Letter

Bern-Carrasco-Johansson relations in $\mathrm{AdS}_5 \times S^3$ and the double-trace spectrum of super gluons

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(Received 2 June 2022; accepted 14 March 2023; published 5 April 2023)

We revisit the four-point function of super gluons in $\mathrm{AdS}_5 \times S^3$ and show how the integrand of a generalized Mellin transform satisfies the $U(1)$ decoupling identity, Bern-Carrasco-Johansson relations and color-kinematic duality, in a way that directly mirrors the analogous flat space relations. We unmix the spectrum of double-trace operators at large N and find all anomalous dimensions at leading order which follow a very simple pattern, as in other theories with hidden conformal symmetries.

DOI: 10.1103/PhysRevD.107.L081901

I. INTRODUCTION

Properties of (quantum) gravity theories and their relation to gauge theories is a primary focus of modern physics. Most of these relations are obscured in a Lagrangian formulation and manifest their full majesty only through observables such as scattering amplitudes. The study of amplitudes has led to a series of impressive and deep results, for example, Bern-Carrasco-Johansson (BCJ) dualities [1] and double-copy constructions [2] in various theories (for a recent review, see [3]). Such properties hint at an underlying common structure between gravity and gauge theories, yet to be fully understood. Most such efforts have related to flat space, mainly because of the difficulties in performing computations in curved backgrounds. However, recent work has begun the exploration of these properties in AdS backgrounds, in Mellin space [4–7], position space [8,9] and momentum space [10,11]. Many of these studies use bootstrap methods that have been successful in exploring supergravity in AdS [12–26], or its string corrections [27–39]. For example, in [5], AdS versions of color-kinematic and double-copy relations were found.

Here we explore these relations, focusing on four-point functions of half-BPS operators dual to the scattering of four super gluons in $\mathrm{AdS}_5 \times S^3$, as computed in [4]. As in [5], we focus on the "reduced" Mellin amplitude. Using ideas developed in [35] for $\mathcal{N} = 4$ super Yang-Mills, we find the amplitude manifests color-kinematics and BCJ relations in a way strikingly similar to flat space. We ascribe this to the existence of an $8d$ conformal symmetry.

We then unmix the spectrum of double-trace operators exchanged in the operator product expansion (OPE) and compute their anomalous dimensions at leading order.

The anomalous dimensions we find are given by a very simple formula, similar to previous results in other backgrounds [19,40,41], and suggest that the hidden conformal symmetry, unavoidably, plays a primary role in constraining the data of these superconformal field theories.

II. $\mathrm{AdS}_5 \times S^3$ MELLIN TRANSFORM AND THE LARGE p FORMALISM

The $\mathrm{AdS}_5 \times S^3$ background arises in two basic stringy setups. One can either consider a stack of N D3-branes probing F-theory 7-brane singularities or a stack of N_F D7-branes wrapping an $\mathrm{AdS}_5 \times S^3$ subspace in the $\mathrm{AdS}_5 \times S^5$ geometry of a stack of N D3-branes. In both cases, the system preserves eight supercharges; therefore, the dual CFT is a $4d$ $\mathcal{N} = 2$ theory with flavor group G_F, which we will keep generic because it is mostly irrelevant for the details considered in this paper. The low-energy degrees of freedom are those of an $\mathcal{N} = 1$ vector multiplet which transforms in the adjoint of G_F. Upon reducing on the sphere, it provides an infinite tower of Kaluza-Klein modes organized in different multiplets. In the dual CFT, the super primaries of these multiplets are half-BPS scalar operators of the form $\mathcal{O}_p^{Ia_1a_2\ldots a_p;\bar{a}_1\bar{a}_2\ldots\bar{a}_{p-2}}$. Here I is the color index, p is the scaling dimension of the operator, $a_1,\ldots,a_p$ are symmetrized $SU(2)_R$ R-symmetry indices and similarly $\bar{a}_i$ are indices of an additional $SU(2)_L$ flavor group; these last two groups realize the isometry group of the sphere S^3. As usual in these contexts, it is convenient to contract the indices with auxiliary bosonic two-component vectors η and $\bar{\eta}$ to keep track of the $SU(2)_R \times SU(2)_L$ indices:

$$\mathcal{O}_p^I \equiv \mathcal{O}_p^{I;a_1a_2\ldots a_p;\bar{a}_1\bar{a}_2\ldots\bar{a}_{p-2}}\eta_{a_1}\ldots\eta_{a_p}\bar{\eta}_{\bar{a}_1}\ldots\bar{\eta}_{\bar{a}_{p-2}}. \qquad (1)$$

In this paper we consider the amplitude of four super gluons, which we denote by

$$G_{\vec{p}}^{I_1I_2I_3I_4}(x_i,\eta_i,\bar{\eta}_i) \equiv \langle \mathcal{O}_{p_1}^{I_1}\mathcal{O}_{p_2}^{I_2}\mathcal{O}_{p_3}^{I_3}\mathcal{O}_{p_4}^{I_4}\rangle. \quad (2)$$

A crucial point here is that the strength of the self-gluon coupling is larger than the coupling of gluons to gravitons [4]. Thus one can perform an expansion in $1/N$ in which gravity is $1/N$ suppressed. Schematically, we have

$$G_{\vec{p}}^{I_1I_2I_3I_4} = G_{\text{disc},\vec{p}}^{I_1I_2I_3I_4} + \frac{1}{N}G_{\text{tree-gluon},\vec{p}}^{I_1I_2I_3I_4} + \cdots. \quad (3)$$

The first "disconnected" term is a sum over products of two-point functions and takes the form of (generalized) free theory. In terms of OPE data, it contains the leading-order contributions to the three-point functions of the external operators with exchanged two-particle operators. We refer to the second term as the "tree-level" amplitude.

The superconformal Ward identities [42] allow us to split the correlator into two parts, each separately respecting crossing symmetry:

$$G_{\text{tree-gluon},\vec{p}}^{I_1I_2I_3I_4} = G_{0,\vec{p}}^{I_1I_2I_3I_4} + \mathcal{P}\mathcal{I}\mathcal{A}_{\vec{p}}^{I_1I_2I_3I_4}. \quad (4)$$

The term $G_{0,\vec{p}}$ contains all contributions due to protected multiplets at this order in $1/N$. The second term contains all the logarithmic terms which arise due to two-particle operators receiving anomalous dimensions. It contains kinematic factors $\mathcal{P}$ and $\mathcal{I}$, due to bosonic and fermionic symmetries, respectively. We define the propagator

$$g_{ij} = \frac{y_{ij}^2}{x_{ij}^2}, \qquad y_{ij}^2 = \langle \eta_i\eta_j\rangle\langle\bar{\eta}_i\bar{\eta}_j\rangle, \quad (5)$$

where $\langle\eta_i\eta_j\rangle = \eta_{ia}\eta_{jb}\epsilon^{ab}$ and $\langle\bar{\eta}_i\bar{\eta}_j\rangle = \bar{\eta}_{i\bar{a}}\bar{\eta}_{j\bar{b}}\epsilon^{\bar{a}\bar{b}}$. We introduce cross-ratios via

$$\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} = U = x\bar{x}, \qquad \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} = V = (1-x)(1-\bar{x}),$$
$$\frac{y_{12}^2y_{34}^2}{y_{13}^2y_{24}^2} = \tilde{U} = y\bar{y}, \qquad \frac{y_{14}^2y_{23}^2}{y_{13}^2y_{24}^2} = \tilde{V} = (1-y)(1-\bar{y}).$$

Note that $y = \frac{\langle\eta_1\eta_2\rangle\langle\eta_3\eta_4\rangle}{\langle\eta_1\eta_3\rangle\langle\eta_2\eta_4\rangle}$ and $\bar{y} = \frac{\langle\bar{\eta}_1\bar{\eta}_2\rangle\langle\bar{\eta}_3\bar{\eta}_4\rangle}{\langle\bar{\eta}_1\bar{\eta}_3\rangle\langle\bar{\eta}_2\bar{\eta}_4\rangle}$. The kinematic factors are then given by

$$\mathcal{P} \equiv \frac{g_{12}^{k_s}g_{14}^{k_t}g_{24}^{k_u}(g_{13}g_{24})^{p_3}}{\langle\bar{\eta}_1\bar{\eta}_3\rangle^2\langle\bar{\eta}_2\bar{\eta}_4\rangle^2}, \qquad \mathcal{I} = (x-y)(\bar{x}-y), \quad (6)$$

where

$$k_s = \frac{p_1+p_2-p_3-p_4}{2}, \qquad k_t = \frac{p_1+p_4-p_2-p_3}{2},$$
$$k_u = \frac{p_2+p_4-p_3-p_1}{2}.$$

Note that, due to the presence of the factor $\mathcal{I}$, the remaining factor $\mathcal{A}_{\vec{p}}$ has equal degrees in y and $\bar{y}$. Moreover, since $\mathcal{A}_{\vec{p}}$ is symmetric under $y,\bar{y}$ exchange, we can write it as a function of $\tilde{U},\tilde{V}$ (as well as U, V and $\vec{p}$).

The function $\mathcal{A}_{\vec{p}}$ admits a very compact and natural representation that extends the well-known Mellin transform [43,44] to the compact space. The transform makes manifest the so-called large p limit [35]—where p refers to the charges—which was found to be very useful in the context of $AdS_5 \times S^5$ [35,37] and $AdS_3 \times S^3$ backgrounds [40]. The generalized Mellin transform is defined via

$$\mathcal{A}_{\vec{p}}^{I_1I_2I_3I_4} = -\oint dsdt \oint d\tilde{s}d\tilde{t}\, U^sV^t\tilde{U}^{\tilde{s}}\tilde{V}^{\tilde{t}}\Gamma\mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4}, \quad (7)$$

where $\mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4} \equiv \mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4}(s,t,\tilde{s},\tilde{t})$. The kernel Γ factorizes into AdS_5 and S^3 contributions and takes the form $\Gamma = \mathfrak{S}\Gamma_s\Gamma_t\Gamma_u$ with

$$\mathfrak{S} = \pi^2\frac{(-)^{\tilde{t}}(-)^{\tilde{u}}}{\sin(\pi\tilde{t})\sin(\pi\tilde{u})}, \qquad \Gamma_s = \frac{\Gamma[-s]\Gamma[-s+k_s]}{\Gamma[1+\tilde{s}]\Gamma[1+\tilde{s}+k_s]} \quad (8)$$

and Γ_t and Γ_u defined similarly. Note that the Mellin variables obey

$$s+t+u = -p_3-1, \qquad \tilde{s}+\tilde{t}+\tilde{u} = p_3-2, \quad (9)$$

which may be used to eliminate u and $\tilde{u}$. Note also that the amplitude $\mathcal{A}_{\vec{p}}$ is polynomial in $\tilde{U}$ and $\tilde{V}$. In fact, the integral over $\tilde{s}$, $\tilde{t}$ becomes a discrete sum over a certain domain that in our case is given by

$$T = \{\tilde{s} \geq \max(0,-k_s),\tilde{t},\tilde{u} \geq 0\}. \quad (10)$$

The contour integral in s and t requires a little care and we will return to this point in the next section. The double integral (7), when combined with the amplitude $\mathcal{M}_{\vec{p}}$ given in the next section, precisely coincides with the result given in [4].

III. BCJ AND COLOR KINEMATICS IN $AdS_5 \times S^3$

Let us consider the field theory amplitude computed in [4] within this formalism. As in [5], we consider the reduced Mellin amplitude $\mathcal{M}_{\vec{p}}$. In the color-factor basis, $\mathcal{M}_{\vec{p}}$ takes the following simple form when written in the large p formalism ($\mathbf{s} = s+\tilde{s}, \mathbf{t} = t+\tilde{t}, \mathbf{u} = u+\tilde{u}$):

$$\mathcal{M}_{\vec{p}}^{I_1I_2I_3I_4} = \frac{n_sc_s}{\mathbf{s}+1} + \frac{n_tc_t}{\mathbf{t}+1} + \frac{n_uc_u}{\mathbf{u}+1}. \quad (11)$$

Here we have

$$n_s = \frac{1}{3}\left(\frac{1}{\mathbf{t}+1} - \frac{1}{\mathbf{u}+1}\right), \qquad c_s = f^{I_1I_2J}f^{I_3I_4J}, \quad (12)$$

and similarly for c_t, n_t, c_u, n_u, where the bold-face variables satisfy

$$\mathbf{s}+\mathbf{t}+\mathbf{u} = -3. \quad (13)$$

Note that the large p limit [35] ensures that the correlator reduces to the flat S matrix (see e.g. [45] for the flat space expression) with the Mandelstam replaced by boldface variables. The large p limit can be seen as a generalization of the usual flat space limit [44] in which only the AdS (Mellin) variables s, t are taken to be large.

In principle, away from large p, nothing would prevent the amplitude from depending on $s, \tilde{s}, \ldots$ separately. However, from (11) we see that in fact the full amplitude $\mathcal{M}$ is just a function of bold-facevariables, as a consequence of a hidden 8d conformal symmetry of the amplitude. This is completely analogous to $\mathrm{AdS}_3 \times S^3$ [40,46–48] and $\mathrm{AdS}_5 \times S^5$ [20,35] backgrounds where the dynamics is also controlled by hidden conformal symmetries.

In fact, due to (13), the Mellin amplitude $\mathcal{M}$ is literally the same function as the flat space amplitude with the Mandelstam variables s, t, and u replaced by $(\mathbf{s}+1)$, $(\mathbf{t}+1)$, and $(\mathbf{u}+1)$, respectively. It follows immediately that all relations obeyed by the flat space amplitudes also apply to $\mathcal{M}$. Note that it is not trivial that this holds; for example, the analogous relation for $\mathrm{AdS}_5 \times S^5$ is $\mathbf{s}+\mathbf{t}+\mathbf{u} = -4$ [35]. As an example of the properties obeyed by $\mathcal{M}$ we have that

$$n_s + n_t + n_u = 0, \qquad c_s + c_t + c_u = 0, \tag{17}$$

which gives an AdS version of the color-kinematic duality, which was already observed in [5]. Note that (17) captures this duality for all Kaluza-Klein modes. This duality is intimately connected to the so-called BCJ relations between color-ordered amplitudes. Recall that the full color-dressed amplitude is

$$\mathcal{M}_{\vec{p}}^{I_1 I_2 I_3 I_4} = \sum_{\mathcal{P}(2,3,4)} \mathrm{Tr}(T^{I_1} T^{I_2} T^{I_3} T^{I_4}) \mathcal{M}_{\vec{p}}(1,2,3,4),$$

where $\mathcal{M}_{\vec{p}}(1,2,3,4)$ are the color-ordered amplitudes and $\mathcal{P}(2,3,4)$ are the permutations of (2, 3, 4). The translation from one basis to the other is

$$\begin{aligned} c_s = {} & \mathrm{Tr}(T^{I_1} T^{I_2} T^{I_3} T^{I_4}) + \mathrm{Tr}(T^{I_1} T^{I_4} T^{I_3} T^{I_2}) \\ & - \mathrm{Tr}(T^{I_1} T^{I_2} T^{I_4} T^{I_3}) - \mathrm{Tr}(T^{I_1} T^{I_3} T^{I_4} T^{I_2}) \end{aligned} \tag{18}$$

and similarly for c_t, c_u. The color-ordered amplitudes then read as follows:

$$\mathcal{M}_{\vec{p}}(1,2,3,4) = \mathcal{M}_{\vec{p}}(1,4,3,2) = \frac{n_s}{\mathbf{s}+1} - \frac{n_t}{\mathbf{t}+1}, \tag{19}$$

and analogously for the others. All the relations obeyed by the flat space color-ordered amplitudes obviously also hold here. For example, from (19), we can see that an analogous $U(1)$ decoupling identity holds:

$$\mathcal{M}_{\vec{p}}(1,2,3,4) + \mathcal{M}_{\vec{p}}(1,2,4,3) + \mathcal{M}_{\vec{p}}(1,3,2,4) = 0. \tag{20}$$

Moreover, it is immediate to see that (a generalization of) BCJ relations also holds:

$$(\mathbf{t}+1)\mathcal{M}_{\vec{p}}(1,2,3,4) = (\mathbf{u}+1)\mathcal{M}_{\vec{p}}(1,3,4,2), \tag{21}$$

where we used the on-shell relation (13). We stress again that the relations (21) capture the appearance of BCJ relations in AdS for *all* Kaluza-Klein modes. Such relations are manifest at level of the reduced Mellin amplitude while they do not hold, at least directly, for the full Mellin amplitude [7]. It is an interesting open question how such relations might extend to higher point amplitudes in AdS and what the role of a reduced Mellin amplitude might be in this regard.

We now return to the issue of the contour in the Mellin integral (7). The presence of poles at $\mathbf{s} = -1$, $\mathbf{t} = -1$ and $\mathbf{u} = -1$ is potentially a problem for the contour of integration. Indeed (13) implies the simultaneous presence of these poles leaves no region in the real s, t plane for the contour to pass through, while separating left-moving and right-moving sequences of poles in the Mellin integrand. Thus the same property which leads to the direct analogy with the flat space amplitudes also leads to a subtlety in returning to position space from Mellin space. For the color-ordered amplitudes, one does not have all three poles present simultaneously. Thus we propose that the correct definition for the contour is tied to the color ordering and we define analogously a color-ordered correlator:

$$\mathcal{A}(1,2,3,4) = -\oint ds dt \oint d\tilde{s} d\tilde{t} U^s V^t \tilde{U}^{\tilde{s}} \tilde{V}^{\tilde{t}} \Gamma \mathcal{M}(1,2,3,4).$$

The contour can now be taken to lie slightly below $\mathbf{s} = -1$ and $\mathbf{t} = -1$. Note then that this introduces a subtlety in interpreting the BCJ relations (21) back in position space, since the left- and right-hand sides of these equations are to be integrated over slightly different contours.

Finally, let us observe that there is also an AdS version of the double-copy prescription [5]. Replacing color with kinematic factors we get

$$\begin{aligned} \mathcal{M}_{\vec{p}}^{I_1 I_2 I_3 I_4} & \underset{c_i \to n_i}{\longrightarrow} \frac{n_s^2}{\mathbf{s}+1} + \frac{n_t^2}{\mathbf{t}+1} + \frac{n_u^2}{\mathbf{u}+1} \\ & = \frac{1}{(\mathbf{s}+1)(\mathbf{t}+1)(\mathbf{u}+1)} \propto \mathcal{M}_{\vec{p}}^{\mathrm{SUGRA}}. \end{aligned} \tag{22}$$

This is nothing but the supergravity (SUGRA) amplitude in $\mathrm{AdS}_5 \times S^5$ [12] rewritten in the large p formalism [35], upon reinterpreting $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$ as the $\mathcal{N} = 4$ variables, i.e. subject to the constraint $\mathbf{u} = -\mathbf{s} - \mathbf{t} - 4$.

IV. LONG DISCONNECTED FREE THEORY

We now examine the anomalous dimensions of the double-trace operators exchanged in the OPE at large N. To do so, we need the superconformal block decompositions of disconnected free theory and of the $\log U$

discontinuity of the tree-level correlator. We also have to deal with the flavor structure of the amplitude. Since this just amounts to considering certain symmetric or antisymmetric combinations built out of the correlator, we postpone the discussion on flavor structures to the end of the next section. More details can be found in [6].

We begin with disconnected free theory. Only representations with a definite parity under $t \leftrightarrow u$ exchange enter the s-channel OPE. Thus we need to decompose the following combinations of disconnected diagrams:

$$G^{\pm}_{\mathrm{disc},pqpq} = \delta_{pq}\frac{g^{p}_{14}g^{p}_{23}}{\langle\bar{\eta}_1\bar{\eta}_4\rangle^2\langle\bar{\eta}_2\bar{\eta}_3\rangle^2} \pm \frac{g^{p}_{13}g^{q}_{24}}{\langle\bar{\eta}_1\bar{\eta}_3\rangle^2\langle\bar{\eta}_2\bar{\eta}_4\rangle^2}. \tag{23}$$

Following [42], we extract the unprotected contribution and decompose it into long superblocks $\mathbb{L}_{\vec{\tau}}$ [50], whose form we give in the Appendix:

$$G^{\pm}_{\mathrm{disc},pqpq}|_{\mathrm{long}} = \sum_{\vec{\tau}} L^{\pm}_{\vec{\tau}}\mathbb{L}_{\vec{\tau}}. \tag{24}$$

We find that the coefficients are particularly simple:

$$L^{\pm}_{\vec{\tau}} = -\frac{\pm 1 + (-1)^{a+l}\delta_{pq}}{(p-1)(q-1)}A_h A_{\bar{h}} B_j B_{\bar{j}} \boldsymbol{\delta}. \tag{25}$$

Here the A and B factors are given, respectively, by

$$A_h = \frac{\Gamma(h+\frac{p-q}{2})\Gamma(h-\frac{p-q}{2})\Gamma(h+\frac{p+q}{2}-1)}{\Gamma(2h-1)\Gamma(h-\frac{p+q}{2}+1)}, \tag{26}$$

$$B_j = \frac{\Gamma(2-2j)[\Gamma(\frac{p+q}{2}+j-1)\Gamma(\frac{p+q}{2}-j)]^{-1}}{\Gamma(1-j+\frac{p-q}{2})\Gamma(1-j-\frac{p-q}{2})} \tag{27}$$

while $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta} = \frac{\delta^{(2)}_{h,j} - \delta^{(2)}_{\bar{h},\bar{j}}}{\delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},\bar{j}}}, \qquad \delta^{(2)}_{h,j} = (h-j)(h+j-1). \tag{28}$$

Here, $h, \bar{h}$ and $j, \bar{j}$ label the conformal and internal representations, respectively. We can also express them in terms of the more common quantum labels $\vec{\tau} = (\tau, b, l, a)$:

$$h = \frac{\tau}{2} + 1 + l, \quad \bar{h} = \frac{\tau}{2}, \quad j = -\frac{b}{2} - a, \quad \bar{j} = -\frac{b}{2}, \tag{29}$$

where τ and l are twist and spin, respectively, and b and a can be seen as the analogs of twist and spin on the sphere. Note the different ways the two internal $SU(2)$ factors enter the coefficients. On the one hand, $SU(2)_L$ only comes in through the function $B_{\bar{j}}$. On the other hand, the decomposition under the R-symmetry group $SU(2)_R$ produces also the function $\boldsymbol{\delta}$ and, in particular, the combination $\delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},\bar{j}}$. It is not difficult to see that this object is the eigenvalue of a Casimir operator acting on the blocks. We refer to [20,41] for more details, where the logic is exactly the same, although the background is different.

V. ANOMALOUS DIMENSIONS AND RESIDUAL DEGENERACY

The details of the computation are similar to the $AdS_5 \times S^5$ case [16,19] and the $AdS_3 \times S^3$ case [40]. The main difference with the $\mathcal{N} = 4$ case is that here double-trace operators have flavor structure. There will then be two types of anomalous dimensions, those of operators exchanged in symmetric or antisymmetric channels.

At large N, the operators acquiring anomalous dimensions are of the schematic form:

$$\mathcal{O}^{\pm}_{pq} = \mathbb{P}^{\pm}_{I_1 I_2}\mathcal{O}^{I_1}_{p}\partial^l \Box^{\frac{1}{2}(\tau-p-q)}\mathcal{O}^{I_2}_{q}, \tag{30}$$

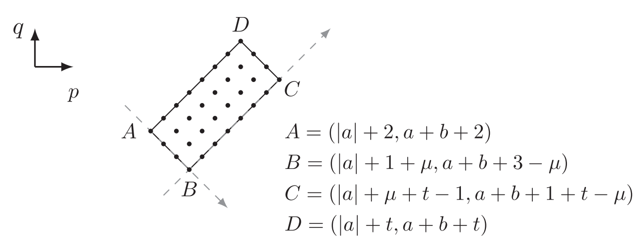
where $\mathbb{P}_{ij}$ is an appropriate projector that projects onto symmetric or antisymmetric representations of the gauge group exchanged in the OPE. For any given quantum numbers $\vec{\tau} = (\tau, b, l, a)$, the number of operators exchanged in the OPE can be represented with the number of pairs (pq) filling a rectangle [19]:

$$R_{\vec{\tau}} = \left\{(p,q)\colon \begin{array}{l} p = i + |a| + 1 + r \\ q = i + a + 1 + b - r \end{array} \middle| \begin{array}{l} i = 1, \ldots, t-1 \\ r = 0, \ldots, \mu - 1 \end{array}\right\}. \tag{31}$$

Thus $R_{\vec{\tau}}$ consists of $d = \mu(t-1)$ lattice points where

$$t \equiv \frac{(\tau - b)}{2} - \frac{(a + |a|)}{2}, \qquad \mu \equiv \begin{cases} \left\lfloor \frac{b+a-|a|+2}{2} \right\rfloor & a + l \,\text{even}, \\ \left\lfloor \frac{b+a-|a|+1}{2} \right\rfloor & a + l \,\text{odd}. \end{cases}$$

The picture below shows an example with $\mu = 4$, $t = 9$:



Let us now consider the OPE at genus zero. This is best cast in a matrix form [16]. First, arrange a $d \times d$ matrix of correlators:

$$\delta_{p_1 p_3}\delta_{p_2 p_4} G^{\pm}_{\mathrm{disc},\vec{p}}|_{\mathrm{long}} + \frac{1}{N}\mathcal{P}(x-y)(\bar{x}-y)\mathcal{A}^{\pm}_{\vec{p}} \tag{32}$$

with the pairs (p_1, p_2) and (p_3, p_4) running over the same $R_{\vec{\tau}}$. Here, we denote by $\mathcal{A}^{\pm}_{\vec{p}}$ the inverse Mellin transform of the following Mellin amplitudes:

$$\begin{aligned}\mathcal{M}^{\pm}_{\vec{p}} &= \frac{1}{2}(\mathcal{M}_{\vec{p}}(1,2,3,4) \pm \mathcal{M}_{\vec{p}}(1,3,4,2)) \\ &= \frac{1}{2}\frac{1}{\mathbf{s}+1}\left(\frac{1}{\mathbf{t}+1} \pm \frac{1}{\mathbf{u}+1}\right).\end{aligned} \tag{33}$$

The OPE equations then read

$$\mathbf{C}^{\pm}_{\vec{\tau}}\mathbf{C}^{\pm T}_{\vec{\tau}} = \mathbf{L}^{\pm}_{\vec{\tau}}, \qquad \mathbf{C}^{\pm}_{\vec{\tau}}\boldsymbol{\eta}^{\pm}_{\vec{\tau}}\mathbf{C}^{\pm T}_{\vec{\tau}} = \mathbf{M}^{\pm}_{\vec{\tau}}. \tag{34}$$

Here $\mathbf{L}^{\pm}_{\vec{\tau}}$ is a (diagonal) matrix of conformal partial waves (CPW) coefficients of disconnected free theory (24), while $\mathbf{M}^{\pm}_{\vec{\tau}}$ is a matrix of CPW coefficients of the $\log U$ discontinuity of $\mathcal{A}^{\pm}_{\vec{p}}$:

$$\mathcal{P}(x-y)(\bar{x}-y)\mathcal{A}^{\pm}_{\vec{p}}|_{\log U} = \sum_{\vec{\tau}} M^{\pm}_{\vec{\tau}}\mathbb{L}_{\vec{\tau}}. \tag{35}$$

Finally, $\boldsymbol{\eta}^{\pm}_{\vec{\tau}}$ is a diagonal matrix of anomalous dimensions and $\mathbf{C}^{\pm}_{\vec{\tau}} = \langle \mathcal{O}_p\mathcal{O}_q\mathcal{K}^{\pm}_{rs}\rangle$ is a matrix of three-point functions with two half-BPS and one double-trace operator. Here, we denote with $\mathcal{K}^{\pm}_{rs}$ the true two-particle operator in interacting theory, that differs from $\mathcal{O}^{\pm}_{pq}$, precisely because there is mixing. Note that, since $\mathcal{A}^{\pm}_{\vec{p}}$ can be written as a function of $\tilde{U}$ and $\tilde{V}$, the $SU(2)_L \times SU(2)_R$ representations contributing to $\mathbf{M}^{\pm}_{\vec{\tau}}$ can be reorganized into $SO(4)$ representations, while this is not so for the disconnected contribution $\mathbf{L}^{\pm}_{\vec{\tau}}$.

It is simple to show, with some linear algebra, that the anomalous dimensions are the eigenvalues of the matrix $\mathbf{M}_{\vec{\tau}}(\mathbf{L}^{\pm}_{\vec{\tau}})^{-1}$. By computing them for various quantum numbers, we find a very simple formula:

$$\eta^{\pm}_{\vec{\tau}} = -\frac{2}{N}\frac{\delta^{(2)}_{h,j}\delta^{(2)}_{\bar{h},j}}{(l^{\pm}_{8d}+1)_4}. \tag{36}$$

Here $l^{\pm}_{8d} = l + 2(p-2) + \frac{1\mp(-1)^{a+l}}{2} - |a|$ can be interpreted as an effective $8d$ spin, defined in analogy to the partial wave decomposition of the flat amplitude in $8d$ [20]. Note that (36) only depends on p, not q, or, in other words, operators on the same vertical line in the rectangle will acquire the same anomalous dimensions.

We conclude by commenting on the flavor structure of the correlator. One way to deal with it is to decompose t, u channel flavor structures (of both disconnected and tree-level correlators) in a basis of representations appearing in the tensor product of two adjoint representations in the s channel. We then read off the coefficients associated to each flavor structure which are of the form

$$G_a^{I_1I_2I_3I_4} \propto G_t^{I_1I_2I_3I_4} \pm G_u^{I_1I_2I_3I_4},$$

where a runs over all symmetric (antisymmetric) representations in adj $\otimes$ adj with the proportionality coefficient depending on the specific group and exchanged representation. Examples of such coefficients are given in [6]. The unmixing procedure can then be consistently carried for each a separately. For the symmetric (antisymmetric) representations the relevant double-trace operators exchanged are of the type $\mathcal{O}^{+}_{pq}$ ($\mathcal{O}^{-}_{pq}$) with the respective anomalous dimensions proportional to $\eta^{+}_{\vec{\tau}}$ ($\eta^{-}_{\vec{\tau}}$). The only antisymmetric representation exchanged is the adjoint.

VI. OUTLOOK AND CONCLUSIONS

In this paper we discussed color-kinematics and BCJ relations between color-ordered amplitudes of super gluons in $\mathrm{AdS}_5 \times S^3$, making use of the large p formalism [35]. We believe this formalism makes clearer the direct parallel with the flat space versions of these relations and that they hold for *all* Kaluza-Klein modes. In turn, this shows, as in flat space, that there is a precise relation between color-kinematic duality and BCJ relations.

We then computed the anomalous dimensions in the large N limit. Due to the $8d$ hidden conformal symmetry, and as for the analogous problems in $\mathrm{AdS}_5 \times S^5$ and $\mathrm{AdS}_3 \times S^3$, the anomalous dimensions exhibit a residual degeneracy, nicely captured by the vertical columns of the rectangular lattice $R_{\vec{\tau}}$ described in (31) and below.

These results open many exciting possibilities. Firstly, as mentioned in the introduction, the knowledge of the anomalous dimensions can be of use in bootstrapping loop corrections, beyond the lowest charge correlator studied in [6], and help in further exploring whether some features of the double-copy relations persist beyond tree level. Moreover, following the procedure described in [49], one can imagine treating the theory of gluons as an effective model and introduce higher-order D^nF^4 interactions, analogous to the higher curvature corrections present for gravitons in e.g. $\mathrm{AdS}_5 \times S^5$. Much like the curvature corrections responsible for completing the Virasoro-Shapiro amplitude in $\mathrm{AdS}_5 \times S^5$ [34,37], such terms will induce a splitting of the residual degeneracy in the anomalous dimensions. Finally, the computation of open and closed string amplitudes in AdS might give a clue on how Kawa-Lewellen-Tye and world-sheet monodromy relations work in a curved spacetime.

ACKNOWLEDGMENTS

We thank F. Aprile, P. Heslop, K. Rigatos and X. Zhou for important feedback and comments on the manuscript. M. S. thanks D. Bufalini, H. Paul and S. Rawash for useful discussions. J. M. D. is supported in part by the ERC Consolidator Grant No. 648630 IQFT. R. G. is supported by an STFC studentship. M. S. is supported by a Mayflower studentship from the University of Southampton.

APPENDIX: SUPERCONFORMAL BLOCKS

We quickly review here the superconformal blocks used in this paper. The long blocks [50], which capture the unprotected multiplets exchanged in the OPE, are the simplest. They are the product of ordinary conformal and internal blocks for both $SU(2)$ factors:

$$\mathbb{L}_{\vec{\tau}} = \mathcal{P}(x-y)(\bar{x}-y)\left(\frac{\tilde{U}}{U}\right)^{p_3}\mathcal{G}_{\tau,l}(x,\bar{x})\mathcal{H}_{b,a}(y,\bar{y}). \tag{A1}$$

Here we have

$$\mathcal{G}_{\tau,l} = \frac{(-1)^l}{(x-\bar{x})U^{\frac{p_{43}}{2}}}(\mathcal{F}^+_{\frac{\tau}{2}+1+l}(x)\mathcal{F}^+_{\frac{\tau}{2}}(\bar{x}) - (x \leftrightarrow \bar{x})),$$
$$\mathcal{H}_{b,a} = \frac{1}{\tilde{U}^{2-\frac{p_{43}}{2}}}\mathcal{F}^-_{-\frac{b}{2}-a}(y)\mathcal{F}^-_{-\frac{b}{2}}(\bar{y}), \quad \text{(A2)}$$

with

$$\mathcal{F}^\pm_h(x) = x^h {}_2F_1\left[h \mp \frac{p_{12}}{2}, h \mp \frac{p_{43}}{2}, 2h\right](x). \quad \text{(A3)}$$

Note, $\mathcal{G}_{\tau,l}(x,\bar{x})$ are the standard $4d$ conformal blocks (up to a shift by 2 in the twist τ) and $\mathcal{H}_{b,a}(y,\bar{y})$ are internal blocks given by the product of two $SU(2)$ spherical harmonics, one for the R-symmetry group $SU(2)_R$ and the other for the flavor group $SU(2)_L$. Finally, τ and l are, respectively, twist and spin, and b and a label the different representations of $SO(4)$ and can be viewed as the analogs of twist and spin on the sphere.

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