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An investigation into wave propagation in hanging chains

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V G Cleante[1], **P J P Gonçalves**[2], **T Waters**[3], **M J Brennan**[2,3], **J P Carneiro Jr.**[2] **and D A Rade**[1]

[1] Division of Mechanical Engineering, Aeronautics Institute of Technology, São José dos Campos, SP 12228-900, Brazil
[2] State University of São Paulo (UNESP), School of Engineering, Bauru, Av. Eng. Luiz Edmundo C. Coube 14-01, Bauru, SP, Brazil
[3] Institute of Sound and Vibration Research, University of Southampton, University Rd, Highfields, Southampton, United Kingdom

E-mail: cleante@ita.br

Abstract. The study introduced in this work is motivated by the prospect of using a hanging chain as an Acoustic Black Hole (ABH) for passive vibration control. An ABH is effectively a waveguide in which a wave slows progressively as it propagates away from the source enabling it to be extinguished with modest damping. The effect can be achieved by engineering inhomogeneity into a structure's geometry or material, the most common realisation being a beam of tapered thickness. This paper proposes an alternative realisation, that of a chain hanging under its own weight. Such a system has a wave speed that naturally decreases to zero, owing to its linear variation in tension, thus overcoming the challenges of constructing precisely shaped beams with vanishingly thin tips. The study of transverse vibration of hanging chains is a classical problem in structural dynamics. The motion of the chain can be described in terms of Bessel or Hankel functions, which are needed to account for the variation in tension along the chain. In this work, the hanging chain problem is revisited from a wave propagation perspective. An expression is derived for the amplitude of the waves in an infinite chain due a point excitation. From which, the spatial behaviour and the receptances of the waves are evaluated, revealing differing characteristics of upward and downward propagating waves. Some experimental results are presented to support the theoretical analysis.

1. Introduction

Mechanical appendages for passive vibration control of structures have been proposed and researched ever since the invention of the vibration absorber by Frahm over a century ago. One device which has attracted significant attention recently is the Acoustic Black Hole (ABH), a structure that attenuates waves as they propagate away from the source [1]. Inhomogeneity in the form of a progressive stiffness reduction slows the wave to near-zero. The shortening wavelength is more readily attenuated by modest passive damping such that reflections are virtually eliminated. A commonly proposed realisation of a ABH is that of a tapered beam of precise thickness profile [2]. Other proposed designs include a plate with thickness reduction in the form of a pit profile [3] and a periodic structure consisting of multiple ABH [4] to enhance the effect. However, the ABH has to be sufficiently long to be effective at low frequencies, making the structure fragile and potentially less practical.

An alternative structure, which may have practical advantages in some applications, is that of a chain hanging vertically under its own weight. The tension varies linearly along its length and the wave speed

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tends to zero at its free end. From the primary work on hanging chains by Bernoulli [5] to the most recent ones, researchers have focussed on determining the natural frequencies and mode shapes of the system. Routh [6] adopted a continuous model of a string to solve the problem in terms of Bessel functions of first and second kinds. Verbin [7] extended this work by investigating the influence of a bead attached at the bottom end of the chain. Investigating a tapered chain with a tip mass Wang [8] showed that the cross-sectional shape of the chain has a profound effect on the natural frequencies. Using a discrete model for the links of the chain, McCreesh et.al. [9] and Weng and Lee [10] found that for problems of small amplitude, the first three vibration modes have a good agreement compared to the continuous model. However, to the authors' knowledge the hanging chain has not been analysed in terms of wave propagation.

This paper focusses on wave propagation in a chain hanging under its own weight. The aim is to understand the physical effect of tension non-uniformity on the dynamic response with the intention to assess the performance of a hanging chain as a passive vibration control device in future work. Based on the work developed by Lee et al. [11], which investigated wave propagation in a tapered rod and beam, the wave equation for the hanging chain is solved in terms of Hankel functions and the wave field along a locally excited chain is evaluated.

2. Wave generation in a hanging chain

An example of the system considered in this paper is shown in figure 1(a). It consists of a hanging chain that is subject to the effects of gravity. This results in tension between the links, which varies linearly from the top of the chain, where it is maximum, to the bottom of the chain where it is zero. Waves generated by an external harmonic force are of interest. The force is applied as shown in figure 1(a). It is given by $F_{\text{ext}}e^{\text{j}\omega t}$ where ω is the angular frequency, t is time and $\text{j}=\sqrt{-1}$, and generates two waves, one propagating in the upward direction given by a^+ and one propagating in the downward direction given by a^-. Figures 1(b) and 1(c) show the direct wave displacement along the hanging chain with non-uniform and uniform tension, respectively. These are discussed in the following sections.

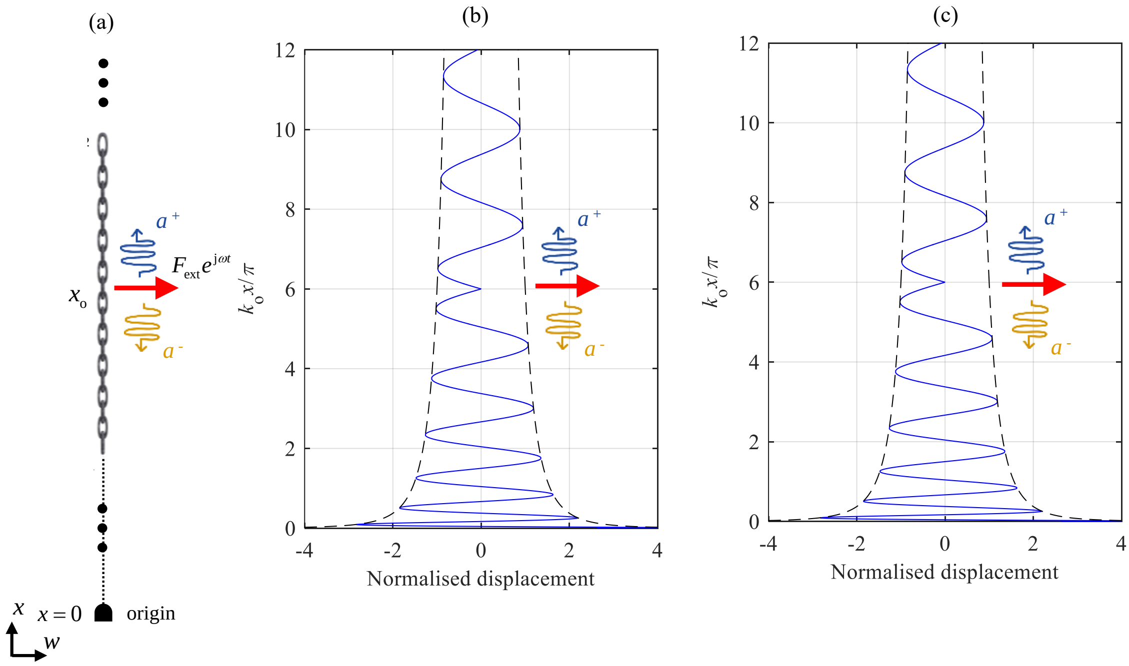

Figure 1. Example of (a) a hanging chain locally excited at any arbitrary point; (b) Snapshot of the normalised displacement of the direct waves along the chain when locally excited by a harmonic force; (c) for comparison, the normalised amplitude of travelling waves if the tension is constant. The black dashed-lines denote the moduli of the wave amplitudes.

2.1. Free wave propagation

Consider first the free wave propagation in the chain, i.e., there is no external excitation. In the following analysis, the chain links are assumed to be very small so the system can be modelled as a continuous string with non-uniform tension and position-dependent tension $T(x)$, such that the equation of motion for free motion is given by [12]

$$\rho\frac{\partial^2 w(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[T(x)\frac{\partial w(x,t)}{\partial x}\right]=0, \tag{1}$$

where ρ is constant and is the chain's mass density per unit length, x is the position in space, w is the lateral displacement, t is time and ∂ denotes partial derivative.

Assuming the tension in the chain due to the weight is given by $T(x)=\rho g x$, in which g is the gravity acceleration and a harmonic motion of the form $w(x,t)=w(x)e^{\mathrm{j}\omega t}$, equation (1) reduces to

$$\frac{d^2w(x)}{dx^2}+\frac{1}{x}\frac{dw(x)}{dx}+k_x^2 w(x)=0, \tag{2}$$

where $k_\mathrm{x}=\omega/c_\mathrm{x}$ is the wavenumber, in which $c_\mathrm{x}=\sqrt{gx}$ is the phase velocity with a non-uniform tension. Note that the subscript x denotes the spatial dependency. A change of variable to $\phi=2k_\mathrm{x}x$, which is the spatial phase, enables equation (2) to be rewritten in the form of Bessel's equation, resulting in

$$\frac{d^2w(\phi)}{d\phi^2}+\frac{1}{\phi}\frac{dw(\phi)}{d\phi}+w(\phi)=0. \tag{3}$$

The solution of equation (3) is a linear combination of up- and down-going waves described by Hankel functions with argument ϕ. Thus, the transverse displacement of the hanging chain at position x is given by

$$w(x)=a^-+a^+, \tag{4}$$

where $a^-=BH_0^{(1)}(2k_\mathrm{x}x)$ and $a^+=AH_0^{(2)}(2k_\mathrm{x}x)$ represent propagating waves, where A and B are constants. The subscript 0 denotes the order, and superscripts (1) and (2) denote Hankel functions of first and second kind, respectively. The internal lateral force can be related to the wave amplitudes by evaluating $F(x)=T(x)\partial w(x)/\partial x$, which gives

$$F(x)=-\rho g k_\mathrm{x}x\frac{H_1^{(2)}(2k_\mathrm{x}x)}{H_0^{(2)}(2k_\mathrm{x}x)}a^+-\rho g k_\mathrm{x}x\frac{H_1^{(1)}(2k_\mathrm{x}x)}{H_0^{(1)}(2k_\mathrm{x}x)}a^-. \tag{5}$$

The relationship between the displacement and force with the propagating waves is given by [13]

$$\begin{Bmatrix} w \\ F \end{Bmatrix}=\begin{bmatrix} 1 & 1 \\ \Phi^+ & \Phi^- \end{bmatrix}\begin{Bmatrix} a^+ \\ a^- \end{Bmatrix}, \tag{6}$$

where

$$\Phi^+=-\rho g k_x x\frac{H_1^{(2)}(2k_x x)}{H_0^{(2)}(2k_x x)} \quad \text{and} \quad \Phi^-=-\rho g k_x x\frac{H_1^{(1)}(2k_x x)}{H_0^{(1)}(2k_x x)}. \tag{7a-b}$$

2.2. Waves due to harmonic force excitation

In this section, the behaviour of wave propagation due to a point force excitation, and the dynamic effects of the tension are considered. An undamped chain is studied.

Consider harmonic force excitation as shown in figure 1(a). Only waves directly generated by the force are determined. From equation (6), the generated wave amplitudes are obtained by applying displacement continuity and force equilibrium conditions at the excitation point x_o. They are given by

$$\frac{a^+}{F_\mathrm{ext}} = \left[-\frac{\mathrm{j}\pi}{2\rho g} H_0^{(1)}\left(2k_\mathrm{o}x_\mathrm{o}\right)\right] H_o^{(2)}\left(2k_\mathrm{x}x\right) \text{ and } \frac{a^-}{F_\mathrm{ext}} = \left[-\frac{\mathrm{j}\pi}{2\rho g} H_0^{(2)}\left(2k_\mathrm{o}x_\mathrm{o}\right)\right] H_o^{(1)}\left(2k_\mathrm{x}x\right). \quad \text{(8a,b)}$$

To illustrate the behaviour of these waves, they are plotted in figure 1(b) for a single frequency of excitation at an arbitrary point in time. They are normalised by $F_\mathrm{ext} / 2\rho g k_\mathrm{o} x_\mathrm{o}$, where k_o is the wavenumber at the excitation point. Also shown in figure 1(b) is the modulus of the wave amplitudes, represented by the dashed line. Note that only the direct waves are considered, the waves reflected from the boundaries are not shown.

It is seen from figure 1(b) that due to the non-uniformity of the tension, the amplitude of the up-going wave reduces as it propagates in the direction where the tension increases. The effect on spatial phase is also evident. Note that as $k_\mathrm{x} \propto 1/\sqrt{x}$, the wave experiences an increase in its wavelength and hence an increase in its phase velocity. The amplitude of the down-going wave increases as it propagates in the direction where the tension reduces, at the same time its wavelength and phase velocity are reduced, which is the typical dynamic behaviour of an ABH system.

It is instructive to compare the waves propagating on a hanging chain with those on a chain (string) in which the tension is constant. In this case the central term in equation (2) is zero, so that the solution to the equation of motion is given by a linear combination of $a_\mathrm{s}^+ = Ae^{-\mathrm{j}k_\mathrm{s}x}$ and $a_\mathrm{s}^- = Be^{\mathrm{j}k_\mathrm{s}x}$ [14], where A and B are constants and $k_\mathrm{s} = \omega/c_\mathrm{s}$ is the wavenumber of the chain with constant tension and $c_\mathrm{s} = \sqrt{T/\rho}$ is the phase velocity. The subscript s denotes the hanging chain with constant tension. The relationship between the force and the waves is simply given by $\Phi_\mathrm{s}^+ = -\mathrm{j}k_\mathrm{s}T$ and $\Phi_\mathrm{s}^- = \mathrm{j}k_\mathrm{s}T$. For a harmonic excitation force $F_\mathrm{ext}e^{\mathrm{j}\omega t}$, the wave amplitudes per unit input force are determined to be

$$\frac{a^\pm}{F_\mathrm{ext}} = -\mathrm{j}\frac{1}{2Tk_\mathrm{s}} \mathrm{e}^{\mp \mathrm{j}k_\mathrm{s}\left(x - x_\mathrm{o}\right)} \quad . \quad (9)$$

An example of the amplitude of the waves normalised by $F_\mathrm{ext}/2Tk_\mathrm{s}$ are shown in figure 1(c). It is clear from figure 1(c) that, when a fictitious hanging chain with constant tension is considered, the upward and downward propagating waves have the same amplitudes, irrespective of position. Further the wave speed is also constant, so the wavenumber, and hence the wavelength is also independent of position.

Having established the spatial behaviour of wave propagation in a hanging chain, attention is now turned to the way in which the generated waves behave as a function of frequency at the point where the force was applied. This yields a wavenumber k_o at x_o. Substituting $k_\mathrm{x}x = k_\mathrm{o}x_\mathrm{o}$ into equations (8a,b), the modulus and phases of the waves normalised by $F_\mathrm{ext}/2\rho g k_\mathrm{o} x_\mathrm{o}$ are plotted in figure 2 as a function of $k_\mathrm{o}x_\mathrm{o}$. Also plotted in figure 2 as black dotted-lines are the modulus and phase of the waves for a fictitious hanging chain with constant tension, equation (9), normalised by $F_\mathrm{ext}/2Tk_\mathrm{s}$.

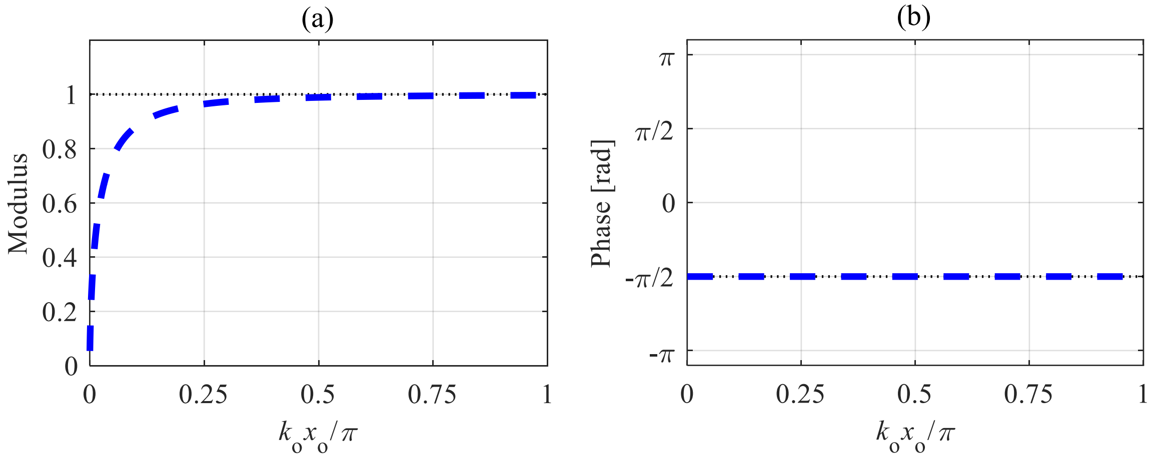

Figure 2. (a) Normalised modulus and (b) phase of an upward- and downward-going (blue dashed-line) waves receptances, equations (8a,b) generated at a local excitation. The thin black dotted line is for waves generated in a hanging chain with constant tension, equation (9).

It can be seen in figure 2(a) that the dynamic effect of the non-uniformity in the tension of the chain is only observed for $k_o x_o/\pi < 0.5$, where the modulus varies from 0 to 1. However, from figure 2(b) it seen that the variation of the tension has no effect on the phase, when comparing it with the case in which the tension is constant. Note that at $x = x_o$ for $k_o x_o \ll 1$, equations (8a,b) asymptote to [15]

$$\frac{a^{\pm}}{F_{\text{ext}}} \approx -\mathrm{j}\frac{1}{2\pi\rho g}\left[\pi^2 + 4\ln\left(k_o x_o\right)^2 + 8\gamma\ln\left(k_o x_o\right) + 4\gamma^2\right], \tag{10}$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Examining equation (10), it can be seen that the receptances of the generated upward and downward waves are negative and imaginary. Thus, the waves receptance have a damping-like characteristic. For $k_o x_o \gg 1$, equations (8a,b) asymptote to [15]

$$\frac{a^{\pm}}{F_{\text{ext}}} \approx -\mathrm{j}\frac{1}{2T_o k_o}, \tag{11}$$

where $T_o = \rho g x_o$. It can be seen that equations (9) and (11) are the same when $x = x_o$ provided the tension of the chain at the excitation point is the same as the constant tension. This shows that the wave receptance is only dependent on the local tension at high frequencies. This is clear in figure 2, where for $k_o x_o/\pi > 0.5$ the generated waves asymptote to the black dotted lines, which are the modulus and phase of the fictitious hanging chain with constant tension.

3. Experimental investigation

An experimental investigation was conducted in the laboratory to investigate the wave velocity of a hanging chain. The experiment involved recording wave propagation along a 1.1 m long hanging chain that was disturbed with an impulse-like input near the top. Using a camera, the peak of the wave was tracked and its position as a function of time was converted to data using the software Tracker Physics [16], after calibrating the video with a reference dimension. Figure 3 presents samples of video frames that illustrates the tracking of the wave peak. It can be seen that the wave peak amplitude increases, while the wavelength decreases as the pulse approaches the bottom of the chain.

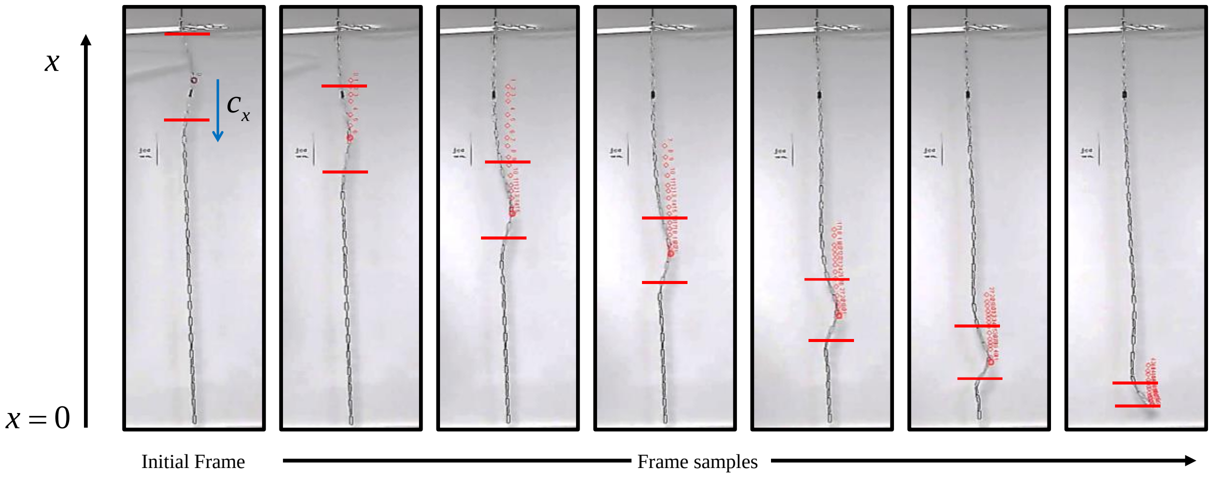

Figure 3. Samples of the video used to record the wave propagation on a handing chain. A disturbance was produced near the top, shown on the left frame. The wavelength is denoted by two horizontal bars.

Considering the theory discussed in section 2, the phase velocity is given by $c_x = dx/dt = -\sqrt{gx}$. The negative sign is due to the direction of the wave, that propagates downwards. The wave peak position relationship can be obtained by integrating the wave velocity, such as

$$\int_0^t dt = -\int_{x_0}^{x} \frac{dx}{\sqrt{gx}} \rightarrow x(t) = x_0 - \left(\sqrt{gx_0}\right)t + \frac{g}{4}t^2, \tag{12}$$

which is only valid for $t < 2\sqrt{x_0/g}$. Figure 4 shows the results data obtained experimentally using the video processing software compared with the theory described in equation (12). The theory and experimental data agree well, validating part of the theory discussed in section 2, showing that the wave velocity tends to zero towards the bottom of the chain.

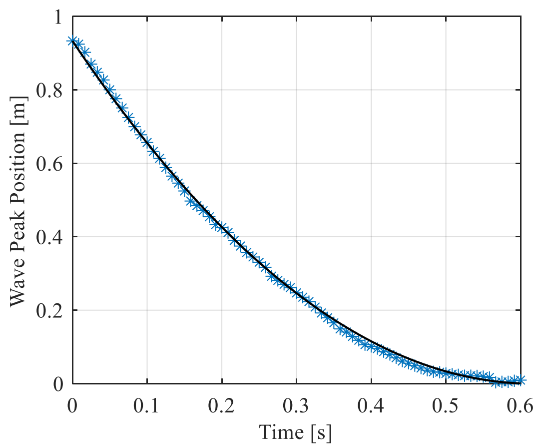

Figure 4. Comparison of the wave peak position as a function of the time. The solid line represents the theory and the markers represent the experimental data.

4. Conclusions

This paper has investigated wave behaviour of a chain hanging vertically. Such a chain has non-uniform tension between the links, which varies linearly from the top, where it is maximum, to the bottom of the chain where it is zero. This has a profound effect on the amplitude and speed of the upand down-going propagating waves in the chain. A wave model has been developed, which was used to analyse the spatial behaviour of the waves, which are described by Hankel functions. It was shown that the amplitude of an up-going wave reduces, and its phase velocity increases, whereas the amplitude of a down-going wave increases with a corresponding decrease in its phase velocity. At the bottom of the chain the tension and the phase velocity are both zero. Using a point harmonic force, it was found that the wave receptance is affected strongly by non-uniformity of the tension at low frequencies, but at higher frequencies it is only affected by the local tension at the excitation position. A laboratory-based experiment using a camera, demonstrated the way in which the phase velocity changes with position along the chain, validating the theoretical prediction.

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