, but we have to keep in mind that the price and dividend at time $t+1$, and the wealth W_{t+1} as well, are random variables, hence investors have to compute their

[3]Another way to interpret this function might be to think that "The last bite is never as satisfying as the first" (Cochrane (2009)).

corresponding conditional expectation E_t and variance V_t (their expected values given that some conditions occurred)[4] based on the past publicly available information (up to time t, when all the variables are known with absolute certainty), like past prices and dividends, so that in fact we aim to maximize $E_t[U(W_{t+1})]$. The question now is to find the expected value of a function (in this case, the utility function U) that depends on a random variable (W_{t+1}). However, if one knows the probability distribution, $f(w)$, of the random variable, then one can use the "law of the unconscious statistician", which asserts

$$E_t[U(W_{t+1})] = \int_{-\infty}^{\infty} U(w)f(w)dw \tag{2.7}$$

where the integral is over all possible values w of W_{t+1}. We choose the particular case where investors assume that the prices and dividends at time $t+1$ are normally distributed, so we can write the wealth W_{t+1} (also normally distributed) as

$$W_{t+1} = E_t[W_{t+1}] + \sqrt{V_t[W_{t+1}]}Z \tag{2.8}$$

where Z is normally distributed with $Z \sim N(0,1)$ (with the probability distribution given by $\varphi(z) = e^{\frac{-z^2}{2}} / \sqrt{2\pi}$). Using equations (2.6) and (2.8), we obtain from equation (2.7) that (see A.1 for complete derivation)

$$E_t[U(W_{t+1})] = E_t[U(Z)] = 1 - e^{-a(E_t[W_{t+1}] - \frac{a}{2}V_t[W_{t+1}])} \tag{2.9}$$

Maximizing this last equation is equivalent to maximizing $(E_t[W_{t+1}] - \frac{a}{2}V_t[W_{t+1}])$, or, equivalently, finding z_t that solves

$$\frac{d}{dz_t}\left\{E_t[W_{t+1}] - \frac{a}{2}V_t[W_{t+1}]\right\} = 0. \tag{2.10}$$

All this is generally summarized by saying that the investors are assumed to be myopic mean-variance maximizers. From equation (2.5) we have

$$E_t[W_{t+1}] = RW_t + z_t E_t[(p_{t+1} + y_{t+1} - Rp_t)] \tag{2.11}$$

where we used the fact that $E_t[W_t] = W_t$ and $E_t[z_t] = z_t$. For the variance $V_t[W_{t+1}]$, we first notice that

$$\begin{aligned} W_{t+1} &= RW_t + z_t\left(E_t[(p_{t+1} + y_{t+1} - Rp_t)] + \sqrt{V_t[(p_{t+1} + y_{t+1} - Rp_t)]}Z\right) \\ &= E_t[W_{t+1}] + z_t\sqrt{V_t[p_{t+1} + y_{t+1} - Rp_t]}Z \end{aligned} \tag{2.12}$$

[4] E_t and V_t should be considered as operators.

with $Z \sim N(0,1)$, and using the fact that $V[X] = E[(X - E(X))^2]$ (where X is the random variable) we get

$$\begin{aligned} V_t[W_{t+1}] &= E_t[(W_{t+1} - E_t[W_{t+1}])^2] \\ &= E_t[(E_t[W_{t+1}] + z_t\sqrt{V_t[p_{t+1} + y_{t+1} - Rp_t]}Z - E[W_{t+1}])^2] \\ &= E_t[(z_t^2 V_t[(p_{t+1} + y_{t+1} - Rp_t)]Z^2)] \\ &= z_t^2 V_t[(p_{t+1} + y_{t+1} - Rp_t)]E_t[Z^2] \\ &= z_t^2 V_t[(p_{t+1} + y_{t+1} - Rp_t)] \end{aligned} \tag{2.13}$$

since $E_t[Z^2] = 1$, so we finally obtain the optimal demand for the risky asset from (2.10) as

$$z_t = \frac{E_t[(p_{t+1} + y_{t+1} - Rp_t)]}{aV_t[(p_{t+1} + y_{t+1} - Rp_t)]} \tag{2.14}$$

From this last equation, we see that investors invest more when they expect the return on the risky asset to become higher and invest less if they expect the asset to be riskier or if they are just more risk averse. Investors' wealth does not appear in this equation, meaning that the demand for the asset is independent of the total wealth accumulated (or initial wealth). For this reason, an investor is said to have a constant absolute rather than relative risk aversion to her wealth (Cochrane (2009)).

2.1.1.2 Heterogeneous beliefs and market clearing mechanism

If different types of traders adopt a strategy 'h' for their forecasts, this would certainly affect the demand for the asset. So, to include this important aspect, we write equation (2.14) in a more general form as

$$z_{h,t} = \frac{E_{h,t}[(p_{t+1} + y_{t+1} - Rp_t)]}{a\sigma^2} \tag{2.15}$$

where $\sigma^2 = V_{h,t}[(p_{t+1} + y_{t+1} - Rp_t)]$ is the conditional variance for trader type 'h', assumed to be equal for all types of traders[5], and, by defining $n_{h,t}$ as the fraction of traders adopting the strategy h at time t (investors can switch from one strategy to another), we can write the aggregate demand for the risky asset at time t as

$$z_{d,t} = \sum_{h=1}^{H} n_{h,t} z_{h,t} \tag{2.16}$$

[5]This assumption is made for analytical tractability and supported by the fact that, in real markets, there is more agreement about assessing whether an asset is risky or not than about its potential return (see Brock and Hommes (1998) and Hommes (2018)).

where the sum is over all the H types of strategies. The next step is to specify a condition that allows us to derive the price dynamics, namely a market clearing mechanism. This model uses the Walrasian auctioneer one[6], whose role is to announce a price, receive orders for the asset at that price and repeat this process until it matches the demand and supply (or, more correctly, select the price that will clear the highest number of trades). If we define z_s as the outside supply of the risky asset, that is, the average supply of shares available per investor, assumed fixed and constant over time[7], then we get the equilibrium of demand and supply as

$$z_{d,t} = z_s. \tag{2.17}$$

Notice that equation (2.17) is equivalent to saying that the excess demand (the difference between demand and supply) for the asset is equal to 0 (this equilibrium has been achieved by the Walrasian auctioneer as discussed above). In a market maker system Bauwens and Giot (2013), the price of the asset is *driven* by the excess demand, which in general can be positive or negative (we will return to this point in section 2.1.3.3 and chapter 3 since we will use this system in our model). In case we assume the supply of outside shares to be zero, $z_s = 0$, then equation (2.16), and using (2.15), leads to the market equilibrium equation

$$Rp_t = \sum_h^H n_{h,t} E_{h,t}[(p_{t+1} + y_{t+1})] \tag{2.18}$$

So the risky asset's price at time t is equivalent to the average expectation (over all the strategies) for the sum of price and dividend at time $t+1$, discounted by the risk-free interest rate. Assuming $z_s = 0$ can be interpreted as ignoring a risk premium $a\sigma^2 z_s$ in equation (2.18) for investors holding the risky asset Hommes (2018)[8]. If $z_s \neq 0$ one can get the same expression for the market equilibrium equation (2.18) if one defines the dividend as $y'_{t+1} = y_{t+1} - a\sigma^2 z_s$ (here the role of the risk premium as a cost is evident), even though this would anyway affect the dynamics of the system.

2.1.1.3 Discrete choice approach and fundamental price

Equation (2.18) is the first component of our dynamical system, and what is left now are the expressions for $n_{h,t}$. We address the derivation of these expressions in depth in

[6]See chapter 2 of Bauwens and Giot (2013) for a description of all possible trading mechanisms in a stock exchange and Chiarella et al. (2009) for a comparison between them.

[7]Generally, it can change over time due to either new shares issued or the splitting of shares Chiarella et al. (2009).

[8]The risk premium is a cost that investors take into account when they decide to invest in an asset that bears some risk and, for this, they want to be compensated (a *premium* for taking the risk)Cochrane (2009). To do this, they subtract this (virtual) cost from their expected return so that with the same demand for the asset with or without a risk premium, the former, if successful, would make more profits than the latter.

the Appendix B.1, while here we give the final form, which is commonly called the *multinomial logit model*:

$$n_{h,t} = \frac{1}{1 + \sum_{i \neq h}^{H} e^{-\beta(u_{h,t-1} - u_{i,t-1})}}. \tag{2.19}$$

where $u_{h,t-1}$ is the *performance function* for the strategy h at time $t-1$ and β is the *intensity of choice*. Simply put, investors choose a strategy instead of another depending on its performance with respect to all other possible options (the performance function can depend on several factors, like profit after investment, agreement with other investors, and so on) and β controls the level of randomness given by the error in the investors' decision (whose cause arises from behavioural biases).

Delayed information The role played by $n_{h,t}$ in this model (and all its extensions to social interactions) is to describe the decision-making process of investors in choosing between different strategies and, consequently, trading the asset with demand $z_{h,t}$. Something that this model, and the other models discussed in the section 2.1.3, fails to take into account is a potential delay on the part of investors in receiving or sending certain information (which could give rise to persistent price growth/decrease or other unexpected phenomena). So, in addition to having the function $n_{h,t}$, there would be another one, $\alpha_{h,t}$, that describes the decision-making process of investors in *sharing* their information with other investors. We will fill this gap by introducing such a function in our model in Chapter 3 and use it for the first time in Chapter 4.

Fundamental value Lastly, we have to be more specific about the expected price and dividends forecasted by each strategy, which depends on the so-called "*fundamental price*" of the risky asset. To get the notion of the fundamental price, let us consider the equation (2.18) again if a market was populated by identical traders, with homogeneous expectations (that is, everyone adopting the same strategy) and assuming $z_s = 0$, then at time t we have

$$p_t = \frac{E_t[p_{t+1}]}{R} + \frac{E_t[y_{t+1}]}{R} \tag{2.20}$$

and at time $t+1$

$$p_{t+1} = \frac{E_{t+1}[p_{t+2}]}{R} + \frac{E_{t+1}[y_{t+2}]}{R}. \tag{2.21}$$

and so on. If we substitute equation (2.21) in (2.20) and use the "Law of iterated expectations", which asserts $E_t[E_{t+1}[X_{t+2}]] = E_t[X_{t+2}]$ (where X_t is a random variable) we get

$$p_t = \frac{E_t[p_{t+2}]}{R^2} + \frac{E_t[y_{t+2}]}{R^2} + \frac{E_t[y_{t+1}]}{R} \tag{2.22}$$

and so, repeating this process for k iterations, we have the general form

$$p_t = \frac{E_t[p_{t+k}]}{R^k} + \sum_{i=1}^{k} \frac{E_t[y_{t+i}]}{R^i}. \tag{2.23}$$

We now assume Hommes (2018) that in a homogeneous, perfectly rational world, traders realize that speculative bubbles do not last forever and, therefore, the price of the risky asset will always converge around a specific value (the fundamental value of the asset), which implies the transversality condition:

$$\lim_{k\to\infty} \frac{E_t[p_{t+k}]}{R^k} = 0. \tag{2.24}$$

As a consequence, for $k \to \infty$, equation (2.23) becomes

$$p_t^* = \sum_{i=1}^{\infty} \frac{E_t[y_{t+i}]}{R^i} \tag{2.25}$$

where p_t^* is the fundamental price, namely the price that would prevail in an efficient market populated only with rational investors, which generally will depend on the stochastic dividend y_t and the discount factor R. We will consider the simplest case of an independent and identically distributed (i.i.d.) dividend process y_t with mean $E_t[y_{t+1}] = \bar{y}$, which implies a constant value for the fundamental price given by

$$p^* = \sum_{i=1}^{\infty} \frac{E_t[y_{t+i}]}{R^i} = \sum_{i=1}^{\infty} \frac{\bar{y}}{R^i} = \frac{\bar{y}}{R-1}. \tag{2.26}$$

2.1.1.4 Adaptive belief system

Now that we know all about the fundamental price, we can get back to our heterogeneous world and focus on the expected value $E_{h,t}[p_{t+1}]$ for each trader type h, which we assume takes the form (for all types of trader)

$$E_{h,t}[p_{t+1}] = E_t[p_{t+1}^*] + f_h(p_{t-1}, p_{t-2}, ..., p_{t-1}^*, p_{t-2}^*, ...). \tag{2.27}$$

Investors believe that the price will deviate from its fundamental value (all investors have the same expectation for this value[9], as well as for the dividend y_t) by some function $f_{h,t} = f_h(p_{t-1}, p_{t-2}, ..., p_{t-1}^*, p_{t-2}^*, ...)$, the *model* used by the investor with

[9] In our opinion, this assumption is too restrictive and we will explain why in Section 2.3.1

strategy 'h', which depend on the prices and fundamental values at previous times. In this case, we can rewrite equation (2.18) as

$$Rp_t = \sum_h^H n_{h,t}(E_t[p^*_{t+1}] + f_{h,t} + E_t[y_{t+1}]) = \sum_h^H n_{h,t}(Rp^* + f_{h,t}), \tag{2.28}$$

where we used $E_t[p^*_{t+1}] = p^*$, $E_t[y_{t+1}] = \bar{y}$ and equation (2.26), and, if we write this last equation in terms of the deviation from the fundamental price

$$x_t = p_t - p^*, \tag{2.29}$$

then we end up with a very compact form for the price dynamics

$$Rx_t = \sum_h^H n_{h,t} f_{h,t}. \tag{2.30}$$

This is the second part of our dynamical system and, together with equation (2.19), we get what Brock and Hommes call the *adaptive belief system*(ABS), that is

$$\begin{cases} Rx_t = \sum_h^H n_{h,t} f_{h,t} \\ n_{h,t} = \frac{1}{1+\sum_{i\neq h}^H e^{-\beta(u_{h,t-1}-u_{i,t-1})}} \end{cases} \tag{2.31}$$

Obviously, this system still needs an explicit form of the model used by the investors of type 'h' ($f_{h,t}$) and their corresponding performance $u_{h,t}$. However, one ends up with a (deterministic) system of nonlinear difference equations whose analytical solution is impossible to derive, so it becomes necessary to adopt qualitative and graphical methods to determine its dynamics. One can perform a stability analysis of the steady states (discovering bifurcations), plot phase diagrams (revealing attractors), or estimate Lyapunov coefficients (which measure the divergence of nearby initial states). These are all analysis tools related to the field of dynamical systems theory.

Social interactions and processes evolving on different time scales As we already pointed out in the introduction chapter, this type of approach is not really meant to account for herding and contagion phenomena, as investors switch between different strategies (which dictate their opinion about the asset's value) only based on how well they perform. As we will see soon (section 2.1.3), very few authors have addressed this issue by including social interactions among investors, and the contribution to this type of extension still remains poor. There is another aspect that none of these models have considered: the time scale in which certain dynamics occur. It can be, for example, that only certain groups of investors switch between different strategies while other investors stick with one strategy only within a period. So, over this period, the investors who only follow one strategy will trade the asset based on how far its price is from its expected value (derived from the adopted strategy), while other investors will adopt different strategies and switch between them based on how they perform *and* the interaction with other investors, potentially trading the asset much more frequently than the former ones (because more active). The system (2.31), and any its extension to social interaction that has been made, does not take all these factors into account, and we will outline an analytically tractable model (in Chapter 3) which will be able to do this. We will make other observations that will support the importance of this aspect in Section 2.2 when we discuss the findings of some works regarding investors trading in a real market.

2.1.2 Brock-Hommes model with fundamentalists and chartists

In this section, we are going to show an example, treated by Brock and Hommes in Brock and Hommes (1998) (section 4.1.2), of a market populated by investors switching between only two strategies, fundamentalists and chartists, whose models are defined as

$$f_{1,t} = 0 \qquad \text{(fundamentalists)} \tag{2.32}$$

$$f_{2,t} = mx_{t-1} \qquad \text{(chartists)} \tag{2.33}$$

Investors using the strategy '1' (fundamentalists) expect the price to be equal to its fundamental value. In fact, from equation (2.27) we have

$$E_{1,t}[p_{t+1}] = E_t[p^*_{t+1}] + f_{1,t} = p^* \tag{2.34}$$

where we used $E_t[p^*_{t+1}] = p^*$, and their corresponding demand (equation(2.15))

$$z_{1,t} = \frac{E_{1,t}[(p_{t+1} + y_{t+1} - Rp_t)]}{a\sigma^2} = \frac{E_{1,t}[p_{t+1}] + E_{1,t}[y_{t+1}] - Rp_t}{a\sigma^2} = \frac{R(p^* - p_t)}{a\sigma^2} \tag{2.35}$$

with $E_{1,t}[y_{t+1}] = \bar{y} = (R-1)p^*$. We see from this last equation that whenever the price p_t is above the fundamental price, the demand takes a negative value, meaning that investors are selling the asset, and the farther the price of the asset is off its fundamental, the larger the amount sold (the opposite when the price is below p^*). Chartists (strategy '2') predict the price to be equal to

$$E_{2,t}[p_{t+1}] = E_t[p^*_{t+1}] + f_{2,t} = p^* + m(p_{t-1} - p^*) = (1-m)p^* + mp_{t-1} \tag{2.36}$$

and several situations may occur depending on the value of m. For $m = 0$, we get the same model used by the fundamentalists, so we will not consider this case (homogeneous expectations). We see from the right-hand side of equation (2.36) that when $0 < m < 1$, the coefficients in front of p^* and p_{t-1} act as proportions, and, because of that, $E_{2,t}[p_{t+1}]$ is derived as a weighted mean of these two prices (if $p_{t-1} > p^*$, then $p_{t-1} > E_{2,t}[p_{t+1}] > p^*$). Chartists here act as 'weak' fundamentalists. They believe the price will converge to its fundamental value but at a slow rate. If $m = 1$, we have simply $E_{2,t}[p_{t+1}] = p_{t-1}$, that is investors predict the price will not change, and for $m > 1$ the price will deviate from p^* (if $p_{t-1} > p^*$, then $E_{2,t}[p_{t+1}] > p_{t-1} > p^*$), in complete contrast with the model used by the fundamentalists. Before looking at the chartists' demand function, it is worth noting that if fundamentalists sell the asset, the chartists have to buy it (and vice versa), as a consequence of the market equilibrium equation (2.18), which appears obvious if we rewrite it as

$$n_{1,t}z_{1,t} + n_{2,t}z_{2,t} = 0 \qquad \text{(from equation (2.16) with } z_s = 0\text{)}. \tag{2.37}$$

We can focus on the case where the price of the asset is always above the fundamental price (if the initial price p_0 is set higher than p^*, the forecasts (2.34) and (2.36) never predict a price below p^*, so $p_t > p^*$ for all t), and analyse the different cases for chartists' demand, given by

$$z_{2,t} = \frac{E_{2,t}[(p_{t+1} + y_{t+1} - Rp_t)]}{a\sigma^2} = \frac{m(p_{t-1} - p^*) - R(p_t - p^*)}{a\sigma^2}. \tag{2.38}$$

Since $z_{1,t} < 0$, we must have $z_{2,t} > 0$, that is the condition

$$\frac{m}{R} > \frac{p_t - p^*}{p_{t-1} - p^*} \tag{2.39}$$

must be satisfied. For $0 < m \leq 1$ (chartists as 'weak' fundamentalists), given that $R > 1$, we have $0 < m/R < 1$, so

$$1 > \frac{p_t - p^*}{p_{t-1} - p^*} \implies p_t < p_{t-1} \tag{2.40}$$

This means that the price p_t will decrease, and because this condition must be valid for all times t, the asset price will inevitably drop to p^* as time goes on. This is true for all $p_0 > p^*$, but one can get the same result for $p_0 < p^*$ (with the price increasing towards p^*) and conclude that p^* is a globally stable steady state, which is not so surprising since all the investors predict the price to converge to its fundamental value.

2.1.2.1 Performance of a strategy

Note that we have not mentioned anything about the performances $u_{h,t}$, which determine the values of the fractions $n_{h,t}$. Things get more complicated when $m > 1$ since the price can go up now and tend to other possible steady states depending on $n_{h,t}$, so we have to decide what factors motivate an investor to choose one strategy over another. A natural way for investors to determine which one could be more successful in the near future is to compare their corresponding past profits. We have already defined the realized profits when we derived the wealth dynamics (2.5), as $\Pi_{t+1} = (p_{t+1} + y_{t+1} - Rp_t)z_t$, but this represents the *potential* profit, since this definition involves random variables. The *past* realized profits of each strategy 'h', after the auctioneer announced the price at time t, is given by

$$\Pi_{h,t} = (p_t + y_t - Rp_{t-1})z_{h,t-1} = (x_t - Rx_{t-1})(\frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}) \tag{2.41}$$

where in the second equation, we skipped some straightforward algebraic steps expressing all in terms of the deviation x_t and models $f_{h,t}$. It is important to remember that these realized profits are the additional profit over that from the risk-free asset. A negative value for $\Pi_{h,t}$ should not necessarily result in a loss in investor's wealth; it may just be a lost opportunity to make more profits. We set for simplicity $a\sigma^2 = 1$ and evaluate the realized profits of each strategy in the case $p_t > p^*(\implies x_t > 0)$ for all t. For fundamentalists, we have

$$\Pi_{1,t} = Rx_{t-1}(Rx_{t-1} - x_t) \tag{2.42}$$

and it is always positive unless

$$Rx_{t-1} < x_t \implies R < \frac{p_t - p^*}{p_{t-1} - p^*} \tag{2.43}$$

that is if the price goes up 'too' fast (i.e. if the price goes up faster than the rate of return from the risk-free asset). This makes sense since a fundamentalist would sell an asset whose value is increasing, which is a bad idea (an investor would waste the opportunity to make more profits by selling the asset at a higher price). We have put quotes on 'too' because a fundamentalist can make profits even if the price increases because selling shares of the risky asset implies buying the risk-free ones, which could be more profitable (it happens when the rate of return of the risk-free asset is higher than the risky one). For chartists, instead, we have

$$\Pi_{2,t} = (x_t - Rx_{t-1})(mx_{t-2} - Rx_{t-1}) \tag{2.44}$$

and this looks harder to analyse because there are three variables involved (x_{t-2}, x_{t-1} and x_t) and two parameters, but finding the condition for positive realized profits is easy knowing that $mx_{t-2} - Rx_{t-1} = z_{2,t-1} > 0$, so that $\Pi_{2,t} > 0$ whenever (2.43) is satisfied, which is the condition for which fundamentalists have negative realized profits (chartists buy the asset and they make more profits as its value increases). In fact, by comparing both strategies, one can find out that:

$$\begin{cases} \Pi_{1,t} > \Pi_{2,t} \iff Rx_{t-1} > x_t \\ \Pi_{1,t} < \Pi_{2,t} \iff Rx_{t-1} < x_t \end{cases} \tag{2.45}$$

Brock and Hommes assume the performance functions $u_{h,t}$ to be equal to the realized profits $\Pi_{h,t}$ adding a cost c whenever investors choose the fundamentalist strategy due to the effort to gather information on economic fundamentals so that we have

$$u_{1,t} = \Pi_{1,t} - c = Rx_{t-1}(Rx_{t-1} - x_t) - c \tag{2.46}$$

$$u_{2,t} = \Pi_{2,t} = (x_t - Rx_{t-1})(mx_{t-2} - Rx_{t-1}) \tag{2.47}$$

If we compare these performances, we get (with $x_t > 0$)

$$\begin{cases} u_{1,t} > u_{2,t} \iff mx_{t-2}(Rx_{t-1} - x_t) > c \\ u_{1,t} < u_{2,t} \iff mx_{t-2}(Rx_{t-1} - x_t) < c \end{cases} \tag{2.48}$$

meaning that not only must the condition (2.45) be satisfied for investors to choose, for example, the fundamentalist strategy, but also the deviation of the price from the fundamental (x_{t-2}) has to be large enough to cover the cost c (we can see here that the mathematical role of the cost c is to avoid the investors adopting the fundamentalist strategy as soon as the price decreases or slowly increases, thus preventing the price of the asset from deviating from its fundamental value too much). If we substitute the equations (2.32), (2.33), (2.46) and (2.47) in the ABS ((2.31)), we finally obtain

$$\begin{cases} x_t = \frac{m}{R} n_{2,t} x_{t-1} \\ n_{2,t} = \frac{1}{1+e^{-\beta[\Delta u_{t-1}]}} \\ \Delta u_t = u_{2,t} - u_{1,t} = m(x_t - Rx_{t-1})x_{t-2} + c \end{cases} \tag{2.49}$$

which is a one dimensional third-order difference equation $x_t = \Phi(x_{t-1}, x_{t-2}, x_{t-3})$.

2.1.2.2 Dynamical system analysis

We are finally ready to analyse our model, whose dynamics will be studied in terms of the intensity of choice β. We will differ from the analysis performed by Brock and Hommes (1998), being more accurate and drawing the same conclusions step by step. Moreover, we will follow Elaydi (2005) and Elaydi (2007) for the definitions and theorems used in discrete dynamical systems theory. First of all, we transform our system (2.49) into a three-dimensional first-order difference equation by defining the new variables $y_t = x_{t-1}$ and $z_t = y_{t-1}$, so that we get the system:

$$\begin{cases} x_t = \frac{m}{R} x_{t-1} \frac{1}{1+e^{-\beta[(x_{t-1}-Ry_{t-1})mz_{t-1}+c]}} \\ y_t = x_{t-1} \\ z_t = y_{t-1} \end{cases} \tag{2.50}$$

which is now a first-order difference equation $(x_t, y_t, z_t) = F(x_{t-1}, y_{t-1}, z_{t-1})$, with the map F given by

$$F(x,y,z) = \left(x \frac{m}{R} \frac{1}{1+e^{-\beta[(x-Ry)mz+c]}}, x, y \right), \tag{2.51}$$

Fixed points The first step of our analysis is to find the steady states (or fixed points) of the map (2.49), that is all those points (or vectors) $x^* \in \Re^3$ such that

$$x^* = F(x^*). \tag{2.52}$$

For the map (2.51), we must find a vector $\vec{x}^* = (x^*, y^*, z^*)$ such that

$$\begin{cases} x^* = x^* \frac{m}{R} \frac{1}{1+e^{-\beta[(x^*-Ry^*)mz^*+c]}} \\ y^* = x^* \\ z^* = y^* \end{cases} \tag{2.53}$$

that is, finding x^* that solves

$$x^* = x^* \frac{m}{R} \frac{1}{1+e^{-\beta[(1-R)m(x^*)^2+c]}}. \tag{2.54}$$

We clearly see from (2.54) that the fundamental value $x^* = 0$ is always a steady state, while for $x^* \neq 0$, we have

$$(x^*)^2 = \frac{1}{\beta m(R-1)} \left[ln \left(\frac{m}{R} - 1 \right) + \beta c \right] \tag{2.55}$$

and it admits a solution if and only if

$$\beta c > ln\left(\frac{R}{m-R}\right) \tag{2.56}$$

For $m < R$, the logarithm in equation (2.55) is not defined, meaning that the fundamental steady state $x^* = 0$ is the only fixed point, for $m > 2R$ the condition (2.56) is always satisfied, since $\beta > 0$, and for $R < m < 2R$ we have bifurcation for $\beta = \beta_1$ equal to

$$c\beta_1 = ln\left(\frac{R}{m-R}\right) \tag{2.57}$$

where two non-fundamental steady states are created (so we have a *pitchfork* bifurcation, Elaydi (2007)). To summarize, we have

- For $m < R$ the only steady state is the fundamental value $x^* = 0$.
- For $R < m < 2R$, if (2.56) is satisfied, then there are three steady states, one fundamental and two non-fundamental ($\pm x^*$) given by the equation (2.55), otherwise the only steady state is the fundamental value.
- For $m > 2R$, there are three steady states, one fundamental and two non-fundamental ($\pm x^*$).

Stability of fixed points The next step is the study of the stability of these fixed points, that is, analysing the behaviour of the system for small deviations from its steady states (which implies a local analysis). To do that, we have to find the eigenvalues of the Jacobian matrix of the function (2.51) calculated at the fixed point x^*, defined as

$$JF(x,y,z)\big|_{x^*} = JF(f_x,f_y,f_z)\big|_{x^*} = \left.\begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial z} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial z} \\ \frac{\partial f_z}{\partial x} & \frac{\partial f_z}{\partial y} & \frac{\partial f_z}{\partial z} \end{pmatrix}\right|_{x^*} \tag{2.58}$$

For our system, we obtain

$$JF(x,y,z) = \begin{pmatrix} \frac{m}{R(1+\psi)}\left(1+\frac{\beta m x z \psi}{1+\psi}\right) & -\frac{\beta m^2 x z \psi}{(1+\psi)^2} & \frac{\beta m^2 x(x-Ry)\psi}{R(1+\psi)^2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.59}$$

with $\psi = exp\left\{-\beta\left[(x-Ry)mz + c\right]\right\}$, and for the fundamental steady state, $x^* = 0$, the Jacobian simply becomes

$$JF(x,y,z)|_{x^*=0} = \begin{pmatrix} \frac{m}{R(1+\psi_f)} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.60}$$

where $\psi_f = exp(-\beta c)$, whose characteristic equation is given by

$$q(\lambda) = \lambda^2 \left[\frac{m}{R(1+\psi_f)} - \lambda \right], \tag{2.61}$$

so we have a double eigenvalue $\lambda_1 = \lambda_2 = 0$ and the eigenvalue

$$\lambda_3 = \frac{m}{R(1+\psi_f)}. \tag{2.62}$$

For a fixed point to be stable, we must have $|\lambda| < 1$ for all eigenvalues (Elaydi (2007)). From equation (2.62), we see that for $m < R$, the fundamental steady state is globally stable ($0 < \psi_f < 1$), for $R < m < 2R$ it is globally stable in case the condition (2.56) is not satisfied and unstable otherwise, and for $m > 2R$ the fixed point is unstable. For the non-fundamental steady states (2.55), the Jacobian is

$$JF(x,y,z)|_{\pm x^*} = \begin{pmatrix} 1+b & -Rb & -b(R-1) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.63}$$

where b is given by

$$b = \frac{m-R}{m(R-1)} \left[ln\left(\frac{m}{R} - 1\right) + \beta c \right] \tag{2.64}$$

and the characteristic equation is

$$q(\lambda) = \lambda^3 - (1+b)\lambda^2 + bR\lambda + b(R-1) \tag{2.65}$$

At the pitchfork bifurcation $\beta = \beta_1$, we have $b = 0$, and the characteristic equation has double eigenvalues $\lambda_1 = \lambda_2 = 0$ and one eigenvalue $\lambda_3 = 1$ (in this case, one cannot establish if the fixed point is stable or not, because the highest eigenvalue is equal to 1, Elaydi (2007)). For $\beta > \beta_1$, $b > 0$ and evaluating the eigenvalues directly becomes hard, so, to establish the stability of these steady states, we have to use Schur-Cohn Criterion (Elaydi (2005)), which states that the zeros of the characteristic polynomial:

$$q(\lambda) = \lambda^3 + q_1\lambda^2 + q_2\lambda + q_3 \tag{2.66}$$

are inside the unit circle ($|\lambda| < 1$) if and only if the following conditions are satisfied:

1. $q(1) > 0$,
2. $q(-1) < 0$,
3. the matrices

$$B^{\pm} = \begin{pmatrix} 1 & 0 \\ q_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & q_3 \\ q_3 & q_2 \end{pmatrix}$$

are positive innerwise ($|B^{\pm}| > 0$).

For the characteristic equation (2.65) we have

$$q(1) = 2b(R-1) > 0 \tag{2.67}$$

$$q(-1) = -2(b+1) < 0 \tag{2.68}$$

so the first and the second conditions are satisfied for all $\beta > \beta_1$. For the third condition, we have

$$|B^{+}| = b^2(R-1)(2-R) + b(2R-1) + 1 > 0 \tag{2.69}$$

which is always true since $1 < R < 2$ [10], and

$$|B^{-}| = -b^2R(R-1) - b(2R-1) + 1 \tag{2.70}$$

is positive if

$$b^2R(R-1) + b(2R-1) - 1 < 0 \tag{2.71}$$

or, equivalently, if

$$b < \frac{1 - 2R + \sqrt{8\left(R - \frac{1}{2}\right)^2 - 1}}{2R(R-1)} \tag{2.72}$$

Using equation (2.64) in (2.72), we can express this condition as a function of β, that is

$$\beta c < \beta_2 c = \frac{m}{2R(m-R)} \left(1 - 2R + \sqrt{8\left(R - \frac{1}{2}\right)^2 - 1}\right) - ln\left(\frac{m}{R} - 1\right) \tag{2.73}$$

so the non-fundamental steady states are stable for $\beta < \beta_2$, where β_2 represents a second bifurcation of our system. For $\beta > \beta_2$, the conditions of the Schur-Cohn criterion are not satisfied, meaning that at least one eigenvalue crosses the unit circle

[10] The second inequality is not assumed by Brock and Hommes, maybe because from an economic point of view, it does not make much sense for a risk-free asset to double or more its value each period, so they might implicitly have taken this for granted.

($|\lambda| > 1$). If this eigenvalue was real, then for $\beta = \beta_2$ we would have $\lambda = \pm 1$, but we know from equation (2.67) and (2.68) that this is impossible, so the eigenvalue must be a complex number $\lambda \in \mathbb{C}$. Therefore, we have two complex eigenvalues (one conjugate of the other) and one real eigenvalue, and for $\beta = \beta_2$ we have a *Neimark-Sacker* bifurcation (Elaydi (2007)). For $m > 2R$ (a case not taken into consideration by Brock and Hommes), we see that the logarithm in equation (2.73) takes positive values, so it might be possible that, for a suitable choice of m and R, one ends up with the condition $\beta_2 < 0$, which implies that the non-fundamental steady states are always unstable. By requiring $\beta_2 > 0$ in equation (2.73) and defining a constant $d \geq 2$ such that $m = dR$, we can establish the *necessary* condition on R for which the non-fundamental steady states are stable, namely

$$R > \rho = \frac{2-\epsilon}{2-\epsilon^2} \qquad \left(\epsilon = \frac{(d-1)ln(d-1)}{d} + 1\right) \tag{2.74}$$

For example, if we choose $d = 2$ ($\Longrightarrow$ $m = 2R$) it is easy to see that the condition (2.74) simply becomes

$$R > 1 \qquad (\text{for } m = 2R) \tag{2.75}$$

and it is always satisfied since $R > 1$ by definition. For $d > 2$, ρ is an increasing function of d, and the lower limit of R must be higher for the non-fundamental steady states to be kept stable. So, we can update our summary by saying that:

- For $m < R$ the only steady state is the fundamental value $x^* = 0$ and it is globally stable.
- For $R < m < 2R$, if $\beta < \beta_1$ the only steady state is the fundamental value (globally stable), if $\beta_1 < \beta < \beta_2$, then there are three steady states, one fundamental (locally unstable) and two non-fundamental ($\pm x^*$) given by the equation (2.55) (locally stable), while for $\beta > \beta_2$ the two non-fundamental steady states become unstable.
- For $m \geq 2R$, there are three steady states, one fundamental (locally unstable) and two non-fundamental $\pm x^*$, which are (locally) stable if the conditions (2.73) and (2.74) are satisfied.

Simulation To conclude this section, we will perform some simulations of this model by setting the parameter values equal to those used by Brock and Hommes in their work (section 4.1.2 in Brock and Hommes (1998)), that is $m = 1.2$, $c = 1$, and $R = 1.1$, and express the results in terms of deviation of the price x_t and difference in fractions, defined as

$$\Delta n_t = n_{1,t} - n_{2,t} = 1 - 2n_{2t} = tanh\left[-\frac{\beta}{2}(\Delta u_t)\right] \tag{2.76}$$

where $\Delta u_t = u_{2,t-1} - u_{1,t-1} = m(x_t - Rx_{t-1})x_{t-2} + c$ is the difference in performance between the two strategies ($u_{1,t-1}$ and $u_{2,t-1}$ are given by equations (2.46) and (2.47)). In Figure 2.2, we show the deviation x_t, the difference Δn_t and the phase space $\Delta n_t - x_t$, for different values of β. We are in the case where $R < m < 2R$, and we know from our analysis that for $0 < \beta < \beta_1$, the price p_t converges to the fundamental value p^* (no matter what the initial conditions are, since it is a global fixed point). In this limit, the difference in performance tends, in the long run, to ($x^* = 0$)

$$\Delta u^* = u_2^* - u_1^* = 1 \tag{2.77}$$

so the difference in fractions becomes

$$\Delta n^* = tanh\left[-\frac{\beta}{2}\right] \tag{2.78}$$

and starting from $\beta = 0$, we have $\Delta n^* = 0$ ($\implies n_1^* = n_2^* = 0.5$), meaning that there are too many fundamentalists for the price to keep on a non-fundamental value, so it drops to p^*. As β increases (see graphs for $\beta = 1.5$ in Figure 2.2), investors choose, wisely, the chartist strategy ($\Delta n^* < 0 \implies n_2^* > 0.5$) because they 'begin to understand' that there is no sense paying a cost ($c = 1$) without any profit. So, as the system converges on its steady state, more investors end up as chartists, and the price tends to the fundamental value more and more slowly, until $\beta \to \beta_1 = ln(11) \approx 2.398$ (from equation (2.57)), where the fraction of chartists reaches a value high enough to prevent the convergence $x_t \to 0$. We are now in the presence of three steady states ($\beta_1 < \beta < \beta_2$), two stable non-fundamental states $\pm x^*$ (given by equation (2.55)) and one unstable (the fundamental one). Whether the system is in one state or another depends on the initial conditions, that is, a positive or negative deviation of the price from the fundamental value p^* at time $t = 0$, namely x_0. So, for $x_0 \neq 0$, the system converges on one of the two stable steady states (if $x_0 > 0$, then $x_t \to +x^*$ as $t \to +\infty$, otherwise $x_t \to -x^*$), and for $x_0 = 0$, the system does not deviate from the fundamental value. As the intensity of choice increases, assuming $x_0 > 0$, the deviation x_t converges to higher values ($\beta = 2.5$ in Figure 2.2), and the difference in performance starts decreasing (or, equivalently, the fundamentalist's performance starts becoming comparable to the chartist's one), since

$$\Delta u^* = u_2^* - u_1^* = \frac{1}{\beta}\left[ln\left(\frac{m}{R} - 1\right)\right] = \frac{ln(11)}{\beta} \tag{2.79}$$

and investors start thinking of a change in strategy as $\beta \to \beta_2 \approx 3.331$ (from equation (2.73)). For $\beta > \beta_2$, the two non-fundamental steady states become two attracting

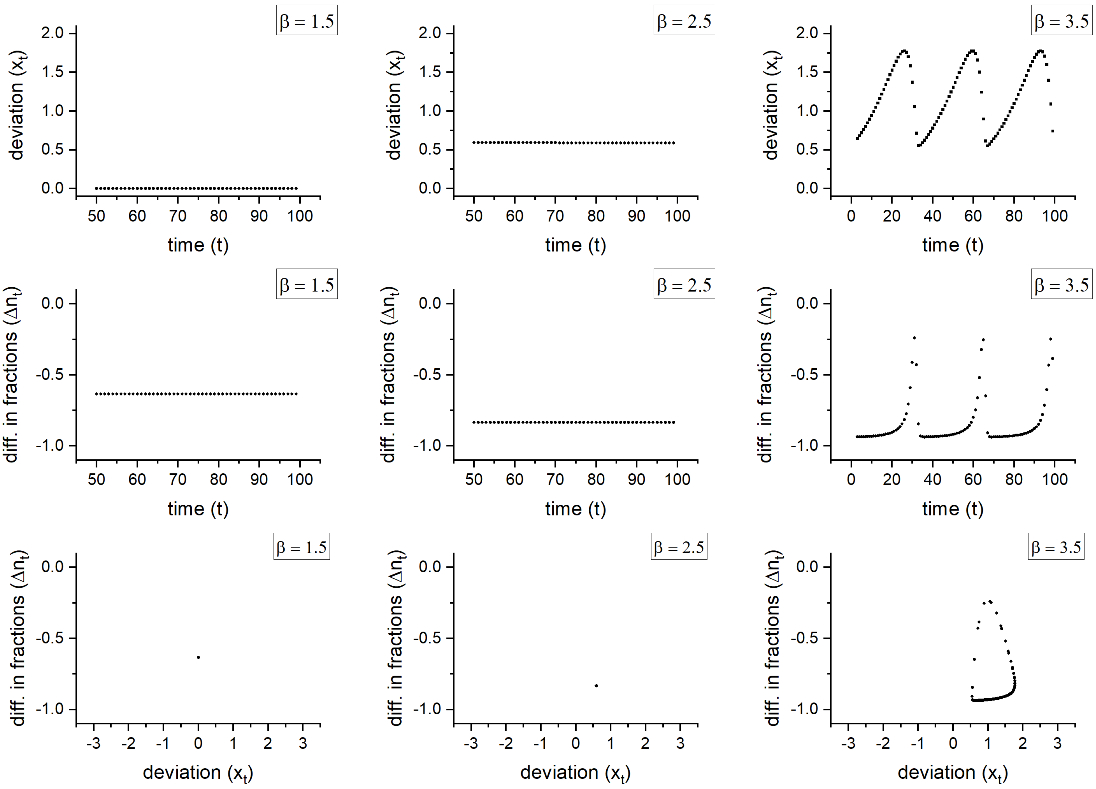


FIGURE 2.2: The deviation x_t, the difference Δn_t and the phase space $\Delta n_t - x_t$, for $\beta = 1.5$, $\beta = 2.5$ and $\beta = 3.5$, using BH model. Here, we show only the case $x_0 > 0$, since for $x_0 < 0$, the deviation x_t evolves anti-symmetrically with respect to the axis $x = 0$). The first 50 time steps have been removed to show only the system's evolution once it has reached stability.

circles in the phase space $\Delta n_t - x_t$ (both x_t and Δn_t start oscillating), and we already knew that, because for $\beta = \beta_2$ a *Neimark-Sacker* bifurcation occurs. Let us focus on the case when $\beta = 3.5$ (Figure 2.2). Here, the system's trajectory in the phase space $\Delta n_t - x_t$ rotates counter-clockwise. When the deviation is close to $x_t \sim 0.5$, almost all investors are chartists (it is not convenient being fundamentalist now since one should pay a cost c to obtain information about how markets work, and the profits are not even high enough to cover it) and, because of that, the price increases (most investors are buying the stock, but with very low demand) up to $x_t \sim 1.8$. At this point, the price is high enough for investors to become fundamentalists and start selling, which causes a drop of the price itself down to $x_t \sim 1$, where the difference in fractions is approximately zero (there are almost the same amount of fundamentalists and chartists). The price keeps falling, and being a fundamentalist no longer brings profits. Investors start adopting the chartist strategy, and the system restarts its cycle.
Figure 2.3 illustrates the bifurcation diagram for the deviation x (that is, the range of values x_t taken in a single simulation) with respect to β, which summarizes what has

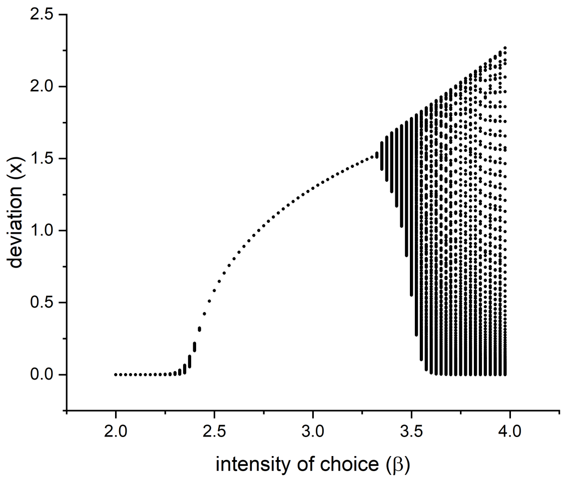


FIGURE 2.3: Bifurcation diagram w.r.t β. The number of time steps for each simulation is $T = 1000$, of which the first $T_c = 400$ time steps have been excluded to remove the initial transition(when the system has not reached the steady state yet).

already been said. It has been built by collecting only the values x_t (with $x_0 > 0$) once the system converged to a steady state, that is, considering only values after a time T_c (transition time) and within a period T, which must be long enough so that x_t takes (almost) all allowed values.

2.1.3 Extension to social interactions

In this section, we will review all past works that extended or modified the Brock and Hommes model, adding social interactions among investors and including a stream of works that differs from this model in many ways. Some of these do not even mention that they deal with social interactions, but since they include 'imitative' behaviour in their models, we will also consider them as a contribution in this context.

2.1.3.1 Mean choice imitation

The first extension was done by Chang (2007) (Chang (2014)), whose primary purpose was to combine social interactions and heterogeneous beliefs in the asset pricing model of Brock and Hommes (1998) in order to study the resulting price dynamics. In his work, following Brock and Durlauf (2001a) (Brock and Durlauf (2001b)), Chang expressed the performance measure $u_{h,t}$ by adding, in addition to the private utility represented by the past realized profits $\Pi_{h,t}$, a social utility given by

$$\Sigma_{h,t} = J w_{h,t} \Delta n_t^e \tag{2.80}$$

where J represents the strength of the social interactions (that is, how much importance investors attach to the strategies adopted by other investors), $w_{h,t}$ is a variable that takes two values, $w_{1,t} = 1$ for strategy '1' and $w_{2,t} = -1$ for strategy '2', and Δn_t^e is the conditional expectation that investors place on the choice of others at time t[11]. The author assumes that Δn_t^e is the last period's mean choice level (difference in fractions)

$$\Delta n_t^e = n_{1,t-1} - n_{2,t-1} \tag{2.81}$$

so that the performance measure $u_{h,t}$ of strategy 'h' at time t becomes

$$u_{h,t} = \Pi_{h,t} + \Sigma_{h,t} = \Pi_{h,t} + J w_{h,t}(n_{1,t-1} - n_{2,t-1}). \tag{2.82}$$

Chang found that the possible steady states of the mean choice level Δn^* and the price dynamics depended not only on the 'exogenous' type of social interactions, tuned by the parameter J, but also on an 'endogenous' one, that arises as a consequence of merging social interactions with heterogeneous beliefs among traders. In a paper published later (Chang (2014)), Chang studied his model focusing on herd behaviour, giving it a specific definition[12], and bubbles (deviation from the fundamental value), showing that herd behaviour arises naturally when the value J is high enough and that even with extremely small bubbles, many investors may engage in herding if the social interactions are strong enough.

2.1.3.2 Local network interactions

A second line of work concerns the embedding of a social network in Brock and Hommes (1998) developed by Panchenko et al. (2007) (Gerasymchuk and Pavlov (2010), Panchenko et al. (2013)). They analyse the effects of different network topologies (fully connected, regular lattice, small-world, random graph), where the nodes represent investors, who can observe the performance measure of a strategy 'h' only if the investors who reside on the nodes directly connected with them adopted that strategy. So, differently from Brock and Hommes (1998), investors might not be able to compare all different strategies. Investors choose, for example, the chartist type (strategy '2') at time t, based on their neighbourhood, with a probability $P_{2,t}^i$ given by

[11] A stream of works adopted a similar approach, that is worth mentioning since they belong to the 'discrete choice approach' category (Franke and Westerhoff (2009)), namely Leombruni et al. (2003), Gallegati et al. (2010), Bischi et al. (2006). Since they are too different from Brock and Hommes (1998) (they assume homogeneous beliefs and no learning mechanisms), we decided not to include them in this review.

[12] That is, a trader engages in herd behaviour at a steady state if her belief choice agrees with the mean choice level in the market.

$$P^i_{2,t} = I_{i,t-1}\Pi_{j\in G_i} I_{j,t-1} + [I_{i,t-1}(1 - \Pi_{j\in G_i} I_{j,t-1}) + (1 - I_{i,t-1})(1 - \Pi_{j\in G_i}(1 - I_{j,t-1}))]\Delta_t \tag{2.83}$$

where $I_{i,t}$ is 1 if the investor i was a chartist in time t (0 if fundamentalist), G_i is investor's i neighbourhood and $\Delta_t = e^{\beta u_{2,t}}/(e^{\beta u_{2,t}} + e^{\beta u_{1,t}})$. Equation (2.83) may take three values: 0, 1 and Δ_t. If at time $t-1$ the investor was a chartist and if there were no fundamentalists in her neighbourhood, then only the first term in (2.83) survives, and it is equal to 1 (so, at time t, investor i will be again a chartist). If, instead, at time $t-1$, the investor was a fundamentalist, and if there were no chartists in her neighbourhood, then all terms are equal to 0 (so, at time t, investor i will be again a fundamentalist). In all other conditions, one has $P^i_{2,t} = \Delta_t$, which happens when there are at least two investors with different strategies in the same neighbourhood (simply said, an investor compares the strategies only if she realizes that her neighbour's strategy is different from the one adopted by her). As pointed out by Panchenko et al. (2013), there are two network topologies where it is possible to reduce the dimensionality of the system ($P^i_{2,t} \rightarrow n_{2,t}$), thus avoiding keeping track of every agent: the fully connected network and the random graph. For a fully connected network, one recovers the Brock and Hommes model, and in a random graph, one can exploit the fact that the neighbourhoods are not clustered, which makes the network-induced dependence between any two nodes fairly small, so the chartists' proportion evolution can be approximated by

$$n_{2,t} = n^{k+1}_{2,t-1} + [1 - n^{k+1}_{2,t-1} - (1 - n_{2,t-1})^{k+1}]\Delta_t \tag{2.84}$$

where k is the average degree of a node. Panchenko et al. performed simulations with different topologies, for which it is difficult to reduce the dimensionality of the system, i.e. regular lattice and small-world, and focused on four different financial strategies (where it is still possible to adopt an analytical approach, but it is more complicated). As a result, they observed that the system's stability depends on the network's parameters and the latency in the information transmission, which is very high for regular lattice and small-world networks, gives rise to greater instabilities and higher deviations in the price dynamics.

2.1.3.3 Heterogeneous fundamentalists and gurus

The next stream of work differs somewhat from the Brock and Hommes model, but it still belongs to the class of heterogeneous agent models, so it is worth discussing them. Naimzada and Ricchiuti (2008) developed a model where the heterogeneity among investors is not given by the different strategies present in a financial market but emerges as a consequence of two experts (e.g. financial advisors), which other investors imitate. Both experts are fundamentalists (simply called *expert 1* and *expert 2*) but with different expectation about the fundamental value (respectively, p_1^* and p_2^*) so that investors' switching mechanisms is driven by the expert's ability, given by the distance between his fundamental value and the current price. The market clearing mechanism adopted by the authors, the equation that determines the evolution of the price, is the *market maker* one, which is one of the most used mechanisms together with the Walrasian auctioneer (the only difference between these two is that the market maker allows the excess demand to be different from zero). In their model, the market maker follows the rule:

$$p_{t+1} = p_t + \mu \left[n_{t+1} z_{1,t} + (1 - n_{t+1}) z_{2,t} \right] \tag{2.85}$$

where $\mu > 0$ is the speed of adjustment, $z_{i,t}$ is the *expert i*'s demand (whose expression is the same as in Brock and Hommes (1998)), and n_{t+1} is the fraction of investors that imitate *expert 1* at time t, given by

$$n_{t+1} = 1 - \frac{(p_1^* - p_t)^2}{(p_1^* - p_t)^2 + (p_2^* - p_t)^2}. \tag{2.86}$$

Equations (2.85) and (2.86) correspond to a one-dimensional non-linear map which was analysed by the authors, showing that (Naimzada and Ricchiuti (2008)) (i) market instability and periodic, or even, chaotic price fluctuations can be generated; (ii) conditions exist under which an expert can drive another expert out of the market; (iii) two experts can survive when the dynamics system either generates a period-doubling bifurcation around an attractor or when a homoclinic bifurcation leads to the merging of the two attractors; (iv) the reaction to misalignment of both market maker and agents plays a central role. This model has been modified several times over the years by the same authors and also by others. In 2009, Naimzada and Ricchiuti published another work (Naimzada and Ricchiuti (2009)) modifying their previous model by changing the switching mechanisms (2.86) to a form very similar to those used by Brock and Hommes (the evolution of the fraction of fundamentalists). In 2014, Gori et al. (2014) developed a continuous time version of Naimzada and Ricchiuti (2008). In 2015, Naimzada and Pireddu (2015) modified the rule used by market maker (2.85), where the excess demand (the second term in the r.h.s. of the equation) has been replaced by a sigmoidal function in order to prevent the price from diverging, and let the fundamental values estimated by the two experts, p_1^* and p_2^*, evolve in time according to a linear combination of the switching mechanism as in Brock and Hommes (1998). Lastly, the two most recent works by Cavalli et al. (2017) and Cavalli et al. (2018) aimed at unifying the models Naimzada and Pireddu (2015) and De Grauwe and Rovira Kaltwasser (2012), which give rise to the same phenomena but using different ways to portray the emergence of, what is called, 'animal spirits'[13] in financial markets, through alternating waves of optimism/pessimism about the fundamental value.

2.1.3.4 Fundamentalists, chartists, and imitators

The last works, published recently, are completely different from the Brock and Hommes framework but are very interesting as heterogeneous agent models that consider imitating investors in the financial market. Brianzoni and Campisi (2020), and Campisi and Tramontana (2020) propose a financial market populated by three types of agents, namely fundamentalists, chartists, and imitators, where the latter submit buying/selling orders according to different trading rules using a 2D piecewise linear discontinuous map. The price is again regulated by a market maker who follows the rules:

$$p_{t+1} = p_t + (z_{1,t} + z_{2,t} + Nz_{3,t}) \tag{2.87}$$

where N represents the number of imitators (the number of chartists and fundamentalists is normalized to 1) and $z_{3,t}$ is the demand function of imitators. The demand functions of fundamentalists and chartists are similar to those used by Brock and Hommes (1998), while imitators look only at the price at time t and $t-1$, such that if p_t is closer than p_{t-1} to the fundamental value, then they conclude that fundamentalists' strategy has been successful and they imitate them at time $t+1$. The demand function z_{3t} is then given by a piecewise linear map

$$z_{3,t} = \begin{cases} \nu(p^* - p_t) + \mu_1 & \text{if } |p^* - p_t| \geq |p^* - p_{t-1}| \\ -\nu(p^* - p_t) + \mu_2 & \text{if } |p^* - p_t| < |p^* - p_{t-1}| \end{cases} \tag{2.88}$$

where the $\nu > 0$ measures the reactivity of imitators to the market signal, μ_1 and μ_2 the costs to imitate the best strategy. The map (2.88) combined with equation (2.87) represents a two-dimensional dynamical system, piecewise linear and discontinuous, which has been analysed by the authors, who showed that new interesting economic scenarios emerge from this system.

Contrarian fundamentalists A financial figure that any of the models discussed above has not considered is the *contrarian fundamentalist*, that is, an investor who expects the asset's price to move away from its fundamental value or deliberately wants it to happen (so as to take advantage of these mispricings), acting as a

[13] It is a term used by Keynes (1937) to describe "a spontaneous urge to action rather than inaction" due to optimism rather than mathematical expectations (so human action moved by instincts rather than logic).

misinforming source for imitators/followers. We will analyse our model by taking this group of investors into account in Chapter 5, along with the interaction between *pure fundamentalists* (who believe the asset's price will move towards its fundamental value and *followers* (who refer to the information shared by both groups).

Final observations

As we have seen in this section, confirming what Dieci and He (2018) and Anufriev et al. (2018) already pointed out, there is an unsatisfactory amount of work regarding heterogeneous agent models in financial markets with social interactions, at least from an analytical point of view. Unfortunately, the lack of possible empirical supports and the complexity of adding parameters while keeping the model analytically tractable have prevented many model-makers from embarking on this feat. In fact, despite being very interesting, all the works cited above lack empirical validation and use a structure of the financial market that is anything but realistic. It is widely known that a stock market is populated by investors, not only heterogeneous in their beliefs but also with regard to social position, like householders, financial corporations, banks and more (Baltakys et al. (2018)). Because of this added heterogeneity, one has to think of a stock market where investors are placed on different networks (with different topologies), based on their social position, with their own social/financial rules[14]. A model that takes these details into account and is, at the same time, analytically tractable depends to some extent on the skill of the developer, but "without an empirical verification of the underlying assumptions [of the model], the assessment of the real amount of heterogeneity present in the market and the detection of its role in price formation dynamics lack empirical support." (Tumminello et al. (2012), p. 2). Tumminello et al. (2012), back in 2012, provided two reasons why such empirical investigations were challenging to carry out: the confidentiality of the investor trading activity and, even when data was available, the complex identification of a group of traders investing similarly (assessing, so, the possible links in the network). Since then, several authors have worked in an attempt to overcome these obstacles by employing techniques able to do that and even go further by investigating how external sources (e.g. news, social networks) affect the behaviour of different categories of investors (Lillo et al. (2014), Siikanen et al. (2018)). With all this new information, it becomes more worthwhile now to engage more in developing more realistic agent-based models that can be compared with empirical data. The hope is that this will finally allow establishing what the relative impact of each financial network component actually is that could destabilize the market, giving rise to bubbles, increasing volatility, and even causing extreme events like stock market crashes. In light of the above, in B.2, we attempted to develop a model of a market

[14] Naimzada and Ricchiuti (2008) are the only authors using two different class of investors, namely financial experts (e.g. fund managers) and imitators (e.g. retail investors).

populated by two boundedly rational categories of investors, institutional and retail, who are heterogeneous in their beliefs and susceptible to social influence. Moreover, we represented the interaction among all traders through a social network, trying out different topologies for both these two categories of investors and possibly a network that connects them (so as to derive a very general form for a fraction of an adopted strategy n_{ht}).

2.2 The ecosystem of a real stock market

In this section, we will discuss the findings of a series of works regarding data analysis of investors' trading activity in the Finnish stock market. The reason for that is twofold: getting more information about the structure of the financial market and focusing on phenomena related to our research question, that is, investor behaviour potentially affected by exogenous sources triggering social biases (e.g. social interaction, mass media, social media) that might have a significant impact on the stock price. The section is divided into two subsections: in the first section, we show a hypothetical illustration of this market (suggested by the outcomes of these works) and discuss its most important aspects, and in the second one, we will draw several conclusions that will lead us to interpret this system differently from how most heterogeneous agent models (HAMs) have done so far. We will formalize our interpretation mathematically in the next chapter.

Beyond social biases Even though social influence is a critical factor in destabilizing financial markets, giving rise to bubbles (or bursts) and, eventually, anomalous volatility, it is also true that investors' intrapersonal biases (e.g. regret, greed) might play an important role in this regard. Actually, until recently, behavioural finance, which analyses the psychological aspects of investors in trading activities, was focused more on the latter than on social biases to explain such phenomena, which go against the *rational* expectation paradigm asserted by mainstream finance[15]. In carrying out our analysis of this system, such as to be sufficiently robust to draw some reliable conclusions, we believe that it is fundamental to take into account *all* possible factors that could affect investors, even if these are not directly related to the topic of our research question. This section, in which we will temporarily digress from our research topic, will lend itself well to showing why this is so important. Our discussion below will provide much evidence of which multiple behavioural biases (or conditions dictated by a specific investment strategy) might have a similar (or

[15] It was only in 2015 when Hirshleifer (2015), who is among the leading exponents of this field, pointed out that it was necessary to start moving to *social* finance (a term he coined) and expand the studies along this direction. Not surprisingly, it was (and still is) more natural for econophysicists, instead, to build their behavioural models based mainly on direct interactions between investors as if they were the 'interacting particles' of a physical system.

opposite) effect on the stock price and, thus, lead to different interpretations of the same result. In line with our research project, we will try to argue and raise questions about the role of social biases in all these occasions and how they could interfere with all other factors. On top of that, we will show that two assumptions underlying HAMs might not be consistent with some of the most recent findings regarding data analysis on investor behaviour. The first is related to investors' estimation of the fundamental value of an asset, and the other one is related to the time scale related to the switching mechanism between different strategies (as we briefly pointed out in Paragraph 2.1.1.4). This will bring us to describe the system slightly differently from what argued by them and, at the same time, more in tune with the aim of this project.

The ecosystem of the Finnish stock market

The data used in these studies are the most extensive and detailed available yet regarding trading transactions at investor-level resolution on a daily basis. These come from the central register of shareholdings for Finnish stocks, provided by Euroclear Finland. The data set includes all the major publicly traded Finnish stocks from January 1995 up to December 2016 and contains more than 100 million transactions from 1.3 million investors directly trading 37 thousand securities (Baltakys et al. (2020)). The investors' anonymized ID characterizes each transaction in the data set, trade and registration dates, security ISIN code, traded volume, investor's sector code, postal code, and investor's birth year and gender for household investors. Using the sector code information, it is possible to classify investors into six main categories: households (private individuals), non-financial corporations (e.g. IKEA, BMW), financial and insurance corporations (e.g. Allianz, HSBC), government (departments and agencies responsible for the administration of economic territory, like the Department for Business, Energy and Industrial Strategy), non-profit institutions (e.g. investment clubs, political parties), and the rest of the world (foreign private individuals or institutional investors, but generally composed of mutual funds, hedge funds, and foreign investment banks). Finnish domestic investors correspond to a separate account ID, while foreign investors can choose to register in a nominee name and, because of that, cannot be uniquely identified since they are pooled together under the custodian's nominee trading account (which belongs to the financial and insurance corporations category). Therefore, a single-nominee registered investor's account holdings may correspond to a large aggregated ownership of several (uncategorizable) foreign investors. In Appendix C.1, we briefly mention all the works that have worked on this data and that we carefully reviewed. From these, we identified several regular patterns and listed a series of 'rules' inferred from these results, thanks to which we depicted a map of what could potentially be the ecosystem of the Finnish (or even a generic) stock market. To show that, we define some terms used to describe the most important aspects of this system, which will be discussed extensively in the following sections:

- By **level of sophistication** of an investor, we mean a measure of their net worth and experience in financial markets.
- By **trading frequency**, we mean the number of days an investor traded a stock during a specific period of time.
- By **trading time delay**, we mean the time interval between the trades of two or more investors trading at the same frequency.
- By **volume traded**, we mean the number of shares of a stock (or asset) bought or sold by an investor.
- By **endogenous factor**, we mean any information that an investor typically uses, based on the trading strategy adopted, to decide when to trade (e.g. stock price, volatility, dividends, gross return of the risk-free asset).
- By **exogenous factor**, we mean any information that can influence an investor's social, cognitive and emotional biases (e.g. neighbour's advice, a news headline, Trump's tweet, weather, the result of a football match).

In Figure 2.4, we show a hypothetical illustration of this system (a *daily snapshot* of all investors who may trade in the stock market), where we classified investors in four different trading frequency classes (following Siikanen et al. (2018)): passive investors (trading, on average, once or twice a month), moderate investors (trading, on average, once a week), active investors (trading, on average, once every two days), highly active investors (daily or intra day investors). We reported some endogenous and exogenous factors (or sources) that affect (or might affect) investors belonging to these classes, along with the average level of sophistication and volume traded within them. As we can see, this system comprises different regions (representing investors trading with the same frequency), and different phenomena and structures may appear in each region. On average, the lower the trading frequency, the higher the number of investors and the less sophisticated (and volume traded), being mostly affected by exogenous factors. The opposite is true at higher frequencies. A specific class of exogenous and endogenous factors will only influence investors in a specific region. Finally, the (daily) stock price is determined by the aggregation of sets of (daily) orders, the nature of which is determined by the (daily) condition of each corresponding region in the system. In the short-term ($\sim$ three months), the system is stable because many parameters that identify it do not change, but in the long-term, it undergoes several evolutions.

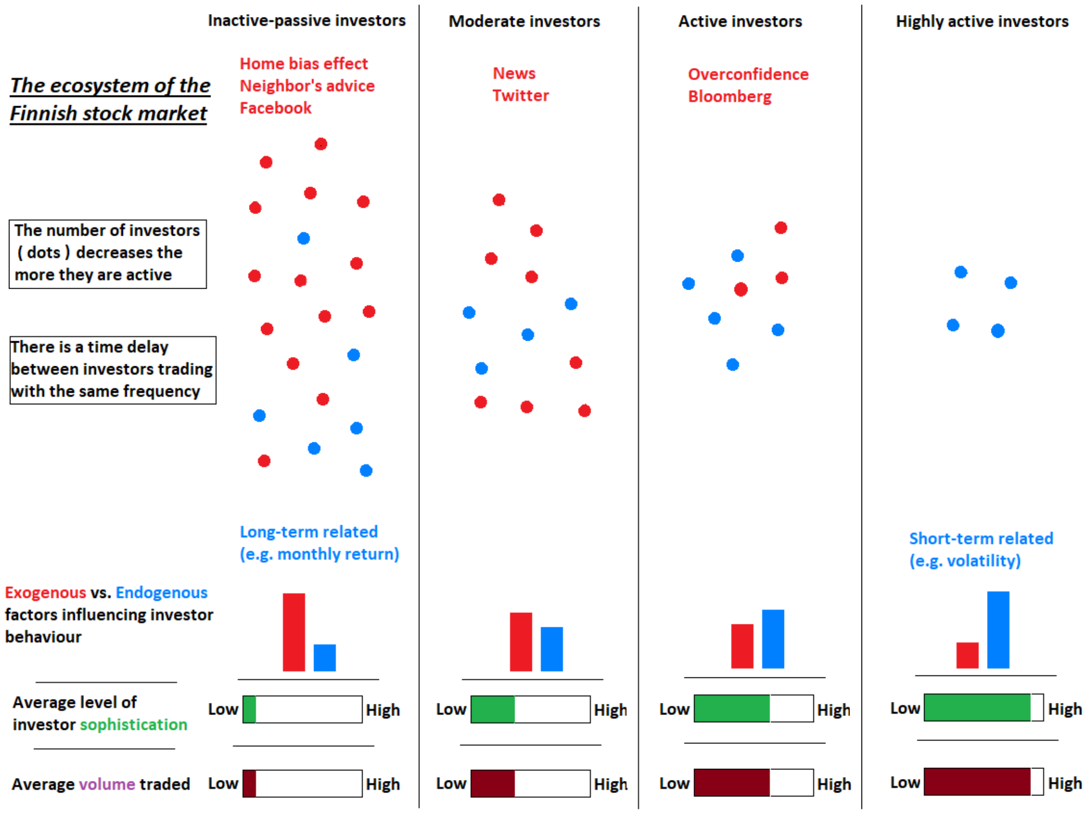


FIGURE 2.4: The Finnish stock market can be described as a complex system with investors and other sources of information (e.g., a company's public announcement website, news, social media) through which endogenous and exogenous factors are disseminated.

Social influence and a novel way to model a financial market Most of these inferences were possible thanks to the outcomes of works on data analysis published very recently, so, to the best of our knowledge, this is the most comprehensive illustration of the Finnish (and potentially a generic) stock market up to now. Several social media (some more reliable than others) and other means of communication (e.g. word-of-mouth) affect different groups of investors trading with different frequencies, and their overall effect plays a crucial role in determining the price dynamics of an asset in a financial market. In an attempt to understand some pricing anomalies while being true to this investor-level data, we will model this system in a novel way (Chapter 3) by taking inspiration from another model that describes (biological) systems which have a strong resemblance with the one shown in Figure 2.4 (we will briefly discuss it in Section 2.2.2.3). In the following sections, we will focus on a set of attributes associated with each investor and try to assert what endogenous or exogenous factors are more likely to affect them. These are the level of sophistication, the trading frequency, the trading time delay, and the volume traded (notice how these can identify with good precision the location of each investor in the complex system illustrated in Figure 2.4). We will discuss their role in influencing the stock price, with particular regard to volatility and bubbles (we will first focus on general aspects, and then we will narrow our discussion to social biases).

Actual and perceived values Before that, it is worth making an observation (that we will refer to in several points below) regarding why an investor trades in a stock market. Investors decide to buy, sell, or hold an asset based on how high the expected return is and how risky the investment is, which is true for any investor (rational or irrational) who participates in a stock market. As we know, in the Brock and Hommes model (and valid in most single-asset pricing models), an investor's demand z_t for an asset having price p_t and paying a dividend y_t at time t is given by the ratio:

$$z_t = \frac{E_t[(p_{t+1} + y_{t+1} - Rp_t)]}{aV_t[(p_{t+1} + y_{t+1} - Rp_t)]} \tag{2.89}$$

where $E_t[(p_{t+1} + y_{t+1} - Rp_t)$ is the expected excess return for the risky asset over the risk-free asset (with a gross return R) at time $t+1$, a is investor's risk aversion (averaged over all investors), and $V_t[(p_{t+1} + y_{t+1} - Rp_t)]$ is the variance of the future return, used to measure the risk associated with the investment. More generally, in a real stock market, investors will estimate the expected return and the risk differently depending on two types of factors: the endogenous ones, affecting the *actual* values (e.g. the dividend, the variance of the expected excess return), and the exogenous ones, changing investors' *perception* about these values. Clearly, these factors will influence the set of attributes used to identify an investor in a complex way. However, we will show that thanks to studies conducted in behavioural finance (along with concepts of mainstream finance), it is possible to classify these factors based on the impact they have on this set (e.g. the class of factors that increase investor trading frequency) and on the category of investors considered (e.g. the class of factors that affects passive households). This will significantly reduce the complexity of this system and will allow us (this is our intention, at least) to describe the dynamics within subsystems of this whole system simply and intuitively (regarding social biases and more), making an analytical and data analysis more viable.

2.2.1 Level of sophistication

The level of sophistication of an investor is a measure of their net worth and experience in financial markets and, typically, tells how well an investor has performed (or performs), based on profits gained, over a period of time T. Grinblatt and Keloharju (2000a) argue that the categories of investors listed above can be (roughly) ranked based on their level of sophistication in the following order: 1) foreign investors, 2) finance and insurance corporations, 3) non-financial corporations,

4) government, 5) non-profit institutions along with large-sized portfolio households, 6) medium-sized portfolio households, 7) small-sized portfolio households. Generally, highly sophisticated investors, on average represented by institutional investors, are weakly affected by exogenous factors and use models present in mainstream finance for their trading decisions. In contrast, unsophisticated investors, on average represented by retail investors, are strongly affected by exogenous factors and their trading decisions are described by models present in behavioural finance.

2.2.1.1 Investor sophistication and behavioural biases

The literature on behavioural finance regarding the relationship between investor sophistication and behavioural biases is very extensive (Nofsinger (2017), Ackert and Deaves (2009)). However, it can be classified into three topics: cognitive, emotional, and social biases. A rather significant fact that we noticed in our review is that each exogenous source alone, triggering social biases (e.g. News, Facebook), is unlikely to destabilize the market since it involves a subset of investors representing a minimal source of the volume traded. So, we believe that their collective impact generates such instabilities (e.g. bubbles, anomalous volatilities) and interdependencies with other behavioural biases (social biases can amplify or mitigate cognitive and emotional biases). This makes the system even more challenging to investigate, especially considering the simultaneous presence of highly sophisticated investors not affected by these, using several strategies based on different asset pricing models and modern portfolio theories.

2.2.1.2 Investor sophistication and social biases: data analysis

Based on the findings of works that performed data analysis on the Finnish data, it is likely that the level of sophistication for many investors should take several months (if not years) to change (Musciotto et al. (2018)). On the other hand, other investors adapt quickly (because of their learning skills), being less affected by behavioural biases as time passes. We will see in the next section how this impacts the structure of the complex system in Figure 2.4. As we have mentioned above, the exogenous sources that have been analysed, potentially triggering social biases within investors in the Finnish market, were mass media (Lillo et al. (2014)), social media (Siikanen et al. (2018)), and social proximity (Shive (2010), Walden (2018), Baltakys et al. (2019)). A criticism that we feel to make against these works is that they considered investor sector code (e.g. household, non-profit institution) instead of their level of sophistication to draw their conclusions. Grinblatt and Keloharju (2000a) were the only authors to classify these categories based on their performance, and we followed this classification in our depiction of the stock market in Figure 2.4. In fact, we believe that a more accurate analysis should focus on this attribute when investigating the correlation between investors' activity and an exogenous source, given that the less sophisticated an investor, the more likely they are to be influenced by such a source[16].

2.2.2 Trading frequency

By trading frequency, we mean the number of days an investor has traded the asset over a period of time T[17]. From Figure 2.4, we see that highly active investors (e.g. daily traders) are typically represented by highly sophisticated investors who trade large volumes of shares and are small in numbers compared to the total number of investors participating in the stock market. Inactive or passive investors (e.g. trading once or twice per month) are typically represented by unsophisticated investors who trade small volumes of shares and are high in numbers compared to the total number of investors. This could suggest that investors are likely to trade more frequently as they get more sophisticated over time (a dot in Figure 2.4 will shift its position to higher frequencies). In fact, if investors consistently perform well for long periods, they will increase their total wealth (or net worth). If their wealth is high enough, they can add more assets to their portfolio (the collection of stocks held by an investor) to further increase it. This leads investors to update their portfolios more frequently (deciding how much to invest in each asset to minimize risks and maximize profits) as they get wealthier and learn more about financial markets, becoming more sophisticated. Actually, we argue that one of the main reasons investors would trade more or less frequently (regardless of any market circumstances) can be attributed to investors' wealth used for trading. Tumminello et al. (2012) partly support this statement by showing that the cumulative probability density for the number of transactions is roughly approximated by a power law with an exponent close to -1 as in Zipf's law, which is also observed in size distributions such as firm sizes (Axtell (2001)) or annual income of companies, Okuyama et al. (1999)). However, the authors point out that this law governs the distribution only up to a specific number of transactions, while for larger numbers, the distribution obeys a different law (even though there is no sign of a bimodal distribution). Another piece of empirical evidence supporting our view that highly sophisticated investors are more active than less sophisticated ones, the main reason for which is related to investor wealth, comes from Baltakys et al. (2020). In their work, they found an increase in an investor's total trading intensity (how frequently an investor trades) associated with more diversified attention to assets. The authors argue that a possible interpretation is that frequent

[16]The level of sophistication can be different for each investor even within the same category of investors (including finance corporations), so the sector code could be misleading when it comes to interpreting the results.

[17]Alternatively, one can define it by considering the number of transactions. However, investors can trade multiple times in a single day, and since the data set's time resolution is limited to daily trading, we will consider the number of trading days instead.

traders rationally aim to diversify their portfolios, with institutional investors being more active than retail investors.

2.2.2.1 Trading frequency and behavioural biases

Some behavioural biases that have been found to affect (or might affect) how frequently investors trade are *overconfidence* (Grinblatt and Keloharju (2009), Graham et al. (2009)) and *familiarity* (or *home bias effect*) (Graham et al. (2009)), both cognitive biases. Overconfidence refers to the psychological bias that leads investors to overestimate their knowledge, underestimate risks, and exaggerate their ability to control events, while familiarity refers to the psychological bias that leads investors to invest in more familiar assets (Nofsinger (2017)). Typically, the more confident the investors, the more frequently they trade, and the more familiar an asset, the more frequently it will be traded. Based on these studies, it is unlikely that overconfidence can be mitigated or amplified by social biases. The opposite, instead, can happen for the familiarity bias.

2.2.2.2 Trading frequency and social biases: data analysis

The trading frequency for many investors could take a few months ($\sim$ three months) to change, if not more (Musciotto et al. (2018)). Only two works used the Finnish data and focused on the relationship between trading frequency and exogenous sources triggering social biases. The first one considers the impact of Facebook (Siikanen et al. (2018)), where the authors found that passive households were most affected by it, and the second one considers the geographical distance (or social proximity) between investors (Baltakys et al. (2019)) finding a negative association with the trade timing similarities. It must be noted that, for the latter work, trading *similarity* does not necessarily mean the same trading *frequency* since two investors could have traded on the same day, but both were trading at a different frequency. So, a more accurate analysis would see investors classified based on their trading frequency and check if social proximity influences these categories of investors differently. One would expect that investors trading at higher frequencies, typically highly sophisticated ones, would be unaffected by this exogenous source, and the opposite for investors who trade at lower frequencies. Other exogenous sources, such as mass media (e.g. News), have not been investigated from this perspective.

2.2.2.3 Trading frequency and volatility

One of the reasons why trading frequency should be an important factor regarding asset price dynamics is because of *flash crashes* (Kirilenko et al. (2017)), that is, extreme

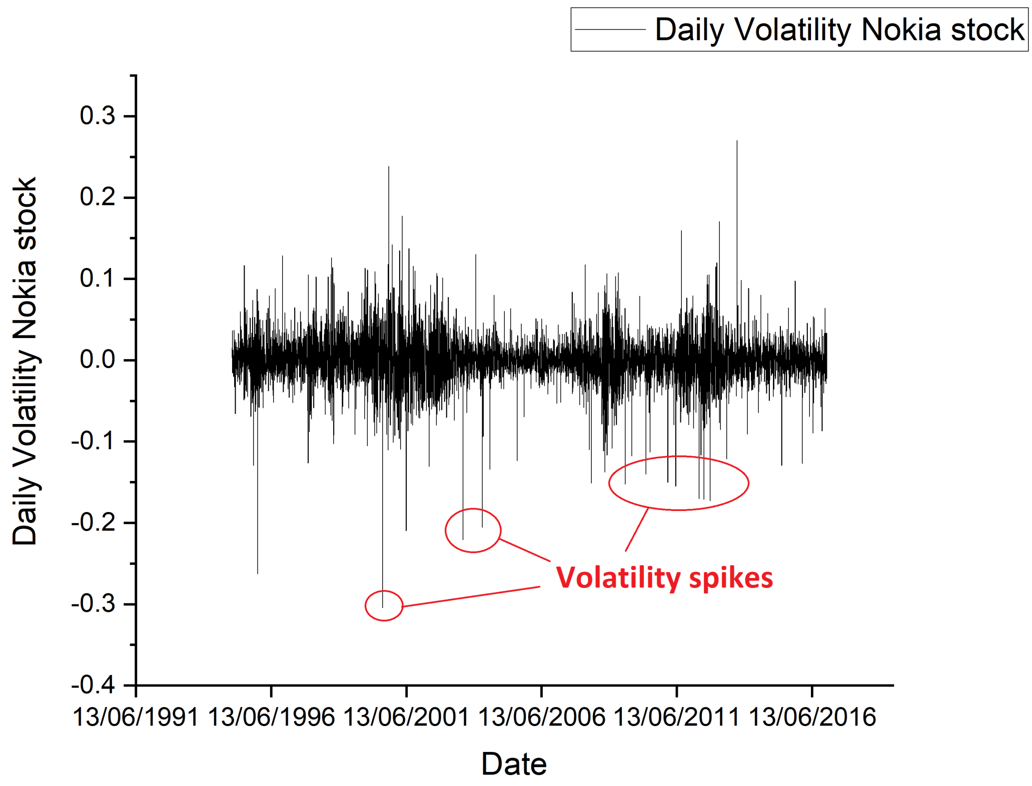


FIGURE 2.5: Nokia daily volatility between 1995-2016: most of these spikes are negative because of selling co-occurrences.

price movements (volatility spikes) within very short time periods (typically few minutes). These phenomena (which are becoming more common and mainly observed in intra-day activity) are still far from being fully understood, but some authors believe that they appear because of investors trading at different frequencies (see Leal et al. (2015) as a first work in this direction, and Staccioli and Napoletano (2021) one of the most recent works). Unfortunately, the Finnish data have a daily time resolution, so one can not investigate these phenomena by using this set. However, these 'crashes', which can cause massive damage to the market participants, can also impact the daily volatility (see Figure 2.5). The complex system in Figure 2.4 shows that investors who belong to different trading frequency classes are generally exposed to different exogenous sources, potentially leading the system to *organize itself* in creating these 'explosive synchronizations' (in the following section, we will show how the system can organize itself by varying the trading time delays for investors belonging to different trading frequency classes). And, here, we did not use the term 'explosive synchronizations' casually. In fact, we referred to the phase transition in the Kuramoto model in complex networks (Rodrigues et al. (2016)), which is a model used to describe synchronization between oscillating components of a system, such as fireflies, lasers, heart cells, and neurons. Porter and Gleeson (2016) show a simple example of this model applied on a star network, a network with a central node (the hub) to which other nodes are connected (the 'leaves'). The leaves are not connected to each other, so they are not aware of other leaves' presence. Let us compare this example with an event that could potentially take place in a real stock market. The frequency ω_i of each oscillator i (investor) is assumed to be proportional to its node degree k_i as $\omega_i \propto k_i^{\beta}$ (where $\beta > 0$). Because we are considering a star network, the hub will oscillate at high frequency (representing a highly sophisticated investor), while the leaves will oscillate at low frequency (less sophisticated investors). As the hub's degree K increases (more leaves are added to the network, as if a higher number of less sophisticated investors, for example, are asking for advice from a financial expert[18]), one can show that it is possible to obtain a discontinuous phase transition, that is, an explosive synchronization (e.g. all investors selling at the same time). An interesting question that we can address here for future considerations is: Does the complex system in Figure 2.4 contain subsystems with similar or more realistic structures (e.g. hierarchical structures), so endangering the stability of the entire market by potentially giving rise to *flash crashes*? Our model outlined in Chapter 3 will take inspiration from Kuramoto's (as stated before in Paragraph 2.2), although its analytical form will slightly differ to be more consistent with what discussed in Section 2.1 regarding the DCA general framework. We will discuss more about the correspondence between our model and Kuramoto's when we draw our conclusions on the first results of our model in Chapter 3.

2.2.3 Trading time delay

Investors who trade at the same frequency can trade on different days, that is, there is a time delay between their trades. If an investor decides to trade the asset at time t_{trade}, it can be seen as a result of either endogenous and/or exogenous factors, such that:

$$t_{trade} = \tau_o(t) + \delta_T(t). \tag{2.90}$$

where $\tau_o(t)$ is the time at which a highly sophisticated investor (unbiased by any exogenous source) would most likely trade the asset and $\delta_T(t)$ is the trading time delay (positive or negative). So, whenever $\delta_T(t) \neq 0$, it means that an exogenous source is probably affecting investor behaviour. One reason why investors would trade at different times, even though they belong to the same trading frequency class, can be associated with their beliefs about the asset's fundamental value, as shown in the following example.

[18] In recent years, there has been an increase in the number of retail investors participating in the stock market (Ranganathan et al. (2018), Frijns et al. (2018)), which may have shifted the balance of the system to structures of "retail-advisors" type.

Example Consider two fundamentalist investors (i_1 and i_2), that is, investors who believe the stock price to converge to its fundamental value, with the difference that they estimate the fundamental value differently ($p_{f,1}$ and $p_{f,2}$). Assume that $p_{f,1} < p_{f,2}$ and the current price p_{t_1} at time t_1 is above both fundamental values, that is $p_{t_1} > p_{f,2} > p_{f,1}$. Both investors had already invested in the stock when the price was below their estimated fundamental values (the stock was undervalued, so they bought the stock), but now they are thinking of selling the asset since they *believe* the stock is overvalued. Let us define Δ, a value equal for both investors, as the threshold for which investor will sell the asset if the stock price increases beyond a specific value, meaning that they will sell the asset if $p_t - p_f > \Delta$. Because of that, investor i_1 will probably sell the asset before investor i_2 given that the difference between the current price and the estimated fundamental value ($p_{t_1} - p_{f,1}$) is higher with respect to what calculated by investor i_2 ($p_{f,1} < p_{f,2} \implies p_{t_1} - p_{f,1} > p_{t_1} - p_{f,2}$). So, investor i_1 will sell the asset at time t_2 when the stock price is such that $p_{t_2} - p_{f,1} > \Delta$, while investor i_2 at time $t_3 > t_2$ when the stock price ($p_{t_3} > p_{t_2}$) will be such that $p_{t_3} - p_{f,2} > \Delta$.

Strategies and reference prices The situation illustrated above frequently happens in the real stock market because investors adopt different strategies to estimate the expected value of an asset. So, more generally, the origin of trading time delays can be interpreted as a consequence of having a market populated by investors with different time horizons for their expectations. In the previous example, investors i_1 and i_2 can also be considered as two chartists, with investor i_1 using the price at time t_1, p_{t_1}, for their trading decisions at time t_2, while investor i_2 considers the price at time t_0, p_{t_0}, that is, they *refer* to different prices (whether unconsciously or not) instead of current one p_t because of endogenous/exogenous factors. This concept is similar to the reference point discussed in Behavioural Finance, which can be, for example, the stock price the investors compare with the current stock price, which they use to determine whether they feel pleasure for a profit or pain for a loss (see Nofsinger (2017) and Ackert and Deaves (2009)). In fact, one can equivalently consider the market populated only by fundamentalists, *as if* they estimate the fundamental value in the same way, but their *reference prices* differ from each other. By referring to the example above, one could argue that the two cases of investors i_1 and i_2, being either fundamentalists or chartists, are indistinguishable (a chartist can be viewed as a fundamentalist whose reference price changes much more frequently). In this case, the heterogeneity occurs in investors' determination of their reference prices, which is related to the strategy adopted. A potential advantage of this perspective is that it does not oblige the modellers to formulate hypotheses (not verifiable by the data) on the details of the strategy adopted by an investor but on other variables that are more easily observable from how an investor trades (e.g. trading frequency, trading time delay, and so on). This is a novel approach in DCA financial modelling, and we will return to this point in Section 2.3.1 since we will use it in our model in the next chapters. In the next section, we will show how most behavioural biases cause trading time delays between investors, *as if they change investors' perception* of the fundamental value (or their reference price).

2.2.3.1 Trading time delay and behavioural biases

Most behavioural biases affect investors in trading an asset with a time delay, either positively or negatively. Below, we list some of those that might influence investors (and how) when they decide to trade a stock. We will (purposely) correlate them with a possible *change in investors' perception* of the fundamental value and/or the risk associated with the stock compared to a more objective estimate (e.g. those estimated by a highly sophisticated investor). Typically, biases can be interpreted as factors affecting both the fundamental value and risk simultaneously and in opposite ways, such as increasing the fundamental value and decreasing the associated risk and vice versa. We refer to Nofsinger (2017) and Ackert and Deaves (2009) for the definitions of these biases.

Positive delays in selling - perceiving the fundamental value higher and/or the risk lower

- *endowment effect* (or *status quo bias*): refers to investors' tendency to hold the investment they have.
- *break-even effect*: refers to investors' tendency to take on higher risk after they face a loss because they see a profit opportunity that could cover the loss.
- *house-money effect*: refers to investors' tendency to take on higher risk after a prior gain because they feel the profit gained does not belong to them (betting with the "house money").
- *disposition effect(1)*: refers to investors' tendency to avoid selling losers (stocks whose price decreased since they were purchased)

Positive delays in buying - perceiving the fundamental value lower and/or the risk higher

- *snakebite effect*: refers to investors' tendency to be less willing to risk investing in a stock because of a recent loss.

Negative delays in selling - perceiving the fundamental value lower and/or the risk higher

- *disposition effect(2)*: refers to investors' tendency to keep selling winners (stocks whose price increased since they were purchased)
- *snakebite effect*

2.2.3.2 Trading time delay and social biases: data analysis

The trading time delay can be affected in many ways and, thus, can vary rapidly, even over short-term periods. We expect that social biases cluster investors into perceiving similar beliefs about an asset's value, leading to trading it with the same time delay (a *bandwagon effect*). Unfortunately, there are no works that carried out a data analysis aimed at investigating the role played by social biases in influencing these delays for Finnish investors (all works are focused on trading co-occurrences). The analysis performed by Baltakys et al. (2019) hints that social proximity could influence investors in trading the asset with the same time delay, so forming *trading clusters*, that is, trading co-occurrences in the same direction (buy or sell). Lillo et al. (2014) and Siikanen et al. (2018) report results suggesting the same conclusions concerning mass media (News) and social media (Facebook). However, there is a lack of analysis aiming to address questions focused on *how* these sources affect investors in trading with different time delays (e.g. how fast do they react to some information, how do they interpret the information). Given the multitude of factors (endogenous and exogenous) that could influence these delays, along with data availability, it is clear that this task was (and still is) out of reach for many researchers.

Trading time delay: bubbles and volatility

Price bubble Many anomalies in price fluctuations can be explained as a consequence of trading time delays. Consider, for example, the price of a stock stable around a value, meaning that the buy and sell order flows are equal. Assume something happens that would lead a fraction of investors to sell the asset with a positive time delay (e.g. investors are unsure about selling the asset or not). Because of that, investors who are willing to buy the asset (demand) will see a drop in the number of investors willing to sell the asset (supply) at the current price. So, to reach the equilibrium between demand and supply again, the price has to be increased (typically, a *market maker* will update the price), and if this positive time delay in selling the asset is persistent, it will give rise to a bubble. We have seen above that several behavioural biases lead to positive time delays in selling an asset that could generate abnormal stock price increases. What role do social biases play from this perspective? Which exogenous sources and the information they disseminate can lead investors to behave this way, and which classes of investors are mainly influenced by it?

Volatility How these delays are distributed over time can also be crucial in explaining many aspects of price volatility. Roughly speaking, volatility measures the intensity of price swings in either direction (upwards and downwards) over a specific period, and the larger the orders, the shorter the time interval in which these are issued, the wider these swings. Volatility is commonly used as a measure of risk associated with an asset, and the higher the volatility, the riskier the investment. It can be generated due to information uncertainty about the fundamental value.

Example As before, consider the price of a stock stable around a value p^*, which corresponds to the fundamental value equal for all investors. At some point, some information δ_t is being disseminated to a fraction of investors n_δ which will change their estimate about the fundamental value from p^* to $p^* + \delta_{i,t}$, with $\delta_{i,t} > 0$ a value increasing over time, subject to investor i's interpretation of the information. Because of that, buy orders are being issued, and the price will increase (the market maker will update the price to trigger the other investors into selling the asset). The value of δ_t will both affect the volume of shares traded and the time in which the order will be issued since investors will refer to a threshold Δ to decide whether to buy ($p^* + \delta_{i,t} - p_t > \Delta_i$, for investor $i \in n_\delta$) or not ($p^* + \delta_{j,t} - p_t < \Delta_j$, for investor $j \in n_\delta$). So, the faster δ_t will go up, the larger the volumes of shares bought, the closer the orders issued in time, and the more rapidly the price will increase. The fraction of investors who have not received any information will still estimate the fundamental value equal to p^*. As soon as they see the price increasing over this value, they think the asset is overvalued and start selling it. However, even these investors will start selling only after the price has exceeded a threshold Δ, leading to different trading time delays, driving the asset's price to a new stable value or decreasing it.

This interplay between investors buying and selling the asset over the same period will generate different intensities in price volatility, and, in this trivial case, δ_t is responsible for generating this destabilization.

Distributions of beliefs More generally, multiple fractions of investors will have access to different sources of information, shaping the distribution of the estimated fundamental values within the system in a very specific way. For example, less sophisticated investors can be affected by the information disseminated through social media (which affects passive households, Siikanen et al. (2018)), changing the shape of this distribution, shifting it from an unimodal to a bimodal or even a multimodal one (or the opposite). The idea is that for each fraction of investors belonging to a trading frequency class (or, more generally, to a subsystem of the entire market), the distribution of the estimated fundamental values will be different and evolve over time. We argue that the shape will be affected mainly by behavioural biases at low frequencies, while at high frequencies, the distribution will be affected mainly by endogenous factors. This motivates the research question outlined in the introduction of this report, that is: In which way does social interaction influence investors' perception of the value of an asset? Can it give rise to abnormal volatilities by spreading investors' beliefs away from each other? Or does it cluster them towards a common value, potentially leading to bubbles or bursts? How investors traded the asset (buy, sell or hold) over a specific period could give us a clue about estimating their corresponding beliefs about the asset's value, and we will discuss it in the next chapters.

2.2.4 Volume traded

The volume traded by an investor is the number of shares of a stock bought or sold at a specific time. As we know, the larger the volume exchanged, the greater its impact on the stock price. To give an idea about real trades that occurred in the Finnish stock market, the exchanged volume distribution for Nokia company between the period from 1999 to 2003 covered approximately ten orders of magnitude, ranging from a dozen to 10,000 million shares traded in a single transaction (Tumminello et al. (2012))(the price of the stock, that is the price of a share, ranged from $10 to $60 during this period). Surprisingly, this attribute (and its relationship with other attributes) has been poorly analysed in all the reviewed works, except in Tumminello et al. (2012), which showed the average exchanged volume from each category of investors and compared it with the number of transactions performed. In Appendix C.2, we briefly discuss the relationship between the volume traded and the trading frequency. However, we will go no further than that in discussing this attribute since we will not focus on it in the following chapters and leave its analysis for future work. Some behavioural biases affect neither how frequently investors trade nor the time delays of their trading but will generally impact the volume and the direction (buy/sell) of their orders. These are related to investors' mood (or sentiment) within a market and, as already noted above, will be discussed in the later works.

2.3 Conclusions

In this last section, we draw some conclusions based on the discussion above that will lead us to interpret this system differently from how most HAMs have done so far, with particular attention to the Brock-Hommes model, by which our work is inspired.

2.3.1 The fundamental value in the Brock-Hommes model

Brock and Hommes argue, and this applies to most HAMs, that agents switch from one strategy to another based on some performance measures. Agents can compare the performance of each strategy thanks to the fact that they have complete information about the fundamental value (determined by the gross return of the risk-free asset, R, and the dividend, y_t). He and Zheng (He and Zheng (2016), p. 65), who focused their attention on information uncertainty, rightly raise the question: "These studies (*which make use of the performance measure*) are limited to explain why not all agents from the same group switch to better-performing strategies with certainty. If the information is complete, should not all agents cluster to the strategy that is expected to perform better?". We know that behavioural biases play an important role in this puzzle, but it is clear that this also has to do with how Brock and Hommes derive the fundamental value. In addition to describing the wealth dynamics based on some restrictive assumptions [19]. Brock and Hommes ignore this aspect, which also affects portfolio optimization (the form of the demand $z_{h,t}$). They also assume the fundamental value p^* given by the discounted sum of expected future dividends. Less sophisticated investors (e.g. retail investors) do not have access to this (costly) information (and, therefore, will look for other sources of information), which means that investigating their dynamics of switching between strategies related to this information might lead to a wrong interpretation of their behaviour. More generally, investors will only switch between strategies they have complete information about (which can be endogenous or exogenous), which is true for all investors. Consider, for example, highly sophisticated investors who estimate the fundamental value p^* given by the discounted sum of expected future dividends (if an asset does not pay any dividends, then, as a possible alternative, the company's future cash flows are considered) that is

$$p_t^* = \sum_{i=1}^{\infty} \frac{E_t[y_{t+i}]}{R^i} \tag{2.91}$$

The fundamental value of an asset is determined by investors' belief about stock *future* dividends, and no one has access to this information. Thus, investors will use different methods to estimate them (e.g. adopting different financial models), which will inevitably lead to heterogeneous beliefs about the fundamental value itself [20]! Furthermore, this heterogeneity is amplified by the presence of investors with different levels of sophistication. So, the *real* fundamental value of an asset estimated by an investor h is generally described by a sum of terms representing endogenous

[19] Recall that, as discussed in the previous section, investor wealth plays a crucial role across different categories of investors since it might determine the trading frequency over a specific period. More specifically, retail investors (low wealth) trade less frequently than institutional investors (high wealth)

[20] He and Zheng (2016) build a model by taking into account this fact.

factors, $end_{i,t}$, and exogenous factors $exo_{j,t}$, that is

$$p^*_{h,t} = \sum_i a_{h,i,t} end_{i,t} + \sum_j b_{h,j,t} exo_{j,t} \tag{2.92}$$

where $a_{h,i,t}$ and $b_{h,j,t}$ represent the weights associated with each endogenous and exogenous factor $end_{i,t}$ and $exo_{j,t}$ included in the 'model' used by investor h at time t. Alternatively, one can assume that the fundamental value is described by the sum of terms representing endogenous factors, but each of these factors can be affected by exogenous sources. However, to make analytical and data analysis more viable, we will assume that a generic investor estimates the fundamental value as in equation (2.92), that is, as a linear combination.

Reference price Taking inspiration from what we showed in Example 2.2.3.2, the fundamental value of an asset $p^*_{h,t}$ estimated by a generic investor h can be alternatively described as

$$p^*_{h,t} = p^* + \delta_{h,t} \tag{2.93}$$

where $\delta_{h,t}$ represents any deviation of the asset's value from its actual fundamental value p^*, as a consequence of an external influence not considered in the fundamental analysis performed on the asset. Notice that (2.93) can be valid for *any* investor trading the asset, even non-fundamentalist ones (like chartists), and it looks very similar to the expected price $E_{h,t}[p_{t+1}]$ from Brock-Hommes model, which for convenience we rewrite here

$$E_{h,t}[p_{t+1}] = E_t[p^*_{t+1}] + f_h(p_{t-1}, p_{t-2}, ..., p^*_{t-1}, p^*_{t-2}, ...). \tag{2.94}$$

In their model, investors believe that the price will deviate from its fundamental value (all investors have the same expectation for this value, which we believe is too restrictive) by some function $f_{h,t} = f_h(p_{t-1}, p_{t-2}, ..., p^*_{t-1}, p^*_{t-2}, ...)$, the *model* used by the investor with strategy 'h', which depend on the prices and fundamental values at previous times. In our model (Chapter 3), instead, $\delta_{h,t}$ will be a function of the current price p_t and investor h's *reference price* $p_{h,t}$, that is the price that they refer instead of current one p_t due to endogenous/exogenous factors (as explained in Paragraph 2.2.3), so that their demand $z_{h,t}$ will be proportional to

$$z_{h,t} \propto p^* - p_{h,t} \tag{2.95}$$

where each group of investors will be described by the function $p_{h,t}$ (see Chapters 4 and 5). As we already said above, this way of describing a financial market (that takes inspiration from the Kuramoto model) is new in the context of the DCA general framework, and it will allow us to perform a satisfactory analytical analysis of the model as well as reproduce realistic price dynamics.

Reference Points in Prospect Theory The concept of reference prices in our model is similar to the *reference points* central to prospect theory (see Nofsinger (2017), Ackert and Deaves (2009), Thaler (2005)). In both frameworks, investors evaluate outcomes relative to a benchmark, which influences their decision-making process. Prospect theory, developed by Kahneman and Tversky (1979), introduces the idea that individuals do not perceive gains and losses from an absolute perspective (as the expected utility theory assumes when it derives investors' demand for the risky asset in equation (2.14)) but rather in relation to a reference point, typically the status quo or a specific expectation level. This reference point is crucial because it determines whether an outcome is perceived as a gain or a loss. Prospect theory also argues that losses are felt more acutely than gains of an equivalent size, a phenomenon known as *loss aversion*. In our model, reference prices serve a similar role by being the benchmarks against which current prices are evaluated. These reference prices are not static; they adapt over time and are influenced by endogenous and exogenous factors, reflecting changes in investor sentiment, market conditions, and external news. This dynamic adjustment can lead to different behaviours among investors, such as the timing of trades and the perception of the asset's value. For example, consider an investor who buys shares of a stock at £100 each, considering this her reference price. If the stock price rises to £150, her reference price might still remain at £100, leading her to perceive a £50 gain. However, suppose she adjusts her reference price to £140 based on recent performance or peer influence. In that case, a subsequent drop to £130 turns a perceived gain into a loss, influencing her decision to sell more quickly than if her reference point remained at £100. This adjustment is akin to the behavioural biases discussed in prospect theory. An investor's willingness to sell a stock when it is losing value can be delayed due to loss aversion — the pain of acknowledging a loss by adjusting the reference price downwards can lead to the irrational holding of assets whose value is decreasing, a common market anomaly known as the *disposition effect* (see Grinblatt and Keloharju (2001c) for an analysis of such phenomenon performed on Finnish investors). Thus, just as in prospect theory, where the reference point can shift based on new information or changes in wealth (Birru (2015)), in our model, reference prices shift based on market dynamics and investor interactions, continuously influencing investment decisions. Both concepts highlight the psychological foundations of financial decisions and the significant impact of cognitive, emotional and social biases on market behaviour.

2.3.2 Fundamental value on different time scales

As recent studies suggest, we also argue that the system is governed by laws determining its evolution on different time scales, as we had already stated in Paragraph 2.1.1.4. In particular, the weights $a_{h,i,t}$ and $b_{h,j,t}$ evolve much more slowly than $end_{i,t}$ and $exo_{j,t}$, meaning that on short-time periods (e.g. hours, days, weeks), investors' estimation about p^* will be determined by those factors used within their current strategy and they will trade depending on whether the asset price is far or close to their expected value, while over more extended periods (e.g. months, years) investors can give more or less weight to other sources of information (for example, by discarding those that have turned out to be less profitable and focusing on those more profitable). From this perspective, the Brock-Hommes model would fit more in describing the evolution of $a_{h,i,t}$ and $b_{h,j,t}$ (through the performance measure and the multinomial logit model, but, over shorter periods, the system obeys different rules. In addition to being very reasonable, this interpretation is also supported by a very recent work performed by Musciotto et al. (2018) who showed, by using the database of the Finnish stock market, the presence of an ecology of groups of investors adopting, using and discarding different investment strategies on time scales ranging from several months to twelve years (they investigated the daily trading decisions of investors investing Nokia stock over a time period of 15 years, from 1995 to 2009). They observe that these time scales are many orders of magnitude larger than those of the financial asset's price dynamics, suggesting that anomalies in price fluctuations (e.g. volatility clusters, bubbles) could arise due to investor trading based on their current strategy and not because of the switching mechanisms, as most HAMs have attempted to demonstrate in recent years. We can conclude by arguing that the degree of *heterogeneity* of the system, given by the presence of multiple strategies over a specific period, will determine the shape of the distribution of the estimated fundamental values and, consequently, give rise to the anomalies observed in the asset price dynamics. However, the degree of *adaptivity*, given by the investors' ability to switch between different strategies, could lead the system to exhibit the same anomalous phenomena, the causes of which may be different depending on the period investigated.

Chapter 3

Social influence over different time scales

The findings summarised in the previous chapter and the topic of our research question lead us to describe the system differently from how most HAMs have done so far. The weights $a_{h,i,t}$ and $b_{h,j,t}$ in equation (2.92) only change over long periods, while on shorter periods, these are more or less constant and each investor trades or not an asset based on their current strategy. As endogenous factors, one can consider the discounted sum of expected future dividends, the asset's past prices (daily, weekly, or monthly averages), or the information used in the Black-Scholes-Merton model (a financial model used for pricing options)[1]. As exogenous factors, one can consider profits gained (triggering, for example, the house-money effect), the investor's geographical location (as a proxy of social interaction) or the number of likes (or tweets) on a company's posts on social media [2]. There are many distinct possibilities, which is one of the main reasons it is challenging to study this system (also considering the inaccessibility of some information, like investors' psychological traits).

[1] Notice that, from the point of view of pure fundamentalists, the information extrapolated by chartists from past prices (e.g. price trends) is classified as *exogenous* since it is not considered in the fundamental analysis performed on the asset.

[2] See Baltakienė et al. (2020) as an example of a possible regression analysis that they used for Finnish investors. Their work is particularly interesting because it is strictly related to our research question (social influence on financial markets), where they attempt to estimate the private information channels based on abnormal trades.

3.1 General framework of the model

3.1.1 Investors' demand

3.1.1.1 Portfolio optimization

First, we assume that the system is populated by investors h who only invest in one risky asset that does not pay any stochastic dividend. In this case, the wealth dynamics is simply described (see equation (2.5) for $R = 1$ and $y_{t+1} = 0$ for any time t) by

$$W_{h,t+1} = W_{h,t} + (p_{t+1} - p_t) z_{h,t} \tag{3.1}$$

After this, we need to specify a utility function $U(W_{h,t+1})$ that measures investors' satisfaction or preference over the allocation of their wealth at time $t + 1$ to this asset and choose the correct value for the demand $z_{h,t}$ to get the highest value of that function. In line with Brock and Hommes' assumption (and as most asset pricing models do), we will adopt a function whose functional form is similar to the CARA utility one, but with the difference that the degree of satisfaction depends on the category of the investor rather than their risk aversion, that is

$$U(W_{h,t+1}) = 1 - e^{-\frac{W_{h,t+1}}{c_h}} \tag{3.2}$$

where c_h contains information related to the size of investor h's portfolio (outside the wealth allocated for the risky asset). For small-size portfolio investors, like households, c_h is also small, while for large-size portfolio investors, like institutional ones, c_h is large. This implies that small-size portfolio investors get satisfied more quickly compared to larger-size portfolio investors over the same wealth level W_{t+1} (see Figure 2.1 for comparison with $a = 1/c_h$). For example, if household investors feel satisfied if their wealth level amounts to £100 to some degree, a financial company's wealth level (e.g. bank) would need to be much higher to reach the same degree of satisfaction. At this point, investors' aim is to choose $z_{h,t}$ that maximizes $U(W_{h,t+1})$ (more specifically, the expected value of this utility function, see equation (2.9), for example) and we assume they are myopic mean-variance maximizers, meaning that they solve (see equation (2.10))

$$\frac{d}{dz_{h,t}} \left\{ E_{h,t}[W_{h,t+1}] - \frac{V_{h,t}[W_{h,t+1}]}{2c_h} \right\} = 0. \tag{3.3}$$

where $E_{h,t}[W_{h,t+1}]$ and $V_{h,t}[W_{h,t+1}]$ are, correspondingly, the conditional expectation and variance on $W_{h,t+1}$. Following the same procedure as we showed in section 2.1.1.1, we obtain the optimal demand for the risky asset as

$$z_{h,t} = \frac{c_h E_{h,t}[(p_{t+1} - p_t)]}{V_{h,t}[(p_{t+1} - p_t)]} = c_h \frac{(p^*_{h,t} - p_t)}{\sigma^2} \tag{3.4}$$

where $p^*_{h,t} = E_{h,t}[(p_{t+1})]$ is the expected price described by equation (2.92) (or takes a specific value based on how investors' beliefs are distributed across the market) and $\sigma^2 = V_{h,t}[(p_{t+1} - p_t)]$ is the conditional variance for trader type 'h', assumed to be equal for all types of traders.

3.1.1.2 Investors' aggregate demand and market clearing mechanism

If we define $n_{h,t}$ as the fraction of traders belonging to group h who trade the asset at time t (we call it *trading mechanism*) and f_h as the fraction of traders belonging to group h compared to all H groups of investors that populate the market, then we can write the aggregate demand for the asset at time t as

$$z_{d,t} = \sum_{h=1}^{H} f_h n_{h,t} z_{h,t} = \sum_{h=1}^{H} d_{h,t} \tag{3.5}$$

where $d_{h,t} = f_h n_{h,t} z_{h,t}$ is group h's net demand and the sum is over all the H groups of investors. Notice that our interpretation of $n_{h,t}$ in our model (whose analytical form will be derived in the next sections) differs from what we see in HAMs, where it represents the fraction of traders adopting the strategy h at time t instead (see Section 2.1.1.3). In fact, such a fraction is described by f_h, which is constant over time, and, combined with c_h, we define

$$\bar{c}_h \stackrel{\text{def}}{=} f_h c_h \tag{3.6}$$

which we call group h's *dominance* over the market. One can think of $\bar{c}_h$ as a comparative parameter between the size of the portfolios of all investors and the size of the groups within the market, which is a convenient way to describe who 'dominates' the market price more. The higher $\bar{c}_h$, the more significant will be group h's contribution to the price trend towards their price belief $p^*_{h,t}$. So, even if c_h is small for small-size portfolio investors compared to large-size portfolio ones, $\bar{c}_h$ could match or even be larger for small-size portfolio investors if the number of investors within this group is higher than in large-size portfolio ones.

Market clearing mechanism To describe the price dynamics, one assumes that the asset price is determined through a market clearing mechanism. Two widely used mechanisms in the literature (mentioned in the previous chapters) are the Walrasian auctioneer (used in the Brock-Hommes model) and the market maker. As already explained in the previous chapter, the role of the Walrasian auctioneer is to announce a price, receive orders for the asset at that price and repeat this process until it matches the demand and supply (or, more correctly, select the price that will clear the highest number of trades). This is equivalent to saying that the asset's excess demand (the difference between demand and supply) equals $z_{d,t} = 0$. On the other hand, in the market maker mechanism, out-of-equilibrium trades ($z_{d,t} \neq 0$) are possible, and a market maker updates the price as

$$p_t = p_{t-1} + \mu \sum_{h=1}^{H} n_{h,t-1} \bar{c}_h (p^*_{h,t-1} - p_{t-1}) = p_{t-1} + \mu z_{d,t-1} \tag{3.7}$$

where μ is the market maker's speed of adjustment to the excess demand (for convenience and without loss of generality, we set $\sigma^2 = 1$). In this case, the market maker announces the price at time t based on order imbalances at the previous time $t-1$. So, if more investors are willing to buy the asset than sell it ($z_{d,t-1} > 0$), then the price has to be increased to trigger investors to sell the asset, while the opposite happens when $z_{d,t-1} < 0$.

Market maker model in the literature The market maker's role in financial markets, as described through various theoretical and empirical studies, justifies its relevance and effectiveness compared to the idealized Walrasian auctioneer (as pointed out by Maureen O'Hara (1998), the latter clearing mechanism is used only in the market for silver in London). The seminal work by Beja and Goldman (1980) highlighted that real asset markets do not operate as pure Walrasian markets and introduced a dynamic model wherein the market maker adjusts prices based on aggregate excess demand, acknowledging that transactions often occur at disequilibrium prices. This recognition of finite adjustment speeds and the allowance for non-equilibrium trading conditions reflected a more accurate depiction of price formation in real markets. Further extending the conceptual framework, Day and Huang (1990) introduced a nonlinear behavioural model that incorporates the role of a market maker similar to a specialist at the New York Stock Exchange. Their model revealed that the market maker's price adjustments could lead to chaotic price fluctuations, mimicking real-world phenomena like bull and bear markets. Following their pivotal work, the market maker's role has been thoroughly examined within the context of HAMs creating complex dynamics (as discussed Chiarella and He (2001), Farmer and Joshi (2002), Lux (1995), Lux and Marchesi (1999), Lux and Marchesi (2000b), and Kirman (1991b)), which can mimic real-world financial phenomena like bubbles and crashes, fat-tailed return distributions, and clustered volatility. Moreover, market makers are not just passive agents but play active roles as liquidity providers and investors. For instance, Wyss (2001) discusses how market makers stabilize markets by temporarily managing inventory to balance order imbalances. Madhavan and Smidt (1993) further elaborated on this dual role, exploring how market makers adjust their inventory levels, acting as both liquidity providers and active investors based on market conditions and portfolio strategies. The complex dynamics introduced by market makers in financial models, illustrated by researchers like Zhu et al. (2009), show that market makers can influence market stability and investor behaviour. Their actions can stabilize and destabilize markets, depending on how they manage their inventory to maximize profits or provide liquidity.

Overall, the literature supports the market maker model as a more realistic representation of financial markets, capable of capturing the multifaceted nature of trading activities. These models allow for a richer analysis of market behaviours and are crucial for understanding market volatility, trends, and cycles, offering a closer approximation to market conditions than the classical Walrasian model. For this reason, we will adopt this framework in our model. We remind that this market clearing mechanism has also been adopted by Naimzada and Ricchiuti (2008), Brianzoni and Campisi (2020), and Campisi and Tramontana (2020) (see equation (2.85) and (2.87)).

3.1.2 Investors' asset price belief

Equation (3.4) corresponds exactly to fundamentalists' demand as described in the Brock-Hommes model if we set $p^*_{f,t}$ in equation (2.92) (where f stands for fundamentalist) as the discounted sum of expected future dividends (no exogenous sources are affecting its value, meaning that $b_{f,j,t} = 0$ for all j in equation (3.4)) and $c_{f,t} = R/a\sigma^2$, that is

$$p^*_{f,t} = \sum_i a_{f,i,t} end_{i,t} + \sum_j b_{f,j,t} exo_{j,t} = p^* \tag{3.8}$$

where

$$\sum_i a_{f,i,t} end_{i,t} = p^* \qquad\qquad \sum_j b_{f,j,t} exo_{j,t} = 0 \tag{3.9}$$

so that

$$z_{f,t} = c_{f,t}(p^*_{f,t} - p_t) = \frac{R}{a\sigma^2}(p^* - p_t) \tag{3.10}$$

3.1.2.1 Investors' reference price

As we suggested in Section 2.3.1, one can alternatively assume that for any group of investors h (even chartists) $p^*_{h,t}$ could be given by

$$p^*_{h,t} = p^* + \delta_{h,t} \tag{3.11}$$

where $\delta_{h,t}$ represents any deviation of the asset's value estimated by h from its (actual) fundamental value p^*, as a consequence of an *external influence* not considered in the fundamental analysis performed on the asset. Among the different choices, $\delta_{h,t}$ can be expressed (or defined) as a function that describes any *shift* in investors h's reference price from the current price p_t to another price $p_{h,t}$ (group h's *actual reference price*), namely

$$\delta_{h,t} \stackrel{\text{def}}{=} p_t - p_{h,t} \tag{3.12}$$

where $p_{h,t}$ now describes the interaction with any exogenous source affecting h's estimation of the asset's value. By using equation (3.11) and (3.12), we have, so,

$$z_{h,t} = \bar{c}_h(p^*_{h,t} - p_t) = \bar{c}_h(p^* + \delta_{h,t} - p_t) = \bar{c}_h(p^* - p_{h,t}) \tag{3.13}$$

meaning that the group h refers to the price $p_{h,t}$ instead of p_t at time t. Thus, in describing the group of investors in Equation (3.8)), we implicitly had

$$p_{f,t} = p_t \implies \delta_{f,t} = 0 \implies \quad \textit{no external influence} \tag{3.14}$$

that is, we assumed that group f's reference price at time t was always equal to the asset price at time t. If we consider, for example, a group of fundamentalists f_2 for which

$$p_{f_2,t} = p_{t-1} \implies \delta_{f_2,t} = p_t - p_{t-1} \tag{3.15}$$

then we would have (from equation (3.13))

$$z_{f_2,t} = \bar{c}_{f_2}(p^* - p_{t-1}) = \frac{\bar{c}_{f_2}}{\bar{c}_f} z_{f,t-1}, \tag{3.16}$$

that is, group f_2 trades the asset at time t in the same way as f_1 did in the previous time step $t-1$, except for a multiplicative constant given by the ratio between their portfolios, *as if* they reacted to the same fundamentals news with a delay. In fact, one of the advantages of describing a group of investors in this way is that we can easily take into account delayed information when they share it with or receive it from other groups. As we highlighted in Paragraph 2.1.1.3, this aspect has not been treated by any other DCA models that considered social interaction among investors, and we will consider it in our model through the help of a function $\alpha_{h,t}$ called the *sharing mechanism*, as we discuss below.

3.1.2.2 Sharing mechanism

More generally, a group of investors may be affected by several sources of information acting as external influences (e.g. different groups of investors sharing their information) and leading them to shift between different reference prices, including the one they had in the previous time step. For example, group h's reference price, $p_{h,t}$, could be described as a combination of its past value, $p_{h,t-1}$, and the asset's current price p_t (gradual update of their reference price), namely,

$$p_{h,t} = \alpha_{h,t} p_t + (1 - \alpha_{h,t}) p_{h,t-1} = p_{h,t-1} + \alpha_{h,t} \left(p_t - p_{h,t-1} \right) \tag{3.17}$$

The previous equation is the *updating* rule for group h's reference price, where $0 \leq \alpha_{h,t} \leq 1$ represents group h's *sharing mechanism* for their reference price and describes the fraction of investors belonging to group h who refers to the current price p_t instead of their past reference price, $p_{h,t-1}$. If we set $\alpha_{h,t} = \alpha_h = 1$, we get equation (3.14), and if we choose $\alpha_{h,t} = \alpha_h = 0$, group h would only refer to its past reference price, that is,

$$p_{h,t} = p_{h,t-1} = p_{h,t-2} = \ldots = p_{h,0} \tag{3.18}$$

where $p_{h,0}$ is group h's initial reference price (a constant, meaning they will be either constantly buying, selling, or holding the asset). If group h interacts with the other groups of investors H that populate the market, then Equation (3.17) can be generalised as

$$p_{h,t} = p_{h,t-1} + \sum_{i \neq h}^{H} \gamma_i \alpha_{i,t-1} \left(p_{i,t-1} - p_{h,t-1} \right) \tag{3.19}$$

$$\sum_{i \neq h}^{H} \gamma_i = 1 \tag{3.20}$$

where $0 < \gamma_i < 1$ is the degree of *reliability* that group h associate with the source of information i, and $0 < \alpha_{i,t-1} < 1$ describes the fraction of investors belonging to group i that shares their information (their reference price $p_{i,t-1}$) with group h at time $t-1$. Notice the similarity of the previous equation to that of market maker update price (3.7), which is in line with what is commonly called a *coupled map* in Complex Systems (Kaneko (2015)), in which the Kuramoto model (our inspirational model) also belongs.

In fact, the previous difference equation reminds us of this model, which describes the motion of N coupled oscillators rotating on a circle, whose equation (in discrete time) is given by

$$\theta_{k,t} = \theta_{k,t-1} + \Omega + \frac{1}{N}\sum_{i=1}^{N} A_{i,k} \sin\left(\theta_{i,t-1} - \theta_{k,t-1}\right) \tag{3.21}$$

where $\theta_{k,t}$ is the phase of the *k-th* oscillator, Ω is the natural frequency equal to all oscillators, and $A_{l,k}$ the coupling constant between the *i-th* and the *k-th* oscillator. In the previous equation, the frequency Ω explicitly appears, while in equation (3.19), it does not. However, one can always make the frequency Ω disappear by analysing the system (3.21) in a rotating framework with frequency Ω. As we suggested at the end of 2.2.2.3, there are strong analogies between some phenomena observed in asset prices and the phase transitions in the Kuramoto model. By analysing this model, one can observe phase transitions called 'explosive synchronisations' (in which all oscillators have the same phase delay), and there could be a sort of correspondence with similar phenomena observed in financial markets related to anomalous price fluctuations, like excess volatility or flash crashes. In our model, this would correspond to a period of time where most investors would adopt the same reference price, which leads them to send orders in the same direction, creating extreme volatility or giving rise to price bubbles. The correspondence between our model and the Kuramoto model is not exact, but it is possible that a related model could describe the price dynamics. In the appendix B.3, we show how we can make this correspondence look more similar by assuming limited supplies/holdings and bounded beliefs.

Behavioural finance and reference prices Behavioural finance teaches us how most behavioural biases can lead investors to refer to different past prices when trading an asset in a specific market. Given that investors may also interact with each other, the exchange of information between different groups of investors can be interpreted as a reference price sharing and, consequently, it will lead investors to change theirs over time. More importantly, social biases may amplify or reduce any cognitive or emotional biases, potentially giving rise to market anomalies we are interested in (e.g., bubbles, excess volatility). The sharing mechanism considers this factor and the fact that groups are willing to share their information (or obtain information) only if some conditions are satisfied. Although we will only consider cases where the sharing mechanism represents the fraction of investors belonging to a group who share their reference price with other groups (social interaction), in principle, nothing prevents us from considering more general cases in which it may include information of a completely different nature, such as a group's tendency to be affected by a specific behavioural bias. On top of that, as we will see at the end of this chapter, the general form of the price dynamics will resemble the one that describes the motion of coupled oscillators rotating on a circle, that is, the Kuramoto model. This aligns with the idea that a related model may describe the price dynamics given the strong analogies between some phenomena observed in asset prices and the phase transitions in the Kuramoto model.

3.1.3 Discrete choice approach - Trading and sharing mechanism

As we have seen in the previous sections, only a fraction $n_{h,t}$ of investors belonging to group h decides to trade a volume of shares $z_{h,t}$ at time t and only a fraction $\alpha_{i,t}$ of investors belonging to group i decides to share their reference price $p_{i,t}$ with group h at time t. In the Brock-Hommes model, the function $n_{h,t}$ is described by equation (2.19), valid when investors switch from one strategy to another and, as we already argued, represents long-term decisions. In our model, this role is played by $\alpha_{i,t}$, while $n_{h,t}$ describes investors making other decisions based on their current strategy in the short term, namely whether to trade (buy or sell) or not (hold) an asset.

3.1.3.1 Trading and sharing threshold

At the beginning of section 2.2.3, we have shown an example of investors trading an asset by referring to a threshold distance Δ between the stock price and their perceived value of the asset beyond which they would trade it. More specifically, for a generic investor h, we had:

$$|p^*_{h,t} - p_t| > \Delta_{\tau,h,t} \qquad \text{trade the asset} \tag{3.22}$$

$$|p^*_{h,t} - p_t| < \Delta_{\tau,h,t} \qquad \text{hold the asset} \tag{3.23}$$

with $0 \leq \Delta_{\tau,h,t} < +\infty$ called *trading threshold* (τ stands for trading). If $p^*_{h,t} - p_t > 0$, the asset is undervalued according to investor h's price belief, and they will buy it. If $p^*_{h,t} - p_t < 0$, the asset is overvalued according to investor h's price belief, and they will sell it. To some extent, we can argue that long-term decisions follow similar rules where a generic investor h gets information about a strategy from investor i (investor i shares their information with investor h), that is

$$|p^*_{i,t} - p_t| > \Delta_{\sigma,i,t} \qquad \text{share the reference price } p_{i,t} \tag{3.24}$$

$$|p^*_{i,t} - p_t| < \Delta_{\sigma,i,t} \qquad \text{do not share the reference price } p_{i,t} \tag{3.25}$$

with $0 \leq \Delta_{\sigma,i,t} < +\infty$ called *sharing threshold* (σ stands for sharing). The previous conditions recall the one expressed in the appendix B.1 (equation (B.1)), to which investors refer when they choose between two or more strategies based on their performance. Here, we will use the same arguments covered back there to derive the probability $P_{h,t}$ for investor h of trading the asset at time t or for investor i of sharing their reference price $p_{i,t}$ replace it with the fraction $n_{h,t}$ and $\alpha_{i,t}$ trading it in a market with infinitely many investors.

Trading threshold Here, we discuss the role of the trading threshold, $\Delta_{\tau,h,t}$, while a similar discourse can be made for the sharing threshold (whose discussion will be deepened in the next chapters). Any investor refers to $\Delta_{\tau,h,t}$, beyond which they would trade the asset with a probability $P_{h,t}$ dictated by the conditions (3.22) and (3.23), which can be expressed more generally as

$$u(|p^*_{h,t} - p_t|) > v(\Delta_{\tau,h,t}) \qquad \text{trade the asset} \tag{3.26}$$

$$u(|p^*_{h,t} - p_t|) < v(\Delta_{\tau,h,t}) \qquad \text{hold the asset} \tag{3.27}$$

where $u(|p^*_{h,t} - p_t|)$ and $v(\Delta_{\tau,h,t})$ are strictly increasing functions of $|p^*_{h,t} - p_t|$ and $\Delta_{\tau,h,t}$, respectively (we will discuss their forms below). We see that the condition (3.26) will be more or less easily satisfied depending on how large $\Delta_{\tau,h,t}$ is, meaning that this parameter is related to how frequently an investor trades (clearly the trading frequency also depends on how fast $p^*_{h,t}$ varies relatively to the current price p_t). So, investors for which $\Delta_{\tau,h,t} \to 0$ could theoretically represent daily (or intra-day) traders, but it could also mean that we are observing the system on a time scale large enough where all investors trade the asset in any time t (months, years), while larger values of $\Delta_{\tau,h,t}$ could correspond to investors who trade less frequently or that we observe the system evolving on a short time scale (weeks, days). We will better specify the right interpretation once we compare the trading threshold $\Delta_{\tau,h,t}$ with the sharing threshold $\Delta_{\sigma,i,t}$ in each case treated in the next chapters. Here again, notice how the conditions (3.26) and (3.27) resemble the fundamentalists' performance measure in the Brock-Hommes model given by equation (2.46). If we reinterpret the Brock-Hommes model within our approach, fundamentalists will trade the asset if their past realized profits become high enough to overcome the cost c.

Discrete choice Investors have to make a *preference* choice dictated by the conditions (3.26) and (3.27), similarly to what happens when investors switch between different strategies in the Brock-Hommes model. So, one would be tempted to adopt the same discrete choice approach in which the probability $P_{h,t}$ for an investor h to trade the asset at time t is given by the *binomial logit model*. However, two important issues need to be stressed in doing this. Firstly, there is no guarantee that the discrete choice approach describes the *trade-hold* case as in the strategy-switch one, and it is good to keep in mind that there is a degree of freedom (the same liberty that Brock and Hommes took in their model) in choosing which model could reasonably represent investor behaviour. Secondly, by adopting the discrete choice approach, one should interpret the intensity of choice (β) differently from what Brock and Hommes did in their model (and in any HAMs). The intensity of choice in a HAM measures the degree of rationality of each investor when they switch between different strategies, *as a consequence of behavioural biases* (and more), potentially leading investors to make errors (e.g. stick to a strategy less profitable than another for too long). However, through the concept of *actual* and *perceived* values that we have introduced here (endogenous factors determine the actual values, while the exogenous factors determine the perceived ones), we transferred all the information that could lead investors to take decisions in a boundedly rational way, from β to $p^*_{h,t}$. So, in principle, as soon as the condition (3.26) is satisfied, investor h would trade the asset with *certainty* (as if they were fully rational, even in making 'wrong' decisions) since we took into account all sources of information that an investor refers to when making a decision (consciously or unconsciously[3]). We outline three possible motivations that could justify the discrete choice approach, regardless of the information held by any investor:

1. Because of human nature: the decision behaviour is probabilistic rather than deterministic (Helbing (2010)). Even if investors were exposed to the same information on multiple occasions, they would make decisions in a probabilistic way. So, what we see from the data (a single experiment, with investors trading at a specific time and volume of shares) is only one possible outcome whose probability is given by the bimodal logit model. The intensity of choice β represents the degree of uncertainty for investor h in making a decision.

2. Because of statistical nature: we focus on groups of investors with access to the same sources of information. So, each investor belonging to a group obeys conditions (3.26) and (3.27) with certainty, but since they will inevitably show some degree of difference from each other, we can only describe their behaviour statistically by using the bimodal logit model. The intensity of choice β represents the degree of uncertainty for the group of investors h (how different are investors within the same group) in making a decision.

3. Because of modelling nature: it is impossible to keep track of all sources of information that affect investors in their decision-making processes. Even if investors obey conditions (3.26) and (3.27) with certainty, modellers have to take into account possible *hidden variables* of which they themselves are ignorant, and that affect investor behaviour, so they can be described only probabilistically by

[3] for example, as a consequence of *anchoring biases* (see Nofsinger (2017))

using the bimodal logit model. The intensity of choice β represents the modellers' uncertainty about investor h's decision-making behaviour.

All these motivations seem reasonable to us, and we can stick to one from now on. So, assuming, for example, that the bimodal logit model describes investor h's behaviour in making decisions, then we have that the probability $P_{h,t}$ of trading the asset at time t is given by

$$P_{h,t} = \frac{e^{\beta_h u(|p^*_{h,t} - p_t|)}}{e^{\beta_{\tau,h} u(|p^*_{h,t} - p_t|)} + e^{\beta_{\tau,h} v(\Delta_{\tau,h,t})}} \tag{3.28}$$

where $\beta_{\tau,h}$ is the degree of *uncertainty* for investor h in making the decision to trade or not the asset.

Functional forms for $u(|p^*_{h,t} - p_t|)$ and $v(\Delta_{\tau,h,t})$ Investors refer to the functions $u(|p^*_{h,t} - p_t|)$ and $v(\Delta_{\tau,h,t})$ when they consider whether or not to trade an asset, as indicated by the conditions (3.26) and (3.27). Brock and Hommes referred to the past realized profit for the performance measure, which is very reasonable and based on common sense since, to avoid going bankrupt, investors have to refer to how well a strategy performs compared to the others. Given that investors focus on other factors over short periods, we will make some assumptions mainly driven (as a starting point) by mathematical induction and convenience. First of all, instead of considering the absolute value of $|p^*_{h,t} - p_t|$, which would force us to investigate the system using piecewise-linear functions and, thus, not differentiable (not very convenient for our analysis), we consider its value squared, $(p^*_{h,t} - p_t)^2$. Secondly, we would expect that, as $(p^*_{h,t} - p_t)^2$ tends to 0, the probability $P_{h,t}$ in equation (3.28) would also tend to 0. Thus, the functional form $u((p^*_{h,t} - p_t)^2)$ should be chosen such that

$$(p^*_{h,t} - p_t)^2 \to 0 \implies u((p^*_{h,t} - p_t)^2) \to -\infty \implies P_{h,t} \to 0 \tag{3.29}$$

A function that suits our case is given by

$$u((p^*_{h,t} - p_t)^2) = ln[(p^*_{h,t} - p_t)^2] \tag{3.30}$$

and we assume the same form for $v(\Delta_{\tau,h,t})$, where

$$\Delta_{\tau,h,t} \to 0 \implies v(\Delta_{\tau,h,t}) \to -\infty \implies P_{h,t} \to 1 \tag{3.31}$$

If $\Delta_{\tau,h,t} = 0$ for all times t, we assume that investor h will always trade the asset, that is, $P_{h,t} = 1$. The conditions (3.26) and (3.27) can be written as

$$ln[(p^*_{h,t} - p_t)^2] > ln(\Delta_{\tau,h,t}) \qquad \text{trade the asset} \tag{3.32}$$

$$ln[(p^*_{h,t} - p_t)^2] < ln(\Delta_{\tau,h,t}) \qquad \text{hold the asset} \tag{3.33}$$

and the binomial logit model takes the form

$$P_{h,t} = \frac{e^{\beta_{\tau,h} ln[(p^*_{h,t} - p_t)^2]}}{e^{\beta_{\tau,h} ln[(p^*_{h,t} - p_t)^2]} + e^{\beta_{\tau,h} ln(\Delta_{\tau,h,t})}} = \frac{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}}}{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}} + \Delta^{\beta_{\tau,h}}_{\tau,h,t}} \tag{3.34}$$

The previous equation is very similar to what Naimzada and Ricchiuti (2008) assume when investors switch between two different fundamental values.

If a group of investors behave in the same way (have similar values of $p^*_{h,t}$, $c_{h,t}$, $\Delta_{\tau,h,t}$, and $\beta_{\tau,h}$), as the number of investors tends to infinity, the probability (3.34) can be replaced by the fraction $n_{h,t}$, the trading mechanism, of those who trade the asset at time t

$$n_{h,t} = \frac{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}}}{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}} + \Delta^{\beta_{\tau,h}}_{\tau,h,t}} \tag{3.35}$$

For now, we assume that the same is true for the fraction $\alpha_{i,t}$, the sharing mechanism, of those who share their reference price $p_{i,t}$ at time t

$$\alpha_{i,t} = \frac{\left[(p^*_{i,t} - p_t)^2\right]^{\beta_{\sigma,i}}}{\left[(p^*_{i,t} - p_t)^2\right]^{\beta_{\sigma,i}} + \Delta^{\beta_{\sigma,i}}_{\sigma,i,t}} \tag{3.36}$$

As we said above, we will discuss the sharing mechanism more deeply in the next chapters when we consider cases with different groups of investors interacting with each other.

3.1.3.2 General model of a financial market with social influence

From Equations (3.7) (market maker price update rule), (3.13) (investors' demand), (3.19) (investors' reference price update rule), (3.20) (investors' degrees of reliability), (3.35) (trading mechanism), and (3.36) (sharing mechanism), we can finally outline our model as a system of difference equations:

$$\begin{cases} p_t = p_{t-1} + \mu \sum_{h=1}^{H} n_{h,t-1} \bar{c}_h (p^* - p_{h,t-1}) \\ p_{h,t} = p_{h,t-1} + \sum_{i \neq h}^{H} \gamma_i \alpha_{i,t-1} \left(p_{i,t-1} - p_{h,t-1}\right) \\ \sum_{i \neq h}^{H} \gamma_i = 1 \\ n_{h,t} = \frac{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}}}{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}} + \Delta_{\tau,h,t}^{\beta_{\tau,h}}} \\ \alpha_{i,t} = \frac{\left[(p^*_{i,t} - p_t)^2\right]^{\beta_{\sigma,i}}}{\left[(p^*_{i,t} - p_t)^2\right]^{\beta_{\sigma,i}} + \Delta_{\sigma,i,t}^{\beta_{\sigma,i}}} \end{cases} \tag{3.37}$$

In the following section (and chapters), we will analyse our model by considering several cases with the interplay between groups of fundamentalists on different time scales, distinguished by their level of sophistication, and identify those critical points beyond which potential temporary bubbles and high price fluctuations arise (stability analysis). Sophisticated investors (e.g. institutional investors) represent the market component with the largest volume traded, so we expect they will predominantly influence the price compared to any other category of investors. However, we are interested to see under which conditions unsophisticated investors (e.g. retail investors) can eventually drive the price away from its fundamental value (bubbles) and give rise to high price fluctuations (anomalous volatility) despite representing a minority in the market (in terms of volume traded).

3.2 A market with only a single group of fundamentalists

As a starting point, we will consider the simplest case: a market populated by a single group of investors. This is a very convenient way to understand the effect the parameters describing a group of investors have on the trading activity of such a group and the price dynamics. As a group of investors, we will first focus on *pure fundamentalists*, whose demand, $z_{f,t}$, is simply given by

$$z_{f,t} = c_f (p^* - p_t) \tag{3.38}$$

where p^* is the asset's fundamental value, that is, their perceived value of the asset estimated by performing a fundamental analysis on the traded asset, and c_f represents their dominance. Their corresponding net demand, $d_{f,t}$, is (see equation (3.5))

$$d_{f,t} = c_f (p^* - p_t) \frac{\left[(p^* - p_t)^2\right]^{\beta_f}}{\left[(p^* - p_t)^2\right]^{\beta_f} + \Delta_f^{\beta_f}} \tag{3.39}$$

where Δ_f is their trading threshold (we dropped the sub-index τ, since there are no other groups that share their information with group f), and β_f is their degree of uncertainty. Since they are the only group of investors in the market, we have

$$z_{d,t} = d_{f,t} \tag{3.40}$$

meaning that the price is updated as

$$p_t = p_{t-1} + \mu c_f (p^* - p_{t-1}) \frac{\left[(p^* - p_{t-1})^2\right]^{\beta_f}}{[(p^* - p_{t-1})^2]^{\beta_f} + \Delta_f^{\beta_f}} \tag{3.41}$$

We can express the previous equation in terms of the price deviation from the fundamental value, $x_t = p_t - p^*$, so we get

$$x_t = x_{t-1} - \mu c_f x_{t-1} \frac{\left[x_{t-1}^2\right]^{\beta_f}}{\left[x_{t-1}^2\right]^{\beta_f} + \Delta_f^{\beta_f}} = x_{t-1} \left(1 - \mu c_f \frac{\left[x_{t-1}^2\right]^{\beta_f}}{\left[x_{t-1}^2\right]^{\beta_f} + \Delta_f^{\beta_f}} \right) \tag{3.42}$$

We will try to extrapolate as much information as possible from this one-dimensional map and exploit it when we add other groups of investors. We will tackle the analysis by first assuming very simple conditions, such as the limit of the trading threshold that tends to $\Delta_f \to +\infty$ and $\Delta_f \to 0$, then we will consider more general cases.

3.2.1 Holding state and system on large-time scale

For $\Delta_f \to +\infty$, investors are in '*holding*' state for any price p_t and the map (3.42) simply becomes

$$x_t = x_{t-1} \tag{3.43}$$

so, the solution of the map is any initial price $x_0 = p_0 - p^*$. For $\Delta_f \to 0$, instead, all investors who belong to the group trade the asset in each time step t ($n_{h,t} = 1$) (again, this could be interpreted as observing the dynamics of the system on a large time-scale in which all investors will have traded the asset between each time step and the map becomes

$$x_t = x_{t-1} \left(1 - \mu c_f\right) \tag{3.44}$$

the general solution of which is Elaydi (2007):

$$x_t = \left(1 - \mu c_f\right)^t x_0 \tag{3.45}$$

where

1. If $\left|1 - \mu c_f\right| < 1$, then $\lim_{t\to\infty} |x_t| = 0$.

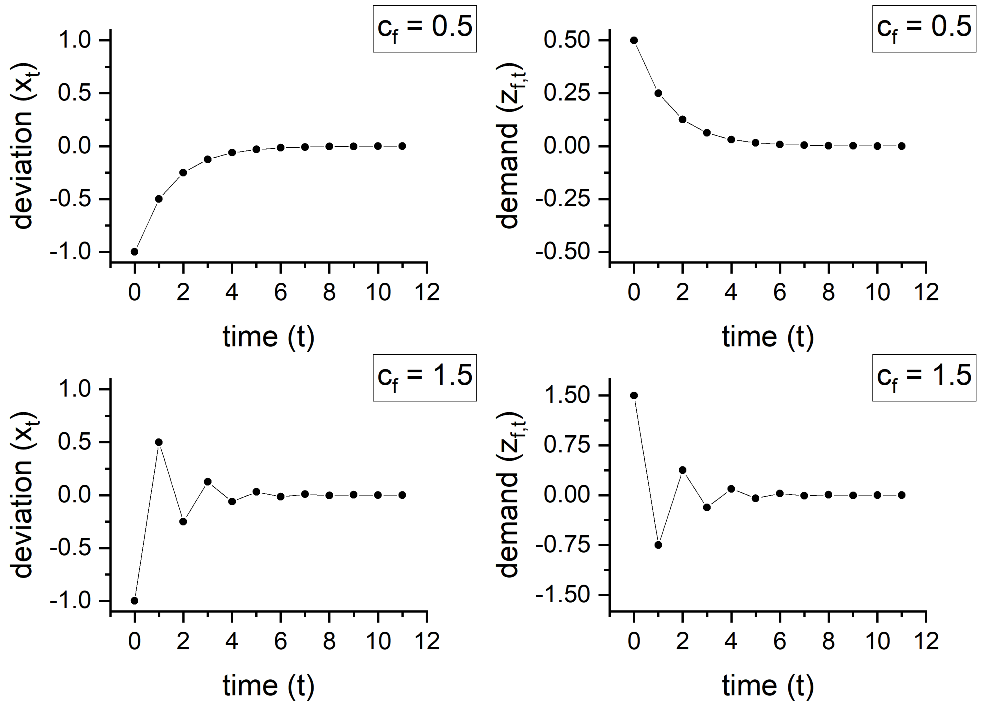


FIGURE 3.1: The deviation x_t and fundamentalists' demand $z_{f,t}$ (equation (3.38)) for two different values of c_f that satisfy the condition $\left|1-\mu c_f\right| < 1$ ($c_f = 0.5$ and $c_f = 1.5$, for $\mu = 1$), with $x_0 = -1$.

2. If $\left|1-\mu c_f\right| > 1$, then $\lim_{t\to\infty} |x_t| = +\infty$.
3. (a) If $1-\mu c_f = 1 \iff \mu c_f = 0$, then $x_t = x_0$ for every t.
   (b) If $1-\mu c_f = -1 \iff \mu c_f = 2$, then $x_t = (-1)^n x_0$ (periodic of period 2).

From here on, we assume that investors' portfolio c_f can only take strictly positive values, that is, $c_f > 0$ (and the same goes for μ), meaning that

$$1-\mu c_f \in (-\infty, 1) \tag{3.46}$$

So, we exclude from our analysis the limit for which $\mu c_f = 0$ (solution 3.(a)). The first solution tells us that the price converges to the fundamental value (Figure 3.1) for

$$\left|1-\mu c_f\right| < 1 \iff -1 < 1-\mu c_f < 1 \iff 0 < \mu c_f < 2 \tag{3.47}$$

For $\mu c_f = 2$, the price does not converge to the fundamental value, but, instead, it will take, periodically, two values: x_0, when t is even, and $-x_0$, when t is odd (top panels of Figure 3.2). Unless the initial price is chosen exactly equal to the fundamental value, $p_0 = p^*$, the price will fluctuate back and forth between two prices, which are equally

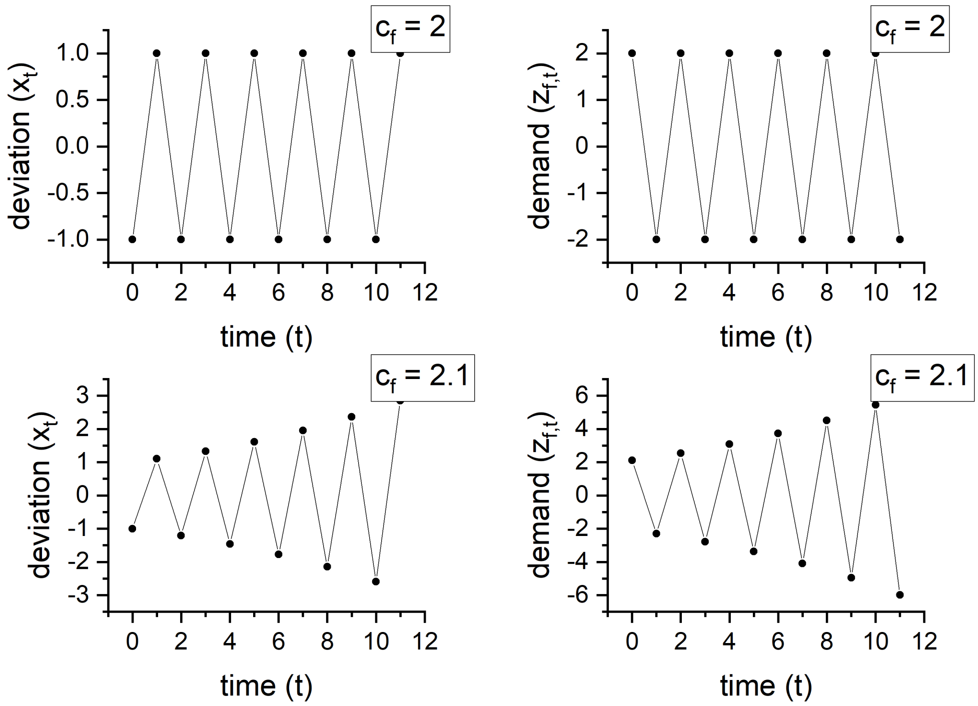


FIGURE 3.2: The deviation x_t and fundamentalists' demand $z_{f,t}$ for two different values of c_f that satisfy the condition of the solution 3.(a) ($c_f = 2$ and $\mu = 1$) and $\left|1 - \mu c_f\right| < -1$ ($c_f = 2.1$ and $\mu = 1$), with $x_0 = -1$.

distant from the fundamental value. As soon as $\mu c_f > 2$, the price diverges to infinity, in an oscillatory way (bottom panels of Figure 3.2). The main reason why this happens is because of investors' unlimited liquidity and supply for trading the asset, which allows them to buy or sell the asset at any price, along with (second reason) the market maker's lack of adaptation into slowing down the price adjustment for such large excess demand (μ is a constant value). The conditions for these phenomena to appear (*exploding* prices) in a stock market are unreal since they can not be observed (if not for a very short period) and, thus, are of little interest. To conclude, this particular case was also important for showing that, in our following analysis, we will have to distinguish between the persistent price fluctuations of *market maker-investor* type (solution for $\mu c_f = 2$), generated by the interaction between the market maker and the groups of investors trading the asset (which can be observed in real markets), from those of *investor-investor* type, generated by the interaction between different groups of investors, which will appear in more complex cases.

3.2.2 System on short-time scale for fixed value of the degree of certainty

Next, we consider the more general case where the trading threshold can take any value (observing the system on a shorter time scale) for $\beta_f = 1$. The map (3.42) takes the form

$$x_t = x_{t-1}\left(1 - \mu c_f \frac{x_{t-1}^2}{x_{t-1}^2 + \Delta_f}\right) \tag{3.48}$$

with

$$n_{f,t} = \frac{x_t^2}{x_t^2 + \Delta_f} \tag{3.49}$$

representing the fraction of fundamentalists trading the asset at time t, and

$$z_{f,t} = -c_f x_t \tag{3.50}$$

their corresponding demand. The map (3.48) can be written as a one dimensional map, $F(x)$,

$$F(x) = x\left(1 - \mu c_f \frac{x^2}{x^2 + \Delta_f}\right) \tag{3.51}$$

We cannot find a general solution of (3.51), so we will look for its steady states, and we will try to analyse their stability.

Fixed points and stability The fixed points x^* of the map (3.51) can be found by solving

$$F(x^*) = x^* \iff x^* = x^*\left(1 - \mu c_f \frac{(x^*)^2}{(x^*)^2 + \Delta_f}\right) \tag{3.52}$$

and one can easily see that the only solution to equation (3.52) is $x^* = 0$, that is, the fundamental value. To study the stability of this fixed point, we take the derivative of the map $F(x)$,

$$\begin{aligned} F'(x) &= \left(1 - \mu c_f \frac{x^2}{x^2 + \Delta_f}\right) - \mu c_f x \left[\frac{2x\left(x^2 + \Delta_f\right) - x^2(2x)}{\left(x^2 + \Delta_f\right)^2}\right] \\ &= 1 - \mu c_f \frac{x^2}{x^2 + \Delta_f} - 2\mu c_f \Delta_f \frac{x^2}{\left(x^2 + \Delta_f\right)^2} \end{aligned} \tag{3.53}$$

and evaluate it at $x^* = 0$, which gives us

$$F'(x)|_{x^*=0} = 1 \tag{3.54}$$

So, the fundamental value is a non-hyperbolic fixed point of the map (3.51). To see if it is stable, we must derive higher-order derivatives of $F(x)$ and evaluate them at this point. We refer to a theorem which can be found in Elaydi (2007):

Theorem 3.1. *Let x^* be a fixed point of a map F such that $F'(x)|_{x^*} = 1$. If $F'(x)$, $F''(x)$, and $F'''(x)$ are continuous at x^*, then the following statements hold:*

1. *If $F''(x)|_{x^*} \neq 0$, then x^* is unstable (or semistable)*[4].
2. *If $F''(x)|_{x^*} = 0$ and $F'''(x)|_{x^*} > 0$, then x^* is unstable.*
3. *If $F''(x)|_{x^*} = 0$ and $F'''(x)|_{x^*} < 0$, then x^* is asymptotically stable.*

In our case, we have

$$\begin{aligned} F''(x) = & -2\mu c_f \Delta_f \left[\frac{x}{\left(x^2+\Delta_f\right)^2} \right] + \\ & - 2\mu c_f \Delta_f \left[\frac{2x\left(x^2+\Delta_f\right)^2 - 4x^3\left(x^2+\Delta_f\right)}{\left(x^2+\Delta_f\right)^4} \right] \\ & = -2\mu c_f \Delta_f \left[\frac{x}{\left(x^2+\Delta_f\right)^2} \right] - 2\mu c_f \Delta_f \left[\frac{2x\left(\Delta_f^2 - x^4\right)}{\left(x^2+\Delta_f\right)^4} \right] \\ & = -2\mu c_f \Delta_f \left[\frac{x}{\left(x^2+\Delta_f\right)^2} \right] - 4\mu c_f \Delta_f^3 \left[\frac{x}{\left(x^2+\Delta_f\right)^4} \right] + \\ & + 4\mu c_f \Delta_f \left[\frac{x^5}{\left(x^2+\Delta_f\right)^4} \right] \end{aligned} \tag{3.55}$$

Therefore, $F''(x)|_{x^*=0} = 0$. To derive $F'''(x)$, we focus on the squared brackets in equation (3.55) and evaluate them in $x^* = 0$, from which we obtain,

$$\frac{d}{dx}\left[\frac{x}{\left(x^2+\Delta_f\right)^2} \right]_{x^*=0} = \left[\frac{\left(x^2+\Delta_f\right)^2 - 4x^2\left(x^2+\Delta_f\right)}{\left(x^2+\Delta_f\right)^4} \right]_{x^*=0} = \frac{1}{\Delta_f^2} \tag{3.56}$$

$$\frac{d}{dx}\left[\frac{x}{\left(x^2+\Delta_f\right)^4} \right]_{x^*=0} = \left[\frac{\left(x^2+\Delta_f\right)^4 - 8x^2\left(x^2+\Delta_f\right)^3}{\left(x^2+\Delta_f\right)^8} \right]_{x^*=0} = \frac{1}{\Delta_f^4} \tag{3.57}$$

$$\frac{d}{dx}\left[\frac{x^5}{\left(x^2+\Delta_f\right)^4} \right]_{x^*=0} = \left[\frac{5x^4\left(x^2+\Delta_f\right)^4 - 8x^6\left(x^2+\Delta_f\right)^3}{\left(x^2+\Delta_f\right)^8} \right]_{x^*=0} = 0 \tag{3.58}$$

From the previous results, we have

[4] A fixed point which is stable either from the left $(x^* - \delta)$ or from the right$(x^* + \delta)$

$$\begin{aligned} F'''(x)|_{x^*=0} &= -2\mu c_f \Delta_f \left(\frac{1}{\Delta_f^2}\right) - 4\mu c_f \Delta_f^3 \left(\frac{1}{\Delta_f^4}\right) + 4\mu c_{s,f} \Delta_f(0) \\ &= -\frac{6\mu c_f}{\Delta_f} \end{aligned} \tag{3.59}$$

So, based on theorem 3.1, we find that the fundamental steady state $x^* = 0$ is asymptotically stable. It is interesting to observe that, from the last equation (3.59), it seems that the higher investors' portfolio c_f, the 'more' asymptotically stable is the fixed point. This finding is counterintuitive compared to what we found in the previous case, when $\Delta_f \to 0$, for which the deviation x_t diverges to infinity as soon as $\mu c_f > 2$, no matter how close the initial value x_0 was to the fundamental value $x^* = 0$. However, we must keep in mind that this observation is valid only locally, and not globally. To show this, we need to look for periodic points $\bar{x}$, such that

$$F^k(\bar{x}) = \bar{x} \tag{3.60}$$

where k represents the number of times the map F is iterated (the point $\bar{x}$, in this case, would be k-periodic point). Based on the previous outcomes, we expect the map to have at least a double periodic point, such that

$$F^2(\bar{x}) = \bar{x} \tag{3.61}$$

Instead of trying to solve the last equation directly, we can observe that the map (3.51) is antisymmetric with respect to x, that is,

$$F(-x) = -F(x) \tag{3.62}$$

meaning that if there is a point $\bar{x}$ for which

$$F(\bar{x}) = -\bar{x} \tag{3.63}$$

then

$$F^2(\bar{x}) = F(-\bar{x}) = -F(\bar{x}) = \bar{x} \tag{3.64}$$

So, any point that satisfies equation (3.63) is a 2-periodic point. To find such values, we just need to solve

$$\bar{x}\left(1 - \mu c_f \frac{(\bar{x})^2}{(\bar{x})^2 + \Delta_f}\right) = -\bar{x} \tag{3.65}$$

which, after some straightforward algebraic calculations, leads to

$$(\bar{x})^2 = \frac{2\Delta_f}{\mu c_f - 2} \tag{3.66}$$

that is,

$$\bar{x} = \pm\sqrt{\frac{2\Delta_f}{\mu c_f - 2}} \tag{3.67}$$

Thus, the orbit of this 2-periodic point is the set

$$O(\bar{x}) = \left\{ -\sqrt{\frac{2\Delta_f}{\mu c_f - 2}}, \sqrt{\frac{2\Delta_f}{\mu c_f - 2}} \right\} \tag{3.68}$$

It is interesting to notice that, for fixed $\mu c_f > 2$, the amplitude of the price swings are proportional to $\sqrt{\Delta_f}$, which, based on one of our assumptions, could set the trading frequency of a group of investors. This could potentially suggest that the intensity of the price swings arising in stock markets over a period might be proportional to a combination of trading frequencies that identifies all the groups of investors trading during that period (e.g. the smaller the price swings (low Δ_f), the more significant is the contribution from high-frequency investors). To see if this point is stable, we need to verify that the map $F^2(x)$ is stable at this point, that is

$$\begin{aligned} \left|\left[F^2(x)\right]'_{\bar{x}}\right| = \left|\left[F^2(\bar{x})\right]'\right| < 1 &\iff \left|\left[F'(\bar{x})\right]\left[F'(F(\bar{x}))\right]\right| < 1 \\ &\iff \left|F'(\bar{x})F'(-\bar{x})\right| < 1 \\ &\iff \left[F'(\bar{x})\right]^2 < 1 \end{aligned} \tag{3.69}$$

where we used the chain rule, equation (3.63) and, from equation (3.53), the fact that the derivative $F'(x)$ is symmetric with respect to x. So, all we need to do is evaluate the square of the derivative at the point $\bar{x}$, which can be easily done if we substitute equation (3.66) into equation (3.53), obtaining

$$\begin{aligned} F'(\bar{x}) &= 1 - 2\left[\frac{\mu c_f \Delta_f}{(\mu c_f - 2)\left(\frac{2\Delta_f}{\mu c_f - 2} + \Delta_f\right)}\right] - 4\left[\frac{\mu c_f \Delta_f^2}{(\mu c_f - 2)\left(\frac{2\Delta_f}{\mu c_f - 2} + \Delta_f\right)^2}\right] \\ &= 1 - 2 - 4\left(\frac{\mu c_f - 2}{\mu c_f}\right) \\ &= -1 - 4\left(\frac{\mu c_f - 2}{\mu c_f}\right) \end{aligned} \tag{3.70}$$

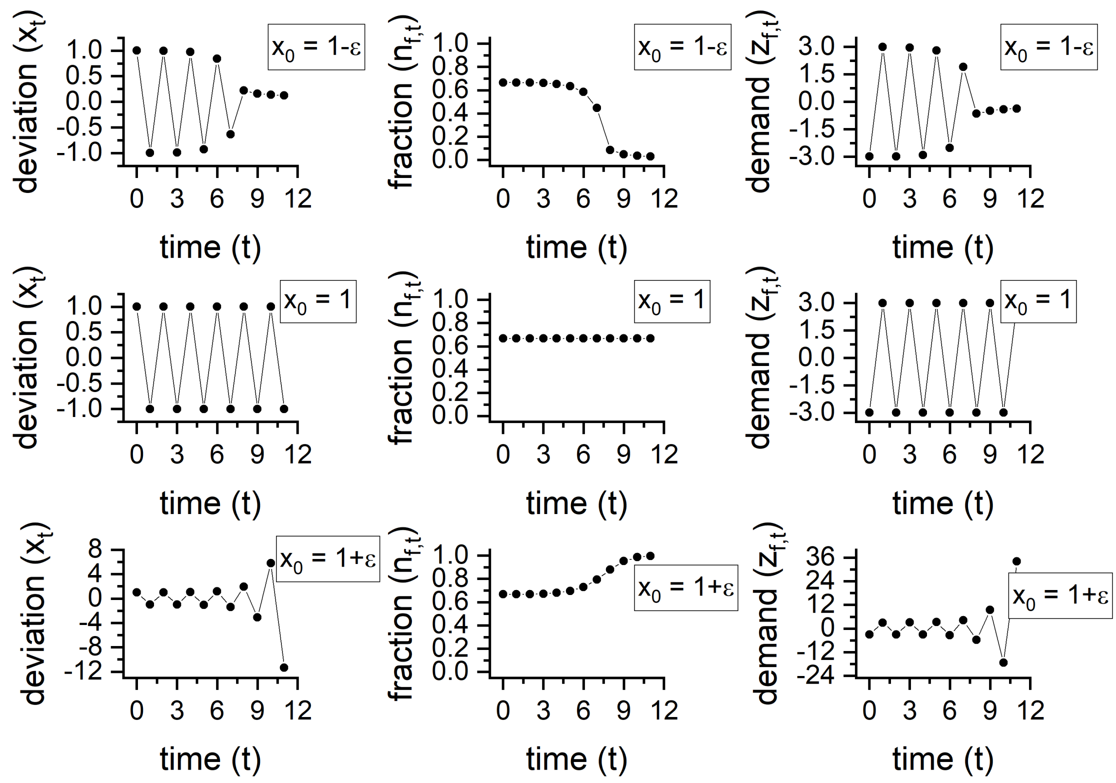


FIGURE 3.3: The deviation x_t, the fraction of fundamentalists $n_{f,t}$, and fundamentalists' demand $z_{f,t}$ for different initial prices $x_0 = \bar{x} + \epsilon$ close to the double period point at $\bar{x} = 1$ (fixed $\mu = 1$ $\Delta_f = 1/2$, and $c_f = 3$ in equation (3.51)) with $\epsilon = 0.001$ (The outcomes for $\bar{x} = -1$ are the same). We see that for $x_0 < 1$ (top panels), the price converges to the fundamental value while the fraction of fundamentalists trading the asset tends to zero, for $x_0 = 1$ (middle panel), the price and investors' demand oscillates periodically (with period 2) and $n_{f,t}$ remains constant (see (3.72)), and for $x_0 > 1$ (bottom panel), the price (and the demand) diverges to infinity with $n_{f,t} \to 1$.

So, for $\mu c_f > 2$, a necessary condition for the existence of the double periodic point (equation (3.67)), we have

$$F'(\bar{x}) < -1 \implies \left[F'(\bar{x})\right]^2 > 1 \tag{3.71}$$

meaning that the double periodic point is unstable. In Figure 3.3, we show the deviation x_t, the fraction of fundamentalists $n_{f,t}$, and fundamentalists' demand $z_{f,t}$ for three different initial prices close to the double periodic point, when the condition $\mu c_f > 2$ is met. We see that for $|x_0| < \bar{x}$, the price converges to the fundamental value while the fraction of fundamentalists trading the asset $n_{f,t}$ (and their demand $z_{f,t}$) tends to zero, for $|x_0| = \bar{x}$ the price (and fundamentalists' demand) oscillates periodically (with period 2) whether the fraction $n_{f,t}$ is constant and equal to (see equation (3.49))

$$n_{f,t} = n_f = \frac{x_0^2}{x_0^2 + \Delta_f} \tag{3.72}$$

and for $|x_0| > \bar{x}$ the price (and the demand) diverges to infinity and $n_{f,t} \to 1$. So, it seems that the basin of attraction of the fundamental steady state of the map (3.51), $W^s(x^* = 0)$, defined as the set

$$W^s(x^* = 0) = \left\{ x : \lim_{k \to +\infty} F^k(x) = x^* = 0 \right\} \tag{3.73}$$

is given by all those points which belong to

$$x \in X^* = \left(-\sqrt{\frac{2\Delta_f}{\mu c_f - 2}}, \sqrt{\frac{2\Delta_f}{\mu c_f - 2}} \right) \tag{3.74}$$

Using the previous set, we can summarise what we said above by illustrating the bifurcation diagram in Figure 3.4, fixed $\Delta_f \geq 0$ and $\mu = 1$, where on the x-axis we have investors' portfolio c_f and on the y-axis the deviation, x, of the asset's price from its fundamental value.

3.2.3 System on short-time scale for any value of the degree of certainty

Most of the results for the general case, the map (3.42), which we rewrite here as

$$F(x) = x \left[1 - \mu c_f \frac{(x^2)^{\beta_f}}{(x^2)^{\beta_f} + \Delta_f^{\beta_f}} \right] \tag{3.75}$$

are similar to those for the case β_f.

Fixed points and stability We still have the fundamental value as a fixed point, and, following the fact that map (3.75) is still antisymmetric with respect to x, a double periodic point which solves the equation

$$\bar{x} \left\{ 1 - \mu c_f \frac{\left[(\bar{x})^2\right]^{\beta_f}}{\left[(\bar{x})^2\right]^{\beta_f} + \Delta_f^{\beta_f}} \right\} = -x^* \tag{3.76}$$

After some straightforward algebraic calculations, this leads to

$$\left[(\bar{x})^2\right]^{\beta_f} = \frac{2\Delta_f^{\beta_f}}{\mu c_f - 2} \tag{3.77}$$

that is,

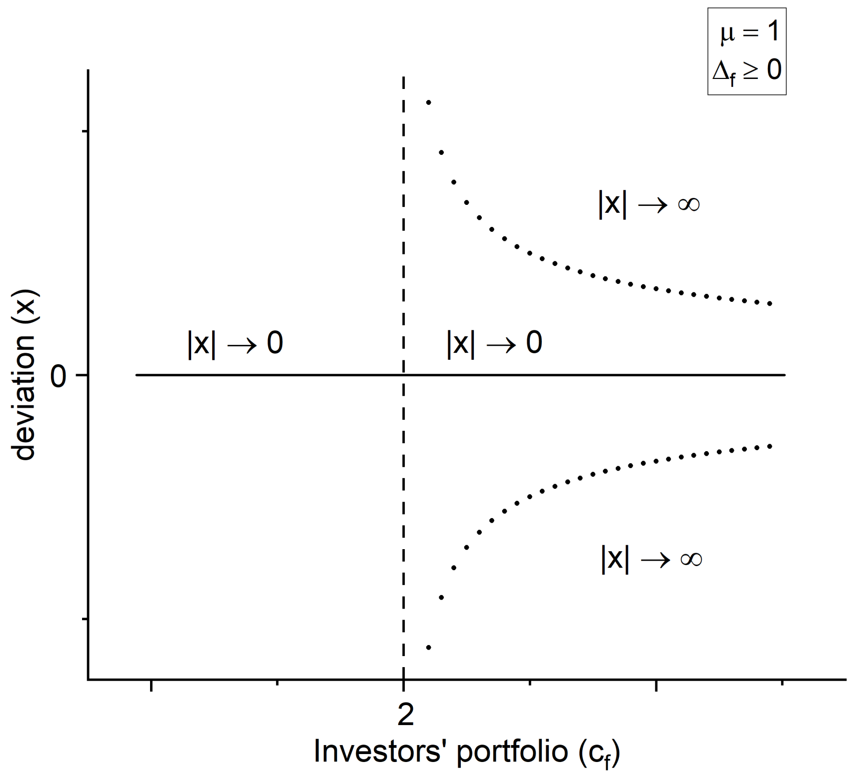


FIGURE 3.4: Bifurcation diagram of the deviation x w.r.t investors' portfolio c_f, fixed $\Delta_f \geq 0$ and $\mu = 1$. The solid line represents the stable fixed point (the fundamental value x^*), while the dotted points the unstable fixed points (double periodic points $\bar{x}$). The vertical dashed line at $c_f = 2$ separates two regions. For $c_f \leq 2$, the fundamental value is the only fixed point of the map (3.51) and globally asymptotically stable, that is, for all initial values $x_0 \in \Re$ we have $|x_t| \to 0$ for $t \to \infty$). For $c_f = 2$, we have a period-doubling bifurcation, so that for $c_f > 2$, in addition to the fundamental steady state (locally asymptotically stable), we have a 2-periodic point whose orbit is given by equation (3.68). In this region, for all initial values x_0 that belong to the set X^* (3.74) we have $|x_t| \to 0$ for $t \to \infty$, while for $x_0 \notin X^*$ we have $|x_t| \to \infty$ for $t \to \infty$.

$$\bar{x} = \pm\sqrt{\Delta_f}\sqrt{\left(\frac{2}{\mu c_f - 2}\right)^{\frac{1}{\beta_f}}} \tag{3.78}$$

In contrast to the case where $\beta_f = 1$, we find some limitations if we want to apply theorem 3.1 in establishing the stability of the fundamental steady state, $x^* = 0$. To show that, we take the derivative of the map (3.75)

$$\begin{aligned}
F'(x) &= \left[1 - \mu c_f \frac{\left(x^2\right)^{\beta_f}}{\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}}\right] + \\
&- \mu c_f x \left\{ \frac{2\beta_f x \left(x^2\right)^{\beta_f - 1} \left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right] - 2\beta_f x \left(x^2\right)^{\beta_f} \left(x^2\right)^{\beta_f - 1}}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^2} \right\} \\
&= \left[1 - \mu c_f \frac{\left(x^2\right)^{\beta_f}}{\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}}\right] - \mu c_f x \left\{ \frac{2\beta_f \Delta_f^{\beta_f} x \left(x^2\right)^{\beta_f - 1}}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^2} \right\} \\
&= 1 - \mu c_f \left\{ \frac{\left(x^2\right)^{\beta_f}}{\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}} \right\} - 2\mu c_f \Delta_f^{\beta_f} \beta_f \left\{ \frac{\left(x^2\right)^{\beta_f}}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^2} \right\}
\end{aligned} \tag{3.79}$$

and evaluate it at $x^* = 0$, which gives us $F'(x)|_{x^*=0} = 1$, meaning that the fundamental value is still a non-hyperbolic fixed point. We proceed, then, to derive higher-order derivatives of the map (3.75) and we will try to evaluate them at the same point. For $F''(x)$, we take the derivative of the curly brackets in equation (3.79), obtaining

$$\frac{d}{dx} \left\{ \frac{\left(x^2\right)^{\beta_f}}{\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}} \right\} = \left\{ \frac{2\beta_f \Delta_f^{\beta_f} x \left(x^2\right)^{\beta_f - 1}}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^2} \right\} \tag{3.80}$$

$$\frac{d}{dx} \left\{ \frac{\left(x^2\right)^{\beta_f}}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^2} \right\} = \left\{ \frac{2\beta_f x \left(x^2\right)^{\beta_f - 1} \left[\Delta_f^{2\beta_f} - \left(x^2\right)^{2\beta_f}\right]}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^4} \right\} \tag{3.81}$$

So, we have

$$\begin{aligned}
F''(x) = -2\mu c_f \beta_f \Delta_f^{\beta_f} &\left\{ \frac{x \left(x^2\right)^{\beta_f - 1}}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^2} \right\} + \\
- 4\mu c_f \beta_f^2 \Delta_f^{\beta_f} &\left\{ \frac{x \left(x^2\right)^{\beta_f - 1} \left[\Delta_f^{2\beta_f} - \left(x^2\right)^{2\beta_f}\right]}{\left[\left(x^2\right)^{\beta_f} + \Delta_f^{\beta_f}\right]^4} \right\}
\end{aligned} \tag{3.82}$$

If we set $\beta_f = 1$, we obtain equation (3.55), but for $\beta_f \neq 1$ we must pay attention on the continuity of $F''(x)$ at $x^* = 0$. For $\beta < 1/2$, the function $F''(x)$ is not defined at this point, because of the term $x \left(x^2\right)^{\beta_f - 1}$. This discontinuity will also appear at higher-order derivatives, even if we choose $\beta_f > 1$, meaning that we can not establish the stability of the map (3.75), unless $\beta_f \in \mathbb{N}^+$. Notice that this does *not* mean that the fundamental value is unstable for other values of β_f than those belonging to this set, but only that we can not apply the theorem 3.1. However, this problem does not arise if we look for the stability of the double periodic point (3.78). In fact, if we plug equation (3.77) into equation (3.79), we get

$$
\begin{aligned}
F'(x) &= 1 - 2\left[\frac{\mu c_f \Delta_f^{\beta_f}}{(\mu c_f - 1)\left(\frac{\Delta_f^{\beta_f}}{\mu c_f - 1} + \Delta_f^{\beta_f}\right)}\right] + \\
&- 4\left[\frac{\mu c_f \Delta_f^{2\beta_f}\beta_f}{(\mu c_f - 1)\left(\frac{\Delta_f^{\beta_f}}{\mu c_f - 1} + \Delta_f^{\beta_f}\right)^2}\right] \\
&= -1 - 4\beta_f\left(\frac{\mu c_f - 1}{\mu c_f}\right)
\end{aligned}
\tag{3.83}
$$

which implies, as it was for $\beta_f = 1$, that the double periodic point is unstable. In Figure 3.5, we show the deviation x_t, the fraction of fundamentalists $n_{f,t}$, and fundamentalists' demand $z_{f,t}$ for different values of β and initial value $x_0 = 1$, when the condition $\mu c_f > 2$ is met. In Figure 3.6, we show a parameter diagram, fixed $\mu = 1$, $\Delta_f \geq 0$, and $c_f \geq 2$, where the x-axis represents investors' degree of certainty β_f and the y-axis the deviation x of the asset's price from its fundamental value.

3.3 Conclusions

In the previous section, we have analysed the case of a market populated by a single group of investors, that is, *pure fundamentalists*, on different time scales whose price evolution in terms of its deviation from the fundamental value p^* is described by equation (3.42). We first considered its simplest form by assuming the trading threshold tending either to infinity ($\Delta_f \to +\infty$) or zero ($\Delta_f \to 0$). In the first case, representing investors in the '*holding*' state for any price p_t, the system simply reduces to equation (3.43), that is, the asset price is constant at any time t. In the second case, all fundamentalists trade the asset at any time t (system evolving over large time scale), and the system becomes as expressed in equation (3.44). We observed three different price behaviours: 1) convergence to the fundamental value (either monotonically or oscillatory) for $0 < \mu c_f < 2$; 2) double-period oscillations around the fundamental value for $\mu c_f = 2$; 3) oscillatory divergence to infinity for $\mu c_f > 2$. The first outcome is something one would expect to observe since only one group of investors is trading in the same way and, thus, having the same price expectation for the asset, that is, the fundamental value p^*. The second and third outcomes (*exploding*

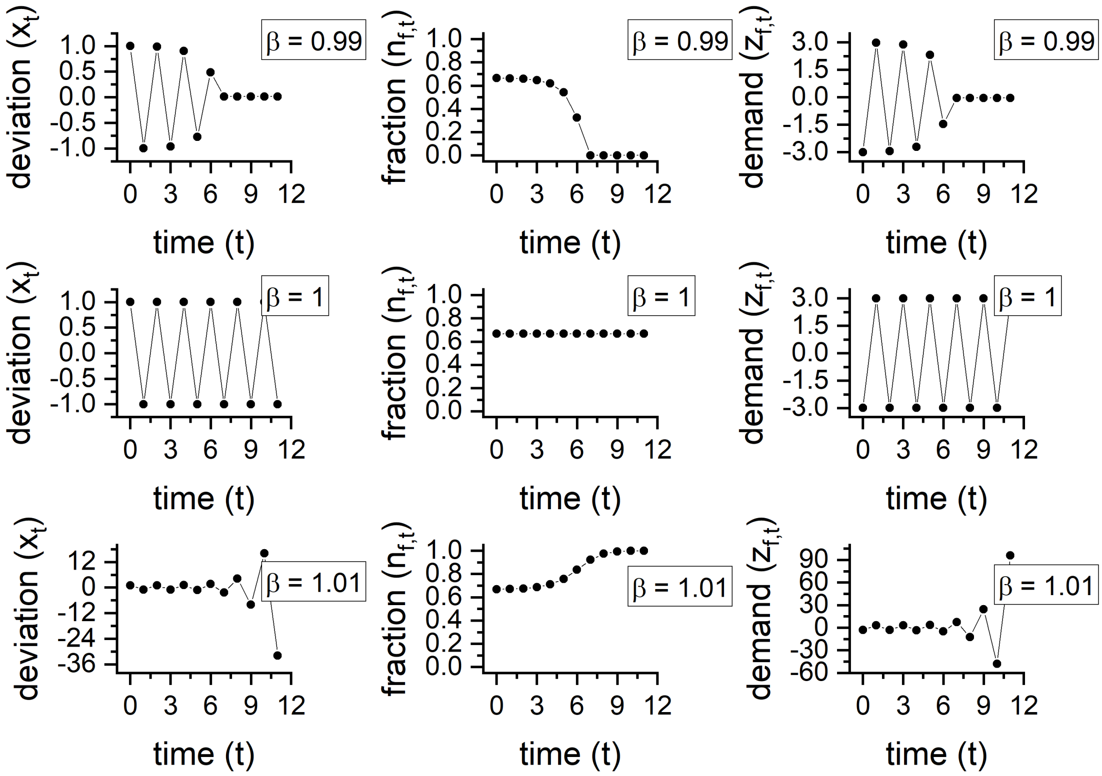


FIGURE 3.5: The deviation x_t, the fraction of fundamentalists $n_{f,t}$, and fundamentalists' demand $z_{f,t}$ for different values of β_f and initial value $x_0 = 1$ (fixed $\mu = 1$ $\Delta_f = 1/2$, and $c_f = 3$ in equation (3.75)). We get very similar outcomes as for $\beta_f = 1$ for different initial values close to $x_0 = 1$ shown in Figure 3.3.

prices) are extreme cases. The assumption of having investors with unlimited liquidity/supply (given by c_f) and the market maker's lack of adaption into slowing down the price adjustment (given by μ) for such large demands contribute to generating these 'unrealistic' phenomena. Next, we focused on the general case in which investors refer now to the trading threshold ($\Delta_f \neq 0$) beyond which they would trade the asset if the deviation of its current price from their perceived value exceeded a certain value (system evolving over short time scale), first by setting their degree of certainty in taking this decision equal $\beta_f = 1$, equation (3.48), and, then, for any generic value ($\beta_f \geq 0$), equation (3.75). We obtained the same price behaviours as we did in the previous cases, with the difference that the price can now converge to the fundamental value for any $\mu c_f > 0$, assuming one chooses the initial price of the asset close enough to its fundamental value. More specifically, for fixed $\Delta_f \neq 0$, the higher the value of μc_f, the closer the initial price p_0 of the asset has to be chosen to its fundamental value in order for the price to converge to p^*. The main reason why we observe this difference between this case and the previous one ($\Delta_f \to 0$) is that, in the latter, *all* investors trade the asset at any time t ($n_{f,t} = 1$), meaning that their (net) demand can easily trigger these price divergences. For $\Delta_f \neq 0$, the fraction of

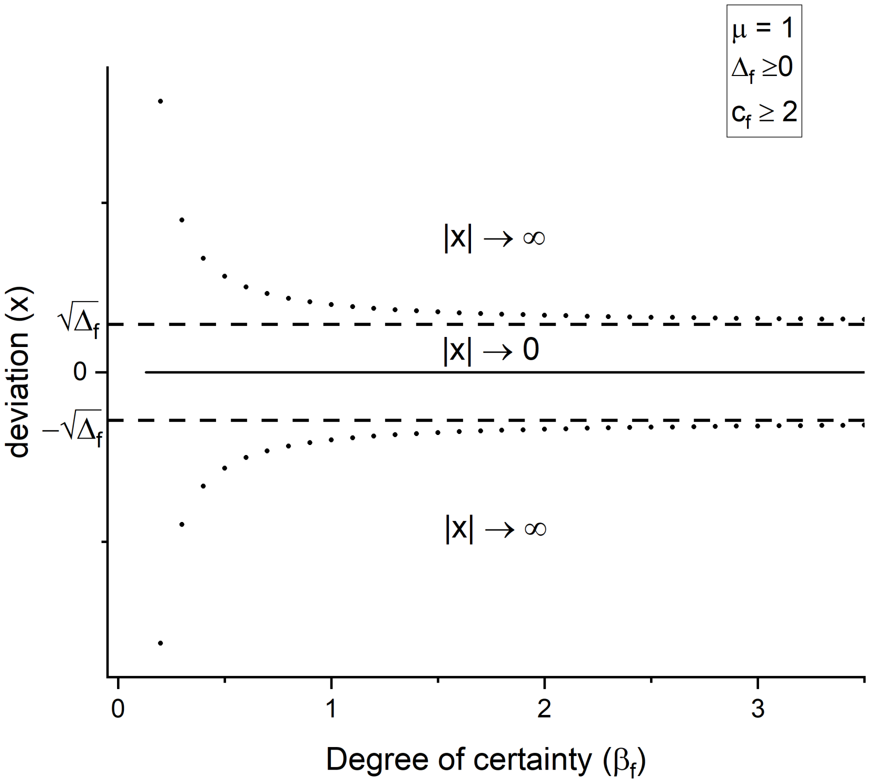


FIGURE 3.6: Parameter diagram of the deviation x w.r.t investors' degree of certainty β_f, fixed $\mu = 1$, $\Delta_f \geq 0$, and $c_f \geq 2$. The horizontal dashed lines at $x = \sqrt{\Delta_f}$ and $x = -\sqrt{\Delta_f}$ represent horizontal asymptotes to which the double-periodic orbits (equation (3.78)) tend for $\beta_f \to \infty$.

fundamentalists trading the asset can vary over time and, eventually, tend to 0 ($n_{f,t} \to 0$) as $t \to \infty$ if one chooses p_0 close enough to p^*, that is, investors will stop trading the asset, thus ensuring the convergence of p_t to p^*. As the degree of certainty β_f tends to infinity ($\beta_f \to +\infty$), for fixed $\mu c_f \neq 0$, the set of initial prices for which the price of the asset converges to its fundamental value narrows down to the interval $I = (-\sqrt{\Delta_f}, \sqrt{\Delta_f})$, outside of which we get double-period oscillations of the price. As we pointed out above, we find it interesting that the amplitude of the price swings tends to be proportional to $\sqrt{\Delta_f}$, which, based on one of our assumptions, could set the trading frequency of a group of investors, suggesting that the intensity of the price swings arising in stock markets over a period might be proportional to a combination of trading frequencies that identify all the groups of investors trading during that period. In the next chapter, we will see what happens if a second group of investors participate in the market and receives information from pure fundamentalists with a delay.

Chapter 4

A market with pure fundamentalists and followers

In the previous chapter, we considered the simplest case of a market populated by one group of investors, that is, *pure fundamentalists*. As we observed there, the presence of such a group gave rise to very simple price dynamics, such as price convergence to the fundamental value and, in extreme (and unrealistic) cases, double-periodic price patterns or price divergences. The only way to observe persistent price fluctuations (market volatility) would be by adding a stochastic term (e.g. noise traders) in the market excess demand, the level of which necessarily has to be of the same size as fundamentalists' demand (something that leads to unsatisfactory results Lux and Alfarano (2016)). In this chapter, we will focus on a more general scenario where a market is populated by pure fundamentalists and *followers*, in which the latter refer to the information shared by the former about the asset's fundamentals with a delay, whose interaction will give rise to richer price dynamics, non-stochastic volatile markets, and one among the many stylised facts present in financial markets, that is, volatility clustering (Cont (2001)). We will start by discussing the variables of our system, that is, the prices to which the groups of investors refer when they trade the asset with respect to its fundamental value (*group's reference price*) and the mechanism that triggers a group of investors to share their information to other investors, leading them to change such reference points over time (*sharing mechanism*). Once introduced two important tools that will allow us to keep track of investors' performances and market volatility (to check how well investors can sustain periods with different price volatilities), we will analyse the system by first considering the simplest case and exploiting the corresponding outcomes to study more complex and analytically intractable cases.

4.1 A market with a second group of fundamentalists: *followers*

First of all, we show again the system of difference equations that describe our financial market derived in the previous chapter (see system (3.37)):

$$\begin{cases} p_t = p_{t-1} + \mu \sum_{h=1}^{H} n_{h,t-1}\bar{c}_h (p^* - p_{h,t-1}) \\ p_{h,t} = p_{h,t-1} + \sum_{i\neq h}^{H} \gamma_i \alpha_{i,t-1} \left(p_{i,t-1} - p_{h,t-1}\right) \\ \sum_{i\neq h}^{H} \gamma_i = 1 \\ n_{h,t} = \frac{\left[(p^*_{h,t}-p_t)^2\right]^{\beta_{\tau,h}}}{\left[(p^*_{h,t}-p_t)^2\right]^{\beta_{\tau,h}} + \Delta_{\tau,h,t}^{\beta_{\tau,h}}} \\ \alpha_{i,t} = \frac{\left[(p^*_{i,t}-p_t)^2\right]^{\beta_{\sigma,i}}}{\left[(p^*_{i,t}-p_t)^2\right]^{\beta_{\sigma,i}} + \Delta_{\sigma,i,t}^{\beta_{\sigma,i}}} \end{cases} \tag{4.1}$$

This time, we consider a market in which we add a group of *pure fundamentalists*, f_1, whose reference price is given by

$$p_{f_1,t} = p_t \tag{4.2}$$

and a second group of *followers*, f_2, who receives the information from the first group f_1 but with a delay, such that their reference price, $p_{f_2,t}$, updates as (see second equation in system (4.1) with $\gamma_{f_1} = 1$)

$$p_{f_2,t} = p_{f_2,t-1} + \alpha_{f_1,t-1} \left(p_{f_1,t-1} - p_{f_2,t-1}\right) \tag{4.3}$$

where $0 \leq \alpha_{f_1,t-1} \leq 1$ represents group f_1's sharing mechanism (given below) for their reference price and describes the fraction of investors belonging to group f_1 that shares their information (their reference price $p_{f_1,t-1}$) at time $t-1$ (which corresponds to p_{t-1}). If we set $\alpha_{f_1,t-1} = \alpha_{f_1} = 1$, then the group f_2 would always refer to the group f_1's past reference price (or, equivalently, f_1 would share their information to f_2 at any time), that is

$$p_{f_2,t} = p_{f_1,t-1} = p_{t-1} \tag{4.4}$$

This condition would see group f_2 behaving almost exactly as f_1 would at all times, as f_1's trading threshold would give enough time to f_2 to update their reference price before f_1 starts trading the asset (except for a time step delay), meaning that this market would not be so different from a single-group market (section 3.2). For $\alpha_{f_1,t-1} = \alpha_{f_1} = 0$ the group f_2 would only refer to its past reference price (or, equivalently, f_1 would never share their information to f_2), that is

$$p_{f_2,t} = p_{f_2,0} \tag{4.5}$$

meaning that if they started buying, selling or holding the asset at time $t = 0$, they would keep doing so for any successive time $t > 0$. In case f_2 is in a 'hold' state ($p_{f_2,0} = p^*$) at all times, then we would have a single-group market (section 3.2). If f_2 is in a 'trade' state ($p_{f_2,0} \neq p^*$), then we would have a market in which both groups would trade the asset in the opposite direction at any time $t > 0$ (for example, if f_2 send buy orders, the asset gets overpriced, and f_1 will, at some point, start sending always sell orders), something that is unlikely to happen in a real context (as investors do not have unlimited supply/liquidity in the long-term). Therefore, it makes more sense if f_2 received the information from f_1, who decides if and when to share it. More specifically, f_2 might refer to group f_1's past reference price depending on how overvalued or undervalued the asset price was from group f_1's perspective (alerting them, for example, to a potential bubble and imminent crash or, vice-versa, a possible surge of the prices in an undervalued market). In fact, we argue that the more distant the asset price is from its fundamental value, the more likely it is that f_1 will share its reference price with f_2.

Group f_1's sharing mechanism Among many possible choices, we choose the functional form of f_1's sharing mechanism to be exactly the same as the fraction of investors belonging to group f_1 that trade the asset at time t, $n_{f_1,t}$, that is,

$$\alpha_{f_1,t} = \frac{|p^* - p_{f_1,t}|^{2\beta_{f_1,\sigma}}}{|p^* - p_{f_1,t}|^{2\beta_{f_1,\sigma}} + \Delta_{f_1,\sigma}^{\beta_{f_1,\sigma}}} \tag{4.6}$$

where $\beta_{f_1,\sigma}$ is the degree of confidence for investors in group f_1 in deciding to share their information and $\Delta_{f_1,\sigma}$ is the *sharing threshold* for f_1's reference price deviation from the fundamental value beyond which ($|p^* - p_{f_1,t}|^{2\beta_{f_1,\sigma}} > \Delta_{f_1,\sigma}^{\beta_{f_1,\sigma}}$) most of them would start sharing their information. The previous equation tells us that the more distant group f_1's reference price is from p^* at time t, the more likely will disseminate their information to f_2 (to ensure a price reversal to its fundamental value), who will shift their reference price to trade as synchronously as possible to f_1. Another argument that could support f_1's intention to share their information is that they can run some risks if they *long* or *short* an asset too early. It is hard to predict how long it will take for the price to reverse its trend from any mispricing, and f_1 could be obliged to prematurely close their position, thus incurring losses. So, from a strategic point of view, it makes sense if they started sharing their information with other groups of investors as soon as they perceive this danger ($|p^* - p_{f_1,t}|^{2\beta_{f_1,\sigma}} > \Delta_{f_1,\sigma}^{\beta_{f_1,\sigma}}$), so as to drive the asset price back to its fundamental value. The choice to adopt the same functional form for group f_1's sharing mechanism as for $n_{f_1,t}$ is dictated not only by being reasonable (investors decide on two different matters by using the asset's price as a reference point.) but also by the need to keep the model analytically tractable. We want to point out that the same discourse could also be made if we considered two groups of (long-term) chartists, where, instead of referring to the asset's fundamental value in their strategy, they estimate the asset's current (and constant) price trend and its *support* and *resistance* level, as the corresponding indicators at which the price is thought to reverse its trend (support $\sim$ undervalued and resistance $\sim$ overvalued).

Price dynamics The price dynamics of this market are described, then, by a two-dimensional system with first-order difference equations (for now, we drop the sub-indices f_i from the parameters, assuming them equal for all groups of fundamentalists, and we add a sub-index τ to the degree of confidence of trading the asset, β_τ, and to the trading threshold, Δ_τ, where τ indicates trading), namely

$$\begin{cases} p_t = p_{t-1} + \mu \left[\bar{c}_{f_1}(p^* - p_{f_1,t-1}) \frac{|p^*-p_{f_1,t-1}|^{2\beta_\tau}}{|p^*-p_{f_1,t-1}|^{2\beta_\tau}+\Delta_\tau^{\beta_\tau}} + \bar{c}_{f_2}(p^* - p_{f_2,t-1}) \frac{|p^*-p_{f_2,t-1}|^{2\beta_\tau}}{|p^*-p_{f_2,t-1}|^{2\beta_\tau}+\Delta_\tau^{\beta_\tau}} \right] \\ p_{f_1,t} = p_t \\ p_{f_2,t} = p_{f_2,t-1} + \left(p_{f_1,t-1} - p_{f_2,t-1}\right) \frac{|p^*-p_{f_1,t-1}|^{2\beta_\sigma}}{|p^*-p_{f_1,t-1}|^{2\beta_\sigma}+\Delta_\sigma^{\beta_\sigma}} \end{cases} \tag{4.7}$$

where $\bar{c}_{f_i}$, represents group f_i's *dominance* over the market. We remind that we define $z_{f_i,t}$ as

$$z_{f_i,t} = \bar{c}_{f_i}(p^* - p_{f_i,t}) \tag{4.8}$$

which represents the demand of a single investor who belongs to group f_i, and group f_i's net demand $d_{f_i,t}$ as

$$d_{f_i,t} = \bar{c}_{f_i}(p^* - p_{f_i,t}) \frac{|p^* - p_{f_i,t}|^{2\beta_\tau}}{|p^* - p_{f_i,t}|^{2\beta_\tau} + \Delta_\tau^{\beta_\tau}} \tag{4.9}$$

We will refer to equation (4.8) and (4.9) in our analysis in later sections. The system (4.7) can be more compactly written as

$$\begin{cases} x_t = x_{t-1} - \mu \left[\bar{c}_{f_1} x_{f_1,t-1} \frac{|x_{f_1,t-1}|^{2\beta_\tau}}{|x_{f_1,t-1}|^{2\beta_\tau}+\Delta_\tau^{\beta_\tau}} + \bar{c}_{f_2} x_{f_2,t-1} \frac{|x_{f_2,t-1}|^{2\beta_\tau}}{|x_{f_2,t-1}|^{2\beta_\tau}+\Delta_\tau^{\beta_\tau}} \right] \\ x_{f_1,t} = x_t \\ x_{f_2,t} = x_{f_2,t-1} + \left(x_{f_1,t-1} - x_{f_2,t-1}\right) \frac{|x_{f_1,t-1}|^{2\beta_\sigma}}{|x_{f_1,t-1}|^{2\beta_\sigma}+\Delta_\sigma^{\beta_\sigma}} \end{cases} \tag{4.10}$$

where $x_t = p_t - p^*$ (the asset's price deviation from the fundamental value) and $x_{f_i,t} = p_{f_i,t} - p^*$ (group f_i's reference price deviation from the fundamental value).

4.1.1 Groups' trading sustainability

In addition to analysing the stability of the market (4.10), we will also investigate groups' sustainability in trading the asset when the market undergoes periods of volatility to see under what circumstances they can, in fact, sustain such events (turning out, eventually, to be profitable) or be driven out of the market by them (leading to the group's bankruptcy or switching to better-performing strategies, as investors do in the Brock-Hommes model). In Chapter 2, we have seen that Brock and Hommes introduce a performance function $\Pi_{h,t}$ to which investors h refer when they have to compare two or more strategies, namely, the past realised profits given by (see equation (2.41))

$$\Pi_{h,t} = (p_t + y_t - Rp_{t-1})z_{h,t-1} \tag{4.11}$$

where y_t is the asset's dividend at time t and R is the gross return of the risk-free asset. In our case, however, we do not consider dividends ($y_t = 0$) or a risk-free asset in the market to which investors refer as a benchmark to optimise their investment ($R = 1$). Moreover, since we are interested in analysing a group's performance, it makes more sense if we consider the group's net demand $d_{h,t}$, as only a fraction $n_{h,t}$ of this group will trade the asset at time t with demand $z_{h,t}$. So, from equation (4.11) we define the group's short term performance $\Pi^s_{h,t}$ as

$$\Pi^s_{h,t} = (p_t - p_{t-1})d_{h,t-1} \tag{4.12}$$

where the 's' index indicates that we are analysing investors' short-term performance (investors' trading activity between two consecutive time steps). The previous function tells us that whenever a group of investors h buys (sells) the asset at time $t-1$ and its price increases (decreases) from p_{t-1} to p_t, that is, $p_t > p_{t-1}$ ($p_t < p_{t-1}$), then they perform well ($\Pi^s_{h,t} > 0$)(they correctly *timed* the market). More generally, we will investigate group's trading sustainability over a period $T = T_{fin} - T_{in}$ (T_{in} is the time where we start keeping track of group's performance, and T_{fin} is the end time), a sum of the past realised profits, by measuring their performance $\Sigma_{\Pi^s,T,h}$ relative to their best and worst possible performance achievable based on their trading activity, that is,

$$\Sigma_{\Pi^s,T,h} = \frac{\sum_{t=T_{in}}^{t=T_{fin}} \Pi^s_{h,t+1}}{\sum_{t=T_{in}}^{t=T_{fin}} |\Pi^s_{h,t+1}|} \tag{4.13}$$

$\Sigma_{\Pi^s,T,h}$ is bounded between $\Sigma_{\Pi^s,MIN} = -1$, the worst possible performance, that is, when each buy (sell) order sent by group h was followed by a price decrease (increase), and $\Sigma_{\Pi^s,MAX} = 1$, the best possible performance, if group h perfectly timed the market in all time steps. Beyond analysing groups' performance in the short-term, we will also consider the long-term behaviour, that is, investors' trading activity with respect to the steady state of the system, the fundamental value. In fact, as we will see, in the long-term, the asset price will oscillate around its fundamental value, so for investors to avoid going bankrupt, their buy orders have to exceed their sell orders for the asset whenever it is undervalued, and vice-versa when overvalued (the average buying price must be lower than the selling price). In this case, the performance function can be defined as

$$\Pi^l_{h,t} = (p^* - p_t)d_{h,t} \tag{4.14}$$

where the 'l' index indicates that we are analysing investors' long-term performance. Equation (4.14) tells that whenever a group of investors h buys (sells) the asset at time t while the asset is undervalued (overvalued), that is, $p^* > p_t$ ($p^* < p_t$), then they perform well ($\Pi^l_{h,t} > 0$). We can take the sum of $\Pi^l_{h,t}$ for each time step over a period T and measure group's relative long-term performance $\Sigma_{\Pi^l,T,h}$, that is,

$$\Sigma_{\Pi^l,T,h} = \frac{\sum_{t=T_{in}}^{t=T_{fin}} \Pi^l_{h,t}}{\sum_{t=T_{in}}^{t=T_{fin}} |\Pi^l_{h,t}|} \tag{4.15}$$

Here, again, $\Sigma_{\Pi^l,T,h}$ is bounded between $\Sigma_{\Pi^l,MIN} = -1$, the worst possible performance, when each buy (sell) order was sent only when the asset was overvalued (undervalued), and $\Sigma_{\Pi^l,MAX} = 1$, the best possible performance, when each buy (sell) order was sent only when the asset was undervalued (overvalued). Notice that both $\Sigma_{\Pi^s,T,h}$ and $\Sigma_{\Pi^l,T,h}$ will be generally hard to track analytically so that we will compute them numerically. It is important to recall that the investors in our system do not refer to any of the aforementioned performance functions in their trading. They do not check if their current strategy performs well or not, as is the case in most heterogeneous agent models, but they trade the asset adopting the same strategy over time (they refer to the same sources of information), which is in line with the fact that we are considering a market over a short period of time (months). We only choose to track the profit because it is informative about strategy performance.

4.1.2 Market volatility

Another aspect we are interested in is how volatile the market is (how intense are the price fluctuations) under circumstances in which the price oscillates around the fundamental value and groups' ability to sustain such events (equation (4.13) and (4.15)). As we did for groups' trading sustainability, we focus on short and long-timescale events, where we consider the absolute value of price changes between two-time steps (short-term price change), that is,

$$\Delta x_{s,t} = |p_t - p_{t-1}| = |x_t - x_{t-1}| \tag{4.16}$$

and the absolute value of price changes with respect to the price in the long-term, which corresponds to the fundamental value (long-term price changes), that is,

$$\Delta x_{l,t} = |p^* - p_t| = |x_t| \tag{4.17}$$

We compute the average value of $\Delta x_{s,t}$ and $\Delta x_{l,t}$ over a period $T = T_{fin} - T_{in}$ as

$$\overline{\Delta} x_{s,T} = \frac{1}{T} \sum_{t=T_{in}}^{t=T_{fin}} \Delta x_{s,t} \tag{4.18}$$

$$\overline{\Delta} x_{l,T} = \frac{1}{T} \sum_{t=T_{in}}^{t=T_{fin}} \Delta x_{l,t} \tag{4.19}$$

and, from these quantities, we compute the *coefficient of market volatility*, v_T, defined as

$$v_T \overset{\text{def}}{=} \frac{\overline{\Delta} x_{s,T}}{2\overline{\Delta} x_{l,T}} \tag{4.20}$$

This coefficient is bounded between $v_{MIN} = 0$, absence of market volatility, that is, when the price takes an infinite time to oscillate between its highest and lowest values, and $v_{MAX} = 1$, extremely volatile market, when the price takes just one-time step to oscillate between its highest and lowest values. Also, here, v_T will generally be hard to characterise analytically, so that we will resort to numerical methods.

4.2 System on large-time scale

Similarly as we did in the previous chapter for the case of a single group of fundamentalists, we will first analyse this system (4.10) in its simplest form, that is, assuming the trading threshold Δ_τ and the sharing threshold Δ_σ to be equal to 0, a 'textbook case' that can be easily analysed and understood, and, then, considering more general cases. This case can be interpreted as if the asset price is always far enough from its fundamental value to trigger all investors to trade it and share information about its mispricings at all times t; that is, we observe the system evolving on a (very) large time scale. The system (4.10) for $\Delta_\tau = 0$ ($n_{f_i,t} = 1 \implies d_{f_i,t} = z_{f_i,t} = -\bar{c}_{f_i} x_{f_i,t}$) and $\Delta_\sigma = 0$ ($\alpha_{f_1,t} = 1$) becomes

$$\begin{cases} x_t = x_{t-1} - \mu \left\{ \bar{c}_{f_1} x_{f_1,t-1} + \bar{c}_{f_2} x_{f_2,t-1} \right\} \\ x_{f_1,t} = x_t \\ x_{f_2,t} = x_{f_2,t-1} + \left(x_{f_1,t-1} - x_{f_2,t-1} \right) = x_{f_1,t-1} = x_{t-1} \end{cases} \tag{4.21}$$

which can be written as a map in matrix form

$$F \begin{pmatrix} x_{f_1} \\ x_{f_2} \end{pmatrix} = \begin{pmatrix} 1 - \mu \, \bar{c}_{f_1} & -\mu \, \bar{c}_{f_2} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{f_1} \\ x_{f_2} \end{pmatrix} \tag{4.22}$$

or, more compactly,

$$F(\vec{x}) = A\vec{x} \tag{4.23}$$

where

$$\vec{x} = \begin{pmatrix} x_{f_1} \\ x_{f_2} \end{pmatrix}; \quad A = \begin{pmatrix} 1 - \mu \, \bar{c}_{f_1} & -\mu \, \bar{c}_{f_2} \\ 1 & 0 \end{pmatrix} \tag{4.24}$$

It is possible to derive the solution of the previous map, given the initial conditions $\vec{x}_0^T = (x_{f_1,0}, x_{f_2,0})$, by iterating it as

$$F^t(\vec{x}_0) = A^t \vec{x}_0 \tag{4.25}$$

where t is the number of iterations. However, to compute the matrix A^t, it is more convenient to write the matrix A in terms of its Jordan normal form, B, given by

$$B = P^{-1}AP \iff PBP^{-1} = A \tag{4.26}$$

where P is a nonsingular matrix. It can be shown that the matrix A, based on its eigenvalues λ_1 and λ_2, can be similar to one of the following matrices (Elaydi (2005), Elaydi (2007)):

1. $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ if $\lambda_1 \neq \lambda_2 \in \mathbb{R}$
2. $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ if $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$
3. $B = \begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1-\lambda_2}{2i} \\ \frac{\lambda_2-\lambda_1}{2i} & \frac{\lambda_1+\lambda_2}{2} \end{pmatrix}$ if $\lambda_1, \lambda_2 (= \bar{\lambda}_1) \in \mathbb{C}$

with $P = (V_1, V_2)$, where V_1 and V_2 are the eigenvectors (or a combination of the two) which correspond to the eigenvalues λ_1 and λ_2, respectively. Computing B^t is very simple. For the first case ($\lambda_1 \neq \lambda_2 \in \mathbb{R}$), we have

$$B^t = \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \tag{4.27}$$

for the second one ($\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$),

$$B^t = \begin{pmatrix} \lambda^t & t\lambda^{t-1} \\ 0 & \lambda^t \end{pmatrix} \tag{4.28}$$

and for the last case ($\lambda_1, \lambda_2 (= \bar{\lambda}_1) \in \mathbb{C}$),

$$B^t = |\lambda_1|^t \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \tag{4.29}$$

where

$$\tan(\omega) = \frac{\lambda_1 - \lambda_2}{i(\lambda_1 + \lambda_2)} \tag{4.30}$$

The solution of the map (4.25) can be easily derived using the relation (4.26) as

$$F^t(\vec{x_0}) = PB^tP^{-1}\vec{x_0} \tag{4.31}$$

Market stability We will first derive the eigenvalues of the matrix A of our system (4.24), whose characteristic polynomial is given by

$$det(A - \lambda I) = \lambda^2 - \left(1 - \mu\,\bar{c}_{f_1}\right)\lambda + \mu\,\bar{c}_{f_2} = 0, \tag{4.32}$$

determine the conditions under which they are real and distinct, real and equal, or complex and conjugate, and derive the exact solution. Then, we will check if the system is stable, that is, if the solution converges to the origin $\vec{x}_o^* = (0,0)$ (the only fixed point of the map) as $t \to +\infty$ for any initial point $\vec{x_0} \in \mathbb{R}^2 \setminus \{\vec{x}_o^*\}$

$$\lim_{t\to+\infty} |F^t(\vec{x_0})| = 0 \quad \text{for any} \quad \vec{x_0} \in \mathbb{R}^2 \setminus \{\vec{x}_o^*\} \quad \Longleftrightarrow \quad \max\{|\lambda_1|, |\lambda_2|\} < 1 \tag{4.33}$$

or unstable (the solution diverges to infinity)

$$\lim_{t\to+\infty} |F^t(\vec{x}_0)| = +\infty \quad \text{for any} \quad \vec{x}_0 \in \mathbb{R}^2 \setminus \{\vec{x}_o^*\} \quad \Longleftrightarrow \quad \max\{|\lambda_1|, |\lambda_2|\} > 1 \quad (4.34)$$

where, for any $\vec{x}^T = (x_{f_1}, x_{f_2}) \in \mathbb{R}^2$, we defined

$$|\vec{x}| \stackrel{\text{def}}{=} |x_{f_1}| + |x_{f_2}| \quad (4.35)$$

There will be cases where the solution neither converges to the origin nor diverges to infinity but orbits around $\vec{x}_o^*$ ($\max\{|\lambda_1|, |\lambda_2|\} = 1$), that is, the asset price oscillates around its fundamental value, representing periods of market volatility. Only under these circumstances will we investigate how well each group of investors performs both in the short-term and long-term (see equations (4.13) and (4.15)) and see if these market conditions (described by the coefficient of market volatility v_T, see equation (4.20)) are sustainable for both groups.

4.2.1 Real and distinct eigenvalues

The equation (4.32) is solved for

$$\lambda_1 = \frac{1 - \mu\,\bar{c}_{f_1}}{2} + \sqrt{\left(\frac{1 - \mu\,\bar{c}_{f_1}}{2}\right)^2 - \mu\,\bar{c}_{f_2}} \quad (4.36)$$

and

$$\lambda_2 = \frac{1 - \mu\,\bar{c}_{f_1}}{2} - \sqrt{\left(\frac{1 - \mu\,\bar{c}_{f_1}}{2}\right)^2 - \mu\,\bar{c}_{f_2}} \quad (4.37)$$

so, we have that for

$$\left(\frac{1 - \mu\,\bar{c}_{f_1}}{2}\right)^2 > \mu\,\bar{c}_{f_2} \quad (4.38)$$

the eigenvalues are real and different from each other. The previous inequality can be rewritten as the product

$$\left(\mu\,\bar{c}_{f_1} - 1 + 2\sqrt{\mu\,\bar{c}_{f_2}}\right)\left(\mu\,\bar{c}_{f_1} - 1 - 2\sqrt{\mu\,\bar{c}_{f_2}}\right) > 0 \quad (4.39)$$

which implies either

$$\mu\,\bar{c}_{f_1} - 1 + 2\sqrt{\mu\,\bar{c}_{f_2}} > 0 \quad \wedge \quad \mu\,\bar{c}_{f_1} - 1 - 2\sqrt{\mu\,\bar{c}_{f_2}} > 0 \quad (4.40)$$

or

$$\mu\,\bar{c}_{f_1} - 1 + 2\sqrt{\mu\,\bar{c}_{f_2}} < 0 \quad \wedge \quad \mu\,\bar{c}_{f_1} - 1 - 2\sqrt{\mu\,\bar{c}_{f_2}} < 0 \tag{4.41}$$

and these relations can be combined as

$$\mu\,\bar{c}_{f_1} > 1 + 2\sqrt{\mu\,\bar{c}_{f_2}} \quad \vee \quad \mu\,\bar{c}_{f_1} < 1 - 2\sqrt{\mu\,\bar{c}_{f_2}} \tag{4.42}$$

The eigenvectors associated with the eigenvalues λ_1 and λ_2 of the matrix A (4.24) in this case are

$$AV_1 = \lambda_1 V_1 \iff V_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}; \quad AV_2 = \lambda_2 V_2 \iff V_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix} \tag{4.43}$$

The nonsingular matrix P and its inverse P^{-1} are, thus, given by

$$P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}; \quad P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \tag{4.44}$$

Using the previous results and equation (4.31), we can derive the solution of the map as

$$F^t(\vec{x}_0) = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.45}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} \lambda_1^{t+1} - \lambda_2^{t+1} & -\lambda_1\lambda_2\left(\lambda_1^t - \lambda_2^t\right) \\ \lambda_1^t - \lambda_2^t & -\lambda_1\lambda_2\left(\lambda_1^{t-1} - \lambda_2^{t-1}\right) \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.46}$$

Assuming the parameters satisfy one of the two relations in (4.42), we see, from the equations (4.36) and (4.37), that for $\mu\,\bar{c}_{f_1} < 1$, both eigenvalues are positive and $\lambda_1 > \lambda_2 > 0$, meaning that λ_1 is the *dominant characteristic root* (the limiting behaviour of the solution (4.45) is determined by this eigenvalue). For $\mu\,\bar{c}_{f_1} > 1$, both eigenvalues are negative and $|\lambda_2| > |\lambda_1| > 0$, so, in this case, λ_2 is the dominant characteristic root. If $\mu\,\bar{c}_{f_1} = 1$, there is no such $\mu\,\bar{c}_{f_2}$ that satisfies one of the two relations (4.42). So, to see if the system is stable for

$$\mu\,\bar{c}_{f_1} < 1 \implies \mu\,\bar{c}_{f_1} < 1 - 2\sqrt{\mu\,\bar{c}_{f_2}} \implies \mu\,\bar{c}_{f_2} < \frac{1}{4} \tag{4.47}$$

we just need to check if λ_1 satisfies the following inequality

$$\lambda_1 < 1 \iff \frac{1-\mu\,\bar{c}_{f_1}}{2} + \sqrt{\left(\frac{1-\mu\,\bar{c}_{f_1}}{2}\right)^2 - \mu\,\bar{c}_{f_2}} < 1 \tag{4.48}$$

or, equivalently,

$$\sqrt{\left(\frac{1-\mu\,\bar{c}_{f_1}}{2}\right)^2 - \mu\,\bar{c}_{f_2}} < \frac{1+\mu\,\bar{c}_{f_1}}{2} \tag{4.49}$$

If we square the previous inequality, we get

$$\left(\frac{1-\mu\,\bar{c}_{f_1}}{2}\right)^2 - \mu\,\bar{c}_{f_2} < \left(\frac{1+\mu\,\bar{c}_{f_1}}{2}\right)^2 \tag{4.50}$$

which leads to, after some straightforward algebraic steps,

$$\bar{c}_{f_1} > -\bar{c}_{f_2} \tag{4.51}$$

which is always satisfied for any $\bar{c}_{f_1}, \bar{c}_{f_2} \in \mathbb{R}^+$. This means that the price never diverges monotonically to infinity (it monotonically decreases to the fundamental value instead, see top panels of Figure 4.1) and, thus, no bubbles can be observed, which makes sense since two groups of fundamentalists populate the market only. For

$$\mu\,\bar{c}_{f_1} > 1 \implies \mu\,\bar{c}_{f_1} > 1 + 2\sqrt{\mu\,\bar{c}_{f_2}} \tag{4.52}$$

the system is stable if λ_2 satisfies the following inequality

$$-1 < \lambda_2 \iff -1 < \frac{1-\mu\,\bar{c}_{f_1}}{2} - \sqrt{\left(\frac{1-\mu\,\bar{c}_{f_1}}{2}\right)^2 - \mu\,\bar{c}_{f_2}} \tag{4.53}$$

that is,

$$\sqrt{\left(\frac{1-\mu\,\bar{c}_{f_1}}{2}\right)^2 - \mu\,\bar{c}_{f_2}} < \frac{3-\mu\,\bar{c}_{f_1}}{2} \tag{4.54}$$

So, if

$$\mu\,\bar{c}_{f_1} < 3 \tag{4.55}$$

we can square the previous inequality and get

$$\mu\,\bar{c}_{f_1} < \mu\,\bar{c}_{f_2} + 2 \tag{4.56}$$

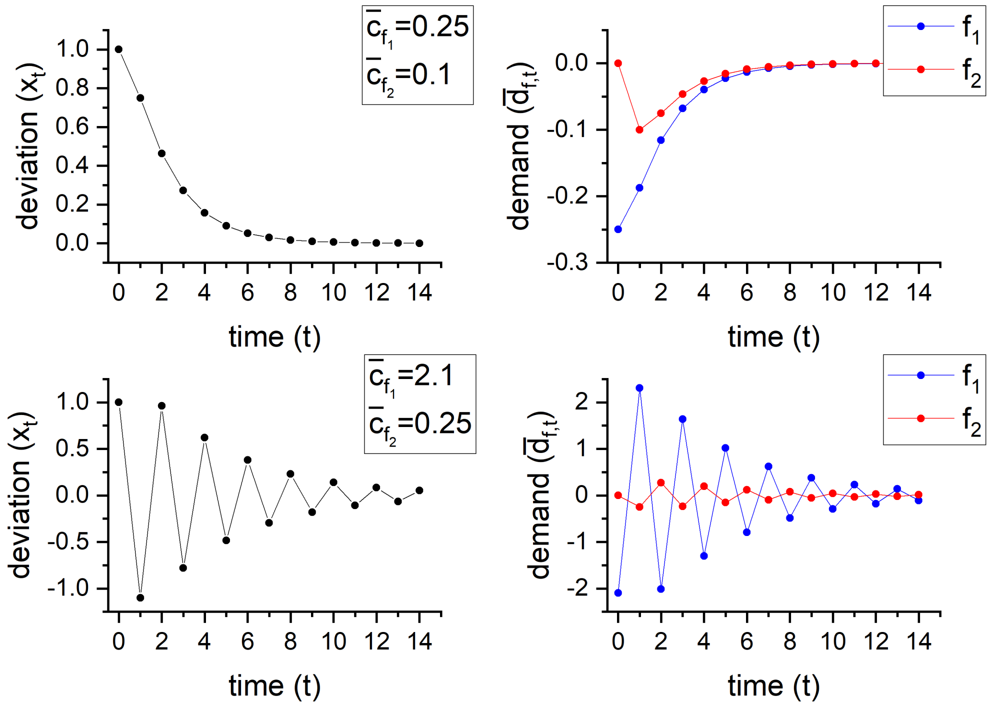


FIGURE 4.1: The deviation x_t and fundamentalists' relative net demand $d_{f_i,t}$ for two different pair of values of $\bar{c}_{f_i}$ (fixed $\mu = 1$) that satisfy the conditions (4.47) and (4.51) (top panels, $\bar{c}_{f_1} = 0.25$ and $\bar{c}_{f_2} = 0.1$), and the conditions (4.52), (4.55) and (4.56) (bottom panels, $\bar{c}_{f_1} = 2.1$ and $\bar{c}_{f_2} = 0.25$), with initial conditions $\vec{x}_0 = (1, 0)$.

For $\bar{c}_{f_2} = 0$, we obtain the same stability condition (3.47) we found in the case where only one group of fundamentalists populates the market, which would have given rise to a double periodic price pattern ($\mu\,\bar{c}_{f_1} = 2$) and price divergences ($\mu\bar{c}_{f_1} > 2$) if violated. However, once group f_2 participates in the market ($\bar{c}_{f_2} > 0$), the stability condition (4.56) allows group f_1 to increase their dominance (or, equivalently, increase the volume of orders) without this leading the market to become unstable (the solution oscillates around the origin and converges to it as $t \to +\infty$). As shown in Figure 4.1 (bottom panels), this happens because group f_2 trades in the opposite direction to group f_1 (if f_1 send buy (sell) orders, then f_2 send sell (buy) orders), meaning that some orders are filled between these groups and, thus, no substantial price adjustment is needed. If, instead,

$$(1 < \mu\,\bar{c}_{f_1} < 3 \quad \wedge \quad \mu\,\bar{c}_{f_1} > \mu\,\bar{c}_{f_2} + 2) \quad \vee \quad \mu\,\bar{c}_{f_1} \geq 3 \tag{4.57}$$

then the market becomes unstable, and, similarly to what was observed in the previous chapter, the solution oscillates and diverges to infinity. Any order generated by the mispricing of the asset (e.g. $x_t > 0$) will lead to higher mispricing in the next time step ($x_{t+1} < -x_t < 0$). Group f_2's volume of orders is not high enough to compensate for group f_1's, and, without an adequate intervention on the speed of

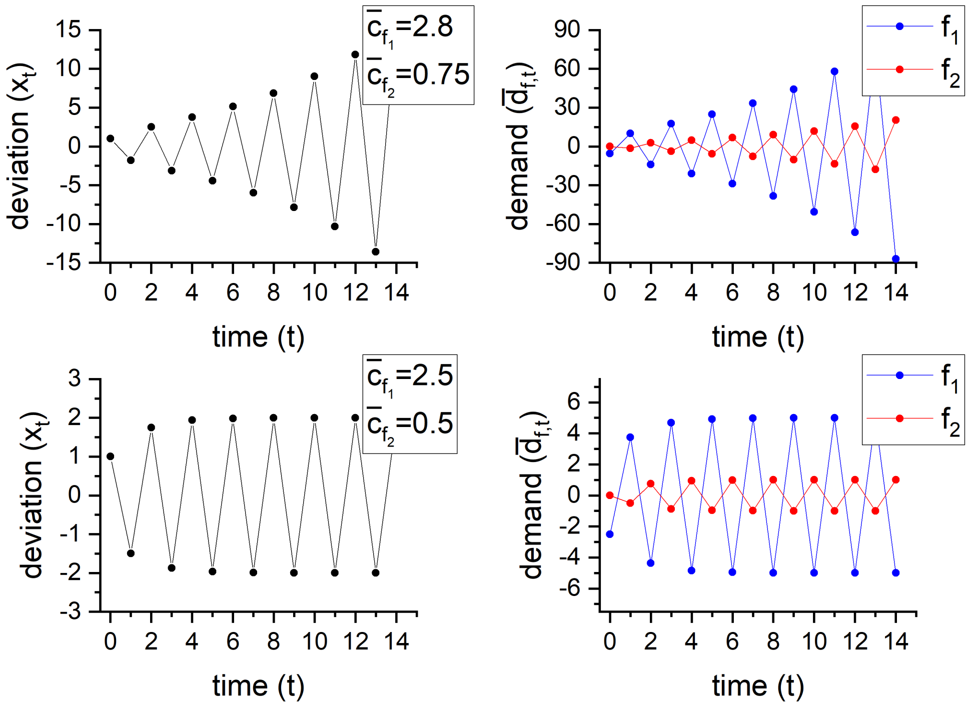


FIGURE 4.2: The deviation x_t and fundamentalists' relative net demand $d_{f_i,t}$ for two different pair of values of $\bar{c}_{f_i}$ (fixed $\mu = 1$) that satisfy the conditions (4.57) (top panels, $\bar{c}_{f_1} = 2.8$ and $\bar{c}_{f_2} = 0.75$), and (4.59) (bottom panels, $\bar{c}_{f_1} = 2.5$ and $\bar{c}_{f_2} = 0.5$), with initial conditions $\vec{x}_0 = (1, 0)$.

price adjustment μ from the market maker, this will inevitably lead the system to exhibit *exploding* prices (top panels of Figure 4.2). Notice that for $\mu\,\bar{c}_{f_2} > 1$, we would necessarily require $\mu\,\bar{c}_{f_1} > 3$ because of the relations (4.42), which leads us to outline more compactly the stability condition (for real and distinct eigenvalues) as

$$|\lambda_1|, |\lambda_2| < 1 \iff \mu\,\bar{c}_{f_1} < \mu\,\bar{c}_{f_2} + 2 \quad \wedge \quad 0 < \mu\,\bar{c}_{f_2} < 1 \tag{4.58}$$

In the special case we have

$$\mu\,\bar{c}_{f_1} = \mu\,\bar{c}_{f_2} + 2 \quad \wedge \quad 0 < \mu\,\bar{c}_{f_2} < 1 \tag{4.59}$$

we obtain, from the eigenvalues λ_1 (4.36) and λ_2 (4.37),

$$-1 < \lambda_1 = -\mu\,\bar{c}_{f_2} < 0 \tag{4.60}$$

and

$$\lambda_2 = -1 \tag{4.61}$$

meaning that the solution will oscillate between two points (double-period orbit)(bottom panels of Figure 4.2). The trading orders of the groups are such as to counterbalance and cause the price to fluctuate around the fundamental value perpetually, giving rise to a volatile market. For sufficiently large times (the transition period needed to reach the attractor), the price will take just one-time step to oscillate between its highest and lowest points, meaning that the coefficient of market volatility will tend to $v_T \to 1$ (extremely volatile market), which has negative consequences on group's f_2 performance, as discussed below.

Groups' sustainability Even though trivial, we have our first case where we can analyse investors' performance, and it will also allow us to highlight some important aspects we will refer to in more complex cases. First of all, the solution of the map, if the condition (4.59) holds, becomes

$$F^t(\vec{x_0}) = \frac{1}{1-\mu\,\bar{c}_{f_2}} \begin{pmatrix} (-\mu\,\bar{c}_{f_2})^{t+1} - (-1)^{t+1} & -\mu\,\bar{c}_{f_2}\left[(-\mu\,\bar{c}_{f_2})^t - (-1)^t\right] \\ (-\mu\,\bar{c}_{f_2})^t - (-1)^t & -\mu\,\bar{c}_{f_2}\left[(-\mu\,\bar{c}_{f_2})^{t-1} - (-1)^{t-1}\right] \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.62}$$

and since we have $(-\mu\,\bar{c}_{f_2})^t \to 0$ as $t \to +\infty$, the solution will tend to

$$F^t(\vec{x_0}) \sim \frac{1}{1-\mu\,\bar{c}_{f_2}} \begin{pmatrix} -(-1)^{t+1} & -\mu\,\bar{c}_{f_2}\left[-(-1)^t\right] \\ -(-1)^t & -\mu\,\bar{c}_{f_2}\left[-(-1)^{t-1}\right] \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.63}$$

$$= \frac{1}{1-\mu\,\bar{c}_{f_2}} \begin{pmatrix} (-1)^t & \mu\,\bar{c}_{f_2}(-1)^t \\ -(-1)^t & -\mu\,\bar{c}_{f_2}(-1)^t \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.64}$$

Groups' reference price deviation from the fundamental value $x_{f_i,t}$ at time t are, thus, given by

$$x_{f_1,t} = x_t = \frac{(-1)^t}{1-\mu\,\bar{c}_{f_2}}\left(x_{f_1,0} + \mu\,\bar{c}_{f_2} x_{f_2,0}\right) \tag{4.65}$$

$$x_{f_2,t} = -\frac{(-1)^t}{1-\mu\,\bar{c}_{f_2}}\left(x_{f_1,0} + \mu\,\bar{c}_{f_2} x_{f_2,0}\right) = -x_t \tag{4.66}$$

and, thus, groups' net demand $d_{f_i,t-1} = z_{f_i,t-1} = -\bar{c}_{f_i} x_{f_i,t-1}$ at time t is simply

$$d_{f_1,t} = -\bar{c}_{f_1}\frac{(-1)^t}{1-\mu\,\bar{c}_{f_2}}\left(x_{f_1,0} + \mu\,\bar{c}_{f_2} x_{f_2,0}\right) \tag{4.67}$$

$$d_{f_2,t} = \bar{c}_{f_2}\frac{(-1)^t}{1-\mu\,\bar{c}_{f_2}}\left(x_{f_1,0} + \mu\,\bar{c}_{f_2} x_{f_2,0}\right) \tag{4.68}$$

with group f_2 trading always in the opposite direction to group f_1. Group f_i's short-term performance between two-time steps is

$$\Pi^s_{f_i,t} = (p_t - p_{t-1})d_{f_i,t-1} = (x_t - x_{t-1})(-\bar{c}_{f_i}x_{f_i,t-1}) \tag{4.69}$$

so, for group f_1 we have ($x_{t-1} = -x_t$ and $x_{f_1,t-1} = -x_t$, see equation (4.65) and (4.66))

$$\Pi^s_{f_1,t} = 2\bar{c}_{f_1}x_t^2 \tag{4.70}$$

meaning that they perform well in all time steps t and, since $|\Pi^s_{f_1,t}| = \Pi^s_{f_1,t}$, we get (see equation (4.13))

$$\Sigma_{\Pi^s,T,f_1} = 1 \tag{4.71}$$

In the long term, we know that, by definition, f_1 will always buy (sell) the asset when it is undervalued (overvalued), so we have (see equation (4.15))

$$\Sigma_{\Pi^l,T,f_1} = 1 \tag{4.72}$$

These results do not come as a surprise, given that the group f_1 predominately drives the price to oscillate below and above the fundamental value (in just one-time step), so both in the short and long term, they perform optimally. On the other hand, group f_2's short-term performance is ($x_{f_2,t-1} = x_t$, see equation (4.66))

$$\Pi^s_{f_2,t} = -2\bar{c}_{f_1}x_t^2 \tag{4.73}$$

and, thus,

$$\Sigma_{\Pi^s,T,f_1} = -1 \tag{4.74}$$

meaning that they always send buy (sell) orders before a price decrease (worst possible performance). Moreover, in the long term, we have

$$\Pi^l_{f_2,t} = (p^* - p_t)z_{f_2,t} = -x_t(-\bar{c}_{f_2}x_{f_2,t}) = -\bar{c}_{f_2}x_t^2 \tag{4.75}$$

which implies

$$\Sigma_{\Pi^l,T,f_2} = -1 \tag{4.76}$$

As we know, group f_2 receives the information when the asset is overvalued or undervalued with a time step delay from group f_1, but as soon as they get it, the price will have already flipped below or above the fundamental value. So, f_2 sends their buy orders only when the asset is overvalued and sell orders when undervalued. In the short-term and long-term, they would keep incurring losses and, thus, be immediately driven out of the market (unless they change to a better-performing strategy by referring to more reliable sources of information). Consequently, this price action is unlikely to be observed realistically. Nonetheless, it is worth pointing out that if group f_2's market dominance were time-dependent, then it would decrease over time ($\bar{c}_{f_2,t} \to 0$ as $t \to +\infty$), either because of $\bar{c}_{f_2,t} \to 0$ (investors losing money) or $f_{f_2,t} \to 0$ (investors abandoning the market). This implies that the condition (4.59) (which is highly sensitive to subtle changes since it contains an equality condition) will eventually be violated and, if neither market maker's speed of adjustment nor group f_1's dominance ($\bar{c}_{f_1}$ or f_1) in this market change over time, the condition (4.57) would then hold now and the price would start diverging to infinity (market instability). So, group f_2's bankruptcy compromises the stability of the market.

4.2.2 Real and equal eigenvalues

Many of the dynamics observed in the previous section also appear when considering the particular case for which

$$\left(\frac{1-\mu\,\bar{c}_{f_1}}{2}\right)^2 = \mu\,\bar{c}_{f_2} \tag{4.77}$$

meaning that the eigenvalues are now real and equal to each other. This is equivalent to have

$$\mu\,\bar{c}_{f_1} = 1 + 2\sqrt{\mu\,\bar{c}_{f_2}} \quad \vee \quad \mu\,\bar{c}_{f_1} = 1 - 2\sqrt{\mu\,\bar{c}_{f_2}} \tag{4.78}$$

and if one of the two conditions hold, then the eigenvalue $\lambda (= \lambda_1 = \lambda_2)$ is equal to

$$\lambda = \frac{1-\mu\,\bar{c}_{f_1}}{2} \tag{4.79}$$

The eigenvector associated with this eigenvalue is

$$AV_1 = \lambda V_1 \iff V_1 = \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \tag{4.80}$$

and, to build the matrix P such that satisfies the relation (4.26), we need to derive the *generalised* eigenvector V_2 of the matrix A, given by

$$AV_2 = \lambda V_2 + V_1 \iff V_2 = \begin{pmatrix} 1+\lambda \\ 1 \end{pmatrix} \tag{4.81}$$

So, we have

$$P = \begin{pmatrix} \lambda & 1+\lambda \\ 1 & 1 \end{pmatrix}; \quad P^{-1} = -\begin{pmatrix} 1 & -(1+\lambda) \\ -1 & \lambda \end{pmatrix} \tag{4.82}$$

and the solution of the map is

$$F^t(\vec{x}_0) = \begin{pmatrix} \lambda & 1+\lambda \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda^t & t\lambda^{t-1} \\ 0 & \lambda^t \end{pmatrix} \begin{pmatrix} -1 & 1+\lambda \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.83}$$

$$= \begin{pmatrix} (t+1)\lambda^t & -t\lambda^{t+1} \\ t\lambda^{t-1} & (1-t)\lambda^t \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.84}$$

For

$$\mu\, \bar{c}_{f_1} < 3 \iff |\lambda| < 1 \tag{4.85}$$

the solution converges to $\vec{x}_o^*$ monotonically, if $\mu\, \bar{c}_{f_1} < 1$ ($\lambda > 0$), or in a oscillatory way, if $\mu\, \bar{c}_{f_1} > 1$ ($\lambda < 0$). For $\mu\, \bar{c}_{f_1} = 1$, which implies $\bar{c}_{f_2} = 0$ (because of the relations (4.78)), the eigenvalue would be equal to $\lambda = 0$ and the price drops immediately to zero at the first time step (top panels of Figure 4.3). If

$$\mu\, \bar{c}_{f_1} > 3 \iff \lambda < -1 \tag{4.86}$$

the price diverges to infinity. This is also true when

$$\mu\, \bar{c}_{f_1} = 3 \implies \mu\, \bar{c}_{f_2} = 1 \iff \lambda = -1 \tag{4.87}$$

since the solution would be proportional to $x_t \sim t\lambda^t$ (bottom panels of Figure 4.3) unless the initial conditions are chosen such that $x_{f_1,0} = -x_{f_2,0}$, that is, the two groups start trading in the opposite direction. In fact, under this condition, all the terms proportional to t in the solution of the map cancel out, and we have

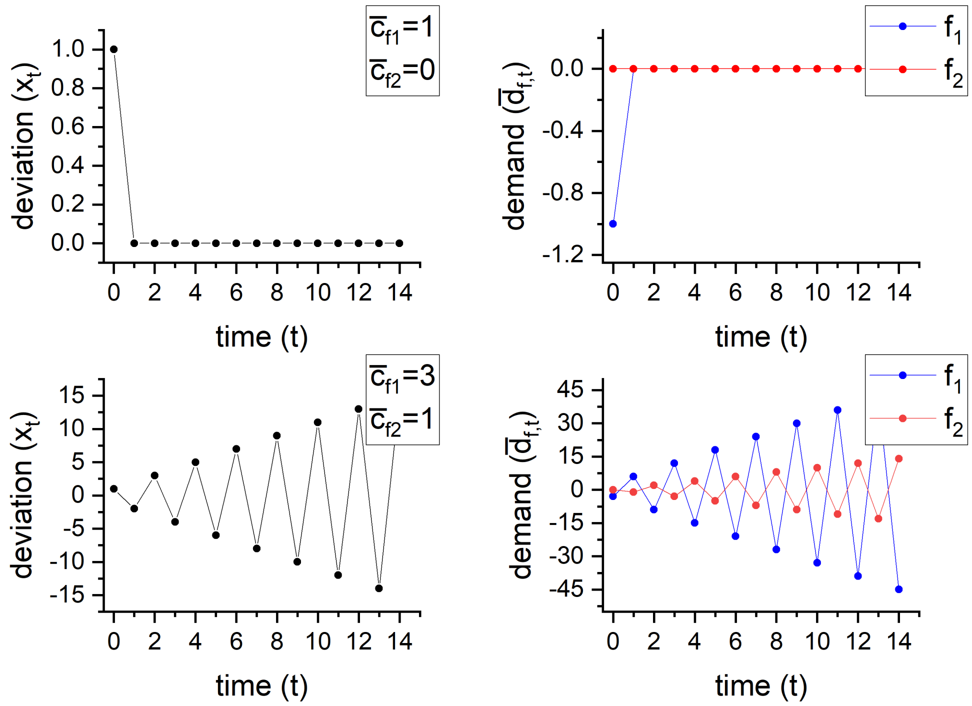


FIGURE 4.3: The deviation x_t and fundamentalists' relative net demand $d_{f_i,t}$ for $\bar{c}_{f_1} = 1$ and $\bar{c}_{f_2} = 0$ (top panels, $\mu = 1$), and $\bar{c}_{f_1} = 3$ and $\bar{c}_{f_2} = 1$ (bottom panels, $\mu = 1$), with initial conditions $\vec{x}_0 = (1, 0)$.

$$F^t(\vec{x}_0) = \begin{pmatrix} (t+1)(-1)^t & -t(-1)^{t+1} \\ t(-1)^{t-1} & (1-t)(-1)^t \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ -x_{f_1,0} \end{pmatrix} \tag{4.88}$$

$$= \begin{pmatrix} x_{f_1,0}(-1)^t \\ -x_{f_1,0}(-1)^t \end{pmatrix} \tag{4.89}$$

which is, again, a 2-period orbit. The same results we got for groups' sustainability and market volatility in the previous section are also valid here, so we will not show them for this case.

4.2.3 Complex eigenvalues

In the last case, we have λ_1 and λ_2 complex numbers (λ_2 is complex conjugate of λ_1) if and only if

$$\left(\frac{1-\mu\,\bar{c}_{f_1}}{2}\right)^2 < \mu\,\bar{c}_{f_2} \tag{4.90}$$

which is satisfied for

$$1 - 2\sqrt{\mu\,\bar{c}_{f_2}} < \mu\,\bar{c}_{f_1} < 1 + 2\sqrt{\mu\,\bar{c}_{f_2}} \tag{4.91}$$

If the previous condition holds, the eigenvalues λ_1 and λ_2 can be written as

$$\lambda_1 = \lambda = a + ib \tag{4.92}$$

$$\lambda_2 = \bar{\lambda} = a - ib \tag{4.93}$$

where a is the real part given by

$$a = \frac{1 - \mu\,\bar{c}_{f_1}}{2} \tag{4.94}$$

and b is the imaginary part given by

$$b = \sqrt{\mu\,\bar{c}_{f_2} - \left(\frac{1 - \mu\,\bar{c}_{f_1}}{2}\right)^2} \tag{4.95}$$

The corresponding eigenvectors associated to the eigenvalues of the matrix A are

$$AV = \lambda_1 V \iff V = V_1 + iV_2 = \begin{pmatrix} a \\ 1 \end{pmatrix} + i \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{4.96}$$

$$A\bar{V} = \lambda_2 \bar{V} \iff \bar{V} = V_1 - iV_2 = \begin{pmatrix} a \\ 1 \end{pmatrix} - i \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{4.97}$$

and the nonsingular matrix P and its inverse are

$$P = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}; \quad P^{-1} = -\frac{1}{b} \begin{pmatrix} 0 & -b \\ -1 & a \end{pmatrix} \tag{4.98}$$

Let

$$\cos(\omega) = \frac{a}{|\lambda|}; \quad \sin(\omega) = \frac{b}{|\lambda|} \tag{4.99}$$

where

$$|\lambda| = \sqrt{a^2 + b^2} = \sqrt{\mu\bar{c}_{f_2}} \tag{4.100}$$

then the solution can be expressed as

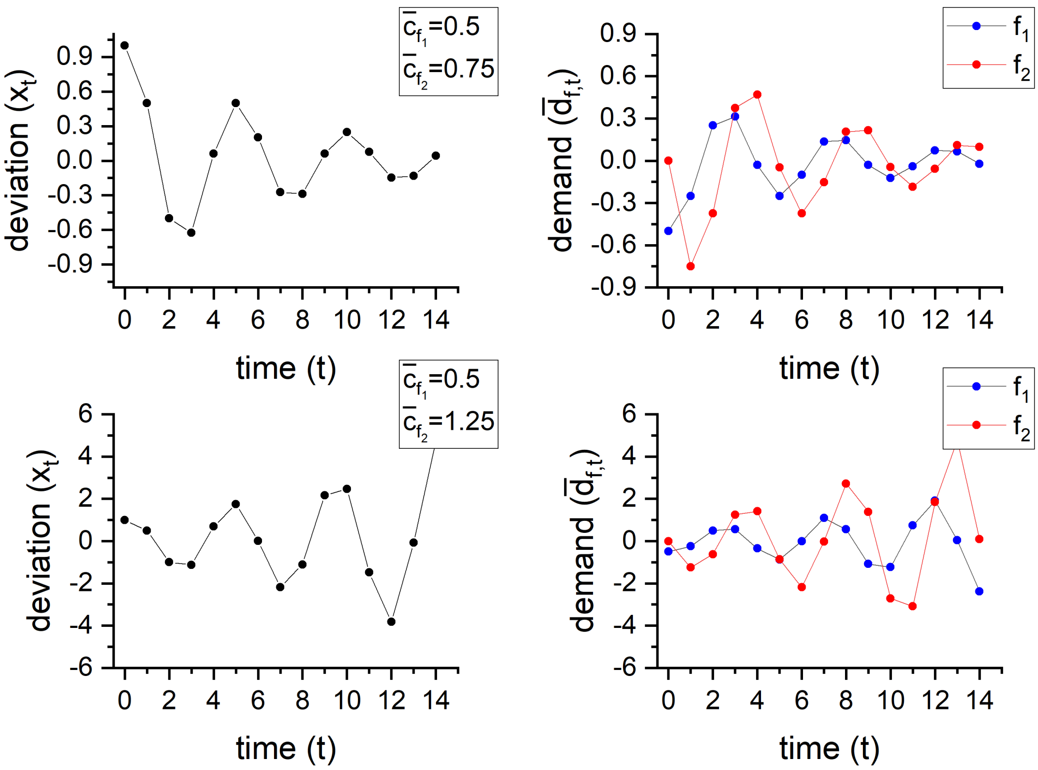


FIGURE 4.4: The deviation x_t and fundamentalists' relative net demand $d_{f_i,t}$ for two different pair of values of $\bar{c}_{f_i}$ (fixed $\mu = 1$) that satisfy the conditions (4.91) and (4.102) (top panels, $\bar{c}_{f_1} = 0.5$ and $\bar{c}_{f_2} = 0.75$), and condition (4.103) (bottom panels, $\bar{c}_{f_1} = 0.5$ and $\bar{c}_{f_2} = 1.25$), with initial conditions $\vec{x}_0 = (1, 0)$.

$$F^t(\vec{x}_0) = |\lambda|^t \begin{pmatrix} \frac{\sin(\omega)\cos(\omega t)+\cos(\omega)\sin(\omega t)}{\sin(\omega)} & -\frac{|\lambda|\sin(\omega t)}{\sin(\omega)} \\ \frac{\sin(\omega t)}{|\lambda|\sin(\omega)} & \frac{\sin(\omega)\cos(\omega t)-\cos(\omega)\sin(\omega t)}{\sin(\omega)} \end{pmatrix} \begin{pmatrix} x_{f_1,0} \\ x_{f_2,0} \end{pmatrix} \tag{4.101}$$

For

$$\mu\bar{c}_{f_2} < 1 \iff |\lambda| < 1 \tag{4.102}$$

the solution oscillates around the origin but converges to it as $t \to +\infty$ (top panels of Fig.4.4), while for

$$\mu\bar{c}_{f_2} > 1 \iff |\lambda| > 1 \tag{4.103}$$

the system is unstable, and the solution diverges to infinity, oscillating around $\vec{x}^*$ (bottom panels of Figure 4.4). Here, investors' demand and, consequently, the price trend is less regular than we observed above (double-period patterns). The main reason for this is that the group f_2's market dominance is now comparable (if not higher) to f_1's. The price is more influenced by f_2's demand, which is proportional to

f_1's delayed demand and because of that, the asset price takes more time to reverse its trend (when overvalued or undervalued), which leads f_2, in some circumstances, to trade almost synchronously (and in the same direction) with f_1 (see right panels in Figure 4.4). In case

$$\mu \bar{c}_{f_2} = 1 \iff |\lambda| = 1 \tag{4.104}$$

the price constantly oscillates around the fundamental value, and the groups' reference price deviation from p^* at time t is

$$x_{f_1,t} = x_t = x_{f_1,0}\cos(\omega t) + \left[\frac{x_{f_1,0}\cos(\omega) - x_{f_2,0}}{\sin(\omega)}\right]\sin(\omega t) \tag{4.105}$$

$$x_{f_2,t} = x_{t-1} = x_{f_2,0}\cos(\omega t) + \left[\frac{x_{f_1,0} - x_{f_2,0}\cos(\omega)}{\sin(\omega)}\right]\sin(\omega t) \tag{4.106}$$

where

$$\cos(\omega) = \frac{1-\mu \bar{c}_{f_1}}{2} \iff \omega = \cos^{-1}\left[\frac{1-\mu \bar{c}_{f_1}}{2}\right] \tag{4.107}$$

$$\sin(\omega) = \frac{\sqrt{(3-\mu \bar{c}_{f_1})(1+\mu \bar{c}_{f_1})}}{2} \tag{4.108}$$

From the conditions (4.91) and (4.104), it follows that $0 \leq \mu \bar{c}_{f_1} < 3$, so ω is limited between

$$\omega(\mu \bar{c}_{f_1} = 0) = \frac{\pi}{3} \leq \omega < \omega(\mu \bar{c}_{f_1} = 3) = \pi \tag{4.109}$$

and the map is periodic of period t_o

$$t_o = \frac{2\pi k}{\omega} \quad \text{for} \quad k \in \left\{a \in \mathbb{N} : \frac{t_o}{6} \leq a < \frac{t_o}{2}\right\} \tag{4.110}$$

where the condition on k comes from the condition (4.109) on ω, since

$$\omega = \frac{2\pi k}{t_o} \implies \frac{\pi}{3} \leq \frac{2\pi k}{t_o} < \pi \iff \frac{t_o}{6} \leq k < \frac{t_o}{2} \tag{4.111}$$

So, we have multiple values of ω and, thus, $\mu \bar{c}_{f_1}$ corresponding to the same period t_o, which are given by

$$\mu \bar{c}_{f_1} = 1 - 2\cos\left(\frac{2\pi k}{t_o}\right) \quad \text{for} \quad k \in \left\{a \in \mathbb{N} : \frac{t_o}{6} \leq a < \frac{t_o}{2}\right\} \tag{4.112}$$

For example, there are three different values of $\mu \bar{c}_{f_1}$ for a periodic solution of period $t_o = 9$ ($k = 2, 3, 4$), given by

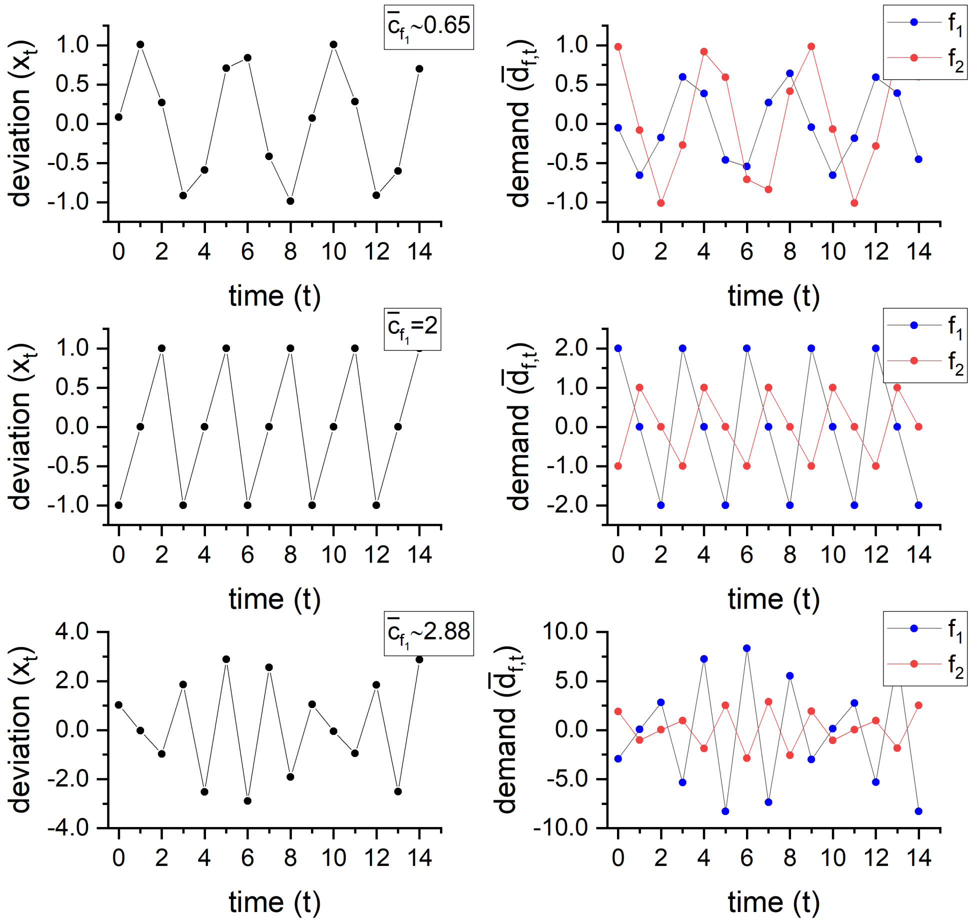


FIGURE 4.5: The deviation x_t and fundamentalists' relative net demand $d_{f_i,t}$ for $\bar{c}_{f_1} \sim 0.65$ (top panels), $\bar{c}_{f_1} = 2$ (middle panels), and $\bar{c}_{f_1} \sim 2.88$ (bottom panels), fixed $\bar{c}_{f_2} = 1$ and $\mu = 1$ with initial conditions $\vec{x}_0 = (1, 0)$. These are all orbits of period $t_o = 9$, and for $\bar{c}_{f_1} = 2$, the orbit is also periodic of period $t_o = 3$.

$$\mu\,\bar{c}_{f_1} = 1 - 2\cos\left(\frac{4\pi}{9}\right) \sim 0.65; \quad \mu\,\bar{c}_{f_1} = 1 - 2\cos\left(\frac{6\pi}{9}\right) = 2; \quad \mu\,\bar{c}_{f_1} = 1 - 2\cos\left(\frac{8\pi}{9}\right) \sim 2.88 \tag{4.113}$$

In Figure 4.5, we show the deviation x_t and investors' relative net demand $\bar{d}_{f,t}$ for each of these values, for fixed $\mu = 1$ and $\bar{c}_{f_2} = 1$(for $\mu\,\bar{c}_{f_1} = 2$, the map is also periodic of period $t_o = 3$). Notice that as $\bar{c}_{f_1}$ increases, the price becomes much more volatile (it takes fewer time steps to oscillate between its highest and lowest values), and investors trade the asset more asynchronously and in the opposite direction, which means that group f_2 tends to buy (sell) when the asset is overpriced (underpriced) (having negative consequences on their performance). The map also contains orbits which are *quasi-periodic*, that is, given $\mu\,\bar{c}_{f_1}$ (e.g. $\omega \in \mathbb{Q}$), there are no values of t_o and k

for which the relation (4.110) can be satisfied. So, in contrast to the case of a market populated by a single group of fundamentalists, here we can observe infinitely many periodic orbits.

Groups' sustainability This time, the system is more complex than we analysed in the previous section concerning the market's volatility and groups' trading sustainability. First of all, let us compute groups' short-term performance when we choose as initial conditions $\vec{x}_0 = (x_{f_1,0}, 0)$, that is, only group f_1 starts trading the asset. From equations (4.105) and (4.106), we get

$$x_t - x_{t-1} = x_{f_1,0}\cos(\omega t) + x_{f_1,0}\left[\frac{\cos(\omega) - 1}{\sin(\omega)}\right]\sin(\omega t) \tag{4.114}$$

and

$$x_{f_1,t-1} = x_{f_2,t} = \frac{x_{f_1,0}}{\sin(\omega)}\sin(\omega t) \tag{4.115}$$

$$\begin{aligned} x_{f_2,t-1} &= \frac{x_{f_1,0}}{\sin(\omega)}\sin\left[\omega(t-1)\right] \\ &= \frac{x_{f_1,0}}{\sin(\omega)}\left[\sin(\omega t)\cos(\omega) - \cos(\omega t)\sin(\omega)\right] \\ &= -x_{f_1,0}\cos(\omega t) + \frac{x_{f_1,0}\cos(\omega)}{\sin(\omega)}\sin(\omega t) \end{aligned} \tag{4.116}$$

so, for group f_1's short-term performance, we have

$$\begin{aligned} \Pi^s_{f_1,t} &= (x_t - x_{t-1})d_{f_1,t-1} = (x_t - x_{t-1})(-\bar{c}_{f_1}x_{f_1,t-1}) = \\ &= \bar{c}_{f_1}\left\{x_{f_1,0}\left[\frac{1-\cos(\omega)}{\sin(\omega)}\right]\sin(\omega t) - x_{f_1,0}\cos(\omega t)\right\}\frac{x_{f_1,0}}{\sin(\omega)}\sin(\omega t) \\ &= \frac{\bar{c}_{f_1}x^2_{f_1,0}}{\sin^2(\omega)}\left\{\left[1-\cos(\omega)\right]\sin^2(\omega t) - \sin(\omega)\sin(\omega t)\cos(\omega t)\right\} \end{aligned} \tag{4.117}$$

and for group f_2's short-term performance

$$\begin{aligned} \Pi^s_{f_2,t} &= (x_t - x_{t-1})d_{f_2,t-1} = (x_t - x_{t-1})(-\bar{c}_{f_2}x_{f_2,t-1}) = \\ &= \frac{\bar{c}_{f_1}x^2_{f_1,0}}{\sin^2(\omega)}\left\{\sin^2(\omega)\cos^2(\omega t) + \cos(\omega)\left[1-\cos(\omega)\right]\sin^2(\omega t) + \right. \\ &\left. - \sin(\omega)\sin(\omega t)\cos(\omega t)\right\} \end{aligned} \tag{4.118}$$

Groups' performance depends now on how dominant they are in the market or, more precisely, on group f_1's market dominance $\bar{c}_{f_1}$, given that group f_2's market dominance is fixed and equal to $\bar{c}_{f_2} = 1$ (without loss of generality, we set $\mu = 1$). Because of the second term in the curly brackets of equation (4.117), it is not guaranteed that f_1 short-term performance is always positive, meaning that their relative short-term performance over all their trades, Σ_{Π^s,T,f_1}, can be less than 1. However, the sum of $\Pi^s_{f_1,t}$ will always tend to a positive value since the second term ($\sim \Sigma_t \sin(\omega t)\cos(\omega t)$) equally includes both negative and positive terms, meaning that they will cancel out over a long period. At the same time, because of quadratic terms in equation (4.118), f_2 short-term performance can take positive values, so, differently from what we obtained in section 7.3.1, Σ_{Π^s,T,f_2} might tend to a positive value as we choose larger periods T. Nonetheless, it is not guaranteed that the sum of $\Pi^s_{f_2,t}$ will tend to a positive value. In fact, as $\bar{c}_{f_1}$ increases, the contribution of the first quadratic term will tend to 0, as (see equation (4.108))

$$\sin(\omega) \to 0 \qquad \text{for} \qquad \bar{c}_{f_1} \to 3 \tag{4.119}$$

while the second quadratic term will tend to a negative value (see equation (4.107))

$$\cos(\omega) \to -1 \qquad \text{for} \qquad \bar{c}_{f_1} \to 3 \tag{4.120}$$

More specifically, we have a critical value at $\bar{c}_{f_1} = 2$, that separates the two regions where $\Sigma_{\Pi^s,T,f_2} > 0$ for $\bar{c}_{f_1} < 2$ and $\Sigma_{\Pi^s,T,f_2} < 0$ for $\bar{c}_{f_1} > 2$, for which

$$\sin^2(\omega) = \frac{3}{4} \quad \wedge \quad \cos(\omega)\left[1 - \cos(\omega)\right] = -\frac{3}{4} \implies \sum_t^T \Pi^s_{h,t} \to 0 \tag{4.121}$$

Notice that because Σ_{Π^s,T,f_i} represents the ratio between the sum of $\Pi^s_{f_i,t}$ and its absolute value over a period T, the term multiplying the curly brackets in equation (4.117) and (4.118) cancels out, which implies that Σ_{Π^s,T,f_i} does not depend on group f_i's portfolio c_{f_i} (and, in this specific case, on the initial condition $x_{f_1,0}$). Next, we consider groups' long-term performance. For f_1 we always have $\Pi^l_{f_1,t} \geq 0$ which implies

$$\Sigma_{\Pi^l,T,f_1} = 1 \tag{4.122}$$

For group f_2, we get

$$\begin{aligned}\Pi^l_{f_2,t} &= (p^* - p_t) z_{f_2,t} = \bar{c}_{f_2} x_t x_{f_2,t} = \\ &= \frac{\bar{c}_{f_2} x^2_{f_1,0}}{\sin^2(\omega)} \left[\cos(\omega)\sin^2(\omega t) + \sin(\omega)\sin(\omega t)\cos(\omega t)\right]\end{aligned} \tag{4.123}$$

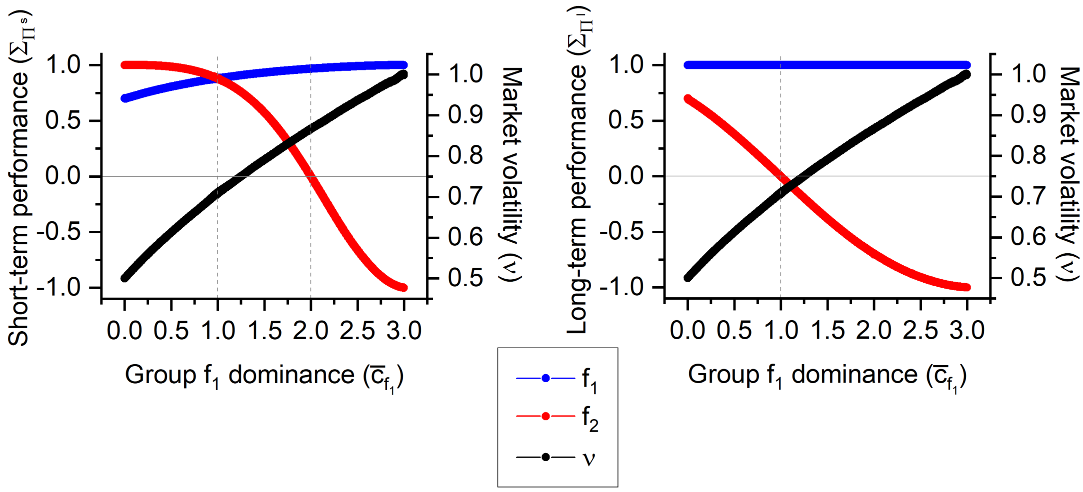


FIGURE 4.6: The limiting behaviour of groups' relative short-term (left panel) and long-term (right panel) performance, and the coefficient of market volatility υ_T (black curve), for $\bar{c}_{f_1} \in [0,3)$. As group f_1's market dominance $\bar{c}_{f_1}$ increases, the market becomes more volatile, and f_1's short-term performance increases as well (the long-term performance is always equal to $\Sigma_{\Pi^l,T,f_1} = 1$), while f_2's performance decreases. For $0 \leq \bar{c}_{f_1} \leq 1$, we have that f_2 performs better than f_1 in the short-term, vice-versa in the long-term, and we have that Σ_{Π^s,T,f_i} and Σ_{Π^l,T,f_i} are positive for both groups. For $1 < \bar{c}_{f_1} \leq 2$, f_1 always performs better than f_2, both in the short-term and long-term. f_2's relative short-term performance is still positive but tends to zero as $\bar{c}_{f_1} \to 2$, while its relative long-term performance is negative as soon as $\bar{c}_{f_1} > 1$. For $2 < \bar{c}_{f_1} < 3$, both Σ_{Π^l,T,f_1} and Σ_{Π^l,T,f_2} are negative.

Here again, whether group f_2's relative long-term performance tends to a positive or negative value depends on the quadratic term of $\Pi^l_{f_2,t}$, which is proportional to $\cos(\omega)$. Thus, the critical value is $\bar{c}_{f_1} = 1$, for which

$$\cos(\omega) = 0 \implies \sum_t^T \Pi^l_{h,t} \to 0 \tag{4.124}$$

and below ($\bar{c}_{f_1} < 1$) and above ($\bar{c}_{f_1} > 1$) which f_2 performs, respectively, positively and negatively.

Groups' performance vs. market volatility To have a complete picture of groups' sustainability when the market undergoes these periods of volatility, in Figure 4.6 we show the limiting behaviour of groups' relative short and long-term performance, and the coefficient of market volatility υ_T (estimated from equation (4.20)), for $\bar{c}_{f_1} \in [0,3)$, in steps of $\delta_{\bar{c}_{f_1}} = 0.001$ (for $\bar{c}_{f_1} \geq 3$, the condition (4.91) is not valid anymore and the system becomes unstable). To get these plots, for each $\bar{c}_{f_1}$, we let the system evolve for $T = 200$ time steps and took the average of Σ_{Π^s,T,f_i}, Σ_{Π^l,T,f_i} and υ_T, starting from different initial conditions (as the performances and the coefficient υ slightly depend on them) $I_{x_f,0} = I_{x_{f_1},0} \times I_{x_{f_2},0} = [-1,1] \times [-1,1]$, in steps of $\delta_{x_f,0} = 0.02$. As we can see, the coefficient v_T has a regular and uptrend behaviour as group f_1's market dominance increases, something we have already observed in the previous case for $t_o = 9$ periodic solutions. The simple explanation is that, as group f_1 becomes more dominant in the market, any deviation of the price from its fundamental value will be more promptly cleared, and, because of f_1's higher demands, the price swings around p^* with stronger fluctuations. Group f_2's relative short-term (long-term) performance decreases and group f_1's one increases for increasing values of $\bar{c}_{f_1}$ ($\Sigma_{\Pi^l,T,f_1} = 1$ for all $\bar{c}_{f_1} \in [0,3)$), which also corresponds to the increase in market volatility. For $0 < \bar{c}_{f_1} \leq 1$, we have that f_2 performs better than f_1 in the short-term, vice-versa in the long-term. We have that Σ_{Π^s,T,f_i} and Σ_{Π^l,T,f_i} are positive for both groups of fundamentalists and, not only are f_1 and f_2 able to sustain these periods of market volatility, these are also profitable for both groups (and the market maker). This suggests that the system (the price action) can hold this state for long periods, as investors would not see the need to switch to better-performing strategies (unless they get, for example, greedy, in which case they would be tempted to increase their dominance in the market). Both groups send most of their buy (sell) orders before a price increase (decrease), although f_2 is better at *timing* the market than f_1, as f_1 send their orders *much before* the asset price reversal to its fundamental value. They also buy (sell) the asset mostly when it is undervalued (overvalued), but f_2 is worse at taking profit from mispricings of the asset than f_1, since they send their orders *much later* than the asset price began its reversal trend to p^*. For $\bar{c}_{f_1} = 1$, both groups perform equally well in the short-term, but in the long-term, group f_2's relative performance tends to 0 (see equation (4.124)). Their buy orders and sell orders are sent when the asset approximately has the same price, that is, the average buying price is equal to the selling price, and no profits are made. For $1 < \bar{c}_{f_1} < 2$, f_1 always performs better than f_2, both in the short-term and long-term. f_2's relative short-term performance is still positive but tends to zero as $\bar{c}_{f_1} \to 2$ (the market becomes more and more volatile), while its relative long-term performance is negative for $\bar{c}_{f_1} > 1$. Even though f_2 still manages to buy (sell) before a price increase (decrease), most of the orders would be sent when the asset is overvalued (undervalued), since as soon as they receive the information from f_1 ("the asset is underpriced (overpriced)"), the price will have already flipped back above (below) p^*. f_2 would be forced, eventually, to switch to better-performing strategies (to avoid going bankrupt) as in the Brock-Hommes model (e.g. by referring to other sources of information) or, even worse, driven out of the market ($\bar{c}_{f_2} \to 0$ if it was time-dependent). So, the system can hold this state only for short periods in a real context (unless there is an inflow of investors taking the place of those exiting the market so that $\bar{c}_{f_2}$ stays constant over time). For $2 \leq \bar{c}_{f_1} < 3$ (highly volatile market), f_2's performance is even worse, where Σ_{Π^s,T,f_2} takes negative values (see equation (4.120) and (4.121)), meaning that the price fluctuates so fast that most of their buy (sell orders) are sent before a price decrease (increase). f_2 would be immediately driven out of the market, so the corresponding

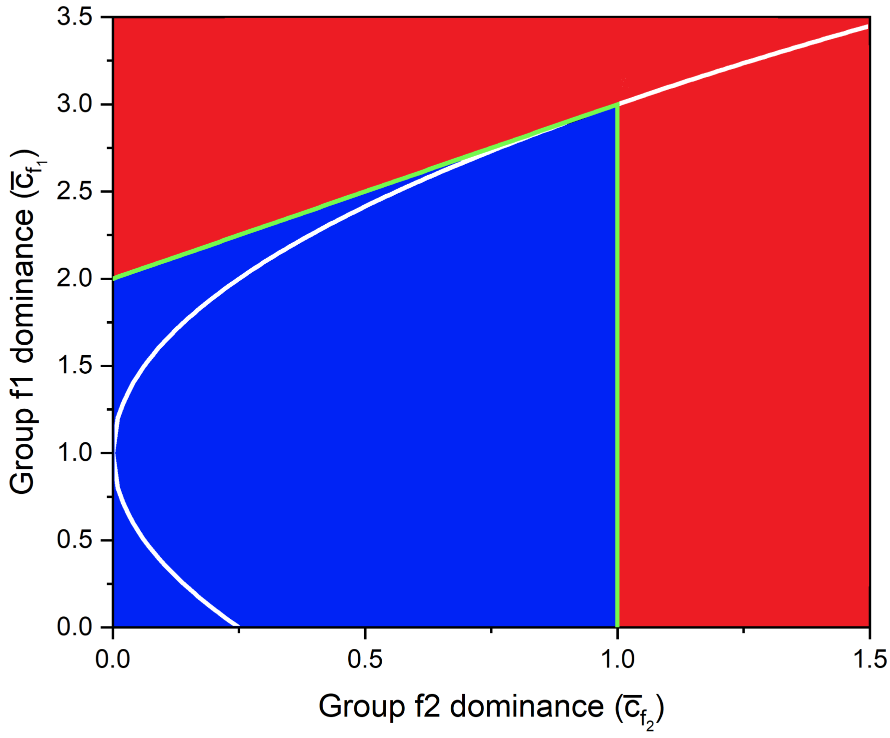


FIGURE 4.7: A parameter diagram showing the stability regions of the market analysed in the previous sections for $\mu = 1$. The white curve, given by equation (4.77), separates the two regions where the eigenvalues λ_1 and λ_2 are real (the left side of the curve) or complex (the right side of the curve). The blue region identifies the pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ in which the system is stable and the orbits converge to the origin $\vec{x}_o^*$, the red region where the system is unstable and the orbits diverge to infinity, and the green one (straight lines) where the orbits oscillate around the origin perpetually (market volatility).

price action is unlikely to be observed in a real context. Notice that, in contrast to what happened in section 7.3.1 (when the conditions (4.42) and (4.59) hold), a decrease in f_2' market dominance $\bar{c}_{f_2}$ when $1 < \bar{c}_{f_1} < 3$ does not lead to market instability (diverging prices), but rather the opposite the system would become stable and the asset price would converge to its fundamental value.

4.2.4 Parameter diagram and summary

Finally, in Figure 4.7, we show a parameter diagram that summarises all the results obtained in the previous sections regarding the stability analysis of the market (we set $\mu = 1$). The white curve, given by equation (4.77), separates the two regions where the eigenvalues λ_1 and λ_2 are real (the left side of the curve) or complex (the right side of the curve). The blue region identifies the pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ in which the system is stable and the orbits (and so the price) converge to the origin $\vec{x}_o^*$ (the fundamental value) and the red region where the system is unstable and the orbits diverge to infinity. Along the green lines ($\mu\bar{c}_{f_1} = \mu\bar{c}_{f_2} + 2$ and $\mu\bar{c}_{f_2} = 1 \wedge 0 \leq \mu\bar{c}_{f_1} < 3$), the orbits oscillate around the origin perpetually (market volatility). For $\mu\bar{c}_{f_1} = \mu\bar{c}_{f_2} + 2$, the price exhibits a double-period pattern (market extremely volatile), where group f_1 performs optimally ($\Sigma_{\Pi^s,T,f_1}, \Sigma_{\Pi^l,T,f_1} > 0$) and group f_2 performs poorly ($\Sigma_{\Pi^s,T,f_2}, \Sigma_{\Pi^l,T,f_2} < 0 \implies$ unsustainable condition), meaning that the conditions of this market are unrealistic. For $\mu\bar{c}_{f_2} = 1$ and $0 \leq \mu\bar{c}_{f_1} < 3$, the price can exhibit infinitely many periodic and aperiodic patterns, and its volatility increases as $\mu\bar{c}_{f_1}$ becomes larger. For $0 < \mu\bar{c}_{f_1} \leq 1$, f_2 performs better than f_1 in the short-term ($\Sigma_{\Pi^s,T,f_2} > \Sigma_{\Pi^s,T,f_1} > 0$), vice-versa in the long-term ($\Sigma_{\Pi^l,T,f_1} > \Sigma_{\Pi^l,T,f_2} > 0$), and these conditions are such as to be sustainable for both groups (realistic market conditions). For $1 < \mu\bar{c}_{f_1} \leq 2$, f_1 performs better than f_2 both in the short and long term, and f_2 performs poorly in the long-term ($\Sigma_{\Pi^l,T,f_2} < 0$), meaning that f_2 would be eventually driven out of the market (unrealistic market conditions). Finally, for $2 < \mu\bar{c}_{f_1} < 3$, f_1 performs almost optimally, while f_2 performs poorly both in the short and long term(unrealistic market conditions).

Summary The results of this 'textbook' case (which we will exploit in the next section) allowed us to understand better the conditions for which it is possible to observe a perpetually volatile market, even if the market is populated by groups of investors who adopt/follow the same strategy (fundamentalists). These price fluctuations typically require the participation of groups of investors who adopt and switch between two or more strategies (Dieci and He (2018)) or the addition of an external stochastic term (e.g. noise traders) in the excess demand (see previous chapter). The main ingredient in our model responsible for giving rise to this simple phenomenon is the presence of *followers*, whose dominance must be comparable (if not higher) than pure fundamentalists' (to have a greater influence on the price trend) to avoid incurring losses and, thus, being driven out of the market. The corresponding conditions imposed on the system to observe this volatile market are very strict, as they must satisfy an equality relation between the market maker's speed of adjustment and followers' dominance (see equation (4.104)), and, as such, are highly sensitive to subtle changes. In the next section, we will see how these conditions are eased by including the trading and sharing threshold, which determines how far the price must be from the asset's fundamental value before most investors belonging to a group are triggered to trade and share information about it.

4.3 System on short-time scale

Next, we consider the case in which the trading and sharing threshold can take a value different from 0. We will only focus on the specific case in which both Δ_τ and Δ_σ

take the same value (Δ_τ=Δ_σ=Δ), meaning that the moment f_1 starts trading the asset, they will also share their information with f_2. Therefore, we observe the evolution of the system on a smaller time scale than in the previous case but still large enough so as to have the followers change their reference price over time. To further simplify our analysis, we will assume that group f_1 is equally confident in deciding on trading the asset and sharing their information as soon as the price crosses the threshold Δ, that is, $\beta_\tau = \beta_\sigma = \beta$. If we take $\beta = 1$, then the map (4.10) takes the form ($x_t = x_{f_1,t}$)

$$\begin{cases} x_{f_1,t} = x_{f_1,t-1} - \mu\left(\bar{c}_{f_1}\frac{x^3_{f_1,t-1}}{x^2_{f_1,t-1}+\Delta} + \bar{c}_{f_2}\frac{x^3_{f_2,t-1}}{x^2_{f_2,t-1}+\Delta}\right) \\ x_{f_2,t} = x_{f_2,t-1} + \left(x_{f_1,t-1} - x_{f_2,t-1}\right)\frac{x^2_{f_1,t-1}}{x^2_{f_1,t-1}+\Delta} \end{cases} \tag{4.125}$$

which can also be written as the map F

$$F(f_{x_{f_1}}, f_{x_{f_2}}) = \begin{pmatrix} x_{f_1} - \mu\left(\bar{c}_{f_1}\frac{x^3_{f_1}}{x^2_{f_1}+\Delta} + \bar{c}_{f_2}\frac{x^3_{f_2}}{x^2_{f_2}+\Delta}\right) \\ x_{f_2} + \left(x_{f_1} - x_{f_2}\right)\frac{x^2_{f_1}}{x^2_{f_1}+\Delta} \end{pmatrix} \tag{4.126}$$

Fixed points and stability We cannot find a general solution of this system, so we will look for its steady states, and we will try to analyse their stability. The fixed points $\vec{x}^* = (x^*_{f_1}, x^*_{f_2})$ of the map (4.125) can be found by solving

$$\begin{cases} x^*_{f_1} = x^*_{f_1} - \mu\left[\bar{c}_{f_1}\frac{(x^*_{f_1})^3}{(x^*_{f_1})^2+\Delta} + \bar{c}_{f_2}\frac{(x^*_{f_2})^3}{(x^*_{f_2})^2+\Delta}\right] \\ x^*_{f_2} = x^*_{f_2} + \left(x^*_{f_1} - x^*_{f_2}\right)\frac{\left(x^*_{f_1}\right)^2}{\left(x^*_{f_1}\right)^2+\Delta} \end{cases} \tag{4.127}$$

For $\bar{c}_{f_1} \neq 0$ and $\bar{c}_{f_2} \neq 0$, the origin $\vec{x}^*_o = (0,0)$ is the only fixed point of the map, while for $\bar{c}_{f_2} = 0$, the fixed points are infinite and of the form $\vec{x}^* = (0, x^*_{f_2})$ with $x^*_{f_2} \in \mathbb{R}$ (the fundamental value is still the only fixed point of the price). We have already analysed the last case since it corresponds to a market with only one group of fundamentalists (see previous chapter). However, it is useful for pointing out that, even if a group of investors does not actively participate in a market by trading the corresponding asset (in this case, group f_2), their perceived value of the asset (or reference price) can still vary over time and, by acting as a source of information, they could still affect other investors. Moreover, if their reference price is constant over time, as to influence investors to trade the asset in one direction only, they may be considered zealots within the system, speculating against or favouring the market. In the next chapters, we will consider such cases as well. To study the stability of the origin, we derive the Jacobian matrix of the function F (4.126), that is,

$$JF(f_{x_{f_1}}, f_{x_{f_2}}) = \begin{pmatrix} \frac{\partial f_{x_{f_1}}}{\partial x_{f_1}} & \frac{\partial f_{x_{f_1}}}{\partial x_{f_2}} \\ \frac{\partial f_{x_{f_2}}}{\partial x_{f_1}} & \frac{\partial f_{x_{f_2}}}{\partial x_{f_2}} \end{pmatrix} \tag{4.128}$$

For our system, we obtain

$$JF(f_{x_{f_1}}, f_{x_{f_2}}) = \begin{pmatrix} 1 - \mu \bar{c}_{f_1} x_{f_1}^2 \left[\frac{x_{f_1}^2 + 3\Delta}{\left(x_{f_1}^2 + \Delta\right)^2} \right] & \mu \bar{c}_{f_2} x_{f_2}^2 \left[\frac{x_{f_2}^2 + 3\Delta}{\left(x_{f_2}^2 + \Delta\right)^2} \right] \\ \frac{x_{f_1}^2}{x_{f_1}^2 + \Delta} + (x_{f_1} - x_{f_2}) \frac{2\Delta x_{f_1}}{\left(x_{f_1}^2 + \Delta\right)^2} & 1 - \frac{x_{f_1}^2}{x_{f_1}^2 + \Delta} \end{pmatrix} \tag{4.129}$$

and for the fundamental steady state, $\vec{x}_o^*$, the Jacobian simply becomes

$$JF(x, y, z)|_{\vec{x}_o^*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.130}$$

that is, the origin is a degenerate point. Because of that, we have to resort to numerical methods to analyse the system (4.125), which one can find in Appendix D.1.

4.3.1 A case study

Here, we discuss more in detail the dynamics of the system (4.125) for the pair of values $\bar{c}_{f_2} = 0.92$ and $\bar{c}_{f_1} = 0.21$. This is a case of great interest since it represents a market that exhibits two types of volatility (see bottom panels of Fig.D.4 in Appendix D.1) and clustering phenomena in price changes (see the bottom-right panel in Fig.D.6 in Appendix D.1). Below, we refer to them as markets in phase '1' and phase '2'.

Market in phase '1' In Figure 4.8, we show the deviation of the price from the fundamental value (top panel), investors' demand ($z_{f_i,t} = -\bar{c}_{f_i} x_{f_i,t}$) (middle panel), and the fraction of investors of each group trading the asset at time t for $T = 200$ time steps, for the market in phase '1'. Both groups of investors trade the asset almost synchronously and in the same direction (if we shifted the time series $z_{f_2,t}$ and $n_{f_2,t}$ one time step forward in Figure 4.8, we would see match with $z_{f_1,t}$ and $n_{f_1,t}$), which clearly plays an important role in maintaining the market volatilities, although it should not be enough considering that in the no-threshold case, for $\bar{c}_{f_2} = 0.92$ and $\bar{c}_{f_1} = 0.21$, the price would converge to the fundamental value (because of low volume of orders) (see diagram Figure 4.6), even more so for $\Delta \neq 0$, since only a fraction of investors in each group trades the asset. The other essential factor that drives the price to fluctuate around p^* is group f_1's sharing mechanism $\alpha_{f_1,t}$, which is nothing more than the fraction of investors belonging to group f_1 that trades the asset $n_{f_1,t}$ ($\alpha_{f_1,t} = n_{f_1,t}$, since $\Delta_\tau = \Delta_\sigma = \Delta$). As a reminder, the sharing mechanism $\alpha_{f_1,t}$ represents the fraction of

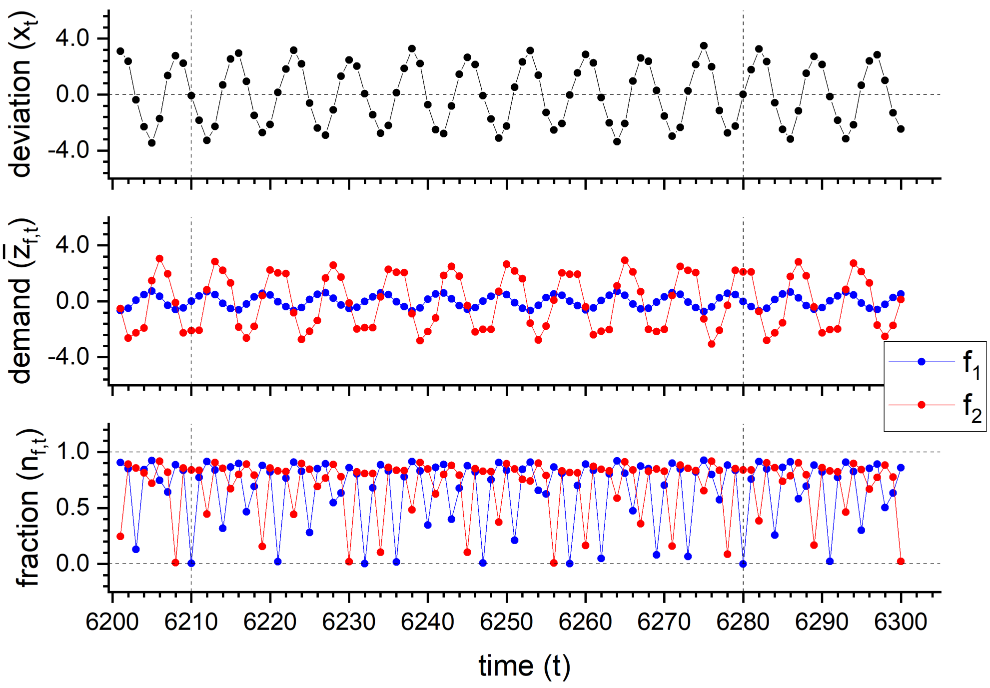


FIGURE 4.8: The deviation of the price from the fundamental value (top panel), investors' demand (middle panel), and the fraction of investors of each group trading the asset at time t for $T = 200$ time steps, for the market in phase '1'. The blue orbits represent time series associated with group f_1, while the red ones with group f_2. The two vertical dashed lines at times $t_1 = 6210$ and $t_2 = 6280$ show two situations in which group f_1 stops trading the asset and, consequently, does not share any information with group f_2 at the next time step, who will keep sending sell orders ($t_1' = 6211$) or buy orders ($t_2' = 6281$)

investors belonging to group f_1 that shares their information (their reference price $p_{f_1,t-1}$) at time $t-1$ (which corresponds to p_{t-1}) and has the function of updating group f_2's reference price depending on how overvalued or undervalued the asset price was from group f_1's perspective, that is (see second equation of the system (4.125)),

$$x_{f_2,t} = x_{f_2,t-1} + \left(x_{f_1,t-1} - x_{f_2,t-1}\right) \frac{x^2_{f_1,t-1}}{x^2_{f_1,t-1} + \Delta} \tag{4.131}$$

If the price is close enough to the fundamental value at time t ($\implies x_{f_1,t} \sim 0$), group f_1 stops trading the asset ($n_{f_1,t} \sim 0$), and, consequently, they do not share any information about its mispricings ($\alpha_{f_1,t+1} \sim 0$, given that f_2 receives the information with one-time step delay). This means that f_2 would keep referring to their previous reference price ($x_{f_2,t} \sim x_{f_2,t-1}$) (sending buy or sell orders) and, thus, driving the price away from p^*. In Figure 4.8, we show two vertical dashed lines at times $t_1 = 6210$ and $t_2 = 6280$ where two of these situations occur (there are several more within these two

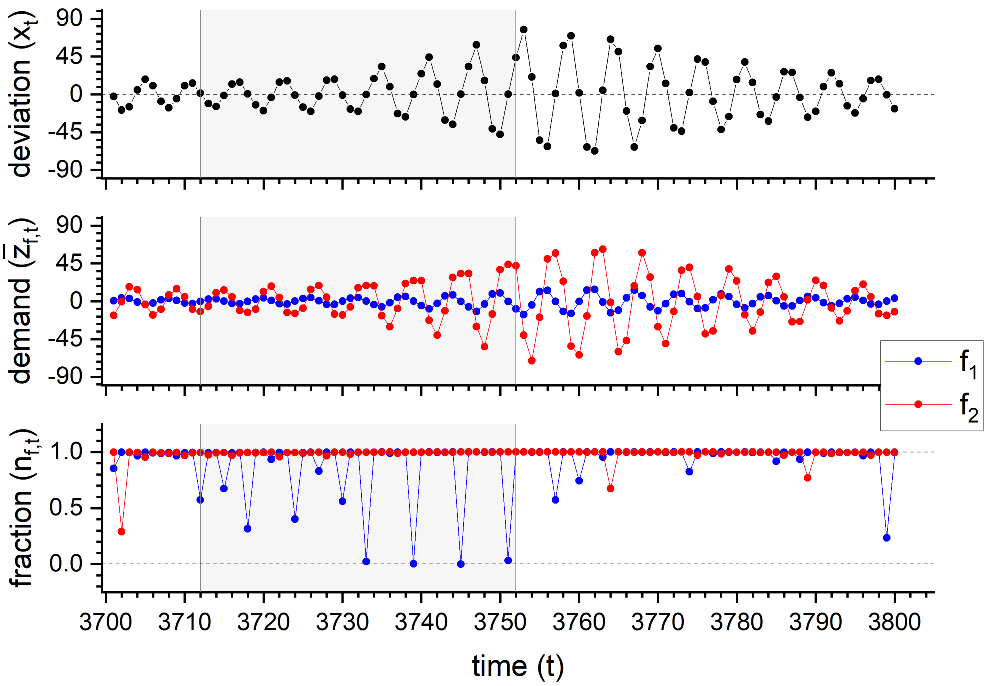


FIGURE 4.9: The deviation of the price from the fundamental value (top panel), investors' demand (middle panel), and the fraction of investors of each group trading the asset at time t for $T = 100$ time steps, for the market in phase '2'. The blue orbits represent time series associated with group f_1, while the red ones with group f_2. Between $t_3 = 3712$ and $t_4 = 3751$, we highlight a series of occurrences (f_1 stops trading the asset) that lead the price to fluctuate around p^* more and more strongly.

periods). So, the quasi-synchronous trading behaviour of both groups [1] and f_2's demand momentum when the price is close to the asset's fundamental value both contribute to price volatility.

Market in phase '2' In Figure 4.9, we show the deviation of the price from the fundamental value (top panel), investors' demand relative to their market dominance (middle panel), and the fraction of investors of each group trading the asset at time t for the market in phase '2'. Here, not only is the market far more volatile than the previous case, but we also observe clustering phenomena in the price dynamics, and, once again, the sharing mechanism plays a crucial role. Starting from $t_1 = 3704$ to $t_2 = 3711$, we see that the price is far enough from the fundamental value to trigger most (if not all) investors to trade the asset at each time step ($n_{f_i,t} \sim 1$). This situation would naturally lead the system (4.125) to behave as in the no-threshold case (system (4.21)), and, therefore, the price would get closer and closer to the fundamental value.

[1] If they traded completely synchronously (only possible if $\alpha_{f_1,t} = 1$ for all times t or if $\Delta_\tau >> \Delta_\sigma$), we would obtain a system similar to the one analysed in the previous chapter, that is, a market with only a group of fundamentalists, whose only realistic solution was the convergence of the price to the fundamental value and (in the extreme case) the double-period price pattern.

Between $t_3 = 3712$ and $t_4 = 3751$, we highlighted a series of events that lead the price to fluctuate around p^* more and more strongly instead. While all investors within group f_2 constantly trade the asset, there are times when a fraction of investors in group f_1 ceases to do so (there are occasions where the price gets very close to the fundamental value). As we discussed above, this leads f_1 to stop sharing their information with f_2 in all these occurrences, which drives the price further away from p^*. Consequently, the higher the asset's mispricing, the higher f_1' demand (and f_2's), which causes a period of (unusual) high volatility in the market. From $t_5 = 3752$ onwards, we see that almost all investors again trade the asset without the price being close to the fundamental value, a condition that *dampens* the period of high volatility, as the system (4.125) approaches to the system (4.21) (no-threshold case). So, as in the previous case, the lack of information shared by group f_1 with group f_2 (as the asset price gets closer to its fundamental value) is behind these anomalies. It is worth noticing that these clusters take place in the r_3 region only (see the bottom left panel in Figure D.6), that is when both groups of investors are highly active.

The system in the presence of noise In this last part, we show some results of introducing a noise term in the system (4.125) as follows

$$\begin{cases} x_{f_1,t} = x_{f_1,t-1} - \mu \left(\bar{c}_{f_1} \frac{x^3_{f_1,t-1}}{x^2_{f_1,t-1}+\Delta} + \bar{c}_{f_2} \frac{x^3_{f_2,t-1}}{x^2_{f_2,t-1}+\Delta} + N_t(0,\sigma_n) \right) \\ x_{f_2,t} = x_{f_2,t-1} + \left(x_{f_1,t-1} - x_{f_2,t-1} \right) \frac{x^2_{f_1,t-1}}{x^2_{f_1,t-1}+\Delta} \end{cases} \tag{4.132}$$

where $N_t(0,\sigma_n)$ takes normally distributed values over time with mean $\overline{N} = 0$ and standard deviation σ_n. We assume this noise term represents noise traders in the market (or random fluctuations of groups' market dominance). One interesting aspect is the persistence of price fluctuations clusters in the presence of noise, even if its level is comparable to investors' demand ($\sigma_n = 1$), as shown in Figure 4.10. As we see, the price fluctuates around the fundamental value much more closely than in the noiseless case (see top panel in Figure 4.9). The noise traders break the series of possible recurrences in which the price can get very close to p^*, which would normally give rise to a chain effect where the asset price gets further and further away from its fundamental value (see explanation above for the noiseless case). However, this does not prevent the price from exhibiting clustering phenomena, as only a single event ($p_t \sim p^*$) is enough to trigger them (see reference lines at $t_5 = 432$ and $t_6 = 462$).

4.4 Conclusions

In this chapter, we have analysed the case of a market populated by *pure fundamentalists*, who estimate the asset's value from the fundamentals analysis, and

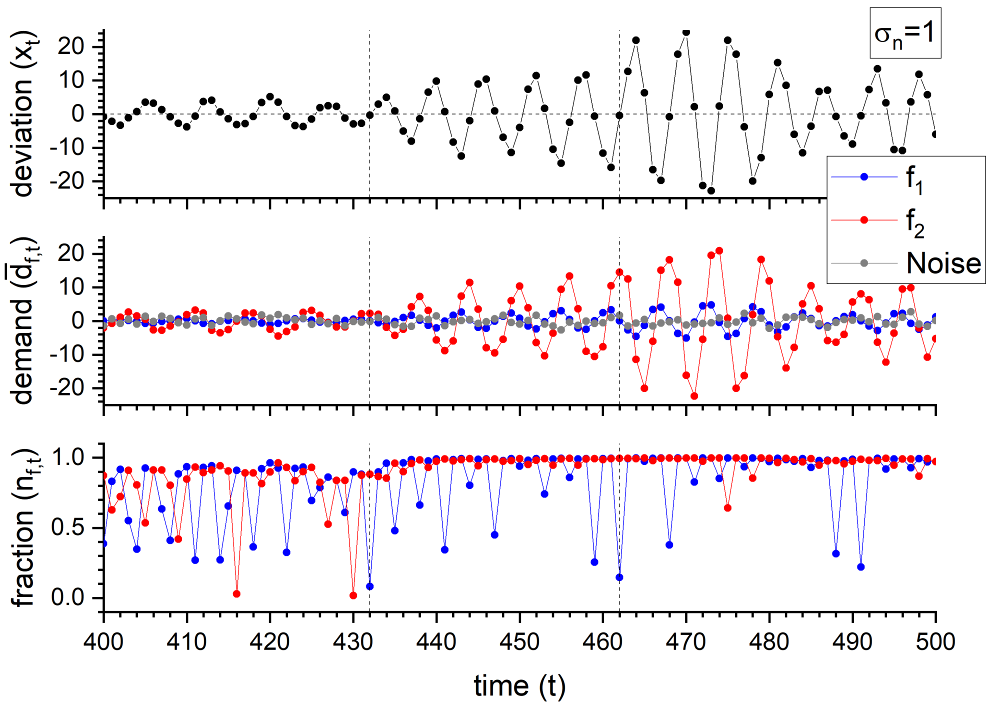


FIGURE 4.10: The deviation of the price from the fundamental value (top panel), investors' net demand relative to their market dominance (middle panel) and the fraction of investors of each group trading the asset at time t (bottom panel) for $T = 100$ time steps when the standard deviation of the noise is equal to $\sigma_n = 1$. The reference lines at $t_5 = 432$ and $t_6 = 462$ indicate two moments in which the price is close enough to p^* to drive the asset price further away from p^* (because of f_2's trade momentum).

followers, who refer to the information shared by the former with a delay. In the first section, we introduced two notions that, despite being simple in their concept, are new in the field of heterogeneous agent models (HAMs): the group's *reference price*, that is, the asset price to which a group of investors refers instead of its current price (as a consequence of the interaction with other groups of investors), and the *sharing mechanism*, which describes the fraction of investors belonging to a group who shares their reference price with another group over time. We put these two notions into practice to describe *followers'* reference price updating rule only (see equation (4.3) and (4.6)), as we assume pure fundamentalists always refer to the asset's current price. We split the analysis of this market, whose price evolution in terms of its deviation from the fundamental value p^* is described by the system (4.10), into two parts. We first considered its simplest form by assuming the trading and sharing threshold equal to zero (section 4.2), that is, when the asset price is far enough from its fundamental value to trigger both groups of investors to trade it and pure fundamentalists to share information about its mispricings with the followers at all times (with one-time step delay), for which the market reduces to the linear system (4.21). We found that when pure fundamentalists predominantly populate the market, the outcomes were no different from those obtained in the previous chapter (as it is obvious to expect), where the market was populated by only one group of fundamentalists (price convergences to the fundamental value, double-period patterns and price divergences). If, instead, the market is mostly populated by followers (the new component of our model), or their dominance is comparable to pure fundamentalists, then we observed a perpetually volatile market (with periodic and quasi-periodic price patterns), a phenomenon that in HAMs is usually explained by means of a switching mechanism adopted by groups of investors between different strategies (Dieci and He (2018)) (or by the presence of noise traders, see the previous chapter). In our model, the price is more influenced by followers' demand, which is pure fundamentalists' delayed demand, and because of that, the asset price takes more time to reverse its trend (when overvalued or overvalued). Under these conditions (which are very strict as they contain an equality relation), the market is sustainable for both groups (they positively perform in the short and long term) if and only if pure fundamentalists are equally or less dominant than followers. If this condition does not hold, the price becomes more volatile since pure fundamentalists more promptly clear any mispricing, and, consequently, the followers would find it harder and harder to keep the pace of price fluctuations. Eventually, followers would incur losses and, thus, be driven out of the market unless they decide to change strategy (referring to more reliable sources of information) as in the Brock-Hommes model. Next, we focused our analysis of a market where investors refer to a trading and sharing threshold (section 4.3), which now are different from zero and equal in value, meaning that as soon as the asset price gets far enough from its fundamental value, pure fundamentalists would start trading it and, almost simultaneously, sharing their information with followers about its mispricing. We found that the conditions that correspond to a volatile market were less restrictive than in the no-threshold case (see parameter diagram D.1 and D.2) and, interestingly, most of these cases represent markets in which both groups of investors perform positively, both in the short and long term, as pure fundamentalists' dominance is generally lower than followers'. Other differences with respect to the no-threshold case are the presence of markets exhibiting multiple price volatilities, volatility clustering (a well-known stylised fact of financial time series, Cont (2001)), and sensitive dependence on initial conditions, which suggests that the system is potentially chaotic (Elaydi (2007)). We considered a case study where we brought out all these aspects, and we observed how pure fundamentalist sharing mechanism plays a critical role in giving rise to these new phenomena, whose function can be summarised as follows: it synchronises investors' demand in the same direction when the asset price is far enough from its fundamental value (pure fundamentalists share their information with followers), reinforcing the asset's price reversal back to its fundamental value, and it desynchronises otherwise (pure fundamentalists do not share their information), leading followers to trade it even if it is close to p^*. By doing so, the asset price perpetually fluctuates around its fundamental value, and if it gets very close to it, a transition occurs where it starts oscillating more and more strongly for a certain period before *damping out*, that is, the market exhibits *volatility clustering*. The same features are observed if we include a noise term in the system (e.g. noise traders), an external factor that, unlike in the previous chapter, turns out to be unnecessary to generate (one of the) realistic properties of financial time series.

Chapter 5

Pure fundamentalists, followers and a misinforming source

In the previous chapter, we considered a market populated by two groups of investors: *pure fundamentalists* and *followers*. Unlike what we observed in the case of a market with pure fundamentalists only (chapter 4), the presence of followers can contribute to persistent price fluctuations (market volatility) without the need for a stochastic term (e.g. noise traders). Moreover, the price trend generated by the demands of these two groups exhibits *volatility clustering* (strong price fluctuations around the asset's fundamental value for a certain period before damping out), one of the stylised facts present in financial markets. However, the price dynamics remain simple, showing mostly sinusoidal trends and, thus, differing quite a bit from what is observed in a real market. In this chapter, we will show how it is possible to generate more realistic price trends by considering a market populated by three groups of investors: one made up of *pure fundamentalists*, *contrarian fundamentalists* and *followers*. This time, followers not only refer to the information shared by pure fundamentalists but will also consider the information shared by contrarian fundamentalists that go against the former ones and, so, act like a misinforming source. The interaction between these three groups will give rise to non-stochastic volatile markets, clustering phenomena, irregular price trends, and bubbles (persistent growth in the asset price above its fundamental value) followed by crashes (sudden asset price decreases much below its fundamental value). Once we discussed the role played by these groups (with much more emphasis on the contrarians), we will analyse the system on a large time scale such that all investors trade the asset at any time but share their information about their *reference price* only if its deviation from the asset's fundamental value is large enough.

5.1 A market with a third group of fundamentalists: *contrarians*

We show again the system of difference equations that describe our financial market derived in the chapter 3 (see system (3.37)):

$$\begin{cases} p_t = p_{t-1} + \mu \sum_{h=1}^{H} n_{h,t-1}\bar{c}_h(p^* - p_{h,t-1}) \\ p_{h,t} = p_{h,t-1} + \sum_{i \neq h}^{H} \gamma_i \alpha_{i,t-1} \left(p_{i,t-1} - p_{h,t-1}\right) \\ \sum_{i \neq h}^{H} \gamma_i = 1 \\ n_{h,t} = \frac{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}}}{\left[(p^*_{h,t} - p_t)^2\right]^{\beta_{\tau,h}} + \Delta_{\tau,h,t}^{\beta_{\tau,h}}} \\ \alpha_{i,t} = \frac{\left[(p^*_{i,t} - p_t)^2\right]^{\beta_{\sigma,i}}}{\left[(p^*_{i,t} - p_t)^2\right]^{\beta_{\sigma,i}} + \Delta_{\sigma,i,t}^{\beta_{\sigma,i}}} \end{cases} \tag{5.1}$$

As we know, any financial market is populated by groups of investors f_i who have different ways to estimate an asset's value over time, which, as explained in chapter 3, can be equivalently interpreted as if they referred to different *reference prices* $p_{f_i,t}$, such that their demand $z_{f_i,t}$ for the asset at time t (whose current price is p_t) can be expressed in terms of the asset's fundamental value p^* as

$$z_{f_i,t} = \bar{c}_{f_i}(p^* - p_{f_i,t}) \tag{5.2}$$

where $\bar{c}_{f_i}$ is group f_i's dominance. So far, we have considered the presence of *pure* fundamentalists f_1, whose demand was given by

$$z_{f_1,t} = \bar{c}_{f_1}(p^* - p_t) \tag{5.3}$$

as they always refer to the asset's current price, that is

$$p_{f_1,t} = p_t \tag{5.4}$$

A second group of *followers*, f_2, whose demand was given by

$$z_{f_2,t} = (p^* - p_{f_2,t}) \tag{5.5}$$

who received the information from the first group f_1 but with a delay, such that their reference price, $p_{f_2,t}$, updated as (see second equation in system (4.1) with $\gamma_{f_1} = 1$)

$$p_{f_2,t} = p_{f_2,t-1} + \alpha_{f_1,t-1}\left(p_{f_1,t-1} - p_{f_2,t-1}\right) \tag{5.6}$$

where $0 \leq \alpha_{f_1,t-1} \leq 1$ was given by

$$\alpha_{f_1,t} = \frac{\left[(p^*_{f_1,t} - p_t)^2\right]^{\beta_{\sigma,f_1}}}{\left[(p^*_{f_1,t} - p_t)^2\right]^{\beta_{\sigma,f_1}} + \Delta^{\beta_{\sigma,f_1}}_{\sigma,f_1,t}} \tag{5.7}$$

and represented group f_1's sharing mechanism for their reference price and described the fraction of investors belonging to group f_1 that shared their information (their reference price $p_{f_1,t-1}$) at time $t-1$ (which corresponds to p_{t-1}). Here, we will consider the more general case in which a misinforming source tries to deceive followers by sending information that goes against the one sent by pure fundamentalists, acting as a *contrarian fundamentalist* (f_m).

Contrarian investors in financial markets A contrarian investor in financial markets typically adopts a strategy against prevailing market trends. This approach is based on the belief that most investors tend to overreact to good and bad news, resulting in stock price movements that do not necessarily reflect the company's long-term fundamentals. By going against the crowd, contrarians often buy stocks when pessimism is at its peak and sell when optimism reaches its height. So, contrarian investing can be highly profitable, particularly in volatile markets (Lo and MacKinlay (1990)). However, it could also result in losses when the underlying analysis of market conditions is incorrect, such as misinterpreting the reasons behind a stock's performance. Markets may continue in their current trend longer than a contrarian investor anticipates, making early or misjudged entries and exits costly (selling the asset too early before it reaches its peak or buying it too late when its price is already declining, as observed for most household investors in the Finnish stock market, see Grinblatt and Keloharju (2000b)). This strategy necessitates a thorough understanding of market psychology, solid fundamental analysis, and the courage to make decisions counter to popular opinion. It is, therefore, natural to expect that, given the risky nature involved in this strategy, these investors could act as sources of misinformation in order for them to succeed in their objective. Chang (2007) was the first author to have extended the Brock-Hommes model by including social interactions among investors (see section 2.1.3.1), and they considered the case where the market was populated by contrarians (or *arbitrageurs*) and noise traders (or *trend chasers*), investors who believe that the price will follow its current trend. In their model, where investors update their beliefs according to the past realised profits and the interactions among them regarding their expectations of the average choice level in the economy, they show that both scenarios described above (contrarians achieving high profits or driven out of the market) may occur depending on the strength of the social interactions among investors.

Contrarian fundamentalists in our model This is all the more true for what we call contrarian fundamentalists, that is, investors who buy a stock when its price is above the fundamental value and sell when it is below it. Such a term is not widely recognised in the literature (see De Grauwe and Grimaldi (2004) and Baur and Glover (2015) for some references to this financial figure, which is assumed to induce explosive price dynamics). Nonetheless, this strategy is something we have already seen in the Brock-Hommes model in Chapter 2 (but also in Day and Huang (1990)). We refer to the case when investors switch from the fundamentalist strategy to the chartist one as soon as the asset price gets close to its fundamental value. As discussed at the end of section 2.1.2.2, when the asset price is close to p^*, investors wisely choose to switch off from the fundamentalist strategy since there is a cost c whenever they choose such a strategy due to the effort to gather information on economic fundamentals (there is no sense in paying it when the profits are not even high enough to cover such a cost). What happens is that they take a position contrary to it, sending buy orders when the price is above p^* and selling orders when it is below p^*, and switch back to it if the price is far enough from the fundamental value. In our model, we describe contrarian fundamentalists as a subgroup of pure fundamentalists who misrepresent their beliefs and only act as a misinforming source for followers to try to deviate the asset price away from its fundamental value if it is too close to it, for the abovementioned reasons. Although not in the context of a financial market, Buechel et al. (2015) developed a dynamic model of opinion formation in social networks in which agents were allowed to misrepresent their beliefs to conform or counter-conform with their peers (as they experience a utility gain to do so), supported by the large empirical evidence of such phenomenon (this behaviour is also known as *normative social influence*, see Asch (1956), Deutsch and Gerard (1955) who first conducted studies on this aspect). In our case, only economic factors drive contrarian fundamentalists to behave this way. However, future works on our model could also consider the type of social influence analysed in the study mentioned above, where a subgroup of pure fundamentalists may also conform to followers' beliefs (e.g. to avoid the discomfort of contradicting them), potentially leading them to go against their main group. An aspect that would further justify adopting such a strategy.

5.1.1 Contrarian fundamentalists

If we take f_m into account, then the group f_2's reference price becomes a weighted sum of their past reference price, f_1's past reference price, and f_m's past reference price, that is

$$p_{f_2,t} = (1-\gamma)\alpha_{f_1,t-1}p_{f_1,t-1} + \gamma\alpha_{f_m,t-1}p_{f_m,t-1} + \left[1-(1-\gamma)\alpha_{f_1,t-1} - \gamma\alpha_{f_m,t-1}\right] p_{f_2,t-1} \tag{5.8}$$

or, alternatively,

$$p_{f_2,t} = p_{f_2,t-1} + (1-\gamma)(p_{f_1,t-1} - p_{f_2,t-1})\alpha_{f_1,t-1} + \gamma(p_{f_m,t-1} - p_{f_2,t-1})\alpha_{f_m,t-1} \tag{5.9}$$

which is the *updating rule for f_2's reference price*. γ (with $0 < \gamma < 1$) represents the degree of *reliability* that group f_2 associates with the source of information f_m, and $0 < \alpha_{f_m,t-1} < 1$ describes the fraction of investors belonging to group f_m that shares their information (their reference price $p_{f_m,t-1}$) at time $t-1$. The role of γ can be understood if we consider the situation where both f_1 and f_m send their information at the same time and with the same 'intensity', that is when $\alpha_{f_1,t-1} \sim \alpha_{f_m,t-1}$. Under these circumstances, it is up to f_2 to decide which source of information is more reliable, having the choice, as extreme cases, to believe f_1 as a completely trustworthy, $\gamma = 0$ (which corresponds to the market analysed in the previous chapter), or that f_m is more trustworthy, i.e. the case $\gamma = 1$.

Contrarian's reference price Since f_m always share their information with f_2 that goes against f_1's, their reference price $p_{f_m,t}$ must be such that

$$p^* - p_{f_m,t} = -(p^* - p_{f_1,t}) \iff p_{f_m,t} = p^* - (p_{f_1,t} - p^*) \tag{5.10}$$

The previous equation tells us that whenever group f_1 informs f_2 about the asset being underpriced (overpriced), f_m would inform them of the opposite (e.g. $p_{f_1,t} > p^* \implies p_{f_m,t} < p^*$). If we substitute equation (5.10) into equation (5.9) and replace every variable with $x_{f_i,t} = p_{f_i,t} - p^*$, then we get

$$x_{f_2,t} = x_{f_2,t-1} + (1-\gamma)(x_{f_1,t-1} - x_{f_2,t-1})\alpha_{f_1,t-1} + \gamma(-x_{f_1,t-1} - x_{f_2,t-1})\alpha_{f_m,t-1} \tag{5.11}$$

Contrarians' sharing mechanism Similarly to what we did in the previous chapter, we assume $\alpha_{f_1,t}$ given by

$$\alpha_{f_1,t} = \frac{|x_{f_1,t}|^{2\beta_\sigma}}{|x_{f_1,t}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \tag{5.12}$$

where β_σ is the degree of confidence for investors in group f_1 in deciding to share their information and Δ_σ is the sharing threshold for f_1's reference price deviation from the fundamental value beyond which ($|p^* - p_{f_1,t}|^{2\beta_\sigma} > \Delta_\sigma^{\beta_\sigma}$) most of them would start sharing their information. As for $\alpha_{f_m,t}$, we first notice that in the simplest case we choose $\alpha_{f_m,t} = \alpha_{f_m} = 0$, we would get the same market that we analysed in the previous chapter, while if we choose $\alpha_{f_m,t} = \alpha_{f_m} = 1$, then f_m would *persistently* send their information to f_2. Although there is nothing wrong if we assume the latter case, we can do better and choose the functional form of $\alpha_{f_m,t}$ so as to mimic investors' behaviour in the Brock-Hommes model when they switch off from the fundamentalist strategy. In this case, f_m's goal is to influence f_2 to manipulate the market when the asset price is close to its fundamental value, without relying on their dominance $\bar{c}_{f_m}$ (which is equal to $\bar{c}_{f_m} = 0$ as they participate only as a source of information). Gathering information on an asset's fundamentals is costly, and the profits one can make when the asset price is close to its fundamental value might not even be enough to cover such costs. So, their goal is to move the asset's price away from its fundamental value when it is too close to it until the asset's price is far enough from p^* and the fundamentalist strategy becomes more profitable from such mispricings. So, assuming contrarian fundamentalists are a subgroup of pure fundamentalists and, thus, have access to the asset's fundamentals, $\alpha_{f_m,t}$ can be formally described as (recalling that $x_t = x_{f_1,t}$)

$$\alpha_{f_m,t} = \frac{\Delta_\sigma^{\beta_\sigma}}{|x_{f,t}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} = 1 - \alpha_{f_1,t} \tag{5.13}$$

which tends to $\alpha_{f_m,t} \to 1$ as $p_t \to p^*$, and $\alpha_{f_m,t} \to 0$ as the price gets far away from the fundamental value (in which case there is no longer a need to manipulate the market, and they aim now for a price correction back to p^*so as to take advantage of these mispricings). The choice of $\alpha_{f_m,t}$ is also dictated by the need to keep the model analytically tractable. By using equations (5.11), (5.12) and (5.13), we can describe the market as a two-dimensional system with first-order difference equations, namely

$$\begin{cases} x_{f_1,t} = x_{f_1,t-1} - \mu\left(\bar{c}_{f_1} x_{f_1,t-1} n_{f_1,t-1} + \bar{c}_{f_2} x_{f_2,t-1} n_{f_2,t-1}\right) \\ x_{f_2,t} = x_{f_2,t-1} + (1-\gamma)\left(x_{f_1,t-1} - x_{f_2,t-1}\right)\alpha_{f_1,t-1} + \gamma\left(-x_{f_1,t-1} - x_{f_2,t-1}\right)\alpha_{f_m,t} \\ n_{f_i,t} = \frac{|x_{f_i,t}|^{2\beta_\tau}}{|x_{f_i,t}|^{2\beta_\tau} + \Delta_\tau^{\beta_\tau}} \\ \alpha_{f_1,t} = \frac{|x_{f_1,t}|^{2\beta_\sigma}}{|x_{f_1,t}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \\ \alpha_{f_m,t} = \frac{\Delta_\sigma^{\beta_\sigma}}{|x_{f_1,t}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \end{cases} \tag{5.14}$$

where β_τ is the degree of confidence for investors in taking the decision to trade the asset and Δ_τ is the trading threshold for the reference price deviation from the fundamental value beyond which ($|p^* - p_{f_i,t}|^{2\beta_\tau} > \Delta_\tau^{\beta_\tau}$) most of them would start trading the asset. As already stated in the introduction of this chapter, we will analyse the system on a large time scale such that all investors trade the asset at any time, that is

$$\Delta_\tau = 0 \implies n_{f_i,t} = 1 \quad \textit{for any time } t \tag{5.15}$$

and share their information about their reference price only if its deviation from the asset's fundamental value is large enough ($\Delta_\sigma \neq 0$).

5.2 Investors with sharing threshold and no trading threshold

Given the condition (5.15), the system (5.14) becomes

$$\begin{cases} x_{f_1,t} = x_{f_1,t-1} - \mu\left(\bar{c}_{f_1} x_{f_1,t-1} + \bar{c}_{f_2} x_{f_2,t-1}\right) \\ x_{f_2,t} = x_{f_2,t-1} + (1-\gamma)\left(x_{f_1,t-1} - x_{f_2,t-1}\right)\alpha_{f_1,t-1} + \gamma\left(-x_{f_1,t-1} - x_{f_2,t-1}\right)\alpha_{f_m,t} \\ \alpha_{f_1,t} = \frac{|x_{f_1,t}|^{2\beta_\sigma}}{|x_{f_1,t}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \\ \alpha_{f_m,t} = \frac{\Delta_\sigma^{\beta_\sigma}}{|x_{f_1,t}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \end{cases} \tag{5.16}$$

which can also be written as the map F

$$F(f_{x_{f_1}}, f_{x_{f_2}}) = \begin{pmatrix} x_{f_1} - \mu\left(\bar{c}_{f_1} x_{f_1} + \bar{c}_{f_2} x_{f_2}\right) \\ x_{f_2} + (1-\gamma)\left(x_{f_1} - x_{f_2}\right)\frac{|x_{f_1}|^{2\beta}}{|x_{f_1}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} + \gamma\left(-x_{f_1} - x_{f_2}\right)\frac{\Delta_\sigma^{\beta_\sigma}}{|x_{f_1}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \end{pmatrix} \tag{5.17}$$

Fixed points The fixed points $\vec{x}_b = (x_{f_1,b}, x_{f_2,b})$ of the map (5.17) can be found by solving the following system

$$\begin{cases} x_{f_1,b} = x_{f_1,b} - \mu\left(\bar{c}_{f_1} x_{f_1,b} + \bar{c}_{f_2} x_{f_2,b}\right) \\ x_{f_2,b} = x_{f_2,b} + (1-\gamma)\left(x_{f_1,b} - x_{f_2,b}\right)\frac{|x_{f_1,b}|^{2\beta_\sigma}}{|x_{f_1,b}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} + \gamma\left(-x_{f_1,b} - x_{f_2,b}\right)\frac{\Delta_\sigma^{\beta_\sigma}}{|x^*_{f_1,b}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \end{cases} \tag{5.18}$$

or, after simplifying some terms,

$$\begin{cases} x_{f_2,b} = -\frac{\bar{c}_{f_1}}{\bar{c}_{f_2}} x_{f_1,b} \\ (1-\gamma)\left(x_{f_1,b} + \frac{\bar{c}_{f_1}}{\bar{c}_{f_2}} x_{f_1,b}\right)|x_{f_1,b}|^{2\beta_\sigma} - \gamma\left(x_{f_1,b} - \frac{\bar{c}_{f_1}}{\bar{c}_{f_2}} x_{f_1,b}\right)\Delta_\sigma^{\beta_\sigma} = 0 \end{cases} \tag{5.19}$$

We can immediately see that the fundamental steady state $\vec{x}_o = (0, 0)$ is a solution. For $x_{f_1,b} \neq 0$, instead, we can divide the second equation by $x_{f_1,b}$ and get, after some straightforward algebraic steps, the simultaneous presence of two non-fundamental steady states $\vec{x}_{b^\pm}$

$$\begin{cases} x_{f_2,b^\pm} = -\frac{\bar{c}_{f_1}}{\bar{c}_{f_2}} x_{f_1,b^\pm} \\ |x_{f_1,b}|^{2\beta_\sigma} = \left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\bar{c}_{f_2}-\bar{c}_{f_1}}{\bar{c}_{f_2}+\bar{c}_{f_1}}\right)\Delta_\sigma^{\beta_\sigma} \iff x_{f_1,b^\pm} = \pm\sqrt{\Delta_\sigma}\left[\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{\bar{c}_{f_2}-\bar{c}_{f_1}}{\bar{c}_{f_2}+\bar{c}_{f_1}}\right)\right]^{\frac{1}{2\beta_\sigma}} \end{cases} \tag{5.20}$$

Let us first focus on the case where pure fundamentalists do not actively participate in the market and only act as a source of information for the followers, which means that $\bar{c}_{f_1} = 0$. Then, the system (5.20) becomes

$$\begin{cases} x_{f_2,b^\pm} = 0 \\ x_{f_1,b^\pm} = \pm\sqrt{\Delta_\sigma}\sqrt{\left[\left(\frac{\gamma}{1-\gamma}\right)\right]^{\frac{1}{\beta_\sigma}}} \iff p_{b^\pm} = p^* \pm \sqrt{\Delta_\sigma}\left(\frac{\gamma}{1-\gamma}\right)^{\frac{1}{2\beta_\sigma}} \end{cases} \tag{5.21}$$

The asymptotic behaviour of the price p_b is equally distant from the fundamental value, being either above it (positive state $\vec{x}_{b^+}$) or below it (negative state $\vec{x}_{b^-}$), and these solutions exist for any fixed value of the parameters $\Delta_\sigma > 0$, $\beta_\sigma > 0$ and $0 < \gamma < 1$. As one would expect, for fixed Δ_σ and β_σ, the more reliable contrarian fundamentalists are considered from the followers ($\gamma \to 1$), the larger these bubbles can get. Interestingly, these mispricings do not depend on f_2's dominance $\bar{c}_{f_2}$. The first equation of system (5.21) tells us that the asymptotic behaviour of the followers is such that they sell and buy the same amount of shares when the asset is either overpriced or underpriced, as we have $\bar{d}_{f_2,b^\pm} = -\bar{c}_{f_2} x_{f_2,b^\pm} = 0$. This means that if the asset is overpriced at time t ($p_t > p^*$), then they mainly send buy orders when $p_t < p_{b^+}$ and mainly send sell orders when $p_t > p_{b^+}$ (vice-versa if the asset is underpriced). This observation already tells us how this limiting case (which gives rise to persistent bubbles) is potentially sustainable for followers (we will see how much in a more detailed analysis below), resulting, therefore, in realistic market conditions. In the more general case where group f_1 trades the asset ($\bar{c}_{f_1} \neq 0$), the non-fundamental steady states now depend on groups' dominance and, more specifically, the solutions $\vec{x}^*_{b^\pm}$ exist if and only if $\bar{c}_{f_2} > \bar{c}_{f_1}$, that is, if followers are more dominant than pure fundamentalists (see second equation of the system (5.20)). From the first equation of the system (5.20), we see that if the price asymptotically tends, for example, to the positive bubble p_{b^+}, the followers will mostly send buy orders ($x_{f_1,b^+} > 0 \implies x_{f_2,b^+} < 0$), as pure fundamentalists will only sell the asset in this case. So, unlike the previous case, there could be cases that cannot occur in a real context (or could last for a short period only) since they would require an infinite amount of supply or liquidity.

Stability analysis To study the stability of these steady states, we first derive the Jacobian matrix of the map (5.17), that is (see appendix A.2.1),

$$JF(f_{x_{f_1}}, f_{x_{f_2}}) = \begin{pmatrix} \frac{\partial f_{x_{f_1}}}{\partial x_{f_1}} & \frac{\partial f_{x_{f_1}}}{\partial x_{f_2}} \\ \frac{\partial f_{x_{f_2}}}{\partial x_{f_1}} & \frac{\partial f_{x_{f_2}}}{\partial x_{f_2}} \end{pmatrix} \tag{5.22}$$

and evaluate it at the fixed points $\vec{x}_o$ and $\vec{x}_{b^\pm}$. After that, we take the determinant of the Jacobian $det(J)$ and its trace $tr(J)$, and consider the following stability conditions:

$$det(J) > -tr(J) - 1 \tag{5.23}$$

$$det(J) > tr(J) - 1 \tag{5.24}$$

$$det(J) < 1 \tag{5.25}$$

In fact, we have that the steady states are (locally) stable if and only if the previous conditions are satisfied (Elaydi (2007)). We will first start with the fundamental steady state and then with the non-fundamental ones.

5.2.1 Stability of the fundamental steady state

The Jacobian matrix calculated at the fundamental steady state becomes (see appendix A.2.2)

$$J_o = JF(f_{x_{f_1}}, f_{x_{f_2}})\big|_{\vec{x}_o^*} = \begin{pmatrix} 1 - \mu\bar{c}_{f_1} & -\mu\bar{c}_{f_2} \\ -\gamma & 1-\gamma \end{pmatrix} \tag{5.26}$$

whose determinant and trace is

$$det(J_o) = (1 - \mu\bar{c}_{f_1})(1-\gamma) - \mu\gamma\bar{c}_{f_2} \tag{5.27}$$

$$tr(J_o) = 2 - \mu\bar{c}_{f_1} - \gamma \tag{5.28}$$

From the first stability condition (5.23), we collect and separate out the common terms of μ from the rest so that we get, after some straightforward algebraic steps,

$$2(2-\gamma) > \mu\left[(2-\gamma)\bar{c}_{f_1} + \gamma\bar{c}_{f_2}\right] \tag{5.29}$$

The LHS of the previous inequality and the term multiplying μ on the RHS are positive, so we can rewrite (5.29) as a condition on μ, that is

$$\mu < \frac{2}{\bar{c}_{f_1} + \left(\frac{\gamma}{2-\gamma}\right)\bar{c}_{f_2}} = \mu_{x_o} \tag{5.30}$$

We first notice that for $\gamma \to 0$ (followers trust pure fundamentalists only), the previous condition becomes

$$\gamma \to 0 \implies \mu < \frac{2}{\bar{c}_{f_1}} \tag{5.31}$$

This condition is the same one we found in the previous chapter, where the market was populated by pure fundamentalists and followers, which, if violated, gave rise to a price pattern with period 2. The interpretation of this is that the market maker does not adequately compensate the excessive demand due to f_1, whose orders, generated by mispricing of the asset ($|x_t| > 0$), lead to higher mispricings. For $0 < \gamma < 1$, the condition (5.30) is stricter than (5.31), meaning that the more the contrarian fundamentalists affect the followers, the more easily they can destabilise the market. In the limit case $\gamma \to 1$, we have

$$\gamma \to 1 \implies \mu < \frac{2}{\bar{c}_{f_1} + \bar{c}_{f_2}} \tag{5.32}$$

so, to ensure the asset price converges to its fundamental value, the market maker has to completely compensate the excess demand generated by group f_1 and f_2 by setting the speed of price adjustment μ below the inverse of the average value of groups' dominance in the market. This allows for dampening any mispricing, even if the contrarian fundamentalists f_m greatly affect group f_2 when the price is close to p^* ($\gamma \sim 1$). From the second stability condition (5.24) we simply get

$$\gamma\mu(\bar{c}_{f_1} - \bar{c}_{f_2}) > 0 \iff \bar{c}_{f_1} > \bar{c}_{f_2} \tag{5.33}$$

Group f_1 has to be more dominant than group f_2 for the fundamental steady state to be asymptotically stable. For $\bar{c}_{f_1} = \bar{c}_{f_2}$ we have a *pitchfork* bifurcation (one of the eigenvalues is equal to $\lambda = 1$), where the two non-fundamental steady states $\vec{x}_{b^\pm}$ are created (see the system (5.20)). Notice that, unlike any stability conditions we have seen so far, the market maker can not prevent this market instability in any way. This makes sense since the market maker can affect how much the price can fluctuate around a value but has no control over the average price, which is decided by investors' expectations of the asset price (fundamental and non-fundamental value). Finally, we have that the third condition is always satisfied since (from the inequality (5.24))

$$-\gamma(1 + \mu\bar{c}_{f_2}) - (1-\gamma)\mu\bar{c}_{f_1} < 0 \tag{5.34}$$

We see that both terms in LHS of the inequality are negative, meaning that there cannot occur any Neimark-Sacker bifurcation.

Orbits with period 2 Assuming the stability condition (5.33) is satisfied, if the condition (5.30) gets violated, then the system undergoes a *period-doubling* bifurcation, that is, the orbits become periodic with period 2 (depending on which initial condition one chooses, the orbit may either flip above and below the fundamental value or diverge eventually to infinity). It can be shown that the form of these periodic fixed points are

$$\begin{cases} x_{f_1,PD^{\pm}} = \pm\sqrt{\Delta_\sigma}\left[\frac{\gamma-(2-\gamma)\tilde{c}}{1-\gamma+(1+\gamma)\tilde{c}}\right]^{\frac{1}{2\beta\sigma}} \\ x_{f_2,b^{\pm}} = \tilde{c}x_{f_1,b^{\pm}} \end{cases} \tag{5.35}$$

where

$$\tilde{c} = \frac{2-\mu\bar{c}_{f_1}}{\mu\bar{c}_{f_2}} \tag{5.36}$$

The numerator and the denominator of the fraction within the square root of the first equation in the system (5.35) must be positive for these solutions to exist (they cannot be both negative). This implies that μ must belong in

$$\mu \in \left(\frac{2}{\bar{c}_{f_1}+\left(\frac{\gamma}{2-\gamma}\right)\bar{c}_{f_2}} = \mu_{x_o}, \frac{2}{\bar{c}_{f_1}-\left(\frac{1-\gamma}{1+\gamma}\right)\bar{c}_{f_2}} \stackrel{\text{def}}{=} \mu_{x_o,PD}\right) \tag{5.37}$$

So, if the speed of price adjustment increases beyond $\mu_{x_o,PD}$, the price (group's demand) with period 2 ceases to exist and it could either undergo other bifurcations or diverge to infinity. From the second equation in the system (5.35), we can conclude that followers' net demand ($\bar{d}_{f_2,t} = -\bar{c}_{f_2}x_{f_2,t}$) is positively proportional to pure fundamentalists' net demand ($\bar{d}_{f_1,t} = -\bar{c}_{f_1}x_{f_1,t}$) if (see equation (5.36))

$$\mu < \frac{2}{\bar{c}_{f_1}} \tag{5.38}$$

meaning that f_2 performs optimally (sending buy orders when the asset price is below its fundamental value and sending sell orders when it's above it only) if the speed of price adjustment is low enough compared to f_1's market dominance $\bar{c}_{f_1}$ (increasing the volatility of price does negatively influence followers' performance, as we already observed in the previous chapter).

5.2.2 Pure fundamentalists more dominant than followers

Here, we will consider some examples of markets where pure fundamentalists are more dominant than followers, so having the fundamental value as the only fixed point of the system (5.16)). For simplicity, we will perform the analysis for fixed $\bar{c}_{f_1} = 1.0$, $\bar{c}_{f_2} = 0.5$, $\Delta_\sigma = 1.0$[1], and $\beta_\sigma = 1.0$, as we change μ and γ.

5.2.2.1 Pure fundamentalists more trustworthy than contrarian fundamentalists

We first consider the case that comes closest to the one analysed in the previous chapter, that is, when followers mostly trust pure fundamentalists, by choosing $\gamma = 0.2$, a value for which the stability condition (5.30) becomes

$$\mu < \mu_{x_o} = \frac{36}{19} \sim 1.895 \tag{5.39}$$

In Figure 5.1, we show the price deviation from the fundamental value $x_t (= x_{f_1,t})$ (top left panel) and investors' net demand $\bar{d}_{f_i,t} = -\bar{c}_{f_i} x_{f_i,t}$ relative to their market dominance (bottom left panel), for fixed $\mu = 1.9$, a value that slightly violates the stability condition (5.39), and $\beta_\sigma = 1.0$ after a transition of $T_{tran} = 10000$ time steps. We choose $\vec{x}_{f,0} = (0.01, 0)$ as the initial state, that is when only group f_1 starts trading the (mispriced) asset and, consequently, share their information (along with the misinforming source f_m) to f_2 at the next time step. The price exhibits the classic pattern with period 2, as we found in the previous chapter. However, differently from that case, we now have followers trading synchronously and in the same direction as pure fundamentalists, that is, buying when the asset is undervalued and selling it when it is overvalued. This is because the price fluctuates back and forth between two values that are close enough to p^* to trigger mostly the contrarian fundamentalist to share their information, as we have (see system (5.16))

$$\alpha_{f_m,t} = 1 - \alpha_{f_1,t} = 0.989 \quad \text{at any time } t \implies (1-\gamma)\alpha_{f_1,t} < \gamma \alpha_{f_m,t} \tag{5.40}$$

meaning that the overall influence of the misinforming source (their trustworthiness, γ, combined with their signal intensity, $\alpha_{f_m,t}$) is greater than the pure fundamentalist one despite followers mostly trusting the latter ($1 - \gamma = 0.8$). Group f_2 receives the information from f_m to buy (sell) the asset when overvalued (undervalued) with a delay, and as soon as they get it, the price will have already flipped below (above) the fundamental value. Consequently, f_2 sends their buy (sell) orders when the asset is undervalued (overvalued), leading them to perform optimally both in the short and

[1] We could choose and consider only a value of the sharing threshold equal to $\Delta_\sigma = 1$, without any loss of generality. In fact, for any other value $\Delta'_\sigma = a\Delta_\sigma$ ($a > 0$), we could always make a variable change $(x_{f_1,t}, x_{f_2,t}) \to (\sqrt{a} x_{f_1,t}, \sqrt{a} x_{f_2,t})$ such that the system (5.16) would end up with investors who would refer to the sharing threshold Δ_σ.

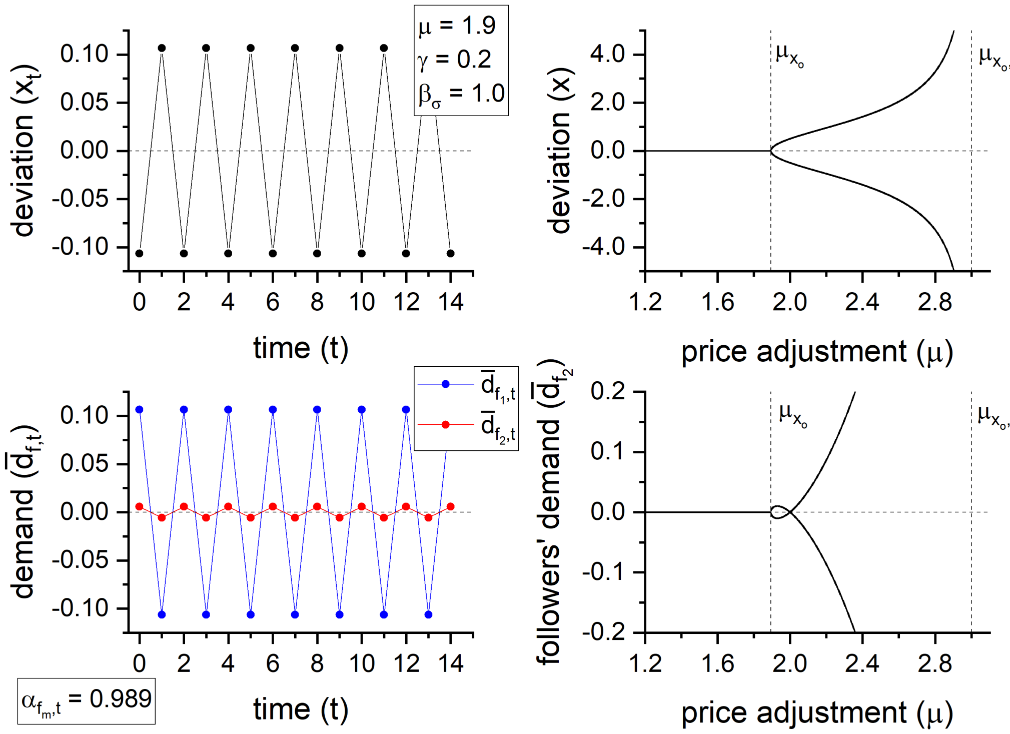


FIGURE 5.1: The deviation of the price from the fundamental value x_t (top left panel) and investors' net demand $\bar{d}_{f,t}$ (bottom left panel), for fixed $\mu = 1.9$, $\gamma = 0.2$, and $\beta_\sigma = 1.0$, and the orbit diagram for the price deviation x (top right panel) and followers' demand $\bar{d}_{f_2}$ (bottom right panel) for different values of μ. For $\mu = 1.9$, followers (red curve) trade synchronously and in the same direction as pure fundamentalists (blue curve), as the price deviates very little from p^* (contrarian fundamentalists greatly affect followers' trading decisions). As μ increases beyond μ_{x_o}, the price (top right panel) fluctuates back and forth between two larger and larger values. After an initial increase (right after $\mu > \mu_{x_o}$), followers' demand (bottom right panel) decreases as pure fundamentalists' become more influential, until $\bar{d}_{f_2} = 0$ (intersection point) when $\mu = 2$. Right after $\mu > 2$, $\bar{d}_{f_2}$ increases again but in the opposite direction to pure fundamentalists. At $\mu = \mu_{x_o,PD}$, the price (and followers' demand) diverges to infinity for any initial state.

long term at any period T. In the right panels of Figure 5.1, we show the orbit diagrams of the price deviation (top right) and followers' demand (bottom right) for $\mu \in [1.2, 3.1]$ in steps of $\delta_\mu = 0.002$, where we kept track of their evolution over a period of time $T_{test} = 2$ (there is no need to take a larger period as we only observe the price converging to a constant value or oscillating with period 2), for different initial states $x_{f_1,0} \in [-1.0, 1.0]$ and $x_{f_2,0} \in [-1.0, 1.0]$ in steps of $\delta_{x_f,0} = 0.25$. For $\mu < \mu_{x_o}$ (see first vertical dashed line), the price converges to the fundamental values, which is stable (solid line), and followers do not trade (the same goes for pure fundamentalists[2]). Right beyond $\mu > \mu_{x_o}$, the fundamental values become unstable (horizontal dashed line), and we see that the price fluctuates back and forth between

[2]Since we have $\bar{c}_{f_1} = 1.0$, the orbit diagram of pure fundamentalists' net demand corresponds exactly to the price deviation one, as $\bar{d}_{f_1} = -x_{f_1,t} = -x_t$

two larger and larger values as μ increases, a sign of the fact that the price becomes more susceptible to the excess demand generated by f_1 and f_2 (because of the market maker). As the price fluctuations increase, the influence of pure fundamentalists on the followers becomes stronger and stronger, and, so, the price deviation to which group f_2 refers shifts from $-x_{f_1,t}$ (f_m's reference price deviation) to $x_{f_1,t}$ (f_1's reference price deviation). This affects followers' net demand $\bar{d}_{f_2,t} = -\bar{c}_{f_2}x_{f_2,t}$, which, after an initial increase, right after $\mu > \mu_{x_o}$, decreases down to $\bar{d}_{f_2,t} = 0$ (holding state, see the intersection point of the orbits in the bottom right panel of Figure 5.1) as the speed of price adjustment gets to $\mu = 2$ (see condition (5.38)). In this specific case, we have, after the system has stabilised (see system (5.16))

$$\alpha_{f_m,t} = 0.8 \implies (1-\gamma)\alpha_{f_1,t} = \gamma\alpha_{f_m,t} \implies x_{f_2,t} = [1-2(1-\gamma)\alpha_{f_1,t}]x_{f_2,t-1} \to 0 \quad (5.41)$$

since $1-2(1-\gamma)\alpha_{f_1,t} < 1$, and

$$|x_{f_1,t}| = \frac{1}{2} \quad (5.42)$$

which represents the price deviation where both f_1 and f_m exert the same influence on f_2, who take no trading position, while if $x_{f_1} \in (-0.5, 0.5)$ then f_m exerts a greater influence than f_1. For $\mu > 2$, pure fundamentalists become more influential than the contrarian fundamentalists as the mispricings become larger, and we obtain the same dynamics that we found in the previous chapter(see the end of section 4.2.1), that is, followers trading synchronously but in the opposite direction to pure fundamentalists (their demand increases again as μ gets larger than 2), so performing poorly (sending buy orders when the asset price is above its fundamental value and sending sell orders when it's below it only) both in the short and long term. For $\mu \geq \mu_{x_{o,PD}}$, the condition of existence for orbits with period 2 gets violated, and we see the price and groups' demand diverging to infinity in an oscillatory way for any initial state.

5.2.2.2 Contrarian fundamentalists more trustworthy than pure fundamentalists

Now, we consider the case where followers mostly trust contrarian fundamentalists by choosing $\gamma = 0.8$, a value for which the stability condition (5.30) becomes

$$\mu < \mu_{x_o} = \frac{3}{2} \quad (5.43)$$

In Figure 5.2, we show the price deviation from the fundamental value x_t (top left panel) and investors' net demand $\bar{d}_{f_i,t}$ relative to their market dominance (bottom left panel), for fixed $\mu = 1.9$ and $\beta_\sigma = 1$, to compare it with the analogous case for fixed $\gamma = 0.2$. This time, we see that the price fluctuates between two values that exceed the

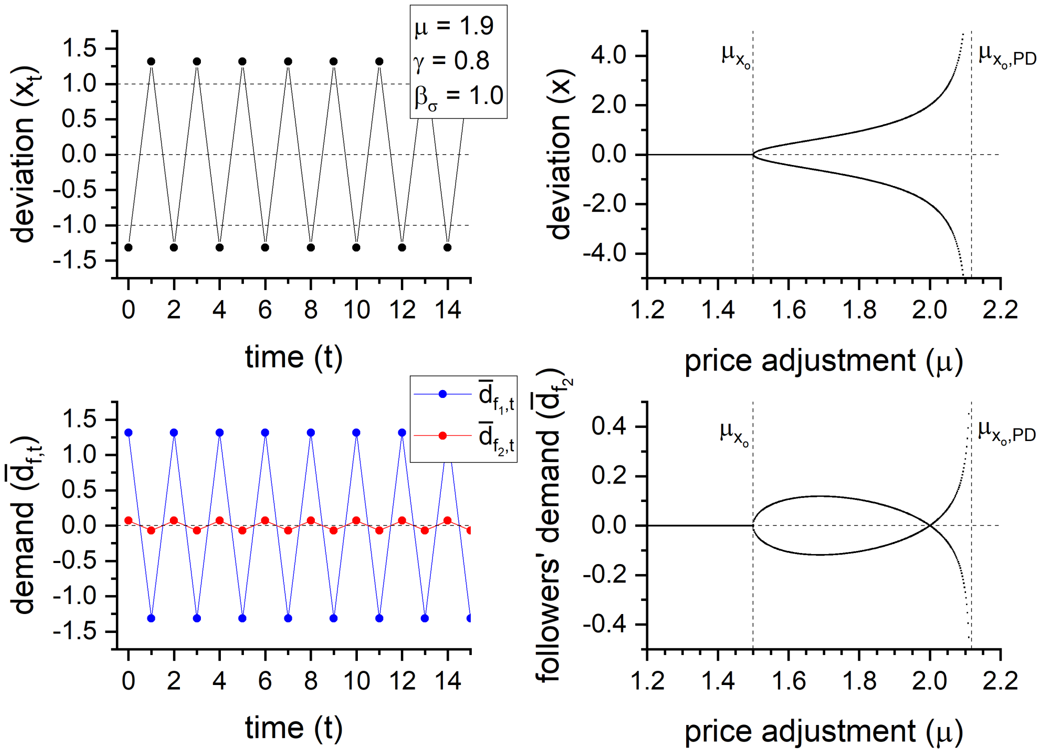


FIGURE 5.2: The deviation of the price from the fundamental value x_t (top left panel) and investors' net demand $\bar{d}_{f,t}$ (bottom left panel), for fixed $\mu = 1.9$, $\gamma = 0.8$, and $\beta_\sigma = 1.0$, and the orbit diagram for the price deviation x (top right panel) and followers' demand $\bar{d}_{f_2}$ (bottom right panel) for different values of μ. The diagrams almost match those for fixed $\gamma = 0.2$ with the difference of having a larger range for μ within which followers trade synchronously and in the same direction with pure fundamentalists (for $\mu > 2$ it gets reversed) and wider price fluctuations. The horizontal dashed lines below and above the fundamental value in the top left panel correspond to the threshold $|x_t| = 1$ beyond which pure fundamentalists are mostly active in sharing their information with their followers.

threshold $|x_t| = 1$ (see the horizontal dashed lines below and above the fundamental value), beyond which pure fundamentalists are mostly active in sharing their information with the followers. However, the overall influence of f_m on f_2 is still greater than f_1, since

$$\alpha_{f_m,t} = 1 - \alpha_{f_1,t} = 0.366 \quad \text{at any time } t \implies (1-\gamma)\alpha_{f_1,t} < \gamma\alpha_{f_m,t} \tag{5.44}$$

which leads group f_2 to trade synchronously and in the same direction with f_1, which is the reason why we see larger price fluctuations than we observed in Figure 5.1. In the right panels of Figure 5.2 we show the orbit diagrams of the price deviation (top right panel) and followers' demand (bottom right panel) for $\mu \in [1.2, 2.1]$ in steps of $\delta_\mu = 0.002$, which almost match those for fixed $\gamma = 0.2$ (see right panels in Figure 5.1), with the difference of having a larger range for $\mu \in (3/2, 2]$ within which followers trade synchronously and in the same direction with pure fundamentalists (for $\mu > 2$ it gets reversed) and (as noticed before) wider price fluctuations.

Summary In summary, when pure fundamentalists are more dominant than followers, the fundamental value is the only fixed point of the system (5.16). If the speed of price adjustment μ is high enough (see condition (5.30)), the price trend is the same as we observed in the previous chapter when the price constantly flips up and down the asset's fundamental value in one single time step (double-period orbit, see conditions (4.59)). However, unlike that case, when μ is not too large, and the asset price fluctuates closely enough to its fundamental value, followers trade synchronously and in the same direction as pure fundamentalists as a consequence of the delayed information shared by the contrarian fundamentalists, whose influence is much greater than pure fundamentalists. This implies that both groups of investors perform optimally, and such market volatility is sustainable for everyone over long periods (even though the price dynamics are very simple). If μ becomes too large (see condition (5.38)), then we obtain the same situation as in the previous chapter, where pure fundamentalists perform optimally and followers poorly (sending buying orders when the price asset is above its fundamental value and vice versa when it is below it). Moreover, the more the follower trusts the misinforming source (i.e., as γ increases), the greater the follower's demand for the asset traded synchronously and in the same direction as pure fundamentalists and the larger the price fluctuations (thanks to the misinforming source, followers can handle greater market volatility). (In the appendix A.2.3, we consider the same market as before, but for fixed $\beta_\sigma = 10.0$, that is, when both f_1 and f_m are more confident in sharing their information with f_2)

5.2.3 Stability of the non-fundamental steady states

In this part, we analyse the case where the system (5.16) also has non-fundamental states as fixed points, which happens as soon as followers become more dominant than pure fundamentalists. The Jacobian matrix (5.22) evaluated at the non fundamental steady states $\vec{x}_{b\pm}$ becomes (see appendix A.2.4)

$$J_b = JF(f_{x_{f_1}}, f_{x_{f_2}})\big|_{\vec{x}_{b\pm}} = \begin{pmatrix} 1 - \mu\bar{c}_{f_1} & -\mu\bar{c}_{f_2} \\ \frac{2\gamma(1-\gamma)\left[\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2) - \bar{c}_{f_1}\bar{c}_{f_2}\right]}{\bar{c}_{f_2}\left[\bar{c}_{f_2} + (1-2\gamma)\bar{c}_{f_1}\right]} & 1 - \frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{\bar{c}_{f_2} + (1-2\gamma)\bar{c}_{f_1}} \end{pmatrix} \tag{5.45}$$

whose determinant and trace is

$$
\begin{aligned}
det(J_b) &= (1-\mu\bar{c}_{f_1})\left[1-\frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}}\right]+\frac{2\mu\gamma(1-\gamma)\left[\beta_\sigma(\bar{c}_{f_2}^2-\bar{c}_{f_1}^2)-\bar{c}_{f_1}\bar{c}_{f_2}\right]}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}} \\
&= 1-\mu\bar{c}_{f_1}-\frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}}+\frac{2\mu\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2-\bar{c}_{f_1}^2)}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}}
\end{aligned}
\tag{5.46}
$$

$$
tr(J_b) = 2-\mu\bar{c}_{f_1}-\frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}} \tag{5.47}
$$

Thus, if we write the first stability condition (5.23) as a condition on μ, we get (see appendix A.2.5)

$$
\mu < \frac{2\left[1-\gamma(1-\gamma)\right]\bar{c}_{f_2}+2(1-2\gamma)\bar{c}_{f_1}}{\bar{c}_{f_1}\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}^2-\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2-\bar{c}_{f_1}^2)} = \mu_{x_b,PD} \tag{5.48}
$$

while for the second stability condition (5.24), we simply have

$$
\bar{c}_{f_2} > \bar{c}_{f_1} \tag{5.49}
$$

and the third stability condition (5.25) becomes

$$
\mu < \frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{2\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2-\bar{c}_{f_1}^2)-\bar{c}_{f_1}\bar{c}_{f_2}-(1-2\gamma)\bar{c}_{f_1}^2} = \mu_{x_b,NS} \tag{5.50}
$$

In Figure 5.3, we plotted $\mu_{x_0}(\bar{c}_{f_2})$ (see the first stability condition (5.30) for the fundamental value), $\mu_{x_b,PD}(\bar{c}_{f_2})$ and $\mu_{x_b,NS}(\bar{c}_{f_2})$ as functions of $\bar{c}_{f_2}$, for fixed $\gamma = 0.2$ (left panel), $\gamma = 0.8$ (right panel) and $\beta_\sigma = 1.0$ (in the appendix A.2.3, we show the stability diagrams $(\bar{c}_{f_2}, \mu)$ when we consider the same market but for fixed $\beta_\sigma = 10.0$, that is, when the sources of information f_1 and f_m are more confident in sharing their information with f_2). First of all, we see that the second stability condition (5.49) is the inverse of (5.33) for the fundamental steady state $\vec{x}_o$, meaning that as soon as followers become more dominant than pure fundamentalists, the non-fundamental steady states $\vec{x}_{b^\pm}$ become stable (assuming (5.48) and (5.50) are also satisfied) and $\vec{x}_o$ unstable (pitchfork bifurcation, see the black vertical line). So, no matter how reliable pure fundamentalists are to followers (excluding the case $\gamma = 0$, the case of the previous chapter), the price will always deviate from the fundamental value (for any initial state excluding the origin itself). Second, if we take $\bar{c}_{f_2} = \bar{c}_{f_1}$, the first stability conditions (5.30) and (5.48) for $\vec{x}_o$ and $\vec{x}_{b^\pm}$ take the same form and are equal to

$$
\mu < \mu_{x_o} = \mu_{x_b,PD} = \frac{2-\gamma}{\bar{c}_{f_2}} \tag{5.51}
$$

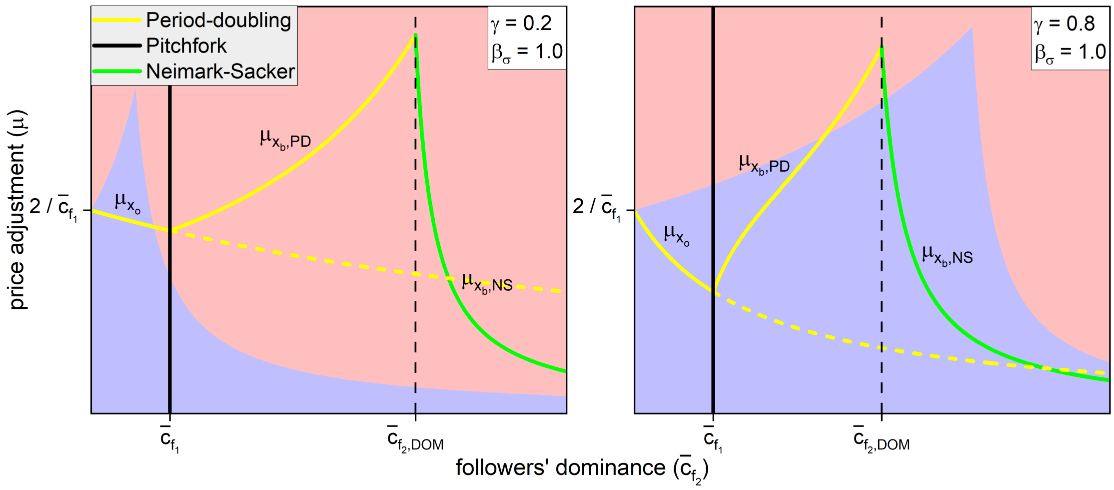


FIGURE 5.3: The parameter space $(\bar{c}_{f_2}, \mu)$ where the stability conditions $\mu_{x_0}(\bar{c}_{f_2})$, $\mu_{x_b,PD}(\bar{c}_{f_2})$ and $\mu_{x_b,NS}(\bar{c}_{f_2})$ are plotted as functions of $\bar{c}_{f_2}$, for fixed $\gamma = 0.2$ (left panel), $\gamma = 0.8$ (right panel) and $\beta_\sigma = 1.0$. The solid yellow curve represents the threshold beyond which the fixed points (the fundamental value, for $\bar{c}_{f_2} < \bar{c}_{f_1}$, and non-fundamental values, for $\bar{c}_{f_2} > \bar{c}_{f_1}$) undergo a period-doubling bifurcation. The dashed yellow curve is the threshold beyond which double-period orbits exist around the fundamental value. The green curve represents a Neimark-Sacker bifurcation, and the vertical solid line the pitchfork bifurcation ($\bar{c}_{f_2} = \bar{c}_{f_1}$). The vertical dashed line is the threshold before which f_1 mostly drives the price, eventually flipping it above and below p^* in a few time steps if μ is high enough, and beyond which f_2 does (the asset price takes more time to reverse its trend and exhibits more irregular patterns). The light blue region in the background represents pairs of values $(\bar{c}_{f_2}, \mu)$ where the system does not exhibit any divergence, while in the light red region, it could.

while the third stability condition (5.50) is always satisfied (see (A.43) and (A.44)). So, we have that if f_2 is equally dominant as (or slightly more dominant than) f_1 (we will shortly see what we mean by 'slightly more dominant'), then the only bifurcation that the system (5.16) exhibits is the period-doubling (PD) one (the yellow curves), either for the fundamental or the non-fundamental steady state. However, as we will see, this bifurcation does not occur for $\vec{x}_o$ once the condition (5.30) is violated (see the yellow dashed line), as the characteristic roots (the eigenvalues that determine the limiting behaviour of the local solution) are both greater than 1 in modulus, meaning that the second stability condition (5.33) may prevail over the first one and the price will only diverge monotonically away from p^* if we chose an initial state close to it. So, even if the system contains orbits with period 2 as solutions (price flipping above and below the fundamental value), these are locally unstable for μ slightly larger than μ_{x_o}. We will also observe that there is a range of values $\mu_{x_o} < \mu < \mu_{x_b,PD}$, for fixed $\bar{c}_{f_2}$ slightly larger than $\bar{c}_{f_1}$, within which the orbits with period 2 and the non-fundamental steady states are simultaneously stable. Apart from this, the market dynamics will not be so different from what we observed previously when $\bar{c}_{f_2} < \bar{c}_{f_1}$, as pure fundamentalists do still mostly drive the price, therefore most conclusions we made back then will be valid here.

Followers *slightly* or *significantly* more dominant than pure fundamentalists
Under these conditions, both the denominator and numerator in RHS of the inequality (5.48) are positive. (It can be easily proven that the numerator is always positive if $\bar{c}_{f_2} > \bar{c}_{f_1}$, see appendix A.2.5.). Given this fact, we see that, from the first stability condition (5.48), as we further increase followers' market dominance $\bar{c}_{f_2}$, the numerator of the fraction increases as well, but the denominator tends to 0 much faster since $\bar{c}_{f_2}$ appears as a negative and quadratic factor. In fact, one can show that there exists a critical value $\bar{c}_{f_2,PD}$ (see (A.39)) such that the denominator equals 0 and beyond which the condition (5.48) is always satisfied, meaning that the non-fundamental steady states no longer show the period-doubling bifurcation. On the other hand, the denominator of the fraction of the third stability condition (5.50), being negative if f_2 is equally dominant as (or slightly more dominant than) f_1, tends to 0 as $\bar{c}_{f_2}$ increases. Also here, there exists a critical value $\bar{c}_{f_2,NS}$ (see (A.48)) such that the denominator equals to 0 and beyond which the condition (5.50) can be violated (the denominator becomes positive), meaning that the non-fundamental steady states may exhibit a Neimark-Sacker bifurcation. Because $\mu_{x_b,PD}(\bar{c}_{f_2})$ and $\mu_{x_b,NS}(\bar{c}_{f_2})$ are, respectively, decreasing and increasing functions of $\bar{c}_{f_2}$, and $\bar{c}_{f_2,NS} < \bar{c}_{f_2,PD}$ (see the appendix A.2.6), there exists an intermediate value $\bar{c}_{f_2,NS} < \bar{c}_{f_2,DOM} < \bar{c}_{f_2,PD}$, for which $\mu_{x_b,PD} = \mu_{x_b,NS}$ (see the vertical dashed line). This separates the two regions in which either (and only) the first stability condition (5.48) can be violated ($\bar{c}_{f_2} < \bar{c}_{f_2,DOM}$) or the third stability (5.50) (see the green curve) can be violated ($\bar{c}_{f_2} > \bar{c}_{f_2,DOM}$) for μ large enough. (There are no 'holes' where the non-fundamental steady states are always stable for any μ.) So, we can now be more specific and say that whenever $\bar{c}_{f_2} \in [\bar{c}_{f_1}, \bar{c}_{f_2,DOM}]$, the followers are *slightly* more dominant than pure fundamentalists, while for $\bar{c}_{f_2} \in [\bar{c}_{f_2,DOM}, +\infty)$ the followers are *significantly* more dominant than pure fundamentalists. As we will see, in this latter case, the price will mostly driven by the followers and the market will exhibit more irregular patterns (as we observed in the previous chapter), because of the delayed information contained in followers' demand.

f_m's trustworthiness The stability curves for $\gamma = 0.2$ and $\gamma = 0.8$ in Figure 5.3 look almost the same with the exception that the system undergoes a period-doubling bifurcation for lower values of μ when contrarian fundamentalists are more trustworthy to followers than pure fundamentalists. However, f_m's trustworthiness does not seem to greatly affect the stability threshold when followers are significantly more dominant than pure fundamentalists (Neimark-Sacker bifurcation), which occurs slightly before as f_m become more reliable than f_1 ($\bar{c}_{f_2,DOM}(\gamma = 0.8) < \bar{c}_{f_2,DOM}(\gamma = 0.2)$). The fact that $\bar{c}_{f_2,DOM}$ increases as followers trust pure fundamentalists more is something we should expect, as for $\gamma \to 0$, the f_1's and f_2's trading behaviour is almost the same (except for the delayed information) and so they can be treated as a unique entity.

Diverging prices In the background of these two parameter spaces, we highlighted the pairs of values $(\bar{c}_{f_2}, \mu)$ in which the system exhibits divergences[3]: in the light blue region, the price can only converge or fluctuate around a value, while in the light red region, we can either have initial states for which the price stabilises around a value or diverges to infinity, or only divergences (all the steady states are unstable). Before discussing this aspect, we observe that the price can only diverge by oscillating above and below the fundamental value (the steady states behave as 'oscillatory' sources) and not monotonically. This is because there is a turning point at which f_2 would start believing pure fundamentalists' feedback, no matter how much they trust the misinforming source ($\gamma < 1$), as we would have $\alpha_{f_1,t} \to 1$ and $\alpha_{f_m,t} \to 0$ (see equation (5.9) or (5.11)). Given this, we initially see that, in both cases, the border that separates the blue and the red region increases as the follower becomes more dominant, which makes sense as the orders sent by f_1 may be cleared by those sent from f_2 (necessarily in the opposite direction) and the risk of observing exploding prices is reduced. This border reaches a maximum peak beyond which the area of the blue region starts decreasing more and more, as only followers start populating the market, meaning that the market maker may not be able to adequately compensate for the excessive demand generated by f_2, unless the price gets updated more slowly. The fact that the blue region is more extended, the more contrarian fundamentalists are trusted makes sense as the stronger the followers' reaction to their feedback (opposite to pure fundamentalists'), the more orders will be cleared between f_2 and f_1. In the following sections, we will first consider the case when followers are slightly more dominant than pure fundamentalists and then when they are significantly more dominant.

5.2.4 Followers slightly more dominant than pure fundamentalists

In Figure 5.4, we illustrate two examples markets when followers are slightly more dominant than pure fundamentalists ($\bar{c}_{f_2} = 1.5 > \bar{c}_{f_1} = 1.0$) by showing the orbit diagram of the price deviation x and followers' net demand $\bar{d}_{f_2}$ for $\mu \in [1.4, 2.1]$ and fixed $\gamma = 0.2$ (f_1 more trustworthy than f_m) (left panels), and for $\mu \in [0.5, 2.1]$ and fixed $\gamma = 0.8$ (right panels) in steps of $\delta_\mu = 0.002$. To get these diagrams, we kept track of x_t and $\bar{d}_{f_2,t}$ (after a transition time of $T_{tran} = 10000$ time steps) over a period of time $T_{test} = 2$, for different initial states $x_{f_1,0} = [-1.0, 1.0]$ and $x_{f_2,0} = [-1.0, 1.0]$ in steps of $\delta_{x_f,0} = 0.1$. The vertical dashed lines at $\mu_{x_o} \sim 1.791$ and $\mu_{x_b,PD} \sim 1.814$ for $\gamma = 0.2$, $\mu_{x_o} \sim 1.0$ and $\mu_{x_b,PD} \sim 1.886$ for $\gamma = 0.8$ represent the thresholds beyond which the first stability conditions (5.30) and (5.48) for $\vec{x}_o$ and $\vec{x}_{b^\pm}$ are violated. The horizontal dashed lines at $x_{f_1,b^\pm} \sim \pm 0.078$ for $\gamma = 0.2$ and $x_{f_1,b^\pm} \sim \pm 0.894$ for $\gamma = 0.8$ in the top left panels, $\bar{d}_{f_2,b^\pm} = -\bar{c}_{f_2} x_{f_2,b^\pm} \sim \pm 0.078$ for $\gamma = 0.2$ and $\bar{d}_{f_2,b^\pm} \sim \pm 0.078$ for $\gamma = 0.8$ in

[3] We computed these regions numerically by choosing an initial state far enough from the steady states ($|\vec{x}_{f,0}| \sim 1e5 >> |\vec{x}_0| \sim |\vec{x}_b^\pm|$) and keeping track of the orbits for $T_{test} = 10000$ time steps. If the norm of the orbit after this time was larger than the initial state, we concluded that there could be divergences.

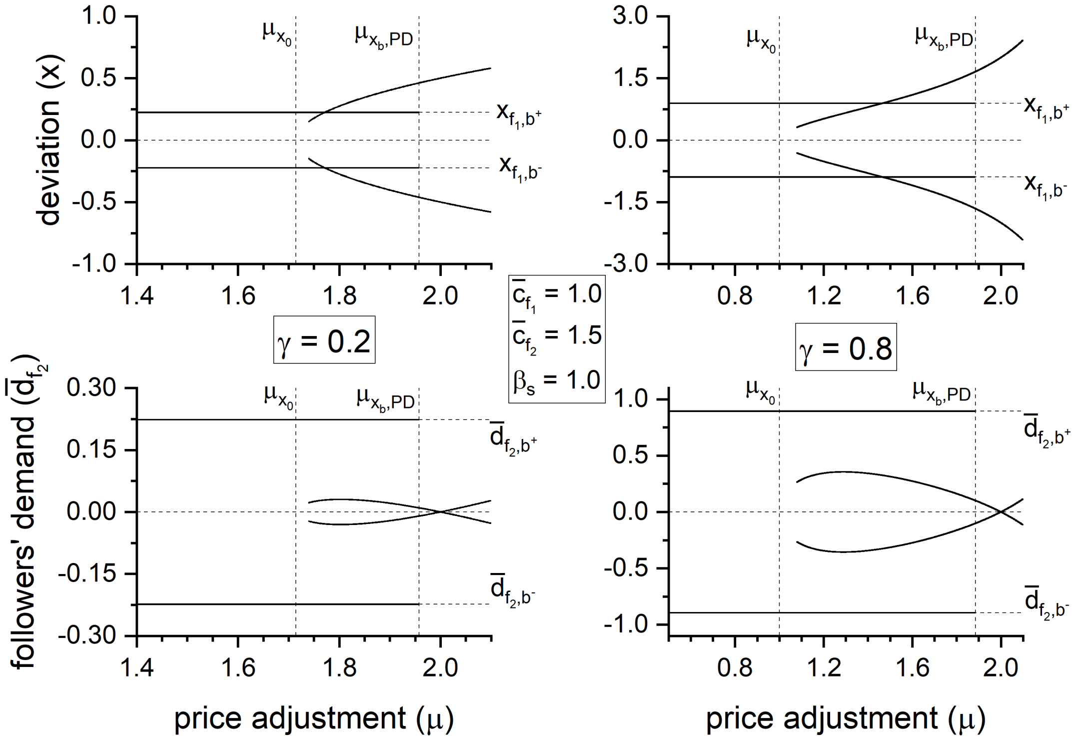


FIGURE 5.4: The orbit diagram of the price deviation x (top panels) and followers' net demand $\bar{d}_{f_2}$ (bottom panels) as we increase μ when $\bar{c}_{f_2} = 1.5 > \bar{c}_{f_1} = 1.0$, for fixed $\gamma = 0.2$ (left panels) and $\gamma = 0.8$ (right panels), with $\beta_\sigma = 1.0$. The vertical dashed lines μ_{x_o} and $\mu_{x_b,PD}$ represent the thresholds beyond which the first stability conditions (5.30) and (5.48) for $\vec{x}_o$ and $\vec{x}_{b\pm}$ are violated ($\mu > \mu_{x_o}$ and $\mu > \mu_{x_b,PD}$, respectively). The horizontal dashed lines at $x_{f_1,b\pm}$ in the top left panels and $\bar{d}_{f_2,b\pm}$ in the bottom left panels represent the non-fundamental steady states of the system. If we choose as initial state a point close to the origin (left panels), the price starts fluctuating above and below p^* only beyond $\mu \sim 1.797 > \mu_{x_o}$. If, instead, the initial state is close to one of the non-fundamental steady states (right panels), then the price fluctuates as soon as $\mu > \mu_{x_b,PD}$.

the bottom left panels represent the non-fundamental steady states of the system (see system (5.20)).

Low values of μ For $\mu < \mu_{x_o}$, we see that the price deviation (thus, pure fundamentalists' net demand, see footnote [2]) and followers' net demand stabilise around a constant value far from the origin (that is, far from the fundamental value and no trading orders) as expected since no stability condition gets violated other than (5.33) (the origin behaves as a *saddle*). More specifically, we have that the price can be either overvalued or undervalued at all times, as pure fundamentalists' constantly send selling or buying orders and followers constantly send buying or selling orders of the same volume (the market maker does not need to change the price of the asset as the orders get completely cleared between f_1 and f_2). Whether the price is above or below the fundamental value depends on the initial conditions. For example, a starting price slightly above p^* leads the misinforming source to inform followers to send buy orders, while pure fundamentalists are mostly inactive, driving the price to higher values. We would have the opposite situation if the price were slightly below p^*. We see that the stronger the trust placed in contrarian fundamentalists by followers, the farther the price deviates from the fundamental value, as we already observed and discussed in our previous examples when $\bar{c}_{f_2} < \bar{c}_{f_1}$ (this is suggested by the solutions in the system (5.20) as well). Realistically, these market conditions cannot last forever, as they would require an infinite amount of supply or liquidity from both groups of investors. Either one group would run out of money, and the price would start decreasing, or the other group would run out of asset shares, and the price would keep rising. So, one can only observe this price (and volume) action over a very short term in a real context.

High values of μ As we increase μ slightly beyond μ_{x_o} (violating the condition (5.30)), we still observe the same market dynamics as before, meaning that, as noticed before, even if the system contains orbits with period 2 as solutions, these are unstable and the price only converges to one of the non-fundamental steady states. Only when $\mu >\sim 1.739$ for $\gamma = 0.2$ and $\mu >\sim 1.079$ for $\gamma = 0.8$ does the price (traders' demand) start flipping above and below the fundamental value (sell and buy orders), in the same way as we observed when $\bar{c}_{f_2} = 0.5 < \bar{c}_{f_1} = 1.0$ where followers traded synchronously and in the same direction as pure fundamentalists. f_1's trading volume is always larger than f_2's one, despite f_2's market dominance being (slightly) larger than f_1's. This happens because of the combined and opposite signals sent by f_1 and f_m to f_2, not entirely convincing the latter to take a specific trading position, which affects their reference price and, thus, demand, while pure fundamentalists (unaffected by any external influence) send their orders with greater determination. Once μ crosses the stability threshold $\mu_{x_b,PD}$, the price does not fluctuate above and below the non-fundamental prices (as happens for $x_{f,0} \sim x_o$ when the condition (5.30) is violated) but, instead, it immediately flips below and above the fundamental value. Also, in this case, one would expect the non-fundamental steady states to become unstable and branch out in two stable curves (orbits with period 2), but the high stability of the already existing orbit with period 2 prevents this. As we further increase the speed of price adjustment, we observe the same orbits as in Figure 5.1 for $\gamma = 0.2$ and Figure 5.2 for $\gamma = 0.8$, where the price fluctuates between two larger and larger values, and followers' demand decreases before increasing again but in the opposite direction to pure fundamentalists for $\mu > 2$ (see condition (5.38)).

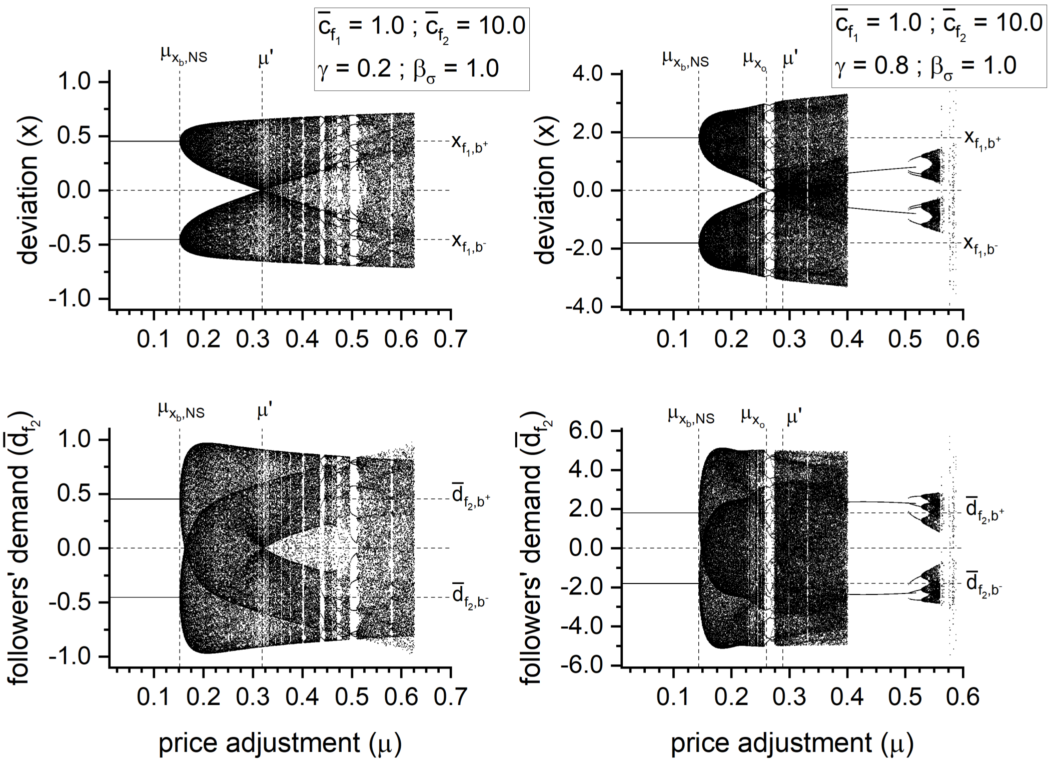


FIGURE 5.5: The orbit diagram of the price deviation x and followers' net demand $\bar{d}_{f_2}$ for increasing values of μ, for fixed $\gamma = 0.2$ (f_1 more trustworthy than f_m) (left panels) and fixed $\gamma = 0.8$ (f_m more trustworthy than f_1) (right panels). For $\mu < \mu_{x_b,NS}$, the price deviation and followers' net demand stabilise around a constant value far from the origin, and the stronger the trust placed in contrarian fundamentalists by followers, the farther the price deviates from the fundamental value. For $\mu > \mu_{x_b,NS}$, the price starts fluctuating around one of the non-fundamental prices with a quasi-periodic orbit, and the price fluctuations around the non-fundamental value become larger and larger as we increase μ. For $\mu = \mu'$ the system undergoes a *homoclinic* bifurcation, and the price will start fluctuating between the two non-fundamental steady states x_{f_1,b^+} and x_{f_1,b^-}. If the degree of reliability γ is high enough, then for $\mu \geq \mu_{x_o}$, the system has as solutions orbits with period 2, which become stable for large values of μ.

5.2.5 Followers significantly more dominant than pure fundamentalists

In the previous section, we have analysed the case where followers were slightly more dominant than pure fundamentalists, that is when $\bar{c}_{f_1} < \bar{c}_{f_2} < \bar{c}_{f_2,DOM}$. In this part, we will consider the case where followers are significantly more dominant than pure fundamentalists, that is when $\bar{c}_{f_2,DOM} < \bar{c}_{f_2}$. In Figure 5.5, we illustrate two examples of markets when followers are significantly more dominant than pure fundamentalists ($\bar{c}_{f_2} = 10.0, \bar{c}_{f_1} = 1.0$) by showing the orbit diagram of the price deviation x and followers' net demand $\bar{d}_{f_2}$ for $\mu \in [0.01, 0.7]$ and fixed $\gamma = 0.2$ (f_1 more trustworthy than f_m) (left panels), and for $\mu \in [0.01, 0.6]$ and fixed $\gamma = 0.8$ (right panels) in steps of $\delta_\mu = 0.001$. To get these diagrams, we kept track of x_t and $\bar{d}_{f_2,t}$ (after a transition time of $T_{tran} = 10000$ time steps) over a period of time $T_{test} = 20$, for different initial states $x_{f_1,0} = [-1.0, 1.0]$ and $x_{f_2,0} = [-1.0, 1.0]$ in steps of $\delta_{x_f,0} = 0.2$. The vertical dashed lines at $\mu_{x_b,NS} \sim 0.152$ for $\gamma = 0.2$ and $\mu_{x_b,NS} \sim 0.144$ for $\gamma = 0.8$ represent the thresholds beyond which the third stability condition (5.49) for $\vec{x}_{b^\pm}$ is violated.

Quasi-periodic orbits around non-fundamental values For $\mu < \mu_{x_b,NS}$, the price deviation (thus, pure fundamentalists' net demand) and followers' net demand stabilise around a constant value far from the origin (see previous paragraph), and the stronger is the trust placed in contrarian fundamentalists by followers, the farther the price deviates from the fundamental value. As soon as $\mu > \mu_{x_b,NS}$, the price starts fluctuating around one of the non-fundamental prices with a quasi-periodic orbit, as we can see in the left panels of Figure 5.6 for $\gamma = 0.2$ and Figure 5.7 for $\gamma = 0.8$, for fixed $\mu = 0.2$. Here, we show the case where we choose an initial state close to one of the non-fundamental steady states $x_{f,0} = (x_{f_1,b^+} + \epsilon, x_{f_2,b^+})$, where $\epsilon = 1\text{e}{-4}$, that is, when the asset is initially overpriced and the pure fundamentalists f_1 send sell orders while the followers f_2 send buy orders. For $\gamma = 0.2$ (top left panel of Figure 5.6), we see that the price deviation x_t takes slightly more time to increase above x_{f_1,b^+} as most of groups' orders get cleared between each other (orders sent in the opposite direction) and decreases slightly faster after x_t deviates far enough from x_{f_1,b^+}. This happens because f_1's overall influence on f_2 becomes larger than the contrarians f_m's as soon as $\alpha_{f_m,t} < 0.8$ (or, equivalently, $\alpha_{f_1,t} > 0.2$), so both f_1 and f_2 send sell orders at the same time. This occurs when $\alpha_{f_m,t}$ ($\alpha_{f_1,t}$) is below (above) the green (blue) horizontal dashed line in the bottom left panel of Figure 5.6. For $\gamma = 0.8$, instead, the rate at which the price increases above and decreases below the non-fundamental price is mostly the same (see top left panel of Figure 5.7), as the volume of f_2's buy orders is larger than f_1's sell orders compared to the previous case. This makes sense since f_m's overall influence on f_2, when the price is already overpriced, is greater for increasing values of γ (see bottom left panels of Figure 5.7) and, thus, they are more persuasive in convincing f_2 to send larger volumes of asset shares.

Average price The horizontal dashed lines in the top left panel of Figure 5.6 and Figure 5.7 represent the fixed point of the non-fundamental price deviation x_{f_1,b^+} and the average of the price deviation $\bar{x}$, defined as

$$\bar{x} = \frac{1}{T_{test}} \sum_{t=1}^{t=T_{test}} x_t \tag{5.52}$$

over a period of time $T_{test} = 20$ time steps. For $\gamma = 0.2$, we see that $\bar{x}$ is slightly below x_{f_1,b^+}, which implies that price deviation spends slightly more time below the non-fundamental steady state than above it, while for $\gamma = 0.8$ we have $\bar{x} \sim x_{f_1,b^+}$. So, the more f_1 are trusted by f_2, the longer the asset price will stay close to its

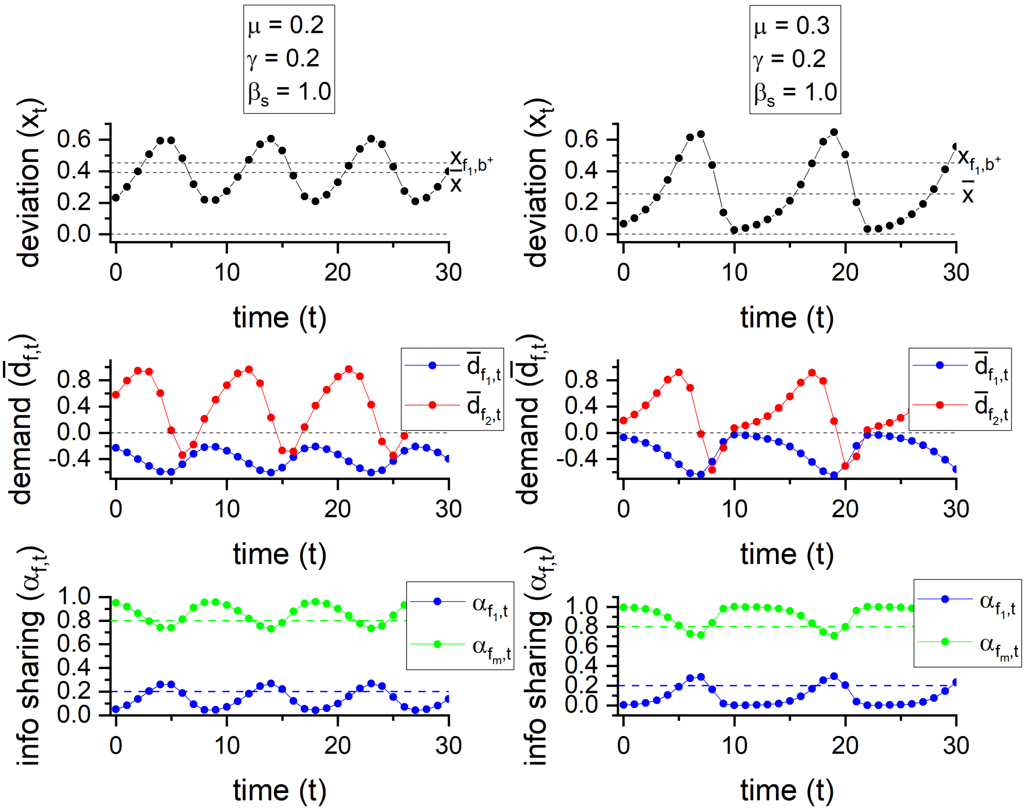


FIGURE 5.6: The price deviation x_t (top panels), groups' net demand $\bar{d}_{f,t}$ (middle panel), and the sharing mechanism $\alpha_{f_i,t}$ for fixed $\mu = 0.2$ (left panels), $\mu = 0.3$ (right panels) and $\gamma = 0.2$, when the asset is initially overpriced and the pure fundamentalists f_1 send sell orders while the followers f_2 send buy orders. The average price $\bar{x}$ is slightly below the non-fundamental steady state x_{f_1,b^+}, which implies that price deviation spends slightly more time below than above it, and the larger the value of μ, the longer the price will stay close to the fundamental value. If $\alpha_{f_m,t}$ is above the green dashed line (or $\alpha_{f_1,t}$ is below the blue dashed line), then f_m's overall influence on f_2 is larger than f_1, and followers will send buy orders (the price slowly increases). If $\alpha_{f_m,t}$ is below the green dashed line, then f_2 will be mostly affected by f_1's feedback and start sending selling orders (the price quickly decreases).

fundamental value before increasing up to and beyond x_{f_1,b^+}, a sign of the fact that followers send lower volumes of buy orders for lower values of γ. If we chose, instead, as the initial price is below the fundamental value and with followers starting to send sell orders, we would observe the same orbits as before but flip with respect to the x-axis.

High values of μ - arise of bubbles and crashes From the orbit diagrams in Figure 5.5, we see that the price fluctuations around the non-fundamental value become larger and larger as we increase μ. For fixed $\gamma = 0.2$ (see the top right panel of Figure 5.6), the price spends more time close p^*, slowly increases up and beyond x_{f_1,b^+} (bubble), then, suddenly drops down close the fundamental values (crash), and the

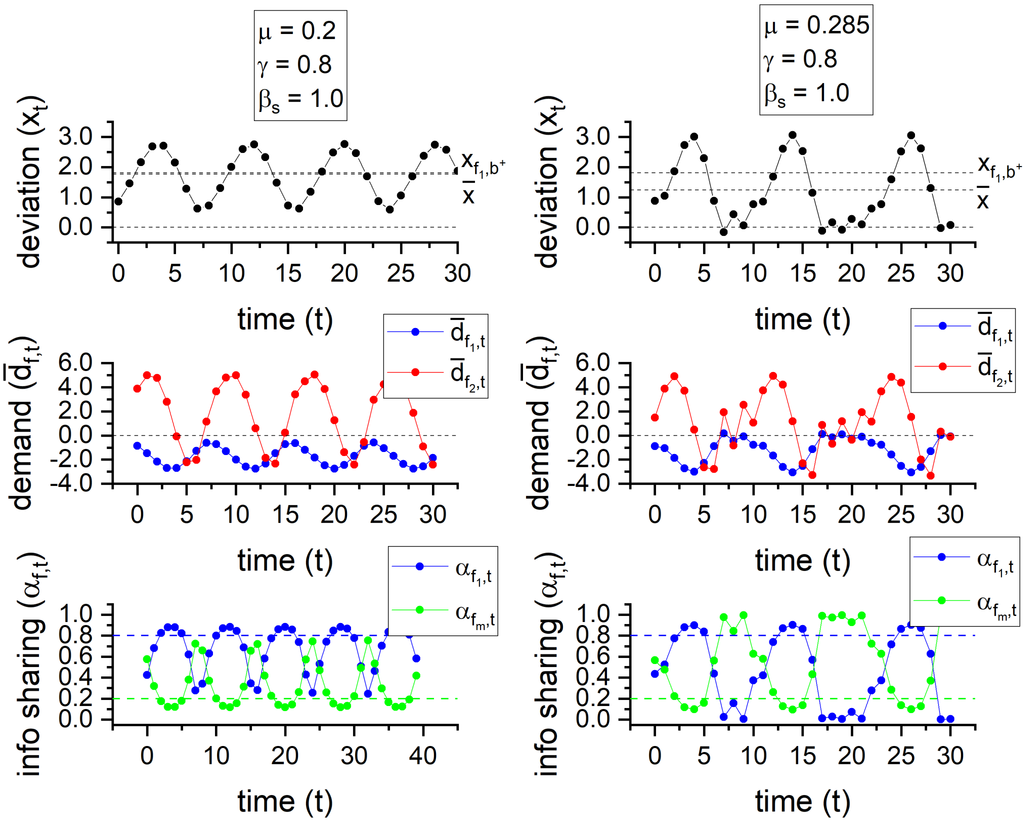


FIGURE 5.7: The price deviation x_t (top panels), groups' net demand $\bar{d}_{f,t}$ (middle panel), and the sharing mechanism $\alpha_{f_i,t}$ for fixed $\mu = 0.2$ (left panels), $\mu = 0.285$ (right panels) and $\gamma = 0.8$, when the asset is initially overpriced and the pure fundamentalists f_1 send sell orders while the followers f_2 send buy orders. For fixed $\mu = 0.2$, the average price $\bar{x}$ almost matches x_{f_1,b^+}, which means that the rate at which the price increases above and decreases below the non-fundamental price is mostly the same. As we increase μ, however, the price will spend more time close to the fundamental value and will flip above and below for a short period of time (f_m's delayed feedback is such to synchronise f_2's orders along with f_1)

cycle repeats itself. Even for fixed $\gamma = 0.8$, the price spends more time close to p^* ($\bar{x} < x_{f_1,b^+}$), but unlike the previous case, the price flips above and below the fundamental value right after the crash, before increasing again. We can observe this in the top right panel of Figure 5.7), where we chose the speed of price adjustment equal to $\mu = 0.285$. This value is larger than $\mu_{x_o} = 0.261$, beyond which the system also has as solutions orbits with period 2, that is, price flipping above and below the fundamental value (see the condition (5.37)). We do not observe this phenomenon for $\gamma = 0.2$, as μ_{x_o} is way beyond the values for which the system is unstable (diverging prices) for any initial condition.

High values of γ - arise of sudden price fluctuations As seen before, for higher values of γ, the price may fluctuate above and below the fundamental value but only briefly (the orbit with period 2 is unstable). In the middle right panel of Figure 5.7, we see that this phenomenon is mainly caused by the followers' orders and only in part by pure fundamentalists. When the price bubble bursts and the price crashes, it does so at such speed to cross the asset's fundamental value in just one-time step after it was overpriced. So, when the price is below p^*, followers' last feedback from the misinforming source is to send buy orders (because of delayed information). This takes the price back above the fundamental value (in just a one-time step) where now the information received from f_m by f_2 is to sell the asset, which brings the price back below p^*. These price fluctuations dampen down as time passes until the price starts to rise again (creation of the bubble).

Homoclininc bifurcation For $\mu = \mu' \sim 0.319$ when $\gamma = 0.2$ and $\mu = \mu' \sim 0.289$ when $\gamma = 0.8$ (values estimated numerically), the system undergoes a *homoclinic* bifurcation, that is, the two limit cycles (the price fluctuations around the non-fundamental steady states) collide with the saddle point (the fundamental value) and the price will start fluctuating between the two non-fundamental steady states x_{f_1,b^+} and x_{f_1,b^-}. In the left panels of Figure 5.8, we show the case for fixed $\gamma = 0.2$, when the speed of price adjustment is slightly beyond μ', that is, $\mu = 0.35$. When the price deviation crashes down from x_{f_1,b^+}, it does at such speed to cross the fundamental value and, if this happens (it does not occur every time after a crash), instead of flipping around it (groups' orders is not enough to bring the price back above p^*), the price keeps decreasing down to x_{f_1,b^-} but at a slower rate before suddenly increasing again once it goes below the non-fundamental price. If we increase μ further (see right panels of Figure 5.8 for fixed $\mu = 0.6$), the price trend becomes more irregular and chaotic (see left panel of Figure 5.9), and the price fluctuations over the short term (between two-time steps) become wider. For fixed $\gamma = 0.8$, as soon as the system undergoes the homoclinic bifurcation (see left panels of Figure 5.10 for $\mu = 0.35$), the price flips above and below the fundamental value each time it gets closer to it while fluctuating between x_{f_1,b^+} and x_{f_1,b^-} (the horizontal dashed lines $x_{f_1,PD}$ below and above the fundamental value in the top left panel of Figure 5.10 represent the distance between the prices of the orbit with period 2). If we increase μ, in this case, we observe irregular patterns and chaotic phenomena only up to $\mu \sim 0.4$, beyond which the price will start fluctuating above and below p^* with period 2 (see right panel of Figure 5.5) where followers trade in the same direction as pure fundamentalists. For higher values of μ, the price will start taking different values whenever it flips above and below p^* (see top right panel of Figure 5.10 for $\mu = 0.55$), a sign of the fact that the system becomes chaotic again. The differences between the $\gamma = 0.2$ and $\gamma = 0.8$ case can be explained as follows. When the followers f_2 trust more the pure fundamentalists f_1 than the misinforming source f_m, the delayed information that they receive when the market maker sets higher values for the speed of price adjustment will mostly lead them to send orders in the opposite direction to f_1 (see left middle panel in Figure 5.10. Thus, most orders get cleared

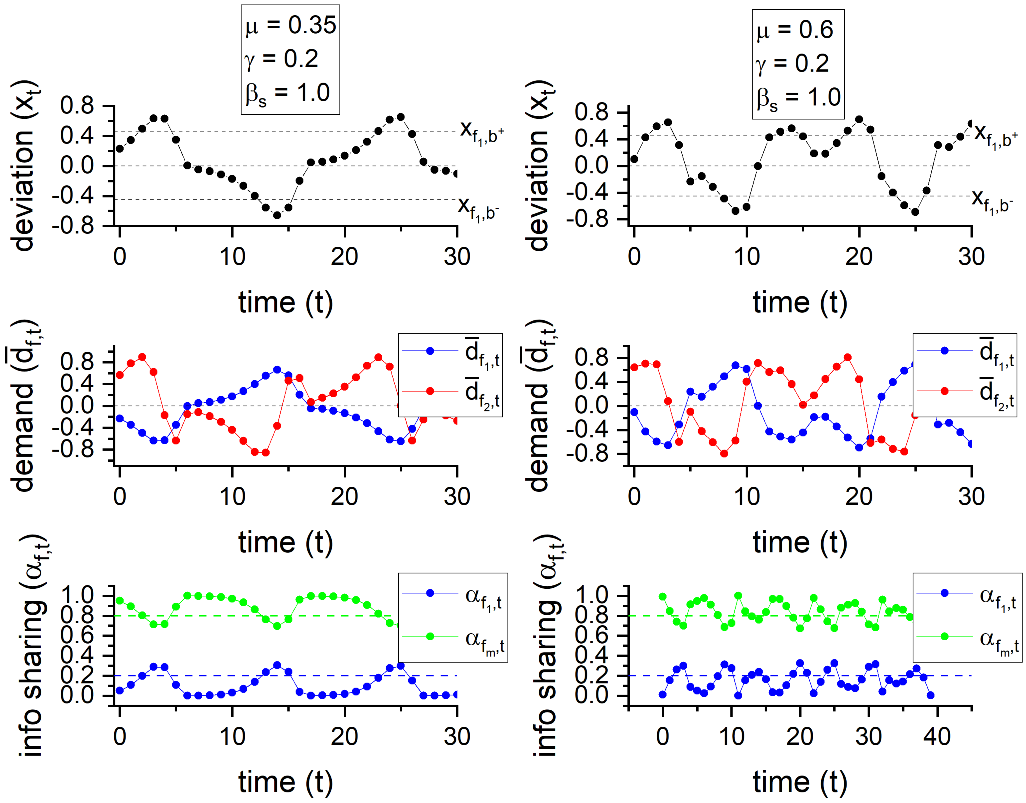


FIGURE 5.8: The price deviation x_t (top panels), groups' net demand $\bar{d}_{f,t}$ (middle panel), and the sharing mechanism $\alpha_{f_i,t}$ for fixed $\mu = 0.35$ (left panels), $\mu = 0.6$ (right panels) and $\gamma = 0.2$. Once $\mu > \mu' \sim 0.319$, the price will start fluctuating between the two non-fundamental steady states x_{f_1,b^+} and x_{f_1,b^-}. For increasing values of μ, the price trend becomes more irregular and chaotic, and its fluctuations over the short term (between two-time steps) become wider.

between each other, and we observe small price fluctuations. When f_2 trust more f_m than f_1, instead, the delayed information will lead f_2 to send orders in the same direction as f_1 (see the right middle panel in Figure 5.10), giving rise to large price fluctuations. For high values of μ, the price fluctuations are such as crossing the fundamental value above and below in just one time (period 2 orbits).

5.2.5.1 Groups' performance

Whether f_1 and f_2 perform positively or negatively over the long term (T_{test}), in all the previous cases, depends on which orders they sent above and below $\bar{x}$: if they buy (sell) when $\bar{x} > x_t$ ($\bar{x} < x_t$) then they perform well, poorly otherwise (the average buying price must be lower than the average selling price). As such, group's f_i relative long-term performance $\Sigma_{\Pi^l, T_{test}, f_i}$ can be defined as

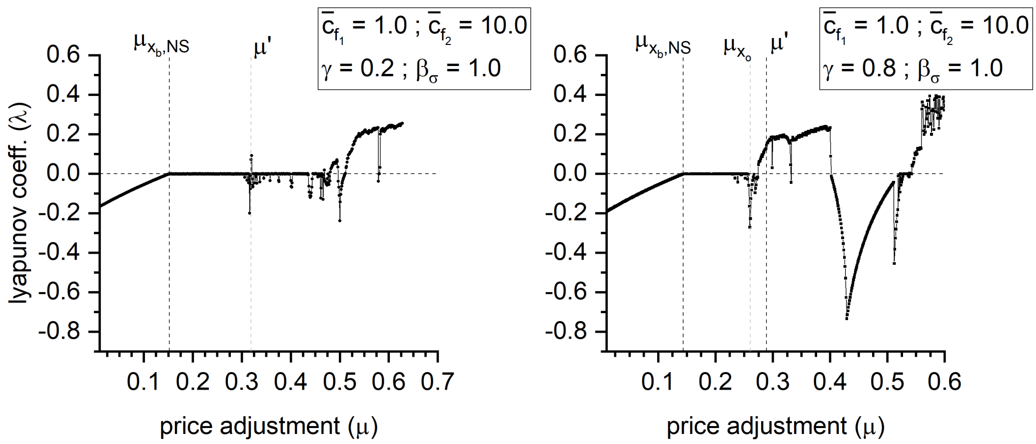


FIGURE 5.9: The Lyapuonv coefficient λ for fixed $\gamma = 0.2$ (left panel) and for fixed $\gamma = 0.8$, for increasing values of μ. For $\gamma = 0.2$, λ is negative for most values of μ, which means that the system does not exhibit any chaotic phenomena, but we observe dependence on initial conditions for large values of μ. Something similar happens for $\gamma = 0.8$, although λ takes negative values as soon as the solution with period 2 becomes stable and then increases again for larger values of μ (the price irregularly flips above and below p^* in one-time step).

$$\Sigma_{\Pi^l,T_{test},f_i} = \frac{\sum_{t=1}^{t=T_{test}} \Pi^l_{f_i,t}}{\sum_{t=1}^{t=T_{test}} |\Pi^l_{f_i,t}|} \tag{5.53}$$

where

$$\Pi^l_{f_i,t} = (\bar{x} - x_t)d_{f_i,t} \tag{5.54}$$

$\Sigma_{\Pi^l,T_{test},f_i}$ is bounded between $\Sigma_{\Pi^l,MIN} = -1$, the worst possible performance, when each buy (sell) order was sent only when the asset price was above (below) $\bar{x}$, and $\Sigma_{\Pi^l,MAX} = 1$, the best possible performance, when each buy (sell) order was sent only when the asset price above was below (above) $\bar{x}$. Group's f_i relative short-term performance $\Sigma_{\Pi^s,T_{test},f_i}$ is still defined as (see the paragraph 'Groups' trading sustainability' in section 4.1.1)

$$\Sigma_{\Pi^s,T_{test},f_i} = \frac{\sum_{t=1}^{t=T_{test}-1} \Pi^s_{f_i,t}}{\sum_{t=1}^{t=T_{test}-1} |\Pi^s_{f_i,t}|} \tag{5.55}$$

where

$$\Pi^s_{f_i,t} = (p_{t+1} - p_t)d_{f_i,t} = (x_{t+1} - x_t)d_{f_i,t} \tag{5.56}$$

In Figure 5.11, we show group f_i's (left panels for f_1 and right panels for f_2) long-term performance (top panels) and short-term performance (bottom panels) for

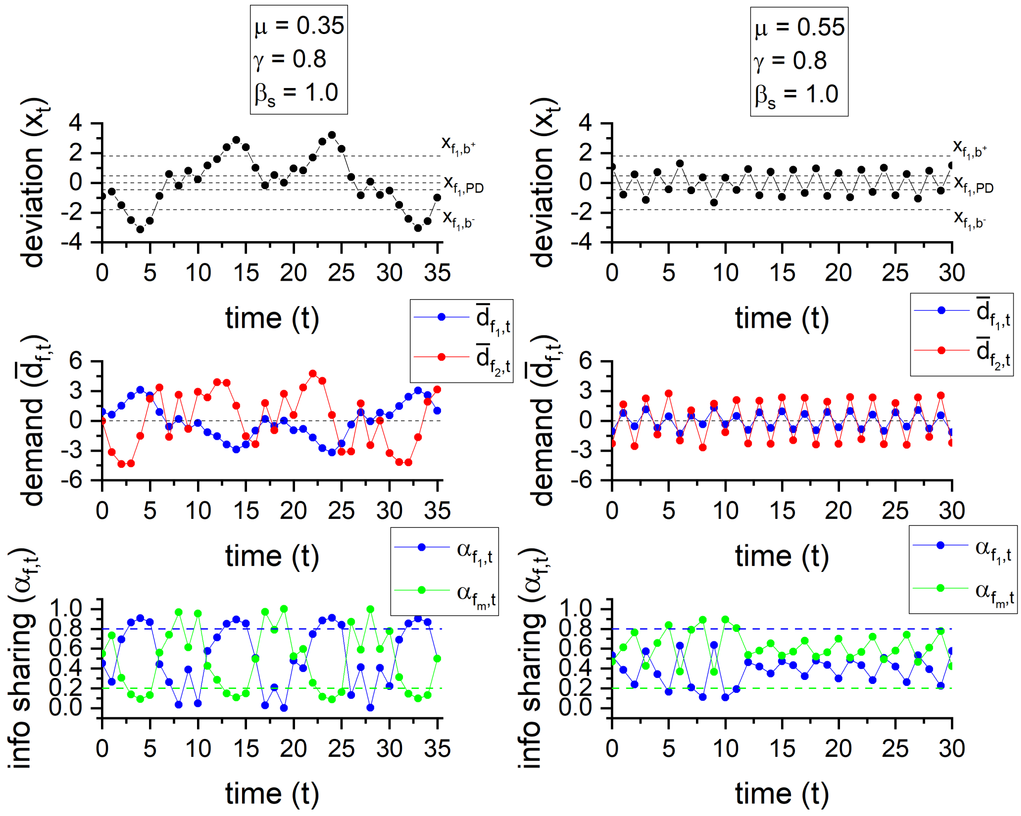


FIGURE 5.10: The price deviation x_t (top panels), groups' net demand $\bar{d}_{f,t}$ (middle panel), and the sharing mechanism $\alpha_{f_i,t}$ for fixed $\mu = 0.35$ (left panels), $\mu = 0.55$ (right panels) and $\gamma = 0.8$. Also here, as it was for $\gamma = 0.2$, once $\mu > \mu' \sim 0.289$, the price will start fluctuating between the two non-fundamental steady states x_{f_1,b^+} and x_{f_1,b^-} and flipping above and below p^* each time it gets closer to it (the horizontal dashed lines $x_{f_1,PD}$ below and above the fundamental value in the top left panel represent the distance between the prices of the orbit with period 2). If we increase μ further, the price will only flip (irregularly) above and below p^* in just one-time step with f_1 and f_2 trading in the same direction.

$\mu \in [0.01, 0.7]$ and fixed $\gamma = 0.2$ (Figure 5.11), in steps of $\delta_\mu = 0.001$ (in the appendix A.2.7 we show group f_i's long-term and short-term performance fixed $\gamma = 0.8$, whose conclusions are the same we will make here). To get these diagrams, we used the same method as for the bifurcation diagrams in Figure 5.5. For $\mu < \mu_{x_b,NS}$, we set the groups' long-term and short-term performance equal to 0, as neither f_1 and f_2 make any profit in this region (both groups constantly send the same orders in the opposite direction).

Performance after Neimark-Sacker bifurcation Once μ crosses $\mu_{x_b,NS}$, where the price starts fluctuating around a non-fundamental price, either above or below p^*, we see that both f_1 and f_2 perform positively in the long-term. However, this performance is fictitious as pure fundamentalists always send orders in the same direction, that is, selling orders when the price is above the fundamental value and

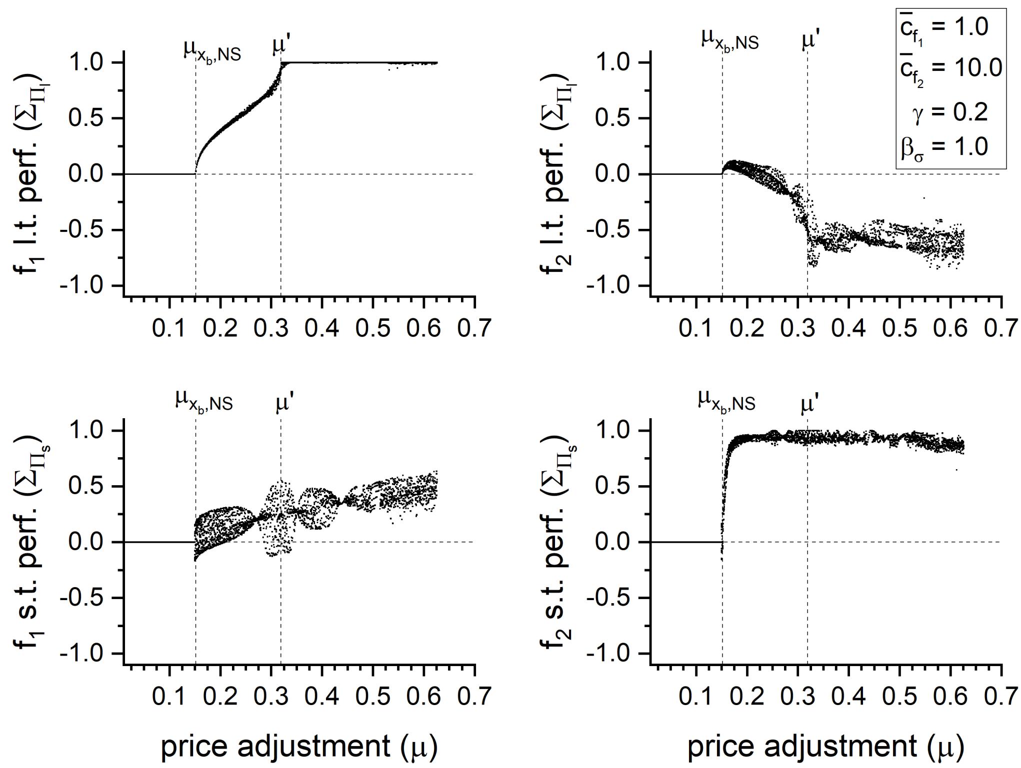


FIGURE 5.11: Group f_i's (left panels for f_1 and right panels for f_2) long-term performance (top panels) and short-term performance (bottom panels) for $\mu \in [0.01, 0.7]$ and fixed $\gamma = 0.2$. For $\mu < \mu_{x_b,NS}$, groups' long-term and short-term performance equal to 0 (neither f_1 and $_2$ make any profit in this region) and once μ crosses $\mu_{x_b,NS}$ both f_1 and f_2 perform positively in the long term (f_1 better than f_2) and short term (f_2 better than f_1). As we increase the speed of price adjustment ($\mu < \mu'$), f_1 performs better and better in the long-term while f_2's performance declines. Once the system undergoes the homoclinic bifurcation ($\mu > \mu'$), pure fundamentalists perform almost optimally in the long-term while f_2 perform negatively, although they are still at timing the market in the short-term.

buying orders when it is below it, so f_1 would either run out of supply or liquidity in the long-term. The reason why f_1 does not perform optimally in the long-term ($\Sigma_{\Pi^l,T_{test},f_1} < 1$) is because they send the same type of orders, either if the price deviation is above or below $\bar{x}$. However, they still manage to perform positively because the amount of orders sent differs depending on how far the price is from the fundamental value. For example, if the price fluctuates above the fundamental value (see left panels of Figure 5.6), the amount of selling orders is larger when the price deviation is above $\bar{x}$ and lower when it is below it, so they are good at timing the market in the long-term (and the same is true when the price fluctuates below p^*). Unlike f_1, f_2 sends both selling and buying orders when the price fluctuates above or below the fundamental value only. When the price fluctuates above p^* (see left panels of Figure 5.6), f_2 sends most of its buying orders when the price deviation is below $\bar{x}$ and selling orders when $x_t > \bar{x}$. Followers also perform positively in the long-term (right after $\mu > \mu_{x_b,NS}$), but less than pure fundamentalists ($\Sigma_{\Pi^l,T_{test},f_2} < \Sigma_{\Pi^l,T_{test},f_1}$)

since they are worse at timing the market. In the short-term (see bottom panels in Figure 5.11 for fixed $\gamma = 0.2$, f_2 perform better than f_1 (and this is true for any value of μ), meaning that they send buy (sell) orders right before a price increase (decrease), although they are bad at taking profit of these moments, while pure fundamentalists try to correct any mispricing of the asset too early, by sending selling (buying) orders way before the price reached the top (bottom) of its cycle. As we increase the speed of price adjustment ($\mu < \mu'$), f_1 performs better and better in the long-term since the price spends more and more time closer to the fundamental value while the time spent at the top or the bottom of its cycle is shorter. So, the fact that f_1 sends its orders way too early before the price reaches these extreme values has less and less negative impact on profits. f_2's long-term performance, instead, declines as μ increases. They send more and more buying (selling) orders when the price fluctuates above (below) the fundamental value and keep doing so even if the price deviation x_t is above (below) $\bar{x}$. They are slow to react to sudden price fluctuations, so as the price increases and decreases (decreases and increases) more rapidly when it reaches the top (bottom), they are worse at taking profits from these by selling right after $x_t > \bar{x}$ (the price fluctuates above p^*) or by buying when $x_t < \bar{x}$ (the price fluctuates below p^*).

Performance after homoclinic bifurcation Once the system undergoes the homoclinic bifurcation ($\mu > \mu'$), we have that $\bar{x} \sim 0$ and the pure fundamentalists perform almost optimally in the long-term ($\Sigma_{\Pi^l,T_{test},f_1} \sim 1$), although they are still not good at timing the market in the short-term (unlike followers) since they send their orders as soon as the asset gets either overpriced or underpriced. We see that the followers perform negatively in the long term now, and the more trust they put in pure fundamentalists, the worse they do. The reason why the misinforming source (counterintuitively) helps followers to reduce their losses in the long and short term is due to delayed feedback when the price fluctuates back and forth the fundamental value for a time, as we saw before (see left panels of Figure 5.10), which lead f_2 to trade in the same direction as f_1.

5.2.5.2 The system in the presence of noise

As already done in the previous chapter, in this last part, we show some results of introducing a noise term in the system (5.16) as follows

$$\begin{cases} x_{f_1,t} = x_{f_1,t-1} - \mu\left(\bar{c}_{f_1} x_{f_1,t-1} + \bar{c}_{f_2} x_{f_2,t-1} + N_t(0,\sigma_n)\right) \\ x_{f_2,t} = x_{f_2,t-1} + (1-\gamma)\left(x_{f_1,t-1} - x_{f_2,t-1}\right)\alpha_{f_1,t-1} + \gamma\left(-x_{f_1,t-1} - x_{f_2,t-1}\right)\alpha_{f_m,t} \\ \alpha_{f_1,t} = \frac{|x_{f_1,t}|^{2\beta\sigma}}{|x_{f_1,t}|^{2\beta\sigma} + \Delta_\sigma^{\beta\sigma}} \\ \alpha_{f_m,t} = \frac{\Delta_\sigma^{\beta\sigma}}{|x_{f_1,t}|^{2\beta\sigma} + \Delta_\sigma^{\beta\sigma}} \end{cases} \tag{5.57}$$

where $N_t(0,\sigma_n)$ takes normally distributed values over time with mean $\overline{N} = 0$ and standard deviation σ_n. We assume this noise term represents noise traders in the market (or random fluctuations of groups' market dominance). As values for the other parameters, we choose those that have shown the most interesting phenomena: periods of bubbles and collapses followed by sudden price fluctuations above and below the fundamental value (see right panels of Figure 5.7). Again, all these phenomena persist in the presence of noise, even if its level is comparable to investors' demand ($\sigma_n = 1$), as shown in Figure 5.12, and the price also fluctuates around both non-fundamental values, something that does not occur in the noiseless case. Finally, we point out that these market conditions are not favourable for followers (see A.2.7), so we expect they will be short-lived in a real context.

5.3 Conclusions

In this chapter, we have analysed the case of a market populated by *pure fundamentalists*, who estimate the asset's value from the fundamentals analysis, *contrarian fundamentalists*, who share information that goes against the one sent by pure fundamentalists, and *followers*, who refer to the information shared by the former two with a delay.

Contrarian fundamentalists In the first section, we introduced *contrarian fundamentalists*, the novelty of our model, by defining their reference price (see equation (5.10)), that we included in the updating rule for f_2's reference price (see equation (5.9)), where we also defined the degree of *reliability*, γ, that group f_2 associates with the source of information f_m (f_m completely untrustworthy for $\gamma = 0$, and completely trustworthy for $\gamma = 1$). We defined their sharing mechanism (see equation (5.13)), and we assumed that, differently from pure fundamentalists, contrarians share their information about their reference price when the asset's price is close to its fundamental value so as to be more effective in influencing followers' trading decision. In the next section, we focused on the case where the system evolves on a large time scale such that all investors trade at any time and share their

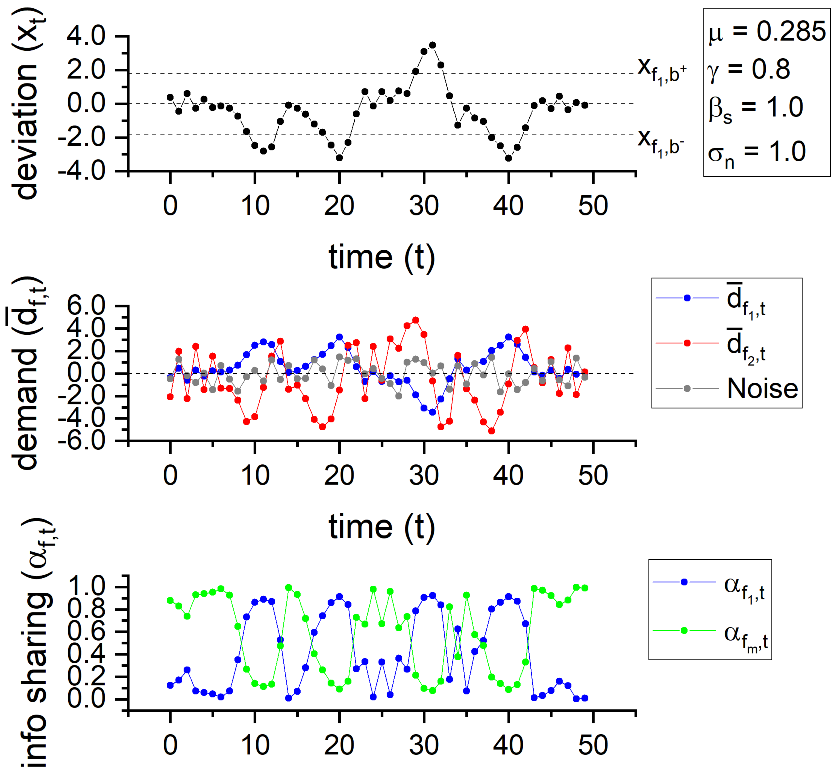


FIGURE 5.12: The deviation of the price from the fundamental value (top panel), investors' net demand relative to their market dominance (middle panel) and the fraction of investors of each group sharing their information about their asset's reference price at time t (bottom panel) for $T = 50$ time steps when the standard deviation of the noise is equal to $\sigma_n = 1$. The horizontal reference lines at x_{f_1,b^+} and x_{f_1,b^-} indicate the two non-fundamental steady states present in the noiseless case.

information about their reference price only if its deviation from the asset's fundamental value is large enough (for which the market reduces to the system (5.16)).

Pure fundamentalists more dominant than followers We first saw that (see section 5.2.2) when pure fundamentalists are more dominant than followers, the fundamental value is the only fixed point of the system (5.16) and, if the speed of price adjustment μ is high enough (see condition (5.30)), the price trend is the same as we observed in the previous chapter when the price constantly flips up and down the asset's fundamental value in one single time step (double-period orbit, see conditions (4.59)). However, unlike that case, when μ is not too large, and the asset price fluctuates closely enough to its fundamental value, followers trade synchronously and in the same direction as pure fundamentalists as a consequence of the delayed information shared by the contrarian fundamentalists, whose influence is much greater than pure fundamentalists. This implies that both groups of investors perform optimally, and such market volatility is sustainable for everyone over long periods (even though the price dynamics are very simple). If μ becomes too large (see condition (5.38)), then we obtain the same situation as in the previous chapter, where pure fundamentalists perform optimally and followers poorly (sending buying orders when the price asset is above its fundamental value and vice versa when it is below it). Moreover, the more the follower trusts the misinforming source (i.e., as γ increases), the greater the follower's demand for the asset traded synchronously and in the same direction as pure fundamentalists and the larger the price fluctuations (thanks to the misinforming source, followers can handle greater market volatility).

Followers *slightly* more dominant than pure fundamentalists Next, we focused on the case when followers were more dominant than pure fundamentalists, in which the market also has non-fundamental steady states along with the fundamental one (see section 5.2.3). Here, we found that there are two regions in which followers can be *slightly* or *significantly* more dominant than pure fundamentalists. The price dynamics when followers are *slightly* more dominant than pure fundamentalists do not differ so much from what we saw in the previous section (when pure fundamentalists are more dominant than followers). However, in this case, we saw that the asset price for low values of the speed of price adjustment, μ, does not converge to its fundamental value, but, instead, it will be constantly overvalued or undervalued at all times (and the stronger the trust placed in contrarian fundamentalists by followers, the farther the price deviates from the fundamental value, as we already observed and discussed in the previous section). In this state, the market maker does not need to change the asset price as the orders get completely cleared between f_1 and f_2, where the former will constantly send selling orders, and the latter will constantly send buying orders when the asset is overvalued (vice versa when the asset is undervalued). For this reason, these market conditions cannot last forever, as they would require an infinite amount of supply or liquidity from both groups of investors. Either one group would run out of money, and the price would start decreasing, or the other group would run out of asset shares, and the price would keep rising. So, one can only observe this price (and volume) action over a very short term in a real context. For high values of μ, we found the same price dynamics as in the previous section, that is, double-period orbits (asset price flipping up and down its fundamental value in one single time step) with followers trading synchronously and in the same direction as pure fundamentalist if μ is low enough, vice versa if the value μ is very high.

Followers *significantly* more dominant than pure fundamentalists The case when followers are *significantly* more dominant than pure fundamentalists is the most interesting part of this chapter since it contains much richer and more realistic price dynamics than observed earlier. As before, if the speed of price adjustment is really low, the price will stabilise around a constant value far from the origin (see above). Once μ crosses a certain threshold, the price starts fluctuating around this value but with quasi-periodic orbits (market cycle), and the higher the value μ, the much longer the asset price will stay close to its fundamental value before slowly rising up to its non-fundamental value (bubble) and suddenly dropping down in few time steps (crash). This happens because most of the groups' orders get cleared between each other before the bubble since the contrarians f_m's influence on f_2 is greater than f_1, so they send orders in the opposite direction) and, once the asset price crosses the non-fundamental value, f_1's overall influence on f_2 becomes larger than f_m's, so both f_1 and f_2 send sell orders at the same time. Moreover, the more f_1 are trusted by f_2 (lower values of γ), the longer the asset price will stay close to its fundamental value before increasing up to and beyond its non-fundamental steady state, a sign of the fact that followers send lower volumes of buy orders for lower values of γ. In this market condition (as soon as the price fluctuates around the non-fundamental values), we saw that both f_1 and f_2 perform positively in the long-term, even though this performance is fictitious as pure fundamentalists always send orders in the same direction, that is, selling orders when the price is above the fundamental value and buying orders when it is below it. So, f_1 would either run out of supply or liquidity in the long term. Unlike f_1, instead, f_2 sends both selling and buying orders when the price fluctuates above or below the fundamental value only and when the price fluctuates above p^*, f_2 sends most of its buying orders when the price deviation is below the average price (of a market cycle) and selling orders when it is above it. If the market maker increases μ further, the asset price spends more time close to its fundamental value before slowly increasing and crashing suddenly, and when followers trust more contrarians than pure fundamentalists, the price will flip above and below the fundamental value right after the crash, before increasing again (a phenomenon is mainly caused by the followers' orders and only in part by pure fundamentalists). In these circumstances, f_1 performs better and better in the long term since the price spends more and more time closer to the fundamental value while the time spent at the top or the bottom of its cycle is shorter. f_2's long-term performance, instead, declines as μ increases. They send more and more buying (selling) orders when the price fluctuates above (below) the fundamental value and keep doing so even if the price deviation x_t is above (below) the average price. Increasing μ further, the price will fluctuate between both non-fundamental values, becoming so volatile that it will negatively affect followers' long-term performance (while pure fundamentalists will always perform positively). The same features are observed if we include a noise term in the system (e.g., noise traders), an additional element that exhibits a price trend that is even more realistic.

Chapter 6

Conclusions

In this thesis, I have set out dynamic asset market models belonging to the Discrete Choice Approach (DCA) modelling framework, in which a financial market is populated by investors whose social interaction is described through a coupled map to investigate the possible destabilizing effects generated by them, with particular regard to excess volatility in price fluctuations and financial bubbles. This map dictated the evolution and dissemination of the perceived value of the asset over time, whose esteem depended on the trading strategy adopted by these investors, while considering potential delays in disseminating such information with processes evolving on different time scales. The results are organised into several chapters.

Literature review In the first part of Chapter 2, I discussed the general framework of a DCA model by following Brock and Hommes's seminal work, and I reviewed all the works that extended this model by considering social interactions among investors. In addition to highlighting the unsatisfactory amount of work regarding this type of extension, I listed three other aspects that these works did not take into account, gaps that we filled in our work: 1) the possibility of a *time delay* in sharing information between investors; 2) the evolution of different processes over *different time scales*, like groups switching between different strategies while other investors stick with only one and trade based on it within a period; 3) the presence of a *misinforming source* that exploits the weak influence of some investors on other groups of investors to drive the price according to their interests. In the second part of Chapter 2, I discussed the findings of a series of works regarding data analysis of investors' trading activity in the Finnish stock market to get more information about the structure of the financial market, focus on phenomena related to our research question, and formalise our interpretations of these findings mathematically in our model. Based on these interpretations, I concluded that the Finnish Stock Market (and potentially any generic one) can be described as a complex system where investors are classified in different trading frequency classes and, within each of these classes, they trade the asset with different trading time delays (or 'phases'). Different endogenous and exogenous factors (the first related to any information that an investor uses based on the strategy adopted, while the second related to any information that influences an investor's behavioural biases, including social biases) affect investors belonging to these classes and give rise to this heterogeneity within trading time delays. In the short-term, the system is stable (investors trade based on their current strategy) because many parameters that identify it do not change, but in the long-term, it undergoes several evolutions (investors switch sources of information), a finding that supports the idea about the evolution of different processes over different time scales (see above). Such a system bears many similarities with other (biological) systems, which are described through *coupled maps*, like the Kuramoto model, and my model takes inspiration from them, which is also a novel way to describe a financial market.

Social influence over different time scales For this purpose, in the first part of Chapter 3, where we outlined the general framework of our model, I introduced three main tools that allowed us to fill the gaps we mentioned earlier and to understand some pricing anomalies present in a financial market while being true to the investor-level data: 1) the *trading mechanism*, a 'short-term' function that describes the fraction of investors who trade the asset with their current strategy; 2) the *sharing mechanism*, a 'long-term' function that describes the fraction of investors who share their information with other groups; 3) the *reference price*, a function that identifies each group of investors in the market and describes the price the investors refer to instead of the current asset price as a consequence of endogenous/exogenous factors. I outlined our model as a system of difference equations (a coupled map, see system (3.37)), which was analysed by considering several cases with the interplay between groups of fundamentalists on different time scales. As a starting point, in the second part of Chapter 3, I considered the simplest case: a market populated by only one group of investors, that is *pure fundamentalists*, who believe the asset price will tend to its fundamental value and always refer to its current price. As expected, the system exhibited very simple price dynamics (both on large and short time scales), such as price convergence to its fundamental value, double-periodic price patterns and price divergences, but it was a very convenient way to understand the effect the parameters describing a group of investors have on the trading activity of such a group and price dynamics.

A market with pure fundamentalists and followers In Chapter 4, I focused on a more general scenario where a market was populated by pure fundamentalists and *followers*, in which the latter refer to the information shared by the former about the asset's fundamentals with a delay (so they do not refer to the asset current price, but to its price at a previous time), whose interaction gave rise to richer price dynamics than in the previous case. I first considered the system evolving on a very large time scale, when both groups trade the asset at any time, and pure fundamentalists share their information with followers at any time. I observed the most interesting scenario when followers were more dominant than pure fundamentalists, in which case the asset price constantly fluctuated around its fundamental value but with *quasi-periodic* orbits, and both groups performed well (meaning that such market conditions are sustainable for both groups and this price dynamics can last for a long time). Although the conditions to observe such dynamics are very strict, it allowed me to understand better the conditions for which it was possible to observe a perpetually volatile market, even if it was only populated by groups of fundamentalists, a phenomenon that in DCA models is usually explained by means of a switching mechanism adopted by groups of investors between different strategies (or the presence of noise traders). Next, I focused our market analysis on a shorter time scale, when both groups trade the asset only if its price is far enough from its fundamental value, and pure fundamentalists would start sharing their information with followers about its mispricing only if they are large enough. I found that the conditions that correspond to a volatile market were less restrictive than in the previous case, and interestingly, most of these cases represented markets in which both groups of investors performed positively (followers still need to be more dominant than pure fundamentalists). Other differences from the previous case are markets exhibiting multiple price volatilities, *volatility clustering* (a well-known stylised fact of financial time series), and sensitive dependence on initial conditions, which suggests that the system is potentially chaotic. I considered a case study where we brought out all these aspects and observed how pure fundamentalists' sharing mechanism plays a critical role in giving rise to these new phenomena. The same features are observed if I included a noise term in the system (e.g. noise traders), an external factor that, unlike in the previous chapter, turns out to be unnecessary to generate (one of the) realistic properties of financial time series.

Pure fundamentalists, followers and a misinforming source In chapter 5, I considered a market populated by three groups of investors: pure fundamentalists, *contrarian fundamentalist*, and followers. Here, followers not only referred to the information shared by pure fundamentalists but also considered the information shared by contrarian fundamentalists that go against the former ones (believing the asset price will move away from its fundamental value) and, so, act like a *misinforming source* (they shared their information with followers only if pure fundamentalists were mostly inactive, that is when the asset price is close to its fundamental value). I analysed the system on a large time scale such that all investors trade the asset at any time but share their information about their reference price only if certain conditions are satisfied, and I found that the system exhibited different price dynamics depending on how dominant followers were with respect to pure fundamentalists. If pure fundamentalists were more dominant than followers, then the price trend was the same as we observed in the previous chapter, when the price could tend to its fundamental value or constantly flip up and down from it in one single time step (double-period orbit), with the difference that the price fluctuations got larger and larger if followers trusted more and more the misinforming source (by means of a parameter we called degree of *reliability*). If followers were slightly more dominant than pure fundamentalists, then the price dynamics did not differ so much from what we saw earlier, but instead of observing the price tending to its fundamental price, it would tend to a non-fundamental one (asset being constantly overvalued or undervalued depending on the initial conditions). In this state, the orders get completely cleared between pure fundamentalists and followers, where the former would constantly send selling orders, and the latter would constantly send buying orders when the asset is overvalued (and vice versa when the asset is undervalued). For this reason, these market conditions cannot last forever, as they would require an infinite amount of supply or liquidity from both groups of investors. Either one group would run out of money, and the price would start decreasing, or the other group would run out of asset shares and the price would keep rising. The case when followers were significantly more dominant than pure fundamentalists was the most interesting part of this chapter since it contained much richer and more realistic price dynamics than observed earlier. If investors' demands are large enough, the price will start fluctuating around non-fundamental values, giving rise to *price bubbles* (the asset price spends some time around its fundamental value before slowing increasing to a non-fundamental price) and *price crashes* (the asset price immediately drops down close to its fundamental value again if it gets too far away from it). And if followers trust contrarians more than pure fundamentalists, then the price may fluctuate above and below the fundamental value (*double-period* pattern) right after the price crash (as a consequence of the delayed information from the misinforming source), but only briefly. After the price fluctuations dampen down as time passes, the price starts rising again, and the market cycle repeats. However, I must point out that these market conditions were not favourable for followers, so I expect they would be short-lived in a real context. Finally, as I did in the previous chapter, I observed the system by including a noise term (e.g. noise traders), which exhibited the same phenomenon and demonstrated *strong robustness* against random fluctuations.

Future work The last chapter ended with analysing my model of a financial market over large-time scales, in which the market was populated by *pure fundamentalists*, *contrarian fundamentalists*, and *followers*. However, I have only scratched the surface of my model, beginning with the simplest cases, and more complex interactions can be considered in future works. More specifically, a very important financial figure that I should take into account is the *technical analyst* or, alternatively called, *chartist*, whose purpose is to analyse the price fluctuations generated by these groups of 'long-term' investors (fundamentalists) and try to predict the price trend from them. I assume these investors to be ’short-term’ ones who trade the asset and/or switch strategies much more frequently than fundamentalists and may amplify overall market volatility (by creating, for example, *flash crashes*). Moreover, I also believe there exist similar groups of investors among them as we have for fundamentalists, with *chartist followers* and/or *contrarian chartists* (another misinforming source), creating, thus, a *fractal* investor structure in the market. Once the analysis on this market is completed, the next step would be comparing the generated price’s statistical properties with those observed in real markets, that is *stylised facts*, being one of the main means to test a financial model against real data sets. A further idea to deepen the analysis of my model would be to consider a *network structure*, an aspect that my model could easily extend to. The stock market can be considered a complex network whose nodes are investors and other sources of information (e.g. the company’s public announcement website, news, and social media), and the links are information channels through which investors’ reference prices are disseminated. Several market structures can be considered where the interplay between endogenous and exogenous sources can be simultaneously analysed, and networks with different topologies, through which investors can influence each other or be influenced by an external source, can be tested to observe their impact on the stability of this system.

Appendix A

Mathematical derivations

A.1 Literature review

Conditional expectation (Section 2.1.1.1)

Using equations (2.6) and (2.8), we obtain from equation (2.7) that

$$\begin{aligned} E_t[U(W_{t+1})] = E_t[U(Z)] &= \int_{-\infty}^{\infty} U(z)\varphi(z)dz \\ &= \int_{-\infty}^{\infty} \left(1 - e^{-a\left(E_t[W_{t+1}] + \sqrt{V_t[W_{t+1}]}z\right)}\right)\left(\frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\right)dz \\ &= 1 - \frac{e^{-aE_t[W_{t+1}]}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\sqrt{V_t[W_{t+1}]}z - \frac{z^2}{2}} dz \\ &= 1 - \frac{e^{-aE_t[W_{t+1}] + \frac{a^2 V_t[W_{t+1}]}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(a\sqrt{V_t[W_{t+1}]}+z)^2}{2}} dz \\ &= 1 - e^{-a(E_t[W_{t+1}] - \frac{a}{2}V_t[W_{t+1}])} \end{aligned} \tag{A.1}$$

where we used the fact that $\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}} dz = \sqrt{2\pi}$.

A.2 Pure fundamentalists, followers and a misinforming source

A.2.1 Analytical form of the Jacobian (Paragraph 5.2)

The elements of the first row of the matrix (5.22) given the map (5.17) are

$$\frac{\partial f_{x_{f_1}}}{\partial x_{f_1}} = 1 - \mu \bar{c}_{f_1} \tag{A.2}$$

$$\frac{\partial f_{x_{f_1}}}{\partial x_{f_2}} = -\mu \bar{c}_{f_2} \tag{A.3}$$

For the elements of the second row, we use the following relations and notations to simply our calculations[1]

$$g(x_{f_1}) = \frac{|x_{f_1}|^{2\beta_\sigma}}{|x_{f_1}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \implies \frac{\partial g(x_{f_1})}{\partial x_{f_1}} = \frac{2\beta_\sigma x_{f_1} |x_{f_1}|^{2\beta_\sigma - 2}}{\left(|x_{f_1}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}\right)^2} = h(x_{f_1}) \tag{A.4}$$

$$\frac{\partial}{\partial x_{f_1}} \left(\frac{\Delta_\sigma^{\beta_\sigma}}{|x_{f_1}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} \right) = \frac{\partial}{\partial x_{f_1}} \left(1 - g(x_{f_1})\right) = -h(x_{f_1}) \tag{A.5}$$

So, for the first element of the second row of the matrix (5.22), we have (see map (5.17))

$$\begin{aligned} \frac{\partial f_{x_{f_2}}}{\partial x_{f_1}} &= (1-\gamma)\left[g(x_{f_1}) + (x_{f_1} - x_{f_2})h(x_{f_1})\right] + \gamma\left[-(1 - g(x_{f_1})) + (x_{f_1} + x_{f_2})h(x_{f_1})\right] \\ &= -\gamma + g(x_{f_1}) + \left[(1-\gamma)(x_{f_1} - x_{f_2}) + \gamma(x_{f_1} + x_{f_2})\right] h(x_{f_1}) \end{aligned} \tag{A.6}$$

while for the second element of the second row we have

$$\begin{aligned} \frac{\partial f_{x_{f_2}}}{\partial x_{f_2}} &= 1 - (1-\gamma)g(x_{f_1}) + \gamma(-1)(1 - g(x_{f_1})) \\ &= 1 - \gamma + (2\gamma - 1)g(x_{f_1}) \end{aligned} \tag{A.7}$$

A.2.2 Jacobian at the fundamental steady state (Section 5.2.1)

For $x_{f_1,b} = 0$ ($\implies$ $x_{f_2,b} = 0$), we have that (see (A.4))

$$g(x_{f_1,b}) = 0 \qquad h(x_{f_1,b})x_{f_1,b} = 0 \tag{A.8}$$

and the Jacobian matrix calculated at the fundamental steady state simply becomes

$$J_o = JF(f_{x_{f_1}}, f_{x_{f_2}})\big|_{\vec{x}_o^*} = \begin{pmatrix} 1 - \mu \bar{c}_{f_1} & -\mu \bar{c}_{f_2} \\ -\gamma & 1 - \gamma \end{pmatrix} \tag{A.9}$$

[1] $\frac{\partial |x|^a}{\partial x} = a x |x|^{a-2}$

A.2.3 More confident groups

Pure fundamentalists more dominant than followers (Section 5.2.2)

Here, we consider the same market as in section 5.2.2, but for fixed $\beta_\sigma = 10.0$, that is, when both f_1 and f_m are more confident in sharing their information with f_2.

Pure fundamentalists more trustworthy than contrarian fundamentalists In the left panels of Figure A.1, we show the orbit diagram for the price deviation x (top left panel) and followers' demand $\bar{d}_{f_2}$ (bottom left panel) for different values of $\mu \in [1.2, 3.1]$ in steps of $\delta_\mu = 0.002$, where we kept track of their evolution over a period of time $T_{test} = 10$ (a larger time window than in the previous case as the price shows more complex dynamics), for different initial states $x_{f_1,0} \in [-1.0, 1.0]$ and $x_{f_2,0} \in [-1.0, 1.0]$ in steps of $\delta_{x_f,0} = 0.25$.

As soon as the stability condition (5.39) is violated ($\mu \sim 1.9$), we still observe the orbit pattern with period 2 but with wider market fluctuations than we had in the previous case ($|x_t| \sim 1$). As the speed of price adjustment reaches $\mu \sim 1.95$, we see that the curves start spreading out, meaning that the price fluctuates above and below the fundamental value but takes different values each time (see top left panel of Figure A.2 for fixed $\mu = 1.98$) and the same goes for groups' net demand (see middle left panel of Figure A.2 for fixed $\mu = 1.98$).

The reason why this happens is due to f_1's and f_m's higher sensitivity in sharing their information when the price deviation is below or above $|x_t| = 1$, as we have

$$|x_t| < 1 \implies |x_t|^{2\beta_\sigma} << \Delta^{\beta_\sigma} = 1 \tag{A.10}$$

$$|x_t| > 1 \implies |x_t|^{2\beta_\sigma} >> \Delta^{\beta_\sigma} = 1 \tag{A.11}$$

meaning that (see bottom panels in Fig.A.2)

$$|x_t| < 1 \implies \alpha_{f_m,t} = 1 - \alpha_{f_1,t} \sim 1 \tag{A.12}$$

$$|x_t| > 1 \implies \alpha_{f_m,t} = 1 - \alpha_{f_1,t} \sim 0 \tag{A.13}$$

Contrarian fundamentalists f_m affect followers' trading decisions more strongly when the price deviation from the fundamental value is below the sharing threshold (that is when the price is close to the fundamental value), leading them to shift more swiftly their reference price in line with f_m's information (and so we also have when pure fundamentalists become more active as $|x_t| > 1$). As a result, the system becomes more and more chaotic by showing a higher sensitive dependence on the initial conditions as the market maker increases the speed of price adjustment μ (see the top

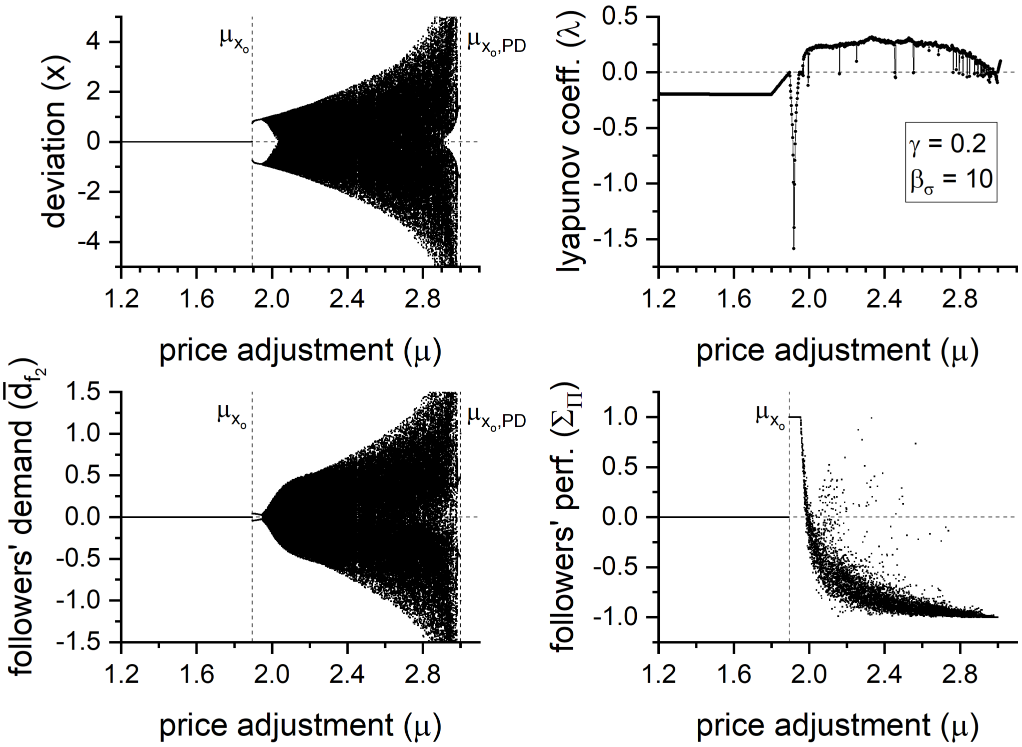


FIGURE A.1: The orbit diagram for the price deviation x (top right panel) and followers' demand $\bar{d}_{f_2}$ (bottom right panel), Lyapunov coefficient (top right panel), and followers' long-term performance (bottom right panel) for fixed $\beta_\sigma = 10.0$, $\gamma = 0.2$ and different values of μ. As the speed of price adjustment increases, the curves start spreading out ($\mu > 1.95$), meaning that the price (demand) fluctuates above and below the fundamental value but takes different values each time. As μ further increases, the curves overlap ($\mu > 2.04$ for x and $\mu > 1.96$ for $\bar{d}_{f_2}$), that is, the price (demand) takes more than one time step before flipping below (above) its fundamental value (hold state), and the system becomes more chaotic, while followers' performance starts to decrease.

right panel in Figure A.1), which drives the mispricings closer and closer to $|x_t| = 1$. We also observe, for certain values of μ, that the orbits converge back in several branches before spreading out again, which represent cases where the orbits are periodic with a period given by the total number of branches, as confirmed by the corresponding (downward spikes) value of the Lyapunov coefficient (absence of chaotic dynamics). Notice that, for $\beta_\sigma = 10.0$, the price deviation where both f_1 and f_m exert the same influence on f_2 (see equation (5.41)) is now

$$|x_{f_1,t}| \sim 0.933 \tag{A.14}$$

so, here, contrarian fundamentalists exert a greater influence than pure fundamentalists on the followers if $x_{f_1} \in (-0.933, 0.933)$ (a wider range of values than in the $\beta_\sigma = 1.0$ case). Beyond certain values of μ, we see that the orbits start to overlap ($\mu > 1.96$ for followers' demand and $\mu > 2.04$ for the price deviation), meaning that

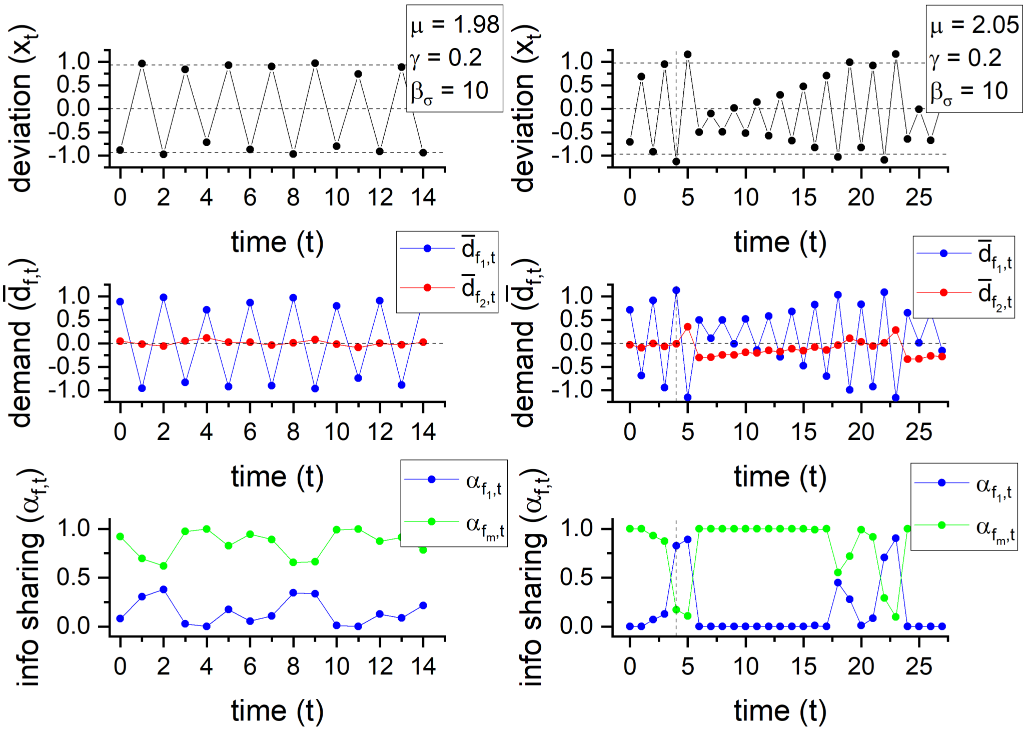


FIGURE A.2: The price deviation x_t (top panels), groups' net demand $\bar{d}_{f,t}$ (middle panel), and the shifting mechanism $\alpha_{f_i,t}$ for fixed $\mu = 1.98$ (left panels) and $\mu = 2.05$ (right panels). The wider market fluctuations and irregular period 2 patterns are due to f_1's and f_m's higher sensitivity in sharing their information with f_2. The horizontal dashed lines above and below the fundamental value ($|x_t| \sim 0.933$) correspond to the thresholds beyond which pure fundamentalists f_1 predominately influence followers (see the vertical dashed line at time $t = 4$ in the right panels as an example).

the asset price may take more than one time step before flipping below (above) its fundamental value at time t if overpriced (underpriced) at time $t-1$. The same applies to followers' net demand (e.g. sending selling orders over consecutive time steps), whose long-term (and short-term) performance starts declining towards values that make their participation in the market unsustainable (see bottom right panel in Figure A.1). In the right panels of Figure A.2, we can observe this dynamic more clearly (where we chose $\mu = 2.05$). Initially, we have that the price (top right panel) fluctuates back and forth between values within $|x_t| < 0.933$ (see the horizontal dashed lines below and above the fundamental value) and, so, the misinforming source f_m mostly affects followers' f_2 trading decisions. As time passes, the price deviates farther and farther away from p^* up to time $t = 4$ (see the vertical dashed line), where the price crosses the threshold (A.14) beyond which pure fundamentalists f_1 predominately influence followers. Consequently, followers start sending orders in the opposite direction to pure fundamentalists (due to delayed information) and because some orders are filled between these groups, the price fluctuations settle back close to the fundamental value, leading group f_m to influence group f_2 again ($t = 6$). However, these fluctuations are now skewed to the downside of p^* due to f_2's

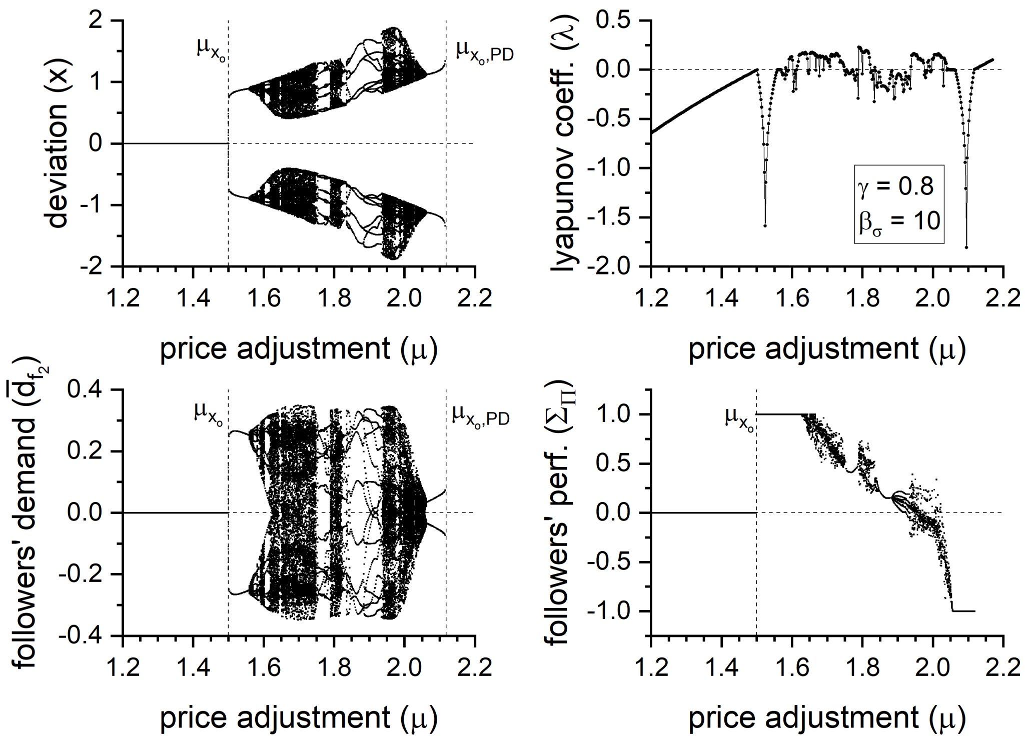


FIGURE A.3: The orbit diagram for the price deviation x (top right panel) and followers' demand $\bar{d}_{f_2}$ (bottom right panel), Lyapunov coefficient (top right panel), and followers' long-term performance (bottom right panel) for fixed $\beta_\sigma = 10.0$, $\gamma = 0.8$ and different values of μ. As in the $\gamma = 0.2$ case, the orbits spread out and overlap as μ increases (while followers' performance decreases more and more), although the regions where the orbits converge back in several branches before spreading out again is larger (periodic orbits, see the corresponding negative Lyapunov coefficients).

tendency to send only sell orders (see right middle panel in Figure A.2), as a consequence of the combined feedback of f_1 (at time $t = 5$) and f_m (at time $t = 6$) advising them to do so. This persistence lasts a few more time steps before a similar cycle starts again. Notice that, in this regime, followers' long-term (and short-term) performance is negative, and thus, this price action is unlikely to be observed in a real context.

Contrarian fundamentalists more trustworthy than pure fundamentalists In the left panels of Fig.A.3, we show the orbit diagram for the price deviation x (top left panel) and followers' demand $\bar{d}_{f_2}$ (bottom left panel) for different values of μ for fixed $\beta_\sigma = 10.0$. Even in this case, we observe orbits spreading out and overlapping (the latter is true for followers' demand only), although the diagrams are less regular than in the previous case for $\gamma = 0.2$ and the range of values for μ within which the orbits converge into several branches is wider.

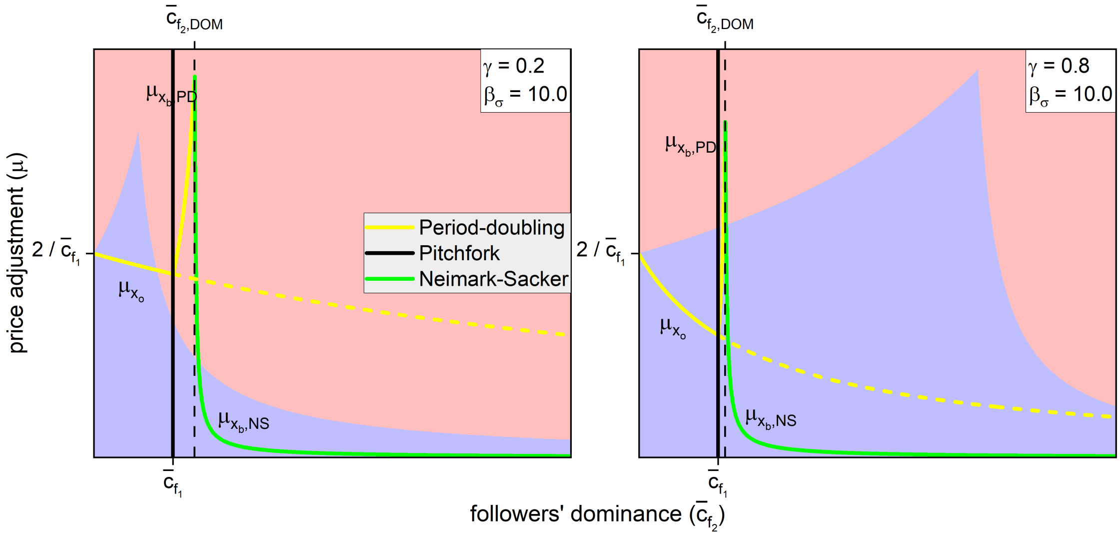


FIGURE A.4: The same stability diagram $(\bar{c}_{f_2}, \mu)$ as in Fig. 5.3 where the stability conditions $\mu_{x_0}(\bar{c}_{f_2})$, $\mu_{x_b,PD}(\bar{c}_{f_2})$ and $\mu_{x_b,NS}(\bar{c}_{f_2})$ are plotted as functions of $\bar{c}_{f_2}$, for fixed $\gamma = 0.2$ (left panel), $\gamma = 0.8$ (right panel) and $\beta_\sigma = 10.0$. The light blue region in the background represents pairs of values $(\bar{c}_{f_2}, \mu)$ where the system does not exhibit any divergence, while in the light red region, it could.

Followers more dominant than Pure fundamentalists (Section 5.2.3)

In the previous example, we assumed f_1 and f_m's confidence degree to equal $\beta_\sigma = 1.0$. In Figure A.4, we show the stability diagrams $(\bar{c}_{f_2}, \mu)$ for fixed $\gamma = 0.2$ (left panel) and $\gamma = 0.8$ (right panel) when we consider the same market but for fixed $\beta_\sigma = 10.0$, that is, when the sources of information f_1 and f_m are more confident in sharing their information with f_2. We see that the region before the pitchfork bifurcation ($\bar{c}_{f_1} > \bar{c}_{f_2}$) is unaffected by this increase, as it should be given that the stability conditions (5.30) and (5.33) do not depend on β_σ. For $\bar{c}_{f_1} < \bar{c}_{f_2}$, instead, the region where the followers are slightly more dominant than pure fundamentalists ($\bar{c}_{f_1} < \bar{c}_{f_2} < \bar{c}_{f_2,DOM}$) shrinks and takes up only a small portion of the entire diagram. We observe similar market dynamics as we did in Figure A.1, Figure 5.2 and in Figure 5.4, with no significant changes. Briefly summarised, for low values of μ, the price tends to one of the non-fundamental values (depending on the initial conditions), which are now further away from p^* and closer to $p_{b^\pm} \sim p^* \pm 1$ (see system (5.20) with $\Delta_\sigma = 1$), with no fluctuations. If μ crosses a certain threshold, the price starts flipping around the fundamental value, and as we further increase μ, the dynamics become more chaotic (the orbits spread out and overlap as in Figure A.1). However, even before the first stability condition (5.48) for $\vec{x}_{b^\pm}$ is violated, once μ becomes large enough, the system becomes extremely unstable, and for most initial conditions the orbits diverge to infinity unless one chooses an initial state that is close to $\vec{x}_{b^\pm}$ (in which case, the price converges to the non-fundamental value). Finally, if μ increases beyond μ_{x_b}, whose value is now greater than in the previous example ($\beta_\sigma = 1.0$) since β_σ appears as a negative factor in the denominator of (5.48) (recalling that the denominator is positive), then the price will diverge to infinity for any initial condition.

A.2.4 Jacobian derivation of the non-fundamental steady states (Section 5.2.3)

To facilitate the computation of the Jacobian at the fixed point $\vec{x}_{b^\pm}$ (see system (5.20)), we define two parameters

$$c_\Delta = \gamma \left(\frac{\bar{c}_{f_1} - \bar{c}_{f_2}}{\bar{c}_{f_2}} \right) \tag{A.15}$$

$$c_\Sigma = (1-\gamma) \left(\frac{\bar{c}_{f_1} + \bar{c}_{f_2}}{\bar{c}_{f_2}} \right) \tag{A.16}$$

From system (5.20), we have that

$$|x_{f_1,b}|^{2\beta_\sigma} = -\frac{c_\Delta}{c_\Sigma} \Delta_\sigma^{\beta_\sigma} \tag{A.17}$$

and, thus,

$$g(x_{f_1,b}) = \frac{|x_{f_1,b}|^{2\beta}}{|x_{f_1,b}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}} = \frac{c_\Delta}{c_\Delta - c_\Sigma} \tag{A.18}$$

$$h(x_{f_1,b})x_{f_1,b} = 2\beta_\sigma \Delta_\sigma^{\beta_\sigma} \frac{|x_{f_1,b}|^{2\beta_\sigma}}{\left(|x_{f_1,b}|^{2\beta_\sigma} + \Delta_\sigma^{\beta_\sigma}\right)^2} = \frac{2\beta_\sigma c_\Delta c_\Sigma}{(c_\Delta - c_\Sigma)(c_\Sigma - c_\Delta)} \tag{A.19}$$

So, for the first element of the second row of the matrix (5.22), we have

$$\begin{aligned} \left.\frac{\partial f_{x_{f_2}}}{\partial x_{f_1}}\right|_{\vec{x}_{b^\pm}} &= -\gamma + g(x_{f_1,b}) + \left[(1-\gamma)\left(x_{f_1,b} - x_{f_2,b}\right) + \gamma(x_{f_1,b} + x_{f_2,b})\right] h(x, b_{f_1}) \\ &= -\gamma + \frac{c_\Delta}{c_\Delta - c_\Sigma} + \left[(1-\gamma)\left(x_{f_1,b} + \frac{\bar{c}_{f_1}}{\bar{c}_{f_2}} x_{f_1,b}\right) + \gamma(x_{f_1,b} - \frac{\bar{c}_{f_1}}{\bar{c}_{f_2}} x_{f_1,b})\right] h(x_{f_1,b}) \\ &= -\gamma + \frac{c_\Delta}{c_\Delta - c_\Sigma} + (c_\Sigma - c_\Delta) x_{f_1,b} h(x_{f_1,b}) \\ &= -\gamma + \frac{c_\Delta}{c_\Delta - c_\Sigma} + 2\beta_\sigma \frac{c_\Delta c_\Sigma}{c_\Delta - c_\Sigma} \end{aligned} \tag{A.20}$$

Now, we substitute c_Δ and c_Σ into the previous equation and after some straightforward algebraic steps, we obtain

$$\left.\frac{\partial f_{x_{f_2}}}{\partial x_{f_1}}\right|_{\vec{x}_{b\pm}} = \frac{2\gamma(1-\gamma)\left[\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2) - \bar{c}_{f_1}\bar{c}_{f_2}\right]}{\bar{c}_{f_2}\left[\bar{c}_{f_2} + (1-2\gamma)\bar{c}_{f_1}\right]} \tag{A.21}$$

For the second element of the second row of the matrix (5.22), we get

$$\begin{aligned}\left.\frac{\partial f_{x_{f_2}}}{\partial x_{f_2}}\right|_{\vec{x}_{b\pm}} &= 1-\gamma+(2\gamma-1)g(x_{f_1}^*) \\ &= 1-\gamma+(2\gamma-1)\frac{c_\Delta}{c_\Delta - c_\Sigma} \\ &= 1-\frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}}\end{aligned} \tag{A.22}$$

The Jacobian matrix calculated at the non-fundamental steady states becomes

$$J_b = JF(f_{x_{f_1}}, f_{x_{f_2}})\big|_{\vec{x}_{b\pm}} = \begin{pmatrix} 1-\mu\bar{c}_{f_1} & -\mu\bar{c}_{f_2} \\ \frac{2\gamma(1-\gamma)\left[\beta_\sigma(\bar{c}_{f_2}^2-\bar{c}_{f_1}^2)-\bar{c}_{f_1}\bar{c}_{f_2}\right]}{\bar{c}_{f_2}\left[\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}\right]} & 1-\frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}} \end{pmatrix} \tag{A.23}$$

A.2.5 Stability conditions (Section 5.2.3)

From the first stability condition (5.23), we get

$$2-\frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}} > \mu\left[\bar{c}_{f_1} - \frac{\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2-\bar{c}_{f_1}^2)}{\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}}\right] \tag{A.24}$$

Since $\bar{c}_{f_2} > \bar{c}_{f_1}$ (a necessary condition for the existence of the non-fundamental steady states), the denominator of the fractions in the previous inequality is always positive, so we can multiply both sides by this term and obtain

$$2\left[\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}\right] - 2\gamma(1-\gamma)\bar{c}_{f_2} > \mu\left[\bar{c}_{f_1}\bar{c}_{f_2} + (1-2\gamma)\bar{c}_{f_1}^2 - \gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2-\bar{c}_{f_1}^2)\right] \tag{A.25}$$

We first show the LHS of (A.25) is always positive, that is,

$$2\left[\bar{c}_{f_2}+(1-2\gamma)\bar{c}_{f_1}\right] - 2\gamma(1-\gamma)\bar{c}_{f_2} > 0 \tag{A.26}$$

which can be rewritten as

$$\bar{c}_{f_2}\left[1-\gamma(1-\gamma)\right] > (2\gamma-1)\bar{c}_{f_1} \tag{A.27}$$

Since

$$1-\gamma(1-\gamma) > 2\gamma-1 \iff (\gamma-2)(\gamma-1) > 0 \tag{A.28}$$

for $0 < \gamma < 1$, we have that the inequality (A.26) is always satisfied. This means that if the RHS of the inequality (A.25) is negative, that is,

$$\bar{c}_{f_1}\bar{c}_{f_2} + (1-2\gamma)\bar{c}_{f_1}^2 - \gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2) < 0 \tag{A.29}$$

then the first stability condition (A.25) is always satisfied. We can rewrite the previous inequality as a quadratic polynomial in $\bar{c}_{f_2}$

$$\gamma(1-\gamma)\beta_\sigma\bar{c}_{f_2}^2 - \bar{c}_{f_1}\bar{c}_{f_2} - [1-2\gamma+\gamma(1-\gamma)\beta_\sigma]\,\bar{c}_{f_1}^2 > 0 \tag{A.30}$$

where we multiplied both sides of the inequality by -1, and we can decompose it as a product of three terms

$$\gamma(1-\gamma)\beta_\sigma\left[\bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_+(\gamma,\beta_\sigma)\right]\left[\bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_-(\gamma,\beta_\sigma)\right] > 0 \tag{A.31}$$

where

$$\Gamma_\pm(\gamma,\beta_\sigma) = \frac{1 \pm \sqrt{1+4\gamma(1-\gamma)\beta_\sigma\left[1-2\gamma+\gamma(1-\gamma)\beta_\sigma\right]}}{2\gamma(1-\gamma)\beta_\sigma} \tag{A.32}$$

We can write the inequality (A.30) as (A.31) because the discriminant of the polynomial in $\bar{c}_{f_2}$ is always positive, that is,

$$1+4\gamma(1-\gamma)\beta_\sigma\left[1-2\gamma+\gamma(1-\gamma)\beta_\sigma\right] > 0 \tag{A.33}$$

In fact, if we write the previous inequality as a quadratic polynomial in β_σ this time, then we get

$$\gamma(1-\gamma)\beta_\sigma^2 + (1-2\gamma)\beta_\sigma + \frac{1}{4\gamma(1-\gamma)} > 0 \tag{A.34}$$

whose discriminant is

$$(1-2\gamma)^2 - 1 = \gamma(\gamma-1) \tag{A.35}$$

which is negative, meaning that the LHS of (A.34) is positive. To see that, one simply chooses $\beta_\sigma = 0$ and shows that the inequality is satisfied, which guarantees that (A.33) is satisfied for any value of β_σ. The third term in (A.31) is positive, that is (recall that $\bar{c}_{f_2} > \bar{c}_{f_1}$),

$$\bar{c}_{f_2} > \bar{c}_{f_1}\Gamma_-(\gamma, \beta_\sigma) \tag{A.36}$$

since

$$\begin{aligned}\Gamma_-(\gamma, \beta_\sigma) < 1 &\iff 1 - 2\gamma(1-\gamma)\beta_\sigma < \sqrt{1 + 4\gamma(1-\gamma)\beta_\sigma\left[1 - 2\gamma + \gamma(1-\gamma)\beta_\sigma\right]} \\ &\iff -4\gamma(1-\gamma)\beta_\sigma < 4\gamma(1-\gamma)\beta_\sigma(1-2\gamma) \\ &\iff \gamma < 1\end{aligned} \tag{A.37}$$

where we squared the first inequality in (A.36) (one can also prove that $\Gamma_-(\gamma, \beta_\sigma) > -1 \implies \bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_-(\gamma, \beta_\sigma) < \bar{c}_{f_2} + \bar{c}_{f_1}$). So, the inequality (A.31) is positive if and only if

$$\bar{c}_{f_2} > \bar{c}_{f_1}\Gamma_+(\gamma, \beta_\sigma) \tag{A.38}$$

or, equivalently,

$$\bar{c}_{f_2} > \bar{c}_{f_1}\left\{\frac{1 + \sqrt{1 + 4\gamma(1-\gamma)\beta_\sigma\left[1 - 2\gamma + \gamma(1-\gamma)\beta_\sigma\right]}}{2\gamma(1-\gamma)\beta_\sigma}\right\} = \bar{c}_{f_2,PB} \tag{A.39}$$

If the condition (A.39) does not hold, then we must directly verify the stability condition (A.25), as the RHS is now positive, so we rewrite it as a condition on μ, that is,

$$\mu < \frac{2\left[1 - \gamma(1-\gamma)\right]\bar{c}_{f_2} + 2(1-2\gamma)\bar{c}_{f_1}}{\bar{c}_{f_1}\bar{c}_{f_2} + (1-2\gamma)\bar{c}_{f_1}^2 - \gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2)} \tag{A.40}$$

or, equivalently,

$$\mu < \frac{2\left[1 - \gamma(1-\gamma)\right]\bar{c}_{f_2} + 2(1-2\gamma)\bar{c}_{f_1}}{\gamma(1-\gamma)\beta_\sigma\left[\bar{c}_{f_1}\Gamma_+(\gamma, \beta_\sigma) - \bar{c}_{f_2}\right]\left[\bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_-(\gamma, \beta_\sigma)\right]} \tag{A.41}$$

For the second stability condition (5.24), we simply have

$$\frac{2\mu\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2)}{\bar{c}_{f_2} - (1-2\gamma)\bar{c}_{f_1}} > 0 \implies \bar{c}_{f_2} > \bar{c}_{f_1} \tag{A.42}$$

which corresponds to the existence condition for the non-fundamental steady states. From the third stability condition (5.25), we get

$$2\gamma(1-\gamma)\bar{c}_{f_2} > \mu\left[2\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2) - \bar{c}_{f_1}\bar{c}_{f_2} - (1-2\gamma)\bar{c}_{f_1}^2\right] \tag{A.43}$$

Here, again, if the RHS of the previous equality is negative, that is,

$$2\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2) - \bar{c}_{f_1}\bar{c}_{f_2} - (1-2\gamma)\bar{c}_{f_1}^2 < 0 \tag{A.44}$$

then the third stability condition (5.25) is always satisfied. We can decompose the LHS of (A.44) as a product of three terms, obtaining

$$2\gamma(1-\gamma)\beta_\sigma \left[\bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_+(\gamma, 2\beta_\sigma)\right] \left[\bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_-(\gamma, 2\beta_\sigma)\right] < 0 \tag{A.45}$$

Notice now that the inequality for $\Gamma_-(\gamma, \beta_\sigma)$ (A.37) does not depend on β_σ, which implies that

$$-1 < \Gamma_-(\gamma, 2\beta_\sigma) < 1 \tag{A.46}$$

So, the third term in (A.45) is still positive, meaning that the second term must be negative to ensure the inequality is being satisfied, that is

$$\bar{c}_{f_2} < \bar{c}_{f_1}\Gamma_+(\gamma, 2\beta_\sigma) \tag{A.47}$$

or, equivalently

$$\bar{c}_{f_2} < \bar{c}_{f_1} \left\{ \frac{1 + \sqrt{1 + 8\gamma(1-\gamma)\beta_\sigma\left[1 - 2\gamma + 2\gamma(1-\gamma)\beta_\sigma\right]}}{4\gamma(1-\gamma)\beta_\sigma} \right\} = \bar{c}_{f_2,NS} \tag{A.48}$$

If the condition (A.48) does not hold, then we must directly verify the stability condition (A.43), as the RHS is now positive, so we rewrite it as a condition on μ, that is,

$$\mu < \frac{2\gamma(1-\gamma)\bar{c}_{f_2}}{2\gamma(1-\gamma)\beta_\sigma(\bar{c}_{f_2}^2 - \bar{c}_{f_1}^2) - \bar{c}_{f_1}\bar{c}_{f_2} - (1-2\gamma)\bar{c}_{f_1}^2} \tag{A.49}$$

or, equivalently,

$$\mu < \frac{\bar{c}_{f_2}}{\beta_\sigma \left[\bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_+(\gamma, 2\beta_\sigma)\right] \left[\bar{c}_{f_2} - \bar{c}_{f_1}\Gamma_-(\gamma, 2\beta_\sigma)\right]} \tag{A.50}$$

A.2.6 Critical threshold (Paragraph 5.2.3)

Here, we show that

$$\bar{c}_{f_2,NS} < \bar{c}_{f_2,PB} \tag{A.51}$$

The previous inequality implies that there exist an intermediate value $\bar{c}_{f_2,NS} < \bar{c}_{f_2,DOM} < \bar{c}_{f_2,PB}$, for which the RHS of the conditions (A.40) and (A.49) are equal to each other. So, we have two regions in which one of the two conditions (A.40) and (A.49) must be necessary violated for μ large enough (second demonstration). First, we have that the partial derivative of $\Gamma_+(\gamma, \beta_\sigma)$ with respect to β_σ is (we use $\Delta(\gamma, \beta_\sigma) = 1 + 4\gamma(1-\gamma)\beta_\sigma [1 - 2\gamma + \gamma(1-\gamma)\beta_\sigma]$)

$$\begin{aligned}
\frac{\partial \Gamma_+(\gamma, \beta_\sigma)}{\partial \beta_\sigma} &= \frac{\frac{\left[4\gamma(1-\gamma)(1-2\gamma)+8\gamma^2(1-\gamma)^2\beta_\sigma\right]2\gamma(1-\gamma)\beta_\sigma}{2\sqrt{\Delta(\gamma,\beta_\sigma)}} - (1+\sqrt{\Delta(\gamma,\beta_\sigma)})2\gamma(1-\gamma)}{4\gamma^2(1-\gamma)^2\beta_\sigma^2} \\
&= \frac{\left[4\gamma(1-\gamma)(1-2\gamma)+8\gamma^2(1-\gamma)^2\beta_\sigma\right]\beta_\sigma - (1+\sqrt{\Delta(\gamma,\beta_\sigma)})2\sqrt{\Delta(\gamma,\beta_\sigma)}}{4\gamma(1-\gamma)\sqrt{\Delta(\gamma,\beta_\sigma)}\beta_\sigma^2} \\
&= \frac{4\gamma(1-\gamma)\left[1-2\gamma+2\gamma(1-\gamma)\beta_\sigma\right]\beta_\sigma - 2\sqrt{\Delta(\gamma,\beta_\sigma)} - 2\Delta(\gamma,\beta_\sigma)}{4\gamma(1-\gamma)\sqrt{\Delta(\gamma,\beta_\sigma)}\beta_\sigma^2} \\
&= \frac{-2\sqrt{\Delta(\gamma,\beta_\sigma)} - 2 - 4\gamma(1-\gamma)(1-2\gamma)\beta_\sigma}{4\gamma(1-\gamma)\sqrt{\Delta(\gamma,\beta_\sigma)}\beta_\sigma^2}
\end{aligned} \tag{A.52}$$

where in the last step we simplified the first term of the numerator with the third term. The derivative (A.52) is always negative, in fact

$$\begin{aligned}
\frac{\partial \Gamma_+(\gamma, \beta_\sigma)}{\partial \beta_\sigma} < 0 &\iff -2\sqrt{\Delta(\gamma,\beta_\sigma)} - 2 - 4\gamma(1-\gamma)(1-2\gamma)\beta_\sigma < 0 \\
&\iff -\sqrt{\Delta(\gamma,\beta_\sigma)} < 1 + 2\gamma(1-\gamma)(1-2\gamma)\beta_\sigma \\
&\iff \Delta(\gamma,\beta_\sigma) > 1 + 4\gamma(1-\gamma)(1-2\gamma)\beta_\sigma + 4\gamma^2(1-\gamma)^2(1-2\gamma)^2\beta_\sigma^2 \\
&\iff 4\gamma^2(1-\gamma)^2\beta_\sigma^2 > 4\gamma^2(1-\gamma)^2(1-2\gamma)^2\beta_\sigma^2 \\
&\iff 1 > (1-2\gamma)^2
\end{aligned} \tag{A.53}$$

and the last inequality is satisfied for any $0 < \gamma < 1$. This means that $\Gamma_+(\gamma, \beta_\sigma)$ is a monotonically decreasing function of β_σ, which implies

$$\begin{aligned}
\Gamma_+(\gamma, 2\beta_\sigma) < \Gamma_+(\gamma, \beta_\sigma) &\iff \bar{c}_{f_1}\Gamma_+(\gamma, 2\beta_\sigma) < \bar{c}_{f_1}\Gamma_+(\gamma, \beta_\sigma) \\
&\iff \bar{c}_{f_2,NS} < \bar{c}_{f_2,PB}
\end{aligned} \tag{A.54}$$

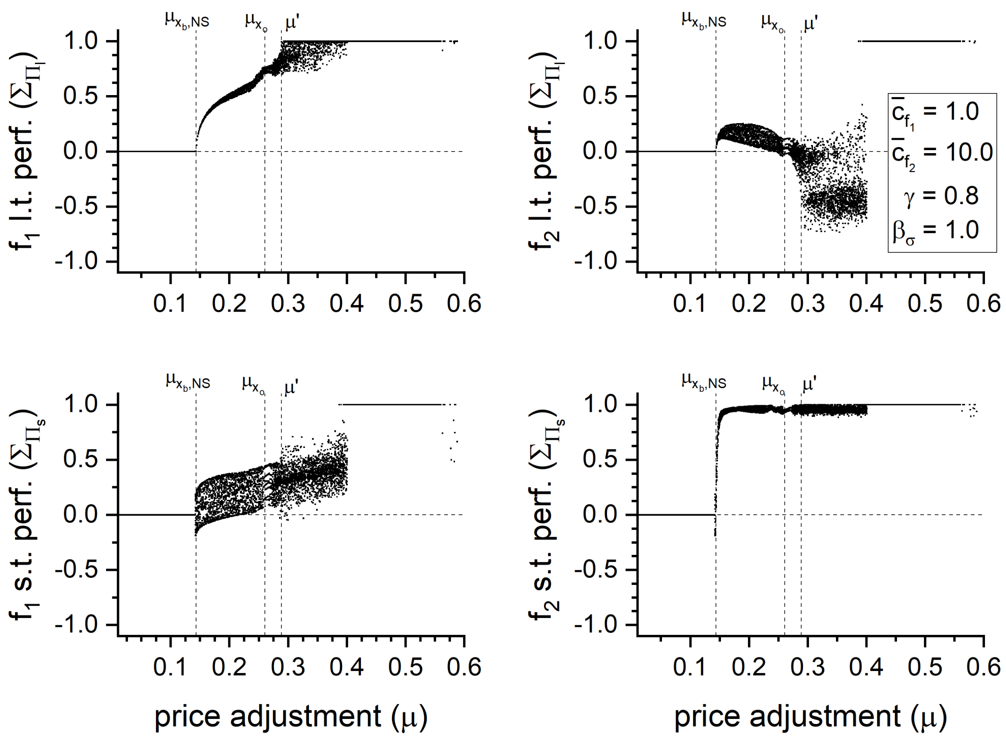


FIGURE A.5: Group f_i's (left panels for f_1 and right panels for f_2) long-term performance (top panels) and short-term performance (bottom panels) for $\mu \in [0.01, 0.6]$ and fixed $\gamma = 0.8$. As for $\gamma = 0.2$, for $\mu < \mu_{x_b,NS}$, groups' long-term and short-term performance equal to 0 (neither f_1 and $_2$ make any profit in this region) and once μ crosses $\mu_{x_b,NS}$ both f_1 and f_2 perform positively in the long term (f_1 better than f_2) and short term (f_2 better than f_1). As we increase the speed of price adjustment ($\mu < \mu'$), f_1 still performs better and better in the long-term while f_2's performance declines but at a slower rate than for $\gamma = 0.2$. Once the system undergoes the homoclinic bifurcation ($\mu > \mu'$), followers perform slightly better in the long-term when the degree of reliability is high thanks to f_m's delayed feedback, which leads f_2 trading in the same direction as f_1 who performs almost optimally.

A.2.7 Groups' performance when followers are *significantly* more dominant than pure fundamentalists and more reliable contrarians (Section 5.2.5.1)

In Figure A.5, we show group f_i's (left panels for f_1 and right panels for f_2) long-term performance (top panels) and short-term performance (bottom panels) for $\mu \in [0.01, 0.6]$ and fixed $\gamma = 0.8$ (Figure A.5) in steps of $\delta_\mu = 0.001$. To get these diagrams, we used the same method as for the bifurcation diagrams in Figure 5.5. For $\mu < \mu_{x_b,NS}$, we set the groups' long-term and short-term performance equal to 0, as neither f_1 and $_2$ make any profit in this region (both groups constantly send the same orders in the opposite direction). Once μ crosses $\mu_{x_b,NS}$, where the price starts fluctuating around a non-fundamental price, either above or below p^*, we see that both f_1 and f_2 perform positively in the long-term. However, this performance is fictitious as pure fundamentalists always send orders in the same direction, that is, selling orders when the price is above the fundamental value and buying orders when it is below it, so f_1 would either run out of supply or liquidity in the long-term. The reason why f_1 does not perform optimally in the long-term ($\Sigma_{\Pi^l,T_{test},f_1} < 1$) is because they send the same type of orders, either if the price deviation is above or below $\bar{x}$. However, they still manage to perform positively because the amount of orders sent differs depending on how far the price is from the fundamental value. For example, if the price fluctuates above the fundamental value (see left panels of Fig. and Fig.), the amount of selling orders is larger when the price deviation is above $\bar{x}$ and lower when it is below it, so they are good at timing the market in the long-term (and the same is true when the price fluctuates below p^*). Unlike f_1, f_2 sends both selling and buying orders when the price fluctuates above or below the fundamental value only. When the price fluctuates above p^* (see left panels of Fig. and Fig.), f_2 sends most of its buying orders when the price deviation is below $\bar{x}$ and selling orders when $x_t > \bar{x}$. Followers also perform positively in the long-term (right after $\mu > \mu_{x_b,NS}$), but less than pure fundamentalists ($\Sigma_{\Pi^l,T_{test},f_2} < \Sigma_{\Pi^l,T_{test},f_1}$) since they are worse at timing the market. In the short-term (see bottom panels in Figure 5.11 for fixed $\gamma = 0.2$ and Figure A.5 for fixed $\gamma = 0.8$), f_2 perform better than f_1 (and this is true for any value of μ), meaning that they send buy (sell) orders right before a price increase (decrease), although they are bad at taking profit of these moments, while pure fundamentalists try to correct any mispricing of the asset too early, by sending selling (buying) orders way before the price reached the top (bottom) of its cycle. As we increase the speed of price adjustment ($\mu < \mu'$), f_1 performs better and better in the long-term (either for fixed $\gamma = 0.2$ or $\gamma = 0.8$) since the price spends more and more time closer to the fundamental value while the time spent at the top or the bottom of its cycle is shorter. So, the fact that f_1 sends its orders way too early before the price reaches these extreme values has less and less negative impact on profits. f_2's long-term performance, instead, declines as μ increases. They send more and more buying (selling) orders when the price fluctuates above (below) the fundamental value and keep doing so even if the price deviation x_t is above (below) $\bar{x}$. They are slow to react to sudden price fluctuations, so as the price increases and decreases (decreases and increases) more rapidly when it reaches the top (bottom), they are worse at taking profits from these by selling right after $x_t > \bar{x}$ (the price fluctuates above p^*) or by buying when $x_t < \bar{x}$ (the price fluctuates below p^*). Once the system undergoes the homoclinic bifurcation ($\mu > \mu'$), we have that $\bar{x} \sim 0$ and the pure fundamentalists perform almost optimally in the long-term ($\Sigma_{\Pi^l,T_{test},f_1} \sim 1$), although they are still not good at timing the market in the short-term (unlike followers) since they send their orders as soon as the asset gets either overpriced or underpriced. We see that the followers perform negatively in the long term now, and the more trust they put in pure fundamentalists, the worse they do. The reason why the misinforming source (counterintuitively) helps followers to reduce their losses in the long and short term is due to delayed feedback when the price fluctuates back and forth the fundamental value for a time, as we saw before (see left panels of Figure 5.10), which lead f_2 to trade in the same direction as f_1.

Appendix B

Extended arguments

B.1 Discrete choice approach (Section 2.1.1.3)

We focus on $n_{h,t}$, that is, derive a rule that establishes how investors switch between multiple strategies over time. We will take a cue from Domencich and McFadden (1975) for that. To start with, one can assume, without restrictions, that an investor chooses a strategy instead of another depending on its performance with respect to all other possible options (the performance can depend on several factors, like profit after investment, agreement with other investors, and so on). So, if we define $u_{h,t-1}$ as the *performance function* for the strategy h at time $t-1$, we can say that an investor will prefer strategy h over all other strategies if

$$u_{h,t-1} > u_{l,t-1} \quad \text{for all} \quad l \neq h \quad \text{with} \quad l, h \in H. \tag{B.1}$$

This relation, however, implies that investors are fully rational and would choose only the strategy such that satisfies (B.1). Thus, we must include a factor representing an error in the investor's decision. This can be done by adding two terms in the performance function, generalized now as

$$\tilde{u}_{h,t-1} = \beta u_{h,t-1} + \varepsilon_{h,t-1} \tag{B.2}$$

where $\varepsilon_{h,t-1}$ is a stochastic term (following a certain probability distribution) related to agents' idiosyncrasies in choosing strategy h or other unobserved factors influencing investors' rational behaviour, and β is the *intensity of choice* parameter, whose function is to control the level of randomness given by $\varepsilon_{h,t-1}$. Because of that, choosing between different strategies occurs with some probability now, and so the relation (B.1) becomes

$$P_{h,t} = \text{Prob}(\tilde{u}_{h,t-1} > \tilde{u}_{l,t-1}) \quad \text{for all} \quad l \neq h \quad \text{with} \quad l = 1, ..., H) \tag{B.3}$$

where $P_{h,t}$ is the probability of choosing strategy h at time t and we write Prob(*condition*) is the probability that the *condition* will occur. We have implicitly assumed that an investor takes some time (one period) before deciding on the next strategy at time t. This last relation can be rewritten using equation (B.2), as

$$P_{h,t} = \text{Prob}(\beta(u_{h,t-1} - u_{l,t-1}) > \varepsilon_{l,t-1} - \varepsilon_{h,t-1} \quad \text{for all} \quad l \neq h \quad \text{with} \quad l = 1, ..., H). \tag{B.4}$$

Before showing the explicit form of equation (B.4), assuming a specific probability density function for the random variables $(\varepsilon_{1,t-1}, ..., \varepsilon_{H,t-1})$, we limit ourselves to the special case with only two strategies and then get the general one. In case $H = 2$, equation (B.4) becomes (for the probability of choosing the strategy '1' at time t)

$$P_{1,t} = \text{Prob}(\beta(u_{1,t-1} - u_{2,t-1}) > \varepsilon_{2,t-1} - \varepsilon_{1,t-1}) := F_X(u) \tag{B.5}$$

which is the definition of the cumulative distribution function (CDF) F of the random variable $X = \varepsilon_{2,t} - \varepsilon_{1,t}$ with argument $u = \beta(u_{1,t-1} - u_{2,t-1})$, defined as the integral

$$F_X(u) := \iint\limits_{x_2 - x_1 < u} f(x_1, x_2) dx_1 dx_2 \tag{B.6}$$

where $f(x_1, x_2)$ is the joint probability density function of $(\varepsilon_{1,t-1}, \varepsilon_{2,t-1})$. If we denote with $f_{\varepsilon_{1,t-1}}(x_1)$ and $f_{\varepsilon_{2,t-1}}(x_2)$ the probability density functions (PDF) corresponding to the random variables $\varepsilon_{1,t-1}$ and $\varepsilon_{2,t-1}$ respectively, and if we assume that the random variables are independent of each other, then we have

$$f(x_1, x_2) = f_{\varepsilon_{1,t-1}}(x_1) f_{\varepsilon_{2,t-1}}(x_2) \tag{B.7}$$

which allows us to write

$$
\begin{aligned}
P_{1,t} = F_X(u) &= \iint\limits_{x_2 - x_1 < u} f_{\varepsilon_{1,t-1}}(x_1) f_{\varepsilon_{2,t-1}}(x_2) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{u+x_1} f_{\varepsilon_{1,t-1}}(x_1) f_{\varepsilon_{2,t-1}}(x_2) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{u+x_1} f_{\varepsilon_{2,t-1}}(x_2) dx_2 \right] f_{\varepsilon_{1,t-1}}(x_1) dx_1 \\
&= \int_{-\infty}^{+\infty} F_{\varepsilon_{2,t-1}}(u + x_1) f_{\varepsilon_{1,t-1}}(x_1) dx_1 \\
&= \int_{-\infty}^{+\infty} F_{\varepsilon_{2,t-1}}(u + x_1) F'_{\varepsilon_{1,t-1}}(x_1) dx_1
\end{aligned}
\tag{B.8}
$$

where we used the relations

$$
F_{\varepsilon_{2,t-1}}(u + x_1) := \int_{-\infty}^{u+x_1} f_{\varepsilon_{2,t-1}}(x_2) dx_2 \tag{B.9}
$$

$$
\frac{dF_{\varepsilon_{1,t-1}}(x_1)}{dx_1} = F'_{\varepsilon_{1,t-1}}(x_1) := f_{\varepsilon_{1,t-1}}(x_1) \tag{B.10}
$$

with $F_{\varepsilon_{1,t-1}}$ and $F_{\varepsilon_{2,t-1}}$ the CDFs corresponding to the random variables $\varepsilon_{1,t-1}$ and $\varepsilon_{2,t-1}$ respectively. At this point, the only step left is to establish what CDF we have to use for the random variables $(\varepsilon_{1,t-1}, \varepsilon_{2,t-1})$. One might be tempted to use a simple normal distribution, but unfortunately, in this case, getting a closed form for $P_{1,t}$ is impossible. Another solution is to use a similar distribution, namely the Gumbel distribution, for which it is possible to have a closed form, with its CDF defined as

$$
F(u) := e^{-e^{-u}}. \tag{B.11}
$$

In fact, by using this distribution for both random variables $(\varepsilon_{1,t-1}, \varepsilon_{2,t-1})$(together with the equation (B.7), this makes them i.i.d.), we finally get, from equation (B.8),

$$
\begin{aligned}
P_{1,t} &= \int_{-\infty}^{+\infty} F_{\varepsilon_{2,t-1}}(u+x_1)F'_{\varepsilon_{1,t-1}}(x_1)dx_1 \\
&= \int_{-\infty}^{+\infty} e^{-e^{-(u+x_1)}}e^{-x_1}e^{-e^{-x_1}}dx_1 \\
&= \int_{-\infty}^{+\infty} e^{-x_1}e^{-e^{-(u+x_1)}-e^{-x_1}}dx_1 \\
&= \int_{-\infty}^{+\infty} e^{-x_1}e^{-e^{-x_1}(1+e^{-u})}dx_1 \\
&= \frac{1}{1+e^{-u}}\int_{-\infty}^{+\infty} e^{-x_1}(1+e^{-u})e^{-e^{-x_1}(1+e^{-u})}dx_1 \\
&= \frac{1}{1+e^{-u}}\left[e^{-e^{-x_1}(1+e^{-u})}\right]_{-\infty}^{+\infty} \\
&= \frac{1}{1+e^{-u}}
\end{aligned}
\tag{B.12}
$$

and given that $u = \beta(u_{1,t-1} - u_{2,t-1})$, we have

$$
P_{1,t} = \frac{1}{1+e^{-\beta(u_{1,t-1}-u_{2,t-1})}} \tag{B.13}
$$

that is, the *binary logit probability model* ($p_{2,t}$ has an equivalent form with '1' replaced by '2' and vice versa). From equation (B.13), if $\beta = 0$, then $P_{1t} = P_{2t} = 1/2$, meaning that a trader chooses a strategy completely at random, reflecting a fully irrational investor behaviour (an investor could also choose completely at random when $u_{1,t-1} - u_{2,t-1} = 0$, that is when both strategies have performed equally well or badly). For $\beta \to +\infty$, instead, one has complete rationality, where investors can evaluate $u_{h,t-1}$ without error and choose the strategy h for which the performance has the highest value. In this limit, $P_{h,t}$ takes only two values: 0 and 1. For the general case, when there are more than two options available to an investor, $P_{1,t}$ can be obtained similarly, with the only difference that the argument of the exponential in equation (B.13) includes a sum over all strategies H

$$
P_{1,t} = \frac{1}{1+\sum_{i=2}^{H} e^{-\beta(u_{1,t-1}-u_{i,t-1})}} \tag{B.14}
$$

and this is called the *multinomial logit model*. Equation (B.14) represents the probability for only one investor of choosing the strategy '1' over all other strategies. Since every investor behaves in the same way, as the number of traders tends to infinity, the probability (B.14) can be replaced by the fraction $n_{1,t}$ of investors choosing the strategy '1' at time t, thus, for a generic strategy 'h', we finally have

$$
n_{h,t} = \frac{1}{1+\sum_{i\neq h}^{H} e^{-\beta(u_{h,t-1}-u_{i,t-1})}}. \tag{B.15}
$$

B.2 Heterogeneous investors and networks (Section 2.1.3.4)

Here, we will attempt to develop a model of a market populated by two boundedly rational categories of investors, institutional and retail, who are heterogeneous in their beliefs and susceptible to social influence. Moreover, we want to represent the interaction among all traders through a social network, trying out different topologies for both these two categories of investors and possibly a network that connects them (so as to derive a very general form for a fraction of an adopted strategy n_{ht}).

The network

First, it is essential to distinguish two types of networks in financial markets: the *information* and the *behavioural* network. In the information network (for which we will try to derive $n_{h,t}$), the nodes represent investors, and the links are information channels, while in the behavioural one, the nodes are still investors, but the links are correlations on investors' behaviour (if two investors are buying/selling the asset similarly (e.g. same volume, order time), then there is a link between them). One can easily build the latter using the former. The motivation for using these two types of networks is due to data availability. It is impossible to directly check the links in an information network of a financial market (for privacy reasons), but one can study their effects on the behavioural one for which data is available. Baltakys (2019) performed a data analysis about shareholders' activities in securities listed in Nasdaq OMX Helsinki Exchange (a period from 1995 to 2017), building the investor networks in terms of their trading behaviour and information transfer. Based on his study, Baltakys identified five groups of investors (non-financial corporations, financial and insurance corporations, general governmental organizations, non-profit organizations, and households) which might interact with each other in several ways. For example, we could think of a household network with word-of-mouth-based information transfer channels in which social interactions influence financial decisions more than rational performance optimization (the opposite in the case of a financial corporation network). This observation suggests the idea of a financial market with disconnected networks, where different 'social' rules are applied. Our model will be based on this framework. If each of these networks G, basic configurations of the system that we call "financial circles", has its own $n_{h,t}^{(G)}$ (with $\sum_h n_{h,t}^{(G)} = 1$), then $n_{h,t}$ is simply given by

$$n_{h,t} = \sum_G n^{(G)} n_{h,t}^{(G)} \tag{B.16}$$

where $n^{(G)}$ is the fraction of G-type networks that make up the system, with $\sum_G n^{(G)} = 1$. If we assume there exists a one-to-one correspondence between the information and behavioural network, the fraction of G-type networks can be used as a comparison parameter between the real data and the model.

Directed hierarchical scale-free network

We will start by considering one plausible investor network structure: a directed hierarchical scale-free network. In this structure, the hubs represent financial advisors, and the directed link indicates how information flows. A typical hierarchical scale-free network is shown in Figure B.1. The central nodes in the fully connected network are hubs and do not interact with other hubs. They influence their neighbourhood (a bi-directional network) and act regardless of their state. The fully connected networks are a group of investors who work together (they share their strategies) and agree to get advice from the same advisor (they might have more than one). We write the probability $P^i_{h,t}$ for an investor i to adopt the strategy h at time t as

$$P^i_{h,t} = P^{i,FC}_{h,t} f(\alpha^2) + P^{i,HB}_{h,t} g(\alpha^2) \tag{B.17}$$

where $P^{i,FC}_{h,t}$ is the probability of choosing strategy h related to the social group and investor's i opinion, $P^{i,HB}_{h,t}$ is the probability of adopting hub's strategy, $f(\alpha^2)$ and $g(\alpha^2)$ are weight functions which depends on a degree of confidence α in investor's i financial advisor, and are such that

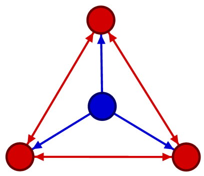
$$f(\alpha^2) + g(\alpha^2) = 1 \tag{B.18}$$

with $0 < f(\alpha^2) < 1$ and $0 < g(\alpha^2) < 1$. In our case, we assume that the fully connected network is more important for an investor (her social group and personal opinion) when α is low, and hubs (financial advisors) are more critical when α is high. So, we can write the weight functions as

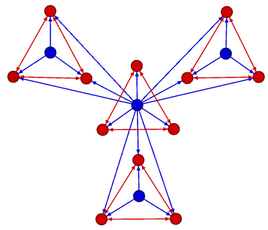
$$f(\alpha^2) = \frac{1}{1 + (\frac{\alpha}{\alpha_0})^2} \tag{B.19}$$

$$g(\alpha^2) = 1 - \frac{1}{1 + (\frac{\alpha}{\alpha_0})^2} \tag{B.20}$$

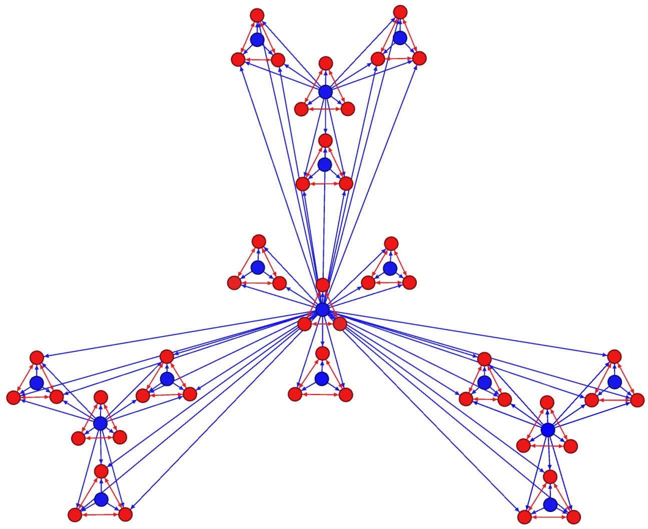
where α_0 is a scale parameter for which, if $\alpha = \alpha_0$, the investor gives equal importance to the fully connected network and hubs. We first focus on $P^{i,FC}_{h,t}$. We follow a similar approach as Chang (2007), where the investor's choice is influenced by a mix of her confidence in evaluating correctly the past realized profit and her propensity to imitate her social group. Thus, we write $P^{i,FC}_{h,t}$ using the multinomial logit model



(A) $n = 0$, $N = 4$



(B) $n = 1$, $N = 16$



(C) $n = 2$, $N = 64$

FIGURE B.1: Construction of a directed hierarchical network (4-node case). The red nodes represent investors who share their strategy with agents within their social group, and the blue nodes are financial advisors. The direction of the edges shows how the information flows (the colour distinguishes the type of information: red for social advice, blue for financial advice). Starting from the network (a)(iteration $n = 0$), we create three copies (generally, the same number of people in the social group) and connect all the nodes of the new social groups (therefore, excluding the new hubs) to the hub of (a) to obtain (b)(iteration $n = 1$). The same procedure can be continued to obtain (c) and larger networks.

$$P_{h,t}^{i,FC} = \frac{e^{\beta u_{h,t-1}^{i} + \gamma s_{h,t-1}^{i}}}{\sum_{l}^{H} e^{\beta u_{l,t-1}^{i} + \gamma s_{l,t-1}^{i}}} \tag{B.21}$$

where β is the investor's personal confidence (or intensity of choice, as in Brock and Hommes (1998), even though it sounds too generic now), γ her propensity to herding, $u_{h,t-1}^{i}$ her past realized profit if she had chosen strategy h at time $t-1$ (H is the total number of strategies present in the market), and $s_{h,t-1}^{i}$ evaluates the group's average agreement to adopt strategy h at time $t-1$, given by

$$s_{h,t-1}^{i} = \frac{1}{n-1} \sum_{j \in G} s_j \tag{B.22}$$

where n is the number of people in the group G and s_j is such that, if investor j adopts strategy h, then $s_j = 1$, otherwise $s_j = -1$. The next step is $P_{h,t}^{i,Hb}$ and this time we try to give a reasonable interpretation by ourselves. We can think of investors as being influenced by financial advisors with an intensity proportional to the number of their connections (the higher the degree of a hub, the more valuable her opinion will be for an investor). Moreover, we impose that only the strategy will be shared. We write $P_{h,t}^{i,HB}$ as a sum over all i's hubs, given by

$$P_{h,t}^{i,HB} = \sum_{H'}^{HUBS} h(k_{H'}) \frac{e^{\delta u_{h,t-1}^{H'}}}{\sum_{l}^{H} e^{\delta u_{l,t-1}^{H'}}} \tag{B.23}$$

where $h(k_{H'})$ is an increasing function of H''s outdegree $k_{H'}$ and δ is the intensity of choice. Financial advisors are not influenced by any other person and adopt their strategy based on their past performance. We choose $h(k_{H'})$ as

$$h(k_{H'}) = \frac{k_{H'}^{\zeta}}{\sum_{H''}^{HUBS} k_{H''}^{\zeta}} \tag{B.24}$$

with $0 \leq \zeta \leq 1$, in this way, a hub's importance is given by its relative proportion to the other hubs. An important note regards the probability of choosing a strategy for financial advisors. Since they are not within a social group and do not get advice from any other financial advisor, we have simply:

$$P_{h,t}^{i,FC} = P_{h,t}^{i,HB} = \frac{e^{\delta u_{h,t-1}^{i}}}{\sum_{l}^{H} e^{\delta u_{l,t-1}^{i}}} \tag{B.25}$$

and (B.17) becomes

$$P_{h,t}^{i} = \frac{e^{\delta u_{h,t-1}^{i}}}{\sum_{l}^{H} e^{\delta u_{l,t-1}^{i}}}. \tag{B.26}$$

We now assume there are only two strategies, fundamentalist (strategy '1') and chartist (strategy '2'), and before obtaining an analytical form for the evolution of the fractions $n_{h,t}$, we will calculate it step by step by performing some simulations.

Simulation

Preliminary tests

Before studying our model, we will perform some preliminary tests by setting the parameters to obtain the Brock and Hommes (BH) model (that is, setting $\alpha = 0$ and $\gamma = 0$ for every agent i and time t) and run some simulations. Just as a reminder, we show again the adaptive belief system (ABS) of the BH model (2.49)

$$\begin{cases} x_t = \frac{m}{R} n_{2,t} x_{t-1} \\ n_{2,t} = \frac{1}{1+e^{-\beta[\Delta u_{t-1}]}} \\ \Delta u_t = m(x_t - R x_{t-1}) x_{t-2} + c \end{cases} \tag{B.27}$$

where x_t is the deviation of the price from the fundamental value p^* at time t, $n_{2,t}$ is the fraction of chartists at time t, m is the chartists' *trend* parameter, R is the gross return of the risk-free asset and c is the cost for adopting the fundamentalist strategy. The parameters will be chosen as those used by BH to perform their numerical analysis ($m = 1.2$, $c = 1$, and $R = 1.1$), expressing the results in terms of x_t and difference in fractions ($\Delta n_t = n_{1,t} - n_{2,t}$).

BH model vs. our model

Figure B.2 shows the comparison between the BH and our model for different values of β in the phase space $\Delta n_t - x_t$ (we have chosen $x_0 > 0$). As we can see, they do not match perfectly, especially for $\beta = 3.5$, and this is due to two reasons: (1) each agent in our model chooses a strategy in a probabilistic way, while in the BH standard model, the fraction of people choosing a strategy evolves deterministically; (2) our model deals with a finite number of investors ($N \approx 17,000$ in our simulations), while in BH model $N \to +\infty$ (this last point is shown in Fig.Figure B.3). Fig.Figure B.4 illustrates the bifurcation diagrams for the deviation x (that is, the range of values x_t taken in a single simulation) with respect to β using the BH model and ours. These have been built by collecting only the values x_t (with $x_0 > 0$) once the system converged to a steady state, that is, considering only values after a time T_c (transition time) and within a period T, which must be long enough so that x_t takes (almost) all allowed values. Here, again, there are some slight differences between our model and BH, especially for values of β within the first and second bifurcation, for the same aforementioned reasons.

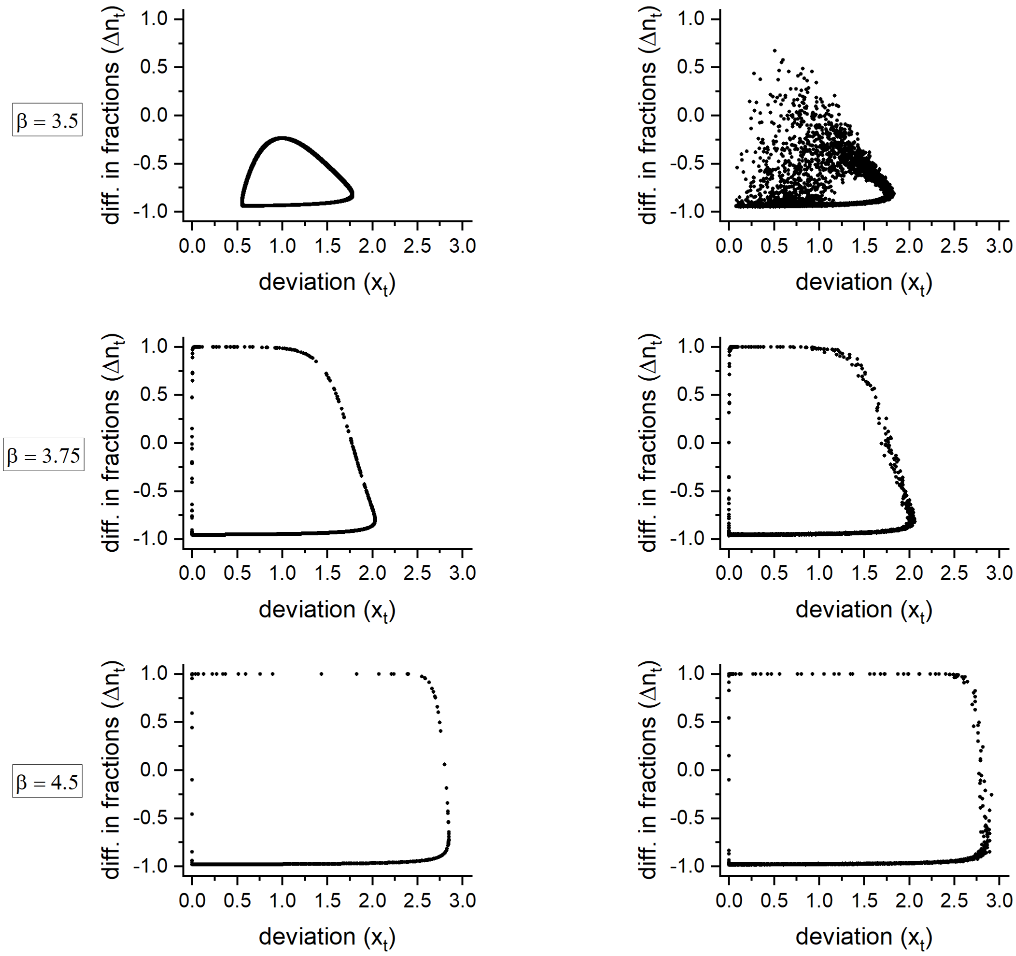


FIGURE B.2: The phase space $\Delta n_t - x_t$, for different values of β with the standard BH model (left) and our model(right).

The effect of social group (γ)

The first parameter we will be testing is the herding propensity γ. Figure B.5 shows the difference in fractions when $\beta = 0$ and γ vary (the price tends to the fundamental value similarly for all γ). For this simulation, we used a network with social groups made up of 5 investors, iterated four times, so the number of agents in our system is equal to $N = (5+1)^{(4+1)} = 7776$. As γ increases, the system goes through different states, starting from random change ($\gamma = 0, 1$) in strategy to a more regular trend ($\gamma = 2, 3$), and ending up with, again, a random behaviour($\gamma = 4, 5$), but with a lower variance compared to γ close to 0. The graphs in Figure B.6 give more information to understand what happens to the system as γ varies. These are diagrams in which each

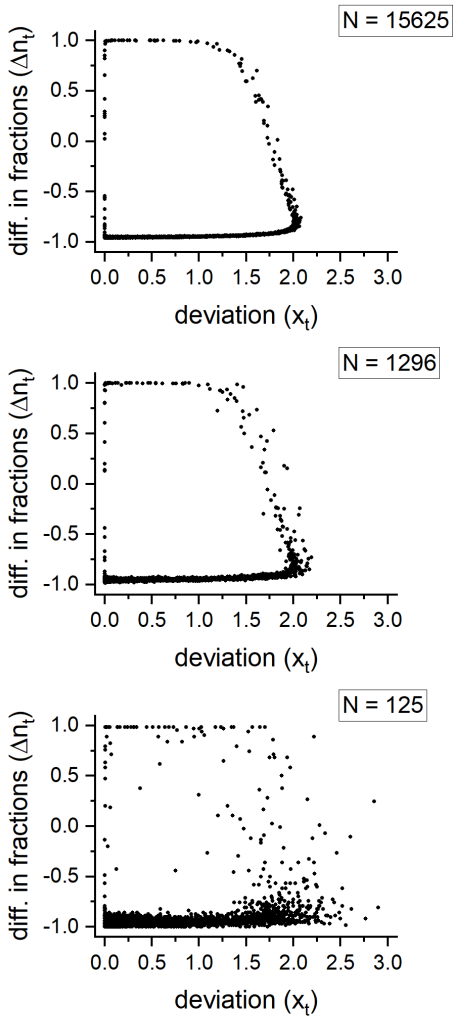


FIGURE B.3: The phase space $\Delta n_t - x_t$, for $\beta = 3.75$ and different number N of investors in our model.

simulation had a total number T of time steps equal to $T = 2000$, and the first $T_c = 1000$ steps have been neglected. The left panel, in Fig.Figure B.6, shows how the average conformity(the conformity in each social group in having the same strategy) increases as herding propensity is higher. For $\gamma = 5$, the average conformity is close to 1, meaning that investors are either all chartists or fundamentalists in each social group. The right panel, in contrast, represents the frequency of changing strategy (per simulation and investor(within a social group!)) as γ varies (one can also interpret it as the probability for an investor of changing her strategy within the time interval $\Delta = T - T_c$), and for $\gamma = 5$, almost no investor changes her opinion during a simulation. Nonetheless, the bottom-right panel in Fig.Figure B.5 shows something different from what one would expect (a straight line since investors should not change their strategy). The reason why the system exhibits random time series in strategy fractions is simply due to financial advisors who keep choosing randomly

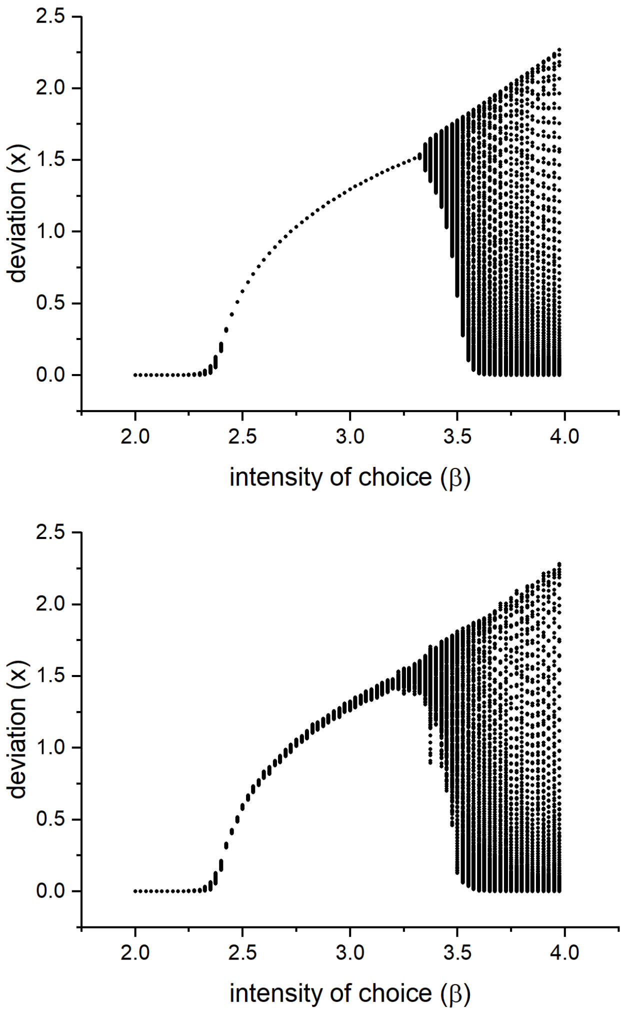


FIGURE B.4: Bifurcation diagram w.r.t β with the standard BH model (top) and our model(bottom). The number of time steps for each simulation is $T = 1000$, of which the first $T_c = 400$ time steps have been excluded in order to remove the initial transition(the system has not reached the steady state yet).

since they do not belong to any social group.

Dimensionality reduction

Intensity of choice(β) vs. Herding propensity(γ)

Now we study what happens when both the investor's rationality and trust in her neighbourhood are involved. We will attempt to derive the analytical form of our model, that is to say, obtain the closed form of the chartists' (or fundamentalists') fraction in time $n_{2,t}$ (or $n_{1,t}$) when the number of investors tends to infinity ($N \to \infty$), being the system made up of financial advisors and those who belong to social groups (social investors). First of all, we note that $\alpha = 0$, so these two types of investors are

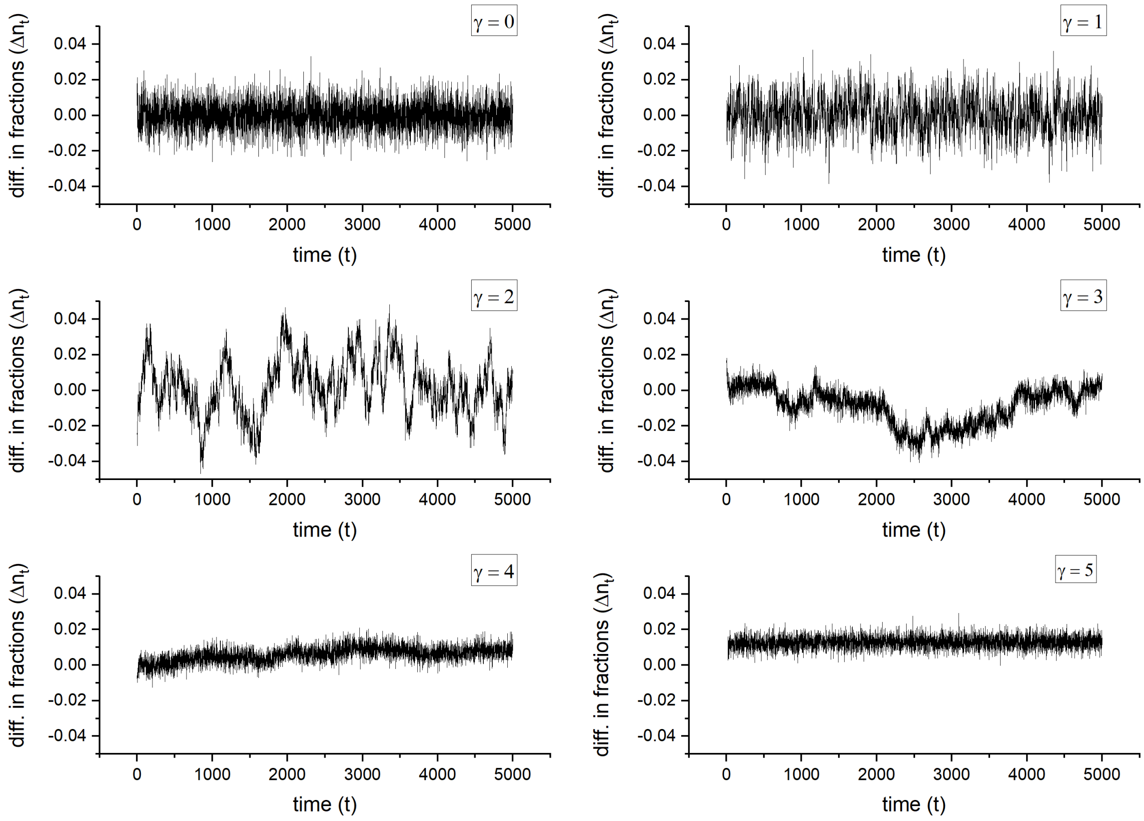


FIGURE B.5: The difference in fractions Δn_t for different values of γ.

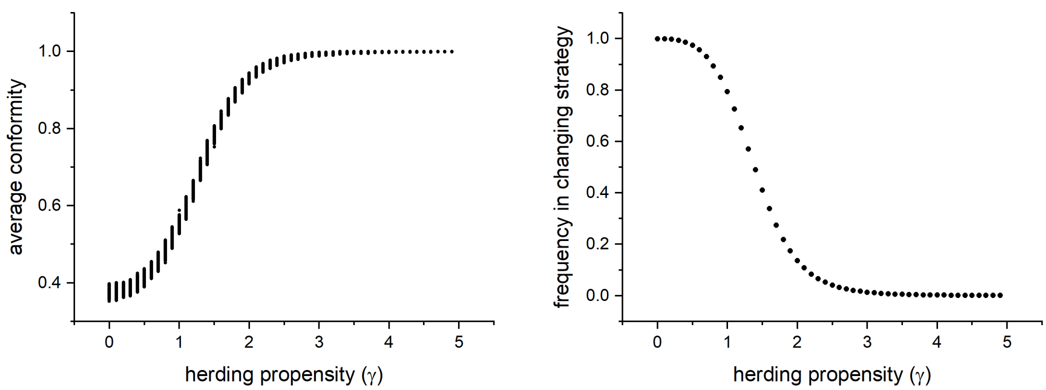


FIGURE B.6: The average conformity and the frequency in changing strategy (per simulation/investor) as γ varies.

disconnected from each other, which means that their corresponding fraction of chartists in time ($n_{2,t}^{HB}$ and $n_{2,t}^{FC}$) do not directly depend on each other, and we can derive their form separately. Secondly, there are three possibilities for this network by which one can tend the number of investors to infinity: (1) keeping fixed the number n of social investors within a group and iterating the procedure, as in Fig.Figure B.1, indefinitely; (2) keeping fixed the number it of iterations and $n \to \infty$; (3) both $n \to \infty$ and $it \to \infty$. We will assume, for now, the first case. Lastly, the number of financial advisors N_{HB} and social investors N_{FC} in our network is given by

$$N_{HB} = (n+1)^{it} \tag{B.28}$$

$$N_{FC} = n(n+1)^{it} \tag{B.29}$$

so their relative fractions in the network are

$$n_{HB} = \frac{N_{HB}}{N} = \frac{1}{n+1} \tag{B.30}$$

$$n_{FC} = \frac{N_{FC}}{N} = \frac{n}{n+1} \tag{B.31}$$

where $N = (n+1)^{it+1}$ is the total number of investors. Given that, $n_{2,t}$ depends linearly on $n_{2,t}^{HB}$ and $n_{2,t}^{FC}$ (equation (B.16)), weighted with coefficients given by equations, respectively, (B.30) and (B.31), the adaptive belief system becomes

$$\begin{cases} x_t = \frac{m}{R} n_{2,t} x_{t-1} \\ n_{2,t} = n_{2,t}^{HB} n_H + n_{2,t}^{FC} n_{FC} = n_{2,t}^{HB} \frac{1}{n+1} + n_{2,t}^{FC} \frac{n}{n+1} \\ n_{2,t}^{HB} = ... \\ n_{2,t}^{FC} = ... \end{cases} \tag{B.32}$$

with the functions $n_{2,t}^{HB}$ and $n_{2,t}^{FC}$ yet to be found (as a reminder, the first equation of this system describes the deviation of the price from the fundamental value in time, discussed in the previous chapter). Let us begin with $n_{2,t}^{HB}$. The probability for a financial advisor to become a chartist at time t is given by equation (B.26) with $h = 2$ (we chose, as well as in the previous sections, $\delta = \beta$ to simplify the calculations), and when $N \to \infty$ one can apply the law of large numbers (the strategies adopted by each investor are independent random variables and identically distributed (i.i.d.)) and get the form

$$n_{2,t}^{HB} = \frac{e^{\beta u_{2,t-1}}}{e^{\beta u_{2,t-1}} + e^{\beta u_{1,t-1}}} \tag{B.33}$$

The next step is deriving $n_{2,t}^{FC}$ from equation (B.21). The difference with (B.26) is for an extra term in the argument of the exponential, the social utility $s_{h,t-1}^{i}$, given by the equation (B.22). Considering the case with 2 strategies, $s_{h,t-1}^{i}$ can be rewritten as

$$s_{2,t-1}^{i} = (n_{2,t-1}^{i} - n_{1,t-1}^{i}) \tag{B.34}$$

$$s_{1,t-1}^{i} = (n_{1,t-1}^{i} - n_{2,t-1}^{i}) \tag{B.35}$$

where $n_{2,t-1}^{i}(n_{1,t-1}^{i})$ is the fraction of chartists(fundamentalists) in the group G from investor's i perspective at time $t-1$. Thus, we can write equation (B.21) as

$$P_{2,t}^{i,FC} = \frac{e^{\beta u_{2,t-1}^{i} + \gamma(n_{2,t-1}^{i} - n_{1,t-1}^{i})}}{e^{\beta u_{2,t-1}^{i} + \gamma(n_{2,t-1}^{i} - n_{1,t-1}^{i})} + e^{\beta u_{1,t-1}^{i} + \gamma(n_{1,t-1}^{i} - n_{2,t-1}^{i})}} \tag{B.36}$$

$$P_{1,t}^{i,FC} = \frac{e^{\beta u_{1,t-1}^{i} + \gamma(n_{1,t-1}^{i} - n_{2,t-1}^{i})}}{e^{\beta U_{2,t-1}^{i} + \gamma(n_{2,t-1}^{i} - n_{2,t-1}^{i})} + e^{\beta u_{1,t-1}^{i} + \gamma(n_{1,t-1}^{i} - n_{2,t-1}^{i})}} \tag{B.37}$$

Differently from equation (B.33), we can not directly use the law of large numbers on equation (B.36) because of the social utility $s_{h,t-1}^{i}$, which can assume different values for each investor i within the same social group at time $t-1$, based on the group's configuration (or, similarly, the strategies adopted by each investor are random variables not identically distributed) and each social group in the network might have different configurations at time $t-1$. Therefore, we have to analyse each possible situation for a fixed number n of investors that make up a social group. We show the simplest case when $n=2$. Here, the probability $P_{2,t}^{i,FC}$ for each social investor in the network can take only two forms, $P_{2,t}^{i,FC,+}$ or $P_{2,t}^{i,FC,-}$, which correspond, respectively, to the case where the investor in the same social group of investor i was a chartist at time $t-1$ ($n_{2,t-1}^{i} - n_{1,t-1}^{i} = 1$) or a fundamentalist ($n_{2,t-1}^{i} - n_{1,t-1}^{i} = -1$), that is (see equation (B.36))

$$P_{2,t}^{i,FC,+} = \frac{e^{\beta u_{2,t-1}^{i} + \gamma}}{e^{\beta u_{2,t-1}^{i} + \gamma} + e^{\beta u_{1,t-1}^{i} - \gamma}} \tag{B.38}$$

$$P_{2,t}^{i,FC,-} = \frac{e^{\beta u_{2,t-1}^{i} - \gamma}}{e^{\beta u_{2,t-1}^{i} - \gamma} + e^{\beta u_{1,t-1}^{i} + \gamma}} \tag{B.39}$$

Whether an investor follows one probability than another depends on the group's configuration at time $t-1$, which can be of type $(2,2)$, $(2,1)$, $(1,2)$ and $(1,1)$ (1=fundamentalist, 2=chartist). Since social groups are disconnected from each other, for a large number of social groups the configurations follow a binomial distribution (for example, if at time $t-1$ the probability of adopting the chartist position for a generic social investor is equal to $P_{2,t-1}^{i,FC}$, the probability he has to end up in a $(2,2)$ configuration is equal to $(P_{2,t-1}^{i,FC})^2$, for a $(2,1)$ configuration is equal to $(P_{2,t-1}^{i,FC})(P_{1,t-1}^{i,FC})$, and so on), thus, the probability $P_{2,t}^{i,FC}$ is given by

$$P_{2,t}^{i,FC} = [(P_{2,t-1}^{i,FC})^2 + P_{2,t-1}^{i,FC} P_{1,t-1}^{i,FC}] P_{2,t}^{i,FC,+} + [P_{1,t-1}^{i,FC} P_{2,t-1}^{i,FC} + (P_{1,t-1}^{i,FC})^2] P_{2,t}^{i,FC,-} \tag{B.40}$$

The coefficients multiplying $P_{2,t}^{i,FC,+}$ and $P_{2,t}^{i,FC,-}$ in the previous equation are the sum of probabilities for an investor to face the situation when her neighbour had a chartist and fundamentalist position at time $t-1$ ($P_{2,t-1}^{i,FC} + P_{1,t-1}^{i,FC} = 1$). Now, we can apply the law of large numbers to obtain the fraction of chartists for the social groups at time t as

$$n_{2,t}^{FC} = [(n_{1,t-1}^{FC})^2 + n_{2,t-1}^{FC} n_{1,t-1}^{FC}] n_{2,t}^{FC,+} + [n_{1,t-1}^{FC} n_{2,t-1}^{FC} + (n_{1,t-1}^{FC})^2] n_{2,t}^{FC,-} \tag{B.41}$$

where $n_{2,t}^{FC,+}$ and $n_{2,t}^{FC,-}$ are given by

$$n_{2,t}^{FC,+} = \frac{e^{\beta u_{2,t-1}+\gamma}}{e^{\beta u_{2,t-1}+\gamma} + e^{\beta u_{1,t-1}-\gamma}} \tag{B.42}$$

$$n_{2,t}^{FC,-} = \frac{e^{\beta u_{2,t-1}-\gamma}}{e^{\beta u_{2,t-1}-\gamma} + e^{\beta u_{1,t-1}+\gamma}} \tag{B.43}$$

One can actually extend the previous calculation for a generic number n and get the general formula for $n_{2,t}^{FC}$

$$n_{2,t}^{FC} = \sum_{k=0}^{n-1} Prob(n-1, k, n_{2,t-1}^{FC}) \left[\frac{e^{\beta u_{2,t-1}+\gamma\left(\frac{2k}{n-1}-1\right)}}{e^{\beta u_{2,t-1}+\gamma\left(\frac{2k}{n-1}-1\right)} + e^{\beta u_{1,t-1}-\gamma\left(\frac{2k}{n-1}-1\right)}} \right] \tag{B.44}$$

where $Prob(n-1, k, n_{2,t-1}^{FC})$ is the probability mass function of the binomial distribution, that is

$$Prob(n-1, k, n_{2,t-1}^{FC}) = \binom{n-1}{k} (n_{2,t-1}^{FC})^k (1 - n_{2,t-1}^{FC})^{n-1-k} \tag{B.45}$$

that is the probability of having k chartists within a group of $n-1$ investors at time $t-1$ if the fraction of chartists among social investors was $n_{2,t-1}^{FC}$. Notice that the fraction $n_{2,t}^{FC}$ in equation (B.44) does not depend on the possible configurations for groups of n investors but on the possible configurations from the perspective of one investor, who keeps track of $n-1$ investors. Finally, the system (B.32) with equations (B.33) and (B.44) describes a market with financial advisors and social investors (disconnected from each other) in a hierarchical scale-free network (the scale-free property does not have an effect here because $\alpha = 0$). We now want to compare the model with a finite number of investors and the system (B.32), when $n = 2, 3, 4, 5$. To do that, we will use heat maps, with the x-axis and y-axis represented, respectively, by β and γ, and the map colours the intensities of the values assumed by a specific variable in each simulation. These maps are conceptually similar to bifurcation diagrams in that the variables under consideration are related to steady states. For this purpose, to detect whether the system reached or less its stability, we will keep track, after a transition time T_c, of the highest and lowest value taken by the deviation x_t,

n	it	N	N_{sg}	N_{conf}
2	7	6561	2187	4
3	6	16384	4096	8
4	5	15625	3125	16
5	4	7776	1296	32

TABLE B.1: The table shows the number of investors within a social group (n), the number of iterations (it), the total number of investors (N), the total number of social groups ($N_{sg} = N_{FC}/n = (n+1)^{it}$)and the number of possible configurations (fundamentalists/chartists) for each group ($N_{conf} = 2^n$)

x_{max} and x_{min}, during a simulation and start recording once these values stop changing (we set a time counter that restarts if x_{max} and x_{min} should vary and stops the simulation once reached a time T_{stop}). This procedure is repeated while both β and γ are being varied. We will start by considering the mean of the deviation $\bar{x}$, defined as

$$\bar{x} = \frac{x_{max} + x_{min}}{2} \tag{B.46}$$

and its range $\sigma_{\bar{x}}$, given by

$$\sigma_{\bar{x}} = \frac{x_{max} - x_{min}}{2} \tag{B.47}$$

which are good enough to highlight the areas of the map where the system behaves differently. Moreover, since the number of investors depends on the number of times the procedure to create the network has been iterated (see equations (B.28) and (B.29)), we will choose the highest value of iterations (it) allowed by our program in order to get the highest number of investors (capped at $N_{MAX} \sim 17000$). In table Figure B.1 are listed some details of our simulations that will help us with some observations. In Figs.Figure B.7, Figure B.8, Figure B.9 and Figure B.10, are shown three maps for each comparison(remember that $m = 1.2$, $c = 1$, and $R = 1.1$): one for $\bar{x}$ (equation (B.46)), $\sigma_{\bar{x}}$ (equation (B.47)) and the total steps T_{tot} needed to reach the steady state. As we can see, the maps with infinite and finite number of investors differ as n increases, giving a good match only for $n = 2, 3$ with some slight differences. The reason for that is most likely due to the low number of social groups present in the simulations so that the possible configurations are no more binomially distributed. In fact, if we take the ratio between the number of social groups (N_{sg}) and the number of possible configurations (N_{conf}) for each case n (table B.1) we see that it decreases as n increases, which highlights a number of 'trials' statistically insufficient to be described by a binomial distribution.

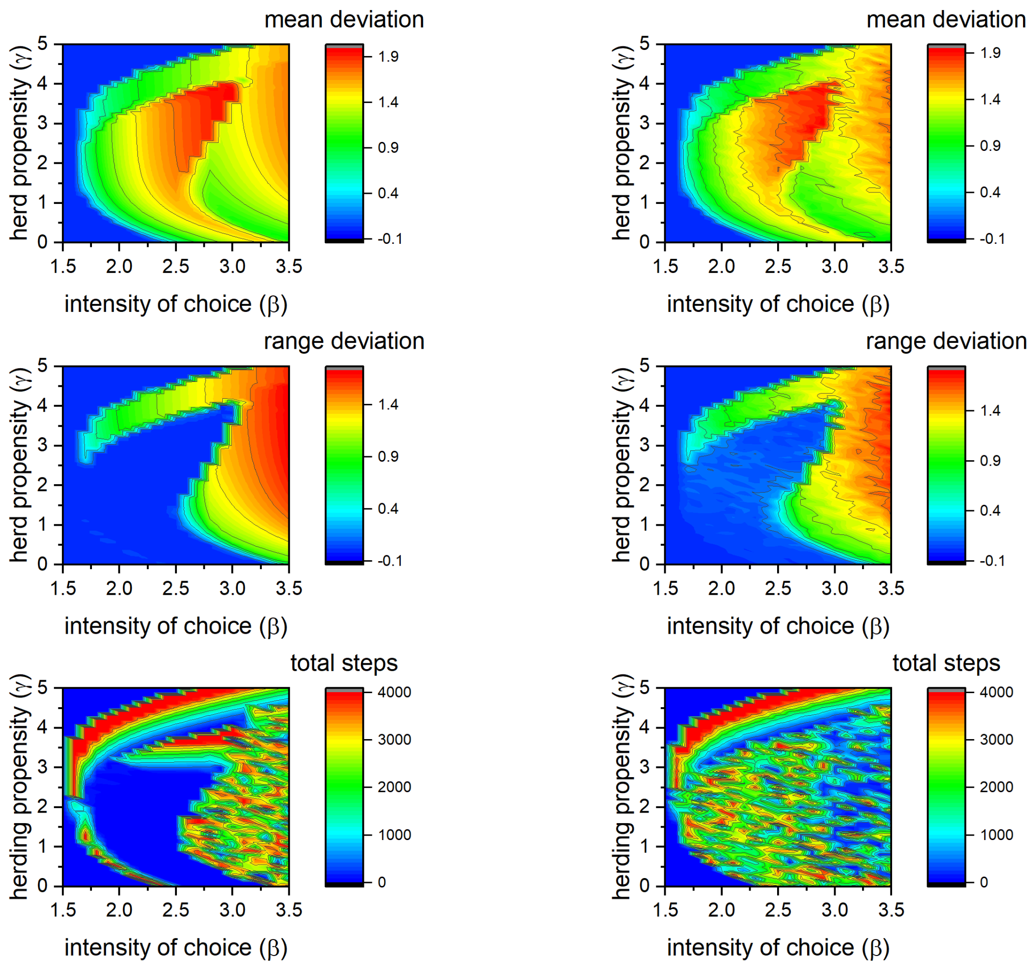


FIGURE B.7: Comparison of heatmaps for $n = 2$ (left maps obtained with system (B.32), right maps with limited number of investors)

B.3 Limited supplies/holdings and bounded beliefs (Section 3.3)

So, as seen in section 3.2, depending on the values taken by the parameters and the initial conditions, the price can either converge to the fundamental value, oscillate periodically (with period 2) between two prices, or diverge to infinity. Clearly, the last outcome is unrealistic, and we already explained that the main reason why this happens is because of investors' unlimited liquidity and supply for trading the asset. In addition to this, there is another feature of our model that leads the system to exhibit such a phenomenon: investors' unbounded price belief about the asset. In our analysis, we implicitly assume (as is implicit in the Brock-Hommes model and in any HAMs) that the difference between the asset fundamental value p^* (investors' price belief for the asset) and the asset current price p_t could take any value, such that

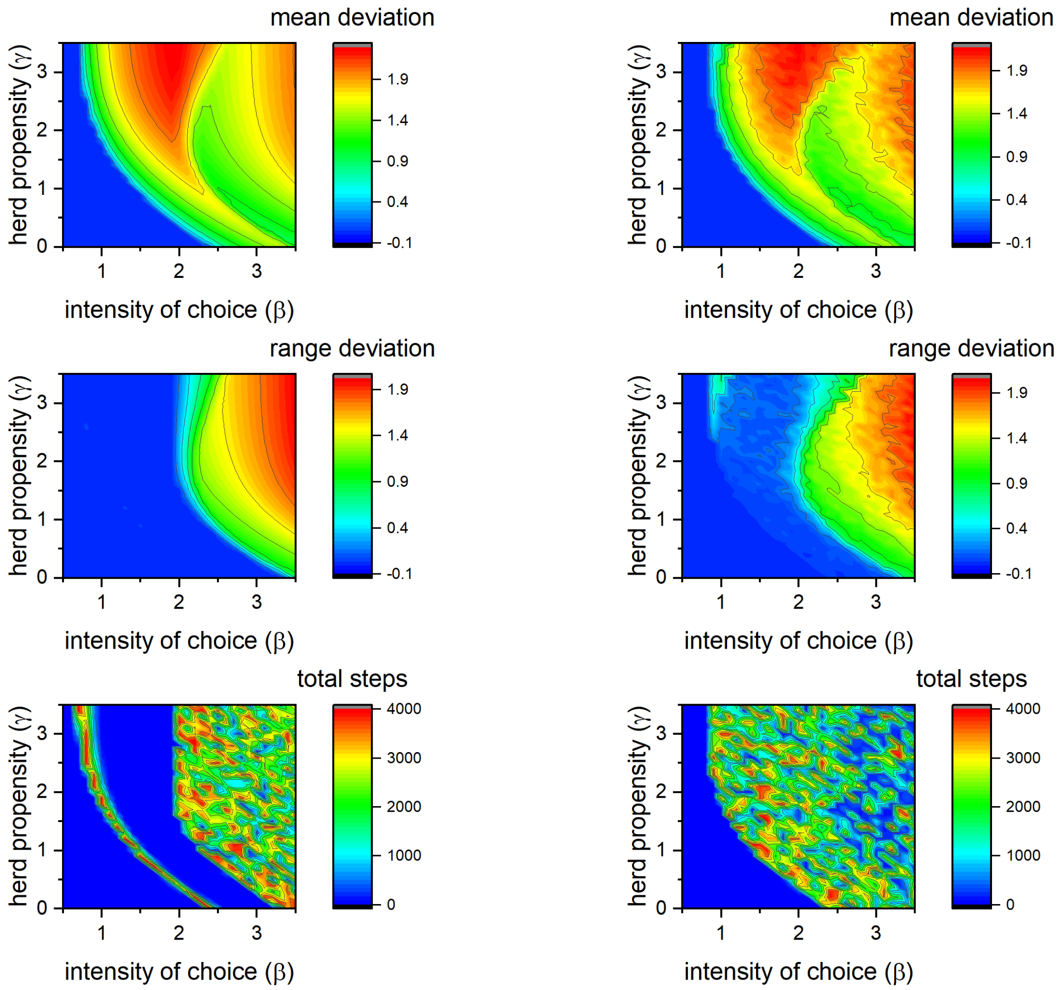


FIGURE B.8: Comparison of heatmaps for $n = 3$ (left maps obtained with system (B.32), right maps with limited number of investors)

$$p_t - p^* = x_t \in (-\infty, +\infty) \tag{B.48}$$

and, in order for this difference to diverge to $\pm\infty$, we must have

$$x_t \to \pm\infty \implies \quad either \quad p^* \to \mp\infty \quad or \quad p_t \to \pm\infty \tag{B.49}$$

However, any investor knows there is a limit on how much the price can drop (it can not go below 0!) or rise, and the same goes for the fundamental value. Therefore, to be more accurate, one should assume the existence of two extremes, $x_{MIN} = x_M$, for which investors would be willing to sell all of their assets, and $x_{MAX} = -x_M$, at which they would be willing to invest all of their liquidity for buying the asset (for simplicity, we set them equally distant from the fundamental value and equal for all investors, since they are fundamentalists). The probability function for investors of trading the asset should be such that

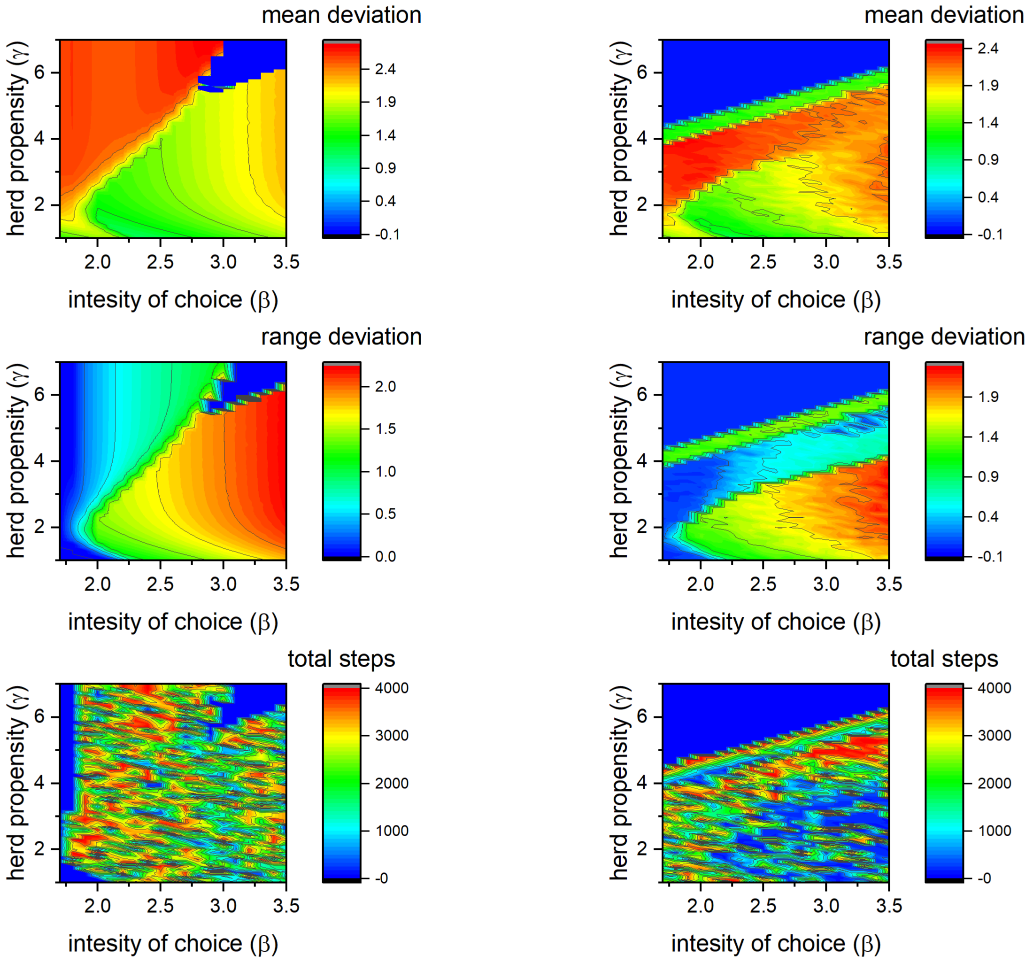


FIGURE B.9: Comparison of heatmaps for $n = 4$ (left maps obtained with system (B.32), right maps with limited number of investors)

$$P(-x_M) = P(x_M) = 1 \quad \textit{and} \quad P(x_{f,t} = 0 \iff p_t = p^*) = 0 \tag{B.50}$$

whose curve shape over the interval $I = [-x_M, x_M]$ would be similar to that of equation (3.35), which is the probability derived from the discrete choice model (i.e. binary logit model). A function that satisfies these conditions is a trigonometric one, that is,

$$P^b_{f,t}(x_{f,t}) = \sin^2\left(\frac{\pi}{2}\frac{x_{f,t}}{x_M}\right) \tag{B.51}$$

where for $x_{f,t} = \pm x_M$ the probability takes value $P^b_{f,t}(\pm x_M) = 1$ (from now on, we use the notation 'b' to denote *bounded price beliefs* and 'ub' for *unbounded price beliefs*). In Figure B.11, we show a comparison between this probability function and the one used in our model (3.35). We set $\beta_f = 2$ and $\Delta_f = (x_M/2)^2$, so that both probabilities are equal to $P^b_{f,t} = P^{ub}_{f,t} = 1/2$ when $x_{f,t} = \pm x_M/2$ and they match reasonably well

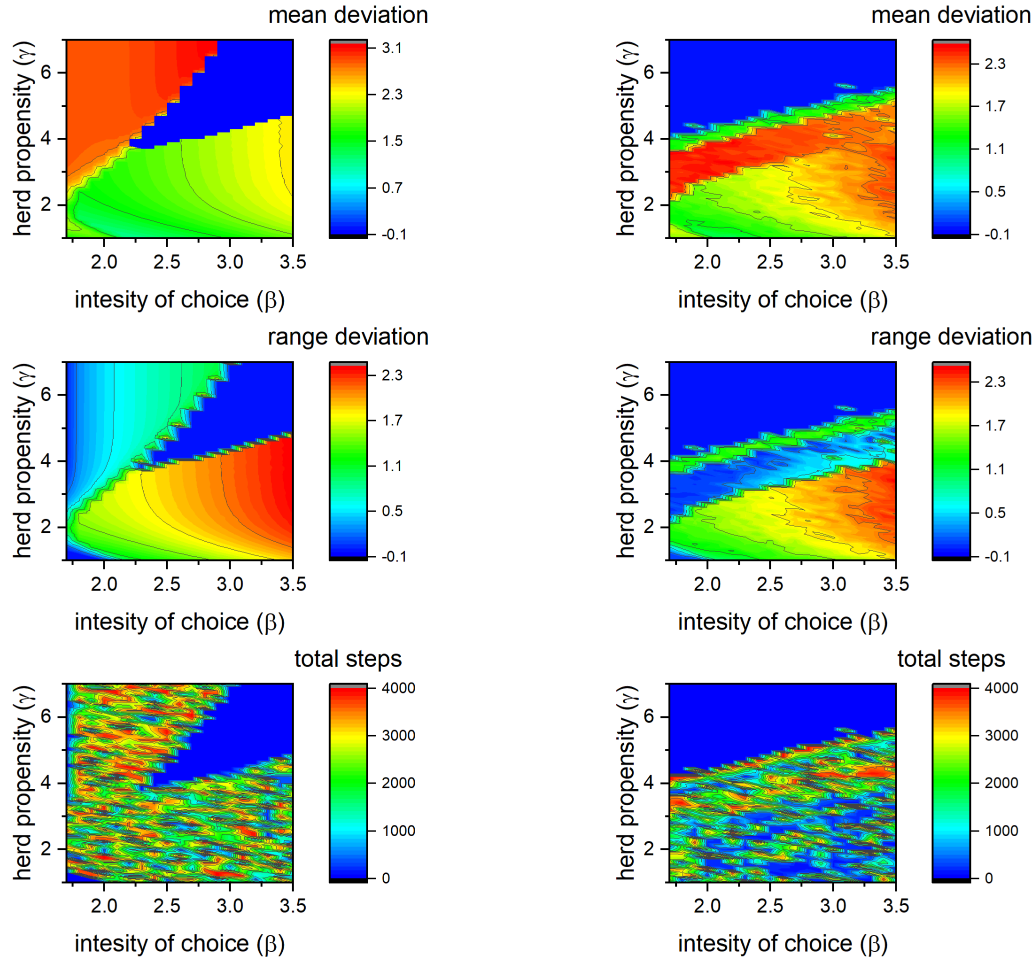


FIGURE B.10: Comparison of heatmaps for $n = 5$ (left maps obtained with system (B.32), right maps with limited number of investors)

over the interval $x_{f,t} \in I = [-x_M, x_M]$, but this choice is arbitrary. It is very important to note that we hardly believe that there exist discrete choice models defined on bounded intervals represented by a combination of trigonometric functions. However, the message we want to deliver here, in order to support our arguments below, is that the *shape* of their curve will likely *resemble* that of a combination of such functions, and this might help us (from an analytical point of view) to shed light on the causes of some anomalous phenomena arising in financial markets (e.g. volatility spikes), which would be otherwise impossible to do if one used a model derived from a better economic foundation. In fact, things get extremely interesting if we apply the same observation we made right above to the case of limited supplies and holdings. Investors in a stock market have limited liquidity ($c_{Max_{liq}}$) and supply ($c_{Max_{hold}}$) of stock shares held, so no matter how much undervalued or overvalued an asset can be, investors (in this case, fundamentalists) will generally have a limit on the number of shares, $z_{f,t}$, they can buy

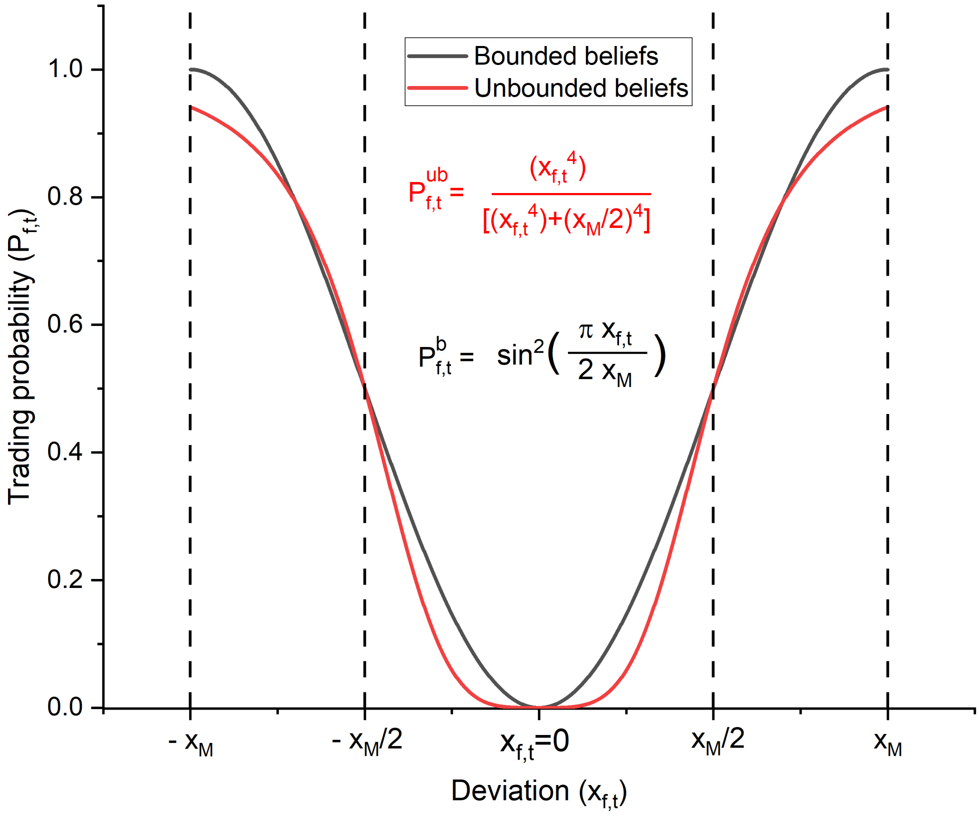


FIGURE B.11: Here, we show a comparison between the probability function (2.39) (black curve) with the one used in our model (2.2)(red curve). We set $\beta_f = 2$ and $\Delta_f = (x_M/2)^2$, so that both probabilities are equal to $P^b_{f,t} = P^{ub}_{f,t} = 1/2$ when $x_{f,t} = \pm x_M/2$ and match reasonably well over the interval $x_{f,t} \in I = [-x_M, x_M]$.

$$z_{f,t} \to c_{Max_{liq}} = c_f \qquad for \qquad x_{f,t} = p_t - p^* << 0 \tag{B.52}$$

or sell

$$z_{f,t} \to -c_{Max_{hold}} = -c_f \qquad for \qquad x_{f,t} = p_t - p^* >> 0 \tag{B.53}$$

where, for simplicity, we assumed that $c_{Max_{liq}} = c_{Max_{hold}} = c_f$. In the case of unbounded price beliefs (ub), the exact form of the demand $d^{ub}_{f,t}$ will generally be complicated, but we would not be so far from being right if we assume that it would look like a hyperbolic tangent function, such that

$$z^{ub}_{f,t} = -c_f \tanh(x_{f,t}) \quad (x_{f,t} \to \pm\infty \implies \tanh(x_{f,t}) \to \pm 1 \implies z^{ub}_{f,t} \to \mp c_f) \tag{B.54}$$

and investors' net demand, $d^{ub}_{f,t}$ (which takes into account their trading threshold, Δ_f), in a market with infinitely many investors ($P^{ub}_{f,t} \to n^{ub}_{f,t}$), could be potentially represented by

$$d_{f,t}^{ub} = z_{f,t}^{ub} n_{f,t}^{ub} = -c_f \tanh(x_{f,t}) \frac{\left(x_{f,t}^2\right)^{\beta_f}}{\left(x_{f,t}^2\right)^{\beta_f} + \Delta_f^{\beta_f}} \tag{B.55}$$

Which kind of function does investors' demand (B.55) look like in case of bounded price beliefs? As before, our choice falls to a trigonometric one, which is again a sine function, that is,

$$z_{f,t}^b = -c_f \sin\left(\frac{\pi}{2}\frac{x_{f,t}}{x_M}\right) \quad \left(x_{f,t} \to \pm x_M \implies \sin\left(\frac{\pi}{2}\frac{x_{f,t}}{x_M}\right) \to \pm 1 \implies z_{f,t}^b \to \mp c_f\right) \tag{B.56}$$

meaning that investors' net demand, $d_{f,t}^b$, would be represented simply by ($P_{f,t}^b \to n_{f,t}^b$)

$$d_{f,t}^b = z_{f,t}^b n_{f,t}^b = -c_f \sin^3\left(\frac{\pi}{2}\frac{x_{f,t}}{x_M}\right) \tag{B.57}$$

In Figure B.12, we show a comparison between $d_{f,t}^{ub}$ and $d_{f,t}^b$, where we again set $\beta_f = 2$ and $\Delta_f = (x_M/2)^2$, so that both aggregate demands match reasonably well over the interval $x_{f,t} \in I = [-x_M, x_M]$. Now, if we want to describe the price dynamics using the demand $d_{f,t}^b$, that is, when only groups of fundamentalists f_i with bounded price beliefs $p_{f_i,t}^*$ and limited supply/liquidity populate the market), then we would get (following system (3.37))

$$p_t = p_{t-1} + \mu \sum_i c_i q_i \sin^3\left[\frac{\pi}{2}\frac{\left(p_{f_i,t-1}^* - p_{t-1}\right)}{z_M}\right] \tag{B.58}$$

where c_i is the maximum liquidity (or supply) available to investor group i to buy (or sell) the asset and q_i their corresponding fraction with respect to the entire market ($\sum_i q_i = 1$). The previous difference equation reminds us of the Kuramoto model, which describes the motion of N coupled oscillators rotating on a circle, whose equation (in discrete time) is given by

$$\theta_{k,t} = \theta_{k,t-1} + \Omega + \frac{1}{N}\sum_{i=1}^{N} A_{i,k} \sin\left(\theta_{i,t-1} - \theta_{k,t-1}\right) \tag{B.59}$$

where $\theta_{k,t}$ is the phase of the *k-th* oscillator, Ω is the natural frequency equal to all oscillators, and $A_{l,k}$ the coupling constant between the *i-th* and the *k-th* oscillator. In the previous equation, the frequency Ω explicitly appears, while in equation (B.58) it does not. However, one can always make the frequency Ω disappear by analysing the system (B.59) in a rotating framework with frequency Ω. As we suggested at the end

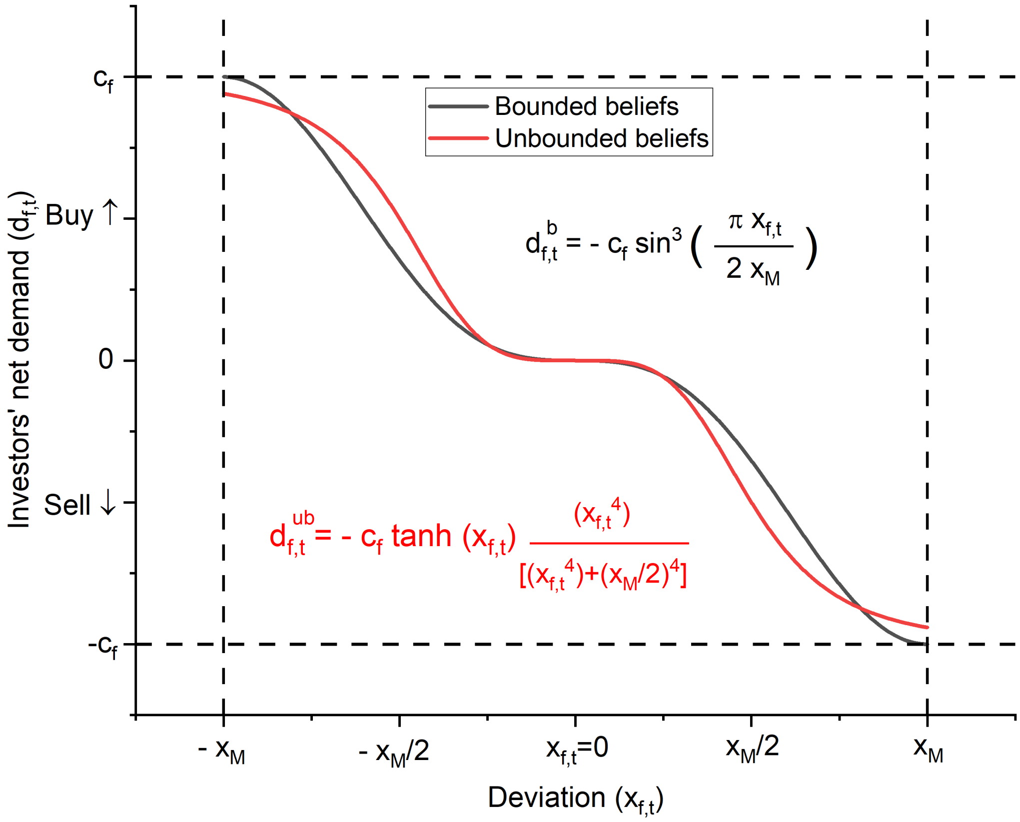


FIGURE B.12: Here, we show the comparison between investors' net demand with unbounded price beliefs, $d^{ub}_{f,t}$ (red curve), and with bounded price beliefs, $d^{b}_{f,t}$ (black curve). We set again $\beta_f = 2$ and $\Delta_f = (x_M/2)^2$, so that both demands match reasonably well over the interval $x_{f,t} \in I = [-x_M, x_M]$.

of section 5.2.2, there are strong analogies between some phenomena observed in asset prices and the phase transitions present in the Kuramoto model. The correspondence between our model and the Kuramoto model is not exact. However, it is possible that our price dynamics could be described by a related model.

Appendix C

Finnish data

C.1 Literature review on data analysis (Section 2.2)

Even here, as in behavioural modelling in financial markets, more than one community investigated this data set, adopting different tools to carry out their analysis. We roughly identified three different research communities: the first made up of financial economists, the second one of econophysicists, and a group who took inspiration from them (and contributed more recently to this literature). We briefly mention all these works, and we will discuss their findings in more detail in the next section when we examine investors' main attributes.

Works related to financial economics

Grinblatt and Keloharju (2000a) were the first authors working on these data and analysed them in order to determine the level of sophistication of each category of investors by relating it with two strategies, momentum and contrarian. Over the years, these authors carried out many other works. They showed which firms investors prefer to invest in Grinblatt and Keloharju (2001b), what the endogenous factors are (e.g. past returns) that make investors trade (Grinblatt and Keloharju (2001a)), how two behavioural biases, namely sensation seeking and overconfidence, influence investor trading activity (Grinblatt and Keloharju (2009)), and how investor intellectual ability (measured from their intelligence quotient) is related to stock market participation and trading behaviour (Grinblatt et al. (2011), Grinblatt et al. (2012)). In addition to them, other authors worked on these data, among them Shive (2010), who tested whether social influence is an important determinant in individual investor trading activity based on an epidemic model, Walden (2018), who used a theoretical model in order to estimate an information network proxy from pairs of investors (Ozsoylev et al. (2013)), and Frijns et al. (2018), who tested an extended version of the Brock and Hommes model by including retail and institutional investors.

Works related to econophysics

In 2012, Tumminello et al. (2012) attempted for the first time an unsupervised identification of clusters of investors (sharing strong similarities in trading activity) in this market (more specifically, those who invested in Nokia stock, the most traded asset in the Finnish market), and other papers took inspiration from them during the following years. Lillo et al. (2014) published a work about how endogenous (stock return and volatility) and exogenous sources (News) influence investors' activity in trading the Nokia stock. Musciotto et al. (2016) investigated the trading behaviour of investors trading the stocks selected to compute the OMXH25 index (a stock market index for the Helsinki Stock Exchange) and, a couple of years later, they extended the work done by Tumminello et al. (2012) by investigating the evolution of clusters of investors over a time period of 15 years and showed that there is an ecology of groups of investors characterized by different attributes and by various investment styles over many years (Musciotto et al. (2018)).

More recent works

More recently, Finnish researchers have analysed this data set and obtained further interesting results. Baltakys et al. (2018) proposed an innovative procedure to identify clusters of investors over single and multiple Finnish stocks. They focused their attention on the trading activity of households and its correlation with another possible exogenous source, namely social interaction (Baltakys et al. (2019)), and investigated how investors distribute their attention across different securities (Baltakys et al. (2020)). Baltakienė et al. (2019) analysed the trading behaviour in several IPO events (that is, when a company starts offering its shares to investors) and attempted to estimate the investors' private information channels (through which investors share their information via, for example, word-of-mouth communication) (Baltakienė et al. (2020)). Siikanen et al. (2018) considered Facebook a possible exogenous factor affecting investor trading decisions on Nokia stock, and Ranganathan et al. (2018) investigated in-depth Nokia stock around the dot-com bubble (the first authors doing that).

C.2 Volume traded and trading frequency (Section 2.2.4)

In Figure 2.4, we show that, typically, the more active an investor, the higher the average volume traded as if the exchanged volume is proportional to how frequently an investor trades (and vice-versa). In fact, Tumminello et al. (2012) report in their work the following statement: "The relation between the exchanged volume and the number of transactions of investors is approximated by a power-law relation with an exponent larger than one (a non-linear power-law fit gives 1.36 for the 2003 data)". From this, one could argue that the volume traded by an investor h at time t, $z_{h,t}$, is proportional to their trading frequency $\omega_{T,h}$ over a time period T as

$$z_{h,t} \propto b\omega_{T,h}^{d} \tag{C.1}$$

where b and d ($d > 1$) are two parameters which will depend on some market attributes. However, it must be noted that the power-law fit in Tumminello et al. (2012) was performed by considering the transactions made by *all* categories of investors together. Thus, one can not conclude anything from their statement, and we now explain why. In their work, Ranganathan et al. (2018) show a non-normalized complementary cumulative distribution function (NNCCDF)[1] for the number of investors $N_{>}(\omega_T)$ who traded the Nokia stock at least on ω_T different days during the period T from 1998 to 2002 ($T \sim 1250$ days). As noted by the same authors, a large number of investors traded a few days and only a low number of investors traded many days during this period, which suggests that the number of investors and the trading frequency are inversely proportional to each other. We attempted to replicate the same distribution as well as possible in Figure C.1, where we show the NNCCDF if the number of investors, $N(\omega_T)$, who traded the stock on ω_T different days, followed a discrete Weibull distribution (Nakagawa and Osaki (1975))(which fits reasonably well if compared to what obtained Ranganathan et al. (2018)), whose probability mass function (PMF), P_I, is given by

$$P_I(i) = q^{i^{\gamma}} - q^{(i+1)^{\gamma}} \tag{C.2}$$

where I is a symbol used to denote the random variable with support $R_I = \{0, 1, 2, ...\}$, $\gamma > 0$ and $0 < q < 1$. In order to get this NNCCDF, we first made the variable change $i = \omega_T - 1$ ($I \rightarrow \Omega_T$) (we suppose that all the investors within the dataset have traded at least once, $\omega_T \geq 1$) and modified equation (C.2) to adapt it for a bounded interval since there is a limit on how frequently an investor has traded, namely $\omega_T \leq T$. So, we defined $P_{\Omega_T}(\omega_T)$ (that satisfies all the properties of a PMF) such that:

[1] The complementary cumulative distribution function (CCDF), $\bar{F}_X(x)$, of a random variable X is equal to $\bar{F}_X(x) = 1 - F_X(x)$, where $F_X(x)$ is the corresponding cumulative distribution function(CDF).

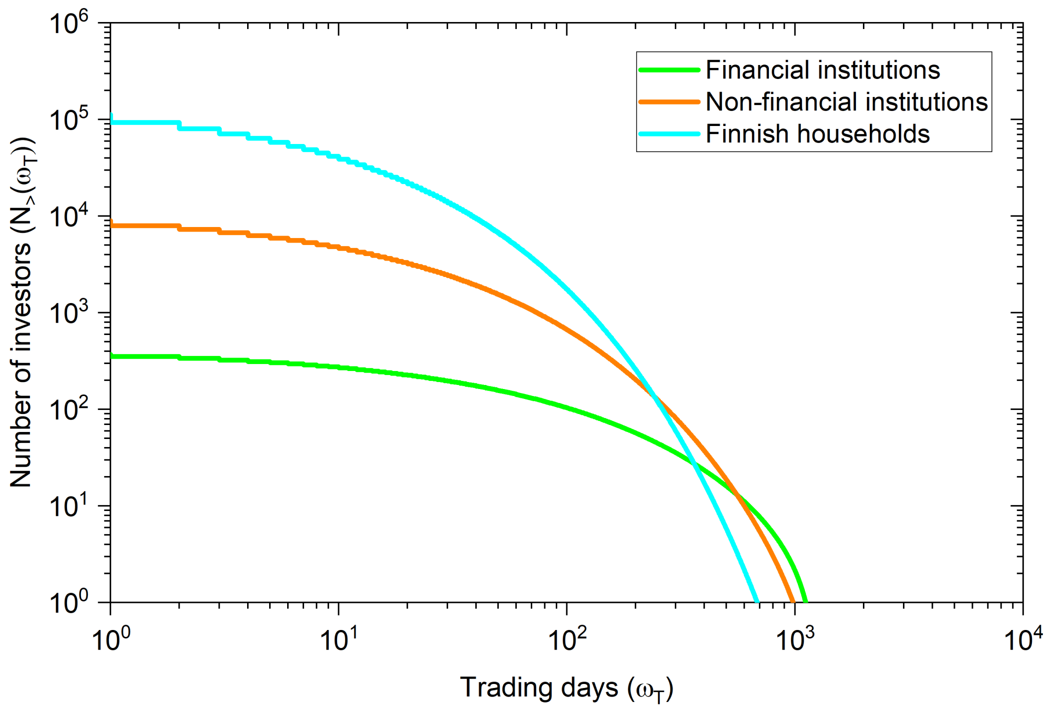


FIGURE C.1: The number of investors $N_>(\omega_T)$ who traded the stock at least on ω_T different days during the period $T \sim 1250$ days, if the number of investors who traded the stock on ω_T different days, $N(\omega_T)$, followed a discrete Weibull distribution.

$$P_{\Omega_T}(\omega_T) = \frac{q^{(\omega_T-1)^\gamma} - q^{(\omega_T)^\gamma}}{\sum_{i=1}^{T}\left[q^{(i-1)^\gamma} - q^{(i)^\gamma}\right]} = \frac{q^{(\omega_T-1)^\gamma} - q^{(\omega_T)^\gamma}}{1 - q^{T^\gamma}} \tag{C.3}$$

with support $R_{\Omega_T} = \{1, 2, 3, ..., T\}$, and derived its corresponding CDF, $F_{\Omega_T}(j)$, which by definition is equal to

$$F_{\Omega_T}(j) = \sum_{\omega_T \leq j} P_{\Omega_T}(\omega_T) = \frac{1 - q^{j^\gamma}}{1 - q^{T^\gamma}} \tag{C.4}$$

with $j \in R_{\Omega_T}$. After that, we obtained the CCDF, $\bar{F}_{\Omega_T}(j)$, given by

$$\bar{F}_{\Omega_T}(j) = 1 - F_{\Omega_T}(j) = \frac{q^{j^\gamma} - q^{T^\gamma}}{1 - q^{T^\gamma}} \tag{C.5}$$

and multiplied it by the total number of investors who traded Nokia stock over the period T for each category of investors (deducible from Tumminello et al. (2012)) to finally obtain the NNCCDF. We set the parameter $\gamma = 0.5$ for all categories of investors, $q_F = 0.87$ for financial institutions, $q_{NF} = 0.75$ for non-financial institutions, and $q_H = 0.63$ for Finnish households. As you can see from Figure C.1, the number of investors decreases as the number of trading days increases, but the decrease is faster for categories of investors considered less sophisticated (e.g. Finnish households) than for highly sophisticated investors (e.g. financial institutions). The 'low trading frequency' region is mostly populated by households, who carry out small exchanges, but within the 'high trading frequency' region, we see that financial institutions, who carry out large exchanges, are more predominant. So, it might very well be that the power-law relation estimated by Tumminello et al. (2012) just reflects the fact that the exchanged volume depends *only* on the category of investors and not on how frequently an investor trades (the volume traded by financial institutions from 1998 to 2003 was around 179,000 millions of shares, while households traded little more than 1,000 millions of shares Tumminello et al. (2012)). Nonetheless, we still believe the exchanged volume depends on the latter, even for investors from the same category. The reason for that is because of the potential profits achievable across different time horizons. For example, the expected price of a typical investor trading on short time intervals (e.g. one week) is closer to the current price compared to the expected price of an investor trading on large time intervals (e.g. one year), meaning that a high-frequency trader will generally invest much more than a low-frequency trader in order to realize some significant profits. A more in-depth analysis of this matter would be very welcome.

Appendix D

Numerical analysis

D.1 Fundamentalists and followers (Section 4.3)

In the following analysis, we will consider only a value of the trading threshold equal to $\Delta = 1$, without any loss of generality. In fact, for any other value $\Delta' = a\Delta$ ($a > 0$), we could always make a variable change $(x_{f_1,t}, x_{f_2,t}) \to (\sqrt{a}x_{f_1,t}, \sqrt{a}x_{f_2,t})$ such that the system would end up with investors who would refer to the trading threshold Δ. So, the global properties of the system are unaffected by the trading threshold.

D.1.1 Asymptotic behaviour of the system

First, we will show a parameter diagram, similar to Fig.4.7, that identifies the regions where the system might converge to the origin, diverge from or oscillate around it. To get this diagram, we let the system evolve for different values of $\bar{c}_{f_1} \in [0, 3.5]$ and $\bar{c}_{f_2} \in (0, 1.5]$ (we set $\mu = 1$), in steps of $\delta_{\bar{c}} = 0.005$. For each pair of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$, we use a set of conditions (outlined below) to establish the asymptotic behaviour of the system for different initial conditions $(x_{f_1,0}, x_{f_2,0}) \in I_{x_{f,0}} \subset \mathbb{R}^2$. The system (4.125) is invariant to a change of sign of the variables $(x_{f_1,t}, x_{f_2,t}) \to (-x_{f_1,t}, -x_{f_2,t})$, so, to choose the set I_0, we can focus either on the right (left) half-plane $x_{f_1,0} \geq 0$ ($x_{f_1,0} \leq 0$) or the upper (lower) half-plane $x_{f_2,0} \geq 0$ ($x_{f_2,0} \leq 0$) in $\mathbb{R}^2$. In our case, we choose the rectangle set on the upper half-plane $I_{x_{f,0}} = [-10, 10] \times [0, 10]$ and, for each pair of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$, let the system evolve in steps of $\delta_{x_{f,0}} = 0.5$ along the coordinates $(x_{f_1,0}, x_{f_2,0}) \in I_0$. This choice allows us to consider the cases where most investors belonging to both groups ($n_{f_i,0} \sim 1$) start trading the asset in the same (and opposite) direction, or if only one group starts trading it (more precisely, with $\Delta = 1$, we have $n_{f_i,0} = 0.\overline{9900}$ for $|x_{f,i}| = 10$). After a transition of $T_{tran} = 10000$ time steps, we observe the system for a further $T_{test} = 200$ time steps in which we keep track of groups' reference price deviation from the fundamental value, $x_{f_1,t}$ and $x_{f_2,t}$, to establish the asymptotic behaviour of the system.

Convergence and divergence condition To confirm whether the system will likely converge to the origin $\vec{x}_o^*$, we check whether the fractions of investors belonging to both groups tend to 0 (as was the case for a market with a single group of fundamentalists), that is,

$$n_{f_1,t} = \frac{x_{f_1,t}^2}{x_{f_1,t}^2 + \Delta} \to 0 \qquad \text{and} \qquad n_{f_2,t} = \frac{x_{f_1,2}^2}{x_{f_1,2}^2 + \Delta} \to 0 \tag{D.1}$$

as t increases, which implies

$$x_{f_1,t}^2 < \Delta \qquad \text{and} \qquad x_{f_2,t}^2 < \Delta \tag{D.2}$$

or, equivalently,

$$|x_{f_1,t}| < \sqrt{\Delta} \qquad \text{and} \qquad |x_{f_2,t}| < \sqrt{\Delta} \tag{D.3}$$

From these conditions, we define the following convergence condition as

$$\max_{t \in T_{test}} \left\{ |\vec{x}_{f,t}| = |x_{f_1,t}| + |x_{f_2,t}| \right\} < \delta_{conv}\sqrt{\Delta} \tag{D.4}$$

where δ_{conv} is a parameter of our choice and describes how strong the convergence condition must be. On the other hand, to see whether the system diverges from the origin, we must necessarily have

$$n_{f_1,t} = \frac{x_{f_1,t}^2}{x_{f_1,t}^2 + \Delta} \to 1 \qquad \text{and} \qquad n_{f_2,t} = \frac{x_{f_1,2}^2}{x_{f_1,2}^2 + \Delta} \to 1 \tag{D.5}$$

as t increases, which implies

$$|x_{f_1,t}| > \sqrt{\Delta} \qquad \text{and} \qquad |x_{f_2,t}| > \sqrt{\Delta} \tag{D.6}$$

Following the same logic as before, we define the divergence condition as

$$\min_{t \in T_{test}} \left\{ |\vec{x}_{f,t}| = |x_{f_1,t}| + |x_{f_2,t}| \right\} > \delta_{div}\sqrt{\Delta} \tag{D.7}$$

where δ_{div} tells how strong the divergence condition must be. For cases where neither the convergence condition (D.4) nor the divergence condition (D.7) hold, we will identify these as periods of market volatility, in which the asset price oscillates around its fundamental value, either periodically or aperiodically. We highlight two main problems that might affect the outcomes of our analysis. Firstly, there might be cases where the system could take a considerable time before starting to converge to (or diverge from) the origin $\vec{x}_o^*$ and we will inevitably end up identifying false negatives (or false positives), since we had to make a reasonable choice of T_{tran} and T_{test} so as to avoid large computational times. This also affects the choice of δ_{conv} and δ_{div} (the smaller T_{tran} and T_{test}, the larger δ_{conv} and the smaller δ_{div} must be chosen), and for our simulations (after some trial tests), we reached a good compromise by setting $\delta_{conv} = 1$ and $\delta_{div} = 50$. Secondly, for some pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$, we might find some false negatives (positives) because of our choice for $I_{x_f,0}$ (which represents only a subset of $\mathbb{R}^2$) and $\delta_{x_{f,0}}$, since the set of initial conditions for which the orbits would diverge to infinity or oscillate around the origin could be outside $I_{x_f,0}$, or its resolution (set by $\delta_{x_{f,0}}$) might not be high enough to detect such orbits.

Parameter diagram Keeping in mind the aforementioned possible outcomes, in Figure D.1 we show the parameter diagram that identifies the stability regions of the system (4.125). First of all, we point out that, for all pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$, there always exist a set of initial conditions for which the orbits converge to the origin that does not include the origin itself (we will see below a case study showing the form of this basin of attraction). So, in contrast to the diagram in Figure 4.7, the blue region identifies the pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ in which the system is stable and the orbits *only* converge to the origin $\vec{x}_o^*$. In the red region the system can also become unstable and the orbits diverge to infinity, in the green one the orbits can also oscillate around the origin perpetually (which are the areas of greatest interest as they represent cases where the market undergoes periods of volatility), and in the yellow region the system can exhibit either divergences or oscillations around the origin, beyond the convergence to $\vec{x}_o^*$. The regions where the orbits of the system (4.125) converge only to $\vec{x}_o^*$ (blue) and diverge from it (red) roughly match those of the diagram in Figure 4.7. We observe more differences for the green regions which appears to be present only below $\bar{c}_{f_1} < 2$ and the straight lines are now replaced by areas, mostly present below $\bar{c}_{f_1} < 1$ and $\bar{c}_{f_2} < 1$. This means that the conditions for which it is possible to observe a volatile market are less restrictive than in the no-threshold case. It is also interesting to notice that these are region are mostly present where both groups performs well (both in the short and long term), as we will show below. The yellow regions are less extended areas than the green ones and mostly present beyond $\bar{c}_{f_2} = 1$, and it can be shown that the orbits of the map (4.125) for pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ in these regions correspond to some solutions of the map for $\Delta = 0$, $\bar{c}_{f_2} = 1$, and $\bar{c}_{f_1} \in [0, 3)$. These solutions persist beyond $\bar{c}_{f_2} = 1$ because only a fraction of investors in each group $n_{f_i,t}$ will trade the asset now ($\Delta \neq 0$) and this will affect group's *effective* dominance in the market, $\bar{c}'_{f_i,t} = \bar{c}_{f_i} n_{f_i,t}$, whose value ($\bar{c}'_{f_i,t} < \bar{c}_{f_i}$) allows the orbits to not diverge to infinity. A more detailed analysis of these regions is not of our primary interest since

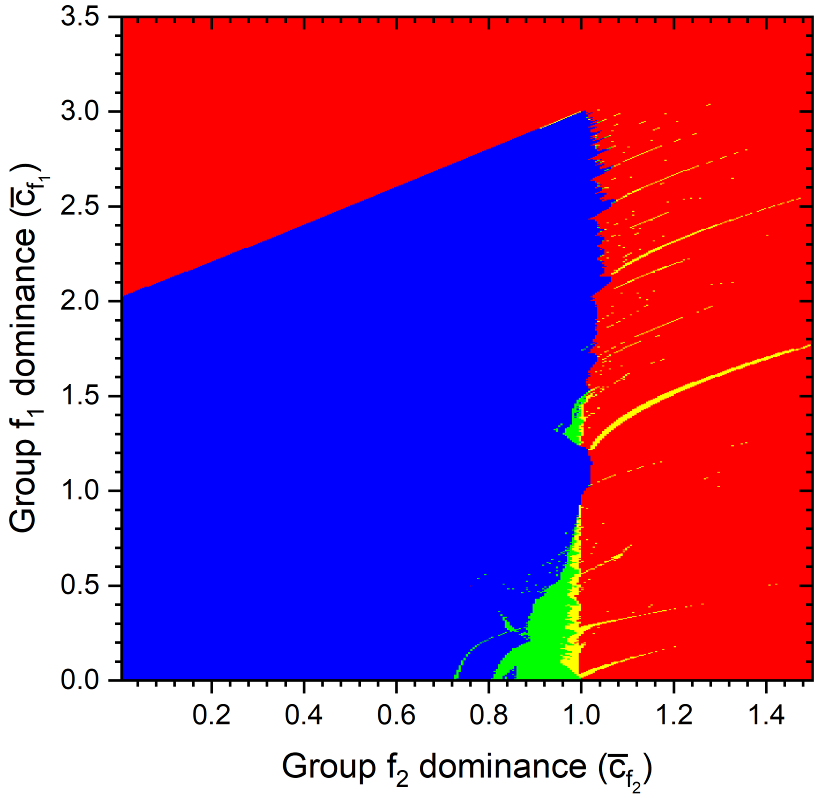


FIGURE D.1: The parameter diagram showing the stability regions of the market described by the system (4.125). For all pair of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$, we have that there exist always a set of initial conditions for which the orbits converge to the origin. The blue region identifies the pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ in which the system is stable and the orbits only converge to the origin $\vec{x}_o^*$, the red region where the system can also become unstable and the orbits diverge to infinity, the green region where the orbits can also oscillate around the origin perpetually (market volatility), and the yellow one where the system can exhibit either divergences or oscillations around the origin, beyond the convergence to $\vec{x}_o^*$.

they represent cases that are unlikely to be observed in a real context, either because these would force f_2 to incur losses (driving them out of the market) ($\bar{c}_{f_1} > 1$) or because the (quasi-)periodic orbits get attracted by the origin or repelled to infinity as one includes a noise term in the system ($\bar{c}_{f_2} > 1$). For this reason, we will deepen our analysis by investigating a restricted region of the diagram Figure D.1, and focus our attention on parameter ranges $\bar{c}_{f_1} \in [0, 1]$ and $\bar{c}_{f_2} \in [0.7, 1.3]$ in steps of $\delta_{\bar{c}} = 0.00125$, shown in Figure D.2. As we will see later, we have that, in this region, both groups of fundamentalists will be able to sustain any circumstance where the market may undergo periods of volatility (the green and yellow region in Figure D.2), similar to what we observed in the previous section ($\Delta = 0$), for $\bar{c}_{f_2} = 1$ and $0 \leq \bar{c}_{f_1} \leq 1$.

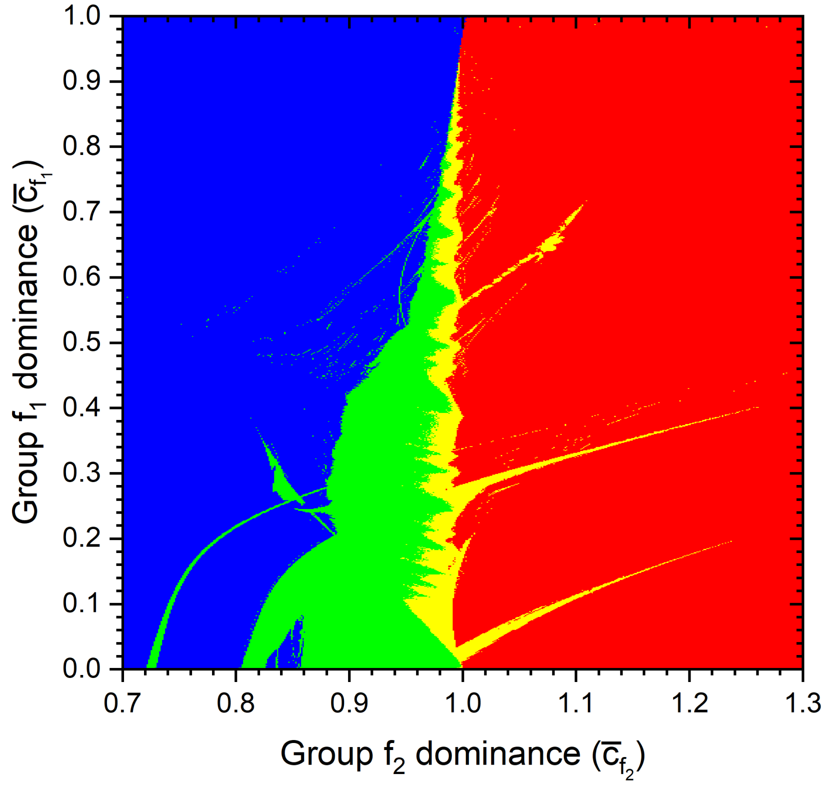


FIGURE D.2: A restricted region of the diagram in Figure D.1 for $\bar{c}_{f_1} \in [0, 1]$ and $\bar{c}_{f_2} \in [0.7, 1.3]$.

D.1.2 Regions with multiple attractors

Here, we will try to identify the regions where the oscillatory solutions of the map (4.125), $\vec{x}_{f,t} = (x_{f_1,t}, x_{f_2,t})$, might orbit around the origin with different amplitudes (market exhibiting multiple price volatilities), and, simultaneously, perform more accurate checks about the convergence and divergence of the solutions. This will allow us to highlight regions with multiple attractors (beyond the fixed point attractor $\vec{x}_o^* = (0, 0)$), in which the properties of the orbits might differ from each other. In this regard, we define four ranges of values where the average of the norm $\overline{|\vec{x}_{f,t}|}$ could stabilise after a period of time T_{test}, that is

$$\begin{cases} \overline{|\vec{x}_{f,t}|} \in r_1, & \text{if} \quad r_{1,MIN} = 0 < \overline{|\vec{x}_{f,t}|} < 1 = r_{1,MAX} \\ \overline{|\vec{x}_{f,t}|} \in r_2, & \text{if} \quad r_{2,MIN} = 1 \leq \overline{|\vec{x}_{f,t}|} < 10 = r_{2,MAX} \\ \overline{|\vec{x}_{f,t}|} \in r_3, & \text{if} \quad r_{3,MIN} = 10 \leq \overline{|\vec{x}_{f,t}|} < 100 = r_{3,MAX} \\ \overline{|\vec{x}_{f,t}|} \in r_4, & \text{if} \quad \overline{|\vec{x}_{f,t}|} \geq 100 = r_{4,MIN} \end{cases} \tag{D.8}$$

The border values of these ranges r_i have been chosen to classify orbits having average values of different orders of magnitude, as we do not expect to observe significant differences in the price dynamics of those with the same order. Since there is a degree of freedom in choosing these values, we started from a range that included all those cases where most investors ended up being inactive after a period T_{test}. In fact, for $\overline{|\vec{x}_{f,t}|} \in r_1$, we have $\overline{|x_{f_i,t}|} < 1$ and, consequently ($\Delta = 1$),

$$\max_{\overline{|\vec{x}_{f,t}|} \in r_1} \{\overline{n_{f_i,t}}\} < \frac{(1)^2}{(1)^2 + \Delta} = 0.5 \iff \text{most investors are inactive for } t \in T_{test} \quad \text{(D.9)}$$

From here, we set the border value of the next range of an order of magnitude higher than the previous one, up to a range representing all those cases where almost all investors were active after a period T_{test}, as $\overline{|\vec{x}_{f,t}|} \in r_4$ implies $\overline{|x_{f_i,t}|} \geq 100$ and, thus,

$$\min_{\overline{|\vec{x}_{f,t}|} \in r_4} \{\overline{n_{f_i,t}}\} \geq \frac{(100)^2}{(100)^2 + \Delta} \sim 1 \iff \text{(almost) all investors trade for } t \in T_{test} \quad \text{(D.10)}$$

The range r_1 and r_4 include cases for which the limiting behaviour of the system must be questioned as they might correspond to solutions where the orbits could eventually converge to the origin (if $\overline{|\vec{x}_{f,t}|} \in r_1$) or diverge from it ($\overline{|\vec{x}_{f,t}|} \in r_4$) if we choose a larger transition period T_{tran}. The range r_2 and level r_3, instead, include solutions whose limiting behaviour is well established and for which we can perform a more detailed analysis (e.g. market volatility, groups' sustainability). However, for any pair of values $(\bar{c}_{f_2}, \bar{c}_{f_1}) \in I_{\bar{c}}$ in which we may find orbits $\vec{x}_{f,t}$ whose average norm would fall within the range of values r_2 or r_3 and, for different initial conditions, *also* within the range of values r_1 or r_4, then further investigations must be needed to establish the correct asymptotic behaviour of the solution. Otherwise, regions for which we observe solutions whose norm stabilizes within range r_2 and r_3 only, then two cases are possible: the average norm is far from the borders of both intervals, that is, there are two different sets of initial conditions (for fixed $\bar{c}_{f_2}$ and $\bar{c}_{f_1}$) for which $r_{2,MIN} \leq \overline{|\vec{x}_{f,t}|} << r_{3,MIN}$ and $r_{3,MIN} << \overline{|\vec{x}_{f,t}|} < r_{3,MAX}$, meaning that there could be potentially a double attractor (concentric to the origin); the average norm is close to the border of both intervals, that is, $\overline{|\vec{x}_{f,t}|} \sim r_{2,MAX} \sim r_{3,MIN}$, meaning that there is only a single attractor or a phase transition to a double attractor[1].

Convergence and divergence check In Figure D.3, we show the regions of the parameter diagram where the average norm of the orbits stabilize within the range r_1 (top left panel), range r_2 (top right panel), range r_3 (bottom left panel), and range r_4 (bottom right panel). The coloured bar on the top indicates how close $\overline{|\vec{x}_{f,t}|}$ is to the

[1]This method only allows detecting at most double attractors. There might be other attractors within the same range, that could be identified by splitting the ranges into even smaller intervals.

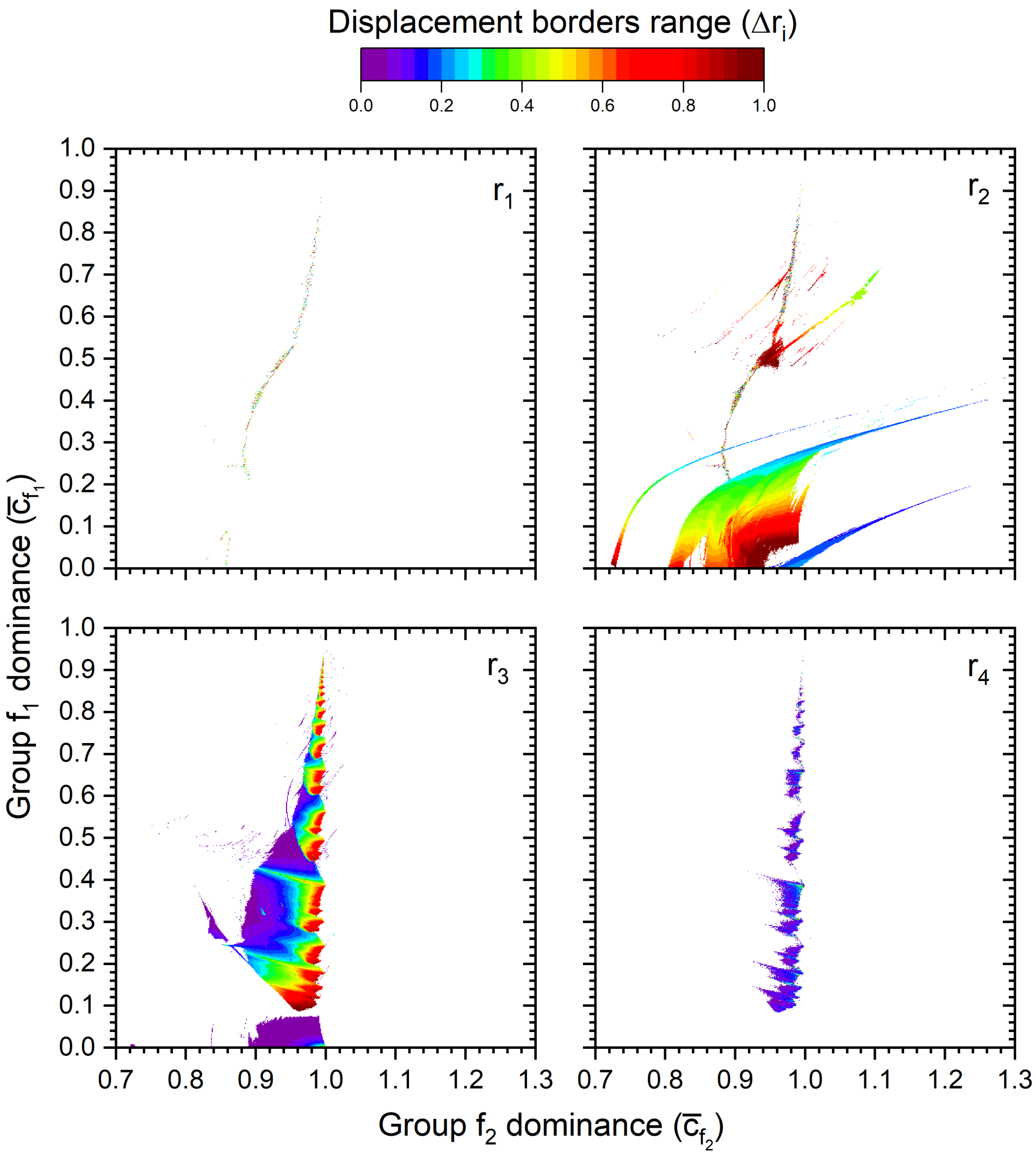


FIGURE D.3: The regions of the parameter diagram Figure D.2 where the average norm of the orbits stabilizes within the range r_1 (top left panel), range r_2 (top right panel), range r_3 (bottom left panel), and range r_4 (bottom right panel). The coloured bar on the top indicates how close $\overline{|\vec{x}_{f,t}|}$ is to the lowest value $r_{i,MIN}$ (violet) and highest value $r_{i,MAX}$ (dark red) in the corresponding range.

lowest value $r_{i,MIN}$ (violet) and highest value $r_{i,MAX}$ (dark red) in the corresponding range, where

$$\Delta r_i = \frac{\overline{|\vec{x}_{f,t}|} - r_{i,MIN}}{r_{i,MAX} - r_{i,MIN}} \iff \overline{|\vec{x}_{f,t}|} = r_{i,MIN} + \Delta r_i (r_{i,MAX} - r_{i,MIN}) \tag{D.11}$$

For example, a violet region for $\overline{|\vec{x}_{f,t}|} \in r_3$ ($\Delta r_3 \sim 0$) corresponds to cases where $\overline{|\vec{x}_{f,t}|} \sim 10$, while a yellow region for $\overline{|\vec{x}_{f,t}|} \in r_2$ ($\Delta r_3 \sim 0.5$) corresponds to cases where $\overline{|\vec{x}_{f,t}|} \sim 5$. Notice that for each pair of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ we let the system evolve for different initial conditions $(x_{f_1,0}, x_{f_2,0}) \in I_0$ and, thus, we had to deal with multiple orbits whose average norm could have been different from each other, even for those stabilizing within a specific range r_i. So, in these plots we show the average value of all norm averages that fell within each range. Although this analysis is not so accurate, it is very effective to bring out the variations in the qualitative behaviour of the system for different values of $(\bar{c}_{f_2}, \bar{c}_{f_1})$, and we will implicitly adopt this method on all occasions where we will show these types of diagrams. As already explained above, the regions highlighted in the range r_1 and range r_4 diagrams allow us to identify the pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ in common to the regions in the range r_2 and range r_3 diagrams where the the system must be further investigated to establish the correct asymptotic behaviour of the orbit. After some tests, we found that the region in the r_1 diagram 'fades away' as we take larger T_{tran}, meaning that the orbits for these pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ are likely to converge to the origin. The same region is also present in the r_2 and r_3 diagram, which, in turn, disappears, and, thus, it will not be included in our subsequent analysis. The region in the r_4 diagram, instead, persists and the average norms of the orbits mostly match those present in the r_3 diagram (the orbits do not diverge to infinity), and, so, these regions will be treated equally (the analysis performed on the r_3 region will also be valid for the r_4 region).

Single and double attractors In Figure D.4, we show the regions of the parameter diagram where the average norm of the orbits stabilizes within the range r_2 (left panels) and range r_3 (right panels), when we exclude the pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ common to r_2 and r_3 (top panels), and when we only include $(\bar{c}_{f_2}, \bar{c}_{f_1})$ common to r_2 and r_3 (bottom panels). The top panels allow us to identify the regions where we may only find orbits corresponding to a single attractor, while the bottom panels where there could be potentially double attractors (or phase transitions from a single to double attractor and vice-versa). For the latter, we make observe that the closer Δr_2 is to 1 (dark red regions in the panel for $r_2 \cap r_3$) and the closer Δr_3 is to 0 (violet region in the panel for $r_3 \cap r_2$), the more likely the orbit corresponds to a single attractor ($\overline{|\vec{x}_{f,t}|} \sim 1$) or a phase transition to a double attractor. Two regions that seem to include such orbits are located within $0.9 < \bar{c}_{f_2} < 1$ and $0 < \bar{c}_{f_1} < 0.08$ (group f_1 participating in the market mainly as a source of information), and $0.48 < \bar{c}_{f_1} < 0.56$ (group f_1

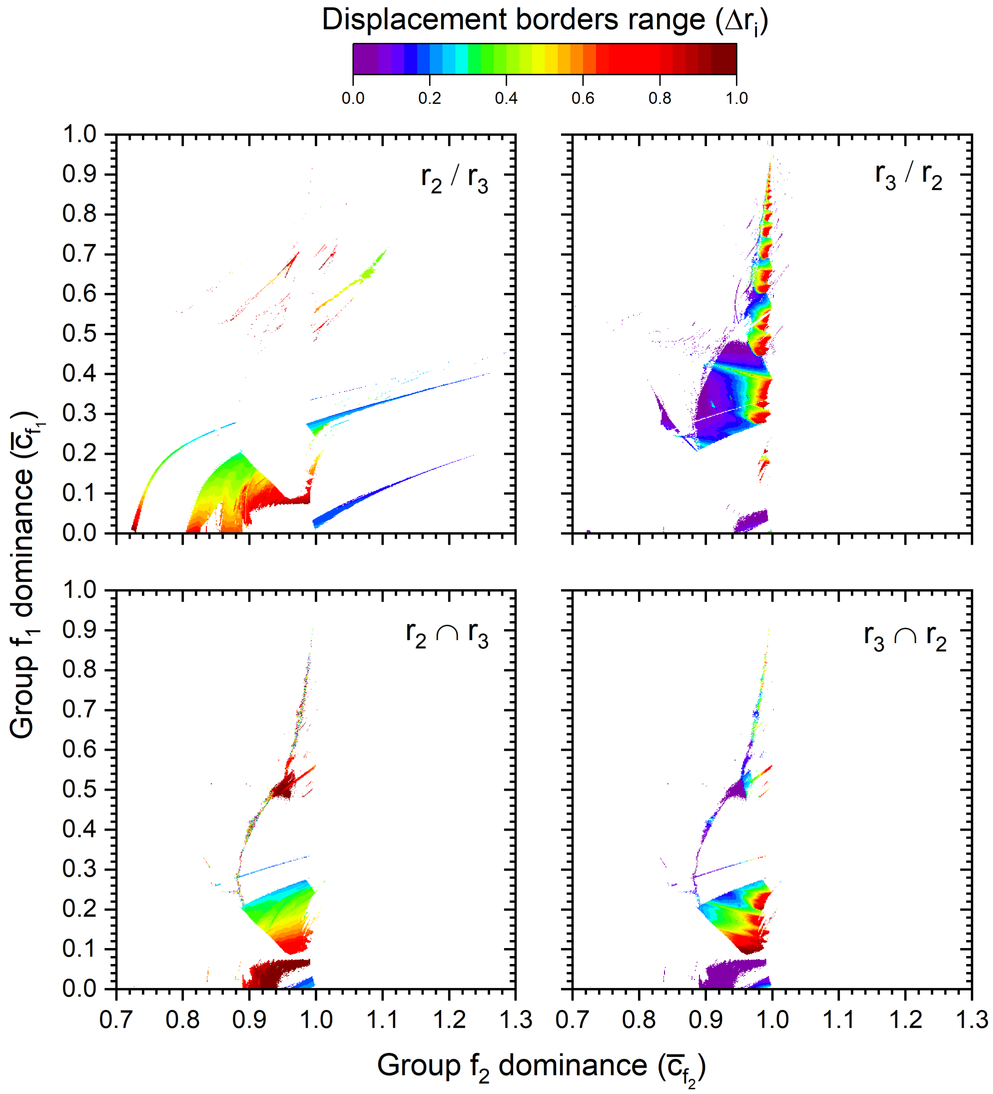


FIGURE D.4: The regions of the parameter diagram D.2 where the average norm of the orbits stabilize within the range r_2 (left panels) and range r_3 (right panels), when we exclude the parameter values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ common to r_2 and r_3 (top panels), and when we only include $(\bar{c}_{f_2}, \bar{c}_{f_1})$ common to r_2 and r_3 (bottom panels). The coloured bar on the top indicates how much close $\overline{|\vec{x}_{f,t}|}$ is to the lowest value $r_{i,MIN}$ (dark blue) and highest value $r_{i,MAX}$ (dark red) in the corresponding range.

participating in the market and trading the asset as well). On the other hand, the more distant Δr_2 and Δr_3 are, correspondingly, from 1 and 0, the more likely the system features a double attractor, as it is for the region within $0.9 < \bar{c}_{f_2} < 1$ and $0.08 < \bar{c}_{f_1} < 0.26$. Below, we will perform a more detailed analysis of a pair of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ within it.

D.1.3 Market volatility and groups' performance

In this part, we will make general observations about the qualitative behaviour of price dynamics, for pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ in the r_2 and r_3 regions. We are interested in identifying anomalies in price fluctuations (e.g clustering phenomena), analysing the sensitive dependence of the system on initial conditions, and testing the stability of the market when a noise term is introduced (e.g. noise traders). As we did in the previous section, we focus both on short and long timescale events, where we consider the absolute value of price changes between two time steps for a solution orbiting in the range r_i (short-term price change), that is (see equation (4.16)),

$$\Delta x_{s,r_i,t} = |p_t - p_{t-1}| = |x_t - x_{t-1}| \tag{D.12}$$

and the absolute value of price changes with respect to the fundamental value (long-term price changes), that is (see equation (4.17)),

$$\Delta x_{l,r_i,t} = |p^* - p_t| = |x_t| \tag{D.13}$$

For each pair of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$, we estimate the average value of $\Delta x_{s,r_i,t}$ and $\Delta x_{l,r_i,t}$ over the period T_{test} (see equation (4.18) and (4.19))

$$\overline{\Delta} x_{s,T_{test},r_i} = \frac{1}{T_{test}} \sum_{t=T_{tran}}^{t=T_{tran}+T_{test}} \Delta x_{s,r_i,t} \tag{D.14}$$

$$\overline{\Delta} x_{l,T_{test},r_i} = \frac{1}{T_{test}} \sum_{t=T_{tran}}^{t=T_{tran}+T_{test}} \Delta x_{l,r_i,t} \tag{D.15}$$

and, from these quantities, we compute the coefficient of market volatility, v_{T_{test},r_i}, defined as (see equation (4.20))

$$v_{T_{test},r_i} \stackrel{\text{def}}{=} \frac{\overline{\Delta} x_{s,T_{test},r_i}}{2\overline{\Delta} x_{l,T_{test},r_i}} \tag{D.16}$$

In order to identify the regions where clustering phenomena in the price dynamics can be observed, we estimate the index of dispersion (for skewed distributions, Bonett and Seier (2006)) of $\Delta x_{s,r_i,t}$, defined by

$$D_{\Delta x_{s,T_{test},r_i}} = \frac{1}{T_{test}} \sum_{t=T_{tran}}^{t=T_{tran}+T_{test}} \frac{|m_{\Delta x_{s,r_i}} - \Delta x_{l,r_i,t}|}{m_{\Delta x_{s,r_i}}} \tag{D.17}$$

where $m_{\Delta x_{s,r_i}}$ is the median of the values $\Delta x_{s,r_i,t}$ observed over the period T_{test}. If the index of dispersion takes a value between $0 < D_{\Delta x_{s,r_i}} < 1$, then it is said that the distribution of $\Delta x_{s,r_i,t}$ is under-dispersed (Cox (2014)), meaning that there are no clustering phenomena, while $D_{\Delta x_{s,r_i}} > 1$ could correspond to the existence of clusters in the price dynamics. To verify if the system shows sensitive dependence on initial conditions, we compute the largest Lyapunov coefficient $\lambda_{r_i,max}$ (Eckmann and Ruelle (2004)), given by

$$\lambda_{r_i,max}(\vec{x}_0, \vec{\delta}_{x_0}) = \lim_{n \to \infty} \frac{1}{t} \ln ||J^t F(\vec{x}_0)\vec{\delta}_{x_0}|| \tag{D.18}$$

where

$$J^t F(\vec{x}_0) = JF^t(\vec{x}_0) JF^{t-1}(\vec{x}_0) ... JF^1(\vec{x}_0) JF(\vec{x}_0) \tag{D.19}$$

with $JF^t(\vec{x}_0)$ the jacobian of the map (see equation (4.129)) after t iterations starting from the initial state $\vec{x}_0$, and $\vec{\delta}_{x_0}$ a small perturbation of the initial point $\vec{x}_0$, $\vec{u}_0 = \vec{x}_0 + \vec{\delta}_{x_0}$. If $\lambda_{r_i,max}(\vec{x}_0, \vec{\delta}_{x_0}) > 0$, then we have that the two orbits starting from $\vec{x}_0$ and $\vec{u}_0$ will exponentially diverge from each other, suggesting that the map is potentially chaotic. In addition to this, we check groups' ability in sustaining periods of market volatility, where we consider group f_i's relative short-term performance, given by (see equation (4.13))

$$\Sigma_{r_i,\Pi^s,T_{test},f_i} = \frac{\sum_{t=T_{tran}}^{t=T_{tran}+T_{test}} \Pi^s_{r_i,f_i,t}}{\sum_{t=T_{tran}}^{t=T_{tran}+T_{test}} |\Pi^s_{r_i,f_i,t}|} \tag{D.20}$$

where

$$\Pi^s_{r_i,f_i,t} = (p_t - p_{t-1}) d_{f_i,t-1} = (x_t - x_{t-1})(-c_{f_i} x_{f_i,t-1}) \frac{x^2_{f_i,t-1}}{x^2_{f_i,t-1} + \Delta_f} \tag{D.21}$$

and group f_i's relative long-term performance, given by (see equation (4.15))

$$\Sigma_{r_i,\Pi^l,T_{test},f_i} = \frac{\sum_{t=T_{tran}}^{t=T_{tran}+T_{test}} \Pi^l_{r_i,f_i,t}}{\sum_{t=T_{tran}}^{t=T_{tran}+T_{test}} |\Pi^l_{r_i,f_i,t}|} \tag{D.22}$$

where

$$\Pi^l_{r_i,f_i,t} = (p^* - p_t) d_{f_i,t} = -x_t(-c_{f_i} x_{f_i,t}) \frac{x^2_{f_i,t}}{x^2_{f_i,t} + \Delta_f} \tag{D.23}$$

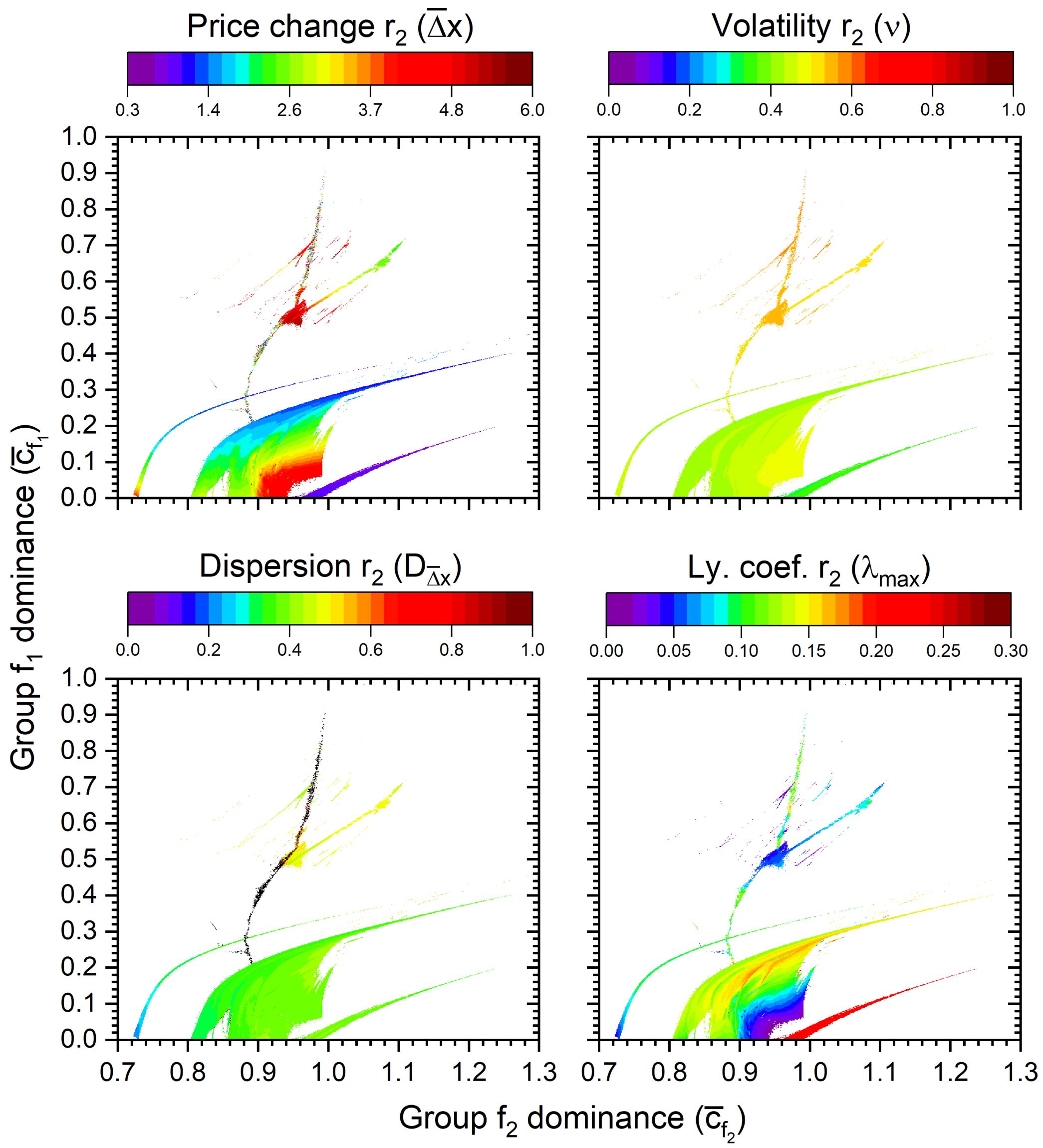


FIGURE D.5: The parameter diagrams of the short-term price change (top left panel), the market volatility (top right panel), the index of dispersion (bottom left panel), and the Liapunov coefficient (bottom right panel) for orbits whose average norm stabilize within range r_2.

Price dynamics In Figure D.5 and Figure D.6, we show the parameter diagrams of the short-term price change (top left panel) (the long-term price change almost follows the same pattern), the market volatility (top right panel), the index of dispersion (bottom left panel), and the Lyapunov coefficient (bottom right panel) for orbits whose average norm stabilizes within ranges r_2 and r_3, respectively. In the r_2 region (in the middle patch but not globally), we see that the average of the price change in the short-term increases as group f_2 becomes more dominant (with a slight increase in the volatility of the market). However, this becomes less significant the more dominant group f_1 is, as the price is constrained to fluctuate closer to the fundamental value

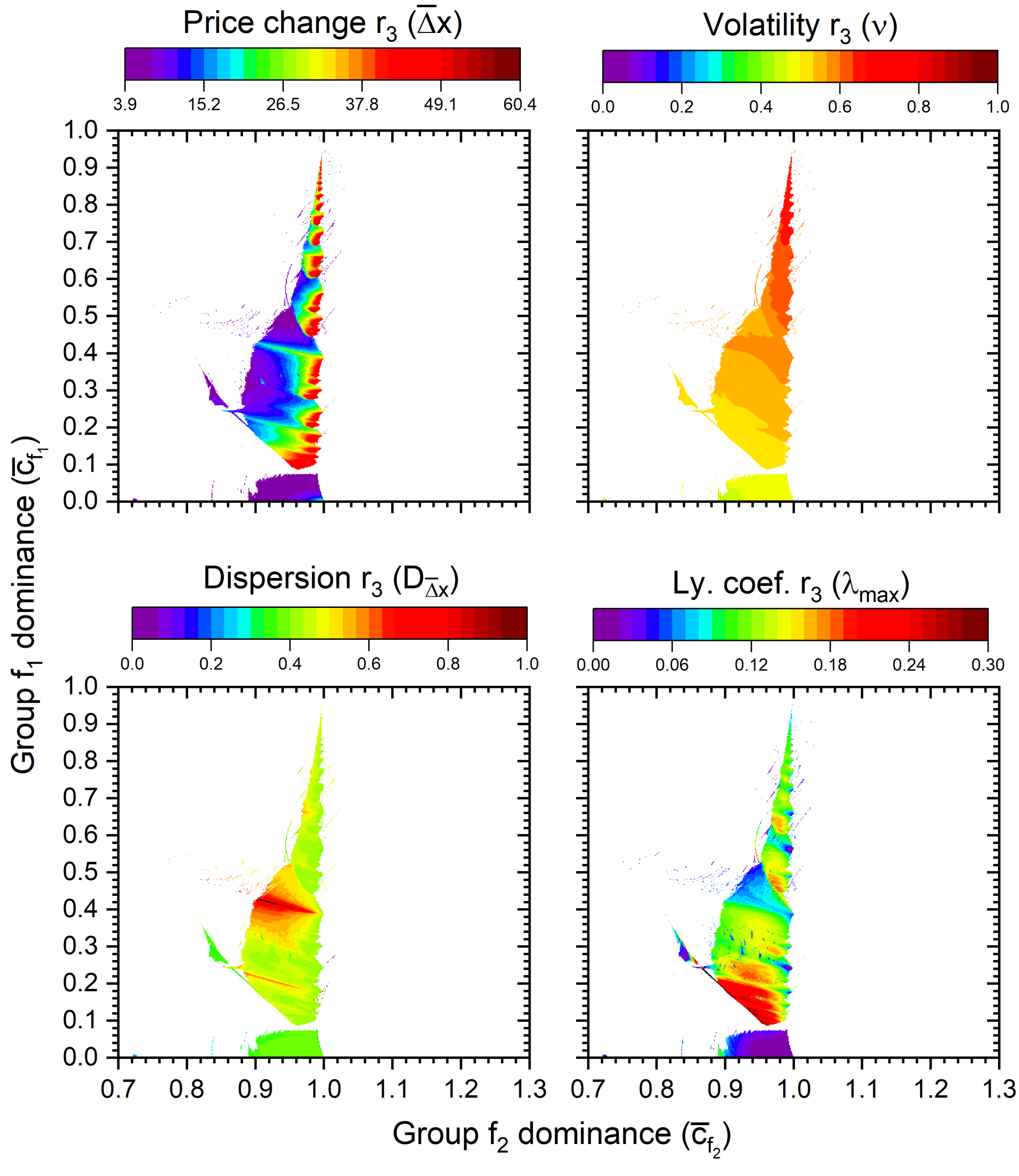


FIGURE D.6: The parameter diagrams of the short-term price change (top left panel), the market volatility (top right panel), the index of dispersion (bottom left panel), and the Liapunov coefficient (bottom right panel) for orbits whose average norm stabilizes within range r_3.

(because of f_1's impact on the price). At this level, the price does not present any clustering phenomena (the black region in the bottom left diagram represents pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$ for which the orbit will eventually converge to $\vec{x}_o^*$), but we observe sensitive dependence on initial conditions ($\lambda_{r_2,max} > 0$) for all pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$. Interestingly, the larger the average price change is in the short-term, the less sensitive the system is to the initial conditions, and vice versa. An increase of f_2's market dominance may lead the system to behave more linearly (no-threshold case) and, thus, in a more predictable way, as the fraction of investors trading the asset tends to $n_{f_2,t} \sim 1$ for $t \in T_{test}$ (the price is, on average, more distant from p^*), while an increase of f_1's dominance (the price is closer to p^*) has the opposite effect. In the r_3 region, the average price change in the short-term shows a less regular pattern than in the r_2 case. We still observe an increase in its value as $\bar{c}_{f_2} \to 1$, but not for all values of $\bar{c}_{f_1}$. The market exhibits a higher degree of volatility (remind that the price changes are of an order of magnitude higher), mostly due to group f_1's dominance. We now see the presence of regions in which the price changes are characterized by clustering phenomena, that is, large price fluctuations are more likely to be followed by large price fluctuations (see 'Volatility clustering' in Cont (2001)), which will be subjects of more detailed study below. As it was in the r_2 region, there is sensitive dependence on initial conditions ($\lambda_{r_3,max} > 0$) for all pairs of values $(\bar{c}_{f_2}, \bar{c}_{f_1})$. The fact that the price changes are such as to lead investors in both groups to trade almost all the time ($n_{f_1,t} \sim 1$ and $n_{f_2,t} \sim 1$ for $t \in T_{test}$) does not significantly impact the Lyapunov coefficient, as happens in the r_2 region. What mainly determines the degree of sensitive dependence on initial conditions is the active participation of both groups in the market.

Groups' performances In Figure D.7 and Figure D.8, we show the parameter diagrams of groups' relative short-term performance (top panels), and long-term performance (bottom panels) for orbits whose average norm stabilizes within ranges r_2 and r_3, respectively. Groups' performances are approximately independent of how large the average norms of the orbits are ($\overline{|\vec{x}_{f,t}|} \in r_2$ or $\overline{|\vec{x}_{f,t}|} \in r_3$), even if they may correspond to cases with different market volatilities (double attractors). Both f_1 and f_2 perform similarly to what they do for $\Delta = 0$, $\bar{c}_{f_2} = 1$ and $\bar{c}_{f_1} \in [0, 1]$ (see last part of section 7.3.3), that is, well in the short and long term, with f_2 better than f_1 in the short-term and, vice versa, f_1 better than f_2 in the long-term. This suggests that both groups trade almost synchronously and in the same direction, sending buy (sell) orders before a price increase (decrease) and when the asset is underpriced (overpriced) (the price fluctuations in both regions r_2 and r_3 are such as to offer enough time for f_2, more prone to incur losses, to react promptly to any asset price reversals to its fundamental value).

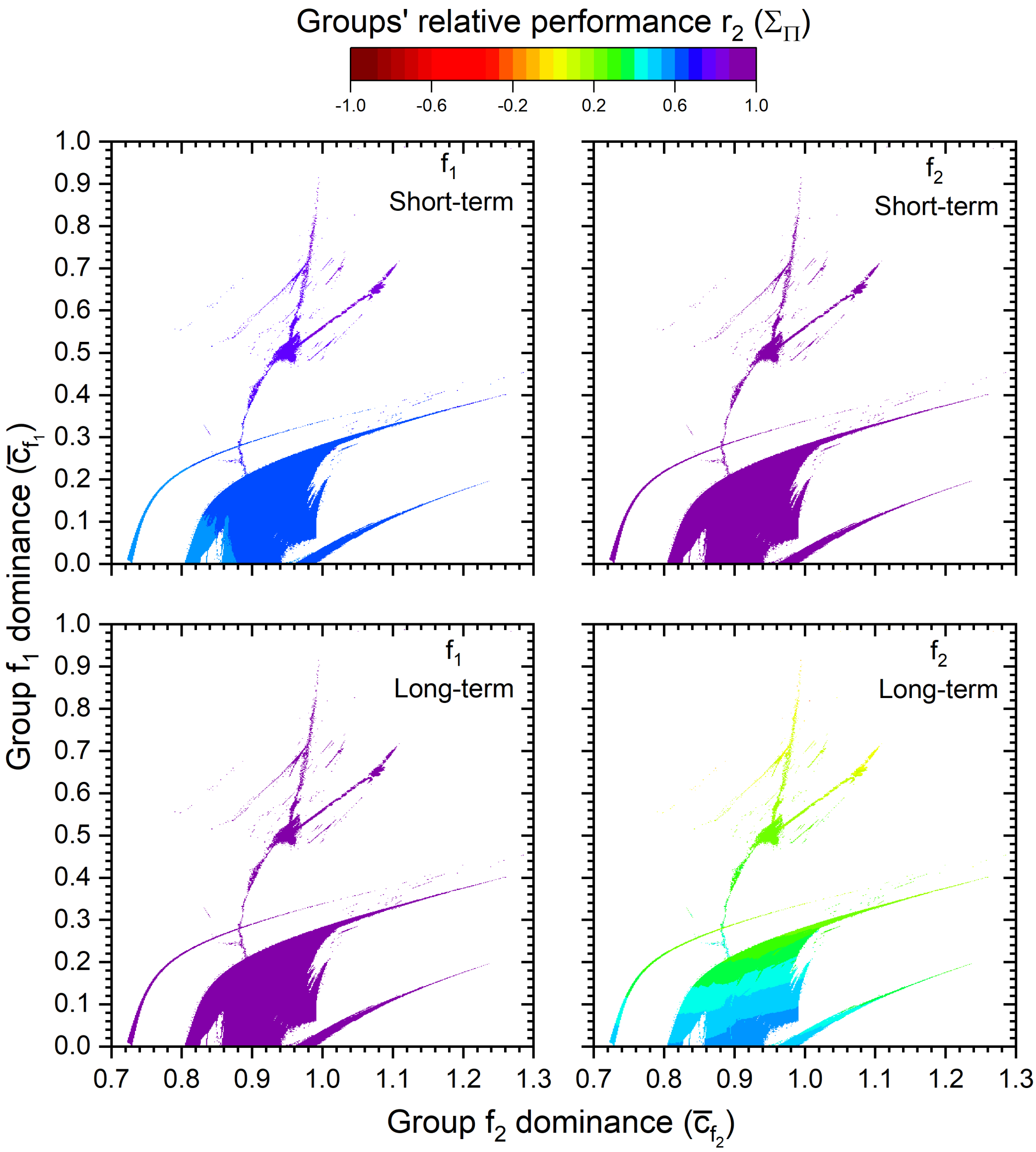


FIGURE D.7: The parameter diagrams of groups' relative short-term performance (top panels), and long-term performance (bottom panels) for orbits whose average norm stabilizes within range r_2.

D.1.4 A case of study (Section 4.3.1)

Basins of attraction and phase portrait We start by showing, in Figure D.9, the basins of attraction of all attractors of the system. In order to get this plot, we let the system evolve for different initial states $x_{f_1,0} \in I_{x_{f_1}} = [-12, 12]$ (x-axis) and $x_{f_2,0} \in I_{x_{f_2}} = [-12, 12]$ (y-axis), in steps of $\delta_{x_{f,0}} = 0.04$. After a transition of $T_{tran} = 10000$ time steps, we observe the system for a further $T_{test} = 200$ time steps in which we keep track of the orbit $\vec{x}_{f,t} = (x_{f_1,t}, x_{f_2,t})$ and take the average value of its norm over the period T_{test}. The coloured bar on the top of the Figure D.9 indicates this

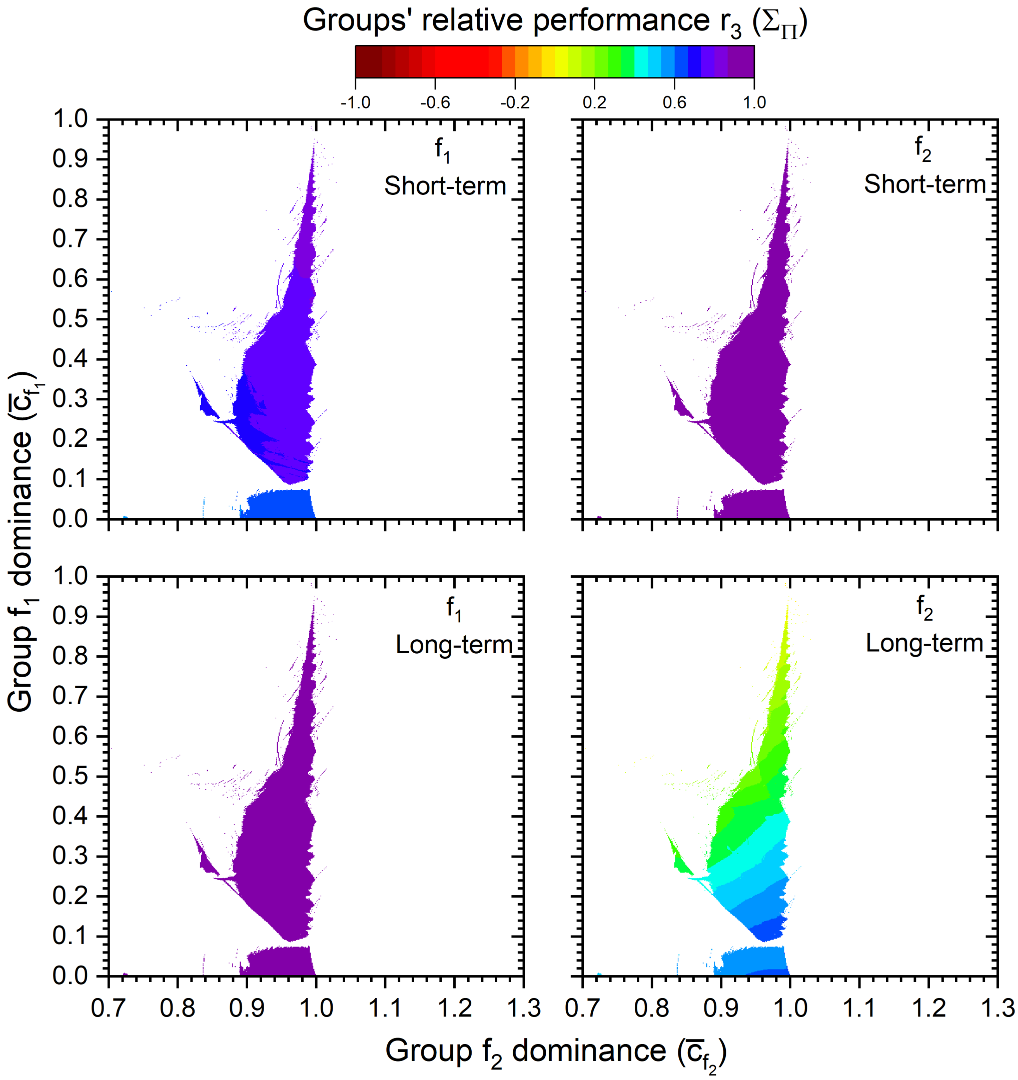


FIGURE D.8: The parameter diagrams of groups' relative short-term performance (top panels), and long-term performance (bottom panels) for orbits whose average norm stabilizes within range r_3.

value for each initial condition in $I_{x_f} \in I_{x_{f_1}} \times I_{x_{f_2}}$. The white region at the center of the diagram is the basin of attraction of the fundamental steady state $\vec{x}_o^* = (0, 0)$, the violet region surrounding it (basin of attractor '1') is that of orbits whose average norms stabilize within the r_2 region, and the outer part (basin of attractor '2') those that stabilize within the r_3 region. The values within the basin of attractor '1' are localised around $\overline{|\vec{x}_f|}_{r_2} \sim 3.8$ and lowly dispersed ($\overline{|\vec{x}_f|}_{r_2,min} \sim 3.78$ and $\overline{|\vec{x}_f|}_{r_2,max} \sim 3.86$), while those within the basin of attractor '2' are localised around $\overline{|\vec{x}_f|}_{r_3} \sim 40$ and highly dispersed ($\overline{|\vec{x}_f|}_{r_3,min} \sim 25$ and $\overline{|\vec{x}_f|}_{r_3,max} \sim 75$). In Figure D.10, we show the phase portrait of the system with some points of the orbits of both attractors[2] and, in the

[2]Starting from an initial state (($x_{f_1,0} = 7, x_{f_2,0} = 0$) for the attractor '1', ($x_{f_1,0} = 0, x_{f_2,0} = 7$) for the attractor '2'), we let the system evolve for $T = 10000$ time steps.

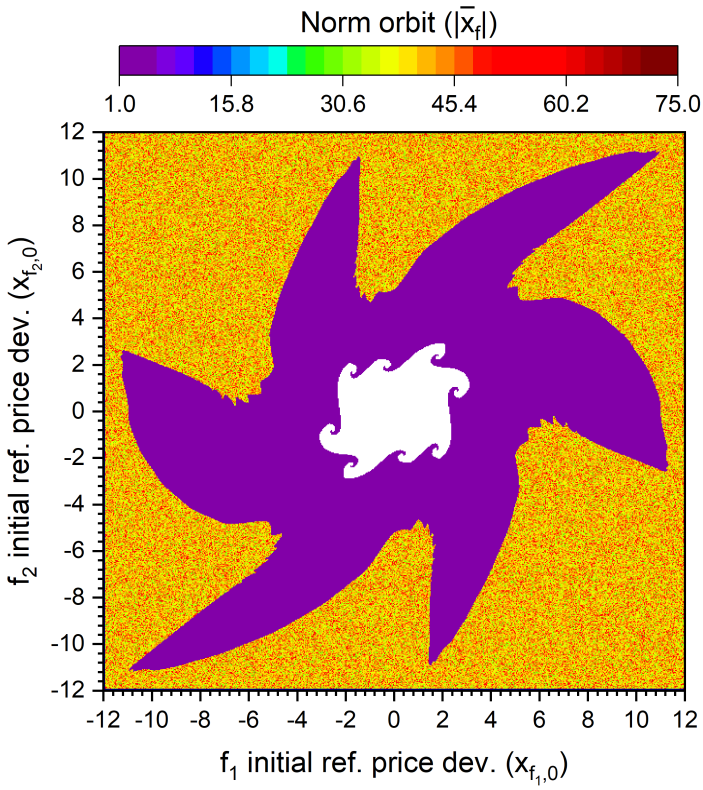


FIGURE D.9: The basins of attraction of all attractors of the system for $\bar{c}_{f_2} = 0.92$ and $\bar{c}_{f_1} = 0.21$. The coloured bar on the top indicates the average norm of the orbit $\vec{x}_{f,t}$ for each initial initial state $x_{f_1,0} \in I_{x_{f_1}} = [-12, 12]$ (x-axis) and $x_{f_2,0} \in I_{x_{f_2}} = [-12, 12]$ (y-axis) over the period T_{test}. The white region at the center of the diagram is the basin of attraction of the fundamental steady state $\vec{x}_o^* = (0, 0)$, the violet region (basin of attractor '1') surrounding it is that of orbits whose average norms stabilize within the r_2 region, and the outer part (basin of attractor '2') those that stabilize within the r_3 region.

background, their corresponding basins (the orbits circle around the origin counterclockwise). The blue points represent the orbit of attractor '1' having the qualitative feature of a (quasi-periodic) limit cycle, while the orbit of attractor '2' (red points) that of a strange attractor. To better understand the causes that lead these orbits (and, therefore, the price) to perpetually fluctuate around the origin (the fundamental value) or exhibit a high dispersion along the trajectories (in case of attractor '2') it is helpful to show investors' trading behaviour within both attractors.

The system in presence of noise In Figure D.11, we show the basins of attraction as we increase the standard deviation σ_n, first for $\sigma_n = 0.01, 0.02, 0.03$ (left panels), and then for $\sigma_n = 0.2, 0.3, 0.4$ (right panels). For low noise values, we see that the basin of attraction of the limit cycle gets easily *absorbed* by the basin of the fundamental steady state, while the basin of the strange attractor remains unchanged. At this level, the average value of 'noise traders' demand' $\bar{d}_N (= \sum_t^T |N_t(0, \sigma_n)| / T)$ is very small

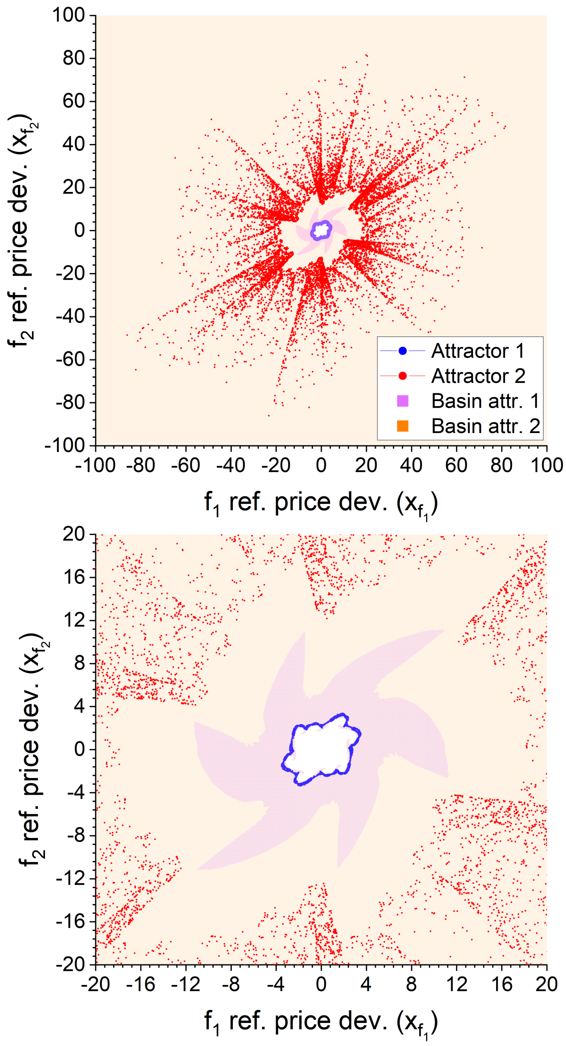


FIGURE D.10: The phase portrait of the system with some points of the orbits of attractor '1' and '2' (the orbits circle around the origin counterclockwise). In the background, we show their corresponding basins, light violet for the attractor '1' and light orange for the attractor '2'. The bottom panel shows a restricted region of the top panel, close to the origin $\vec{x}_o^*$. The blue points represent the orbit of attractor '1' (limit cycle), while the red points the orbit of attractor '2' (strange attractor).

compared to the other two groups, representing only $\sim 0.5\%$ of group f_1's demand and $\sim 0.1\%$ of group f_2's demand when the limit cycle disappears for $\sigma_n = 0.03$. The basin of attraction of $\vec{x}_o^*$ persists as we further increase the standard deviation of the noise up to $\sigma_n = 0.2$, beyond which it starts getting absorbed by the strange attractor one now. We also see that the average norm of the orbits within the r_3 region decreases, a sign of the fact that the orders sent by f_1 and f_2 are filled between the noise traders too, whose demand represents now $\sim 10\%$ of group f_1's demand and $\sim 2\%$ (for $\sigma_n = 0.4$).

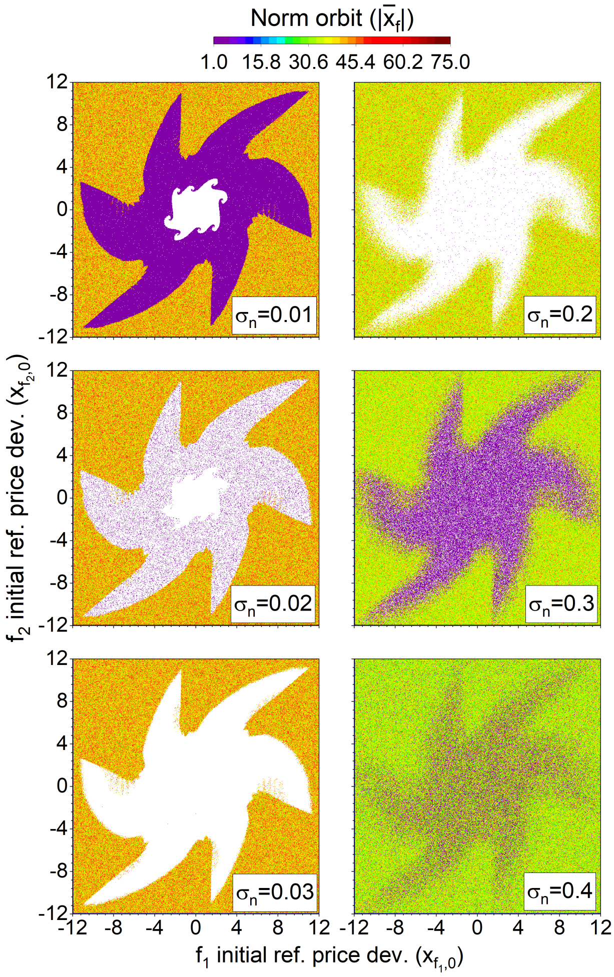


FIGURE D.11: The basins of attraction as we increase the standard deviation σ_n, first for $\sigma_n = 0.01, 0.02, 0.03$ (left panels) and then for $\sigma_n = 0.2, 0.3, 0.4$ (right panels). For low noise values, the basin of attraction of the limit cycle gets *absorbed* by the basin of the fundamental steady state, while the basin of the strange attractor remains unchanged. The basin of attraction of $\vec{x}_o^*$ persists as we further increase the standard deviation of the noise up to $\sigma_n = 0.2$, beyond which it starts getting absorbed by the strange attractor one now.

Appendix E

Computational Codes

E.1 Code for time series plots (price, demand, fraction)

```
#include<random>    //libraries
#include<omp.h>
#include<chrono>
#include<bits/stdc++.h>
#include<stdlib.h>
#include<time.h>
#include<math.h>
#include<iostream>
using namespace std;

#define noise_err 1

#define in_st_nf 0

#if noise_err

long double err=1.0;


mt19937_64 rng;   //casual number generator
normal_distribution<long double> prob(0,err);

#endif

const int T_tran=10000;
const int T_tran_test = 5000; // 0<T_tran_test<T_tran
const int T_test=50;
const int T=T_tran+T_test;


long double mu = 0.285;
long double x[T];

long double b_t_f1=1.0;
long double b_t_f2=1.0;
long double b_s_f1=1.0;
long double b_s_f2=1.0;
```

```
long double d_t_f1=0;
long double d_t_f2=0;
long double d_s_f1=1.0;
long double d_s_f2=1.0;

long double g=0.8;

long double c_f1_b=1.0;
long double c_f2_b=10.0;

long double c = (g/(1.0-g))*((c_f2_b-c_f1_b)/(c_f1_b+c_f2_b));

long double fp1 = sqrtl(powl(c,(1.0/b_s_f1)));
long double fp2 = -fp1*c_f1_b/c_f2_b;

long double c2 = (2.0-mu*c_f1_b)/(mu*c_f2_b);

long double pfp1 = sqrtl(powl((g-(2-g)*c2)/(1-g+(1+g)*c2),1/b_s_f1));
long double pfp2 = pfp1*c2;

#if in_st_nf
long double x_f1_in = fp1+0.0001;
long double x_f2_in = fp2;
#endif // in_st_nf

#if !in_st_nf
long double x_f1_in=0.0000001;
long double x_f2_in=0;
#endif

long double z_f1_b[T];
long double z_f2_b[T];
long double d_f1_b[T];
long double d_f2_b[T];
long double z_n[T];
long double x_f1[T];
long double x_f2[T];
long double n_f1[T];
long double n_f2[T];
long double a_f1[T];
long double a_f2[T];

ofstream xp;
ofstream x_f;
ofstream z_f_b;
ofstream d_f_b;
ofstream n_f;
ofstream a_f;

long double frac_t_f(long double x, long double d_t_f, long double b_t_f)
{
if(d_t_f==0)
{
    return 1.0;
}
    else
    {
    long double xsq = (x)*(x);
```

```
    long double b_t_fxsq = powl(xsq,b_t_f);
    long double b_t_fd_t_f = powl(d_t_f,b_t_f);
    long double sum_b_t_fxsq_b_t_fd_t_f = b_t_fxsq + b_t_fd_t_f;

    return b_t_fxsq/sum_b_t_fxsq_b_t_fd_t_f;
    }
}

long double frac_s_f(long double x, long double d_s_f, long double b_s_f)
{
    if(d_s_f==0)
    {
       return 1.0;
    }
    else
    {
    long double xsq = (x)*(x);
    long double b_s_fxsq = powl(xsq,b_s_f);
    long double b_s_fd_s_f = powl(d_s_f,b_s_f);
    long double sum_b_s_fxsq_b_s_fd_s_f = b_s_fxsq + b_s_fd_s_f;

    return b_s_fxsq/sum_b_s_fxsq_b_s_fd_s_f;
    }
}


int main()
{
#if noise_err

uint64_t timeSeed = chrono::high_resolution_clock::now().time_since_epoch().count();
seed_seq ss{uint32_t(timeSeed & 0xffffffff), uint32_t(timeSeed>>32)};
// it initializes the casual number generator with a time-dependent seed rng.seed(ss);

    #endif // noise_err

    xp.open("price.txt", ios::out);
    x_f.open("reference_f.txt", ios::out);
    z_f_b.open("demand_f.txt", ios::out);
    d_f_b.open("netdemand_f.txt", ios::out);
    n_f.open("fraction_tr.txt", ios::out);
    a_f.open("fraction_sh.txt", ios::out);

    #if noise_err
    long double noise=prob(rng);
    #endif // noise_err


    x[0]=x_f1_in;
    x_f1[0]=x[0];
    x_f2[0]=x_f2_in;
    z_f1_b[0]=-c_f1_b*x_f1[0];
    z_f2_b[0]=-c_f2_b*x_f2[0];

    #if noise_err
    z_n[0]=-noise;
    #endif // noise_err

    #if !noise_err
```

```
z_n[0]=0;
#endif


n_f1[0]=frac_t_f(x_f1[0], d_t_f1, b_t_f1);
n_f2[0]=frac_t_f(x_f2[0], d_t_f2, b_t_f2);
d_f1_b[0]=z_f1_b[0]*n_f1[0];
d_f2_b[0]=z_f2_b[0]*n_f2[0];
a_f1[0]=frac_s_f(x_f1[0], d_s_f1, b_s_f1);
a_f2[0]=1-frac_s_f(x_f1[0], d_s_f2, b_s_f2);

for(int i=1; i<T_tran; i++)
{

    #if noise_err
    x[i] = x[i-1] - mu*(c_f1_b*x_f1[i-1]*n_f1[i-1]+c_f2_b*x_f2[i-1]*n_f2[i-1]+noise);
    #endif // noise_err

    #if !noise_err
    x[i] = x[i-1] - mu*(c_f1_b*x_f1[i-1]*n_f1[i-1]+c_f2_b*x_f2[i-1]*n_f2[i-1]);
    #endif

    x_f1[i]=x[i];
    x_f2[i]=x_f2[i-1]+(1.0-g)*(x_f1[i-1]-x_f2[i-1])*a_f1[i-1]-g*(x_f1[i-1]+x_f2[i-1])*a_f2[i-1];
    z_f1_b[i]=-c_f1_b*x_f1[i];
    z_f2_b[i]=-c_f2_b*x_f2[i];

    #if noise_err
    noise=prob(rng);
    z_n[i]=-noise;
    #endif // noise_err

    #if !noise_err
    z_n[i]=0;
    #endif

    n_f1[i]=frac_t_f(x_f1[i], d_t_f1, b_t_f1);
    n_f2[i]=frac_t_f(x_f2[i], d_t_f2, b_t_f2);
    d_f1_b[i]=z_f1_b[i]*n_f1[i];
    d_f2_b[i]=z_f2_b[i]*n_f2[i];
    a_f1[i]=frac_s_f(x_f1[i], d_s_f1, b_s_f1);
    a_f2[i]=1-frac_s_f(x_f1[i], d_s_f2, b_s_f2);

}


 for(int i=T_tran; i<T; i++)
 {
    #if noise_err
    x[i] = x[i-1] - mu*(c_f1_b*x_f1[i-1]*n_f1[i-1]+c_f2_b*x_f2[i-1]*n_f2[i-1]+noise);
    #endif // noise_err

    #if !noise_err
    x[i] = x[i-1] - mu*(c_f1_b*x_f1[i-1]*n_f1[i-1]+c_f2_b*x_f2[i-1]*n_f2[i-1]);
    #endif

    x_f1[i]=x[i];
    x_f2[i]=x_f2[i-1]+(1.0-g)*(x_f1[i-1]-x_f2[i-1])*a_f1[i-1]-g*(x_f1[i-1]+x_f2[i-1])*a_f2[i-1];
    z_f1_b[i]=-c_f1_b*x_f1[i];
```

```
        z_f2_b[i]=-c_f2_b*x_f2[i];

        #if noise_err
        noise=prob(rng);
        z_n[i]=-noise;
        #endif // noise_err

        #if !noise_err
        z_n[i]=0;
        #endif

        n_f1[i]=frac_t_f(x_f1[i], d_t_f1, b_t_f1);
        n_f2[i]=frac_t_f(x_f2[i], d_t_f2, b_t_f2);
        d_f1_b[i]=z_f1_b[i]*n_f1[i];
        d_f2_b[i]=z_f2_b[i]*n_f2[i];
        a_f1[i]=frac_s_f(x_f1[i], d_s_f1, b_s_f1);
        a_f2[i]=1.0-frac_s_f(x_f1[i], d_s_f2, b_s_f2);

     }

x_f<<"time"<<"\t"<<"f1_refdev"<<"\t"<<"f2_refdev"<<"\t"<<"fp1"<<"\t"<<"fp2"<<"\t";
x_f<<"pfp1"<<"\t"<<"pfp2"<<"\t"<<"aver_pdev"<<"\t"<<"aver_pdev_displ_st"<<endl;
d_f_b<<"time"<<"\t"<<"f1_netd"<<"\t"<<"f2_netd"<<"\t"<<"noise_netd"<<endl;

 for(int i=T_tran; i<T; i++)
    {
xp<<i-T_tran<<"\t"<<x[i-1]<<endl;
x_f<<i-T_tran<<"\t"<<x_f1[i-1]<<"\t"<<x_f2[i-1]<<"\t";
x_f<<fp1<<"\t"<<fp2<<"\t <<pfp1<<"\t<<pfp2<<endl;
z_f_b<<i-T_tran<<"\t"<<z_f1_b[i-1]<<"\t"<<z_f2_b[i-1]<<"\t"<<z_n[i-1]<<endl;
d_f_b<<i-T_tran<<"\t"<<d_f1_b[i-1]<<"\t"<<d_f2_b[i-1]<<"\t"<<z_n[i-1]<<endl;
n_f<<i-T_tran<<"\t"<<n_f1[i-1]<<"\t"<<n_f2[i-1]<<endl;
a_f<<i-T_tran<<"\t"<<a_f1[i-1]<<"\t"<<a_f2[i-1]<<endl;
     }

    return 0;
}
```

E.2 Code for diagram plots (orbit, Lyapunov, performance)

```
#include<math.h>     //libraries
#include<fstream>
#include<fstream>
#include<random>
#include<omp.h>
#include<chrono>
#include<bits/stdc++.h>
#include<stdlib.h>
#include<time.h>
#include<iostream>
#include <ctime>

using namespace std;

#define foll_dem 1
```

```
const int T_tran=10000;
const int T_tran_test = 5000; // 0<T_tran_test<T_tran
const int T_test=28;
const int T=T_tran+T_test;

long double e_neg=powl(10.0,-100.0);
long double e_pos=powl(10.0,+100.0);
long double ln_e_neg = logl(e_neg);
long double ln_e_pos = logl(e_pos);

long double g=0.8;
long double bs=1.0;

long double mu_in = 0.01;
int Dmu = 600;
long double dmu = 0.001;

long double c_f1_b=1.0;
long double c_f2_b=10.0;

long double x_f1_st=-2.0;
int Dx_f1 = 10;
long double dx_f1 = 0.4;

long double x_f2_st=2.0;
int Dx_f2 = 10;
long double dx_f2 = 0.4;

int nth = 10;
int nth_tot_w = (Dmu)/nth;

ofstream mu_l, mu_xf1, mu_xf2, mu_af1, mu_sp_pf1;
ofstream mu_lp_pf1, mu_sp_pf2, mu_lp_pf2, mu_pdaver, mu_pdaver_stdispl, mu_param;

int main()
{
mu_l.open("mu_vs_ly.txt", ios::out);
mu_xf1.open("mu_vs_xf1.txt", ios::out);
mu_xf2.open("mu_vs_xf2.txt", ios::out);
mu_af1.open("mu_vs_af1.txt", ios::out);

mu_sp_pf1.open("mu_vs_sp_pf1.txt", ios::out);
mu_lp_pf1.open("mu_vs_lp_pf1.txt", ios::out);

mu_sp_pf2.open("mu_vs_sp_pf2.txt", ios::out);
mu_lp_pf2.open("mu_vs_lp_pf2.txt", ios::out);

mu_pdaver.open("mu_vs_pdaver.txt", ios::out);
mu_pdaver_stdispl.open("mu_vs_pdaver_stdispl.txt", ios::out);
mu_param.open("mu_param.txt", ios::out);

mu_param<<"T_tran"<<"\t"<<"T_test"<<"\t"<<"x_f1_st"<<"\t"<<"dx_f1"<<"\t"<<"x_f2_st"<<"\t"<<"dx_f2";
mu_param<<"\t"<<"g"<<"\t"<<"bs"<<"\t"<<"c_f1_b"<<"\t"<<"c_f2_b"<<endl;
mu_param<<T_tran<<"\t"<<T_test<<"\t"<<x_f1_st<<"\t"<<dx_f1<<"\t"<<x_f2_st<<"\t"<<dx_f2;
mu_param<<"\t"<<g<<"\t"<<bs<<"\t"<<c_f1_b<<"\t"<<c_f2_b<<endl;

    int iter=0;
    int step = 1;
```

```
auto start = chrono::steady_clock::now();

time_t timetoday;
 time(&timetoday);
 cout << "Simulation started at: " << asctime(localtime(&timetoday))<<endl<<endl;

for(int o=0; o<Dmu; o++){

long double mu = mu_in + o*dmu;

mu_xf1<<mu<<"\t";
mu_xf2<<mu<<"\t";
mu_af1<<mu<<"\t";

 mu_l<<mu<<"\t";

 mu_sp_pf1<<mu<<"\t";
 mu_lp_pf1<<mu<<"\t";

 mu_sp_pf2<<mu<<"\t";
 mu_lp_pf2<<mu<<"\t";

 mu_pdaver<<mu<<"\t";
 mu_pdaver_stdispl<<mu<<"\t";

 #pragma omp parallel for collapse(2) schedule (dynamic,1)
for(int o=0; o<Dx_f1; o++){
        for(int p=0; p<Dx_f2; p++){

long double J[2][2];
long double Ju[2];
long double Ju_max;
long double Ju_temp[2];
long double Ju_norm;
long double Ju_norm_add;

long double sum_sp_f1=0;
long double sum_sp_f2=0;
long double sum_abs_sp_f1=0;
long double sum_abs_sp_f2=0;

long double sum_lp_f1=0;
long double sum_lp_f2=0;
long double sum_abs_lp_f1=0;
long double sum_abs_lp_f2=0;

long double sp_f1=0.0;
long double sp_f2=0.0;
long double S_sp_f1=0.0;
long double S_sp_f2=0.0;

long double lp_f1=0.0;
long double lp_f2=0.0;
long double S_lp_f1=0.0;
long double S_lp_f2=0.0;

long double h;

long double x_f1_in = x_f1_st + o*dx_f1;
```

```
long double x_f2_in = x_f2_st - p*dx_f2;

if ((x_f1_in<powl(10.0,-5.0))&&(x_f1_in>-powl(10.0,-5.0)))
{
    x_f1_in = x_f1_in+0.00001;
}

long double x_f1_pre = x_f1_in;
long double x_f2_pre = x_f2_in;

long double xsq = x_f1_pre*x_f1_pre;
long double x_f1_2bs = powl(xsq,bs);

long double a_f1_pre = x_f1_2bs/(x_f1_2bs+1.0);
long double a_f2_pre = 1.0-a_f1_pre;

long double x_f1_post;
long double x_f2_post;

long double x_f1[T_test+1];
long double x_f2[T_test+1];

long double aver_pdev_displ_st = 0;
long double aver_pdev = 0;

long double c=0;
bool div=false;

int i=1;

    while (i<T_tran)
{

    x_f1_post = x_f1_pre - mu*(c_f1_b*x_f1_pre+c_f2_b*x_f2_pre);
    x_f2_post=x_f2_pre+(1-g)*(x_f1_pre-x_f2_pre)*a_f1_pre+g*(-x_f1_pre-x_f2_pre)*a_f2_pre;

    x_f1_pre=x_f1_post;
    x_f2_pre=x_f2_post;

    xsq = x_f1_pre*x_f1_pre;
    x_f1_2bs = powl(xsq,bs);

    a_f1_pre = x_f1_2bs/(x_f1_2bs+1.0);
    a_f2_pre = 1.0-a_f1_pre;

    if(i>=T_tran_test)
    {

       h=2.0*bs*x_f1_pre*x_f1_2bs/((xsq)*(x_f1_2bs+1.0)*(x_f1_2bs+1.0));

       J[0][0] = 1.0 - mu*c_f1_b;
       J[0][1] = - mu*c_f2_b;
       J[1][0] = -g+a_f1_pre+((1.0-g)*(x_f1_pre-x_f2_pre)+g*(x_f1_pre+x_f2_pre))*h;
       J[1][1] = 1.0-g+(2.0*g-1.0)*a_f1_pre;

        if(i==T_tran_test)
        {
            Ju[0]=x_f1_pre/(1000.0);
            Ju[1]=x_f2_pre/(1000.0);
```

```
        Ju_max=0;
        Ju_norm_add=0;

    for (int j = 0; j< 2; j++)
        {
            Ju_temp[j]=Ju[j];
        }

        for (int j = 0; j < 2; j++) {

        Ju[j]=0;

    for (int k = 0; k < 2; k++) {
        Ju[j] += J[j][k] * Ju_temp[k];
    }
}

    }
    else
    {
        for (int j = 0; j < 2; j++)
        {
            Ju_temp[j]=Ju[j];
        }

        for (int j = 0; j < 2; j++) {

        Ju[j]=0;

    for (int k = 0; k < 2; k++) {
        Ju[j] += J[j][k] * Ju_temp[k];
    }

    if (abs(Ju[j])>Ju_max)
    {
        Ju_max=abs(Ju[j]);
    }

}

//divergence check
if((Ju_max<e_neg)||(Ju_max>e_pos))
{

    if (Ju_max<e_neg)
    {
       for (int j = 0; j < 2; j++)
        {
            Ju[j]=Ju[j]/e_neg;
        }

        Ju_norm_add += ln_e_neg;
    }
    else if (Ju_max>e_pos)
    {
       for (int j = 0; j < 2; j++)
        {
            Ju[j]=Ju[j]/e_pos;
```

```
                }

                 Ju_norm_add += ln_e_pos;
            }

            }

        }

        Ju_max=0;

        }

        i = i + 1;

        if (x_f1_pre>1000*abs(x_f1_st))
        {
            i=T_tran;
            div=true;
        }
    }

    if(!div)
    {

     while (i<T)
    {
        x_f1_post = x_f1_pre - mu*(c_f1_b*x_f1_pre+c_f2_b*x_f2_pre);
        x_f2_post = x_f2_pre+(1-g)*(x_f1_pre-x_f2_pre)*a_f1_pre+g*(-x_f1_pre-x_f2_pre)*a_f2_pre;

                if(i==T_tran)
                {
                    aver_pdev = aver_pdev + x_f1_pre;
                    x_f1[0]=x_f1_pre;
                    x_f2[0]=x_f2_pre;
                }

          aver_pdev_displ_st = aver_pdev_displ_st + abs(x_f1_post-x_f1_pre);
          aver_pdev = aver_pdev + x_f1_post;
          x_f1[i-T_tran+1]=x_f1_post;
          x_f2[i-T_tran+1]=x_f2_post;

          c++;

        //short-term performance
        sp_f1 = -(x_f1_post-x_f1_pre)*x_f1_pre;
        sp_f2 = -(x_f1_post-x_f1_pre)*x_f2_pre;
        sum_sp_f1 = sum_sp_f1 + sp_f1;
        sum_sp_f2 = sum_sp_f2 + sp_f2;
        sum_abs_sp_f1 = sum_abs_sp_f1 + abs(sp_f1);
        sum_abs_sp_f2 = sum_abs_sp_f2 + abs(sp_f2);

        if(sum_sp_f1==0)
        {
          S_sp_f1=0;
        }
        else
        {
          S_sp_f1 = sum_sp_f1/sum_abs_sp_f1;
```

```
}

 if(sum_sp_f2==0)
{
  S_sp_f2=0;
}
else
{
  S_sp_f2 = sum_sp_f2/sum_abs_sp_f2;
}

x_f1_pre=x_f1_post;
x_f2_pre=x_f2_post;

xsq = x_f1_pre*x_f1_pre;
x_f1_2bs = powl(xsq,bs);

a_f1_pre = x_f1_2bs/(x_f1_2bs+1.0);
a_f2_pre = 1.0-a_f1_pre;

#pragma omp critical
{
    if(abs(x_f1_pre)<e_neg)
    {
mu_xf1<<0<<"\t";
#if foll_dem
mu_xf2<<0<<"\t";
#endif // foll_dem

#if !foll_dem
mu_xf2<<0<<"\t";
#endif // foll_dem

mu_af1<<0<<"\t";
    }
    else
    {
        mu_xf1<<x_f1_pre<<"\t";
#if foll_dem
mu_xf2<<-c_f2_b*x_f2_pre<<"\t";
#endif // foll_dem

#if !foll_dem
mu_xf2<<x_f2_pre<<"\t";
#endif // foll_dem

mu_af1<<a_f1_pre<<"\t";
    }

}

h=2.0*bs*x_f1_pre*x_f1_2bs/((xsq)*(x_f1_2bs+1.0)*(x_f1_2bs+1.0));

J[0][0] = 1.0 - mu*c_f1_b;
J[0][1] = - mu*c_f2_b;
J[1][0] = -g+a_f1_pre+((1.0-g)*(x_f1_pre-x_f2_pre)+g*(x_f1_pre+x_f2_pre))*h;
J[1][1] = 1.0-g+(2.0*g-1.0)*a_f1_pre;

 for (int j = 0; j < 2; j++)
```

```
            {
                Ju_temp[j]=Ju[j];
            }

            for (int j = 0; j < 2; j++) {

            Ju[j]=0;

        for (int k = 0; k < 2; k++) {
            Ju[j] += J[j][k] * Ju_temp[k];
        }

        if (abs(Ju[j])>Ju_max)
        {
            Ju_max=abs(Ju[j]);
        }
    }

    if((Ju_max<e_neg)||(Ju_max>e_pos))
    {

        if (Ju_max<e_neg)
        {
           for (int j = 0; j < 2; j++)
            {
                Ju[j]=Ju[j]/e_neg;
            }

            Ju_norm_add += ln_e_neg;
        }
        else if (Ju_max>e_pos)
        {
           for (int j = 0; j < 2; j++)
            {
                Ju[j]=Ju[j]/e_pos;
            }

             Ju_norm_add += ln_e_pos;
        }
        }

    Ju_max=0;

    i = i + 1;

    if ((x_f1_pre>1000*abs(x_f1_st))||div)
    {
        i=T;
        div=true;
    }
}

if(!div)
{
aver_pdev = aver_pdev/(c+1.0);
aver_pdev_displ_st = aver_pdev_displ_st/c;

if (abs(aver_pdev_displ_st)<powl(10.0,-10.0))
{
```

```
    S_lp_f1=0;
    S_lp_f2=0;
    S_sp_f1=0;
    S_sp_f2=0;
}
else{
for (int j=0; j<T_test+1; j++)
{

lp_f1=(x_f1[j]-aver_pdev)*x_f1[j];
lp_f2=(x_f1[j]-aver_pdev)*x_f2[j];

sum_lp_f1 = sum_lp_f1 + lp_f1;
sum_lp_f2 = sum_lp_f2 + lp_f2;
sum_abs_lp_f1 = sum_abs_lp_f1 + abs(lp_f1);
sum_abs_lp_f2 = sum_abs_lp_f2 + abs(lp_f2);

if(sum_lp_f1==0)
    {
      S_lp_f1=0;
    }
    else
    {
      S_lp_f1 = sum_lp_f1/sum_abs_lp_f1;
    }

     if(sum_lp_f2==0)
    {
      S_lp_f2=0;
    }
    else
    {
      S_lp_f2 = sum_lp_f2/sum_abs_lp_f2;
    }
}
}

Ju_norm=0;

    for (int j = 0; j < 2; j++)
    {
        Ju_norm += Ju_norm + Ju[j]*Ju[j];
    }

    Ju_norm = (logl(Ju_norm))/(2.0)+Ju_norm_add;

    Ju_norm = Ju_norm/(T_tran-T_tran_test+T_test);

    #pragma omp critical
    {

  mu_l<<Ju_norm<<"\t";

  mu_sp_pf1<<S_sp_f1<<"\t";
  mu_lp_pf1<<S_lp_f1<<"\t";

  mu_sp_pf2<<S_sp_f2<<"\t";
  mu_lp_pf2<<S_lp_f2<<"\t";
```

```
       if(abs(aver_pdev)<e_neg)
       {
        mu_pdaver<<0<<"\t";
       }
       else
       {
        mu_pdaver<<aver_pdev<<"\t";
       }


      if(abs(aver_pdev_displ_st)<e_neg)
       {
         mu_pdaver_stdispl<<0<<"\t";
       }
       else
       {
         mu_pdaver_stdispl<<aver_pdev_displ_st<<"\t";
       }
     }

    }
    }

 }
}

    mu_xf1<<endl;
    mu_xf2<<endl;
    mu_af1<<endl;

    mu_l<<endl;

    mu_sp_pf1<<endl;
    mu_lp_pf1<<endl;

    mu_sp_pf2<<endl;
    mu_lp_pf2<<endl;

    mu_pdaver<<endl;
    mu_pdaver_stdispl<<endl;

    iter = iter + 1;

     if ((iter%nth_tot_w)==0)
     {
         auto end = chrono::steady_clock::now();
         cout<<step<<" "<<"out of "<<nth<<" steps done"<<" - Elapsed time in seconds: ";
         cout<< chrono::duration_cast<chrono::seconds>(end - start).count()<< " sec"<<endl;
         step = step + 1;
     }
    }

    return 0;
}
```

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