Data-driven analysis and control of 2D Fornasini-Marchesini models

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***Abstract*—We study data-driven analysis and control of 2D Fornasini-Marchesini second models. We give necessary and sufficient conditions for the data to be informative for identification, and state a 2D "fundamental lemma". We propose a data-driven approach for stability verification and state-feedback stabilization via LMIs.**

I. Introduction

Data-driven control is a fast-growing area of theoretical research in control with important practical applications. Research has almost exclusively been focused on systems evolving over the independent variable 'time' (called '1D systems' in the following). Research in data-driven control for systems evolving over $n > 1$ independent variables (called 'nD systems' in the following) has been much less intensive.

Of central importance in 1D data-driven control is "sufficient informativity" and parametrizations of system trajectories from a "sufficiently informative" data. Such concepts are related to "persistency of excitation" and the "fundamental lemma" and its generalizations (see [10], [12], [13], [9]). One of the reasons for the lack of progress in nD data-driven control research may be that similar concepts and results are missing. We started to fill this gap in [8], where a "fundamental lemma" for autonomous (no inputs) quarter-plane causal systems was stated. In this paper we consider $2D$ *open* systems (i.e. with inputs) in the Fornasini-Marchesini (FM) second representation. Every "quarter-plane causal" $2D$ system is representable by such models (see [4]), so our results are not overly restrictive. We assume that input and state are directly measurable; this is a standard assumption in $1D$ data-driven control (see [2]). Our contributions are the following:

- In Section III we provide necessary and sufficient conditions for the data to be informative for identification;
- In Section IV we prove a $2D$ version of the "fundamental lemma";
- In Section V we show how to obtain a system representation from sufficiently informative data;
- In Section VI we give data-based sufficient conditions to verify the stability of a FM second model;
- In Section VII we provide a data-based design procedure to stabilize FM second models by state-feedback.

We summarize some preliminary material on FM models and their stability in Section II. In Section VIII we discuss the limitations of our work and current research directions.

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Notation

We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ respectively the set of natural, integer and real numbers, and by $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$. $\mathbb{R}^n$ denotes the space of n-dimensional vectors with real entries. $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices with real entries; $\mathbb{R}^{n\times\infty}$ the set of real matrices with n rows and an infinite number of columns; and $\mathbb{R}^{n\times\bullet}$ the set of matrices with n rows and a finite (unspecified) number of columns. The transpose of a matrix M is denoted by $M^\top$ and its pseudoinverse by $M^\dagger$; the image of M is denoted by $\mathrm{im}(M)$. If $A, B \in \mathbb{R}^{\bullet\times m}$ we define $\mathrm{col}(A,B) := \begin{bmatrix} A^\top & B^\top \end{bmatrix}^\top$. The spectral radius of M is denoted by $\rho(M)$. Positive- and negative-definiteness of matrices are denoted by $\succ 0$ and $\prec 0$, respectively. Given a subspace $\mathcal{V}$, its orthogonal subspace is denoted by $\mathcal{V}^\perp$.

$\mathbb{R}[z_1,z_2]$ is the ring of polynomials with real coefficients in the indeterminates z_i, $i=1,2$, and $\mathbb{R}^{n\times m}[z_1,z_2]$ the ring of $n\times m$ matrices with entries in $\mathbb{R}[z_1,z_2]$. Given $S \subset \mathbb{R}[z_1,z_2]$, we denote by $\langle S\rangle$ the module generated by the elements of S. The same notation is used for modules of the ring $\mathbb{R}^{1\times m}[z_1,z_2]$ of polynomial row vectors with m entries.

We denote by $(\mathbb{R}^q)^{\mathbb{Z}^2}$ the set of q-real valued doubly-indexed sequences: $(\mathbb{R}^q)^{\mathbb{Z}^2} := \{w : \mathbb{Z}^2 \to \mathbb{R}^q\}$. We denote by σ_i, $i=1,2$, the shifts on $(\mathbb{R}^q)^{\mathbb{Z}^2}$:

$$(\sigma_1 w)(i,j) := w(i+1,j)\ ,\ (i,j) \in \mathbb{Z}^2\ ,$$

and analogously for σ_2. We define the reverse shifts σ_i^{-1}, $i = 1,2$ on $(\mathbb{R}^q)^{\mathbb{Z}^2}$ by

$$\left(\sigma_1^{-1}w\right)(i,j) := w(i-1,j) \text{ and } \left(\sigma_2^{-1}w\right)(i,j) := w(i,j-1)\ .$$

We denote the composition of σ_1 and σ_2^{-1} by $\sigma := \sigma_1 \circ \sigma_2^{-1}$.

II. Background material

A. Fornasini-Marchesini second models

The state equation of a Fornasini-Marchesini second model (abbreviated FM in the rest of the paper)

$$\sigma_1\sigma_2 x = A_1\sigma_2 x + A_2\sigma_1 x + B_1\sigma_2 u + B_2\sigma_1 u\ ,$$

can equivalently be written as

$$\sigma_1 x = A_1 x + A_2\sigma x + B_1 u + B_2\sigma u\ . \tag{1}$$

We associate with (1) the set of trajectories

$$\mathfrak{B} := \left\{ \mathrm{col}(x,u) : \mathbb{Z}^2 \to \mathbb{R}^{n+m} \mid \mathrm{col}(x,u) \text{ satisfies (1)} \right\} . \tag{2}$$

We associate with (1) the $n \times (n+m)$ polynomial matrix in the indeterminates z_1, $z = z_1 z_2^{-1}$ defined by

$$R(z_1, z) := \begin{bmatrix} z_1 I_n - A_1 - zA_2 & B_1 + zB_2 \end{bmatrix} . \tag{3}$$

The highest degree in z_1 in (3), denoted by $\ell(\mathfrak{B})$, is $\ell(\mathfrak{B}) = 1$.

We also consider the system (1) without inputs:

$$\sigma_1 x = A_1 x + A_2 \sigma x . \tag{4}$$

B. Stability of FM second models

An exhaustive review of stability results for 2D systems is given in [1]. We recall here two results concerning FM second models (see Sections 4.1, 4.2 and 4.4 *ibid.*); while only *sufficient* conditions, they are often used for the design of stabilizing controllers, since they are easier to work with than necessary and sufficient conditions (see [5]).

Proposition 1. *If* $\exists\ Q_i = Q_i^\top \in \mathbb{R}^{n \times n}$, $i = 1, 2$ *such that*

$$\begin{aligned} 0 &\prec Q_i \ , \ i = 1, 2 \qquad (5) \\ 0 &\succ \begin{bmatrix} A_1 (Q_1 + Q_2) A_1^\top - Q_1 & A_1 (Q_1 + Q_2) A_2^\top \\ A_2 (Q_1 + Q_2) A_1^\top & A_2 (Q_1 + Q_2) A_2^\top - Q_2 \end{bmatrix} \end{aligned}$$

then the FM second model (4) *is stable.*

Proof. This is Corollary 1 p. 5 of [1], from [3]. □

We give another sufficient condition for stability of (4).

Proposition 2. *If* $\exists\ X = X^\top, Y = Y^\top \in \mathbb{R}^{n \times n}$ *such that*

$$\begin{aligned} 0 &\prec X \ , \ 0 \prec Y \\ 0 &\succ \begin{bmatrix} -X & 0 & YA_1^\top \\ 0 & X - Y & YA_2^\top \\ A_1 Y & A_2 Y & -Y \end{bmatrix} \end{aligned} \tag{6}$$

then the FM second model (4) *is stable.*

Proof. This is Theorem 2 p. 5 of [1], from [11]. □

III. Informativity for identification for FM second models

We denote by $\mathcal{L}_k$ the k-th *diagonal line* in $\mathbb{Z} \times \mathbb{Z}$:

$$\mathcal{L}_k := \{(i,j) \in \mathbb{Z}^2 \mid i + j = k\} \ , \ k = 0, \dots, N \ ,$$

and by $\mathcal{L}_{0:N}$ the $N+1$ consecutive diagonal lines

$$\mathcal{L}_{0:N} := \{(i,j) \in \mathbb{Z}^2 \mid 0 \leqslant i + j \leqslant N\} \ .$$

We measure a (state,input) trajectory $\mathrm{col}(\hat{x}, \hat{u}) \in \mathfrak{B}$; we denote by $\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_k}$ the restriction of $\mathrm{col}(\hat{x}, \hat{u}) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^{n+m}$ to $\mathcal{L}_k$ and by $\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_{0:N}}$ the restriction of $\mathrm{col}(\hat{x}, \hat{u})$ to $\mathcal{L}_{0:N}$. We call $\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_{0:N}}$ the *data*; from them we define the *data set* $\mathcal{D}_{0:N}(\mathrm{col}(\hat{x}, \hat{u}))$ by

$$\mathcal{D}_{0:N}(\mathrm{col}(\hat{x}, \hat{u})) := \mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_{0:N}} \ . \tag{7}$$

Given $j \in \mathbb{N}$, we denote by $\mathcal{H}_j(\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_k})$ the block-Hankel matrix defined by

$$\mathcal{H}_j(\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_k}) := \tag{8}$$
$$\begin{bmatrix} \dots & \mathrm{col}(\hat{x}, \hat{u})_{k-1,1} & \mathrm{col}(\hat{x}, \hat{u})_{k,0} & \dots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{k,0} & \mathrm{col}(\hat{x}, \hat{u})_{k+1,-1} & \dots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{k+1,-1} & \mathrm{col}(\hat{x}, \hat{u})_{k+2,-2} & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{k+j-1,-j+1} & \mathrm{col}(\hat{x}, \hat{u})_{k+j,-j} & \dots \end{bmatrix} .$$

Each column of (8) consists of $j+1$ consecutive values of $\mathrm{col}(\hat{x}, \hat{u})_{\mathcal{L}_k}$; the i-th column is the σ-shift of the $(i-1)$-th one. Thus, the matrix (8) is a block-Hankel matrix corresponding to the 1D-sequence $\mathrm{col}(\hat{x}, \hat{u})_{\mathcal{L}_k}$ operated on with the shift σ.

We define the *data matrix* $\mathbb{D}_N(\mathrm{col}(\hat{x}, \hat{u}))$ by

$$\mathbb{D}_N(\mathrm{col}(\hat{x}, \hat{u})) := \begin{bmatrix} \mathcal{H}_N(\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_0}) \\ \mathcal{H}_{N-1}(\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_1}) \\ \vdots \\ \mathcal{H}_0(\mathrm{col}(\hat{x}, \hat{u})_{|\mathcal{L}_N}) \end{bmatrix} . \tag{9}$$

Example 1. We use the lattice depicted in Fig. 1 to show how $\mathbb{D}_N$ defined by (9) is constructed for the case $N = 2$. Note the color coding for the three lines $\mathcal{L}_0$ (green), $\mathcal{L}_1$ (blue) and $\mathcal{L}_2$ (red). The first column of the matrix in (10) corresponds to

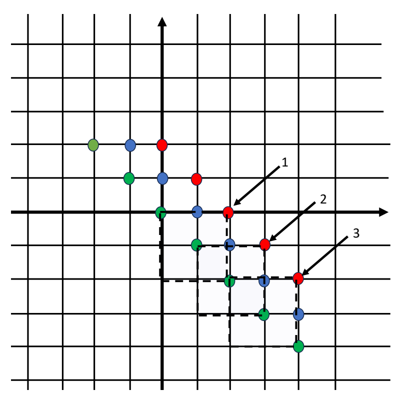

Fig. 1. Construction of $\mathbb{D}_2$ for Example 1.

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the "equilateral triangle" of $\mathbb{Z}^2$ indicated by "1" in the figure; the second column, to the triangle indicated by "2"; the third one, to that indicated by "'3" in Figure 1.

$$\begin{bmatrix} \dots & \mathrm{col}(\hat{x}, \hat{u})_{0,0} & \mathrm{col}(\hat{x}, \hat{u})_{1,-1} & \mathrm{col}(\hat{x}, \hat{u})_{2,-2} & \dots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{1,-1} & \mathrm{col}(\hat{x}, \hat{u})_{2,-2} & \mathrm{col}(\hat{x}, \hat{u})_{3,-3} & \dots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{2,-2} & \mathrm{col}(\hat{x}, \hat{u})_{3,-3} & \mathrm{col}(\hat{x}, \hat{u})_{4,-4} & \dots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{1,0} & \mathrm{col}(\hat{x}, \hat{u})_{2,-1} & \mathrm{col}(\hat{x}, \hat{u})_{3,-2} & \dots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{2,-1} & \mathrm{col}(\hat{x}, \hat{u})_{3,-2} & \mathrm{col}(\hat{x}, \hat{u})_{4,-3} & \dots \\ \dots & \mathrm{col}(\hat{x}, \hat{u})_{2,0} & \mathrm{col}(\hat{x}, \hat{u})_{3,-1} & \mathrm{col}(\hat{x}, \hat{u})_{4,-2} & \dots \end{bmatrix} \tag{10}$$

Each column of $\mathbb{D}_N$ is built by "unfolding" $\mathrm{col}(\hat{x}, \hat{u})$ on a triangle with apex at $(k+N, -k)$ and side length $N+1$. □

The matrices (8) and (9) have an infinite number of columns and a finite number of rows, respectively $(j+1)(n+m)$ and $(n+m)\sum_{j=0}^{N}(j+1) = (n+m)\frac{(N+1)(N+2)}{2}$.

We denote by $\mathcal{N}(\mathcal{D}_{0:N}(\operatorname{col}(\widehat{x},\widehat{u})))$ the set of left-annihilators of $\mathcal{D}_N(\operatorname{col}(\widehat{x},\widehat{u}))$, defined by

$$\left\{\eta \in \mathbb{R}^{1\times(n+m)}\left[z_1,z\right] \mid \eta(\sigma_1,\sigma)\operatorname{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_{0:N}}=0\right\}\ , \quad (11)$$

and by $\langle\mathcal{N}(\mathcal{D}_{0:N}(\operatorname{col}(\widehat{x},\widehat{u})))\rangle$ the module generated by its elements. The module of annihilators of $\mathfrak{B}$ is denoted by $\mathcal{N}(\mathfrak{B})$; given a kernel representation (3) of $\mathfrak{B}$, $\mathcal{N}(\mathfrak{B})$ consists of all polynomial combination of the rows of $R(z_1,z_1z_2^{-1})$:

$$\mathcal{N}(\mathfrak{B})=\left\{vR\left(z_1,z_1z_2^{-1}\right)\mid v\in\mathbb{R}^{1\times n}\left[z_1,z_1z_2^{-1}\right]\right\}\ .$$

Every annihilator of $\mathfrak{B}$ also annihilates $\operatorname{col}(\widehat{x},\widehat{u})$; consequently $\mathcal{N}(\mathfrak{B})\subseteq\langle\mathcal{N}(\mathcal{D}_{0:N}(\operatorname{col}(\widehat{x},\widehat{u})))\rangle$. The issue of whether the converse inclusion holds, i.e. whether the data $\operatorname{col}(\widehat{x},\widehat{u})$ is "sufficiently rich" to allow the identification of *all* annihilators of the system, is at the core of the following definition.

Definition 1. *The data* $\mathcal{D}_{0:N}(\operatorname{col}(\widehat{x},\widehat{u}))$ *are* informative for identification *if* $\langle\mathcal{N}(\mathcal{D}_{0:N}(\operatorname{col}(\widehat{x},\widehat{u})))\rangle=\mathcal{N}(\mathfrak{B})$.

The following result is analogous to the first part of Theorem 2 in [8], and generalizes it to the case of open systems.

Theorem 1. *Define* $\mathfrak{B}$ *by* (2). *Let* $\operatorname{col}(\widehat{x},\widehat{u})\in\mathfrak{B}$. *Assume that* $N\geqslant\ell(\mathfrak{B})=1$. *The following statements are equivalent:*

1) The data are informative for system identification;
2) $\mathcal{N}(\mathfrak{B})=\langle\mathcal{N}(\mathcal{D}_{0:N}(\operatorname{col}(\widehat{x},\widehat{u})))\rangle$*;*
3) $\operatorname{rank}(\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u})))=n(N+1)+m\frac{(N+1)(N+2)}{2}$.

Proof. (1) $\Longleftrightarrow$ (2) follows from Definition 1.

To prove (2) $\Longrightarrow$ (3), recall that $N\geqslant\ell(\mathfrak{B})=1$, and write $N=1+r$, with $r\geqslant0$. Note that $\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u}))$ has $d:=(n+m)\frac{(N+1)(N+2)}{2}$ rows. Assume first that $r=0$; then

$$\mathbb{D}_1(\operatorname{col}(\widehat{x},\widehat{u}))=\begin{bmatrix}\mathcal{H}_1(\operatorname{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0})\\ \mathcal{H}_0(\operatorname{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_1})\end{bmatrix}\in\mathbb{R}^{3(n+m)\times\infty}\ .$$

By assumption the module of annihilators of $\mathfrak{B}$ is isomorphic to the left kernel of $\mathbb{D}_1(\operatorname{col}(\widehat{x},\widehat{u}))$; it follows that every left-annihilator of $\mathbb{D}_1(\operatorname{col}(\widehat{x},\widehat{u}))$ is a linear combination of the linearly independent n rows of

$$\begin{bmatrix}A_1 & B_1 & A_2 & B_2 & -I_n & 0_{n\times m}\end{bmatrix}\ . \quad (12)$$

Consequently $\operatorname{rank}(\mathbb{D}_1(\operatorname{col}(\widehat{x},\widehat{u})))=3(n+m)-n=2n+3m$ and the claim is verified. For $r=1$, $\mathbb{D}_2(\operatorname{col}(\widehat{x},\widehat{u}))$ equals

$$\begin{bmatrix}\mathcal{H}_2(\operatorname{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0})\\ \mathcal{H}_1(\operatorname{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_1})\\ \mathcal{H}_0(\operatorname{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_2})\end{bmatrix}\in\mathbb{R}^{6(n+m)\times\infty}$$
$$=\begin{bmatrix}\dots & \operatorname{col}(\widehat{x},\widehat{u})_{-1,1} & \operatorname{col}(\widehat{x},\widehat{u})_{0,0} & \dots\\ \dots & \operatorname{col}(\widehat{x},\widehat{u})_{0,0} & \operatorname{col}(\widehat{x},\widehat{u})_{1,-1} & \dots\\ \dots & \operatorname{col}(\widehat{x},\widehat{u})_{1,-1} & \operatorname{col}(\widehat{x},\widehat{u})_{2,-2} & \dots\\ \dots & \operatorname{col}(\widehat{x},\widehat{u})_{0,1} & \operatorname{col}(\widehat{x},\widehat{u})_{1,0} & \dots\\ \dots & \operatorname{col}(\widehat{x},\widehat{u})_{1,0} & \operatorname{col}(\widehat{x},\widehat{u})_{2,-1} & \dots\\ \dots & \operatorname{col}(\widehat{x},\widehat{u})_{1,1} & \operatorname{col}(\widehat{x},\widehat{u})_{2,0} & \dots\end{bmatrix}\ .$$

Given the assumption 2), the structure of $\mathbb{D}_2(\operatorname{col}(\widehat{x},\widehat{u}))$ and the shift-invariance of $\mathfrak{B}$, each left annihilator of $\mathbb{D}_2(\operatorname{col}(\widehat{x},\widehat{u}))$ is a linear combination of the rows of the 3 matrices

$$\begin{matrix}\begin{bmatrix}A_1 & B_1 & A_2 & B_2 & 0 & -I_n & 0 & 0\end{bmatrix}\\ \begin{bmatrix}0 & A_1 & B_1 & A_2 & B_2 & 0 & -I_n & 0\end{bmatrix}\\ \begin{bmatrix}0 & 0 & 0 & A_1 & B_1 & A_2 & B_2 & -I_n\end{bmatrix}\ .\end{matrix} \quad (13)$$

The last n annihilators in (13) are computed from those of $\mathbb{D}_1(\operatorname{col}(\widehat{x},\widehat{u}))$ by preceding them by $3(n+m)$ zeros (the number of rows of $\mathcal{H}_2(\operatorname{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0})$); they are obtained from the laws of the system (1) applied to samples of $\operatorname{col}(\widehat{x},\widehat{u})$ on $\mathcal{L}_1$ and $\mathcal{L}_2$. The first $2n$ annihilators in (13) are due to the laws of the system (1) applied to samples of $\operatorname{col}(\widehat{x},\widehat{u})$ on $\mathcal{L}_0$ and $\mathcal{L}_1$. It follows that $\operatorname{rank}(\mathbb{D}_{\ell(\mathfrak{B})+1}(\operatorname{col}(\widehat{x},\widehat{u})))=6(n+m)-3n=3n+6m$, verifying the claim for $N=2$, i.e. $r=1$.

This argument shows that for the cases $r=0$ and $r=1$

$$\begin{aligned}\dim\text{left ker }\mathbb{D}_r(\operatorname{col}(\widehat{x},\widehat{u})) &= \dim\text{left ker }\mathbb{D}_{r-1}(\operatorname{col}(\widehat{x},\widehat{u}))\\ &+ (r+1)n\ ,\end{aligned} \quad (14)$$

where we define $\dim$ left ker $\mathbb{D}_{-1}(\operatorname{col}(\widehat{x},\widehat{u})):=0$.

We now show that (14) holds also for $r\geqslant2$ by constructing a basis for the set of left annihilators of $\mathbb{D}_{r+1}$ as follows. Precede each basis element of the set of left annihilators of $\mathbb{D}_r$ by $(r+2)(n+m)$ zeroes; we obtain $\dim$ left ker $\mathbb{D}_r(\operatorname{col}(\widehat{x},\widehat{u}))$ linearly independent left annihilators of $\mathbb{D}_{r+1}$. Additional elements of left ker $\mathbb{D}_{r+1}(\operatorname{col}(\widehat{x},\widehat{u}))$ can be constructed from the linear combination of $\sigma^h\operatorname{col}(\widehat{x},\widehat{u})_{\mathcal{L}_0}$ and $\sigma^{h+1}\operatorname{col}(\widehat{x},\widehat{u})_{\mathcal{L}_0}$, $h=0,\dots,r$, with $\sigma^h\operatorname{col}(\widehat{x},\widehat{u})_{\mathcal{L}_1}$, according to the dynamics (1). There are $(r+1)n$ linearly independent such annihilators.

Now denote $A(\sigma):=A_1+A_2\sigma$ and $B(\sigma):=B_1+AB_2\sigma$; it follows from (1) that $\operatorname{col}(\widehat{x},\widehat{u})\in\mathfrak{B}$ if and only if for every $k,p\in\mathbb{N}$ with $k>p$ it holds that

$$\widehat{x}_{\mathcal{L}_k}=A(\sigma)^{k-p}\widehat{x}_{\mathcal{L}_p}+\sum_{j=p}^{k-1}A(\sigma)^{k-1-j}B(\sigma)\widehat{u}_{\mathcal{L}_j}\ .$$

Thus, every left annihilator of $\mathbb{D}_{r+1}(\operatorname{col}(\widehat{x},\widehat{u}))$ is a linear combination of the elements constructed with the procedure just described. We proved the equality (14) also for $r\geqslant2$.

Conclude that the number of linearly independent left annihilators of $\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u}))$ equals $\sum_{r=0}^{N-1}(r+1)n=n\frac{N(N+1)}{2}$ and consequently that $\operatorname{rank}(\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u})))$ equals

$$\frac{(N+1)(N+2)}{2}(n+m)-n\frac{N(N+1)}{2}\ ,$$

which is easily seen to be equal to $n(N+1)+m\frac{(N+1)(N+2)}{2}$; statement 3 of the Theorem is proved.

To prove the implication (3) $\Longrightarrow$ (2), note that $\operatorname{col}(\widehat{x},\widehat{u})$ satisfies (1). Consequently left $\ker\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u}))$ contains all constant vectors associated with linear combination of the rows of the coefficient matrices of (3) and its shifts. That is, the rows of (12) are contained in left $\ker\mathbb{D}_1(\operatorname{col}(\widehat{x},\widehat{u}))$, those of (13) are contained in left $\ker\mathbb{D}_2(\operatorname{col}(\widehat{x},\widehat{u}))$, and so forth for every $N\geqslant1$. Using assumption 3) we conclude that there are no other left-annihilators of $\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u}))$. The polynomial matrices associated with the bases (12), (13) and so forth generate the module $\mathcal{N}(\mathfrak{B})$. Consequently,

$$\mathcal{N}(\mathfrak{B})\supseteq\langle\text{left }\ker\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u}))\rangle\ .$$

Statement 2 of the Theorem is proved. ☐

Remark 1. If the data are informative, the rank of *some* finite submatrix of $\mathbb{D}_N(\operatorname{col}(\widehat{x},\widehat{u}))$ satisfies 3) in Theorem 1. ☐

We illustrate the result of Theorem 1 with a numerical example. In this and all other examples in this paper, we generate (input, state) data for systems (1) as follows. We randomly generate 98 values of $x_{\mathcal{L}_0}$, $u_{\mathcal{L}_0}$, $u_{\mathcal{L}_1}$ and $u_{\mathcal{L}_2}$, and compute the corresponding values of $x_{\mathcal{L}_i}$, $i = 1, 2$ using (1).

Example 2. We define

$$A_1 := \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \; A_2 := \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \; B_1 := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; B_2 := \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The first 7 singular values of the matrix $\mathbb{D}_1$ are

$$713070, 7.7964, 3.4980, 3.0897, 2.8103, 2.6836, 2.3185\;,$$

and the last two are $1.1769 \cdot 10^{-15}$ and $8.6754 \cdot 10^{-16}$. As stated in statement 3 of Theorem 1, $\text{rank}(\mathbb{D}_1) = 7$.

$\mathbb{D}_2$ has 19 rows and its last 7 singular values have order 10^{-14} or smaller, while the 12th equals 1.6366. Consequently $\text{rank}(\mathbb{D}_2) = 19 - 7 = 12$, as stated in Theorem 1. □

Remark 2. Roesser models are an alternative class of representations for quarter-plane causal $2D$ systems, equivalent to FM ones (see Section 3 of [6]). An approach analogous to that illustrated in this section can be developed also for such representations. Details will be given elsewhere. □

In the following sections we state some consequences of Theorem 1 relevant for data-driven control applications.

IV. A 'FUNDAMENTAL LEMMA' FOR FM SECOND MODELS

Let $N \in \mathbb{N}$, $N \geqslant 1$, and define $d := \frac{(N+1)(N+2)}{2}(n+m)$. Given $\text{col}(\widehat{x}, \widehat{u}) \in \mathfrak{B}$, we define its *$N$-unfolding at* $(k, -k)$ by

$$f_k(x,u) := \begin{bmatrix} \text{col}(\widehat{x},\widehat{u})_{k,-k} \\ \vdots \\ \text{col}(\widehat{x},\widehat{u})_{k+N,-k-N} \\ \text{col}(\widehat{x},\widehat{u})_{k,-k+1} \\ \vdots \\ \text{col}(\widehat{x},\widehat{u})_{k+N,-k-N+1} \\ \vdots \\ \text{col}(\widehat{x},\widehat{u})_{k+N,-k} \end{bmatrix} \in \mathbb{R}^d \;.$$

Note that every column of $\mathbb{D}_{0:N}(\text{col}(\widehat{x}, \widehat{u}))$ is an unfolding. We define the set of *N-unfoldings of* $\mathfrak{B}$ (see Example 1) by

$$\mathcal{V} := \left\{ v \in \mathbb{R}^d \mid \exists \, \text{col}(\widehat{x},\widehat{u}) \in \mathfrak{B}, k \in \mathbb{Z} \text{ s.t. } v = f_k(\text{col}(\widehat{x},\widehat{u})) \right\}. \tag{15}$$

Since $\mathfrak{B}$ is linear, $\mathcal{V}$ is a subspace of $\mathbb{R}^d$.

We show that informativity for identification implies that the image of the data matrix coincides with $\mathcal{V}$.

Corollary 1. *Define $\mathfrak{B}$ by (2). Let $\text{col}(x, u) \in \mathfrak{B}$. Assume that $N \geqslant \ell(\mathfrak{B}) = 1$ and that the data are informative for system identification. Define $\mathcal{V}$ by (15). Then*

$$\text{im}\,(\mathbb{D}_{0:N}(\text{col}(x,u))) = \mathcal{V}\;. \tag{16}$$

Proof. Since $\mathfrak{B}$ is linear, the inclusion $\text{im}\,(\mathbb{D}_{0:N}(\text{col}(x,u))) \subseteq \mathcal{V}$ is satisfied. To prove the claim we show that informativity for identification implies the converse inclusion: any unfolding of a trajectory of $\mathfrak{B}$ belongs to $\text{im}\,(\mathbb{D}_{0:N}(\text{col}(x,u)))$.

Choose any $\text{col}(\widehat{x}, \widehat{u}) \in \mathfrak{B}$ and $k \in \mathbb{N}$, and construct the corresponding unfolding $f_k\,(\text{col}(\widehat{x}, \widehat{u}))$. Since $\text{col}(\widehat{x}, \widehat{u}) \in \mathfrak{B}$ and $N \geqslant 1$, $f_k\,(\text{col}(\widehat{x}, \widehat{u}))$ is annihilated by the coefficient vector of any element in $\mathcal{N}(\mathfrak{B})$. Because of the equivalence of statements 1) and 2) in Theorem 1, $f_k\,(\text{col}(\widehat{x}, \widehat{u}))$ is annihilated by every vector orthogonal to the columns of $\mathbb{D}_{0:N}(\text{col}(x, u))$. Since $\text{im}\,(\mathbb{D}_{0:N}(\text{col}(x,u))) = (\text{left ker } \mathbb{D}_{0:N}(\text{col}(x,u)))^{\perp}$, we conclude that $f_k\,(\text{col}(\widehat{x}, \widehat{u})) \in \text{im}\,(\mathbb{D}_{0:N}(\text{col}(x,u)))$. □

V. A DATA-BASED OPEN LOOP SYSTEM REPRESENTATION

The following result is analogous to Theorem 1 of [2].

Corollary 2. *Let $\text{col}(\widehat{x}, \widehat{u}) \in \mathfrak{B}$. Assume that $N \geqslant \ell(\mathfrak{B}) = 1$. Assume that the data are informative for system identification. Then $\mathcal{H}_1(\text{col}(\widehat{x}, \widehat{u})_{|\mathcal{L}_0})$ has full row rank.*

Denote by $H_- \in \mathbb{R}^{(n+m)\times\bullet}$ a full rank submatrix of $\mathcal{H}_1(\text{col}(\widehat{x}, \widehat{u})_{|\mathcal{L}_0})$ consisting of consecutive columns, and by $H_+ \in \mathbb{R}^{(n+m)\times\bullet}$ the corresponding submatrix of $\mathcal{H}_0(\text{col}(\widehat{x}, \widehat{u})_{|\mathcal{L}_1})$. Partition $H_+ =: \begin{bmatrix} H_{+,x} \\ H_{+,u} \end{bmatrix}$, with $H_{+,x} \in \mathbb{R}^{n\times\bullet}$, $H_{+,u} \in \mathbb{R}^{m\times\bullet}$. Denote by $H_-^{\dagger}$ a right-inverse of H_-; then

$$H_{+,x}H_-^{\dagger} = \begin{bmatrix} A_1 & B_1 & A_2 & B_2 \end{bmatrix}. \tag{17}$$

Proof. We show that the data being sufficiently informative implies that $\mathcal{H}_1(\text{col}(\widehat{x}, \widehat{u})_{|\mathcal{L}_0})$ has full rank. From statement 3) of Theorem 1 we know that

$$\mathbb{D}_1(\text{col}(\widehat{x},\widehat{u})) = \begin{bmatrix} \text{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0} \\ \sigma\,\text{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0} \\ \text{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_1} \end{bmatrix} \in \mathbb{R}^{3(n+m)\times\infty}\;,$$

has rank $2n+3m$. From the proof of the implication 2) $\Longrightarrow$ 3) we know that a basis for left ker $\mathbb{D}_1(\text{col}(\widehat{x}, \widehat{u}))$ is given by the rows of the matrices (13). Consequently, $\begin{bmatrix} \text{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0} \\ \sigma\,\text{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0} \end{bmatrix} = \mathcal{H}_1(\text{col}(\widehat{x}, \widehat{u})_{|\mathcal{L}_0})$ has full row rank $2(n+m)$.

Use the equation (1) to conclude that

$$\text{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_1} = \begin{bmatrix} A_1 & B_1 & A_2 & B_2 \end{bmatrix} \mathcal{H}_1(\text{col}(\widehat{x},\widehat{u})_{|\mathcal{L}_0})$$

and consequently that $H_{+,x} = \begin{bmatrix} A_1 & B_1 & A_2 & B_2 \end{bmatrix} H_-$. Now postmultiply both sides of the last equation by $H_-^{\dagger}$. The fact that A_i, B_i $i = 1, 2$ are uniquely defined by these equations follows from the fact that the data are sufficiently informative and consequently the set of left-annihilators of $\mathfrak{B}$ defined in (2) is uniquely determined by the data. □

Example 3. We apply (17) to the data generated in Example 2. We obtain the model associated with

$$\begin{aligned} \widehat{A}_1 &:= \begin{bmatrix} 1.6168 \cdot 10^{-15} & 1.0000 \\ 2.0000 & -8.7430 \cdot 10^{-15} \end{bmatrix} \\ \widehat{A}_2 &:= \begin{bmatrix} -7.0777 \cdot 10^{-16} & 1.0000 \\ 9.0000 & 1.3600 \cdot 10^{-15} \end{bmatrix} \\ \widehat{B}_1 &:= \begin{bmatrix} -3.4694 \cdot 10^{-16} \\ 1.0000 \end{bmatrix} \text{ and } \widehat{B}_2 := \begin{bmatrix} 1.0000 \\ 1.0000 \end{bmatrix}, \end{aligned} \tag{18}$$

that coincides up to machine-precision with the data-generating one, see Example 2. □

As shown in Example 3, the state- and input matrices obtained applying formula (17) to real data are only *estimates* of the corresponding matrices in (1), since the system of linear equations $H_{+,x} = \begin{bmatrix} A_1 & B_1 & A_2 & B_2 \end{bmatrix} H_-$ is numerically solved only in a least-squares sense. In the following we use the $\widehat{A}_i$, $\widehat{B}_i$, $i = 1,2$ symbols to distinguish such estimates from the actual matrices A_i, B_i, $i = 1,2$ of the model.

VI. Data-driven stability analysis of FM models

We apply Proposition 1 and Proposition 2 to the model obtained from the data using formula (17) and obtain two data-driven sufficient conditions for the stability of FM models.

Proposition 3. *Let* $\mathrm{col}(\widehat{x}, \widehat{u}) \in \mathfrak{B}$. *Assume that* $N \geqslant \ell(\mathfrak{B}) = 1$. *Assume that the data are informative for system identification. With the same notation as in Corollary 2, partition*

$$H_{+,x} H_-^\dagger =: \begin{bmatrix} A_1 & B_1 & A_2 & B_2 \end{bmatrix} .$$

If there exist $Q_i = Q_i^\top \in \mathbb{R}^{n \times n}$, $Q_i > 0$, $i = 1,2$ *such that*

$$0 > \begin{bmatrix} A_1 (Q_1 + Q_2) A_1^\top - Q_1 & A_1 (Q_1 + Q_2) A_2^\top \\ A_2 (Q_1 + Q_2) A_1^\top & A_2 (Q_1 + Q_2) A_2^\top - Q_2 \end{bmatrix} \tag{19}$$

then the data-generating model is stable.

Proof. Follows from Proposition 1 and Corollary 2. □

Example 4. Consider the stable system described by

$$\begin{aligned} A_1 &:= \begin{bmatrix} 0 & 0.001 \\ 0.002 & 0 \end{bmatrix} , \; A_2 := \begin{bmatrix} 0 & 0.001 \\ 0.009 & 0 \end{bmatrix} \\ B_1 &:= \begin{bmatrix} 0 & 1 \end{bmatrix}^\top , \; B_2 := \begin{bmatrix} 1 & 1 \end{bmatrix}^\top . \end{aligned}$$

We generate data following the same procedure used in Example 2. Using (17) we compute numerical estimates $\widehat{A}_i$ and $\widehat{B}_i$, $i = 1,2$ of the state- and input matrices:

$$\begin{aligned} \widehat{A}_1 &:= \begin{bmatrix} 5.9674 \cdot 10^{-15} & 0.001 \\ 0.002 & 6.9389 \cdot 10^{-17} \end{bmatrix} \\ \widehat{A}_2 &:= \begin{bmatrix} -3.3307 \cdot 10^{-16} & 0.001 \\ 0.009 & -3.6082 \cdot 10^{-16} \end{bmatrix} \\ \widehat{B}_1 &:= \begin{bmatrix} 9.7145 \cdot 10^{-17} \\ 1.0000 \end{bmatrix} \text{ and } \widehat{B}_2 := \begin{bmatrix} 1.0000 \\ 1.0000 \end{bmatrix} , \end{aligned} \tag{20}$$

which equal A_i, B_i, $i = 1,2$ up to numerical precision. We solve the LMIs (19) with `Yalmip` (see [7]), and obtain

$$Q_1 = \begin{bmatrix} 0.5018 & 0 \\ 0 & 0.4982 \end{bmatrix}, \; Q_2 = \begin{bmatrix} 0.5094 & 0 \\ 0 & 0.4096 \end{bmatrix}.$$

We conclude that the data-generating system is stable. □

Proposition 4. *Let* $\mathrm{col}(\widehat{x}, \widehat{u}) \in \mathfrak{B}$. *Assume that* $N \geqslant \ell(\mathfrak{B}) = 1$. *Assume that the data are informative for system identification. With the same notation as in Corollary 2, partition* $H_{+,x} H_-^\dagger =: \begin{bmatrix} A_1 & B_1 & A_2 & B_2 \end{bmatrix}$. *If there exist* $X = X^\top, Y = Y^\top \in \mathbb{R}^{n \times n}$ *such that* $X > 0$, $Y > 0$ *and*

$$0 > \begin{bmatrix} -X & 0 & Y A_1^\top \\ 0 & X - Y & Y A_2^\top \\ A_1 Y & A_2 Y & -Y \end{bmatrix} \tag{21}$$

then the data-generating model is stable.

Proof. Follows from Proposition 1 and Corollary 2. □

Example 5. We consider the same data and matrices as in the previous example and we apply the LMI test (21) with `Yalmip`. We obtain

$$X = \begin{bmatrix} 35.3797 & 0 \\ 0 & 35.3809 \end{bmatrix}, \; Y = \begin{bmatrix} 58.9644 & 0 \\ 0 & 58.9681 \end{bmatrix}.$$

We conclude that the data-generating system is stable. □

Remark 3. *Necessary and sufficient* LMI conditions for stability of FM models are stated in [3]; such conditions can also be cast in a data-driven framework, analogously to what we did with Propositions 1 and 2. It is well known (see Remark 1 p. 1512 of [3]) that such conditions are not easy to apply for controller synthesis. Since our aim is to provide effective data-driven control design techniques, we do not investigate any further LMI characterizations of stability. □

VII. Data-driven state-feedback stabilization of FM models

The problem of state-feedback stabilization of (1) consists in finding a gain $K \in \mathbb{R}^{m \times n}$ such that the closed-loop system

$$\sigma_1 x = (A_1 - B_1 K) x + (A_2 - B_2 K) \sigma x , \tag{22}$$

is stable (see Section 8 of [6]). We use Propositions 1 and 2 to compute a stabilizing gain K.

Proposition 5. *If* $\exists \; Q_i > 0$, $i = 1,2$ *and* $F \in \mathbb{R}^{m \times n}$ *such that*

$$\begin{bmatrix} Q_1 & 0 & A_1(Q_1 + Q_2) - B_1 F \\ 0 & Q_2 & A_1(Q_1 + Q_2) - B_2 F \\ \star & \star & Q_1 + Q_2 \end{bmatrix} > 0 ;$$

then $K := F(Q_1 + Q_2)^{-1}$ *stabilizes the system* (22).

Proof. Substitute $A_{i,K} := A_i - B_i K$ for A_i, $i = 1,2$, in (5) and rewrite it as

$$\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} - \begin{bmatrix} A_{1,K} \\ A_{2,K} \end{bmatrix} (Q_1 + Q_2) \begin{bmatrix} A_{1,K}^\top & A_{2,K}^\top \end{bmatrix} > 0 .$$

Schur-complement and obtain the equivalent LMI

$$\begin{bmatrix} Q_1 & 0 & A_{1,K}(Q_1 + Q_2) \\ 0 & Q_2 & A_{2,K}(Q_1 + Q_2) \\ \star & \star & Q_1 + Q_2 \end{bmatrix} > 0 .$$

The claim follows writing $A_{i,K}(Q_1 + Q_2) = A_i(Q_1 + Q_2) - B_i K(Q_1 + Q_2)$, $i = 1,2$ and defining $F := K(Q_1 + Q_2)$. □

Proposition 6. *If* $\exists \; X = X^\top > 0, Y = Y^\top > 0 \in \mathbb{R}^{n \times n}$ *and* $F \in \mathbb{R}^{m \times n}$ *such that*

$$\begin{bmatrix} X & 0 & (A_1 Y - B_1 F)^\top \\ 0 & X - Y & (A_2 Y - B_2 F)^\top \\ A_1 Y - B_1 F & A_2 Y - B_2 F & Y \end{bmatrix} > 0 ,$$

then $K := F Y^{-1}$ *is a stabilizing gain for* (22).

Proof. Substitute $A_i - B_i K$ for A_i, $i = 1,2$, in (6). Rewrite the expression as

$$0 > \begin{bmatrix} -X & 0 & \star \\ 0 & X - Y & \star \\ A_1 Y - B_1 K Y & A_2 Y - B_2 K Y & -Y \end{bmatrix} .$$

Define $F := KY$; the claim follows. $\square$

Example 6. Consider the system (1) defined by

$$A_1 := \begin{bmatrix} 0.3274 & -1.5873 \\ 0.6571 & -1.3715 \end{bmatrix},\ A_2 := \begin{bmatrix} -0.5735 & 2.0319 \\ 0.1638 & -0.3335 \end{bmatrix}$$
$$B_1 := \begin{bmatrix} -0.6504 & 0.3515 \end{bmatrix}^\top,\ B_2 := \begin{bmatrix} 0.7259 & 0.4319 \end{bmatrix}^\top.$$

This system is unstable since $\rho(A_2) = 1.0427 > 1$.

We generate informative data following the same procedure as in Example 2. Using formula (17), we obtain a model $\hat{A}_i$, $\hat{B}_i$, $i = 1, 2$, that coincides up to machine-precision with the data generating one. We apply Proposition 5, and obtain the following solution to the LMIs:

$$Q_1 = \begin{bmatrix} 1.2433 & 0.4068 \\ 0.4068 & 0.1449 \end{bmatrix},\ Q_2 = \begin{bmatrix} 1.4361 & 0.3826 \\ 0.3826 & 0.1324 \end{bmatrix},$$

$F = \begin{bmatrix} 1.3250 & 0.3202 \end{bmatrix}$, from which we compute

$$K = F(Q_1 + Q_2)^{-1} = \begin{bmatrix} 0.9554 & -1.5644 \end{bmatrix}.$$

It can be checked that $\rho(A_1 - B_1K) = 0.5678$ and $\rho(A_2 - B_2K) = 0.9938$.

Applying Proposition 6, we obtain

$$X = \begin{bmatrix} 0.7718 & 0.2052 \\ 0.2052 & 0.0707 \end{bmatrix},\ Y = \begin{bmatrix} 2.4696 & 0.7348 \\ 0.7348 & 0.2673 \end{bmatrix},$$

$F = \begin{bmatrix} 1.2246 & 0.2939 \end{bmatrix}$, from which we compute

$$K' = FY^{-1} = \begin{bmatrix} 0.9274 & -1.4501 \end{bmatrix}.$$

Then $\rho(A_1 - B_1K') = 0.1895$ and $\rho(A_2 - B_2K') = 0.8214$.

VIII. Conclusions and further work

We investigated some issues relevant to $2D$ data-driven control: the definition of "sufficient informativity", establishing a "fundamental lemma", and data-based stability analysis and state-feedback stabilization. Without loss of generality, we worked with Fornasini-Marchesini second models of quarter-plane causal $2D$-systems.

The assumption that the state is measurable is a serious limitation to the applicability of our approach to realistic problems. While direct measurement of the state is a standard premise in $1D$ data-driven control, it postulates an insight into the system structure that is at odds with a truly data-driven point of view, where control problems should be stated and solved at the level of *external* variables. A pressing issue of current research is extending the notions of informativity and persistency of excitation, and proving an analogous of the fundamental lemma, when only input-output data are available.

Notwithstanding the limitations due to the assumption of state-measurability, the results presented in this paper are of potential interest in several areas. FM and Roesser models (see Remark 2) are also used in Iterative Learning Control and in Repetitive Control (see [14], [15], [16]) and the results presented in this paper may provide alternatives to current data-driven approaches in those areas (see [17], [18]).

Corollary 1 provides a parametrization of all "unfoldings" of system trajectories in terms of a basis for $\mathrm{im}\,(\mathbb{D}_N(\mathrm{col}(\widehat{x}, \widehat{u})))$. This result opens up the possibility of performing data-driven simulation for $2D$ systems (on this see also Section V of [8]). Moreover, such parametrization puts within reach the data-driven solution of finite-horizon $2D$ quadratic control problems and the data-driven solution of $2D$ model predictive control problems. Ongoing research in these directions will be presented elsewhere.

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