EXPECTED UTILITY: WEAK AND STRONG

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In memory of Professor Tapan Mitra on the occasion of his 75th birthday.

Abstract. In this paper we show that, given a monotone preference order that satisfies weak independence, the expected utility model (as a *weak* or *strong* representation) can be directly obtained from intuitive conditions. In particular, a weak expected utility representation can be constructed by using *substitutability*, while the stronger counterpart can be derived by using *solvability*. We provide simple examples which demonstrate that our characterization of the expected utility preferences is tight. Unlike many other expected utility characterizations, our proofs are constructive making them especially useful when testing the model.
Keywords. Expected utility; Monotonicity; Independence; Substitutability; Solvability; Continuity.

1. Introduction

The expected utility (EU) model is the main workhorse for decision making under uncertainty with wide range of theoretical and empirical applications. Two key implications of the EU model are well-known, independence and continuity. In their seminal work on the theory of games, von Neumann and Morgenstern [20] were the first to show that a strong independence (IND) axiom and an order-theoretic Archimedean continuity (aCON) are together sufficient to obtain the EU model as representation of a given preference order. Herstein and Milnor [9], on the other hand, used a weaker independence (hm-IND; an independence condition restricted to the indifference subsets of the preference order with mixture-weights fixed at $1/2$), while they required a stronger topological mixture continuity (mCON) to characterize the EU preferences.

In this paper, we consider both type of axioms (independence and continuity) in their weaker forms while we confine our analysis to monotone preference orders. Specifically, we use the hm-IND axiom but we replace mCON with weaker axioms, substitutability (SUB) or solvability (SOL), and show that a preference order over lotteries admits an expected utility representation if it satisfies monotonicity (MON). The SUB axiom requires that any outcome should be exchangeable with a lottery comprising of the worst and the best outcomes. The SOL axiom requires that outcomes, not only in isolation (as in SUB) but also when they together form a lottery, should be exchangeable with a lottery comprising of the two extreme outcomes. The MON axiom is defined with respect to the first order stochastic dominance relation reflecting

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Although von Neumann and Morgenstern [20] provided the first expected utility representation result, the independence axiom was implicit in their result. The first explicit statement of the independence axiom appeared in Marschak [13], Nash [15], and Malinvaud [12]; see, e.g., Hammond [8] and Bleichrodt et al [5] for details.

the intuitive idea that the more likely to receive preferred outcomes the better the decision maker becomes.

We establish the EU representation in two steps. We first identify the axioms under which the preference order $\succsim$ over the set of lotteries P (defined over a given finite set of outcomes X) admits a weak expected utility representation $u : P \to \mathbb{R}$; that is, for any $p, q \in P$, $p \succ q$ if $u(p) > u(q)$. A weak expected utility representation allows the decision maker to choose among lotteries a best lottery as long as the lotteries do not yield the same utility, while it becomes indecisive whenever there are two or more best lotteries. We then show when the weak representation $u : P \to \mathbb{R}$ becomes a strong representation; that is, for any $p, q \in P$, $p \succ q$ if and only if $u(p) > u(q)$. A strong expected utility representation not only allows the decision maker to choose a best lottery, but also lets her declare indifferences whenever there are two or more best lotteries. In other words, a strong representation ensures that there is no room left for improvement in terms of preference maximization allowing the decision maker to be decisive for sure in a suitably given choice problem.

Our first main result (Theorem 1) shows that MON, hm-IND, and SUB are sufficient for a weak expected utility representation $u : P \to \mathbb{R}$. A counter example (Example 4.1) shows that these axioms are not necessary for weak expected utility representation (even when hm-IND is strengthened to IND). Another counter example (Example 4.2) shows that these axioms are not sufficient for strong expected utility representation. Finally, our second main result (Theorem 2), by replacing SUB with its stronger counterpart SOL, shows that MON, hm-IND, and SOL are both necessary and sufficient for a strong expected utility representation $u : P \to \mathbb{R}$.

Behavioral implications of the EU model have been tested in several experimental settings since the first critique of the EU model provided by Allais [2]. There have been some recent experimental findings which directly or indirectly test the EU axioms. Among these work, Agranov and Ortoleva [1] use the multiple price list method documenting the range of violations of SUB and SOL; Nielsen and Rehbeck [16] use simple lottery choices documenting the frequency of violations of MON and IND; Dembo et al. [7] apply revealed preference techniques with contingent commodity budget sets documenting the extent and prevalence of violations of preference maximization that satisfies MON and/or IND. Our axiomatic results are useful in interpreting these recent experimental findings, as well as providing some rationale for understanding why some of the EU axioms are violated more often than the others. We briefly discuss the connection of our work to this recent literature after providing our representation results.

The rest of the paper is organized as follows. In Section 2, we introduce our framework. Section 3 provides a brief overview of some of the earlier expected utility theorems mentioned above. In Section 4, we provide our analysis of the expected utility by employing different forms of independence and continuity axioms. In Section 5, we discuss the relation of our theoretical results with some recent experimental results documenting violations of several axioms of the EU model including monotonicity, independence, and continuity (as substitutability or solvability). Proofs of all results are provided in an Appendix.

See Starmer [19] for a review of this early literature. See Ozbek [17] for an account of recent theoretical literature on risk preferences which can accommodate Allais type paradoxes.

2. FRAMEWORK

Let I denote the set $\{1,2,...,n\}$ and let $I_0 = I \cup \{0\}$, where $n \geq 2$. In the following, X is a finite set of $n+1$ prizes, with typical elements $x_i \in X$ for $i \in I_0$ called *outcomes*; P is the set of all probability distributions on X with typical elements $p,q,r \in P$ called *lotteries*. With slight abuse of notation, we denote a lottery yielding an outcome $x \in X$ for sure by $x \in P$. We denote by p_i the probability of outcome x_i under lottery p. For any $\alpha \in [0,1]$, let $p\alpha q$ denote a *mixed-lottery*, which is the mixture of lotteries p and q. That is, $p\alpha q$ is the lottery $r \in P$ such that $r_i = \alpha p_i + (1-\alpha)q_i$ for all $i \in I_0$.

Our primitive is a binary relation $\succsim$ on the set of lotteries, with asymmetric part denoted $\succ$ and symmetric part denoted $\sim$. We interpret this binary relation $\succsim$ as the DM's risk preferences and assume that it is a preference order (i.e., a complete and transitive binary relation). We assume that the outcomes are ordered such that $x_n \succ p \succ x_0$ for any $p \in P$ with $p_i > 0$ for some $i \in I \setminus \{n\}$. Without loss of generality assume that $x_i \succ x_j$ if and only if $i > j$.

We say that $\succsim$ has a *weak representation* if there exists a real-valued function $f : P \to \mathbb{R}$ such that $p \succ q$ whenever $f(p) > f(q)$. We say that $\succsim$ has a *partial representation* if there exists a real-valued function $f : P \to \mathbb{R}$ such that $f(p) > f(q)$ whenever $p \succ q$. We say that f is a *strong representation* whenever $p \succ q$ if and only if $f(p) > f(q)$; that is, f is a strong representation if it serves both as a weak and a partial representation. Note that whenever for a given preference order $\succsim$ there is a weak representation f and a partial representation g, then there must exist some *weakly increasing* transformation $\phi : \mathbb{R} \to \mathbb{R}$ such that $f(p) = \phi(g(p))$ for all $p \in P$. Whenever ϕ is *strictly increasing*, then both f and g become a strong representation for the preference order $\succsim$.

In order to illustrate these different types of representations, consider the following example.

Example 2.1. Let $\succsim$ be a preference order such that for any $p,q \in P$,

$$p \succsim q \text{ if } \left(\lfloor u(p) \rfloor , \lceil u(p) \rceil \right) \geq_L \left(\lfloor u(q) \rfloor , \lceil u(q) \rceil \right),$$

where $\geq_L$ is the lexicographic order defined on $\mathbb{R}^2$, $u(.)$ is a utility function defined on P, $\lfloor . \rfloor$ is the floor function, and $\lceil . \rceil$ is the ceiling function. $\diamondsuit$

In Example 2.1 above, we have $p \succ q$ whenever $\lfloor u(p) \rfloor > \lfloor u(q) \rfloor$, and so the function $f(.) = \lfloor u(.) \rfloor$ is a weak representation for the preference order. Similarly, since we have $u(p) > u(q)$ whenever $p \succ q$, the function $g(.) = u(.)$ is a partial representation for the preference order. The transformation between the weak representation $\lfloor u(.) \rfloor$ and partial representation $u(.)$ is obviously the floor function $\lfloor . \rfloor$, which is weakly increasing. It is easy to verify that the function $h(.) = \lfloor u(.) \rfloor + \lceil u(.) \rceil$ provides a strong representation for the preference order.

Finally, let U denote the set of normalized utilities that are monotone with respect to the DM's preferences $\succsim$ over X; that is, $x_i \succ x_j$ if and only if $u_i > u_j$. More formally, let $U = \{u \in \mathbb{R}^{n+1} : u_0 = 0, u_n = 1, u_i > u_j \text{ iff } x_i \succ x_j\}$. For any $p \in P$ and $u \in \mathbb{R}^{n+1}$, let $u(p)$ denote the product $u \cdot p \in \mathbb{R}$. In particular, when $u \in U$, let $u(p) \in [0,1]$ denote the expected utility of p under u. We say a (resp. weak, partial, or strong) representation function f is a (resp. weak, partial, or strong) *expected utility representation* if there exists $u \in U$ such that for all $p \in P$,

Equivalently, f is a strong representation whenever for all $p,q \in P$, we have $p \succsim q$ if and only if $f(p) \geq f(q)$.

This is true since a preference order $\succsim$ is complete, for any $p,q \in P$ we have $g(p) \geq g(q)$ implies $p \succsim q$ which implies $f(p) \geq f(q)$.

we have $f(p) = u(p)$. Note that whenever there is a normalized utility $u \in U$ providing (weak, partial, or strong) expected utility representation, then it must be unique in U.

3. Axioms and facts

In this section, we briefly review some of the well known expected utility theorems.

von Neumann-Morgenstern expected utility theorem. The following axiom is the key behavioral implication of the expected utility model.

Axiom 1 (Independence, IND)**.** For any $p,q,r \in P$ and $\alpha \in (0,1)$, $p \succ q$ (resp. $p \sim q$) implies $p\alpha r \succ q\alpha r$ (resp. $p\alpha r \sim q\alpha r$).

Independence says that mixing two lotteries with a common lottery should not alter the preference for any mixture weight or common lottery used. In addition to IND, a continuity axiom is needed to establish an expected utility representation. The following continuity axiom is arguably the simplest one used in the literature for this purpose.

Axiom 2 (Archimedean Continuity, aCON)**.** For any $p,q,r \in P$, $p \succ q \succ r$ implies $p\alpha r \succ q$ and $q \succ p\beta r$ for some $\alpha, \beta \in (0,1)$.

Archimedean continuity states that there is no lottery so good (resp. bad) that when mixed with a lottery worse (resp. better) than another lottery, the mixture is always better (resp. worse) than the intermediate lottery. As is well known, these two axioms imply an expected utility representation.

Fact 1 (von Neumann-Morgenstern)**.** A preference order $\succsim$ satisfies IND and aCON if and only if it has a strong expected utility representation.

This result, given by von Neumann and Morgenstern [20], provided the first axiomatic foundation for the expected utility model.

Herstein-Milnor expected utility theorem. Another well-known expected utility theorem uses a topological mixture continuity axiom instead (rather than the order-theoretic aCON) while it weakens the independence requirement.

Axiom 3 (Mixture Continuity, mCON)**.** For any $p,q,r \in P$, the sets $\{\alpha : p\alpha r \succsim q\}$ and $\{\alpha : q \succsim p\alpha r\}$ are closed.

Mixture continuity implies that the preference ordering is continuous in probability distributions by requiring above two sets to be closed with respect to the standard topology. To establish an expected utility representation, mCON is associated with an independence condition.

Axiom 4 (Herstein-Milnor Independence, hm-IND)**.** For any $p,q,r \in P$, $p \sim q$ implies $p1/2r \sim q1/2r$.

The Herstein-Milnor independence axiom fixes the mixture weight at $1/2$ and requires independence to hold only for the indifference relation. Herstein and Milnor [9] showed that these two axioms imply an expected utility representation.

Fact 2 (Herstein-Milnor)**.** A preference order $\succsim$ satisfies hm-IND and mCON if and only if it has a strong expected utility representation.

We see a tradeoff between the von Neumann and Morgenstern [20] and Herstein and Milnor [9] axiomatizations of the expected utility model. The former result uses an order-theoretic continuity, but requires a stronger independence, while the latter result uses a weaker independence, but requires a topological continuity. In particular, IND implies hm-IND and mCON implies aCON. In the next section, we will use the weaker independence axiom, hm-IND, and employ weak order-theoretic continuity axioms (substitutability and solvability) similar to aCON. Our goal is to show that a monotone preference order admits an expected utility representation whenever it satisfies these weaker independence and continuity axioms over lotteries.

4. Analysis

In this section, we obtain a weak and a strong expected utility representation. For these two results, we consider a monotonic preference order. Let $\rhd$ denote the first-order stochastic dominance relation associated with $\succsim$, which is a strict partial order such that $p \rhd q$ if $\sum_{x \succsim z} p(x) \geq \sum_{x \succsim z} q(x)$ for all $z \in X$ and $\sum_{x \succsim z} p(x) > \sum_{x \succsim z} q(x)$ for some $z \in X$.

Axiom 5 (Monotonicity, MON)**.** For any $p, q \in P$, $p \rhd q$ implies $p \succ q$.

Monotonicity is an intuitive condition reflecting the idea that the more likely to receive preferred outcomes (in the sense of the first order stochastic dominance) the better. It is well-known that lottery p first order stochastically dominates lottery q if and only if $u(p) \geq u(q)$ for all expected utilities (with strict inequality for some expected utility), and so MON is clearly an implication of the expected utility representation.

4.1. Weak expected utility.

In establishing an expected utility representation, both von Neumann and Morgenstern [20] and Herstein and Milnor [9] use a continuity condition in addition to an independence axiom. The reason is that even the stronger IND alone is not enough to guarantee a representation for a given preference order over lotteries.

Example 4.1. Let $\succsim$ be a preference order such that for any $p, q \in P$,

$$p \succsim q \quad \text{if} \quad (\,u(p)\,,\,v(p)\,) \geq_L (\,u(q)\,,\,v(q)\,),$$

where $\geq_L$ is the lexicographic order defined on $\mathbb{R}^2$, and $u, v \in U$ are two expected utility functions defined on P. ◇

Preference order $\succsim$ given in Example 4.1 satisfies IND, but clearly it has no representation whenever u and v are two distinct expected utility functions in U. Recall that $U = \{u \in \mathbb{R}^{n+1} : u_0 = 0, u_n = 1, u_i > u_j \text{ iff } i > j\}$. As such, when $u \neq v$, there must be some $i \in I = \{1, \ldots, n-1\}$ such that $u_i \neq v_i$. Hence, there can be no lottery $q \in P$ with $q_n + q_0 = 1$ that is indifferent to the lottery x_i. In other words, for a given preference order satisfying IND non-representability can

For a discussion of various versions of independence and continuity axioms and their relations to each other for decision making under uncertainty, see Ozbek [18].

Despite its normative appeal, violations of MON are well-documented (see section 5). One possible rationale for violations of MON can be due to ”probability weighing” as in the Prospect Theory of Kahneman and Tversky [10], or due to ”salience of the outcomes” as in the Salience Theory proposed by Bordalo et al [4], or due to ”complexity aversion” as suggested by Bernheim and Sprenger [3].

be due to not permitting substitution possibilities between the likelihoods of different outcomes. The following substitutability axiom formalizes this idea.

Axiom 6 (Substitutability, SUB)**.** For any $i \in I_0$, there exists $q \in P$ with $q_n + q_0 = 1$ such that $x_i \sim q$.

Substitutability axiom requires that any outcome should be exchangeable with a lottery comprising of the two extreme outcomes (i.e., x_0 and x_n). Thus, for instance, the preference order $\succsim$ given in Example 4.1 can satisfy the substitutability axiom only when when $u = v$, in which case we observe that $\succsim$ has an expected utility representation. This observation suggests that adding the SUB axiom might lead to an expected utility representation. The following result shows that this intuition is true only in a weak sense whenever IND is replaced by MON and hm-IND.

Recall that $u \in U$ is a weak expected utility representation of $\succsim$ if for all $p, q \in P$, we have $u(p) > u(q)$ implies $p \succ q$.

Theorem 1 (WEU)**.** If a preference order $\succsim$ satisfies MON, hm-IND, and SUB, then it has a weak expected utility representation for some $u \in U$.

Theorem 1 shows that when MON and hm-IND hold, then the SUB axiom is enough to guarantee a weak expected utility representation. That means a decision-maker whose preference order satisfies these three axioms should be able to make choices among lotteries based on their expected utilities as long as the lotteries do not yield the same expected utility. However, note that although the SUB axiom is sufficient to obtain a weak representation when supplied with MON and hm-IND (as shown by Theorem 1), SUB is not necessary to have a weak representation even when hm-IND is replaced with the stronger counterpart IND (as demonstrated by Example 4.1).

Proof sketch: In showing Theorem 1, we first construct a set of utility weights (i.e., $u(x_i)$ for each $i \in I_0$) by using the SUB axiom. Specifically, by SUB, we can find for each $i \in I \setminus \{n\}$, some $r^i \in P$ such that $x_i \sim r^i$ with $r^i_n + r^i_0 = 1$. By MON, we must have $r^i_n, r^i_0 > 0$. Thus, if we let $u_i = r^i_n$ for all $i \in I \setminus \{n\}$, and let $u_0 = 0$ and $u_n = 1$, we have $u \in U$. Given two lotteries p, q with $u(p) > u(q)$, we then construct, by using the hm-IND axiom iteratively, two lotteries $\hat{p}$ and $\hat{q}$ such that (i) $\hat{p} \sim p$ and $\hat{q} \sim q$ and (ii) $\hat{p} \rhd \hat{q}$. By MON, we obtain $p \succ q$.

4.2. **Strong expected utility.** It is natural to ask if the expected utility function $u \in U$ constructed in Theorem 1 can serve as a strong representation. In other words, do we also have $u(p) > u(q)$ whenever $p \succ q$ for any given two lotteries $p, q \in P$? The following example shows that this is not necessarily the case.

Example 4.2. Let $\succsim$ be a preference order such that for any $p, q \in P$,

$$p \succsim q \ \text{ if } \ (u(p), \phi(p_1 - q_1)) \geq_L (u(q), 0),$$

where $\geq_L$ is the lexicographic order defined on $\mathbb{R}^2$, $u \in U$ is an expected utility function, and $\phi : \mathbb{R} \to \{-1, 0, 1\}$ is an indicator function. Specifically, $\varphi(r) = 1$ if $r \in \mathbb{A}$, $\varphi(r) = 0$ if $r \in \mathbb{Q}$,

Substitution has the flavour of a solvability axiom, which typically asserts that solutions exist to certain classes of equations. See, e.g., Krantz et al. [11] for a general discussion and examples of these type of axioms.

and $\varphi(r) = -1$ if $r \in \mathbb{B}$, where $\mathbb{Q}$ is the set of rational numbers while $\mathbb{A}$ and $\mathbb{B}$ decompose the set of irrationals $\mathbb{I}$ into two sets satisfying the following properties: (i) $\mathbb{A} = -\mathbb{B}$, (ii) $a, a' \in \mathbb{A}$ implies $a + a' \in \mathbb{A}$, and (iii) $a \in \mathbb{A}$ and $r \in \mathbb{Q}$ implies $a + r \in \mathbb{A}$. $\diamondsuit$

It is clear that the preference order $\succsim$ given in Example 4.2 satisfies MON, hm-IND, and SUB. Moreover, u is a weak representation for this preference order. However, the preference order has no strong representation due to its lexicographic nature. As such, allowing for substitution between outcomes is not sufficient to obtain a strong representation. One needs a stronger condition than the SUB axiom. We will use the following solvability axiom for this purpose.

Axiom 7 (Solvability, SOL)**.** For any $p \in P$, there exists $q \in P$ with $q_n + q_0 = 1$ such that $p \sim q$.

Clearly, SOL directly implies SUB. The SOL axiom requires that outcomes should be exchangeable with a lottery comprising of the two extreme outcomes not only in isolation, but also when they together form a lottery. The preference order given in Example 4.2 does not allow for exchanges between any two lotteries p and q whenever $p_1 - q_1 \notin \mathbb{Q}$. As such, using the SOL axiom should eliminate these type of exotic preference orders.

In fact, SOL together with MON imply that there must be another function $v : P \to \mathbb{R}$ with $\{v(x_i)\} \in U$ that can serve as a strong representation for the preference order $\succsim$ in addition to the weak representation $u \in U$ constructed in Theorem 1. Thus, there must exist a weakly increasing transformation $\phi : \mathbb{R} \to \mathbb{R}$ such that $u(p) = \phi(v(p))$ for all $p \in P$. The following result shows that ϕ indeed must be a strictly increasing transformation, and therefore u becomes a strong representation.

Theorem 2 (SEU)**.** If a preference order $\succsim$ satisfies MON, hm-IND, and SOL, then it has a strong expected utility representation for some $u \in U$.

It is clear that the three axioms, MON, hm-IND, and SOL, are also necessary to have a strong expected utility representation with some $u \in U$. Theorem 2 shows that when SOL and MON hold, then the hm-IND axiom is enough to guarantee a strong expected utility representation. That means a decision-maker whose preference order satisfies these three axioms can just rely on expected utilities of lotteries to make choices, as well as to declare indifferences.

Proof sketch: Using SOL, for any $p \in P$ we can let $v(p) = \alpha_p$ where $p \sim x_n \alpha_p x_0$ for some $\alpha_p \in [0, 1]$. By MON, $v : P \to \mathbb{R}$ provides a strong representation for the preference order. Since SOL implies SUB, by Theorem 1 we can construct an expected utility function $u \in U$ that serves as a weak representation. But then for any $p \in P$, we must have $u(p) = u(x_n v(p) x_0)$ implying that $u(p) = v(p)$. This shows that u is a strong expected utility representation.

5. Discussion

Although the expected utility model has normatively appealing implications, it is not without a criticism. Starting with the critiques of Allais [2], a tremendous literature tests the EU model

For a proof of the existence of these two sets decomposing the set of irrationals, see Mitra and Ozbek [14].

Although both SUB and SOL are novel axioms, a stronger form of these axioms has been employed in the literature. In particular, to characterize an implicit expected utility model Dekel [6] uses a stronger solvability (sSOL) axiom: for any $p, q, r \in P$, $p \succ q \succ r$ implies $p\alpha r \sim q$ for some $\alpha \in (0, 1)$. Ozbek [18] shows that mCON directly implies both aCON and sSOL. Although we are not aware of whether aCON implies SOL or SUB, it is worth noting that the preference order given in Example 4.2 satisfies SUB, but fails aCON.

in experimental settings. As a consequence of the findings in these experiments, a strand of theoretical literature has grown proposing alternative non-expected utility models to rationalize violations of the EU model implications. Overwhelming majority of both the experimental and theoretical literature on non-expected utility preferences have focused on the strong independence axiom. A recent line of experimental work points out that violations of other key implications of the EU model also warrant attention.

Among these work, Dembo et al. [7] apply revealed preference techniques with contingent commodity budget sets documenting the extent of violations of preference maximization that satisfies SUB (and SOL), MON, and IND; Nielsen and Rehbeck [16] use simple lottery choices documenting the prevalence of violations of MON and IND; Agranov and Ortoleva [1] use the multiple price list method documenting the range of violations of SUB and SOL. Our axiomatic results are useful in interpreting these recent experimental findings, as well as providing some rationale for understanding why some of the EU axioms are violated more often than others.

Dembo et al. [7] find out that while all of their subjects violate MON, a significant portion of their subjects (65 subjects out of 141, which is around 45%) does not violate other EU axioms conditional on violating MON and SUB (or SOL). This is intuitive given our weak and strong expected utility representations, since hm-IND is easy to state, and perhaps therefore, it is an easy to internally process and satisfy axiom. To this point, Nielsen and Rehbeck [16] document that although overwhelming majority of their subjects (around 90% to be more precise) state that they want their decisions to satisfy MON, when given choices most subjects (around 85%) violated MON. However, Nielsen and Rehbeck [16] also document that while majority of subjects (around 80%) stated that they want their decisions to satisfy the Branch Independence (which is weaker than IND but stronger than hm-IND), when given choices only a minority of them (around 25%) violated the Branch Independence. Clearly MON requires a deeper understanding in action than hm-IND (or SUB or SOL). Having said this, violations of SUB and SOL are also prevalent. On this issue, Agranov and Ortoleva [1] document that most subjects (around 80%) violate SUB or SOL by reporting a range of randomizations between outcomes and two outcome lotteries. Agranov and Ortoleva [1] also note that violations of MON and SUB (or SOL) are strongly related. In particular, overwhelming majority of subjects who violate MON (over 90%) does violate SUB (or SOL). Dembo et al. [7] report a similar finding that only a minority of their subjects (11 subjects out of 141, which is around 8%) violate SUB (or SOL) without violating MON.

In sum, violations of the EU model implications are prevalent which brings its practical use into question. However, as documented by Nielsen and Rehbeck [16], individuals also find the EU axioms (including MON and IND) normatively appealing and most of them are willing to change their choices when they are shown their mistakes conflicting with their normative view. As such, individuals can be helped when making choices under uncertainty. On this, our representation results suggest that first the utility $u(x_i)$ of each outcome $x_i \in X$ can be elicited from preferences of the individuals, and then using the vector of utilities $\{u(x_i)\}_{i=0}^n$, the value

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The experimental literature discussed here does not formally state the SUB and SOL axioms, but rather it documents indirectly the violations of these axioms through the observation of either (i) ranges of randomizations over outcomes and lotteries (as in Agranov and Ortoleva [1]) or (ii) violations of rationalizability by a utility function that is monotone over outcomes (as in Dembo et al. [7]).

of each lottery $p \in P$ can be calculated. Based on these calculations, the individuals can then make a more informed choice.

APPENDIX A.

A.1. **hm-IND implies b-IND.** In this section, we provide a preliminary result that we use to prove our main results. We can define a stronger form of the hm-IND axiom by requiring mixture weights to be any *binary rational* number between 0 and 1, not just $1/2$. A binary (or dyadic) rational is a rational number that can be expressed as a fraction whose denominator is a power of two (e.g., $\frac{1}{2}$, or $\frac{5}{8}$, or $\frac{27}{32}$, etc.). The set of binary rationals is dense in reals. Moreover, a binary rational number has a finite binary representation. In particular, a binary rational between 0 and 1 can be expressed as a convex combination of 0 and 1 in a finite number of steps with equal weights (on each end) at each step. This means for any binary rational $\alpha \in (0,1)$, the mixture lottery $p\alpha q$ can be expressed as a convex combination of p and q in a finite number of steps with equal weights (on each end) at each step.

Axiom 8 (Binary Independence, b-IND)**.** For any $p,q,r \in P$ and binary rational $\alpha \in (0,1)$, $p \sim q$ implies $p\alpha r \sim q\alpha r$.

The following result shows that hm-IND implies the b-IND axiom.

Proposition 1. If a preference order $\succsim$ satisfies hm-IND, then it must satisfy b-IND.

Proof. Let $\succsim$ be a preference order satisfying hm-IND. Suppose $p,q,r,s \in P$ such that $p \sim q$ and $r \sim s$. We observe that $p1/2r \sim q1/2s$. This is true since by hm-IND, we have $p1/2r \sim q1/2r$ and $r1/2q \sim s1/2q$. By transitivity, we must have $p1/2r \sim s1/2q$, and so $p1/2r \sim q1/2s$.

Now let $p,q,r \in P$ and let $\alpha \in (0,1)$ be a binary rational. Suppose $p \sim q$. We want to show that $p\alpha r \sim q\alpha r$. We proceed in an algorithmic fashion. Let $\alpha_1 = 1/2$. Then we have $p\alpha_1 r \sim q\alpha_1 r$. If $\alpha = \alpha_1$, we are done. If $\alpha > \alpha_1$, then we have $p1/2(p\alpha_1 r) \sim q1/2(q\alpha_1 r)$ by the above observation. In this case, let $\alpha_2 = 1/2 \times 1 + 1/2 \times \alpha_1$. If, however, $\alpha < \alpha_1$, then we have $(p\alpha_1 r)1/2r \sim (q\alpha_1 r)1/2r$ by hm-IND. In this case, let $\alpha_2 = 1/2 \times 1 + 1/2 \times \alpha_1$. In either case, if $\alpha = \alpha_2$ we are done. If $\alpha > \alpha_2 > \alpha_1$, then we have $p1/2(p\alpha_2 r) \sim q1/2(q\alpha_2 r)$ by the above observation. In this case, let $\alpha_3 = 1/2 \times 1 + 1/2 \times \alpha_2$. If $\alpha_2 > \alpha > \alpha_1$, then we have $(p\alpha_2 r)1/2(p\alpha_1 r) \sim (q\alpha_2 r)1/2(q\alpha_1 r)$ by hm-IND. In this case, let $\alpha_3 = 1/2 \times \alpha_2 + 1/2 \times \alpha_1$. If $\alpha_1 > \alpha_2 > \alpha$, then we have $(p\alpha_2 r)1/2r \sim (q\alpha_2 r)1/2r$ by hm-IND. In this case, let $\alpha_3 = 1/2 \times \alpha_2 + 1/2 \times 0$. If $\alpha_1 > \alpha > \alpha_2$, then we have $(p\alpha_1 r)1/2(p\alpha_2 r) \sim (q\alpha_1 r)1/2(q\alpha_2 r)$ by the above observation. In this case, let $\alpha_3 = 1/2 \times \alpha_1 + 1/2 \times \alpha_2$. If $\alpha_3 = \alpha$, we are done. Otherwise, proceed as in previous steps. It is clear that at each step k, the mixture weight α_k gets closer to α. Since α is a binary rational, in a finite number of steps (say m) the mixture weight will satisfy the equality $\alpha_m = \alpha$ yielding that $p\alpha r \sim q\alpha r$. □

A.2. **Proofs of main results.** In this section, we provide proofs for our results given in the main text.

Proof of Theorem 1. Let $\succsim$ be a preference order satisfying hm-IND, SUB, and MON. Note that, by SUB, we can find for each $i \in I \setminus \{n\}$, some

$$r^i \in P \tag{A.1}$$

such that $x_i \sim r^i$ with $r^i_n + r^i_0 = 1$. By MON, we must have $r^i_n, r^i_0 > 0$.

Let $u_i = r_n^i$ for all $i \in I \setminus \{n\}$, and let $u_0 = 0$ and $u_n = 1$. Clearly, we have $0 < u_i < 1$ for all $i \in I \setminus \{n\}$. Moreover, by MON we must have $u_j > u_i$ whenever $x_j \succ x_i$. As such, we have $u \in U$.

We want to show that $u \in U$ defined above provides a *weak* expected utility representation for the preference order; that is, $u(p) > u(q)$ implies $p \succ q$ for all $p, q \in P$. For each $i \in I \setminus \{n\}$ let $d_i \in (0,1)$ be a *binary rational* defined recursively satisfying the following property: given a binary rational $d_k \in (0,1)$ for each $k \in I \setminus \{i,n\}$, $d_i \in (0,1)$ is such that,

$$0 < \left(\prod_{j=1}^{n-1} d_j\right)(p_i - q_i) + \left((1-d_i)\prod_{j=i+1}^{n-1} d_j\right)(p_i^i - q_i^i) < \left(\prod_{j=1}^{n-1} d_j\right)\left(\frac{(u(p)-u(q))/n}{u_i}\right) \quad \text{(A.2)}$$

where $p_i^i = 0$ and $q_i^i = 1$ if $p_i > q_i$ and $p_i^i = 1$ and $q_i^i = 0$ if $q_i \geq p_i$. Notice that since $(p_i - q_i)(p_i^i - q_i^i) \leq 0$, a binary rational $d_i \in (0,1)$ satisfying (A.2) always exists.

Using these binary rationals, we will now construct two lotteries, $\hat{p}$ and $\hat{q}$ such that (i) $\hat{p} \sim p$ and $\hat{q} \sim q$, and (ii) $\hat{p} \rhd \hat{q}$. The latter, by MON, will imply that $\hat{p} \succ \hat{q}$, so that the former will imply $p \succ q$, completing the proof.

Define two lotteries $\hat{p}$ and $\hat{q}$ such that

$$\hat{p} = \left(\prod_{j=1}^{n-1} d_j\right) p + \ldots + \left((1-d_i)\prod_{j=i+1}^{n-1} d_j\right) p^i + \ldots + (1-d_{n-1})p^{n-1}$$

and

$$\hat{q} = \left(\prod_{j=1}^{n-1} d_j\right) q + \ldots + \left((1-d_i)\prod_{j=i+1}^{n-1} d_j\right) q^i + \ldots + (1-d_{n-1})q^{n-1}$$

where for all $i \in I \setminus \{n\}$, $p^i = r^i$ and $q^i = x^i$ if $p_i > q_i$, and $p^i = x_i$ and $q^i = r^i$ if $q_i \geq p_i$, where r^i is defined as in Eq. (A.1). Note that by iterative application of b-IND (given that b-IND is implied by hm-IND by Proposition 1), we have $\hat{p} \sim p$ and $\hat{q} \sim q$ showing (i) above. We now show that for all $i \in I$, we must have $\hat{p}_i > \hat{q}_i$, and so, by MON, $\hat{p} \succ \hat{q}$.

To see this, first note that for any $i \in I \setminus \{n\}$,

$$\hat{p}_i = \left(\prod_{j=1}^{n-1} d_j\right) p_i + \left((1-d_i)\prod_{j=i+1}^{n-1} d_j\right) p_i^i > \left(\prod_{j=1}^{n-1} d_j\right) q_i + \left((1-d_i)\prod_{j=i+1}^{n-1} d_j\right) q_i^i = \hat{q}_i$$

by the first inequality in Eq. (A.2) above. Moreover, by the second inequality in Eq. (A.2), we have for all $i \in I \setminus \{n\}$,

$$\left(\prod_{j=1}^{n-1} d_j\right) u_i(p_i - q_i) + \left((1-d_i)\prod_{j=i+1}^{n-1} d_j\right) u_i(p_i^i - q_i^i) < \left(\prod_{j=1}^{n-1} d_j\right) \frac{u(p)-u(q)}{n}.$$

Using definition of $u_i = r_n^i$ above and rearranging the terms, we derive for all $i \in I \setminus \{n\}$,

$$\left(\prod_{j=1}^{n-1} d_j\right)\left[u_i(p_i - q_i) - \frac{u(p)-u(q)}{n}\right] < \left((1-d_i)\prod_{j=i+1}^{n-1} d_j\right)(p_n^i - q_n^i).$$

Then, summing above inequalities over all $i \in I \setminus \{n\}$, adding $\left(\prod_{j=1}^{n-1} d_j\right)(p_n - q_n)$ to both sides, and using the definition of $\hat{p}_n$ and $\hat{q}_n$, we receive

$$\frac{u(p)-u(q)}{n} < \hat{p}_n - \hat{q}_n$$

which implies that $\hat{p}_n - \hat{q}_n > 0$. Since we have $\hat{p}_i > \hat{q}_i$ for all $i \in I$, by MON, $\hat{p} \succ \hat{q}$, and so $p \succ q$ as desired. $\square$

Proof of Theorem 2. Let $\succsim$ be a preference order satisfying MON, hm-IND, and SOL. Since SOL implies SUB, by Theorem 1, we can construct an expected utility function $u \in U$ that serves as a weak representation. By SOL and MON, for any $p \in P$ there is a unique mixture weight $\alpha_p \in [0,1]$ such that $p \sim x_n \alpha_p x_0$. Let $v : P \to \mathbb{R}$ be the function such that $v(p) = \alpha_p$ for each $p \in P$. By MON, v provides a strong representation for the preference order $\succsim$ such that $\{v(x_i)\}_{i=0}^n \in U$.

By definition, we have $p \sim x_n v(p) x_0$ for any $p \in P$. However, we must also have $u(p) = u(x_n v(p) x_0)$. Otherwise, we would have either $p \succ x_n v(p) x_0$ or $x_n v(p) x_0 \succ p$ since u is a weak representation, a contradiction. But then, using the linearity of u, we obtain $u(p) = v(p)u(x_n) + (1 - v(p))u(x_0)$ implying that $u(p) = v(p)$ since $u(x_n) = 1$ and $u(x_0) = 0$. This shows that u provides a strong expected utility representation. $\square$

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