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Faculty of Social Sciences
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Semigroups have layers: a generalisation of stratified semigroups

by

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A thesis for the degree of
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Abstract

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This thesis is a work of two parts. In the first part, we attempt to study the structure of $E-$inversive (also known as $E-$dense) semigroups. As these semigroups can be viewed as a generalisation of regular semigroups, our approach aims to adapt methods used to understand regular semigroups to the $E-$dense setting. Our first method is a geometric approach inspired by work due to K. S. S. Nambooripad, while the second is based on T. S. Blyth's work on inverse transversals. We also briefly examine these results in the context of some simple wreath products, motivated by the Krohn-Rhodes decomposition of finite semigroups.

Due to a number of factors, including the Covid-19 pandemic, our focus then shifts to a generalisation of work by Pierre Grillet on stratified semigroups. The main body of this second part of the thesis consists largely of joint work with James Renshaw, namely the following papers, elements of which also appear in the introductory chapter:

[1] James Renshaw & William Warhurst, *Semilattices of Stratified Semigroups*, preprint, available at arXiv:2305.11535 [math.GR], 2023.

[2] James Renshaw & William Warhurst, *The multiplicative semigroup of a Dedekind domain*, preprint, available at arXiv:2309.02831 [math.GR], 2023.

In [1], we introduce stratified extensions as a generalisation of Grillet's stratified semigroups, which we describe these in terms of ideal extensions of semigroups. While many commonly studied semigroups (such as monoids or regular semigroups) are stratified extensions only in a fairly trivial sense, we provide a number of interesting examples of semigroups which can be decomposed as semilattices of stratified semigroups.

In [2], we continue this work by first showing that the multiplicative semigroup of any commutative ring can be viewed as a semilattice of some semigroups. We then show that if the ring is a Dedekind domain then the semilattice consists of the group of units and a stratified extension of the trivial group and hence the multiplicative semigroup can be viewed as a semilattice of stratified extensions. Further, if the ring is any quotient of a Dedekind domain, the multiplicative semigroup is again a semilattice of stratified extensions, with potentially a much more complex structure than in the non-quotient case.

Contents

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

[1] James Renshaw & William Warhurst, *Semilattices of Stratified Semigroups*, preprint, available at arXiv:2305.11535 [math.GR], 2023.

[2] James Renshaw & William Warhurst, *The multiplicative semigroup of a Dedekind domain*, preprint, available at arXiv:2309.02831 [math.GR], 2023.

Signed:.. Date:..................

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My thanks go out to Jim, for the support and guidance over these four years and without whom this journey would never have begun, let alone reached this point.

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To my family, who continue to tolerate my steadfast failure to get a real job.

And last but not least, to the tiny queer people who live in my computer. Thank you for helping me find myself.

Notation

$A(s)$	the set of associates of s
$\mathrm{Base}(S)$	the base of a semigroup S
$\mathcal{D}$	Green's relation
$\mathcal{D}_S$	Green's relation on S
D_s	the $\mathcal{D}$-class containing s
$E(S)$	the set of idempotents of S
$\mathcal{H}$	Green's relation
$\mathcal{H}_S$	Green's relation on S
H_s	the $\mathcal{H}$-class containing s
$\mathcal{J}$	Green's relation
$\mathcal{J}_S$	Green's relation on S
J_s	the $\mathcal{J}$-class containing s
$\ker\phi$	the kernel of the map ϕ
$\mathcal{L}$	Green's relation
$\mathcal{L}_S$	Green's relation on S
L_s	the $\mathcal{L}$-class containing s
$\mathcal{R}$	Green's relation
$\mathcal{R}_S$	Green's relation on S
R_s	the $\mathcal{R}$-class containing s
$\mathrm{Reg}(S)$	the set of regular elements of S
S_i	the $i-$th layer of S
$V(s)$	the set of inverses of s
$W(s)$	the set of weak inverses of s

Chapter 1

Background

In this chapter we introduce many of the background results in semigroup theory underpinning the results throughout this thesis. We begin in Section 1.1 with basic definitions of a semigroup as well as equivalence relations and congruences, and also introduce the notion of a semilattice of semigroups. This content draws heavily from John Howie's *Fundamentals of Semigroup Theory* [18], as does the first part of Section 1.2 which introduces the concept of regularity. In Section 1.2.2 we look at $E-$inversive semigroups as a generalisation of regularity. An overview of these semigroups can be found in Mitsch [24]. Section 1.3 introduces the concept of ideal extensions as defined by Clifford and Preston [4] and Section 1.4 covers Grillet's work on stratified semigroups [14].

1.1 Semigroups

1.1.1 Basic definitions

Let S be a set equipped with a binary operation, that is, a map $S \times S \to S$ written as $(x, y) \mapsto x * y$. We say $(S, *)$ is a *semigroup* if $*$ is associative, i.e.

$$(x * y) * z = x * (y * z).$$

Where the multiplication on the semigroup is obvious from context, we shall refer simply to the semigroup S and denote multiplication by juxtaposition, so $x * y$ takes the form xy. As with other algebraic structures, we call a subset A of S which is closed under the multiplication on S a *subsemigroup* of S. If the subsemigroup A is also a group under the inherited multiplication of S then A is called a *subgroup* of S.

If a semigroup S has the property that, for all $x, y \in S$,

$$xy = yx$$

then we say S is a *commutative semigroup*. If S contains an element 1 such that, for all $x \in S$,

$$1x = x1 = x$$

then 1 is an *identity element* and we say S is a *monoid*. If S contains an element 0 such that, for all $x \in S$,

$$0x = x0 = 0$$

then 0 is a *zero element*. It is easy to see that if a semigroup contains either an identity or a zero then it must be unique. We can also define one-sided analogues of these elements. For example, an element $e \in S$ is a *left zero* if, for all $x \in S$,

$$ex = e.$$

We donte by S^1 the semigroup obtained by adjoining an identity element to S if it did not already contain one, and likewise S^0 for the same construction with a zero element.

Example 1.1.1. *Any group is trivially also a semigroup, as the definition of a group includes the same properties as that of a semigroup along with the existence of inverses and identity elements. The remaining examples are not groups.*

Example 1.1.2. *The natural numbers* $\mathbb{N}$ *(with or without* 0*) form a semigroup under either of addition or multiplication.*

Example 1.1.3. *A* left zero semigroup *is a semigroup in which every element is a left zero, i.e. for any elements x, y of a left zero semigroup S we have $xy = x$.*

Example 1.1.4. *Let $B = \mathbb{N}_0 \times \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of natural numbers including* 0*. Define multiplication on B by $(a, b)(x, y) = (a - b + \max(b, x), y - x + \max(b, x))$. Since $\max(b, x)$ is at least as large as both b and x, both coordinates are non-negative so the operation is closed. It can also be shown that the operation is associative and hence B is a semigroup, called the* bicyclic semigroup*. An alternative way to view this semigroup is as the monoid with two generators, p and q, under the relation $qp = 1$. Note that unlike a group presentation, this does not imply that $pq = 1$.*

If A and B are subsets of a semigroup S, we denote by AB the subset

$$\{ab \mid a \in A, b \in B\}.$$

Associativity of this set-wise multiplication follows immediately from associativity in S, and so we may freely make reference to constructions such as ABC or A^3. Note in this latter case the distinction between a set such as A^3 and the set $\{a^3 \mid a \in A\}$ which is a subset of A^3. In particular, for $i \geq 2$, S^i denotes the set of elements of S which can be written as a product of i elements. If I is a subset of S such that

$$SI \subseteq I$$

then I is called a *left ideal* of S. Similarly, if

$$IS \subseteq I$$

then I is called a *right ideal* of S. If I is both a left and a right ideal then it is called an *ideal*. A left ideal I is called a *principal left ideal* if there exists $a \in S$ such that

$$I = S^1 a = \{a\} \cup Sa.$$

Similarly I is called a *principal right ideal* if

$$I = aS^1 = \{a\} \cup aS$$

and a *principal ideal* if

$$I = S^1 a S^1 = \{a\} \cup Sa \cup aS \cup SaS.$$

A map

$$f : S \to R$$

between two semigroups is called a *morphism* if it preserves multiplication, i.e.

$$f(xy) = f(x)f(y)$$

for all $x, y \in S$. As usual, a bijective morphism is called an *isomorphism*.

1.1.2 Relations and congruences

A *binary relation* ρ on a semigroup S is a subset of the cartesian product $S \times S$. Two elements $x, y \in S$ are related if $(x, y) \in \rho$, also written as $x\rho y$. If, for all $x, y, z \in S$, ρ satisfies

1. $(x, x) \in \rho$ (reflexivity)
2. $(x, y) \in \rho \Rightarrow (y, x) \in \rho$ (symmetry)
3. $(x, y), (y, z) \in \rho \Rightarrow (x, z) \in \rho$ (transitivity)

then ρ is an *equivalence relation*. We say ρ is *left compatible* if, for all $x, y, z \in S$,

$$(x, y) \in \rho \Rightarrow (zx, zy) \in \rho.$$

A *right compatible* relation is defined in an analagous way. A relation is called compatible if, for all $x, y, x', y' \in S$,

$$(x, y), (x', y') \in \rho \Rightarrow (xx', yy') \in \rho.$$

A left (resp. right) compatible equivalence relation is called a *left (resp. right) congruence* and a compatible equivalence relation is called a *congruence*. It can be shown [18, Proposition 1.5.1] that a relation is a congruence if and only if it is both a left congruence and a right congruence.

Given a congruence ρ on a semigroup S, we can define a binary operation on the set of equivalence classes S/ρ in a natural way by letting

$$(x)\rho(y)\rho = (xy)\rho$$

for all $x, y \in S$. This is well-defined precisely because ρ is a congruence, as if $(x)\rho = (x')\rho$ and $(y)\rho = (y')\rho$ then $x\rho x'$ and $y\rho y'$. Hence $xy\rho x'y'$ and so

$$(xy)\rho = (x'y')\rho$$

as required.

Congruences on a semigroup play a similar role to normal subgroups of a group or two-sided ideals of a ring, in that they determine the homomorphic images of the semigroup. If S and T are semigroups and ϕ is a homomorphism from S to T then we can define the *kernel* of ϕ as the relation $\{(a, b) \in S \times S \mid \phi(a) = \phi(b)\}$, denoted by $\ker \phi$. We then have the first isomorphism theorem:

Theorem 1.1.5. *Let S and T be semigroups and let $\phi : S \to T$ be a homomorphism. Then the kernel of ϕ is a congruence and the image of ϕ is isomorphic to $S/\ker \phi$.*

An important family of relations in semigroup theory are collectively known as Green's equivalences, after J. A. Green [13]. There are five such equivalences on a semigroup S, denoted by $\mathcal{L}_S$, $\mathcal{R}_S$, $\mathcal{J}_S$, $\mathcal{H}_S$ and $\mathcal{D}_S$, and we shall define them in that order. In most instances the semigroup S will be apparent from context and so we shall allow ourselves to omit the subscripts. The equivalence class of $s \in S$ under each of the relations is denoted by L_s, R_s, J_s, H_s and D_s respectively.

The relation $\mathcal{L}$ is defined by the rule that $x\mathcal{L}y$ if and only if x and y generate the same principal ideal, i.e. $S^1x = S^1y$. The relation $\mathcal{R}$ is defined symmetrically, so $x\mathcal{R}y$ if and only if $xS^1 = yS^1$. These definitions admit an alternative characterisation, which will often prove to be more useful to work with in practice.

Proposition 1.1.6 ([18, Proposition 2.1.1])**.** *Let $x, y \in S$. Then $x\mathcal{L}y$ if and only if there exist $u, v \in S^1$ such that $ux = y$ and $vy = x$. Similarly, $x\mathcal{R}y$ if and only if there exist $s, t \in S^1$ such that $xs = y$ and $yt = x$.*

We also have the two-sided analogue in the form of the $\mathcal{J}$ equivalence, with $x\mathcal{J}y$ if and only if $S^1xS^1 = S^1yS^1$. In a similar manner to $\mathcal{L}$ and $\mathcal{R}$ it can be shown that $x\mathcal{J}y$ if and only if there exist $u, v, s, t \in S^1$ such that $x = uyv$ and $y = sxt$.

The remaining relations are based on the relationship between $\mathcal{L}$ and $\mathcal{R}$. The easier one to define is $\mathcal{H}$, the intersection of $\mathcal{L}$ and $\mathcal{R}$. In other words, $x\mathcal{H}y$ if and only if $x\mathcal{L}y$ and $x\mathcal{R}y$. Finally, $\mathcal{D}$ is the join of $\mathcal{L}$ and $\mathcal{R}$. In other words, $x\mathcal{D}y$ if and only if we can reach y from x through some series of $\mathcal{L}$ and $\mathcal{R}$ relations, e.g.

$$x\mathcal{L}x_1\mathcal{R}x_2\mathcal{L}\ldots\mathcal{L}x_n\mathcal{R}y.$$

Fortunately this definition is greatly simplified by the fact that $\mathcal{L}$ and $\mathcal{R}$ commute [18, Proposition 2.1.3], so there exists $c \in S$ such that $x\mathcal{L}c\mathcal{R}y$ if and only if there exists $d \in S$ such that $x\mathcal{R}d\mathcal{L}y$. Hence $x\mathcal{D}y$ if and only if there exists $c \in S$ such that $x\mathcal{L}c\mathcal{R}y$.

Note that for any semigroup we have

$$\mathcal{H} \subseteq \mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}.$$

The exact relationship between the relations can vary dramatically. For example, in a group G we have

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = G \times G$$

while in a right zero semigroup S we have $\mathcal{R} = S \times S$ but $\mathcal{L}$ is the identity relation.

The structure of a $\mathcal{D}$-class can be visualised using so-called 'eggbox' diagrams in which each row represents an $\mathcal{R}$-class, each column represents an $\mathcal{L}$-class, and correspondingly each cell represents an $\mathcal{H}$-class. Hence the diagram

a, b		c
d		e

tells us, among other things, that $a\mathcal{H}b$, $a\mathcal{R}c$, $a\mathcal{L}d$ and $a\mathcal{D}e$.

An important result regarding the $\mathcal{H}$-classes of a semigroup is commonly known as Green's Theorem .

Theorem 1.1.7 ([18, Theorem 2.2.5]). *If H is an $\mathcal{H}$-class in a semigroup S then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S.*

A particular consequence of this theorem is that any $\mathcal{H}$-class can have at most one idempotent.

1.1.3 Semilattices

An element e of a semigroup S is called *idempotent* if $e^2 = e$. The set of all idempotent elements of S is denoted by $E(S)$. If a semigroup T is such that $E(T) = T$ then T is called a *band*. If $E(S)$ is a subsemigroup of S then it forms a band and so we say S has a band of idempotents.

There is a natural partial order on the idempotents $E(S)$ given by $e \leq f$ if and only if

$$ef = fe = e.$$

To see that this is a partial order, note that $ee = e$ so $e \leq e$. If $a \leq b$ and $b \leq a$ then

$$a = ab = ba = b.$$

Finally, if $a \leq b$ and $b \leq c$ then

$$ac = (ab)c = a(bc) = ab = a$$

and similarly $ca = a$ so $a \leq c$.

Let X be a partially ordered set and let $a, b \in X$. We say x is a *lower bound* of a and b if $x \leq a$ and $x \leq b$. If, in addition, $y \leq x$ for any lower bound y of a and b we say x is the *greatest lower bound* of a and b. A partially ordered set in which every pair of elements a and b has a greatest lower bound is called a *semilattice*.

If E is a commutative band then E is a semilattice with respect to the natural partial order, with ef as the greatest lower bound of e and f. To see this, note first that $e(ef) = ef$ and $(ef)f = ef$ so $ef \leq e, f$. Suppose $g \leq e, f$ and consider gef. Since $g \leq e$ we have $ge = g$ and since $g \leq f$ we have $gf = g$. Hence $gef = gf = g$ and $g \leq ef$ so ef is the greatest lower bound of e and f. We therefore call a commutative band a semilattice, and if $E(S)$ is a semilattice we say S has a semilattice of idempotents.

Let S be a semigroup and Y a semilattice and suppose that for each $\alpha \in Y$ there exists a subsemigroup S_α of S satisfying the following conditions:

1. $S = \bigcup_{\alpha \in Y} S_\alpha$
2. $S_\alpha \cap S_\beta = \emptyset$ for all $\alpha \neq \beta \in Y$
3. $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$.

Then S is called a *semilattice of semigroups* and write $S = \mathcal{S}[Y; S_\alpha]$. This construction is most useful when the subsemigroups S_α all share some particular property, in which case we refer to the semilattice with that property. For example, we might have a semilattice of groups, or a semilattice of finite semigroups, or a semilattice of left zero semigroups, and so on.

If $S = \mathcal{S}[Y; S_\alpha]$ then there is a natural homomorphism $\phi : S \to Y$ given by $\phi(x) = \alpha$ if $x \in S_\alpha$. The semigroups S_α are then clearly the congruence classes of $\ker \phi$. Conversely, if ϕ is a homomorphism from S onto some semilattice Y then $S = \mathcal{S}[Y; S_\alpha]$ where $S_\alpha = \{x \in S \mid \phi(x) = \alpha\}$.

The benefit of decomposing a semigroup into a semilattice of semigroups (versus, say, merely a union of semigroups) is that we gain some degree of control over the global multiplicative structure. If $x \in S_\alpha$ and $y \in S_\beta$ then we know $xy \in S_{\alpha\beta}$. The limitation here is that we have no way of knowing where in $S_{\alpha\beta}$ the element xy lies. One way to address this lies in the concept of a strong semilattice.

Let Y be a semilattice and for each $\alpha \in Y$ let S_α be a semigroup with $S_\alpha \cap S_\beta = \emptyset$ for every $\alpha, \beta \in Y$. For each $\alpha \geq \beta \in Y$ let $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a homomorphism such that

1. $\phi_{\alpha,\alpha}$ is the identity map on S_α for all $\alpha \in Y$
2. for all $\alpha \geq \beta \geq \gamma$ the composition of $\phi_{\alpha,\beta}$ and $\phi_{\beta,\gamma}$ is equal to $\phi_{\alpha,\gamma}$.

Then we can define a multiplication on $S = \cup_{\alpha \in Y} S_\alpha$ by

$$xy = \phi_{\alpha,\alpha\beta}(x)\phi_{\beta,\alpha\beta}(y)$$

where $x \in S_\alpha$ and $y \in S_\beta$. A semigroup of this form is called a *strong semilattice of semigroups* and we write $S = \mathcal{S}[Y; S_\alpha; \phi_{\alpha,\beta}]$.

Example 1.1.8. *A semigroup S is called a* rectangular band *if, for every $a, b, c \in S$, $abc = ac$. Every band decomposes as a semilattice of rectangular bands [22, Theorem 1].*

Example 1.1.9. *A* Clifford semigroup *is a semigroup S in which $E(S)$ is a semilattice and every element of S lies in some subgroup of S. A semigroup S is a Clifford semigroup if and only if S is a semilattice of groups $\mathcal{S}[Y; G_\alpha]$. Moreover, every semilattice of groups is a strong semilattice, with the map $\phi_{\alpha,\beta}$ given by*

$$\phi_{\alpha,\beta}(x) = e_\beta x$$

where e_β is the identity of the group G_β. Hence S is a Clifford semigroup if and only if $S = \mathcal{S}[Y; G_\alpha; \phi_{\alpha,\beta}]$ [18, Theorem 4.2.1].

1.2 Regularity

1.2.1 Regular semigroups

Let S be a semigroup. An element $s \in S$ is called *regular* if there exists $t \in S$ such that $sts = s$. The set of regular elements of S is denoted by $\mathrm{Reg}(S)$. The semigroup S is called regular if every element of S is regular. A related concept is that of an inverse. If s is an element of a semigroup S then we say that $s' \in S$ is an *inverse* of s if

$$ss's = s \text{ and } s'ss' = s'.$$

It is clear from this definition any element $s \in S$ with an inverse is necessarily regular. In fact s has an inverse if and only if s is regular: if $sts = s$ let $s' = tst$. Then

$$ss's = ststs = sts = s \text{ and } s'ss' = tststst = tstst = tst = s'$$

and hence s' is an inverse of s. The set of inverses of an element s is denoted by $V(s)$. From the symmetry of the definition it is clear that $s \in V(t)$ if and only if $t \in V(s)$. Note that if $s' \in V(s)$ then $ss'ss' = ss'$ and $s'ss's = s's$ so $ss', s's \in E(S)$.

The name inverse suggests a similarity to the inverses of group theory, and indeed it is the case that a group inverse is an inverse under this definition. In fact, for any element g of a group G we have $V(g) = \{g^{-1}\}$. More generally, however, inverses need not be unique. For an example of the opposite extreme, consider a right zero semigroup S. For any $s, t \in S$ we have $sts = s$ and $tst = t$ by definition, and hence every element of S is an inverse of every other element.

If every element of a semigroup S has a unique inverse, then S is called an *inverse semigroup*. It is easy to see that a sufficient condition for a regular semigroup S to be an inverse semigroup is if $E(S)$ is a semilattice: if $s', s^* \in V(s)$ then

$$s' = s'ss' = s'ss^*ss' = s'ss'ss^* = s'ss^*$$

since ss' and ss^* commute. By a similar argument, $s^* = s'ss^*$ and so $s' = s^*$. In fact, this is also a necessary condition.

Theorem 1.2.1 ([18, Theorem 5.1.1]). *Let S be a regular semigroup. Then S is an inverse semigroup if and only if $E(S)$ is a semilattice.*

Proof. Note that if $e, f \in E(S)$ commute then $efef = eeff = ef$ so $ef \in E(S)$. Hence if all idempotents of S commute then $E(S)$ is necessarily a subsemigroup of S and in particular is a semilattice. It remains to show that if S is an inverse semigroup then its idempotents commute. Let $e, f \in E(S)$ and let $z \in V(ef)$ be the unique inverse of ef. Then

$$(ef)(fze)(ef) = efzef = ef$$

and

$$(fze)(ef)(fze) = f(zefz)e = fze$$

so $fze \in V(ef)$ and furthermore $fze \in E(S)$. Since inverses are unique, $fze = z$ so z is idempotent and hence $zzz = z$ and $z \in V(z)$. Then $ef, z \in V(z)$ so $ef = z$ and $ef \in E(S)$. A similar argument shows that $fe \in E(S)$. Then

$$(ef)(fe)(ef) = efef = ef$$

and

$$(fe)(ef)(fe) = fefe = fe$$

so $fe \in V(ef)$. But ef is idempotent so $ef \in V(ef)$ and hence $ef = fe$ and the idempotents commute. □

Let $s \in S$ be a regular element of S and let $t \in L_s$. Then there exist $u, v \in S$ such that $us = t$, $vt = s$. Hence

$$t = us = uss's = ts's = t(s'v)t$$

where s' is an inverse of s, and so t is regular. It follows that if s is regular then every element of L_s is regular. The same argument can be applied to R_s, and hence we have proved

Proposition 1.2.2 ([18, Proposition 2.3.1])**.** *If s is a regular element of a semigroup S then every element of D_s is regular.*

Let $s \in S$ and let $s' \in V(s)$. Then $(ss')s = s$ so $s\mathcal{R}ss'$. Similarly, $s'(ss') = s'$ so $s'\mathcal{L}ss'$. Then $s\mathcal{R}ss'\mathcal{L}s'$ and hence every inverse of s lies in the $\mathcal{D}$-class D_s. The eggbox diagram

s		ss'
$s's$		s'

visualises the relationship between s, its inverse s', and the related idempotents ss' and $s's$. An immediate consequence of this arrangement is that in a regular $\mathcal{D}$-class, every $\mathcal{L}$- and $\mathcal{R}$-class contains an idempotent.

Suppose $s', s^* \in V(s)$ and $s'\mathcal{H}s^*$. It can be easily seen from the above discussion that $ss'\mathcal{H}ss^*$ and $s's\mathcal{H}s^*s$ and so, since idempotents are unique in $\mathcal{H}$-classes, $ss' = ss^*$ and $s's = s^*s$. Hence

$$s' = s'ss' = s^*ss^* = s^*$$

and so each $\mathcal{H}$-class contains at most one inverse of s.

Example 1.2.3. *The full transformation semigroup $\mathcal{T}(X)$ on a set X is the set of all maps $f : X \to X$ under composition. Let $f, g \in \mathcal{T}(X)$ such that if $a \in \mathrm{Im}(f)$ then $g(a) = b$ where $f(b) = a$. Then $fgf(x) = f(x)$ for all $x \in X$ so $fgf = f$. Hence $\mathcal{T}(X)$ is regular.*

Example 1.2.4. *Let S be the bicyclic semigroup. It can be easily verified that $(i,j)(j,i)(i,j) = (i,j)$ for any $i, j \in \mathbb{N}^0$ and hence $(j,i) \in V((i,j))$. The idempotents of S are the elements of the form (i,i) which can be readily seen to form a commutative subsemigroup, and hence the bicyclic semigroup is an inverse semigroup with $V((i,j)) = \{(j,i)\}$.*

1.2.2 $E-$inversive semigroups

A semigroup S is said to be $E-$*inversive* if for all $s \in S$ there exists $t \in S$ such that $st \in E(S)$. If S is $E-$inversive and $E(S)$ is a semilattice then S is called $E-$*dense*. Note that some authors omit this distinction and use the terms $E-$dense and $E-$inversive interchangably.

Note that this definition need not be one-sided. Let S be an $E-$inversive semigroup and let $s \in S$. There exists $t \in S$ such that $st \in E(S)$. Then

$$(tsts)^2 = tststs = t(st)^3 s = tsts$$

and so $(tst)s \in E(S)$. Since $s(tst) = st \in E(S)$ we have that S is $E-$inversive if and only if there exists $x \in S$ such that $sy, ys \in E(S)$.

We can generalise the idea of an inverse by relaxing the symmetry in the definition. We have seen that if, for some $s \in S$, there exists x such that $sxs = s$ then s necessarily has an inverse in the form of xsx. As such, we adopt the other half of the definition. If there exists $x \in S$ such that

$$xsx = x.$$

we say that x is a *weak inverse* of s. The set of all weak inverses of s is denoted by $W(s)$. If t is a weak inverse of s, then s is called an *associate* of t. The set of associates of t is denoted by $A(t)$ and clearly $t \in W(s)$ if and only if $s \in A(t)$. Note that weak inverses and associates cover both sides of the definition of an inverse, so $V(s) = W(s) \cap A(s)$. The term 'associate' here has no relation to the use of the term in ring theory, which makes an appearance later in this thesis.

As above, if $s' \in W(s)$ then $ss's \in V(s')$ and so every weak inverse is regular. Conversely, $V(s) \subseteq W(s)$. Hence if s is regular then for any $s' \in V(s)$, $s \in V(s') \subseteq W(s')$ and so s is a weak inverse. We have therefore shown that the set of all weak inverses in a semigroup S, denoted by $W(S)$, is exactly the set of regular elements $\mathrm{Reg}(S)$.

It is easy to check that, in a similar fashion to inverses, if s' is a weak inverse of s then both ss' and $s's$ are idempotents. Hence if every element of S has a weak inverse then S is an $E-$inversive semigroup. Conversely, let S is $E-$inversive and $s, t \in S$ such that $st \in E(S)$. Then

$$(tst)s(tst) = t(st)^3 = tst$$

and so $tst \in W(s)$. Hence a semigroup S is $E-$inversive if and only if every element of S has a weak inverse.

Let $s, t \in S$ and $(st)' \in W(st)$. Then $(st)' = (st)'st(st)'$ and so $t(st)' = t(st)'st(st)'$ and hence $t(st)' \in W(s)$. Similarly $(st)'s \in W(t)$. Hence

$$(st)' = ((st)'s)(t(st)') \in W(t)W(s)$$

and so $W(st) \subseteq W(t)W(s)$. It is not true in general that $W(st) = W(t)W(s)$, however this equality does hold under certain conditions.

Proposition 1.2.5 ([27, Proposition IV.3.1])**.** *Let S be a semigroup. The following are equivalent.*

1. *$E(S)$ is a band.*
2. *$V(t)V(s) \subseteq V(st)$.*
3. *$W(t)W(s) \subseteq W(st)$.*

Proof. *(1) implies (2).* Let $s' \in V(s), t' \in V(t)$. Then $s's, tt' \in E(S)$ so $tt's's \in E(S)$. Hence

$$tt's'stt's's = tt's's$$

and by multiplying by t' on the left and s' on the right and simplifying inverses we get

$$t's'stt's' = t's'$$

so $t's' \in W(st)$. A similar construction shows that $st \in W(t's')$ and so $t's' \in V(st)$ as required.

(2) implies (3). Let $s' \in W(s), t' \in W(t)$. Then $ss's \in V(s')$ and $tt't \in V(t')$. Hence $ss'stt't \in V(t's')$ and so

$$t's' = (t's')(ss'stt't)(t's') = (t's')(st)(t's')$$

as $s's, tt' \in E(S)$. Hence $t's' \in W(st)$ as required.

(3) implies (1). For any idempotent e we have $eee = e$ so $e \in W(e)$. Then if $e, f \in E(S)$ we have $ef \in W(e)W(f) \subseteq W(fe)$ so

$$ef = ef(fe)ef = efef$$

and hence $ef \in E(S)$ as required. □

There is a relationship between the $\mathcal{J}$-class containing s and the $\mathcal{J}$-classes containing its weak inverses. The relation $\leq$ on the $\mathcal{J}$-classes here is given by $J_a \leq J_b$ if $S^1aS^1 \subseteq S^1bS^1$.

Lemma 1.2.6. *Let S be a semigroup. If $s' \in W(s)$ then $J_{s'} \leq J_s$.*

Proof. Since $s' \in W(s)$, $s' = s'ss'$ and so $J_{s'} = J_{s'ss'}$. By [18, Equation 2.1.4] we have $J_{s'} = J_{s'ss'} \leq J_s$. □

Example 1.2.7. *Since $V(s) \subseteq W(s)$ every regular semigroup is $E-$inversive.*

Example 1.2.8. *For a more detailed example using a regular semigroup, let S be the bicyclic semigroup. Then $W((i,j)) = \{(j+n, i+n) \mid n \in \mathbb{N}_0\}$. Note that the inverse of (i,j) is obtained when $n = 0$.*

Example 1.2.9. *Let $S = \langle a \mid a^m = a^n \rangle$ be a monogenic semigroup. Such a semigroup has exactly one idempotent, say a^i. Then every element of S can be written (possibly not uniquely) as a^j where $1 \leq j < 2i$. Let $k = 2i - j$. Then $a^k a^j a^k = a^{4i-j} = a^{2i-j} = a^k$ so $a^k \in W(a^j)$ and S is an $E-$inversive semigroup. It can be seen that S is not regular in general since a has no inverse if $m, n > 1$.*

1.3 Ideal extensions

Let S and T be semigroups, with T containing a zero. A semigroup Σ is called an *ideal extension* of S by T if it contains S as an ideal and the Rees quotient Σ/S is isomorphic to T. Grillet and Petrich define an extension as *strict* if every element of $\Sigma \setminus S$ has the same action on S as some element of S [15, Definition 2.1] and *pure* if no element of $\Sigma \setminus S$ does [15, Proposition 2.10]. They also showed that any extension of an arbitrary semigroup S is a pure extension of a strict extension of S.

Proposition 1.3.1 ([15, Proposition 2.4])**.** *Every extension of S is strict if and only if S has an identity.*

Let S and T be disjoint semigroups, with T containing a unique zero 0. A *partial homomorphism* from T to S is a map $f : T \setminus \{0\} \to S$ such that for all $x, y \in S, f(xy) = f(x)f(y)$ whenever $xy \neq 0$.

We adopt the convention used by Clifford and Preston [4, Section 4.4] that elements of $T \setminus \{0\}$ are denoted by capital letters and elements of S by lowercase letters. A partial homomorphism from $T \setminus \{0\}$ to S given by $A \mapsto \overline{A}$ defines an extension $\Sigma = S \bigcup T \setminus \{0\}$ with multiplication given by

1. $A * B = \begin{cases} AB & AB \neq 0 \\ \overline{A}\,\overline{B} & AB = 0 \end{cases}$
2. $A * s = \overline{A}s$
3. $s * A = s\overline{A}$
4. $s * t = st$

where $A, B \in T \setminus \{0\}$ and $s, t \in S$ [4, Theorem 4.19]. From parts (2) and (3) above, all extensions defined in this way are strict. Under certain conditions on the semigroup S, all strict extensions of S are defined by partial homomorphisms.

Let S be a semigroup and let $a, b \in S$. We say that a and b are *interchangeable* if $a \neq b$ and

$$\forall x \in S, ax = bx \text{ and } xa = xb.$$

A semigroup is called *weakly reductive* if it contains no interchangeable elements. Notice that every monoid is weakly reductive.

Theorem 1.3.2 ([15, Theorem 2.5])**.** *Let S be weakly reductive. Then every strict extension of S is determined by a partial homomorphism, and conversely.*

1.4 Stratified semigroups

Recall that for any semigroup S, S^i denotes the set $\{x_1...x_i \mid x_1, ..., x_i \in S\}$ of words of length i in S. A semigroup S with zero is called *stratified* if

$$\bigcap_{i>0} S^i = \{0\}.$$

If S is a semigroup without zero, S is called stratified if the semigroup S^0 is stratified. Note that since $0 \in (S^0)^i$ for any $i \in \mathbb{N}$ and $0s = s0 = 0$ for all $s \in S$ we have

$$(S^0)^i = S^i \cup \{0\}.$$

Hence if S is a semigroup without zero we may equivalently say that S is stratified when

$$\bigcap_{i>0} S^i = \emptyset.$$

In our later generalisation of Grillet's definition, we refer to the intersection $\bigcap_{i>0} S^i$ as the *base* of the semigroup, and denote it by $\text{Base}(S)$. Under this definition, a semigroup is stratified if its base is either empty or contains only 0.

The *layers* of S are the sets $S_i = S^i \setminus S^{i+1}$ where $i \in \mathbb{N}$. As $S^{i+1} \subseteq S^i$ for all $i \in \mathbb{N}$, an element $x \in S$ can lie in at most one layer of S. If x does not lie in any layer of x then $x \in S^i$ for all $i \in \mathbb{N}$ and hence $x \in \cap_{i>0} S^i$. It follows that in a stratified semigroup S, every non-zero element lies in exactly one layer. Hence we can define a map $\lambda : S \setminus \{0\} \to \mathbb{N}$ by

$$\lambda(x) = i \text{ when } x \in S_i$$

which we call the *depth function* on S.

Layers of S may be empty. If $S_k = \emptyset$ then since $S^{k+1} \subseteq S^k$ we have $S^k = S^{k+1}$. But then

$$S^{k+2} = SS^{k+1} = SS^k = S^{k+1}$$

and hence, proceeding inductively,

$$S^k = S^{k+1} = S^{k+2} = S^{k+3} = \dots .$$

It then follows that

$$\bigcap_{i>0} S^i = S \cap S^2 \cap \dots \cap S^k = S^k$$

since $S^k \subseteq S^i$ for all $1 \leq i < k$. If $x \in S$ then $x^k \in S^k$ and hence S^k cannot be empty, so if S is a stratified semigroup with an empty layer S_k it must be the case that $S^k = \{0\}$. Hence S is a nilpotent semigroup, and further the nilpotency index is the least integer j such that S_j is empty.

Every finite stratified semigroup is necessarily nilpotent. All finite semigroups are periodic, so for every $x \in S$ there exists some $i \in \mathbb{N}$ such that x^i is an idempotent. Since for any idempotent $e \in E(S)$ we have $e = ee$, $e \in S^k$ for all $k \in \mathbb{N}$. Hence the only idempotent in a stratified semigroup is zero, and so in a finite stratified semigroup for every x there exists $i \in \mathbb{N}$ such that $x^i = 0$.

If $x \in S_k$ then $x \in S^k$ so can be written as a product of k elements of S, $x_1 ... x_k$. Suppose that $x_i \in S^2$ for some $i \in \{1, \dots, k\}$, so $x_i = y_1 y_2$ for some $y_1, y_2 \in S$. Then

$$x = x_1 \dots x_{i-1} y_1 y_2 x_{i+1} \dots x_k$$

is a product of $k+1$ elements so $x \in S^{k+1}$, a contradiction. Hence $x_i \notin S^2$ and so $x_i \in S_1 = S \setminus S^2$ for all $i \in \{1, \ldots, k\}$. Since every non-zero element of S can be written in this way, we have therefore proved

Proposition 1.4.1 ([14, Proposition 1.1])**.** *The top layer S_1 of a stratified semigroup S is the smallest generating subset of S. Every element of S_k is the product of m elements of S_1.*

Note here that we mean generated as either a semigroup or a semigroup with zero as appropriate. A stratified semigroup S with zero is not generated as a semigroup by S_1 alone if it contains no zero divisors.

Example 1.4.2. *A free semigroup is stratified without zero. Every word of the form $x_1 \ldots x_k$ lies in the k-th layer.*

Example 1.4.3. *The nilpotent monogenic semigroup $S = \langle x \mid x^i = x^{i+1} \rangle$ is stratified with zero. It has $i-1$ non-empty layers each of which consists of a single element.*

Example 1.4.4. *Any non-trivial regular semigroup is* not *stratified, since for every $x \in S$ and $x' \in V(x)$ we have $x = x(x'x)^i$ and hence $x \in S^{i+1}$ for all $i \in \mathbb{N}$.*

Chapter 2

The structure of $E-$dense semigroups

Our aim in this chapter is to develop a structure theory for $E-$dense semigroups. We explore a number of avenues for developing this theory, inspired by approaches to similar results in regular semigroups and finite semigroups, both of which are fully contained under the umbrella of $E-$dense semigroups. After a few preliminary results, Section 2.2 aims to generalise work by Nambooripad [26] on constructing a groupoid from a given regular semigroup. Section 2.3 builds on results from this section by examining $E-$dense semigroups through the relationship between weak inverses and the natural partial order. Section 2.4 looks at a different method, instead seeking to generalise Blyth's work on inverse transversals of regular semigroups [1]. Finally, Section 2.5 explores the use of wreath products to construct $E-$dense semigroups, motivated by the Krohn-Rhodes decomposition of finite semigroups.

2.1 Preliminaries

Let S be a semigroup. There is a partial order on the idempotents of S given by $e \leq f$ if and only if

$$ef = fe = e.$$

If the semigroup contains a zero, it is clear that $0 \leq e$ for any idempotent e. An idempotent which is minimal in the set of non-zero idempotents with respect to this partial order is called *primitive*.

This partial order can be extended to every element of the semigroup to give the *natural partial order* [23, Theorem 3]. For $a, b \in S$ we say $a \leq b$ if and only if there exist $x, y \in S$ such that

$$a = xb = by \text{ and } a = xa = ay.$$

If $x, y \in E(S)$ then we need only check that $a = xb = by$, as then

$$xa = xxb = xb = a$$

and similarly for ay.

Suppose a is regular and $a \leq b$. Then there exists $x \in S$ such that $xb = xa = a$. For any inverse $a' \in V(a)$ we have

$$(aa'x)^2 = aa'(xa)a'x = aa'aa'x = aa'x$$

and so $aa'x$ is idempotent. Further, $aa'xb = aa'a = a$. By similar argument, if $by = ay = a$ then $ya'a$ is an idempotent such that $bya'a = a$. Hence if a is regular then $a \leq b$ if and only if there exist $e, f \in E$ such that $a = eb = bf$.

The *closure* of a subset A of S is the set

$$\{s \in S \mid s \geq a \text{ for some } a \in A\}$$

and is denoted by $A\omega$. Similarly, the *order ideal* generated by A is the set

$$\{s \in S \mid s \leq a \text{ for some } a \in A\}.$$

A semigroup S is called *simple* if it has no proper (two-sided) ideals. A simple semigroup which contains a primitive idempotent is called *completely simple*. A full description of completely simple semigroups is given by the Rees Theorem.

Theorem 2.1.1 ([18, Theorem 3.3.1])**.** *Let G be a group, let I, Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G. Let $S = I \times G \times \Lambda$ and define a multiplication on S by*

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

Then S is a completely simple semigroup. Conversely, every completely simple semigroup is isomorphic to a semigroup constructed in this way.

The semigroup S is denoted by $\mathcal{M}(G; I, \Lambda; P)$. This structure theorem allows us to immediately deduce many properties of a completely simple semigroup S. Suppose that (i, g, λ) is idempotent. Then

$$(i, g, \lambda) = (i, g, \lambda)^2 = (i, gp_{\lambda i}g, \lambda)$$

and hence $g = gp_{\lambda i}g$ in G. Hence $g = p_{\lambda i}^{-1}$ and so every idempotent of S has the form $(i, p_{\lambda i}^{-1}, \lambda)$.

It is easy to see from the definition of the multiplication on S that the $\mathcal{R}$-classes consist of those elements with the same first coordinate, and likewise the $\mathcal{L}$-classes consist of those with the same third coordinate. It follows that the $\mathcal{H}$ classes consist of elements which agree in both coordinate, whence each $\mathcal{H}$-class is isomorphic to the group G.

A semigroup S is called *completely regular* if and only if every element of S lies in a subgroup. Clearly all completely simple semigroups are completely regular. In fact, completely regular semigroups are built up of completely simple semigroups.

Theorem 2.1.2 ([18, Theorem 4.1.3])**.** *A semigroup S is completely regular if and only if it is a semilattice of completely simple semigroups.*

Note that a completely simple semigroup in which idempotents commute is a group, and hence when $E(S)$ is a semilattice this theorem gives us the semilattice structure of a Clifford Semigroup as described in Example 1.1.9.

2.2 A geometric approach

In [26] Nambooripad showed an equivalence between the category of regular semigroups and the category of inductive groupoids by providing a method to construct a groupoid from any given regular semigroup. We aim to use a similar construction in order to investigate $E-$dense semigroups via a geometric structure.

2.2.1 Construction of $\mathcal{S}$

Let S be an $E-$dense semigroup with semilattice of idempotents E. We construct a semicategory $\mathcal{S}$ from S in the following way:

- For each $e \in E$ there is a vertex, also denoted by e.
- For each $s \in S$ and weak inverse $s' \in W(s)$ there is a map from the vertex ss' to the vertex $s's$, denoted by (s, s').

The composition of two maps (x, x') and (y, y') exists if $x'x = yy'$ and is given by $(xy, y'x')$. It is easy to verify that under the given condition this is a map from xx' to $y'y$. This is well-defined as if $(y, y') = (z, z')$ we have $y = z$ and $y' = z'$ so $xy = xz$ and $y'x' = z'x'$. Associativity of this composition of maps follows from associativity in the underlying semigroup, as

$$[(x, x')(y, y')](z, z') = ([xy]z, z'[y'x']) = (x[yz], [z'y']x') = (x, x')[(y, y')(z, z')].$$

An immediate divergence from Nambooripad's construction is that we do not necessarily have an identity map at each vertex. In the regular case, the identity on a vertex e is given by (e, e). In this construction, however,

$$(ss', ss')(s, s') = (ss's, s'ss')$$

and so (ss', ss') is a left identity if and only if s is a weak inverse of s', or equivalently if s' is an inverse of s.

This issue can be avoided by working with the monoid S^1, in which case $(1, e)$ is the identity map on the vertex e. Appending an identity to a semigroup which was not originally a monoid seems to have little effect on the semicategory structure as other than the identity maps on each vertex it adds only a single vertex with no additional maps.

With no identity we also do not have inverse maps in the usual groupoid sense, and indeed even when identity maps are present we do not necessarily have such inverses. However, as the weak inverses form an inverse semigroup the map

$$(x, x') : xx' \to x'x$$

gives rise to a map

$$(x', (x')^{-1}) : x'(x')^{-1} \to (x')^{-1}x'.$$

As idempotents commute,

$$x'(x')^{-1} = x'xx'(x')^{-1} = x'(x')^{-1}x'x = x'x$$

and similarly $(x')^{-1}x' = xx'$. Hence this is a map from $x'x$ to xx'. This map is a weak inverse of (x, x'), in the sense that

$$(x', (x')^{-1})(x, x')(x', (x')^{-1}) = (x', (x')^{-1}).$$

2.2.2 Structure of $\mathcal{S}$

A natural question to ask at this point is whether we can determine exactly which maps exist between a given pair of vertices. Our first observation is that the initial and terminal vertices of a map (x, x') depend only on the weak inverse x'.

Lemma 2.2.1. *Let S be an $E-$dense semigroup with semilattice of idempotents E, $x, y \in S$, and $z \in W(x) \cap W(y)$. Then the maps (x, z) and $y, z)$ of $\mathcal{S}$ share initial and terminal vertices.*

Proof. Let $z \in W(x) \cap W(y)$ and consider xz. As E is a semilattice, we have

$$xz = x(zyz) = (xz)(yz) = (yz)(xz) = y(zxz) = yz$$

and similarly $zx = zy$. Hence the initial vertices of the maps (x, z) and (y, z) are equal, as are their terminal vertices. □

This fact motivates the following lemma.

Lemma 2.2.2. *Let S be an $E-$dense semigroup with semilattice of idempotents E and let $\mathcal{S}$ be the semicategory constructed as in Section 2.2.1. Let x' and y' be weak inverses of x and y respectively, giving rise to maps (x, x') and (y, y'). Then*

1. *(x, x') and (y, y') share an initial vertex if and only if $x'\mathcal{L}y'$.*
2. *(x, x') and (y, y') share a terminal vertex if and only if $x'\mathcal{R}y'$.*
3. *(x, x') and (y, y') are maps between the same directed pair of vertices if and only if $x'\mathcal{H}y'$.*
4. *(x, x') and (y, y') lie in the same connected component of $\mathcal{S}$ if and only if $x'\mathcal{D}y'$.*

Proof. We first show that the conditions on the maps of $\mathcal{S}$ imply the conditions on the relations of S.

1. If (x, x') and (y, y') share an initial vertex, we have $xx' = yy'$. Then
$$x' = x'xx' = (x'y)y'$$
and similarly
$$y' = y'yy' = (y'x)x'$$
so $x'\mathcal{L}y'$.
2. By a dual argument, if the maps share a terminal vertex we have $x'x = y'y$ and so $x'\mathcal{R}y'$.
3. By the previous two results, if the maps share initial and terminal vertices then $x'\mathcal{H}y'$.
4. If the maps (x, x') and (y, y') lie in the same connected component of $\mathcal{S}$ there is a path of maps connecting xx' and yy'. By observing that $x'\mathcal{D}(x')^{-1}$, the weak inverses along that path form a chain of $\mathcal{D}$ relations and so $x'\mathcal{D}y'$.

For the converses, note that by Lemma 2.2.1 it is sufficient to find any y such that the map (y, y') satisfies the required conditions.

1. If we have $x'\mathcal{L}y'$, then there exists $s \in S$ such that $sy' = x'$. Then

$$y'xsy' = y'xx' = y'$$

as $y'\mathcal{L}x'\mathcal{L}xx'$ and so y' is a weak inverse of xs. As $xsy' = xx'$, the initial vertex of the map (xs, y') is xx' as required. Hence, by Lemma 2.2.1, for any $y \in S$ such that $y' \in W(y)$ we have that the initial vertex of (y, y') is $yy' = xsy' = xx'$.

As before, the second and third statements follow by a dual argument and the combination of the previous two results respectively.

4. If $x'\mathcal{D}y'$, there exists $s \in S$ such that $x'\mathcal{L}s\mathcal{R}y'$. Since s lies in a regular $\mathcal{D}$-class it is regular and hence is a weak inverse of some element of S. Then by the previous results, maps arising from s share an initial vertex with maps arising from x' and a terminal vertex with maps arising from y', and hence all of these maps lie within the same connected component of $\mathcal{S}$.

□

As a corollary, we can easily characterise how taking inverses respects Green's relations by observing that maps arising from $(x')^{-1}$ point in the opposite direction to those arising from x'.

Corollary 2.2.3. *Let x and y be regular elements of S. Then*

1. *$x\mathcal{L}y$ if and only if $x^{-1}\mathcal{R}y^{-1}$.*
2. *$x\mathcal{R}y$ if and only if $x^{-1}\mathcal{L}y^{-1}$.*
3. *$x\mathcal{H}y$ if and only if $x^{-1}\mathcal{H}y^{-1}$.*
4. *$x\mathcal{D}y$ if and only if $x^{-1}\mathcal{D}y^{-1}$.*

Having established that the location of a map in $\mathcal{S}$ depends only on the weak inverse, we turn our attention to investigating the structure of the maps between a given directed pair of vertices. We first show that for a given $x \in S$ there can be at most one map (x, x') between any given directed pair of vertices in $\mathcal{S}$. We will later see that we can prove a stronger version of this result by taking a different approach.

Lemma 2.2.4. *Let S be an $E-$dense semigroup with semilattice of idempotents E and let $\mathcal{S}$ be the semicategory constructed in Section 2.2.1. Let x' and x^* be weak inverses of x such that the maps (x, x') and (x, x^*) share initial and terminal vertices. Then $x' = x^*$. In particular, if H is a regular $\mathcal{H}$-class of S with $s, t \in H$ then $A(s) \cap A(t) = \emptyset$ unless $s = t$.*

Proof. By construction we have $xx' = xx^*$ and $x'x = x^*x$. Hence $x'\mathcal{L}xx' = xx^*$ and $x^*\mathcal{R}x^*x = x'x$. As any idempotent is a right identity on its $\mathcal{L}$-class and a left identity on its $\mathcal{R}$-class [18, Proposition 2.3.3], we have $x' = x'xx^* = x^*$ as required.

Now let H be a regular $\mathcal{H}$-class of S, $s, t \in H$, and suppose $x \in A(s) \cap A(t)$. Then since $s\mathcal{H}t$ the maps (x, s) and (x, t) share initial and terminal vertices by Lemma 2.2.2. Hence $s = t$. □

Let S be an $E-$dense semigroup with semilattice of idempotents E and let H_s be the $\mathcal{H}$-class containing s. By definition, $A(H_s) = \bigcup_{x \in H_s} A(x)$. Lemma 2.2.4 shows that the sets forming this union are in fact a partition of $A(H_s)$. The following lemma gives another characterisation of this subset of S.

Lemma 2.2.5. *Let S be an $E-$dense semigroup with semilattice of idempotents E. For each weak inverse $s \in S$ we have $A(H_s) = H_{s^{-1}}\omega$.*

Proof. Let $x \in A(H_s)$. Then x has a weak inverse x' such that $x' \in H_s$. Then $x \geq xx'x = (x')^{-1} \in H_{s^{-1}}$ by Corollary 2.2.3 so $x \in H_{s^{-1}}\omega$.

Conversely, suppose $x \in H_{s^{-1}}\omega$. Then there exists some $t \in H_{s^{-1}}$ such that $x \geq t$. As t is an element of a regular $\mathcal{H}$-class it is regular, and so has an inverse t^{-1}. By Corollary 2.2.3, $t^{-1} \in H_s$. We claim that $t^{-1}xt^{-1} = t^{-1}$ and hence $x \in A(H_s)$.

Since $x \geq t$, there exists an idempotent e in S such that $t = xe = te$. Then

$$\begin{aligned} t^{-1} &= t^{-1}tt^{-1}tt^{-1} \\ &= t^{-1}xet^{-1}tt^{-1} \\ &= t^{-1}xt^{-1}tet^{-1} \text{ (as } e \text{ and } t^{-1}t \text{ commute)} \\ &= t^{-1}xt^{-1}tt^{-1} \\ &= t^{-1}xt^{-1} \end{aligned}$$

as required. □

For each idempotent e in S, we have that $e^{-1} = e$. Hence for $\mathcal{H}$-classes containing an idempotent Lemma 2.2.5 can be rewritten as the following.

Corollary 2.2.6. *For each idempotent $e \in S$ we have $A(H_e) = H_e\omega$.*

Let S be an $E-$dense semigroup with semilattice of idempotents E and let $e \in E$. For every $s \in H_e\omega$, there exists a unique $s' \in W(s) \cap H_e$ by Lemma 2.2.4and Lemma 2.2.5. Since $s'\mathcal{H}e$, by Lemma 2.2.2 it follows that (s, s') and (e, e) share initial and terminal vertices in $\mathcal{S}$, namely the vertex corresponding to e. Conversely by Lemma 2.2.2 any other loop on this vertex (x, y) must satisfy $y\mathcal{H}e$ and hence $y \in W(x) \cap H_e$ and so

$y \in A(H_e) = H_e\omega$. Hence there is a one-to-one correspondence between elements of $H_e\omega$ and loops on the vertex e given by $x \mapsto (x, x')$ where $x \in H_e\omega$ and $x' \in W(x) \cap H_e$.

If $s, t \in H_e\omega$ and the maps (s, s') and (t, t') are loops on the vertex e, then their composition $(st, t's')$ is also a loop on the vertex e. Hence $st \in H_e\omega$ with $t's' \in W(st) \cap H_e$ and so $H_e\omega$ is a subsemigroup of S. By definition, every element of $H_e\omega = A(H_e)$ has a weak inverse in H_e, and since idempotents commute in S they commute in $H_e\omega$. We have therefore proved the following.

Theorem 2.2.7. *Let S be an $E-$dense semigroup with semilattice of idempotents E. For each $e \in E$, $H_e\omega$ is an $E-$dense subsemigroup of S.*

2.2.3 Examples

In this section we describe the semicategory construction on a number of examples. We also give the subsemigroups $H_e\omega$ as described in Theorem 2.2.7

Example 2.2.8. *Let S be a finite monogenic semigroup with generator s. S has only one idempotent, namely the identity of its kernel, and each element of S has a unique weak inverse. The semicategory generated by S consists therefore of a single vertex and a single loop on this vertex for each element $s \in S$, labelled by (s, s').*

For a concrete example, consider the monogenic semigroup given by $\langle a \mid a^6 = a^3 \rangle$. The single idempotent of this semigroup is the element a^3 and the loops on this vertex are the maps (a, a^5), (a^2, a^4), (a^3, a^3), (a^4, a^5), and (a^5, a^4). The only idempotent map here is (a^3, a^3), but this is clearly not an identity map as, for example, $(a^3, a^3)(a, a^5) = (a^4, a^5)$. In this case $H_{a^3}\omega$ is the whole of S as every element has a weak inverse lying in the kernel.

Example 2.2.9. *Let B be the bicyclic semigroup, i.e. $\mathbb{N}^0 \times \mathbb{N}^0$ with multiplication given by*

$$(a, b)(c, d) = (a - b + \max(b, c), d - c + \max(b, c)).$$

The idempotents of B are the elements of the form (i, i) and for each $(i, j) \in B$ we have

$$W((i, j)) = \{(j + n, i + n) \mid n \in \mathbb{N}^0\}$$

with (j, i) being the unique inverse of (i, j). As this is an inverse semigroup, every element is a weak inverse of some other element.

The maps arising from the weak inverse (i, j) have initial vertex (j, j) and terminal vertex (i, i). Since each element (i, j) is a weak inverse of precisely $\min(i, j) + 1$ elements, there are exactly that many maps from the vertex (i, i) to the vertex (j, j). For example, from $(1, 1)$ to $(2, 2)$ there are two maps, namely $((1, 2), (2, 1))$ and $((0, 1), (2, 1))$. For an idempotent (i, i), *the subsemigroup* $H_{(i,i)}\omega$ *is precisely the set* $\{(j,j) \in B \mid j \leq i\}$. *For example,* $H_{(2,2)}\omega = \{(0,0),(1,1),(2,2)\}$.

Example 2.2.10. *Let* S *be a finite monogenic semigroup with generator* s *and* T *be an isomorphic copy of* S *with generator* t. *Let* U *be a semigroup isomorphic to* $W(S)$ *and* $W(T)$. *We form the amalgamated free product* $S *_U T$. *In other words, this semigroup consists of words in* s *and* t *under the condition that for* $s^i \in W(S)$ *we have* $s^i = t^i$, *with* t^i *being the corresponding element of* $W(T)$. *Successive applications of this relation can be used to write any word containing an element of either kernel as a single element. Hence* $W(S *_U T)$ *is isomorphic to* $W(S)$. *Each word* $s^{e_1}t^{e_2}...s^{e_n}$ *has a unique weak inverse given by* $s^{e_n'}...t^{e_2'}s^{e_1'}$. *As in Example 2.2.8, this semigroup has exactly one idempotent* e *and every element has a weak inverse in* H_e *so* $H_e\omega = S *_U T$.

This construction gives us a family of examples of infinite, non-commutative $E-$dense semigroups which are not regular. The semicategory produced by this semigroup is very similar to that of Example 2.2.8, with the exception that there are infinitely many loops. In fact, the semicategory from Example 2.2.8 can be found as a subsemicategory by picking out the loops whose first term is a power of s.

We can be further generalised by replacing S and T with two copies of any $E-$dense semigroup for which the set of weak inverses is unitary, as it is this property which allows us to conclude that any word containing a weak inverse of the original semigroup can be rewritten as some such weak inverse.

2.3 The natural partial order

2.3.1 The relationship between A, W and $\leq$

In Lemma 2.2.5 we showed that $A(H_s) = H_{s^{-1}}\omega$. We can in fact make some stronger statements about how weak inverses relate to the natural partial order.

Lemma 2.3.1. *Let* S *be an* $E-$*dense semigroup with semilattice of idempotents* E. *Let* $s \in S$ *be a regular element. Then* $t \in A(s)$ *if and only if* $s^{-1} \leq t$. *Further,* $u \in W(s)$ *if and only if* $u \leq s^{-1}$.

Proof. Let t be an associate of s. Then $s^{-1} = tst \leq t$. Conversely, suppose $s^{-1} \leq t$ so $s^{-1} = et = es^{-1}$ for some $e \in E$. Then

$$s = ss^{-1}s = ses^{-1}s = ss^{-1}se = se$$

as e and $s^{-1}s$ commute. So

$$sts = (se)ts = s(et)s = ss^{-1}s = s$$

and hence $s \in W(t)$. If u is a weak inverse of s then it is regular. Then by the previous argument, $u^{-1} \leq s$ and so $u \leq s^{-1}$ as s is regular. Conversely, if $u \leq s^{-1}$ then $u = es^{-1}$ for some idempotent e. It can be seen that se is an inverse for u and so u is regular. Again by the previous argument, we have $u \in W(s)$. □

This approach allows for an alternative proof of [29, Lemma 1.4, part 6].

Lemma 2.3.2. *Let S be an $E-$dense semigroup with semilattice of idempotents E and let $s \in S$. Then*

$$W(W(s)) = \{t^{-1} \mid t \in W(s)\}.$$

Further, $W(W(W(s))) = W(s)$.

Proof. Let $x \in \{t^{-1} \mid t \in W(s)\}$. Then

$$x^{-1} \in W(x) \subseteq W(W(y))$$

as required. Conversely, let $x \in W(W(s))$. Then there exists $s' \in W(S)$ such that

$$x \leq (s')^{-1} \leq s.$$

Then $x \leq s$ so by Lemma 2.3.1 we have $x^{-1} \in W(s)$ and hence

$$x \in \{t^{-1} \mid t \in W(s)\},$$

Now, $W(W(W(s)))$ is the union of $W(W(s'))$ over all $s' \in W(s)$. We have

$$\bigcup_{s' \in W(s)} W(W(s')) = \bigcup_{s' \in W(s)} \{t^{-1} \mid t \in W(s')\} = \{t^{-1} \mid t \in W(W(s))\}.$$

Applying the previous argument to $W(W(s))$ once more gives the desired result. □

This method is slightly stronger than the original, as it shows that the effect of successive applications of W beyond the first is to repeatedly take the inverse of every element.

We can also use this characterisation to examine repeated applications of A, but the results are less dramatic. If s is not regular then $A(s)$ is the empty set, and if s is regular $A(s)$ is the closure of the set $\{s^{-1}\}$. In either case, it is clear that $A(A(s)) = A(s)$.

2.3.2 The structure of $W(s)$

Let S be an $E-$dense semigroup with semilattice of idempotents E and let $s \in S$. We know from [29, Lemma 1.4, part 2] that $W(s)$ forms a semilattice under the natural partial order. Further, if s is regular, $s' \in W(s)$ and $t \in S$ such that $t \leq s'$ then the results of the previous section show that $t \in W(s)$ and so the semilattice $W(s)$ forms an order ideal of S. In fact, this property holds even when s is not a regular element of S.

Proposition 2.3.3. *Let S be an $E-$dense semigroup with semilattice of idempotents E and let $s \in S$. If $s' \in W(s)$ and $t \leq s'$ then $t \in W(s)$.*

Proof. As $t \leq s'$ there exists $e \in E$ such that $t = es'$. Then

$$tst = es'ses' = ees'ss' = es' = t$$

as e and $s's$ commute and so $t \in W(s)$ as required. □

To further explore the structure of $W(s)$, we first need to establish a stronger version of Lemma 2.2.4.

Lemma 2.3.4. *Let S be an $E-$dense semigroup with semilattice of idempotents E. Let $s \in S$ and $s', s^* \in W(s)$. If $ss' = ss^*$ then $s' = s^*$. Similarly, if $s's = s^*s$ then $s' = s^*$.*

Proof. As $ss' = ss^*$ and idempotents commute, we have

$$s' = s'ss' = s'ss^* = s'ss'(ss^*) = s'(ss^*)ss'.$$

Rebracketing gives

$$s'(ss^*)ss' = s's(s^*s)s' = s^*ss'ss' = s^*ss' = s^*ss^* = s^*$$

as required. □

This argument shows that, given a particular $s \in S$ and $e \in E$, the semicategory $\mathcal{S}$ has at most one map of the form (s, s') with initial vertex e. Likewise, there is at most one map of the form (s, s^*) with terminal vertex e. Note that if both maps exist then $s' = s^*$ if and only if $(s, s') = (s, s^*)$ is a loop on e.

We can now show that the semilattice $W(s)$ is isomorphic to a sublattice of the semilattice of idempotents of S.

Lemma 2.3.5. *Let S be an $E-$dense semigroup with semilattice of idempotents E and let $s \in S$. The map $\lambda_s : W(s) \to E$ given by $\lambda_s(s') = ss'$ is a monomorphism, with respect to the semilattice structure on $W(s)$.*

Proof. By the previous result, if $\lambda_s(s') = \lambda_s(s^*)$ then $s' = s^*$ and so the map is injective. It remains to show that it respects the multiplication on the semilattice. Let $s', s^* \in W(s)$. By [29, Lemma 1.4, part 2], their meet in the semilattice is given by $s'ss^*$. Hence

$$\lambda_s(s')\lambda_s(s^*) = ss'ss^* = \lambda_s(s'ss^*)$$

as required. □

This result suggests that by understanding the behaviour of the set of idempotents we can understand the weak inverses. For example, if we know that the set of idempotents forms a chain (such as in the bicyclic semigroup) we can conclude that $W(s)$ will also form a chain for every s in S.

2.4 Weak inverse transversals

This approach is inspired by Blyth's work on regular semigroups [1]. Under certain conditions on a regular semigroup, we can pick out a subset containing exactly one inverse for every element. This subset forms an inverse subsemigroup, and properties of this subsemigroup can give us information about the larger semigroup. We aim to emulate this with weak inverses, and so the first step is to determine when a weak inverse transversal exists. We say a subset S° of S is a *weak inverse transversal* of S if for every element $s \in S$ we have $|W(s) \cap S^\circ| = 1$. The single element of $W(s) \cap S^\circ$ is denoted by s°.

2.4.1 The commutative case

We begin by restricting ourselves to $E-$dense semigroups. We will see later how to generalise this result to some $E-$inversive semigroups without commuting idempotents. We prove the following theorem.

Theorem 2.4.1. *Let S be an $E-$dense semigroup with semilattice of idempotents E. A weak inverse transversal S° exists if and only if $W(s)$ contains a minimal element for every s in S. Further, if S° exists, it forms a subgroup of S.*

Note that if a minimal element of $W(s)$ exists it is necessarily unique. To see this, let s', s^* be minimal elements of $W(s)$. Then by [29, Lemma 1.4, part 2] we have $s'ss^* \in W(s)$ with $s'ss^* \leq s', s^*$. Since these are both minimal, it must be the case that $s' = s'ss^* = s^*$.

Of interest here is that some of the early work on inverse transversals (for example [2]) looked at semigroups where every element had a maximal inverse, showing an almost dual relation with our minimal weak inverses.

Recall that with commuting idempotents, each regular element has a unique inverse and for any $s, t \in S$ we have $W(st) = W(t)W(s)$. Also recall the following useful lemma, as well as Lemmas 2.3.1 and 2.3.5. We include a useful corollary to the latter two.

Lemma 2.4.2 ([18, p. 152]). *Let S be a semigroup with semilattice of idempotents $E(S)$ and let a and b be regular elements of S. Then $a \leq b$ if and only if $a^{-1} \leq b^{-1}$.*

Corollary 2.4.3. *Let S be an $E-$dense semigroup with semilattice of idempotents E. Then $W(s)$ contains a minimal element for all s in S if and only if S contains a minimal idempotent.*

Proof. Let e be an idempotent and consider another idempotent f such that $f \leq e$. Then f is a weak inverse of e, and so if S has no minimal idempotent then $W(e)$ has no minimal element. Conversely if S has a minimal idempotent then $W(s)$ has a minimal element by Lemma 2.3.5 as the monomorphism preserves order by definition. □

We can now build up the results required to prove Theorem 2.4.1.

Lemma 2.4.4. *Let S be an $E-$dense semigroup with semilattice of idempotents E and weak inverse transversal S°. Then*

$$s^{\circ\circ} = (s^\circ)^{-1}.$$

Proof. Suppose S has a weak inverse transversal S° and consider an element s. By definition, $s^\circ \in W(s)$, $s^{\circ\circ} \in W(s^\circ)$ and $s^{\circ\circ\circ} \in W(s^{\circ\circ})$. By applying Lemmas 2.4.2 and 2.3.1 we can construct the following chains of inequalities.

$$s \geq (s^\circ)^{-1} \geq s^{\circ\circ} \geq (s^{\circ\circ\circ})^{-1}$$

$$s^\circ \geq (s^{\circ\circ})^{-1} \geq s^{\circ\circ\circ}$$

It then follows from Lemma 2.3.1 that $s^{\circ\circ\circ} \in W(s)$ and so by uniqueness we must have that $s^\circ = s^{\circ\circ\circ}$ and hence $s^{\circ\circ} = (s^\circ)^{-1}$. □

We can now prove the first part of Theorem 2.4.1.

Proof of Theorem 2.4.1. Let S be an $E-$dense semigroup with semilattice of idempotents E and weak inverse transversal S°. Suppose there exists $s' \in W(s)$ such that $s' \leq s^\circ$. Consider s'°. As $s'^\circ \in W(s')$, by Lemmas 2.4.2 and 2.3.1 we have $s'^\circ \leq (s')^{-1} \leq (s^\circ)^{-1}$ and hence $s'^\circ \in W(s^\circ)$. But by Lemma 2.4.4, $s^{\circ\circ} = (s^\circ)^{-1}$, so by uniqueness $s'^\circ = (s^\circ)^{-1}$. Then $(s')^{-1} = (s^\circ)^{-1}$ and so $s' = s^\circ$, and hence s° is minimal in $W(s)$.

Conversely, suppose every element has a minimal weak inverse. Let s' be the minimal weak inverse of s and suppose that $s' \in W(t)$. If there exists $t' \in W(t)$ such that $t' \leq s'$

then by Lemma 2.3.1 $t' \in W(s)$, contradicting the minimality of s' in $W(s)$. Hence no such t' can exist, so s' is minimal in $W(t)$. Letting $s^\circ = s'$ so S° is the set of minimal weak inverses therefore gives us that $|W(s) \cap S^\circ| = 1$ as required. □

This now allows us to establish another lemma which will be useful for proving the second part of Theorem 2.4.1.

Lemma 2.4.5. *Let S be an $E-$dense semigroup with semilattice of idempotents E and weak inverse transversal S° and let $s \in S$. Then $ss^\circ = s^\circ s = e^\circ$, where e° is the minimal idempotent. Conversely, if $s' \in W(s)$ and $ss' = e^\circ$ or $s's = e^\circ$ then s' is minimal in $W(s)$.*

Proof. Suppose $ss^\circ = e \geq f$ for some $e, f \in E$. We want to show that $s^\circ f \leq s^\circ$. One direction is obvious, so it remains to show that $s^\circ f = us^\circ$ for some $u \in E$. Let $u = s^\circ f s$. This is idempotent as

$$(s^\circ f s)(s^\circ f s) = s^\circ s s^\circ f f s = s^\circ f s$$

and we have $(s^\circ f s)s^\circ = s^\circ s s^\circ f = s^\circ f$ as required. Hence $s^\circ f \leq s^\circ$, so by minimality $s^\circ f = s^\circ$. Then

$$e = ss^\circ = ss^\circ f = ef = f$$

and so e is the minimal idempotent. A similar argument holds for $s^\circ s$.

For the converse, suppose $s' \in W(s)$ and $ss' = e^\circ$. As s° is minimal, $s^\circ \leq s'$. By Lemma 2.3.5, $ss^\circ \leq e^\circ$, and by minimality $ss^\circ = e^\circ$. Hence by Lemma 2.3.5 $s' = s^\circ$ as required. □

The following corollary is also convenient to include here.

Corollary 2.4.6. *Let S be an $E-$dense semigroup with semilattice of idempotents E and weak inverse transversal S°. Then for all $s \in S$, $s^\circ e^\circ = e^\circ s^\circ = s^\circ$ where e° is the minimal idempotent.*

Proof. By Lemma 2.4.5, $s^\circ e^\circ = s^\circ s s^\circ = s^\circ$, and similarly $e^\circ s^\circ = s^\circ$. □

We can now show that S° forms a subgroup of S.

Proof of Theorem 2.4.1 (continued). We first show S° is closed under the multiplication inherited from S. Let s° and t° be the minimal weak inverses of s and t respectively. $s^\circ t^\circ$ is then a weak inverse of ts. Then by Lemma 2.4.5 and Corollary 2.4.6

$$tss^\circ t^\circ = te^\circ t^\circ = tt^\circ = e^\circ$$

and so $s^\circ t^\circ$ is minimal.

Associativity follows from associativity in S, and Corollary 2.4.6 gives the existence of an identity, so it remains to show that each element of S° has an inverse. By Lemma 2.4.4, for each s° the semigroup inverse $(s^\circ)^{-1} = s^{\circ\circ}$ is in S°. Then by Lemma 2.4.5, $s^\circ s^{\circ\circ} = e^\circ$ as required. □

We can be more specific about the subgroup of S forming the weak inverse transversal S°.

Proposition 2.4.7. *Let S be an $E-$dense semigroup with semilattice of idempotents E. If e° is the minimal idempotent of S, then $S^\circ = e^\circ S e^\circ$, the local subsemigroup of S with respect to e°.*

Proof. As e° acts as an identity on S° we clearly have $s^\circ = e^\circ s^\circ e^\circ$ so $S^\circ \subseteq e^\circ S e^\circ$. For the reverse inclusion, by Lemma 2.4.5 we have

$$e^\circ s e^\circ = ss^\circ ss^\circ s = ss^\circ s = (s^\circ)^{-1} = s^{\circ\circ}$$

so $e^\circ S e^\circ \subseteq S^\circ$. □

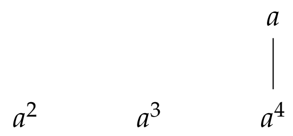
This structure gives us some information about the multiplication on S. By Lemma 2.3.1, if $t \in A((s^\circ)^{-1})$ then $s^\circ \leq t$. Since $s^\circ \in A((s^\circ)^{-1})$ it must be minimal in this set. The sets of the form $A((s^\circ)^{-1})$ are disjoint, as if not we would have some element s for which $|W(s) \cap S^\circ| \geq 1$. Further, since $W(st) = W(t)W(s)$ it follows that the product of two elements must lie in a set of associates dependent on the sets containing the original two elements. More precisely, if $a \in A(s)$ and $b \in A(t)$ then $ab \in A(ts)$. In the following examples, the diagrams illustrate the order structure of these sets. Specifically, the connected components represent the disjoint sets $A((s^\circ)^{-1})$, while connected elements

indicate that $s \leq t$. The lowest elements on each connected component therefore form the weak inverse transversal S°.

Example 2.4.8. *Let S be given by the presentation $\langle a | a^5 = a^2 \rangle$. The only idempotent is a^3, and the sets of weak inverses are as follows.*

s	$W(s)$
a	$\{a^2\}$
a^2	$\{a^4\}$
a^3	$\{a^3\}$
a^4	$\{a^2\}$

Clearly S° must be $\{a^2, a^3, a^4\}$, isomorphic to the cyclic group of order 3. The order structure is then as follows:



i.e. $S^\circ = \{a^2, a^3, a^4\}$, $A((a^2)^{-1}) = \{a^2\}$, $A((a^3)^{-1}) = \{a^3\}$, and $A((a^4)^{-1}) = \{a, a^4\}$ with $a^4 \leq a$.

Going in the other direction, we can start from the same order structure and recover the original semigroup S. Consider the following order structure, isomorphic to the one above.

We can begin to fill out a multiplication table, starting with the multiplication on S°.

	1	α	α^{-1}	β
1	1	α	α^{-1}	
α	α	α^{-1}	1	
α^{-1}	α^{-1}	1	α	
β				

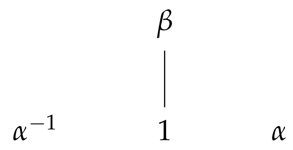
Most of the remaining products can be determined by the order structure alone. For example, $(\alpha\beta)^\circ = \beta^\circ\alpha^\circ = \alpha^{-1}\alpha^{-1} = \alpha$ so $\alpha\beta$ must lie in $A(\alpha)$, and hence is equal to α^{-1}. The only products that cannot be found in this way are 1β and $\beta 1$, as they must lie in $A(\alpha)$ and so there are two choices. However, since $\alpha \leq \beta$, we must have that $e\beta = \alpha$ and $\beta f = \alpha$ for some idempotents e and f, and 1 is the only idempotent available. Hence we can complete the multiplication table as follows.

	1	α	α^{-1}	β
1	1	α	α^{-1}	α
α	α	α^{-1}	1	α^{-1}
α^{-1}	α^{-1}	1	α	1
β	α	α^{-1}	1	α^{-1}

By noticing that β generates the whole semigroup and $\beta^2 = \alpha^{-1} = \beta^5$, we can see that we do indeed recover the original semigroup.

Unfortunately we cannot in general determine the semigroup uniquely from the order structure, as the next example shows.

Example 2.4.9. *Again, let S° be the cyclic group of order 3 and consider the following order structure.*



We can follow the same process as above to obtain most of the multiplication table.

	1	α	α^{-1}	β
1	1	α	α^{-1}	
α	α	α^{-1}	1	α
α^{-1}	α^{-1}	1	α	α^{-1}
β		α	α^{-1}	

Here we need to use a different method to find 1β and $\beta 1$. Since $1 = \alpha\alpha^{-1}$, associativity gives us that $1\beta = \alpha(\alpha^{-1}\beta) = 1$ and similarly $\beta 1 = 1$. However, when it comes to β^2 we run into an issue. From the group structure, the product must be either 1 or β, but both choices give a valid semigroup.

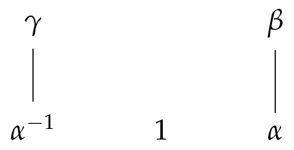
	1	α	α^{-1}	β
1	1	α	α^{-1}	1
α	α	α^{-1}	1	α
α^{-1}	α^{-1}	1	α	α^{-1}
β	1	α	α^{-1}	β

	1	α	α^{-1}	β
1	1	α	α^{-1}	1
α	α	α^{-1}	1	α
α^{-1}	α^{-1}	1	α	α^{-1}
β	1	α	α^{-1}	1

The difference between the two semigroups given in this example is a question of which elements are regular. If we assume that β is regular, we are forced to complete the multiplication table by setting $\beta^2 = \beta$. On the other hand, assuming that β is not regular forces us to choose $\beta^2 = 1$. This was not an issue in the previous example as β could not be regular, as if it were it would contradict Lemma 2.3.5.

Our final example shows that even knowing which elements are regular is not always enough to uniquely determine a smeigroup from the order structure.

Example 2.4.10. *Once more, let S° be the cyclic group of order 3 and consider the following order structure.*



We can use our previous techniques to get the following multiplication table, missing only β^2 and γ^2.

	1	α	α^{-1}	β	γ
1	1	α	α^{-1}	α	α^{-1}
α	α	α^{-1}	1	α^{-1}	1
α^{-1}	α^{-1}	1	α	1	α
β	α	α^{-1}	1		1
γ	α^{-1}	1	α	1	

There are three possibilities here.

Case 1: $\beta^2 = \gamma$. Then $\gamma^2 = \alpha$ and we get a monogenic semigroup with β as the generator.

Case 2: $\gamma^2 = \beta$. Then $\beta^2 = \alpha^{-1}$ and we get a monogenic semigroup with γ as the generator, isomorphic to Case 1.

Case 3: $\beta^2 = \alpha^{-1}$ and $\gamma^2 = \alpha$. This is an amalgam of the (isomorphic) monogenic semigroups generated by β and γ with core given by their respective kernels.

In all three cases β and γ are non-regular elements. Even Green's relations are identical in all three cases, suggesting that what differentiates the possibilities is something deeper. This, along with the fact that the ordering structure breaks down in the more general case, suggests that the natural partial order is ultimately not the right tool to use when studying the structure of these semigroups.

As a final observation, the presence of a zero often causes issues when studying weak inverses and $E-$dense semigroups. As a zero is trivially a weak inverse of every element, an $E-$dense semigroup can be constructed from any semigroup by adding a zero, regardless of whether the original semigroup had any weak inverses at all. Also note that as we are working with commuting idempotents, any left (or right) zero is unique and hence is also a right (or left) zero, so we may assume the zero is a unique two-sided zero and denote it by 0. If S is a semigroup with 0, it is easy to check that for any s in S, $0 \leq s$. We therefore have that $S^\circ = \{0\}$ and so the transversal appears to tell us nothing about the larger semigroup. It is possible to adapt the definition of a weak inverse transversal in a manner analogous to that of $0-$inversive semigroups [24]. Our focus, however, will remain with our original definition.

2.4.2 Completely simple kernels

That the weak inverse transversal forms a group can be viewed as a special case of a more general result of Gigoń, who approached the problem from the direction of primitive (i.e. minimal) idempotents. Gigoń proved that an $E-$inversive semigroup containing a primitive idempotent necessarily has a completely simple kernel containing that idempotent, and further that the kernel is in fact the union of the *local subsemigroups* of each primitive idempotent. The local subsemigroup of an element $a \in S$ is given by aSa, and if a is a primitive idempotent this forms a group with a as the identity. We reproduce Gigoń's result below.

Theorem 2.4.11 ([10, Theorem 2.5]). *An $E-$inversive semigroup S has a completely simple kernel if and only if it contains a primitive idempotent. Moreover, in that case,*

$$K_S = \bigcup\{eSe \mid e \in PE(S)\}$$

where $PE(S)$ is the set of primitive idempotents of S and each eSe is a group.

To compare this to Theorem 2.4.1, let S be an $E-$dense semigroup with semilattice of idempotents E such that E contains a minimal idempotent e. As E is a semilattice this minimial idempotent is unique, so by Gigoń's result the kernel is the local subgroup of e. Alternatively, applying Theorem 2.4.1 and Corollary 2.4.3 to S tells us that it has a weak inverse transversal S° containing e which by Proposition 2.4.7 is also the local subgroup of e.

In this case, the kernel forms exactly the group obtained as a weak inverse transversal of S. This correspondence motivates attempting to find a weak inverse transversal in $E-$inversive semigroups with a completely simple kernel. Before we look at transversals, we recall the definition of a completely simple semigroup and give a couple of results showing another way in which they may be of interest with regards to $E-$inversive semigroups.

Recall that S is a completely simple semigroup if and only if $S = \mathcal{M}(G; I, \Lambda; P)$. Recall also that S consists of exactly one $\mathcal{D}$-class in which every $\mathcal{H}$-class is isomorphic to the group G. We can now prove two results regarding the interaction between weak inverses and inverses.

Proposition 2.4.12. *Let S be an $E-$inversive semigroup. For all $s \in S$, every weak inverse of s is an inverse of s if and only if S is a completely simple semigroup.*

Proof. Let S be an $E-$inversive semigroup and suppose $W(s) = V(s)$. If $e, f \in E(S)$ are idempotents such that $e \leq f$ then we have $efe = e$ and so $e \in W(f)$. Hence $e \in V(f)$ and so $fef = f$. As $e \leq f$, we also have $fef = e$ and hence $e = f$. Therefore every idempotent in S must be primitive. As S must also clearly be regular, it follows that S is a completely simple semigroup [18, Theorem 3.3.3].

For the converse, let S be a completely simple semigroup. Suppose $(j, h, \mu) \in W((i, g, \lambda))$. Then

$$(j, h, \mu) = (j, hp_{\mu i}gp_{\lambda j}h, \mu)$$

and so $h = p_{\lambda j}^{-1}g^{-1}p_{\mu i}^{-1}$. Then

$$(i, g, \lambda)(j, h, \mu)(i, g, \lambda) = (i, gp_{\lambda j}p_{\lambda j}^{-1}g^{-1}p_{\mu i}^{-1}p_{\mu i}g, \lambda) = (i, g, \lambda)$$

and hence every weak inverse of (i, g, λ) is also an inverse. □

We can extend this further to identify a necessary condition for an element of a semigroup satisfy $W(s) = V(s)$, as well as conditions under which this becomes an exact characterisation. If s is non-regular, we must have $W(s) = V(s) = \emptyset$ and so the problem is equivalent to identifying where the semigroup fails to be $E-$inversive. For the regular elements, and by extension for all elements of an $E-$inversive semigroup, the following result holds.

Proposition 2.4.13. *If a regular element s of an $E-$inversive semigroup S satisfies $W(s) = V(s)$ then S has a completely simple kernel K_S containing s.*

If K_S is completely simple and additionally for every idempotent $e \in E(S)$ there is a primitive idempotent f such that $f \leq e$, then every element of K_S satisfies $W(s) = V(s)$.

Proof. Let s be an element of S such that $W(s) = V(s)$ and $s' \in V(s)$. Consider $e \in E(S)$ such that $e \leq s's$. Then

$$es'ses' = es'ss' = es'$$

so $es' \in W(s)$. Hence $es' \in V(s)$ so $s = ses's = se$ and $s's = s'se = e$, so $s's$ is a primitive idempotent. It then follows that S has a completely simple kernel K_S containing $s's$ by Theorem 2.4.11. Since $s' \in V(s)$, $s\mathcal{L}s's$ and so $s \in K_S$.

Now let S be a semigroup with a completely simple kernel K_S such that for every $f \in E(S)$ there exists a primitive idempotent e such that $e \leq f$. Let $s \in K_S$ and consider $s' \in W(s)$. Then there exists a primitive idempotent $e \leq s's$ and by the previous argument $es' \in W(s)$. We have $es'\mathcal{R}e$ so every ideal containing e also contains es'. Since e is primitive we have $e \in K_S$ by Theorem 2.4.11, so $es' \in K_S$ and by Proposition 2.4.12 $es' \in V(s)$. Hence, as before, $e = s's$ so $s' = es' \in V(s)$. □

The requirement that every idempotent be bounded below by a primitive idempotent is not a particularly restrictive condition to impose on the already broad class of

$E-$inversive semigroups. Some obvious examples that satisfy the condition include every finite semigroup, every group, and every ($E-$inversive) semigroup with finitely many idempotents, among others. This property is key to finding a weak inverse transversal, as we will see in the following section. Before we look at weak inverse transversals, we have the following corollary to Proposition 2.4.13.

Corollary 2.4.14. *Let S be an $E-$inversive semigroup in which every idempotent is bounded below by a primitive idempotent. Let $s \in K_S$ and $s' \in W(s)$. Then $s' \in K_S$.*

Proof. Since $s' \in W(s)$ we have $s' \in V(s)$. Then $s'\mathcal{D}s$ and so $s' \in K_S$. □

2.4.3 Weak inverse transversals

As stated above, our goal for this section is to show that an $E-$inversive semigroup in which every idempotent is bounded below by a primitive idempotent has a weak inverse transversal. More specifically, we will prove the following proposition.

Theorem 2.4.15. *Let S be an $E-$inversive semigroup in which every idempotent is bounded below by a primitive idempotent. Then every $\mathcal{H}$-class of the kernel K_S acts as a weak inverse transversal S° of S.*

Since by construction S contains a primitive idempotent, K_S is completely simple by Theorem 2.4.11 and so each $\mathcal{H}$-class of K_S is isomorphic to the same group, meaning as in the commutative case (see Proposition 2.4.7) weak inverse transversals constructed in this way are (isomorphic to) a uniquely determined group. We will see later that, unlike in the commutative case, it is also possible in general to find a weak inverse transversal S° which is not closed under multiplication. It should also be noted that in a semigroup containing zeroes we run into similar issues as in the less general case. As we no longer require idempotents to commute, S can now contain multiple left or right zeroes rather than merely a unique two sided zero. If S does contain zeroes, then K_S consists of exactly those zeroes, each in a separate $\mathcal{H}$-class. Then S° is once again the trivial group.

Our first step on the path to Theorem 2.4.15 is to show that every element of S does in fact have a weak inverse in K_S, following a similar argument as in the previous section.

Lemma 2.4.16. *Let S be an $E-$inversive semigroup in which every idempotent is bounded below by a primitive idempotent. Then $W(s) \cap K_S \neq \emptyset$ for every s in S.*

Proof. Let $s \in S$ and $s' \in W(S)$. Then ss' is idempotent and so there exists a primitive idempotent e such that $e \leq ss'$. Then

$$s'ess'e = s'ss'e = s'e$$

so $s'e$ is a weak inverse of s. Further, $ss'e = e$ so $e\mathcal{L}s'e$. As e is primitive it lies in K_S by Theorem 2.4.11 and so $s'e$ also lies in K_S. Hence $s'e \in W(s) \cap K_S$ as required. □

Having shown that every element of S has a weak inverse in K_S, we would like to be able to conclude that this weak inverse is unique. Unfortuantely, as we will see shortly, this is about as far from true as possible. We can however make the following observation, which holds for any semigroup.

Lemma 2.4.17. *Let S be a semigroup and s an element of S. If s' and s^* are both weak inverses of s such that $s'\mathcal{H}s^*$ then $s' = s^*$.*

Proof. Let $s \in S$ and consider $s', s^* \in W(s)$ such that $s'\mathcal{H}s^*$. Then $s'.s\mathcal{R}s'$ so $s's\mathcal{R}s^*$. Since $s's$ is idempotent, we have $s'ss^* = s^*$. Similarly, $ss^*\mathcal{L}s'$ and so $s's^* = s'$ and hence $s' = s^*$. □

Armed with this lemma, to obtain a weak inverse transversal we now need only to show that some $\mathcal{H}$-class of K_S contains a weak inverse for every element of S. Theorem 2.4.1 makes a stronger claim, that this is true of every $\mathcal{H}$-class of K_S.

Lemma 2.4.18. *Let S be an $E-$inversive semigroup in which every idempotent is bounded below by a primitive idempotent. Then for every element s of S, every $\mathcal{H}$-class of K_S contains a weak inverse $s' \in W(s)$.*

Proof. Let s be an element of S. By Lemma 2.4.16 there exists a weak inverse s' of s lying in K_S. Let $e = ss'$, so $e\mathcal{L}s'$ and let r be any idempotent such that $r\mathcal{R}e$.

Then $r = eu$ for some $u \in S^1$ and so $s'rss'r = s'rer = s'r$. Hence $s'r$ is a weak inverse of s. Since $r\mathcal{R}e$ we have $s'r\mathcal{R}s'e = s'ss' = s'$. We also have $ss'r = er = r$ so $s'r\mathcal{L}r$. The egg-box diagram for this arrangement is the following.

s'		$s'r$
e		r

As K_S is completely simple there will be one such r for every $\mathcal{H}$-class in the $\mathcal{R}$-class of e. Hence we can find a weak inverse $s'r$ of s in every $\mathcal{H}$-class in the $\mathcal{R}$-class of s'.

Note that if $a \in K_S$, $b \in S$ such that $a\mathcal{H}b$ then $b \in K_S$. Further, $a\mathcal{H}_{K_S}b$ where $\mathcal{H}_{K_S}$ denotes the $calH$-relation on K_S when viewed as a semigroup in its own right.

Following a similar argument, we have a weak inverse ls' for every idempotent l lying in the $\mathcal{L}$-class of $f = s's$. Again there is one such l for every $\mathcal{H}$-class in the $\mathcal{L}$-class of f, and so we get a weak inverse ls' of s in every $\mathcal{H}$-class in the $\mathcal{L}$-class of s'. Putting these results together fills in weak inverses of s in a "plus shape" of K_S centred on s', as shown in the following egg-box diagram, where $r, r^* \in R_e$ and $l, l^* \in L_e$.

	ls'	
$s'r$	s'	$s'r^*$
	l^*s'	

Our original choice of s' as a weak inverse of s was entirely arbitrary, and so we could repeat the process using, for example, $s'r$ as our starting point. This would allow us to fill in a "plus shape" of weak inverses centred on $s'r$. If we keep iterating this process along an $\mathcal{R}$-class, clearly we will fill in the corresponding $\mathcal{L}$-class at each iteration, and so we will be able to place a weak inverse in every $\mathcal{H}$-class of K_S. □

Combining these lemmas proves Theorem 2.4.15. Choose any $\mathcal{H}$-class of K_S. By Lemma 2.4.18, this $\mathcal{H}$-classcontains at least one weak inverse for every element of S while Lemma 2.4.17 tells us that it contains at most one weak inverse for every element of S. Hence the $\mathcal{H}$-class contains exactly one weak inverse for each element of S and so is a weak inverse transversal as required.

Weak inverse transversals in the form of $\mathcal{H}$-classes of a completely simple kernel are not the only possible sets that contain exactly one weak inverse of every element. We can see this with a simple example. Let $S = R \times G$ where $R = \{a, b\}$ is a right zero semigroup and $G = \{1, g\}$ is a group. Then the associates of $(a, 1)$ are $(a, 1)$ and $(b, 1)$ and the associates of (b, g) are (a, g) and (b, g) so the two elements form a weak inverse transversal despite not being $\mathcal{H}$-related. Note that unlike weak inverse transversals arising from the kernel, in this instance the subset $S^\circ = \{(a, 1), (b, g)\}$ is not a subsemigroup.

2.5 Wreath products

The goal for this section is a partial exploration of a method of constructing additional examples of $E-$inversive semigroups. This particular method was motivated by the Krohn-Rhodes decomposition of finite semigroups, which states that every finite semigroup is a divisor of a wreath product of finite simple groups and copies of U_2 [8]. For simplicity we restrict ourselves for the moment to examining the structure of a simpler class of wreath products, including how it relates to the results of the preceding section.

Let A and S be semigroups and let A^S be the set of maps from S to A. This set forms a semigroup under pointwise multiplication, i.e. if $f, g \in A^S$ then for all $x \in S$,

$$(x)(fg) = (x)f(x)g$$

where the multiplication on the right is taking place in A. We define a left action $\varphi : S \times A^S \to A^S$ by

$$(s, f) \mapsto {}^s f$$

where $(x)\,{}^s f = (xs)f$ for all $x \in S$. The *wreath product* $A \wr S$ is then given by the semidirect product $A^S \rtimes_\varphi S$. Elements have the form (f, s) with $f \in A^S$ and $s \in S$, with multiplication given by $(f, s)(g, t) = (f\,{}^s g, st)$.

We are interested in wreath products where $A = U_2$, where U_n is the monoid formed by adjoining an identity to the n element right zero semigroup (so U_2 consists of an identity and two right zeroes denoted by 1, a and b respectively), and $S = G$ is any group. For ease of notation, G will be written additively, so $(f, s)(g, t) = (f\,{}^s g, s + t)$.

2.5.1 Idempotents

We first consider the idempotents of $U_2 \wr G$. Given an idempotent (f, s), we need $(f\,{}^s f, s + s) = (f, s)$. From the second coordinate, it is clear that we need $s = 0$. Then $f\,{}^0 f = ff$ and so we need f to be idempotent in ${U_2}^G$. As the multiplication in ${U_2}^G$ is pointwise and every element of U_2 is idempotent, every element of ${U_2}^G$ is also idempotent. The idempotents of $U_2 \wr G$ are then exactly the elements of the form $(f, 0)$. An immediate consequence is that the idempotents form a band, and in fact are isomorphic to ${U_2}^G$.

We can also look at the natural partial order on the idempotents. Recall that for any semigroup, there is a natural partial order on the idempotents where $e \leq f$ if and only if $ef = fe = e$. Due to the isomorphism above, we can work in ${U_2}^G$ and translate our results back to the idempotents of $U_2 \wr G$.

Let $e, f \in {U_2}^G$ and consider $(x)(ef) = (x)e(x)f$ for $x \in G$. As the non-identity elements of U_2 are right zeroes, there are two cases. If $(x)f = 1$ then $(x)(ef) = (x)e$, and otherwise $(x)(ef) = (x)f$. Then $ef = e$ if and only if for all $x \in G$ either $(x)f = (x)e$ or $(x)f = 1$. For $fe = e$ there are again two cases. Either $(x)e = 1$, in which case $(x)f = (x)e = 1$, or $(x)e$ is a right zero so $(x)(fe) = (x)e$. From these results it follows that if $e \leq f$ then for every $x \in G$ either $(x)e = (x)f$ or $(x)f = 1$. This can be easily verified to be a sufficient condition for $e \leq f$.

These results are summarised in the following lemma.

Lemma 2.5.1. *The idempotents of $U_2 \wr G$ are the elements of the form $(f,0)$ for any $f \in {U_2}^G$ and form a band isomorphic to ${U_2}^G$. Under the natural partial order, $(f,0) \leq (g,0)$ if and only if for all $x \in G$ either $(x)g = (x)f$ or $(x)g = 1$.*

Let $g \in {U_2}^G$ be such that for every $x \in G$, $(x)g \neq 1$ and suppose there exists $f \in {U_2}^G$ such that $(f,0) \leq (g,0)$. Then by Lemma 2.5.1 we have $(x)f = (x)g$ for all $x \in G$ so $(f,0) = (g,0)$ and hence $(g,0)$ is a primitive idempotent. Additionally, for any $(g,0) \in U_2 \wr G$ we can define $f \in {U_2}^G$ such that

$$(x)f = \begin{cases} (x)g & (x)g \neq 1 \\ a & (x)g = 1 \end{cases}$$

to obtain a primitive idempotent $(f,0) \leq (g,0)$. Hence every idempotent of $U_2 \wr G$ is bounded below by a primitive idempotent, and so by Theorem 2.4.15 $U_2 \wr G$ has a weak inverse transversal.

2.5.2 Green's relations

We now describe the $\mathcal{L}$- and $\mathcal{R}$-class structure of $U_2 \wr G$. A useful observation is that for every element of $U_2 \wr G$ there is a natural way to pick out a particular idempotent in its $\mathcal{L}$-class, and similarly for its $\mathcal{R}$-class.

Let $\mathbb{1} \in {U_2}^G$ be the constant map to 1. This is clearly an identity in the semigroup ${U_2}^G$. Moreover, ${}^s\mathbb{1} = \mathbb{1}$ for any $s \in G$, as is the case for any constant map. Then

$$(f,s)(\mathbb{1},-s) = (f\,{}^s\mathbb{1}, s-s) = (f\mathbb{1},0) = (f,0)$$

and

$$(f,0)(\mathbb{1},s) = (f\,{}^0\mathbb{1}, 0+s) = (f\mathbb{1},s) = (f,s)$$

and so $(f,s)\mathcal{R}(f,0)$.

Performing similar products on the left, we see that

$$(\mathbb{1},-s)(f,s) = ({}^{-s}f,0)$$

and

$$(\mathbb{1},s)({}^{-s}f,0) = (f,s)$$

so $(f,s)\mathcal{L}({}^{-s}f,0)$, or equivalently $({}^{s}f,s)\mathcal{L}(f,0)$. This allows us to consider only the idempotents as, for example, $(f,s)\mathcal{R}(g,t)$ if and only if $(f,0)\mathcal{R}(g,0)$.

Suppose $(f,0)\mathcal{L}(g,0)$. Then $(f,0)=(u,s)(g,0)$ and $(g,0)=(v,t)(f,0)$ for some $(u,s),(v,t)\in U_2\wr G$. Clearly from the second coordinate $s=t=0$ and so $f=ug$ and $g=vf$ in ${U_2}^G$. From these equations we can see that if $(x)f\neq 1$ then $(x)g=(x)f$ and if $(x)g\neq 1$ then $(x)f=(x)g$. The only other possibility is that $(x)f=(x)g=1$, so in all cases we have $(x)f=(x)g$ and so $f=g$.

It follows that each $\mathcal{L}$-class contains exactly one idempotent. We have a separate $\mathcal{L}$-class of $U_2\wr G$ for each map f in ${U_2}^G$, consisting of the elements $({}^sf,s),s\in G$. Equivalently, $(f,s)\mathcal{L}(g,t)$ if and only if ${}^{-s}f={}^{-t}g$.

Now suppose $(f,0)\mathcal{R}(g,0)$. Then $(f,0)=(g,0)(u,s)$ and $(g,0)=(f,0)(v,t)$ for some $(u,s),(v,t)\in U_2\wr G$. Again the second coordinate tells us $s=t=0$ and so $f=gu$ and $g=fv$ in ${U_2}^G$. If $(x)f=1$ then $(x)(gu)=1$ so $(x)g=(x)u=1$, and similarly if $(x)g=1$ then $(x)f=(x)v=1$. In other words, $(x)f=1$ if and only if $(x)g=1$ so $(1)f^{-1}=(1)g^{-1}$. This condition is sufficient to show $(f,0)\mathcal{R}(g,0)$ since if it holds we have $f=gf$ and $g=fg$.

The elements lying in the $\mathcal{R}$-class of (f,s) then have the form (g,t) where $(x)g=1$ if and only if $(x)f=1$ and t is any element of G. In other words, $(f,s)\mathcal{R}(g,t)$ if and only if $(1)f^{-1}=(1)g^{-1}$.

We can of course combine these results to characterise the $\mathcal{H}$- and $\mathcal{D}$-classes. The $\mathcal{H}$-class of (f,s) consists of the permutations $({}^tf,t+s)$ such that $(1)f^{-1}=(1)({}^tf)^{-1}$, or equivalently $(1)f^{-1}=(1)f^{-1}+t$. For the $\mathcal{D}$-classes, we have $(f,s)\mathcal{D}(g,t)$ if and only if $(1)g^{-1}=(1)f^{-1}+u$ for some $u\in G$.

Applying this to our knowledge of the primitive idempotents above, we can see that the kernel of $U_2\wr G$ contains exactly the elements of the form (f,s) where $xf\neq 1$ for all $x\in G$ and s is any element of G. A weak inverse transversal of $U_2\wr G$ consists of any $\mathcal{H}$-class of the kernel, and therefore the elements of the form $({}^sf,s)$ for a particular such $f\in {U_2}^G$. It can be easily seen that this group is isomorphic to the original group G.

2.5.3 Weak inverses

Suppose (f,s) is a weak inverse of (g,t). Then $(f,s)(g,t)(f,s)=(f,s)$ so

$$(f\,{}^sg\,{}^{s+t}f,s+t+s)=(f,s).$$

From the second coordinate we must have $s=-t$ so this simplifies to

$$(f\,{}^{-t}gf,-t)=(f,-t).$$

Now if $(x)f \neq 1$ we have $(x)(f^{-t}gf) = (x)f$ as required. If $(x)f = 1$ then $(x)(f^{-t}gf) = (x)^{-t}g$ so we need $(x)^{-t}g = 1$. Then (f,s) is a weak inverse of (g,t) if and only if $s = -t$ and $(1)f^{-1} \subseteq (1)({}^{-t}g)^{-1}$, or equivalently $(1)f^{-1} + t \subseteq (1)g^{-1}$.

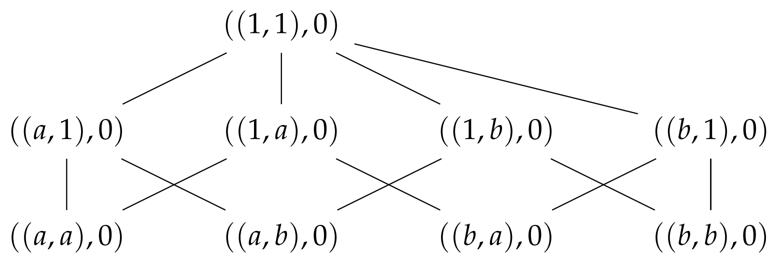
As every $\mathcal{D}$-class contains an idempotent, $U_2 \wr G$ is regular and so we should also ask what the inverses look like. By reversing the equations above, (g,t) is a weak inverse of (f,s) if and only if $t = -s$ and $g\,{}^{t}fg = g$. Then $(1)g^{-1} \subseteq (1)({}^{t}f)^{-1}$ or equivalently $(1)g^{-1} \subseteq (1)f^{-1} + t$, and so (f,s) and (g,t) are inverses if and only if $s = -t$ and $(1)f^{-1} + t = (1)g^{-1}$.

2.5.4 Example

To illustrate the above results we show how they apply to a simple example, $U_2 \wr C_2$ where $C_2 = \{0, s\}$ is the two element cyclic group. Specifically, we identify the idempotent structure and the $\mathcal{D}$-class structure, and identifying the possible weak inverse transversals. For notation in this example we we will write the map $f : C_2 \to U_2$ as the pair $((0)f, (s)f)$.

The idempotents of $U_2 \wr C_2$ are of course the elements of the form $(f, 0)$, and the order structure on these idempotents is given in the following diagram, where

indicates that $e \leq f$.



For the $\mathcal{D}$-class structure we have the following egg-box diagrams. The third $\mathcal{D}$-class is the kernel containing each primitive idempotent, and it can be easily verified that each of its $\mathcal{H}$ classes contains a weak inverse of every element of $U_2 \wr C_2$.

Here the $\mathcal{D}$-classes correspond to the possible sizes of $1f^{-1}$, but we can see how larger groups lead to more $\mathcal{D}$-classes by looking at $U_2 \wr C_4$ where $C_4 = \{0, s, 2s, 3s\}$. Let $f, g \in {U_2}^{C_4}$ be such that $1f^{-1} = \{0, s\}$ and $1g^{-1} = \{0, 2s\}$. Then $(f,t)\mathcal{D}(g,u)$ if and

((1,1),0) ((1,1),s)

((1,a),0)	((1,a),s)	((1,b),0)	((1,b),s)
((a,1),s)	((a,1),0)	((b,1),s)	((b,1),0)

((a,a),0) ((a,a),s)	((a,b),0) ((b,a),s)	((b,a),0) ((a,b),s)	((b,b),0) ((b,b),s)

only if $\{0,s\} + v = \{0,2s\}$ for some $v \in C_4$. No such v exists, so they must lie in separate $\mathcal{D}$-classes despite the preimages having the same size.

For a weak inverse transversal, by Theorem 2.4.15 we may take the $\mathcal{H}$-class of any primitive idempotent. For example, if we take $(U_2 \wr C_2)^\circ = \{((a,a),0),((a,a),s)\}$ then we have

$$A(((a,a),0)) = \{(f,0) \mid f \in {U_2}^{C_2}\}$$

and similarly

$$A(((a,a),s)) = \{(f,s) \mid f \in {U_2}^{C_2}\}$$

clearly encompassing every element of ${U_2}^{C_2}$.

Chapter 3

Semilattices of stratified extensions

In this chapter we generalise Grillet's introduction of stratified semigroups [14] by examining the situation where $\cap_{m>0} S^m$ is neither $\{0\}$ nor the empty set. For details of Grillet's definitions, see Section 1.4. After some basic definitions and preliminary results in Section 3.1, in Section 3.2 we introduce the concept of a *stratified extension* as a generalisation of Grillet's stratified semigroups, and we provide a number of interesting results on the overall structure of such semigroups. In Section 3.3 our focus is on semigroups in which every regular $\mathcal{H}-$class contains an idempotent. We show that group-bound semigroups with this property are semilattices of stratified extensions of completely simple semigroups and describe the semilattice structure. Finally in Section 3.4 we look at strict extensions of Clifford semigroups and show amongst other things that strict stratified extensions of Clifford semigroups are semilattices of stratified extensions of groups.

3.1 Preliminaries

An element s in a semigroup S is called *eventually regular* if there exists $n \geq 1$ such that s^n is regular. A semigroup is *eventually regular* if all of its elements are eventually regular. It is clear that eventually regular semigroups are $E-$inversive. A semigroup S is called *group-bound* if for every $s \in S$, there exists $n \geq 1$ such that s^n lies in a subgroup of S. Clearly group-bound semigroups are eventually regular. If S is eventually regular and each regular $\mathcal{H}-$class is a group then S is group-bound.

A semigroup S is called *Archimedean* if for any $a, b \in S$ there exists $n \in \mathbb{N}$ such that $a^n \in SbS$.

Theorem 3.1.1 ([32, Theorem 3])**.** *Let S be a group-bound semigroup. Then S is a semilattice of Archimedean semigroups if and only if every regular $\mathcal{H}-$class of S is a group.*

Let S be a semigroup with 0. We say that an element $x \in S$, is *nilpotent* if there is $n \in \mathbb{N}$ such that $x^n = 0$. The semigroup S is called *nilpotent* if every element of S is nilpotent. The semigroup S is called *nilpotent with degree* $n \in \mathbb{N}$ if $S^n = \{0\}$. Note that Grillet [14] and Shevrin [32] call nilpotent semigroups *nilsemigroups*, whereas they refer to nilpotent semigroups with a finite degree as simply nilpotent.

3.2 Stratified extensions

Let S be a semigroup (not necessarily stratified) and define the *base* of S to be the subset $\mathrm{Base}(S) = \bigcap_{m>0} S^m$. We shall say that a semigroup S is a *stratified extension* of $\mathrm{Base}(S)$ if $\mathrm{Base}(S) \neq \emptyset$. The reason for this name will become apparent later. Clearly $\mathrm{Base}(S)$ is a subsemigroup of S. When $\mathrm{Base}(S)$ is a trivial subgroup then S is a stratified semigroup. A stratified semigroup S is not in general a stratified extension as we may have $\mathrm{Base}(S) = \emptyset$, however if S is a stratified semigroup then S^0 is also stratified and is a stratified extension with trivial base. Further, S is called a *finitely stratified extension* if there exists $m \in \mathbb{N}$ such that $S^m = S^{m+1} = \mathrm{Base}(S)$. The smallest such m is called the *height* of S and where necessary we shall refer to S as a *finitely stratified extension with height* m. If for every s in S there is an $m \in \mathbb{N}$ such that $s^m \in \mathrm{Base}(S)$ then S is a *nil-stratified extension*. All finitely stratified extensions are nil-stratified extensions, but it is easy to demonstrate that not all nil-stratified extensions are finitely stratified extensions.

A finitely stratified extension is a stratified extension over the same base, since $S^m = S^{m+1}$ implies $S^n = S^m$ for all $n \geq m$ and so $\bigcap_{k>0} S^k = S^m$, where m is the height of S. The converse is not true since, for example, if S is a free semigroup with a zero adjoined, then S is a stratified extension with trivial base but not a finitely stratified extension. It is clear that a (finitely) stratified extension has a unique base.

Clearly, for all $m \geq 1$, $S^{m+1} \subseteq S^m$ and so we define the *layers* of S as the sets $S_m = S^m \setminus S^{m+1}$, $m \geq 1$. Every element of $S \setminus \mathrm{Base}(S)$ lies in exactly one layer, and if $s \in S_m$ then m is the *depth* of s. The layer S_1 generates every element of $S \setminus \mathrm{Base}(S)$ and is contained in any generating set of S. However $\mathrm{Base}(S) \not\subseteq \langle S_1 \rangle$ in general. For example, let S be a semigroup with 0, with no zero divisors. Then $0 \in \mathrm{Base}(S)$ but $0 \notin \langle S_1 \rangle$.

Since $\mathrm{Base}(S) \subseteq S^m$ for any $m \in \mathbb{N}$, we have an alternative characterisation for the elements of $\mathrm{Base}(S)$. Any $s \in S$ lies in $\mathrm{Base}(S)$ if and only if s can be factored into a product of m elements for any $m \in \mathbb{N}$, i.e. $s = a_1 a_2 \ldots a_m$ for some $a_i \in S$. This characterisation gives us some immediate properties of $\mathrm{Base}(S)$ as a subsemigroup of S.

Lemma 3.2.1. *Let S be a semigroup and let $s \in S$. If $s \in Ss \cup sS \cup SsS$ then $s \in \text{Base}(S)$.*

Proof. It follows that for $m \geq 1$, $s = x^m s$ or $s = sy^m$ or $s = x^m sy^m$ and so the result follows from the previous observation. □

Corollary 3.2.2. *Suppose that S is a semigroup.*

1. *Every submonoid of S is a submonoid of* $\text{Base}(S)$.
2. $\text{Reg}(S) \subseteq \text{Base}(S)$. *Hence if S is regular,* $\text{Base}(S) = S$.
3. $E(S) = E(\text{Base}(S))$.
4. *If $s \in S \setminus \text{Base}(S)$ then $|J_s| = 1$, where J_s is the $\mathcal{J}-$class of s.*

To see (4) notice that if $a \mathcal{J} b$ and $a \neq b$ then we have $a = ubv$ for some $u, v \in S^1$ and since $a \neq b$ we have u and v not both equal to 1. Similarly $b = sat$ with $s, t \in S^1$ not both equal to 1 and hence $a \in Sa \cup aS \cup SaS$. The converse is not true, since for example in a semigroup with zero we have $J_0 = \{0\}$ but $0 \in \text{Base}(S)$.

If follows immediately that the class of stratified extensions contains the class of semigroups with regular elements and hence in particular the classes of monoids, finite semigroups and regular semigroups. However, not every semigroup is a stratified extension. Consider for example a semigroup with a length function (i.e. a function $l : S \to \mathbb{N}$ such that for all $x, y \in S, l(xy) = l(x) + l(y)$). If T is the subsemigroup of elements with non-zero length, then the elements of T^m each have length at least m. Hence the elements of length exactly m lie in $T_m \not\subseteq \text{Base}(T)$ and so the base is empty. In particular, a free semigroup is not a stratified extension, nor is the semigroup of polynomials of degree ≥ 1 over any ring, under multiplication.

This property allows us to prove the following results, justifying the names of stratified, nil-stratified, and finitely stratified extensions.

Lemma 3.2.3. *Let S be a stratified extension. Then* $\text{Base}(S)$ *is an ideal of S.*

Proof. For any $u, v \in S^1$, $t \in \text{Base}(S)$ and $m > 3$, we have $t \in S^{m-2}$ so $utv \in S^m$ and hence $utv \in \text{Base}(S)$. □

Hence we can regard S as being an ideal extension of $\text{Base}(S)$ by $S/\text{Base}(S)$ and note that $S/\text{Base}(S)$ is a stratified semigroup with 0. If S is a nil-stratified extension then it follows that for every $s \in S$ there exists $m \in \mathbb{N}$ such that $s^m \in \text{Base}(S)$. Hence in the Rees quotient $S/\text{Base}(S)$, $s^m = 0$ and so $S/\text{Base}(S)$ is nilpotent and S is an ideal extension by a nilpotent stratified semigroup. Recall that the nilpotency degree of a semigroup is the smallest value m such that every product of m elements is zero. It is easy to see that if the nilpotency degree of $S/\operatorname{Base}(S)$ is m then the height of S is m and so S is a finitely stratified extension. Conversely, any nilpotent semigroup S of finite nilpotency degree m is a stratified semigroup with $S^m = \{0\}$. We have hence proved the following.

Proposition 3.2.4. *Let S be a stratified extension. Then*

1. *S is an ideal extension of* $\operatorname{Base}(S)$ *by a stratified semigroup with* 0*.*
2. *If S is a nil-stratified extension then it is an ideal extension of* $\operatorname{Base}(S)$ *by a nilpotent semigroup.*
3. *If S is a finitely stratified extension then it is an ideal extension of* $\operatorname{Base}(S)$ *by a nilpotent semigroup of finite degree.*

The converses of these results do not hold. To see this, let S be a free semigroup and T be the two element nilpotent semigroup. Then T is a stratified semigroup with 0 but an extension of S by T is not a stratified extension. Further, T is a nilpotent semigroup of finite degree and an extension of S^0 by T is a stratified extension, but is not a finitely stratified nor nil-stratified extension.

Proposition 3.2.5. *Let S be a stratified extension.*

1. *If S is a nil-stratified extension then* $\operatorname{Base}(S)$ *is periodic if and only if S is periodic;*
2. *If S is a nil-stratified extension then* $\operatorname{Base}(S)$ *is eventually regular if and only if S is eventually regular;*
3. $\operatorname{Base}(S)$ *is $E-$inversive (resp. E-dense) if and only if S is $E-$inversive (resp. $E-$dense).*

So a stratified extension with a periodic base is $E-$dense.

Proof. The first two statements are easy to deduce. For the third, let $\operatorname{Base}(S)$ be $E-$inversive and let $s \in S$. Then for any $t \in \operatorname{Base}(S), ts \in \operatorname{Base}(S)$ and so there exists $u \in \operatorname{Base}(S)$ such that $uts \in E(\operatorname{Base}(S)) = E(S)$ and so S is $E-$inversive. Conversely suppose that S is $E-$inversive. Since $W(S) = \operatorname{Reg}(S) \subseteq \operatorname{Base}(S)$, then $\operatorname{Base}(S)$ is $E-$inversive. For $E-$dense, note that as every idempotent of S lies in the base, $E(S) = E(\operatorname{Base}(S))$. □

Notice that periodic ⇒ eventually regular ⇒ $E-$inversive ⇒ stratified.

There is in general little control over the base as a stratified extension can be constructed with any given semigroup as its base. However, finitely stratified extensions allow us to place restrictions on the semigroup forming the base.

Proposition 3.2.6. *Let T be any semigroup. There exists a stratified extension S with base T.*

Proof. Let R be any semigroup and let $S = R \dot\cup T$. Define a binary operation $*$ on S by $r_1 * r_2 = r_1 r_2$ for $r_1, r_2 \in R$, $t_1 * t_2 = t_1 t_2$ for $t_1, t_2 \in T$, and $r * t = t * r = t$ for $r \in R$ and $t \in T$. It is easy to verify that this operation is associative and so $(S, *)$ is a semigroup. Then $T \subseteq \bigcap_{m>0} (R \dot\cup T)^m$ and so S is a stratified extension. Moreover, if we choose R such that $\bigcap_{m>0} R^m = \emptyset$, for example $R = A^+$, a free semigroup, we see that $\bigcap_{m>0} (R \dot\cup T)^m = T$ and so we can obtain a stratified extension with base T. □

In contrast, the possible bases for a finitely stratified extension are much more restricted. Let S be a finitely stratified extension with $T = \text{Base}(S)$ and consider T^2. There exists $m \in \mathbb{N}$ such that $T = S^m$, so $T^2 = S^{2m}$. But by definition $S^m = S^{m+1} = S^{m+2} = \cdots = S^{2m}$ and so $T^2 = T$. A semigroup T satisfying $T^2 = T$ is said to be *globally idempotent* and so the base of a finitely stratified extension is globally idempotent. Note also that if S is globally idempotent, then S is a finitely stratified extension in a trivial sense, with base S and height 1.

Proposition 3.2.7. *A semigroup S is a finitely stratified extension if and only if it is an ideal extension of a globally idempotent semigroup by a nilpotent semigroup of finite degree.*

Proof. We need only justify the converse. Let Σ be an ideal extension of a globally idempotent semigroup S by a nilpotent semigroup T of finite degree m. Then $\Sigma^m = S$ and $S = S^2$ so $\Sigma^m = \Sigma^{2m}$ and as each $\Sigma^i \subseteq \Sigma^{i+1}$ it follows that $\Sigma^m = \Sigma^{m+1}$. Hence Σ is a finitely stratified extension with base S and height m. □

This is still a very broad class of semigroup, including among its members every monoid and every regular semigroup. It should also be noted that a globally idempotent semigroup need not contain idempotents, the Baer-Levi semigroup being one such example.

Proposition 3.2.8. *There exists a finitely stratified extension of height h, for any $h \in \mathbb{N}$.*

Proof. Let G be any finite cyclic group of order r and let S be the monogenic semigroup of index h and period r. Then it is reasonably clear that S is a finitely stratified extension with base G and height h. □

Let S be a semigroup and let ρ be a congruence on S. It is easy to see that for any $m \in \mathbb{N}$ we have $(S/\rho)^m = S^m/\rho$. Hence if S is a stratified extension then S/ρ is also a stratified extension, with $\text{Base}(S/\rho) = \text{Base}(S)/\rho$. Further, if S is a finitely stratified or nil-stratified extension then so is S/ρ.

Let S_i be a family of semigroups. Then $(\prod_{i\in I} S_i)^m = \prod_{i\in I} {S_i}^m$. Hence if each S_i is a stratified extension, the product $\prod_{i\in I} S_i$ is also a stratified extension with $\text{Base}(\prod_{i\in I} S_i) = \prod_{i\in I} \text{Base}(S_i)$. If I is a finite set and each S_i is a nil-stratified extension then so is $\prod_{i\in I} S_i$. Similarly if each S_i is a finitely stratified extension then so is $\prod_{i\in I} S_i$. To see that we cannot remove the condition $|I| < \infty$, let $I = \mathbb{N}$ and for each $i \in I$ let S_i be a finitely stratified (and hence nil-stratified) extension of height i. Then $\prod_{i\in I} S_i$ is a stratified extension but is neither a finitely stratified extension nor a nil-stratified extension.

Subsemigroups of (finitely, nil-) stratified extensions are not necessarily (finitely, nil-) stratified extensions. For example the bicyclic semigroup is a finitely stratified extension (in fact globally idempotent) but contains $(\mathbb{N}, +)$ as a subsemigroup which is free and hence not even a stratified extension.

The class of (finitely, nil-) stratified semigroups therefore does not form a variety. However, we have proved the following theorem.

Theorem 3.2.9. *Let S be a (finitely, nil-) stratified extension and let S_i for $i \in I$ be a family of stratified semigroups.*

1. *If ρ is a congruence on S then S/ρ is a (finitely, nil-) stratified extension with base* $\text{Base}(S)/\rho$*;*
2. *the direct product $\prod_{i\in I} S_i$ is a stratified extension with base $\prod_{i\in I} \text{Base}(S_i)$;*
3. *if $|I| < \infty$ and each S_i is a (finitely, nil-) stratified extension then the direct product $\prod_{i\in I} S_i$ is a (finitely, nil-) stratified extension with base $\prod_{i\in I} \text{Base}(S_i)$.*

By part (1) of this theorem, any homomorphic image of a stratified extension is a stratified extension. This holds true even when the morphism involved is not surjective, though we lose some precision regarding the base of the second semigroup.

Proposition 3.2.10. *Let S, T be semigroups and $f : S \to T$ a morphism. Then for all $i \in \mathbb{N}$, $S^i \subseteq f^{-1}(T^i)$ and so in particular,* $\text{Base}(S) \subseteq f^{-1}(\text{Base}(T))$.

Proof. Let $x \in S^i$ so $x = x_1 \dots x_i$ for some $x_1, \dots, x_i \in S$. Then

$$f(x) = f(x_1 \dots x_i) = f(x_1) \dots f(x_i) \in T^i$$

so $x \in f^{-1}(T^i)$ and hence $S^i \subseteq f^{-1}(T^i)$. The second result follows immediately. □

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice of stratified extensions. Then

$$\bigcup_{\alpha\in Y} \text{Base}(S_\alpha) \subseteq \bigcap_{m>0} S^m$$

and so S is a stratified extension.

In the case of finitely stratified extensions, we can construct a semilattice of finitely stratified extensions which is not a finitely stratified extension. Let $Y = \mathbb{N} \cup \{0\}$ be a semilattice under the multiplication $ij = 0$ for all $i, j \in Y$ with $i \neq j$. For each $i \in \mathbb{N}$ let S_i be a finitely stratified extension with height i and let S_0 be globally idempotent. Let S be the union of each S_i as a semilattice of semigroups over Y. Then $S^m = S_0 \cup \bigcup_{i \in \mathbb{N}} {S_i}^m$. If $i > m$ then there are elements in ${S_i}^m$ which are not in ${S_i}^{m+1}$ and so $S^m \neq S^{m+1}$ for any $m \in \mathbb{N}$.

3.3 Semilattices of group bound semigroups

Let S be a semigroup such that every regular $\mathcal{H}$-class contains an idempotent. Equivalently, every regular element of S lies in some subgroup of S. Such a semigroup is called a *s*trongly $2-$chained semigroup, following the definition given in [21, Corollary 1.2]. We define a relation ρ on S by $s\rho t$ if and only if for every $\mathcal{D}$-class D of S we have

$$W(s) \cap D \neq \emptyset \iff W(t) \cap D \neq \emptyset.$$

Clearly ρ is an equivalence relation. We will show that ρ is in fact a congruence, and moreover that S/ρ is a semilattice.

We begin by establishing some properties of such semigroups.

Lemma 3.3.1. *Let S be a strongly $2-$chained semigroup.*

1. *Every regular $\mathcal{D}$-class of S is a completely simple subsemigroup of S.*
2. *Let $s' \in W(s)$. Every $\mathcal{H}$-class of $D_{s'}$ contains a weak inverse of s.*

Proof. These are fairly straightforward.

1. Let D be a regular $\mathcal{D}-$class and let $a, b \in D$ and let e be the idempotent lying in $L_a \cap R_b$. Then $ab\mathcal{L}eb\mathcal{R}ee = e$. Hence $ab \in D$. To see that D is a completely simple semigroup, first note that D is a union of groups and hence completely regular. Let $\mathcal{L}_D, \mathcal{R}_D, \mathcal{D}_D$ be the respective Green's relations on D as a semigroup. By [18, Theorem 2.4.2] we have that $\mathcal{L}_D$ and $\mathcal{R}_D$ are exactly the restrictions of $calL$ and $\mathcal{R}$ to D. As D is a $\mathcal{D}$ class, it follows that for any $s, t \in D$, $s\mathcal{D}_D t$ and so there exist $u, v \in D$ such that $s = utv$. Hence $s \in DtD$ and so D has no proper ideals. Then D is a simple, completely regular semigroup so by [18, Theorem 3.3.2] it is completely simple.

2. Let D be the (regular) $\mathcal{D}-$class containing s' and let r be an idempotent such that $r\mathcal{R}ss'$. Then $ss'r = r$ and $rss' = ss'$, and it follows that $s'r \in W(s)$ and $s'\mathcal{R}s'r\mathcal{L}r$.

r	ss'		
$s'r$	s'		

Let $I = S/\mathcal{R}$ and $\Lambda = S/\mathcal{L}$ and as is normal denote the $\mathcal{R}-$classes as R_i $(i \in I)$, the $\mathcal{L}-$classes as L_λ $(\lambda \in \Lambda)$ and the $\mathcal{H}-$class $R_i \cap L_\lambda$ as $H_{i\lambda}$. Suppose $s' \in R_j$ and $ss' \in R_i$. For each $\lambda \in \Lambda$, let $r_{i\lambda}$ be the idempotent in $H_{i\lambda}$ so that we produce a weak inverse of s, $s'_{j\lambda}$, in $H_{j\lambda}$. Let $s'_{j\lambda}s \in L_\mu$, and for each $k \in I$ let $l_{k\mu}$ be the idempotent in $H_{k\mu}$ and note that, using a similar argument to above, $l_{k\mu}s'_{j\lambda} \in W(s)$ and $l_{k\mu}s'_{j\lambda} \in H_{k\lambda}$.

$r_{i\lambda}$			
$l_{k\mu}s'_{j\lambda}$			$l_{k\mu}$
$s'_{j\lambda}$			$s'_{j\lambda}s$

□

The second point allows us to give an equivalent definition of ρ: $s\rho t$ if and only if for every $\mathcal{H}$-class H of S we have

$$W(s) \cap H \neq \emptyset \iff W(t) \cap H \neq \emptyset.$$

Note that part (2) of this lemma and its proof bear similarity to Lemma 2.4.18. We can generalise these results in the following proposition, the proof of which follows the same format.

Proposition 3.3.2. *Let S be a semigroup and D a completely simple subsemigroup of S. If $s \in S$ has a weak inverse $s' \in D$ then every $\mathcal{H}_S$-class of D contains a weak inverse of s.*

The next result is key in what follows.

Lemma 3.3.3. *Let S be a strongly 2−chained semigroup and let $s, t \in S$. For any $\mathcal{D}$-class D of S we have $W(st) \cap D \neq \emptyset$ if and only if $W(s) \cap D \neq \emptyset$ and $W(t) \cap D \neq \emptyset$.*

Proof. Let $s' \in W(s) \cap D$ and suppose $W(t) \cap D \neq \emptyset$. Then $s's$ is an idempotent lying in D. By Lemma 3.3.1(2) t has a weak inverse in every $\mathcal{H}$-class of D, so let t' be the weak inverse of t lying in the $\mathcal{H}$-class of $s's$. Then $t's'stt's' = t'tt's' = t's'$. By Lemma 3.3.1(1) $t's' \in D$ and so $W(st) \cap D \neq \emptyset$.

Conversely let $(st)' \in W(st) \cap D$. Then $t(st)'st(st)' = t(st)'$ and so $t(st)'$ is a weak inverse of s. As $t(st)'\mathcal{L}(st)'$ we have $W(s) \cap D \neq \emptyset$. Similarly we have $(st)'s \in W(t) \cap D \neq \emptyset$. □

Corollary 3.3.4. *Let S be a strongly 2−chained semigroup and let $s, t \in S$. Then $s\rho s^2$ and $st\rho ts$.*

Corollary 3.3.5. *Let S be a strongly 2−chained semigroup. Either S is E−inversive or the set $\{s \in S | W(s) = \emptyset\}$ is an ideal of S.*

We can now prove the following theorem.

Theorem 3.3.6. *Let S be a strongly 2−chained semigroup. Then the relation ρ is a congruence and S/ρ is a semilattice.*

Proof. Let $a, b, c, d \in S$ such that $a\rho b$ and $c\rho d$ and let D be a $\mathcal{D}$-class of S. By Lemma 3.3.3 $W(ac) \cap D \neq \emptyset$ if and only if $W(a) \cap D \neq \emptyset$ and $W(c) \cap D \neq \emptyset$. As $a\rho b$ and $c\rho d$ this latter condition is equivalent to $W(b) \cap D \neq \emptyset$ and $W(d) \cap D \neq \emptyset$ which is in turn equivalent to $W(bd) \cap D \neq \emptyset$ by Lemma 3.3.3. It follows that $ac\rho bd$ and so ρ is a congruence. That S/ρ is a semilattice follows from Corollary 3.3.4. □

We can now prove some results about the structure of S.

Lemma 3.3.7. *Let S be a strongly 2−chained semigroup and let $s, t \in \mathrm{Reg}(S)$. Then $s\rho t$ if and only if $s\mathcal{D}t$.*

Proof. From Lemma 3.3.3 it follows that all of Green's relations are contained in ρ. To see this suppose that $(s, t) \in \mathcal{J}$. Then there exists $u, v \in S^1$ such that $s = utv$. So for every $\mathcal{D}$−class D, if $W(s) \cap D \neq \emptyset$ then $W(utv) \cap D \neq \emptyset$. Hence by Lemma 3.3.3 $W(t) \cap D \neq \emptyset$. By a dual argument we then deduce that $W(s) \cap D \neq \emptyset$ if and only if $W(t) \cap D \neq \emptyset$ and so $(s, t) \in \rho$.

As s is regular it has an inverse which lies in the same $\mathcal{D}$-class and so $W(s) \cap D_s \neq \emptyset$. Hence $W(t) \cap D_s \neq \emptyset$ and by Lemma 3.3.1 there exists $t' \in W(t)$ such that $t'\mathcal{L}s$. By a similar argument there exists $s' \in W(s)$ such that $s'\mathcal{R}t$. Then

$$s\mathcal{L}t'\mathcal{R}t't\mathcal{L}st\mathcal{R}ss'\mathcal{L}s'\mathcal{R}t$$

and so $s\mathcal{D}t$ as required. □

It follows that for each ρ-class S_α either S_α has no regular elements or the regular elements in S_α are contained within a single $\mathcal{D}$-class and hence by Lemma 3.3.1 form a completely simple subsemigroup of S_α. In the latter case S_α is an $E-$inversive semigroup as by definition of ρ each element has a weak inverse lying in the regular $\mathcal{D}$-class. Since each $\mathcal{J}$-class is contained within a ρ-class, it also follows that the regular $\mathcal{J}$-classes of S are exactly the regular $\mathcal{D}$-classes.

Lemma 3.3.8. *Let S be a strongly $2-$chained semigroup and let $x \in S$. Then $x\rho$ is an $E-$inversive subsemigroup of S if and only if $x\rho$ contains a regular element.*

Proof. One way round is obvious. That $x\rho$ is a subsemigroup of S follows from the fact that ρ is a semilattice. Let $y \in x\rho$ be regular. Then there exists $y' \in W(y) \cap D_y$ and so for any $z \in x\rho$ there exists $z' \in W(z) \cap D_y$. Since $D_y \subseteq x\rho$ then $x\rho$ is $E-$inversive. □

Lemma 3.3.9. *Let S be an $E-$inversive semigroup such that* $\mathrm{Reg}(S)$ *is a completely simple semigroup. Then* $\mathrm{Reg}(S)$ *is an ideal of S.*

Proof. Let $s \in \mathrm{Reg}(S)$ and $t \in S$. Let $t' \in W(t)$ and let $\mathcal{H}$ be Green's $\mathcal{H}-$relation on $\mathrm{Reg}(S)$. As $\mathrm{Reg}(S)$ is completely simple every regular $\mathcal{H}$-class contains an inverse of s so we may choose $s' \in V(s)$ such that $s'\mathcal{R}tt'$. Then $t's'stt's' = t's'ss' = t's'$ and $stt's'st = ss'st = st$. Hence st is regular and so $\mathrm{Reg}(S)$ is a right ideal. A dual argument shows $\mathrm{Reg}(S)$ is a left ideal and hence an ideal. □

Lemma 3.3.10. *Let S be a strongly $2-$chained semigroup, let $\alpha \in S/\rho$, let $S_\alpha = \rho^{\natural^{-1}}(\alpha)$ and let $s \in S_\alpha$. Then for all $\beta \in S/\rho$, S_β contains a weak inverse of s if and only if S_β contains a regular element and $\beta \leq \alpha$.*

Proof. Suppose $\beta \leq \alpha$ and S_β contains regular elements. Let $t \in S_\beta$. Then $st \in S_{\alpha\beta} = S_\beta$. As S_β contains a regular element it is $E-$inversive by Lemma 3.3.8, and so there exists $(st)' \in W(st) \cap S_\beta$. Then $t(st)'st(st)' = t(st)'$ so $t(st)' \in W(s) \cap S_\beta$ as required. Conversely, let $s' \in W(s) \cap S_\beta$. Clearly s' is regular, and $s' = s'ss' \in S_{\beta\alpha\beta} = S_{\alpha\beta}$ so $\alpha\beta = \beta$ and hence $\beta \leq \alpha$ as required. □

From the perspective of stratified extensions, we cannot say anything about these semigroups in general. For example, a free semigroup S and a group G are both strongly $2-$chained semigroups, but $\mathrm{Base}(S) = \emptyset$ and $\mathrm{Base}(G) = G$. One condition that allows us to make more precise statements is to require that S is a group-bound semigroup. Note that group-bound implies eventually regular, and if S is a strongly $2-$chained semigroup the converse also holds.

We will show that applying our results to a semigroup which is also group-bound gives the same decomposition as that in Theorem 3.1.1.

If S is a group-bound semigroup and $e \in E(S)$ then let H_e denote the largest subgroup of S containing e. The set of elements s such that $s^n \in H_e$ for some $n \in \mathbb{N}$ is denoted by K_e. This is well defined in the sense that if $s^n \in H_e$ we have $s^m \in H_e$ for all $m > n$ [32, Lemma 1]. It also follows that the sets K_e partition S. In general K_e is not a subsemigroup of S [32, Proposition 7] and in addition in a group bound semigroup $\mathcal{D} = \mathcal{J}$ [32, Lemma 4]. As is usual, J_s will denote the $\mathcal{J}-$class of s.

The following result is important in what follows.

Lemma 3.3.11. *Let S be an eventually regular strongly $2-$chained semigroup and let $e, f \in E(S)$. If $s \in K_e$ then J_e is the greatest $\mathcal{J}$-class containing a weak inverse of s. Moreover, if $e\mathcal{J}f$ and $s \in K_e$ and $t \in K_f$ then $(s,t) \in \rho$.*

Proof. Let S be a semigroup satisfying the conditions stated. As S is eventually regular and strongly $2-$chained then S is group-bound. Let $s \in K_e$ for some idempotent e so that there exists $n \in \mathbb{N}$ such that $s^n \in H_e$. Then

$$(s^n(s^{n+1})^{-1})s(s^n(s^{n+1})^{-1}) = s^n(s^{n+1})^{-1}e = s^n(s^{n+1})^{-1}$$

where $(s^{n+1})^{-1}$ is the inverse of s^{n+1} in H_e. Therefore s has a weak inverse in H_e and hence in J_e.

Now let $s' \in W(s)$ and notice that s' is regular and so lies in a group H_f, say. By Lemma 3.3.1 every $\mathcal{H}$-class of J_f contains a weak inverse of s. Let s'' be a weak inverse of s such that $s''\mathcal{L}s's$ and note that $s'' \in D_f = J_f$. Then as $s''s's = s''$ we have

$$s''s's^2s''s' = s''ss''s' = s''s',$$

so $s''s' \in W(s^2)$ and by Lemma 3.3.1, $s''s' \in J_f$. We can proceed inductively as follows. Let $s''' \in W(s) \cap L_{s''s's^2}$ so that $s'''s''s's^2 = s'''$ and

$$s'''s''s's^3s'''s''s' = s'''ss'''s''s' = s'''s''s'.$$

Hence $s'''s''s' \in W(s^3) \cap J_f$.

We see then that there is a weak inverse of s^n in J_f for any $n \in \mathbb{N}$. In particular, since $s \in K_e$, we can choose n large enough such that $s^n \in H_e \subseteq J_e$. Let s^* be the associated weak inverse of s^n in J_f. Then by Lemma 1.2.6 we have $J_f = J_{s^*} \leq J_{s^n} = J_e$. Consequently if $s \in K_e$ then J_e is the greatest $\mathcal{J}$-class containing a weak inverse of s.

Now let $s \in K_e$ and $t \in K_f$ as in the statement of the lemma. We can assume that s and t are regular. To see this, let $n \in \mathbb{N}$ be the minimum value such that $s^n \in H_e$ and note that if $(s^n)'$ is a weak inverse of s^n then $s^{n-1}(s^n)'$ is a weak inverse of s with $s^{n-1}(s^n)'\mathcal{L}(s^n)'$. This, along with the previous argument, shows that s has a weak inverse in a $\mathcal{J}-$class J if and only if the regular element s^n has a weak inverse in J.

Let J be a $\mathcal{J}-$class containing a weak inverse s' of s. If $t\mathcal{L}s$ then $ts'\mathcal{L}ss'$ and so $ts' \in J$. Then, since J is regular, there exists $r \in J$ such that $ts'r \in J$ is an idempotent, and so $s'rts'r \in J$ is a weak inverse of t. By a similar argument if $t\mathcal{R}s$ there is a weak inverse of t in J and so if $s\mathcal{J}t$ there is a weak inverse of t in J. A dual argument then gives the opposite direction, and the result follows from the definition of ρ. □

Note that each $\mathcal{H}$-class of S contains at most one weak inverse of s: if $s', s^* \in W(s)$ with $s'\mathcal{H}s^*$ then $s's\mathcal{R}s'\mathcal{R}s^*\mathcal{R}s^*s$. As $\mathcal{L}$ is a right congruence we also have $s's\mathcal{L}s^*s$. Since $s's$ and s^*s are idempotents it follows that $s's = s^*s$ and by a similar argument $ss' = ss^*$. Then $s' = s'ss' = s^*ss' = s^*ss^* = s^*$.

Proposition 3.3.12. *Let S be a strongly $2-$chained semigroup. If S is group-bound then S is a semilattice of Archimedean semigroups of the form $K_{J_e} = \bigcup_{f\in E(J_e)} K_f$ for $e \in E(S)$.*

Proof. Let $e \in E(S)$ and define $K_{J_e} = \bigcup_{f\in E(J_e)} K_f$. Let $s, t \in K_{J_e}$ and notice that $s \in K_f, t \in K_g$ for some $f, g \in J_e$, so that by Lemma 3.3.11, $(s, t) \in \rho$. Conversely, if $(s, t) \in \rho$ then there exists $e, f \in E(S)$ such that $s \in K_e \subseteq K_{J_e}, t \in K_f \subseteq K_{J_f}$. By Lemma 3.3.11, J_e is the greatest $\mathcal{J}-$class containing a weak inverse of s and J_f is the greatest $\mathcal{J}-$class containing a weak inverse of t. Since $(s, t) \in \rho$ it easily follows that $J_e = J_f$ and so $s, t \in K_{J_e} = K_{J_f}$. Hence the sets K_{J_e} are the $\rho-$classes and so partition S and since ρ is a semilattice then the result follows.

For each $e, f \in E(S)$ it follows that there exists $g \in E(S)$ such that $K_{J_e}K_{J_f} \subseteq K_{J_g}$. Since $e \in K_{J_e}$ and $f \in K_{J_f}$ then $ef \in K_{J_g}$. In addition there exists a uniquely determined $h \in E(S)$ such that $ef \in K_h \subseteq K_{J_h}$ and so $K_{J_g} = K_{J_h}$.

To see that K_{J_e} is an Archimedean semigroup, let $a, b \in K_{J_e}$. Then there exist $m, n \in \mathbb{N}$ such that $a^m, b^n \in J_e$ and so, as J_e is a regular $\mathcal{D}$-class and hence completely simple by Lemma 3.3.1, we have

$$a^m \in J_e b^n J_e \subseteq K_{J_e} b^n K_{J_e} \subseteq K_{J_e} b K_{J_e}$$

as required. □

Note that a decomposition into a semilattice of Archimedean semigroups is necessarily unique: Let $S = \mathcal{S}[Y; S_\alpha] = \mathcal{S}[Y'; S_a]$ be two Archimedian semilattice decompositions of the semigroup S. If $s, t \in S$ lie in the same subsemigroup S_α where $\alpha \in Y$ and $s \in S_a, t \in S_b$ where $a, b \in Y'$, then there exist $n \in \mathbb{N}$ and $u, v \in S$ such that $s^n = utv$ and $a \leq b$. Similarly $b \leq a$ and so $s, t \in S_a$ and the two semilattices, Y and Y', are isomorphic. We have hence recovered the same decomposition as Shevrin (Theorem 3.1.1) in this case.

It is clear from the above structure that these semigroups are group-bound and since it is straightforward to check that $\mathrm{Reg}(K_{J_e}) = J_e$, then the regular elements form a completely simple subsemigroup.

The converse of Proposition 3.3.12 does not hold in general as an Archimedean semigroup need not contain regular elements and hence a semilattice of Archimedean semigroups may not be group-bound. It is enough, however, to require that each Archimedean semigroup contains a regular element.

Corollary 3.3.13. *Let S be a strongly $2-$chained semigroup. Then S is group-bound if and only if $S = \mathcal{S}[Y; S_\alpha]$ is a semilattice of Archimedean semigroups S_α with $\mathrm{Reg}(S_\alpha) \neq \emptyset$.*

Proof. Clearly if S is group-bound then every subsemigroup contains a regular element. Conversely, let $s \in S$. Then $s \in S_\alpha$ for some α and let $t \in \mathrm{Reg}(S_\alpha)$. Since S_α is an Archimedean semigroup, there exists $n \in \mathbb{N}$ such that $s^n \in S_\alpha t S_\alpha$. Since S_α contains a regular element, by Lemma 3.3.8 it is $E-$inversive. Then by Lemma 3.3.9 $\mathrm{Reg}(S_\alpha)$ is an ideal of S_α and hence $s^n \in \mathrm{Reg}(S_\alpha) \subseteq \mathrm{Reg}(S)$. □

Proposition 3.3.14. *Let S be a semigroup. Any two of the following implies the third.*

1. *S is group-bound*
2. *S is a strongly $2-$chained semigroup*
3. *S is a semilattice of Archimedean semigroups S_α with $\mathrm{Reg}(S_\alpha) \neq \emptyset$.*

Proof. By Corolary 3.3.13 we have (1) and (2) imply (3) and (2) and (3) imply (1). The remaining implication follows from Theorem 3.1.1. □

We now turn our attention to describing the subsemigroups K_{J_e} at each vertex of the semilattice. Since each semigroup contains regular elements, they are all stratified extensions with a base consisting of at least the regular elements. From [32, Proposition 3] each K_{J_e} is an ideal extension of the completely simple semigroup J_e by a nilpotent semigroup. If this nilpotent semigroup is stratified then K_{J_e} is a nil-stratified extension with base J_e. Not every nilpotent semigroup is stratified, however, as illustrated by the following example.[1]

Example 3.3.15. *Let $S = \mathcal{P}(\mathbb{N})$ with multiplication $A \circ B$ given by*

$$A \circ B = \begin{cases} A \cup B & A \cap B \neq \emptyset \\ \emptyset & \textit{otherwise} \end{cases}$$

[1]This example was kindly provided by the anonymous referee of [31].

so that S is a semigroup with zero given by $\emptyset$. Then $A^2 = \emptyset$ for every $A \in S$ so S is a nilpotent semigroup. It is easy to check that for every $i \in \mathbb{N}$, S^i consists of $\emptyset$ and all subsets of $\mathbb{N}$ with at least i elements. Hence $\mathrm{Base}(S)$ *contains $\emptyset$ and every subset of $\mathbb{N}$ with infinite cardinality, so S is not a stratified semigroup (though it is a stratified extension).*

Proposition 3.3.16. *Let S be an eventually regular semigroup such that* $\mathrm{Reg}(S)$ *is completely simple and suppose S is a finitely stratified extension. Then* $\mathrm{Base}(S) \setminus \mathrm{Reg}(S)$ *is either empty or infinite.*

Proof. Suppose $s_0 \in \mathrm{Base}(S) \setminus \mathrm{Reg}(S) \neq \emptyset$. Since S is a finitely stratified extension, $\mathrm{Base}(S)$ is a globally idempotent subsemigroup so $s_0 = s_1 t_1$ for some $s_1, t_1 \in \mathrm{Base}(S)$. If s_1 is regular then as $\mathrm{Reg}(S)$ is an ideal, s_0 is regular giving a contradiction. Further, if $s_1 = s_0$ then $s_0 = s_0 t_1 = s_0 {t_1}^n$ for any $n \in \mathbb{N}$. We can choose n such that ${t_1}^n$ is regular, so s_0 is again regular giving a contradiction. Hence s_1 is an element of $\mathrm{Base}(S) \setminus \mathrm{Reg}(S)$ not equal to s_0. By a similar argument, $s_1 = s_2 t_2$ where $s_2 \in \mathrm{Base}(S) \setminus \mathrm{Reg}(S)$ and s_2 is not equal to s_0 nor s_1. Proceeding inductively we deduce that the set $\{s_0, s_1, s_2, \dots\}$ is an infinite subset of $\mathrm{Base}(S) \setminus \mathrm{Reg}(S)$. □

It follows that any finite strongly $2-$chained semigroup is a semilattice of finitely stratified extensions with completely simple bases.

Theorem 3.3.17. *A semigroup S is a finite strongly $2-$chained semigroup if and only if $S = [Y; S_\alpha]$ is a finite semilattice of finite semigroups S_α where each S_α is a finitely stratified extension of a completely simple semigroup.*

Proof. To see that the converse is true, let $s \in S$ be a regular element, so that there exists α such that $s \in S_\alpha$. Let s' be an inverse of s (within S) with $s' \in S_\beta$ for some β. Then $s = ss's \in S_\alpha S_\beta S_\alpha \subseteq S_{\alpha\beta} \cap S_\alpha$, and so $S_\alpha = S_{\alpha\beta}$. Similarly $s' = s'ss' \in S_{\alpha\beta} \cap S_\beta$ and so $S_\alpha = S_\beta$. It follows that s is regular within S_α and so $s \in \mathrm{Base}(S_\alpha)$ and is therefore $\mathcal{H}-$related to an idempotent as required. □

3.4 Strict extensions of Clifford semigroups

This section makes use of the notation of Clifford and Preston [4, Section 4.4], and in particular that relating to ideal extensions determined by partial homomorphisms. A Clifford semigroup is a completely regular inverse semigroup. It is well known that a Clifford semigroup S decomposes as a semilattice of groups $S = \mathcal{S}[Y; G_\alpha]$. We begin by showing that a strict extension Σ of a Clifford semigroup S has a semilattice structure isomorphic to that of the Clifford semigroup itself.

Lemma 3.4.1. *Let $S = \mathcal{S}[Y; G_\alpha]$ be a Clifford semigroup. An ideal extension of S is strict if and only if it is determined by a partial homomorphism.*

Proof. Let $a, b \in S$ be such that $ax = bx$ and $xa = xb$ for all $x \in S$. As S is a Clifford semigroup $a \in G_\alpha$ and $b \in G_\beta$ for some $\alpha, \beta \in Y$. Let e, f be the identities of G_α, G_β respectively. Then $a = ea = eb$ and so $\alpha \leq \beta$. Similarly, $b = fb = fa$ so $\beta \leq \alpha$ and so $\alpha = \beta$ and $e = f$. Then $a = ea = eb = b$ and hence S is weakly reductive. The result then follows from Theorem 1.3.2. □

Lemma 3.4.2. *Let Σ be a strict extension of a Clifford semigroup $S = \mathcal{S}[Y; G_\alpha]$ by a semigroup T defined by a partial homomorphism $A \mapsto \overline{A}$ and let $\Sigma_\alpha = G_\alpha \cup \{A \in T \setminus \{0\} | \overline{A} \in G_\alpha\}$ for each $\alpha \in Y$. Define a relation $\sim$ on Σ by $s \sim t$ if and only if $s, t \in \Sigma_\alpha$ for some $\alpha \in Y$. Then $\sim$ is a congruence and $\Sigma/\sim$ is a semilattice isomorphic to Y.*

Proof. Clearly $\sim$ is an equivalence relation. To prove $\sim$ is a congruence and that $\Sigma/\sim \cong Y$ we show that $\sim$ is the kernel of the homomorphism $\theta : \Sigma \to Y$ where if $s \in \Sigma_\alpha$ then $\theta(s) = \alpha$. Note that if $A \in T \setminus \{0\}$ then $\theta(A) = \theta(\overline{A})$. We have four cases to consider:

1. If $s, t \in S$ then $\theta(s)\theta(t) = \theta(st)$ follows from the semilattice structure of S.
2. If $s \in S$ and $A \in T \setminus \{0\}$ then $\theta(s)\theta(A) = \theta(s)\theta(\overline{A}) = \theta(s\overline{A}) = \theta(sA)$, where the last two equalities follow from the first case and multiplication in a strict extension respectively.
3. The case for $\theta(A)\theta(s)$ follows similarly.
4. If $A, B \in T \setminus \{0\}$ then $\theta(A)\theta(B) = \theta(\overline{A})\theta(\overline{B}) = \theta(\overline{A}\,\overline{B})$ by the first case. Then if $AB = 0$ in T we have $\theta(AB) = \theta(\overline{A}\,\overline{B})$ and if $AB \neq 0$ in T we have $\theta(AB) = \theta(\overline{AB}) = \theta(\overline{A}\,\overline{B})$. In either case $\theta(A)\theta(B) = \theta(AB)$.

Hence θ is a homomorphism as required and $\sim$ is clearly its kernel. □

Theorem 3.4.3. *Every strict extension Σ of a Clifford semigroup S by a semigroup T is a semilattice of extensions of groups. Conversely, if $\Sigma = \mathcal{S}[Y; \Sigma_\alpha]$ is a semilattice of extensions Σ_α of groups G_α and $S = \bigcup_{\alpha \in Y} G_\alpha$ is an ideal of Σ then Σ is a strict extension of the Clifford semigroup S.*

Proof. By Lemma 3.4.2, Σ is a semilattice of semigroups Σ_α defined via a partial homomorphism $A \mapsto \overline{A}$ from $T \setminus \{0\} \to S$. The restriction of this map to $\Sigma_\alpha \setminus G_\alpha$ gives a partial homomorphism defining the ideal extension Σ_α of the group G_α.

Conversely, let Σ be a semilattice of semigroups Σ_α where each Σ_α is an ideal extension of a group G_α by a semigroup T_α and $S = \bigcup_{\alpha\in Y} G_\alpha$ is an ideal of Σ. It follows that S is a Clifford semigroup and Σ is an ideal extension of S by $T = \Sigma/S$, where T can equivalently be viewed as $\{0\} \cup \bigcup_{\alpha\in Y} T_\alpha \setminus \{0\}$. As G_α has identity e_α the extension Σ_α is determined by the partial homomorphism $A \mapsto Ae_\alpha$ $(= e_\alpha A)$ (Proposition 1.3.1 and Theorem 1.3.2). The union of these maps is then a map $\varphi : T \setminus \{0\} \to S$ such that $\varphi(A) = Ae_\alpha$ for each $A \in T_\alpha \setminus \{0\}$. We will show that φ is a partial homomorphism and that it defines the ideal extension Σ. For clarity, the multiplication determined by φ will be denoted by $\circ$, multiplication within T by $*$, and the original multiplication of the semilattice Σ by juxtaposition.

Let $A, B \in T \setminus \{0\}$ such that $A * B \neq 0$ and assume $A \in T_\alpha$, $B \in T_\beta$ so that $A * B \in T_{\alpha\beta}$. Then $\varphi(A)\varphi(B) = Ae_\alpha(Be_\beta) = A(Be_\beta)e_\alpha = ABe_{\alpha\beta} = \varphi(AB)$ as required.

This partial homomorphism determines an ideal extension of S consisting of the same set Σ under the multiplication $\circ$ defined by

1. $s \circ t = st$
2. $A \circ B = \begin{cases} AB & \text{if } A * B \neq 0 \\ \varphi(A)\ \varphi(B) & \text{otherwise} \end{cases}$
3. $A \circ s = \varphi(A)s$
4. $s \circ A = s\varphi(A)$

where $A, B \in T \setminus \{0\}$ and $s, t \in S$. We show that in all cases, this multiplication is equivalent to the original multiplication on Σ. The first condition and the first part of the second condition do not require proof. For the second part of the second condition, let $A \in T_\alpha \setminus \{0\}$ and $B \in T_\beta \setminus \{0\}$ with $A * B = 0$ so $AB \in G_{\alpha\beta}$. Then

$$A \circ B = \varphi(A)\varphi(B) = Ae_\alpha(Be_\beta) = A(Be_\beta)e_\alpha = ABe_{\alpha\beta} = AB$$

as required. For the third condition, let $A \in T_\alpha \setminus \{0\}$ and $s \in G_\beta$ with $As \in G_{\alpha\beta}$. Then

$$A \circ s = \varphi(A)s = Ae_\alpha(se_\beta) = A(se_\beta)e_\alpha = Ase_{\alpha\beta} = As$$

as required. The fourth condition follows a dual argument. Hence φ determines the extension Σ and so it is a strict extension of S. □

Corollary 3.4.4. *Let Σ be a strict stratified extension of a Clifford semigroup S. Then Σ is a semilattice of stratified extensions of groups.*

Proof. Let Σ be a strict extension of a Clifford semigroup S by a stratified semigroup T. By Theorem 3.4.3, Σ is a semilattice of semigroups Σ_α, each of which is an ideal extension of a group G_α by a subsemigroup of T containing zero. It can be easily verified that such a subsemigroup is also stratified, and hence Σ is a semilattice of stratified extensions of groups. □

The converse of Corollary 3.4.4 does not hold in general as each T_α being a stratified semigroup does not guarantee that T is itself a stratified semigroup. For example, let $Y = \{a, b\}$ with $a \leq b$. For each $\alpha \in Y$ let G_α be a group, T_α a free semigroup with adjoined zero, and Σ_α an ideal extension of G_α by T_α. For $s \in T_a$ and $t \in T_b$ let $st = ts = s$. Along with the fact that $S = G_a \cup G_b$ is an ideal of Σ, this defines a multiplication on the semilattice $\Sigma = \Sigma_a \cup \Sigma_b$. Each T_α is a stratified semigroup so each Σ_α is a stratified extension of a group, however $T = \Sigma/S$ is not stratified, as $\bigcap_{i\geq 1} T^i \cong T_a$. A sufficient, but clearly not necessary, condition under which T will always be stratified is if T is finite.

As an example of the above construction, consider the following. Let $n \in \mathbb{N}$ and let $N = \{1, \ldots, n\}$. Let $S = G_1^0 \times \ldots \times G_n^0$ be a direct product of $0-$groups G_i^0, $i \in N$. For $s = (a_1, \ldots, a_n) \in S$ define $\text{dom}(s) = \{i \in N | a_i \neq 0\}$.

Let $m \in \mathbb{N}$ and define a relation ρ_m on $(\mathbb{N}, +)$ by

$$\rho_m = 1_{\mathbb{N}} \cup \{(x, y) \in \mathbb{N} \times \mathbb{N} | x, y \geq m\}.$$

Then it is easy to check that S is a Clifford semigroup over the semilattice of possible domains $\text{dom}(s)$ with meet given by intersection of sets, and hence a strong semilattice of groups. Further, ρ_m is a congruence on $\mathbb{N}$ and $\mathbb{N}/\rho_m$ is a finite monogenic semigroup with trivial kernel. For simplicity, we shall identify $\mathbb{N}/\rho_m$ with $\{1, \ldots, m\}$, in the obvious way. Let T' be the semigroup of all partial maps from N to $\mathbb{N}/\rho_m$ with binary operation $*$ given by $(f * g)(x) = f(x) + g(x)$ when both are defined and undefined otherwise. Let $I \subseteq T'$ be the set of maps whose image is $\{m\}$. It can be readily seen that I is an ideal of T' and $T = T'/I$ is a nilpotent semigroup.

For each $i \in N$ pick an element $g_i \in G_i$ and let $\alpha_i : T \setminus \{0\} \to G_i^0$ be the partial homomorphism given by

$$\alpha_i(f) = \begin{cases} g_i^{f(i)} & f(i) \text{ is defined} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\alpha : T \setminus \{0\} \to S$ given by $\alpha(f) = (\alpha_1(f), \ldots, \alpha_n(f))$ is a partial homomorphism defining an ideal extension Σ of S by T.

Notice that $s\mathcal{J}t$ if and only if $\text{dom}(s) = \text{dom}(t)$. It follows that the semilattice structure of S is defined in terms of the power set of N (i.e. $\text{dom}(st) = \text{dom}(s) \cap \text{dom}(t)$). Let S_M be the $\mathcal{J}$-class of S with $\text{dom}(s) = M$ for

$s \in S_M$. Then $T_M = \alpha^{-1}(S_M)$ is the set of maps in $T \setminus \{0\}$ whose domain is exactly M. The set $T_M^0 = T_M \cup \{0\}$ is a subsemigroup of T and is nilpotent. The restriction of α to T_M then gives a partial homomorphism from T_M^0 to S_M which defines an ideal extension Σ_M of the group S_M by T_M^0. It can then be shown that Σ is a semilattice of these semigroups Σ_M.

Chapter 4

The multiplicative semigroup of a Dedekind domain

This chapter covers a further example of a semigroup which decomposes as a semilattice of stratified semigroups, namely the multiplicative semigroups of both Dedekind domains and quotients thereof. After a few preliminaries, in Section 4.2 we consider the multiplicative structure of commutative rings in a more general way and describe the $\mathcal{J}-$classes in terms of certain annihilators. We then show that the multiplicative semigroup can be viewed as a semilattice of semigroups. In Section 4.3 we specialise to Dedekind domains and show that the subsemigroups of the semilattice are stratified extensions of groups. In Section 4.4, we consider quotients of Dedekind domains and demonstrate by using prime factorisations of ideals, that the multiplicative structure is a finite Boolean algebra of stratified extensions of groups and give a 'recipe' for constructing both the semilattice and the stratified subsemigroups. Section 4.5 then presents some interesting examples.

4.1 Preliminaries

For details of basic definitions and results in ring theory, we refer the reader to [5] and [6]. An ideal I of a ring R is *prime* if I is a proper ideal and for all $a, b \in R$, if $ab \in I$ then either $a \in I$ or $b \in I$. A *domain* is a ring with no non-zero zero-divisors and a *Dedekind domain* is a commutative domain in which every non-zero proper ideal can be factored into a product of prime ideals. If $I, J \trianglelefteq R$ are ideals of a commutative ring R then we say that I *divides* J and write $I|J$ if and only if there exists $H \trianglelefteq R$ with $I = JH$. Then R is a Dedekind domain if and only if

$$\text{for all } I, J \trianglelefteq R, J \subseteq I \text{ if and only if } I|J.$$

A *principal ideal domain* is a commutative domain in which every ideal is principal. A Dedekind domain is a principal ideal domain if and only if it is a unique factorisation domain. If R is a Dedekind domain and $\{0\} \neq I \trianglelefteq R$ is a non-zero ideal of R then R/I is a principal ideal ring. A Dedekind domain is Noetherian and as such every non-zero, non-unit element can be factorised into a product of irreducible elements. The following elementary properties of ideals will be used implicitly in some of what follows.

Lemma 4.1.1. *Let I, J be ideals of R. Then*

1. $IJ \subseteq I \cap J$.
2. $I \cup J \subseteq I + J$
3. $I \subseteq J \iff I + J = J$.
4. $IJ + J = J$.

4.2 Rings as semilattices of semigroups

Throughout the remainder of this chapter we assume R is a commutative ring with unity.

We will show that the multiplicative semigroup of R is a semilattice of semigroups, and investigate this structure further in Section 4.3. Note that as R is commutative then on the multiplicative semigroup of R, Green's relations, $\mathcal{H} = \mathcal{R} = \mathcal{L} = \mathcal{D} = \mathcal{J}$ coincide. Recall that for any ring R, the quotient $R/(0)$ is naturally isomorphic to R itself via the map $x \mapsto x + (0)$. Some of the results below take place within $R/(0)$ and we could make use of this isomorphism to recast them in R instead. However we have chosen not to do this explicitly.

Let D be the set of all ideals of R. It is easy to see that, under the usual addition and multiplication of ideals, D forms a semiring (i.e. an algebraic structure satisfying all of the usual ring properties with the exception that additive inverses need not exist) with additive identity (0) and multiplicative identity $(1) = R$. Let $\delta : R \to D$ be given by $\delta(x) = (x)$. This is clearly a semiring homomorphism.

Proposition 4.2.1. *The kernel of δ is Green's $\mathcal{J}-$relation and hence $\mathcal{J}$ is a congruence on R.*

Proof. Let $x, y \in R$ such that $\delta(x) = \delta(y)$. Then the principal ideals RxR and RyR are equal, so $x\mathcal{J}y$. Conversely if $x\mathcal{J}y$ then $\delta(x) = RxR = RyR = \delta(y)$. □

Let $x \in R$ and let

$$\overline{x} = \mathrm{Ann}(x) = \{y \in R \mid xy = 0\}$$

be the annihilator of x, which is clearly an ideal of R. Let $R_{\overline{x}} = R/\overline{x}$ and let $U_{\overline{x}}$ be the group of units of this quotient. For $y \in R$, we denote by $[y]_{\overline{x}}$ the coset $y + \overline{x}$ and consider the set $xU_{\overline{x}}$. We will see that this set is essentially the $\mathcal{J}-$class of R containing x. Note that $x\overline{x} = \{xy \mid y \in R, xy = 0\} = \{0\} = (0)$.

Lemma 4.2.2. *Let $x \in R$ and let $V_x = \{u \in R \mid \exists v \in R, xuv = x\}$. Then*

1. *$xU_{\overline{x}} \subseteq xR_{\overline{x}} \subseteq R/(0)$,*
2. *For $[u]_{\overline{x}}, [v]_{\overline{x}} \in R_{\overline{x}}$, $x[u]_{\overline{x}} = x[v]_{\overline{x}}$ if and only if $[u]_{\overline{x}} = [v]_{\overline{x}}$,*
3. *V_x is a submonoid of R and $u \in V_x$ if and only if $[u]_{\overline{x}} \in U_{\overline{x}}$.*

Proof.
1. For any $u \in R$ we have $x[u]_{\overline{x}} = x(u + \overline{x}) = xu + x\overline{x} = xu + (0) \in R/(0)$ and so $xR_{\overline{x}} \subseteq R/(0)$.
2. Let $x[u]_{\overline{x}} = x[v]_{\overline{x}}$. Then $xu - xv \in (0)$ and so $x(u - v) = 0$. Hence $u - v \in \overline{x}$ and so $[u]_{\overline{x}} = [v]_{\overline{x}}$. The converse is obvious.
3. That V_x is a submonoid of R is fairly clear. Suppose that $[u]_{\overline{x}} \in U_{\overline{x}}$ so that there exists $[v]_{\overline{x}} \in U_{\overline{x}}$ such that $[u]_{\overline{x}}[v]_{\overline{x}} = [1]_{\overline{x}}$. Then $uv - 1 \in \overline{x}$ and so $xuv = x$. Conversely, if $u, v \in R$ such that $xuv = x$ then $x[u]_{\overline{x}}[v]_{\overline{x}} = xuv + (0) = x + (0) = x[1]_{\overline{x}}$ and so from part (2) it follows that $[u]_{\overline{x}} \in U_{\overline{x}}$.

□

Theorem 4.2.3. *Let $x, y \in R$. Then $x\mathcal{J}y$ if and only if $y + (0) \in xU_{\overline{x}}$. Consequently the sets $xU_{\overline{x}}$ are the $\mathcal{J}-$classes of $R/(0)$.*

Proof. Suppose $y + (0) \in xU_{\overline{x}}$. Then $y + (0) = x[u]_{\overline{x}} = xu + (0)$ for some $u \in V_x$, and so $y = xu$. Since $[u]_{\overline{x}}$ is a unit, there exists $v \in V_x$ such that $xuv = x$ and so $yv + (0) = xuv + (0) = x + (0)$ and hence $x = yv$ and so $x\mathcal{J}y$.

Now let $x\mathcal{J}y$. Then there exists $u \in R$ such that $xu = y$. Then $y + (0) = xu + (0) = x[u]_{\overline{x}}$. Further, there exists $v \in R$ such that $x = yv$. Then $x = xuv$ and so from Lemma 4.2.2(3), $[u]_{\overline{x}} \in U_{\overline{x}}$ and $y + (0) \in xU_{\overline{x}}$. □

From this and the fact that $\mathcal{J} = \ker(\delta)$ we immediately deduce

Corollary 4.2.4. *Let $x, y \in R$. Then*

$$x\mathcal{J}y \text{ if and only if } xU_{\overline{x}} = yU_{\overline{y}} \text{ if and only if } (x) = (y).$$

Notice that $x\mathcal{J}_R y$ if and only if $(x+(0))\,\mathcal{J}_{R/(0)}\,(y+(0))$ and that the sets $xU_{\overline{x}}$ are the $\mathcal{J}-$classes of $R/(0)$. Consequently, if $x_1+(0), x_2+(0) \in xU_{\overline{x}}$ then

$$x_1 \;\mathcal{J}_R\; x_2 \;\mathcal{J}_R\; x$$

and so since $\mathcal{J}_R = \ker\delta$

$$(x_1) = (x_2) = (x).$$

Note also that since $u \in V_x$ if and only if $[u]_{\overline{x}} \in U_{\overline{x}}$ then $y \in xV_x$ if and only if $y+(0) \in xU_{\overline{x}}$. It follows that xV_x is the image of $xU_{\overline{x}}$ under the natural isomorphism from $R/(0)$ to R and hence is the $\mathcal{J}$-class of R containing x. Our reason for working with $xU_{\overline{x}}$ rather than xV_x is due to Lemma 4.2.2(2): if $u, v \in V_x$ and $xu = xv$ then it is not necessarily the case that $u = v$. For example, if $e \in R$ is a non-unit idempotent then $1, e \in V_e$ and $e1 = ee = e$ but $e \neq 1$. This cancellation property is required for the following theorem.

Theorem 4.2.5. *Let R be a ring and let $x \in R$. The set $xU_{\overline{x}}$ is a subsemigroup of $R/(0)$ if and only if $\delta(x)$ is an idempotent. It is in fact a subgroup which is isomorphic to $U_{\overline{x}}$.*

Proof. Let $x_1+(0), x_2+(0) \in xU_{\overline{x}}$. Then from above, $x_1\mathcal{J}x_2\mathcal{J}x$ and so since $\mathcal{J}$ is a congruence, it follows that $x_1x_2\mathcal{J}x^2$ and hence from Corollary 4.2.4, $x_1x_2+(0) \in x^2U_{\overline{x^2}}$. But if $xU_{\overline{x}}$ is a subsemigroup of $R/(0)$ then $x_1x_2+(0) \in xU_{\overline{x}}$ and so $xU_{\overline{x}} \cap x^2U_{\overline{x^2}} \neq \emptyset$. Consequently $xU_{\overline{x}} = x^2U_{\overline{x^2}}$ and so $\delta(x) = (x) = (x^2) = \delta(x)^2$ by Corollary 4.2.4.

Conversely, if $(x) = (x^2)$ then $x\mathcal{J}x^2$ and so in particular $x+(0)$ and $x^2+(0)$ belong to the same $\mathcal{H}$-class of $R/(0)$, $xU_{\overline{x}}$, and hence this $\mathcal{H}$-class is a group.

If $(x) = (x^2)$ then $x = x^2k$ for some $k \in R$ and so $x \in V_x$. Now define $\phi : xU_{\overline{x}} \to U_{\overline{x}}$ by $\phi(x[u]_{\overline{x}}) = [xu]_{\overline{x}}$. By Lemma 4.2.2(3) this map is well-defined and it is clearly onto. In addition

$$\begin{aligned}\phi(x[u]_{\overline{x}}x[v]_{\overline{x}}) &= \phi(x^2[uv]_{\overline{x}}) = \phi(x[xuv]_{\overline{x}})\\ &= [x^2uv]_{\overline{x}} = ([xu]_{\overline{x}})\,([xv]_{\overline{x}})\\ &= \phi(x[u]_{\overline{x}})\phi(x[v]_{\overline{x}}),\end{aligned}$$

and so ϕ is a morphism. Finally, if $\phi(x[u]_{\overline{x}}) = \phi(x[v]_{\overline{x}})$ then $[xu]_{\overline{x}} = [xv]_{\overline{x}}$ and so $[x]_{\overline{x}}[u]_{\overline{x}} = [x]_{\overline{x}}[v]_{\overline{x}}$. Hence $[u]_{\overline{x}} = [v]_{\overline{x}}$ since $[x]_{\overline{x}} \in U_{\overline{x}}$. Therefore $x[u]_{\overline{x}} = x[v]_{\overline{x}}$ as required. □

For any $I \in D$, let $E_I = \{J \in E(D) \mid J \subseteq I\}$. This set is non-empty since $(0) \in E(D)$ and $(0) \subseteq I$ for any $I \in D$. We claim that

$$\varepsilon(I) = \sum_{J \in E_I} J$$

is the greatest element of E_I with respect to subset inclusion. It is easy to see that $\varepsilon(I) \in D$ and for every $J \in E_I$, $J \subseteq \varepsilon(I)$. It remains to show that $\varepsilon(I) \in E(D)$ and $\varepsilon(I) \subseteq I$.

A general element of $\varepsilon(I)$ has the form $e_1 + \ldots + e_n$ where each e_i lies in some ideal $J_i \in E_I$. Hence every element of $\varepsilon(I)$ is an element of I and so $\varepsilon(I) \subseteq I$. As $J_i \in E(D)$, $e_i \in J_i J_i$. Then

$$e_1 + \cdots + e_n \in J_1 J_1 + \ldots + J_n J_n \subseteq (J_1 + \ldots + J_n)(J_1 + \ldots + J_n).$$

Since each $J_i \in E_I$, $J_1 + \ldots + J_n \subseteq \varepsilon(I)$ so $\varepsilon(I) \subseteq \varepsilon(I)\varepsilon(I)$. The reverse inclusion holds for any ideal and so $\varepsilon(I) \in E(D)$ as required.

This construction clearly describes a well-defined map $\varepsilon : D \to E(D)$. For each $e \in E(D)$ let

$$D_e = \varepsilon^{-1}(e).$$

Proposition 4.2.6. *The multiplicative semigroup of D is a semilattice of semigroups $\mathcal{S}[E(D); D_e]$.*

Proof. As R is commutative, D is also commutative and hence $E(D)$ is a semilattice. If $I \in E(D)$ then clearly I is the greatest element of E_I so $\varepsilon(I) = I$ and ε is a surjection onto the semilattice $E(D)$. It remains to show that ε is a homomorphism.

Let $I, J \in D$ and $K \in E(D)$. If $K \in E_I \cap E_J$ then $K \subseteq I$ and $K \subseteq J$. Then $K = KK \subseteq IJ$ so $K \in E_{IJ}$. Conversely, if $K \in E_{IJ}$ then $K \subseteq IJ$. But $IJ \subseteq I$ so $K \subseteq I$ and $K \in E_I$. In a similar way, $K \in E_J$ and hence $K \in E_I \cap E_J$ and $E_{IJ} = E_I \cap E_J$.

It is easily seen that for all $L \in D$, $E_L = E_{\varepsilon(L)}$ and hence

$$E_{\varepsilon(IJ)} = E_{IJ} = E_I \cap E_J = E_{\varepsilon(I)} \cap E_{\varepsilon(J)} = E_{\varepsilon(I)\varepsilon(J)}.$$

As $E(D)$ is a semilattice $\varepsilon(I)\varepsilon(J) \in E(D)$ and so it follows that $\varepsilon(IJ) = \varepsilon(I)\varepsilon(J)$ as required. □

For each $e \in E(D)$ let $R_e = (\varepsilon\delta)^{-1}(e)$.

Theorem 4.2.7. *The multiplicative semigroup of R is a semilattice of semigroups $\mathcal{S}[\mathrm{Im}(\varepsilon\delta); R_e]$.*

Proof. It is clear that $\varepsilon\delta$ is a surjective homomorphism from R onto its image. Since $\mathrm{Im}(\varepsilon\delta)$ is a subsemigroup of the semilattice $E(D)$, it is also a semilattice. □

We now want to consider the nature of the semigroups D_e and R_e for a specific type of ring.

4.3 Dedekind domains

Let R be a Dedekind domain. We wish in the next section to consider quotients of Dedekind domains but we first make some observations about Dedekind domains in general. While the semigroup structure of these rings is not too complex, it is interesting in its own right. We will show that R is a semilattice of stratified extensions of groups. More specifically, R is a semilattice of two semigroups; its group of units and a stratified extension of the trivial group. Recall that D is the collection of all ideals of R and that $\varepsilon\delta(x)$ is the largest idempotent ideal contained in (x).

Proposition 4.3.1. *The idempotents of D are R and (0).*

Proof. As R is a Dedekind domain, every non-zero proper ideal I of R can be factorised uniquely as a product of prime ideals, so $I = X_1 \dots X_n$ for some prime ideals $X_i \trianglelefteq R$. If $I \in E(D)$ then $I = I^2 = X_1 \dots X_n X_1 \dots X_n$ is another factorisation of I into prime ideals. This contradicts the uniqueness of the factorisation, and so I cannot be idempotent. Hence there are no idempotent non-zero proper ideals of R and so $E(D) = \{R, (0)\}$. □

Clearly for any ideal I of R we have $(0) \subseteq I \subseteq R$ and so $\varepsilon(I) = R$ if $I = R$ and $\varepsilon(I) = (0)$ otherwise. Note also that $R = \varepsilon\delta(1)$ and $(0) = \varepsilon\delta(0)$ so $\varepsilon\delta$ is surjective, and in addition D_R is the trivial semigroup and hence is vacuously a stratified extension of a group.

Proposition 4.3.2. *The subsemigroup $D_{(0)}$ is a stratified extension of the trivial group $\{(0)\}$.*

Proof. Let I be a non-zero proper ideal of R so $I \in D_{(0)}$. Suppose I factors uniquely as a product of n prime ideals, $I = X_1 \dots X_n$. Each X_i is a non-zero proper ideal of R so lies in $D_{(0)}$ and hence $I \in {D_{(0)}}^n$. If $I \in {D_{(0)}}^{n+1}$ then $I = Y_1 \dots Y_{n+1}$ for some $Y_i \in D_{(0)}$. Since I is non-zero, clearly each Y_i is non-zero and so factors as a product of prime ideals. But then I can be written as a product of at least $n+1$ prime ideals, contradicting the uniqueness of the previous factorisation. Hence $I \notin {D_{(0)}}^{n+1}$ and so $I \in {D_{(0)}}^n \setminus {D_{(0)}}^{n+1}$. As this holds for every non-zero ideal of R, we have $\mathrm{Base}(D_{(0)}) = \{(0)\}$ and hence D is a stratified extension of the trivial group. □

Theorem 4.3.3. *Let R be a Dedekind domain. Then R is a semilattice of stratified extensions of groups. In particular, the semilattice is the two element semilattice $R > (0)$, R_R is the group of units of R, and $R_{(0)}$ is a stratified extension of the trivial group.*

Proof. Since $\varepsilon\delta$ is a surjection, by Theorem 4.2.7, R is a semilattice of semigroups $\mathcal{S}[E(D); R_e]$. Clearly $\delta(x) = R$ if and only if x is a unit of R, and so R_R is exactly the group of units of R. For $R_{(0)}$, by Proposition 3.2.10,

$$\text{Base}(R_{(0)}) \subseteq \delta^{-1}(\text{Base}(D_{(0)})) = \delta^{-1}((0)) = \{0\}.$$

Since 0 is idempotent we have $0 \in \text{Base}(R_{(0)})$ and so $\text{Base}(R_{(0)}) = \{0\}$ and hence $R_{(0)}$ is a stratified extension of the trivial group. □

Note that $\text{Base}(R_{(0)}) = \delta^{-1}(\text{Base}(D_{(0)}))$. In general the layers within the stratified structure of D_0 and within R_0 will not be the same. However,

Proposition 4.3.4. *Let R be a Dedekind domain and let $i > 1$. Then ${R_{(0)}}^i = \delta^{-1}({D_{(0)}}^i)$ if and only if R is a principal ideal domain.*

Proof. Note that by Proposition 3.2.10, ${R_{(0)}}^i \subseteq \delta^{-1}({D_{(0)}}^i)$ is always true for any commutative ring R.

Let R be a principal ideal domain and let $x \in \delta^{-1}({D_{(0)}}^i)$. Then $\delta(x) = (x)$ can be factorised as a product of i principal ideals $(x) = (x_1)\dots(x_i) = (x_1 \dots x_i)$ with each $(x_j) \in D_{(0)}$. Hence for $1 \le j \le i$, $x_j \in R_{(0)}$ and so $x_1 \dots x_i \in R_{(0)}$. Since $\delta(x) = \delta(x_1 \dots x_i)$ we have $x \mathcal{J} x_1 \dots x_i$ and so $x = x_1 \dots x_i u$ for some $u \in R$. Since

$$\varepsilon\delta(x_i u) = \varepsilon\delta(x_i)\varepsilon\delta(u) = (0)\varepsilon\delta(u) = (0)$$

then $x_i u \in R_{(0)}$ and it follows that $x \in {R_{(0)}}^i$.

For the converse, note that since R is a Dedekind domain it is Noetherian and hence every non-zero, non-unit element can be factorised into a product of irreducible elements. It follows that every irreducible element of R is prime if and only if R is a unique factorisation domain and hence a principal ideal domain. Hence if R is not a principal ideal domain there exists some $x \in R_{(0)}$ such that x is irreducible but not prime (recall that R_R consists of units which are not irreducible). Since x is irreducible it cannot be written as a product of two non-unit elements of R and hence $x \notin {R_{(0)}}^2$. Since x is not prime, (x) is not a prime ideal and so has a unique factorisation as a product of prime ideals $X_1 \dots X_n$ for some $n > 1$. In particular, $(x) \in {D_{(0)}}^2$ and so ${R_{(0)}}^2 \neq \delta^{-1}({D_{(0)}}^2)$. It is then an easy matter to extend this for all $i \geq 2$. □

Notice then that when R is a principal ideal domain, the i-th layer of $R_{(0)}$ is the preimage of the i-th layer of $D_{(0)}$.

Corollary 4.3.5. *Let R be a Dedekind domain. An element $x \in R$ is prime if and only if (x) lies in the first layer of $D_{(0)}$. Additionally, if $x \in R$ is prime then x lies in the first layer of $R_{(0)}$. The converse holds only when R is a PID.*

4.4 Quotients of Dedekind domains

Let S be a Dedekind domain, $A \trianglelefteq S$ and let $R = S/A$. We will demonstrate that R is a semilattice of stratified extensions of groups. Note that when $A = (0), R \cong S$ and this case has effectively been considered in Section 4.3. When $A = S$ then $R = \{0\}$ and this situation is trivial. Hence we shall assume in what follows that $S \neq A \neq (0)$.

Let D_A be the set of ideals of S containing A and define an operation $*$ on D_A such that $X * Y = XY + A$. Then

$$(X * Y) * Z = (XY + A) * Z = (XY + A)Z + A = XYZ + AZ + A = XYZ + A$$

and similarly $X * (Y * Z) = XYZ + A$ and so $*$ is associative. Note that for every $I \in D_A, I + A = I$.

The following is well known (see for example [5, Third Isomorphism Theorem, Page 303]), but as the result is normally presented as an isomorphism $D_A \to D$, we feel the proof is useful to present here.

Lemma 4.4.1. *The map $\Phi : D \to D_A$ given by $\Phi(I) = \bigcup_{X \in I} X$ is an isomorphism.*

Proof. Note that $x + A \in I$ if and only if $x \in \Phi(I)$.

We first show Φ is well defined. Let $x, y \in \Phi(I)$. Then $x + A, y + A \in I$ so $x + y + A \in I$ and hence $x + y \in \Phi(I)$. Similarly for any $z \in S, z + A \in R$ so $xz + A \in I$ and $xz \in \Phi(I)$ and hence $\Phi(I)$ is an ideal of S. Since $0 + A \in I$, $A \subseteq \Phi(I)$ and so $\Phi(I) \in D_A$.

To see that Φ is injective, if $\Phi(I) = \Phi(J)$ then we have

$$x + A \in I \Leftrightarrow x \in \Phi(I) \Leftrightarrow x \in \Phi(J) \Leftrightarrow x + A \in J$$

so $I = J$. For surjectivity, let I be an ideal of S containing A. Then $J = \{x + A \mid x \in I\}$ is clearly an ideal of R and $\Phi(J) = I$.

Finally we show that Φ is a homomorphism. If $x \in \Phi(I) * \Phi(J) = \Phi(I)\Phi(J) + A$ then $x = x_1y_1 + \ldots + x_ny_n + a$ where $a \in A$, $x_i + A \in I$ and $y_i + A \in J$ for each $i \in \{1, \ldots, n\}$. Then $x_1y_1 + \ldots + x_ny_n + A \in IJ$ so $x_1y_1 + \ldots + x_ny_n \in \Phi(IJ)$ and $x_1y_1 + \ldots + x_ny_n + a \in \Phi(IJ) + A = \Phi(IJ)$. Hence $\Phi(I) * \Phi(J) \subseteq \Phi(IJ)$. For the

reverse inclusion let $x \in \Phi(IJ)$ so $x + A \in IJ$ and
$x + A = (x_1 + A)(y_1 + A) + \ldots + (x_n + A)(y_n + A) = x_1y_1 + \ldots + x_ny_n + A$. Then
$x - (x_1y_1 + \ldots + x_ny_n) \in A$ so $x = x_1y_1 + \ldots + x_ny_n + a$ for some $a \in A$. Hence
$x \in \Phi(I)\Phi(J) + A = \Phi(I) * \Phi(J)$. Therefore $\Phi(I) * \Phi(J) = \Phi(IJ)$ and Φ is a homomorphism. □

Notice that if $K \in D_A$ then $\Phi^{-1}(K) = K/A$.

Since S is a Dedekind domain, every nonzero proper ideal factors into a product of prime ideals. Hence for all $I \in D, I \neq R$, $\Phi(I) = P_1P_2 \ldots P_n$ for some prime ideals P_i of S. Then $\Phi(I) = \Phi(I) + A = P_1 \ldots P_n + A$ since $A \subseteq \Phi(I)$. For each $1 \leq i \leq n$,
$A \subseteq \Phi(I) \subseteq P_i$ so $P_i \in D_A$. Hence
$\Phi(I) = P_1 \ldots P_n + A = P_1 * \ldots * P_n = \Phi(X_1) * \ldots * \Phi(X_n)$ where
$X_i = \Phi^{-1}(P_i) = P_i/A$ and so $I = X_1 \ldots X_n$. Note that this is not necessarily a unique factorisation, as for example (4) as an ideal of $\mathbb{Z}_{12}$ can be written as $(2)(2)$ or as $(2)(2)(2)$. It is however a factorisation into prime ideals.

Lemma 4.4.2. *The ideal I is a prime ideal of R if and only if $\Phi(I)$ is a prime ideal of S.*

Proof. Suppose $\Phi(I)$ is a prime ideal of S and let $(x + A)(y + A) = xy + A \in I$. Then $xy \in \Phi(I)$ and so without loss of generality $x \in \Phi(I)$. Hence $x + A \in I$ and so I is a prime ideal. Conversely, suppose I is a prime ideal of R and let $xy \in \Phi(I)$. Then $xy + A \in I$ so without loss of generality $x + A \in I$ and hence $x \in \Phi(I)$ so $\Phi(I)$ is prime. □

The following lemma shows that the factorisation of I into $\Phi^{-1}(P_1) \ldots \Phi^{-1}(P_n)$ is a *minimal prime factorisation*, in the sense that any other prime factorisation of I must include each of these factors.

Lemma 4.4.3. *Let $I \in D$ be such that $\Phi(I)$ has a unique prime factorisation $P_1 \ldots P_n$. If $X_1 \ldots X_m$ is a prime factorisation of I then $m \geq n$ and, up to reordering factors, $X_i = \Phi^{-1}(P_i)$ for $i \in \{1, \ldots, n\}$.*

Proof. By definition,

$$\Phi(I) = \Phi(X_1) * \ldots * \Phi(X_m) = \Phi(X_1) \ldots \Phi(X_m) + A$$

so $\Phi(X_1) \ldots \Phi(X_m) \subseteq \Phi(I)$ and hence $\Phi(I)$ divides $\Phi(X_1) \ldots \Phi(X_m)$ as S is a Dedekind domain. Then $P_1 \ldots P_nQ = \Phi(X_1) \ldots \Phi(X_m)$ for some ideal Q of S so, by uniqueness of prime factorisations in S, we have $m \geq n$ and, reordering if necessary, $P_i = \Phi(X_i)$ for each $i \in \{1, \ldots, n\}$. Applying Φ^{-1} to each equality then gives the desired result. □

Suppose A has prime factorisation $P_1^{e_1} \dots P_n^{e_n}$ ($e_i > 0$). By definition, any ideal $\Phi(I) \in D_A$ has $A \subseteq \Phi(I)$ and so $\Phi(I)$ divides A and hence $\Phi(I) = P_1^{f_1} \dots P_n^{f_n}$ where $0 \leq f_i \leq e_i$. In particular, if P is a prime ideal of S then $A \subseteq P$ if and only if $P = P_i$ for some $i \in \{1, \dots, n\}$. Let

$$A_i = \Phi^{-1}(P_i)$$

for each $i \in \{1, \dots, n\}$. Then $A_1, \dots, A_n$ are precisely the prime ideals of R and any $I \in D$ has minimal prime factorisation $A_1^{f_1} \dots A_n^{f_n}$, for some $f_i \geq 0$. Notice that the primes $A_1, \dots, A_n$ are unique with respect to this construction, by Lemma 4.4.3.

Note here that we adopt the convention $P_1^0, \dots, P_n^0 = S$ and $A_1^0, \dots, A_n^0 = R$, i.e. that the empty powers of primes are the identity elements of D_A and D respectively.

Lemma 4.4.4. *Let $I \in D$. If I has prime factorisation $A_1^{g_1} \dots A_n^{g_n}$ then the minimal prime factorisation of I is given by $A_1^{f_1} \dots A_n^{f_n}$ where $f_i = \min(e_i, g_i)$. Hence a prime factorisation is minimal if and only if $0 \leq g_i \leq e_i$ for all $i \in \{1, \dots, n\}$.*

Proof. Let $I = A_1^{g_1} \dots A_n^{g_n}$. Then

$$\begin{aligned} \Phi(I) &= P_1^{g_1} * \dots * P_n^{g_n} \\ &= P_1^{g_1} \dots P_n^{g_n} + A \\ &= P_1^{g_1} \dots P_n^{g_n} + P_1^{e_1} \dots P_n^{e_n}. \end{aligned}$$

Note that we can factorise

$$P_1^{g_1} \dots P_n^{g_n} + P_1^{e_1} \dots P_n^{e_n} = P_1^{\min(g_1, e_1)} \dots P_n^{\min(g_n, e_n)} (J + K)$$

for some ideals J, K of S such that J and K have no common factors. Then, as $J, K \subseteq J + K$ by Lemma 4.1.1, $J + K$ is a common factor of J and K and hence $J + K = S$. Then

$$P_1^{g_1} \dots P_n^{g_n} + P_1^{e_1} \dots P_n^{e_n} = P_1^{\min(g_1, e_1)} \dots P_n^{\min(g_n, e_n)}$$

is the unique prime factorisation of $\Phi(I)$ so $A_1^{\min(g_1, e_1)} \dots A_n^{\min(g_n, e_n)}$ is the minimal prime factorisation of I. □

Corollary 4.4.5. *Let $I = A_1^{i_1} \dots A_n^{i_n}$ and $J = A_1^{j_1} \dots A_n^{j_n}$ be minimal prime factorisations of $I, J \in D$. The minimal prime factorisation of IJ is*

$$A_1^{\min(i_1 + j_1, e_1)} \dots A_n^{\min(i_n + j_n, e_n)}.$$

We can now apply our methods from Section 4.2, and in particular Proposition 4.2.6, to find the semilattice structure of D.

Lemma 4.4.6. *Let $I \in D$ with minimal prime factorisation $A_1^{f_1} \dots A_n^{f_n}$. Then $I \in E(D)$ if and only if $f_i \in \{0, e_i\}$ for all $i \in \{1, \dots, n\}$.*

Proof. Let $I \in D$ have minimal prime factorisation $A_1^{f_1} \dots A_n^{f_n}$ so I^2 has minimal prime factorisation $A_1^{\min(2f_1, e_1)} \dots A_n^{\min(2f_n, e_n)}$. If $f_i = 0$ then $\min(2f_i, e_i) = 0 = f_i$ and if $f_i = e_i$ then $\min(2f_i, e_i) = e_i = f_i$ so $I^2 = A_1^{f_1} \dots A_n^{f_n} = I$.

Conversely, if $I^2 = I$ then $\min(2f_i, e_i) = f_i$ for all $i \in \{1, \dots, n\}$. Then if $f_i \leq e_i/2$ we have $2f_i = f_i$ so $f_i = 0$ and if $f_i > e_i/2$ we have $f_i = e_i$. Hence $f_i \in \{0, e_i\}$ for all $i \in \{1, \dots, n\}$. □

Let $N = \{1, \dots, n\}$. The previous lemma shows that an idempotent e is entirely determined by which A_i have a non-zero power f_i, and hence we have a bijection $\Lambda : E(D) \to \mathcal{P}(N)$ given by $\Lambda(e) = \{i \in N | f_i = e_i\}$. If $\mathcal{P}(N)$ is equipped with the operation of union of sets it then becomes a semilattice and Λ can easily seen to be an order isomorphism.

Note that for every $I \in D$ there exists $e \in E(D)$ such that I has minimal prime factorisation $\prod_{i \in \Lambda(e)} A_i^{f_i}$ where $0 < f_i \leq e_i$ for all $i \in \Lambda(e)$. In fact $I \in D_e$ if and only if its minimal prime factorisation can be written in this way. To see this, it is sufficient to observe that for any prime ideal A_i we have $\varepsilon(A_i) = A_i^{e_i}$ as ε is a homomorphism.

Proposition 4.4.7. *Let D_e be a subsemigroup of D for some $e = \prod_{j \in \Lambda(e)} A_j^{e_j} \in E(D)$. Let $I \in D_e$ with minimal prime factorisation $\prod_{j \in \Lambda(e)} A_j^{f_j}$ for $0 < f_j \leq e_j$ and suppose that $I \neq e$. Then for each $i \geq 1$, $I \in {D_e}^i$ if and only if $\min\{f_j | f_j \neq e_j\} \geq i$.*

Note that $\{f_j | f_j \neq e_j\}$ is non-empty since $I \neq e$. Since e is idempotent, $e \in {D_e}^i$ for all $i \in \mathbb{N}$.

Proof. By definition, $\prod_{j \in \Lambda(e)} A_j$ divides every element of D_e so $\prod_{j \in \Lambda(e)} A_j^i$ divides every element of ${D_e}^i$. Then for every $I \in {D_e}^i$ there exists a prime factorisation $\prod_{j \in \Lambda(e)} A_j^{g_j}$ with $g_j \geq i$. By Lemma 4.4.4 the minimal prime factorisation of I, $\prod_{j \in \Lambda(e)} A_j^{f_j}$, has $f_j = \min(e_j, g_j)$ so we have $f_j = g_j \geq i$ for every $f_j \neq e_j$ and hence $\min\{f_j | f_j \neq e_j\} \geq i$.

For the converse, suppose I has minimal prime factorisation $\prod_{j \in \Lambda(e)} A_j^{f_j}$ such that $\min\{f_j | f_j \neq e_j\} \geq i$. Then for each $j \in \Lambda(e)$ either $f_j = e_j$ or $i \leq f_j < e_j$. Let $g_j = \max(f_j, i)$ and $J = \prod_{j \in \Lambda(e)} A_j^{g_j}$. Then $J = \left(\prod_{j \in \Lambda(e)} A_j\right)^{i-1} \left(\prod_{j \in \Lambda(e)} A_j^{g_j - (i-1)}\right)$ so, as $g_j - (i-1) > 0$, $J \in {D_e}^i$. If $i < f_j < e_j$ then $g_j = f_j < e_j$. Otherwise, $g_j > f_j = e_j$. In either case, $\min(g_j, e_j) = f_j$ and hence I and J have the same minimal prime factorisation, so $I = J$ and $I \in {D_e}^i$. □

Corollary 4.4.8. *The ideal* $I = \prod_{j\in\Lambda(e)} A_j^{f_j}$ *with* $0 < f_j \leq e_j$ *lies in the i-th layer of* D_e, ${D_e}^i \setminus {D_e}^{i+1}$, *if and only if* $\min\{f_j | f_j \neq e_j\} = i$.

Corollary 4.4.9. *For* $e \in E(D)$, $\operatorname{Base}(D_e) = \{e\}$ *and the subsemigroup* D_e *is a stratified semigroup with zero.*

We can now easily prove the main theorem.

Theorem 4.4.10. *Let S be a Dedekind domain and* $A \trianglelefteq S$ *an ideal of S. If* $R = S/A$ *and D is the semiring of ideals of R, then the multiplicative semigroup of R is a semilattice* $\mathcal{S}[E(D); R_e]$ *of stratified extensions of groups.*

Proof. If $A = \{0\}$ then the result follows from Theorem 4.3.3, while if $A = S$ the result is trivial. Henceforth, assume that $\{0\} \neq A \neq S$.

That R is the given semilattice follows immediately from Theorem 4.2.7 and the observation that as every ideal of a quotient of a Dedekind domain is principal, the map $\varepsilon\delta$ is a surjection.

Let $e \in E(D)$ and consider $\operatorname{Base}(R_e)$. By Proposition 3.2.10,

$$\operatorname{Base}(R_e) \subseteq \delta^{-1}(\operatorname{Base}(D_e)) = \delta^{-1}(\{e\}).$$

Since every ideal of R is principal, there exists some $x \in R$ such that $\delta(x) = (x) = e$ and hence $\operatorname{Base}(R_e) \subseteq \delta^{-1}(\{e\}) = xU_{\overline{x}}$. By Theorem 4.2.5, $xU_{\overline{x}}$ is a group and hence by Corollary 3.2.2, $xU_{\overline{x}} \subseteq \operatorname{Base}(R_e)$ and so $\operatorname{Base}(R_e) = xU_{\overline{x}}$ and R_e is a stratified extension of a group. □

It is clear from Proposition 3.2.10 that ${R_e}^i \subseteq \delta^{-1}({D_e}^i)$. In fact, we have equality.

Proposition 4.4.11. *Let* $e = \prod_{j\in\Lambda(e)} A_j^{e_j} \in E(D)$. *Then* ${R_e}^i = \delta^{-1}({D_e}^i)$.

Proof. It remains to show that $\delta^{-1}({D_e}^i) \subseteq {R_e}^i$. Let $x \in R_e$ be such that $\delta(x) \in {D_e}^i$. Then $\delta(x)$ has minimal prime factorisation $\prod_{j\in\Lambda(e)} A_j^{f_j}$ where $\min\{f_j | f_j \neq e_j\} = i$. Let $g_j = \max(f_j, i)$. Then $\prod_{j\in\Lambda(e)} A_j^{g_j}$ is a prime factorisation of $\delta(x)$ with $g_j \geq i$ for every $j \in \Lambda(e)$.

Since S is a Dedekind domain, every ideal of R is principal so there exists some $a_j \in R$ such that $\delta(a_j) = (a_j) = A_j$ for every $j \in \Lambda(e)$. Let $y = \prod_{j\in\Lambda(e)} a_j^{g_j}$. Clearly $\delta(y) = \delta(x)$ and so $x\mathcal{J}y$ and hence $x = yu$ for some $u \in R$. Then

$$x = \left(\prod_{j\in\Lambda(e)} a_j\right)^{i-1} \left(u \prod_{j\in\Lambda(e)} a_j^{g_j-(i-1)}\right).$$

Clearly $\varepsilon\delta(\prod_{j\in\Lambda(e)} a_j) = e$, so if $\varepsilon\delta(u\prod_{j\in\Lambda(e)} a_j^{g_j-(i-1)}) = e$ then $x \in {R_e}^i$ as required. Suppose otherwise, so $\varepsilon\delta(u\prod_{j\in\Lambda(e)} a_j^{g_j-(i-1)}) = f$ for some $f \in E(D)$ with $f \neq e$. As each $g_j - (i-1) > 0$, $\varepsilon\delta(\prod_{j\in\Lambda(e)} a_j^{g_j-(i-1)}) = e$ and so e divides f and hence $ef = f$. But then $\varepsilon\delta(x) = e^{i-1}f = f$, a contradiction. Hence $\varepsilon\delta(u\prod_{j\in\Lambda(e)} a_j^{g_j-(i-1)}) = e$ and so $x \in {R_e}^i$ and $\delta^{-1}({D_e}^i) \subseteq {R_e}^i$. □

Corollary 4.4.12. *Let $x \in R_e$. Then x lies in the i-th layer of R_e if and only if $\delta(x)$ lies in the i-th layer of D_e.*

Corollary 4.4.13. *Let $A \trianglelefteq S$ be a non-zero proper ideal with prime factorisation $P_1^{e_1} \dots P_n^{e_n}$ and $A_i = P_i/A$. Then*

1. *R_e is a group if and only if $e = \prod_{i\in\Lambda(e)} A_i$.*
2. *R is a semilattice of groups if and only if $e_1 = \dots = e_n = 1$.*
3. *R is a semilattice of groups if and only if $R_{(0)}$ is a group.*
4. *$E(D)$ is a chain if and only if $n = 1$, in which case it is the two element semilattice.*

Here (1)-(3) follow from the observation that if e is square-free then $D_e = \{e\}$ and hence $\mathrm{Base}(R_e) = R_e$. An interesting consequence of these results is that S/A is a field if and only if A is prime.

Proposition 4.4.14. *Let R be a quotient of a Dedekind domain. Then R is a strong semilattice of semigroups if and only if it is a semilattice of groups.*

Proof. It is well known (see, for example, [18, Theorem 4.2.1]) that a semilattice of groups is a strong semilattice. For the converse, suppose R is a strong semilattice of semigroups. For any $e \in E(D)$ we have $R \geq e$ and so there exists a morphism $\phi_{R,e} : R_R \to R_e$ such that $xy = \phi_{R,e}(x)y$ for any $x \in R_R$ and $y \in R_e$. Since $1 \in R_R$ we have $x = 1x = \phi_{R,e}(1)x$ for every $x \in R_e$. Then $x \in {R_e}^2$ so ${R_e}^2 = R_e$ and hence R_e is a group. As this holds for every $e \in E(D)$, R is a semilattice of groups. □

We can summarise the construction of the semilattice of semigroups with this short 'recipe'. First we note that we can reduce the amount of calculation required by making use of the following result.

Proposition 4.4.15. *Let S be a Dedekind domain, A an ideal of S, and $R = S/A$. For all $x \in R$, $xV_x = xU$ where U is the group of units of R.*

Proof. Note that when $A = S$ the result is trivial and when $A = (0)$ we have $V_x = U$ by cancellativity as R is a domain. We assume henceforth that A is a non-zero proper ideal.

It is readily apparent that $U \subseteq V_x$ for all $x \in R$ and hence $xU \subseteq xV_x$. For the reverse inclusion, we note that it is well known that R is then a principal ideal ring and so, by [19, Lemma 2.1 and Theorem 12.3], if $(a) = (b)$ then $a = bu$ for some $u \in U$. It is easy to see that if $y \in xV_x$ then $(x) = (y)$ and so $y \in xU$ as required. □

Let S be a Dedekind domain and $A \trianglelefteq S$. If $A = S$ then $R = S/A$ is the trivial ring. If $A = (0)$ then $R \cong S$ and so by Theorem 4.3.3 we have a semilattice of two semigroups. One is R_R, the group of units, while the other is $R_{(0)}$, a stratified extension of the trivial group. In the latter case, the elements of layer i are precisely those which can be factorised as a product of i irreducible elements.

Otherwise, let $(0) \neq A \neq S$ be a proper non-zero ideal of S and let $A = P_1^{e_1} \dots P_n^{e_n}$ be the unique factorisation of A into a product of prime ideals of S. Then the semilattice is order isomorphic to $\mathcal{P}(N)$ and each subset $K \subseteq N$ is associated with an idempotent $e \in E(D)$. The subsemigroup R_e is then a stratified extension of a group where the group is $\text{Base}(R_e) = \delta^{-1}(e)$.

To calculate R_e and $\text{Base}(R_e)$ in a practical setting, we proceed as follows. First, the two easy cases are when $K = \emptyset$, in which case $e = (1 + A)$ and Base_{R_e} is the group of units of R, while if $K = N$ then $e = (0 + A)$ and the group consists of only the zero of R. Suppose now that $\emptyset \subset K \subset N$ and let $A_i = \Phi^{-1}(P_i) = P_i/A$. Then $e = \prod_{i \in K} A_i^{e_i}$ and since R is a principal ideal ring, if $e = (x + A)$ then $\text{Base}(R_e) = \{xv + A\}$ where $v + A \in V_{x+A}$, and so by Proposition 4.4.15, $\text{Base}(R_e) = (x + A)R_R$. To determine the stratified structure of R_e, note that if $e_i = 1$ for all $i \in K$ then R_e is a group and so there are no layers. If at least one of the $e_i > 1$ and if for a given subset K and a collection $f_i, i \in K$

$$\prod_{i \in K} A_i^{f_i} = (y + A)$$

and where $0 < f_i \leq e_i$ is such that $j = \min\{f_i \mid f_i \neq e_i, i \in K\}$, then $\{yv + A \mid v + A \in V_{y+A}\} = (y + A)R_R$ is a subset of the j-th layer, and moreover the j-th layer consists of the union of all such subsets. Note that if $A_i = (a_i + A)$ then $y + A = \prod_{i \in K}(a_i^{f_i} + A)$.

4.5 Examples

In this short section we illustrate the above theory by considering a number of examples of Dedekind domains and examining the semilattice and stratified structure of the multiplicative semigroup of both the domain and of a typical quotient of the domain.

At the more trivial end of the spectrum, suppose that $S = F$, a field. Then every non-zero element is a unit, so we have $S_{(0)} = \{0\}$ and $S_S = F^{\times}$. In other words, the multiplicative semigroup of a field is simply a group with zero as expected.

4.5.1 The integers

As a more interesting example, let $S = \mathbb{Z}$, the ring of integers. For any $n \in \mathbb{Z}$, the sets $nU_{\overline{n}}$ are (isomorphic to) $\{n, -n\}$ and the units in $\mathbb{Z}$ are of course ± 1 and so S_S is the two element group. We know from Theorem 4.3.3 that $S_{(0)}$ is a stratified extension of the trivial group and the layered structure of $S_{(0)}$ is then easy to establish. The first layer of $S_{(0)}$ consists of every prime integer p. The second layer contains all products pq of exactly 2 (not necessarily distinct) primes p and q, and in general, layer n consists of all products of exactly n (not necessarily distinct) primes.

Given that $\mathbb{Z}$ is a principal ideal domain, then all ideals of $\mathbb{Z}$ are of the form (n) for some $n \in \mathbb{Z}$. If $R = S/(n)$ then of course $R = \mathbb{Z}_n$ the ring of integers modulo n. To reduce pedantry we will assume that $\mathbb{Z}_n = \{1, \ldots, n\}$. We know from Theorem 4.4.10 that R is a semilattice of stratified extensions of groups, $\mathcal{S}[E(D); R_e]$ and that $E(D) \cong \mathcal{P}(K)$ where $K = \{1, \ldots, k\}$ and where $n = p_1^{e_1} \ldots p_k^{e_k}$ is the prime factorisation of n in $\mathbb{Z}$.

First, note that if $(e) \in E(D)$ then we can assume, without loss of generality, that $e = \prod_{i \in I} p_i^{e_i} \in \mathbb{Z}_n$ where $I = \Lambda((e)) \in \mathcal{P}(K)$. The base of R_e is $\text{Base}(R_e) = eU_{\overline{e}}$ and using Theorem 4.2.5 and Lemma 4.2.2, we deduce that $\mathbb{Z}_n/(\overline{e}) \cong \mathbb{Z}_{n/e}$ and that

$$eU_{\overline{e}} \cong U_{n/e}$$

where $U_{n/e}$ is the group of units in $\mathbb{Z}_{n/e}$. If (e) is square-free then $R_e = U_{n/e}$ otherwise R_e is a stratified extension of $U_{n/e}$ with height $m = \max\{e_j | j \in \Lambda(e)\} - 1$. In this case, the structure of the individual layers of R_e is more complicated to describe is general, but essentially if x is in the i-th layer of R_e, $1 \leq i \leq m$, then

$$x = \prod_{j \in \Lambda(e)} p_j^{g_j} u$$

where $u \in U_n$ and $0 < g_j \leq e_j$ and $\min\{g_j | g_j \neq e_j\} = i$. Note that $\{g_j | g_j \neq e_j\} \neq \emptyset$ as otherwise $x \in \text{Base}(R_e)$.

As an example, if $n = 12 = 2^2 \times 3$, then

$$E(D) = \{(12), (4), (3), (1)\}$$

and we have four subsemigroups

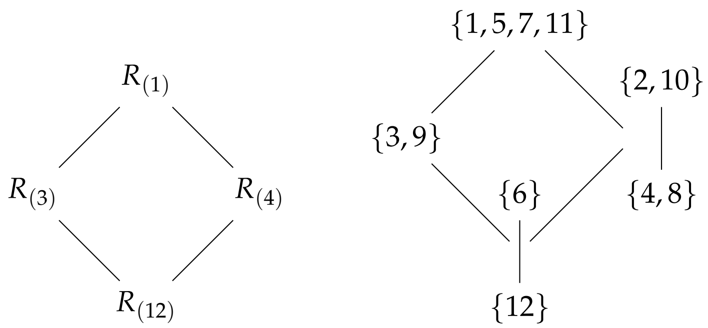
$R_{(12)} = \{6, 12\}$ where $\text{Base}(R_{(12)}) = \{12\}$.

$R_{(4)} = \{2, 4, 8, 10\}$ where $\text{Base}(R_{(4)}) = \{4, 8\}$ and $\{2, 10\}$ forms layer 1.

$R_{(3)} = \{3, 9\}$ which is a group.

$R_{(1)} = \{1, 5, 7, 11\}$ which is the group of units mod 12.

The semilattice structure can be pictured as



Notice that the semilattice will always be a finite Boolean algebra and the stratification structure is wholly dependent on the prime power factorisation of n.

4.5.2 The $p-$adic integers

Let S be the p-adic integers. There are a number of ways to view $p-$adic numbers but we consider S to consist of formal sums

$$S = \left\{ \sum_{i \geq 0} a_i p^i \mid 0 \leq a_i \leq p-1 \right\}$$

with arithmetic performed in the usual formal manner. For more detail we refer the reader to [12]. The expression $\sum_{i \geq 0} a_i p^i$ is also known as the *$p-$adic expansion* of the relevant number.

It is easy to demonstrate that the units in S are the elements where $a_0 \neq 0$ in the $p-$adic expansion and that non-unit elements have the form $p^k u$ where u is a unit of S and $k \in \mathbb{N}$. It is well-known that S forms a principal ideal domain and so from Theorem 4.3.3 we deduce that S is a (2-element) semilattice of stratified extensions of groups. $S_{(1)}$ is the group of units and $\text{Base}(S_{(0)}) = \{0\}$. It follows from the definition of S that the proper non-zero ideals are those of the form (p^k) for $k \in \mathbb{N}$, and so clearly $D_{(0)}$ is isomorphic to the infinite monogenic semigroup with zero. Since S is a principal ideal domain, it follows from Proposition 4.3.4 that ${S_{(0)}}^i = \delta^{-1}({D_{(0)}}^i)$ for all $i \in \mathbb{N}$ and so the i-th layer of $S_{(0)}$ consists of exactly the elements of the form $p^i u$ where u is a unit.

Every non-zero proper ideal $A \trianglelefteq S$ has the form $(p^k) = (p)^k$ for some $k \in \mathbb{N}$. This means that S/A is isomorphic to the ring of integers modulo p^k. Clearly (p) is a prime ideal and so $R = S/A$ is a 2-element semilattice of stratified extensions of groups, consisting of the group of units $R_{(1+A)}$ and the semigroup $R_{(0+A)}$. The latter is a stratified semigroup with zero and $k-1$ non-zero layers. For each $1 \leq i \leq k-1$ the i-th layer consists of elements of the form $p^i u + A$ where u is a unit of S.

4.5.3 Rings of algebraic integers

We now consider rings consisting of algebraic integers and as a specific example we shall consider the ring $S = \mathbb{Z}[\sqrt{-5}]$. It is well know that rings of this nature are Dedekind domains but are not always principal ideal domains. In fact, $2 + \sqrt{-5}$ is an example of an element which can easily be shown to be irreducible but not prime. If A is an ideal of $\mathbb{Z}[\sqrt{-5}]$ define a 'norm' on S/A by
$N(z + A) = (z + A)(\overline{z} + A) = z\overline{z} + A$, where $\overline{z}$ is the conjugate of z. It is easy to check that N is multiplicative and that $z + A$ is a unit in S/A if $N(z + A) = \pm 1 + A$.

From section 4.3, S is a 2-element semilattice of the group of units, $S_{(1)} = \{1, -1\}$, and a stratified semigroup with 0, $S_{(0)}$. Since

$$2 + \sqrt{-5} = 9 \times (-2) + (-1 + 4\sqrt{-5}) \times (-\sqrt{-5}),$$

$$(-1 + 4\sqrt{-5}) = (2 + \sqrt{-5})^2 \text{ and } 9 = (2 - \sqrt{-5})(2 + \sqrt{-5}),$$

it follows that $(3, 2 + \sqrt{-5})^2 = (9, -1 + 4\sqrt{-5}) = (2 + \sqrt{-5})$ and so although $2 + \sqrt{-5}$ is in the first layer of $S_{(0)}$ (being irreducible), $(2 + \sqrt{-5})$ is not in the first layer of $D_{(0)}$. The layer structure of $S_{(0)}$ is not so easy to determine, as clearly $a + b\sqrt{-5}$ is in the i-th layer of $S_{(0)}$ if and only if it can be written as a product of i irreducible elements.

However determining the structure of a quotient of S is slightly easier, as we need only factorise a single ideal of S into a product of prime ideals. As an illustrative example, let us consider

$$A = (10, 5 + 5\sqrt{-5}) = (2, 1 + \sqrt{-5})(5, \sqrt{-5})^2 \qquad (*)$$

with $P_1 = (2, 1 + \sqrt{-5})$ and $P_2 = (5, \sqrt{-5})$ and let $R = S/A$. It is easy to show that $(2, 1 + \sqrt{-5})$ and $(5, \sqrt{-5})$ are both prime ideals of $\mathbb{Z}[\sqrt{-5}]$. In fact

$$(2, 1 + \sqrt{-5}) = \{a + b\sqrt{-5} \mid a \equiv b \bmod 2\} \text{ and } (5, \sqrt{-5}) = \{5a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\},$$

while

$$A = (10, 5 + 5\sqrt{-5}) = \{5a + 5b\sqrt{-5} \mid a \equiv b \bmod 2\}.$$

It is easy to check that the ring R has cardinality 50. In what follows, we shall frequently simplify the notation by working modulo A and write the element $a + b\sqrt{-5} + A$ of R as simply $a + b\sqrt{-5}$. We shall also assume a particular set of residues by taking $0 \leq a \leq 9$ and $0 \leq b \leq 4$.

Note from the comments preceding Proposition 4.4.7 that $|E(D)| = |\mathcal{P}(\{1,2\})|$ and it can then be easily verified that

$$A_1 = \Phi^{-1}(P_1) = (2, 1+\sqrt{-5})/A = (6) \text{ and } {A_2}^2 = \Phi^{-1}({P_2}^2) = (5, \sqrt{-5})^2/A = (5)$$

and so by Lemma 4.4.6

$$E(D) = \{(0), (1), (5), (6)\}.$$

We now apply the results of Theorem 4.4.10 and Corollary 4.4.12. It follows that $R_{(1)}$ and $R_{(6)}$ are groups and $R_{(0)}$ and $R_{(5)}$ are stratified extensions of groups, each with a height of 1. By Proposition 4.4.15, the group $\text{Base}(R_e)$ for each $e \in E(D)$ is equal to $xU_{\overline{x}}$ where $(x) = e$ and hence isomorphic to $U_{\overline{x}}$.

In practical terms, $R_{(1)} = R_R$ is the group of units of R, and using norms we can deduce that $|R_R| = 20$ and in fact

$$R_R = \{a + b\sqrt{-5} \mid a \not\equiv b \bmod 2, a \not\equiv 0 \bmod 5\}.$$

For $R_{(5)}$, it follows that $\text{Base}(R_{(5)}) = \{5v \mid v \in R_R\} = 5R_R = \{5\}$. To find the elements in layer 1 of $R_{(5)}$ we note that $(5, \sqrt{-5})/A = (\sqrt{-5})$ and so layer 1 is

$$\delta^{-1}((\sqrt{-5})) = \{(\sqrt{-5})v \mid v \in R_R\} = \{\sqrt{-5}, 3\sqrt{-5}, 7\sqrt{-5}, 9\sqrt{-5}\}.$$

For $R_{(6)}$, it follows that

$$\text{Base}(R_{(6)}) = 6R_R = \{a + b\sqrt{-5} \mid a \equiv b \bmod 2, a \not\equiv 0 \bmod 5\}.$$

Note that $|\text{Base}(R_{(6)})| = 20$ also.

Finally, the layer 1 in $R_{(0)}$ can be calculated in the same way as for $R_{(5)}$ using the fact that $(2, 1+\sqrt{-5})/A\ (5, \sqrt{-5})/A = (5+\sqrt{-5})$. It then follows easily that the first layer of $R_{(0)}$ is

$$(5+\sqrt{-5})R_R = \{2\sqrt{-5}, 4\sqrt{-5}, 5+\sqrt{-5}, 5+3\sqrt{-5}\}.$$

4.5.4 Integers revisited

For a final example we return to a less complicated ring in order to demonstrate a more complicated layer structure. Let $S = \mathbb{Z}$, $A = (6000) = (2)^4(3)(5)^3$ and $R = S/A$. Working modulo A, let $e \in E(D)$ be the ideal $(2000) = (2)^4(5)^3$. Then R_e is a stratified extension of a group with height 3. Applying our previous results, we see that $\text{Base}(R_e) = \delta^{-1}((2000)) = \{2000u \mid u \in R_R\} = 2000R_R$.

For the layers, note that layer 1 of D_e consists of $(2)(5)$, $(2)^2(5)$, $(2)^3(5)$, $(2)^4(5)$, $(2)(5)^2$ and $(2)(5)^3$. Layer 1 of R_e is hence the union of $10R_R$, $20R_R$, $40R_R$, $80R_R$, $50R_R$ and $250R_R$.

Proceeding in a similar fashion, layer 2 of R_e is the union of $100R_R$, $200R_R$, $400R_R$ and $500R_R$, while layer 3 is simply the set $1000R_R$.

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