Coupled unsteady actuator disc and linear theory of an oscillating foil propulsor: outline of equations

This document outlines the full set of equations used to couple cycle-averaged actuator disc theory to Garrick's function, following the method presented in the paper "Coupled unsteady actuator disc and linear theory of an oscillating foil propulsor" [1] by the authors of this document. The full outline and derivation of Garrick's theory for foils in pitch and heave, and with trailing edge flap actuation, can be found in Garrick's paper [2]. Below we use only the expression pertaining to an aerofoil in combined heave and pitch. Only the final equations are presented; for the derivations, validation and control volumes used, the reader is referred to the original paper.

List of variables

The input variables to be specified are as follows:

b	Aerofoil half-chord, m
U_∞	Farfield freestream velocity, m/s
ω	Pitch and/or heave oscillation frequency, rad/s
h_0	Heave amplitude, m
α_0	Pitch amplitude, rad
a	Distance to centre of pitch from the mid-chord, non-dimensionalised by b
ψ	Phase angle between the heaving and pitching motion, rad
ρ	Fluid density, kg/m^3
$\mathcal{A}$	Maximum peak-to peak amplitude of motion, m

The model outputs are as follows:

F_x	Cycle-averaged aerofoil thrust, N
C_{Tg}	$= \frac{2F_x}{\rho U_\infty^2 \mathcal{A}}$ Actuator disc (global) thrust coefficient
C_T	$= \frac{F_x}{\rho U_\infty^2 b}$ Foil thrust coefficient
$\overline{W}_f$	Cycle-averaged foil input power, W
C_{Pg}	$= \frac{2\overline{W}_f}{\rho U_\infty^3 \mathcal{A}}$ Actuator disc (global) power coefficient
C_P	$= \frac{\overline{W}_f}{\rho U_\infty^3 b}$ Foil power coefficient
η_g	$= \frac{U_\infty F_x}{\overline{W}_f} = \frac{C_{Tg}}{C_{Pg}} = \frac{C_T}{C_P}$ Global propulsive efficiency

Additional key variables used in the control volume analysis are defined as follows:

α_2	$= \frac{U_{foil}}{U_\infty}$	Flow acceleration parameter at the foil
α_4	$= \frac{U_{exit}}{U_\infty}$	Flow acceleration parameter at the control volume exit plane
k_g	$= \frac{\omega b}{U_\infty}$	Global reduced frequency
k_f	$= \frac{\omega b}{\alpha_2 U_\infty}$	Reduced frequency at the foil
k_e	$= \frac{\omega b}{\alpha_4 U_\infty}$	Reduced frequency at the control volume exit plane

Garrick theory for foils in combined pitch and heave

Aerofoil forces

Cycle-averaged thrust and power are given by Garrick as:

$$F_x = \pi \rho b \omega^2 \left[A_1 h_0^2 + A_2 \alpha_0^2 + 2A_4 \alpha_0 h_0 \right] \tag{1}$$

$$\overline{W}_f = \pi \rho b^2 \frac{\omega^3}{k_f} \left(B_1 h_0^2 + B_2 \alpha_0^2 + 2B_4 \alpha_0 h_0 \right) \tag{2}$$

The variables A_1, A_2, A_4, B_1, B_2 and B_4 are given as follows:

$$A_1 = B_1 - C_1 \tag{3}$$

$$A_2 = B_2 - C_2 \tag{4}$$

$$A_4 = B_4 - C_4 \tag{5}$$

$$B_1 = F \tag{6}$$

$$B_2 = b^2 \left\{ \frac{1}{2}\left(\frac{1}{2} - a\right) - \left(\frac{1}{2} + a\right) \left[F\left(\frac{1}{2} - a\right) + \frac{G}{k_f} \right] \right\} \tag{7}$$

$$B_4 = \frac{b}{2} \left[\left(\frac{1}{2} - 2aF + \frac{G}{k_f} \right) \cos(\psi) - \left(\frac{F}{k_f} - G \right) \sin(\psi) \right] \tag{8}$$

The variables C_1, C_2 and C_4 are:

$$C_1 = \frac{2}{\pi k_f D} \tag{9}$$

$$C_2 = \frac{2b^2}{\pi k_f D} \left[\frac{1}{k_f^2} + \left(\frac{1}{2} - a \right)^2 \right] \tag{10}$$

$$C_4 = \frac{2b}{\pi k_f D} \left[-\frac{1}{k_f} \sin(\psi) + \left(\frac{1}{2} - a \right) \cos(\psi) \right] \tag{11}$$

The above expressions are functions of variables F and G, respectively the real and imaginary parts of the Theodorsen function [3]:

$$F + iG = \frac{H_1^{(2)}(k_f)}{H_1^{(2)}(k_f) + iH_0^{(2)}(k_f)}. \tag{12}$$

$H^{(2)}$ is the Hankel function of the second kind, defined as:

$$H_v^{(2)}(k_f) = J_v(k_f) - iY_v(k_f) \tag{13}$$

where J_v and Y_v are the Bessel functions of the first and second kinds respectively, with v denoting their order ($v = 0$ or 1 in this case), taking the foil reduced frequency k_f as the argument. Finally, the variable D is given by:

$$D = [J_1(k_f) + Y_0(k_f)]^2 + [Y_1(k_f) - J_0(k_f)]^2 \tag{14}$$

Wake circulation

For the control volume analysis below we require expressions for the wake circulation $\gamma(x,t)$ at time t and streamwise location x, at a location far downstream of the aerofoil on the control volume exit face. It is given by Garrick, based on Theodorsen, as:

$$\gamma(x,t) = A_0\cos(k_e\frac{x}{b}) + B_0\sin(k_e\frac{x}{b}) \tag{15}$$

$$A_0 = 4\left[\zeta_1\sin(\omega t) - \zeta_2\cos(\omega t)\right] \tag{16}$$

$$B_0 = 4\left[\zeta_1\cos(\omega t) + \zeta_2\sin(\omega t)\right] \tag{17}$$

Note that while the circulation is used in the derivation of the final cycle-averaged control volume equations, in the final equations below only the A_0 term is required, in the time-averaged and squared form $\overline{A_0^2}$. This is given by:

$$\overline{A_0^2} = 8\left[\zeta_1^2 + \zeta_2^2\right] \tag{18}$$

The variables ζ_1 and ζ_2 are functions of the aerofoil kinematics:

$$\zeta_1 = (BK - AJ) \tag{19}$$

$$\zeta_2 = (AK + BJ) \tag{20}$$

$$J = \frac{J_1(k_f) + Y_0(k_f)}{D} \tag{21}$$

$$K = \frac{Y_1(k_f) - J_0(k_f)}{D} \tag{22}$$

$$A = \alpha_2 U_\infty \alpha_0\cos(\psi) - b\left(\frac{1}{2} - a\right)\alpha_0\omega\sin(\psi) \tag{23}$$

$$B = \alpha_2 U_\infty \alpha_0\sin(\psi) + h_0\omega + b\left(\frac{1}{2} - a\right)\alpha_0\omega\cos(\psi) \tag{24}$$

Note that in all the above expressions the phase of the heave motion relative to time $t = 0$ is assumed to be $\psi_{heave} = 0$.

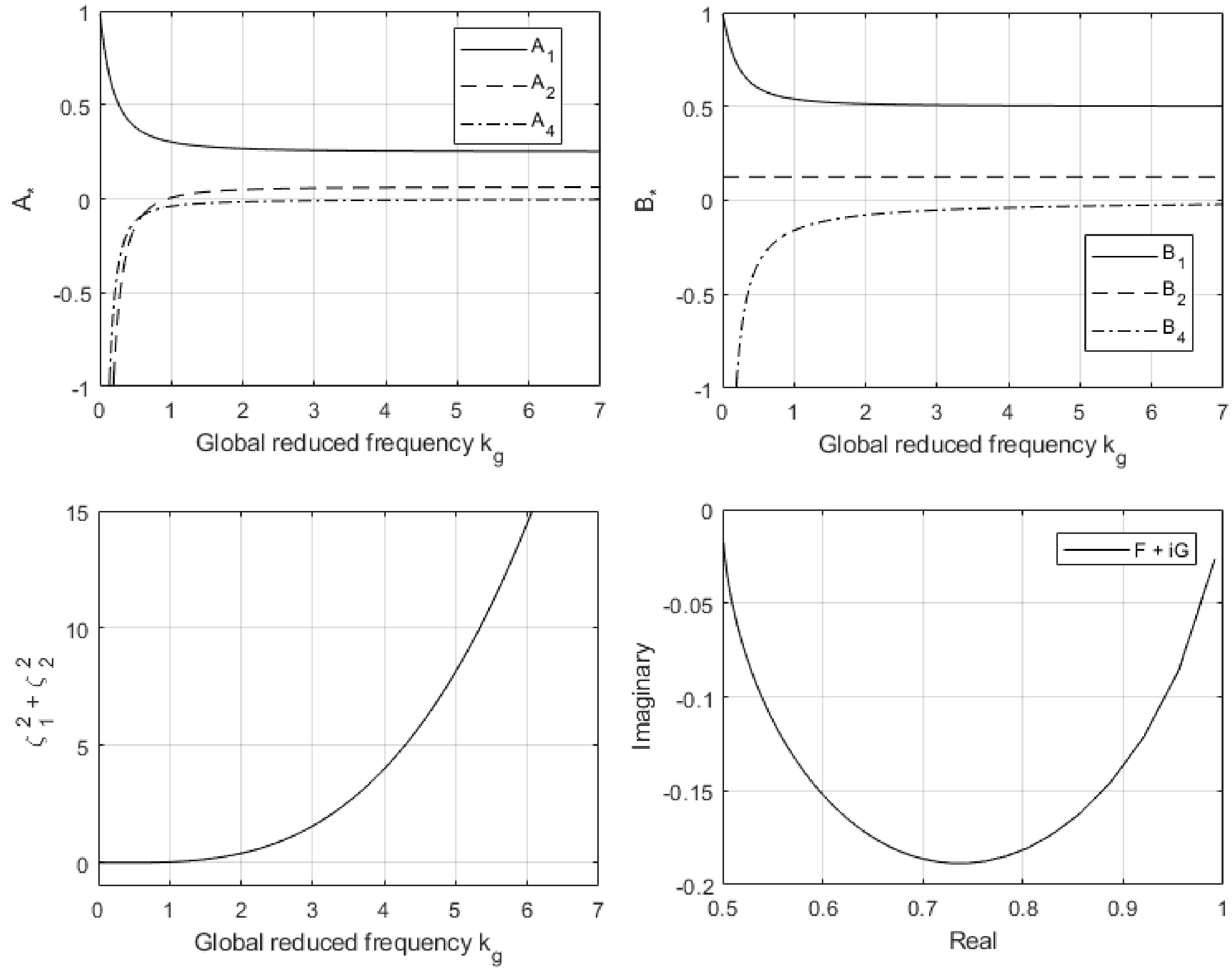

Figure 1: Key variables evaluated for $h_0 = 0.2$, $\alpha_0 = 0.1878$, $\psi = 90°$, $b = 0.5$ and $a = -0.5$ over the range $0 \leq k_g \leq 7$, for $\alpha_2 = \alpha_4 = 1$ and $U_\infty = 1$. a) Variables required to evaluate thrust in Equation 1. b) Variables required to evaluate power in Equation 2. c) Kinematic terms required to evaluate η_{am} in Equation 28. d) The Theodorsen function (Equation 12).

Local flow acceleration

In Garrick theory the local flow acceleration induced by the aerofoil thrust is not accounted for, meaning that $\alpha_2 = \alpha_4 = 1$ and $k_f = k_e = k_g$ in all the above expressions in the original paper. For the present work, note that expressions relating to wake circulation amplitude above use k_f, since the circulation strength is determined by flow conditions at the foil. In expressions relating to wake vortex location, k_e is used since the point of interest is the control volume exit face far downstream, where the vortex spacing is determined by the local velocity $\alpha_4 U_\infty$.

Cycle-averaged control volume analysis

Governing equations

The governing equations for the cycle-averaged unsteady actuator disc analysis are derived in the paper as:

$$C_{Tg} = 2\alpha_2 \left(\alpha_4 - 1\right) \tag{25}$$

$$C_{Tg} = \eta_l \left[\alpha_4^2 \eta_{am} - 1\right]. \tag{26}$$

Equation 25 is equivalent to the steady-flow actuator disc theory result, while Equation 26 differs through the terms η_l and η_{am}, respectively defined as:

$$\eta_l = \frac{\alpha_2 U_\infty F_x}{\overline{W}_f} \tag{27}$$

$$\eta_{am} = 1 + \frac{1}{2\alpha_2 k_g} \frac{\overline{A_0^2} b}{U_\infty^2 \mathcal{A}}. \tag{28}$$

Both expressions are analytically derived in the published paper. The steady actuator disc equations for a propulsor are recovered when setting $\eta_l = \eta_{am} = 1$. The actuator disc area $\mathcal{A}$ is calculated from the foil kinematics as the swept area of the foil during one motion cycle.

Solving the actuator disc equations

Equations 1, 2, 25, 26, 27 and 28 represent a system of 6 equations with 6 unknowns: α_2, α_4, F_x, $\overline{W}_f$, η_l and η_{am}. Due to the complex interdependency of the variables, the system of equations is solved iteratively, taking as initial guess the result obtained analytically by the original Garrick function by setting $\alpha_2 = \alpha_4 = 1$. In the published paper the MATLAB function *fsolve* is used for the iterative solution, until convergence of all variables.

Note that the steady-flow actuator disc system of equations, recovered by setting $\eta_l = \eta_{am} = 1$, is robust and converges within a few iteration steps. However, the unsteady actuator disc system derived above does not converge easily, and is especially difficult to converge near $F_x = 0$. Simple iterative methods were not able to converge the cycle-averaged equations, but the *fsolve* function in MATLAB was successful. Improved convergence was achieved by first solving the steady actuator disc equations, using the cycle-averaged thrust from Equation 1, and then using this as the starting guess for the iterative solution of the unsteady actuator disc equations.

References

[1] A. S. M. Smyth, T. Nishino and A. N. Zurman-Nasution (2024) *Coupled unsteady actuator disc and linear theory of an oscillating foil propulsor*, Journal of Fluid Mechanics Rapids.

[2] I. E. Garrick (1937) *Propulsion of a flapping and oscillating airfoil*, NACA Report no. 567.

[3] T. Theodorsen (1935) *General theory of aerodynamic instability and the mechanism of flutter*, NACA Report no. 496.