Bootstrapping string theory on $\mathrm{AdS}_5 \times S^5$

J. M. Drummond, H. Paul, and M. Santagata

School of Physics and Astronomy, University of Southampton, Highfield, SO17 1BJ, United Kingdom

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We make an ansatz for the Mellin representation of the four-point amplitude of half-BPS operators of arbitrary charges at order $\lambda^{-\frac{5}{2}}$ in an expansion around the supergravity limit. Crossing symmetry and a set of constraints on the form of the spectrum uniquely fix the amplitude and double-trace anomalous dimensions at this order. The results exhibit a number of natural patterns which suggest that the bootstrap approach outlined here will extend to higher orders in a simple way.

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I. INTRODUCTION

Recently, great progress has been made in understanding the structure of amplitudes in anti–de Sitter space by imposing consistency of the boundary conformal field theory. A particular case has been the focus of many investigations, namely $\mathcal{N} = 4$ super-Yang-Mills theory on the boundary, which corresponds to type-IIB superstrings interacting in the $\mathrm{AdS}_5 \times S^5$ bulk [1].

Physical quantities depend on the gauge coupling g and the gauge group, which we take to be $SU(N)$. The holographic relation between the bulk and boundary theories implies that the spectrum of the conformal field theory is drastically simplified in the supergravity regime $0 \ll \lambda \ll N$, where $\lambda = g^2 N$ is the 't Hooft coupling. In this limit, the spectrum is given by single-particle half-BPS operators and their multitrace products, while other operators, corresponding to excited string states, acquire infinite scaling dimensions in the limit and decouple.

We study four-point functions of single-particle operators in a double expansion in $1/N$ and $\lambda^{-\frac{1}{2}}$ around the supergravity limit. The leading large-N contributions to the operator product expansion (OPE) come from a degenerate spectrum of double-trace operators [2–12].

The supergravity contribution to the four-point functions has a compact Mellin representation [13,14]. The mixing between the double-trace operators can be resolved, yielding a very simple formula for the leading contributions to their anomalous dimensions [4,7]. In fact, the degeneracy is not fully lifted in supergravity. The residual partial degeneracy can be understood in terms of a surprising ten-dimensional conformal symmetry [15].

Recent papers have explored the structure of string corrections to the tree-level supergravity amplitudes. Constraints from the flat-space limit [8] and results derived using localization [16] allowed a family of correlation functions to be fixed at the first two nontrivial orders, $\lambda^{-\frac{3}{2}}$ and $\lambda^{-\frac{5}{2}}$. The order-$\lambda^{-\frac{3}{2}}$ corrections can be determined for every half-BPS four-point function from the relevant term in the flat-space Virasoro-Shapiro amplitude [17]. From these results, it was found in Ref. [17] that the double-trace spectrum reflected the ten-dimensional symmetry structure, even when taking into account the $\lambda^{-\frac{3}{2}}$ corrections. We explore this feature further here, generalizing the results of previous papers to determine all half-BPS four-point functions up to order $\lambda^{-\frac{5}{2}}$.

We use an ansatz for the Mellin amplitude as a function of the external charges and minimal assumptions about the form of the corrections to the spectrum. Combined with crossing symmetry and OPE consistency, the above is sufficient to determine the $\lambda^{-\frac{5}{2}}$ corrections to the correlation functions, as well as the spectrum and three-point functions of the double-trace operators. The results reveal many beautiful features that are suggestive of a general pattern which should allow the method to be simply extended to yet higher orders in $\lambda^{-\frac{1}{2}}$. As observed in Ref. [17], we find that the ten-dimensional effective spin determines which operators receive string corrections to their dimensions and three-point functions. Moreover, at order $\lambda^{-\frac{5}{2}}$, we find that the partial degeneracy is broken at finite twist in a way consistent with other general features of the spectrum and suggestive of a general structure.

II. HALF-BPS FOUR-POINT FUNCTIONS

We recall that $\mathcal{N} = 4$ super-Yang-Mills theory has a spectrum of single-particle half-BPS operators given by

$$\mathcal{O}_p(x, y) = y^{R_1} \ldots y^{R_p} \mathrm{tr}(\phi_{R_1} \ldots \phi_{R_p})(x) + \cdots. \quad (1)$$

Here, $y^2 = 0$, and we omit $1/N$ suppressed multitrace contributions determined by the condition that $\mathcal{O}_p$ should be orthogonal to all multitrace operators [7].

Here, we focus on four-point functions of such operators which, due to superconformal symmetry, have the form

$$\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle = \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{free}} + \mathcal{P}\mathcal{I}\mathcal{H}. \tag{2}$$

The first term on the rhs is the contribution from free theory where $g = 0$. The second term contains the factors $\mathcal{P}$ and $\mathcal{I}$, given in Eqs. (A5) and (A7), and $\mathcal{H}(x,\bar{x};y,\bar{y})$, which encodes the dynamical contribution to the correlator. It depends on conformal and $su(4)$ cross ratios:

$$U = x\bar{x} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad V = (1-x)(1-\bar{x}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},$$
$$\frac{1}{\sigma} = y\bar{y} = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \qquad \frac{\tau}{\sigma} = (1-y)(1-\bar{y}) = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}. \tag{3}$$

Here, we are concerned only with the leading large-N contribution to $\mathcal{H}$ corresponding to tree-level string amplitudes. This term admits an expansion in $\lambda^{-\frac{1}{2}}$,

$$\mathcal{H} = \frac{1}{N^2}[\mathcal{H}^{(0)} + \lambda^{-\frac{3}{2}}\mathcal{H}^{(3)} + \lambda^{-\frac{5}{2}}\mathcal{H}^{(5)} + \cdots]. \tag{4}$$

The leading term $\mathcal{H}^{(0)}$ in the above expansion was determined for all external charges $\langle p_1p_2p_3p_4\rangle$ in Refs. [13,14], extending previous results (see, e.g., Refs. [18–20]) and verified by more recent supergravity analyses [21–23].

As in Refs. [13,14], we will use a Mellin representation,

$$\mathcal{H}^{(n)} = \int \frac{ds}{2}\frac{dt}{2} U^{\frac{s+p_3-p_4}{2}} V^{\frac{t-p_2-p_3}{2}} \Gamma \mathcal{M}^{(n)}(s,t;\sigma,\tau),$$
$$\Gamma = \Gamma\left[\frac{p_1+p_2-s}{2}\right]\Gamma\left[\frac{p_3+p_4-s}{2}\right]\Gamma\left[\frac{p_1+p_4-t}{2}\right]$$
$$\times\Gamma\left[\frac{p_2+p_3-t}{2}\right]\Gamma\left[\frac{p_1+p_3-u}{2}\right]\Gamma\left[\frac{p_2+p_4-u}{2}\right], \tag{5}$$

where the Mandelstam-type variables s, t, u obey

$$s+t+u = 2\Sigma - 4, \qquad \Sigma = \frac{1}{2}(p_1+p_2+p_3+p_4). \tag{6}$$

We give the Mellin amplitude of Refs. [13,14] in Eq. (A1) in the Appendix. The most important point here is that it reduces to the flat-space supergravity amplitude in the large s, t, u limit,

$$\mathcal{M}^{(0)} \to B(\sigma,\tau)/(stu), \qquad B(\sigma,\tau) = \sum_{i,j}\mathcal{N}_{ijk}\sigma^i\tau^j, \tag{7}$$

where the coefficients $\mathcal{N}_{ijk}$ are given in Eq. (A3). In fact, the large s, t, u limit at each order in $\lambda^{-\frac{1}{2}}$ is controlled by the flat-space Virasoro-Shapiro amplitude $\mathcal{V}$ [8,16,24,25],

$$\mathcal{V} = \exp\left\{\sum_{n\geq 1}\frac{2\zeta_{2n+1}}{2n+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})\right\}. \tag{8}$$

The precise relation between $\mathcal{V}$ and $\mathcal{M}$ requires an integral which gives the leading large s, t, u behavior:

$$\mathcal{M}^{(3)} \to 2^{-3}(\Sigma-1)_3 B(\sigma,\tau)\times 2\zeta_3, \tag{9}$$

$$\mathcal{M}^{(5)} \to 2^{-5}(\Sigma-1)_5 B(\sigma,\tau)\times \zeta_5(s^2+t^2+u^2). \tag{10}$$

The poles in the factor Γ are due to unprotected double-trace operators exchanged in the OPE. The remaining poles in the supergravity Mellin amplitude $\mathcal{M}^{(0)}$ are due to long single-trace contributions (or excited string state contributions), which must cancel against corresponding contributions present in the free-theory term in Eq. (2), since they should be absent from the supergravity spectrum. The $\lambda^{-\frac{1}{2}}$ corrections should then have no such poles and are therefore polynomial in s, t, u [8,16,24,25]. It follows [17] that the result of Eq. (9) for $\mathcal{M}^{(3)}$ is in fact complete. The limit in Eq. (10) for $\mathcal{M}^{(5)}$, however, only determines the quadratic terms and does not specify additional linear and constant contributions in s and t,

$$\mathcal{M}^{(5)} = \zeta_5[2^{-5}(\Sigma-1)_5 B(\sigma,\tau)(s^2+t^2+u^2)$$
$$+\alpha(\sigma,\tau)s + \beta(\sigma,\tau)t + \gamma(\sigma,\tau)]. \tag{11}$$

The coefficients α, β, γ are currently only known for external charges $\langle 22qq\rangle$ [8,16] and, up to a single free parameter, $\langle 23q-1q\rangle$ [17], in which cases there is no dependence on σ and τ. In the case of $\langle 22qq\rangle$, we have

$$B = \frac{2^5 q^2}{(q-2)!}, \qquad \alpha = \frac{(q)_5 2q^2(q-2)}{(q-2)!}, \qquad \beta = 0,$$
$$\gamma = -\frac{q(q)_4}{(q-2)!}2(q^4+9q^3+10q^2-20q-25). \tag{12}$$

To describe an ansatz for $\mathcal{M}^{(5)}$, it is helpful to parametrize the charges as

$$\langle p_1p_2p_3p_4\rangle = \langle p-mpq-nq\rangle. \tag{13}$$

We use the $su(4)$ blocks $Y_{[aba]}(\sigma,\tau)$ [Eq. (A8)] instead of working with monomials in σ and τ,

$$\alpha(\sigma,\tau) = (\Sigma-1)_4\sum_{a,b}B_{a,b}\tilde{\alpha}_{a,b}Y_{[aba]}(\sigma,\tau), \tag{14}$$

and similarly for β, while for γ we replace $(\Sigma-1)_4$ with $(\Sigma-1)_3$. We have included an explicit factor,

$$B_{a,b} = \frac{pq(p-m)(q-n)(\Sigma-2)!b!(b+1)!(b+2+a)}{(p+r_1)!(p-r_2-2-a)!(q+r_3)!(q-r_4-2-a)!r_1!r_2!r_3!r_4!}, \tag{15}$$

where we use the notation

$$r_1 = \frac{b-m}{2}, \qquad r_2 = \frac{b+m}{2}, \qquad r_3 = \frac{b-n}{2},$$
$$r_4 = \frac{b+n}{2}. \tag{16}$$

The factor $B_{a,b}$ is in part motivated by the fact that

$$B(\sigma,\tau) = 8\sum_b B_{0,b} Y_{[0b0]}(\sigma,\tau), \tag{17}$$

and also by the fact that for each $su(4)$ channel $[a,b,a]$ we can consistently make a polynomial ansatz for $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ as a function of p and q for each required value of m and n. Based on the observed structure of the $\langle 22qq\rangle$ amplitude, we allow $\tilde{\alpha}$ and $\tilde{\beta}$ to be quadratic and $\tilde{\gamma}$ to be quartic in p and q.

Consistency with the $\langle 22qq\rangle$ results and crossing symmetry imposes many constraints among the free parameters of the ansatz but cannot fix it uniquely. To discuss the additional constraints we will impose, it is helpful to recall some facts about the double-trace spectrum.

III. THE DOUBLE-TRACE SPECTRUM

At leading order in the large-N expansion, only double-trace multiplets are exchanged in the OPE. The primaries take the form

$$\mathcal{O}_{pq} = \mathcal{O}_p \partial^l \Box^{\frac{1}{2}(\tau-p-q)} \mathcal{O}_q, \qquad (p<q). \tag{18}$$

For a given twist τ, spin l, and $su(4)$ channel $[a,b,a]$, all the operators in Eq. (18) are degenerate at leading order in large N. We parametrize the unprotected ones as in [7]

$$p = i+a+1+r, \qquad q = i+a+1+b-r,$$
$$i = 1,\ldots,(t-1), \qquad r = 0,\ldots,(\kappa-1), \tag{19}$$

where we use the notation

$$t \equiv (\tau-b)/2-a, \quad \kappa \equiv \begin{cases} \lfloor\frac{b+2}{2}\rfloor & a+l \text{ even}, \\ \lfloor\frac{b+1}{2}\rfloor & a+l \text{ odd}. \end{cases} \tag{20}$$

For each $\vec{\tau} = (\tau,l,a,b)$, there are $d=\kappa(t-1)$ degenerate operators which mix, and we denote the range of values of (p,q) by $\mathcal{D}_{\vec{\tau}}$. We will label the eigenstates $\mathcal{K}_{pq}$, with p and q parametrized by i and r as above. The mixing problem can be addressed by considering the OPE. If we arrange a $(d\times d)$ matrix of correlators with the pairs (p_1,p_2) and (p_3,p_4) ranging over the same set $\mathcal{D}_{\vec{\tau}}$, we have

$$O(N^0): \quad \langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle^{\text{long}}_{\text{free}} = \sum_{\vec{\tau}} A_{\vec{\tau}} \mathbb{L}_{\vec{\tau}},$$
$$O(N^{-2}): \quad \mathcal{PIH}|_{\log u} = \sum_{\vec{\tau}} M_{\vec{\tau}} \mathbb{L}_{\vec{\tau}}, \tag{21}$$

where $A_{\vec{\tau}}$ and $M_{\vec{\tau}}$ are matrices of coefficients and $\mathbb{L}_{\vec{\tau}}$ is the superblock for long multiplets given in Eq. (A4) in the Appendix. The coefficients $A_{\vec{\tau}}$ are independent of λ, while $M_{\vec{\tau}}$ receives contributions at all orders where the corresponding $\mathcal{M}^{(n)}$ is nonzero.

The matrices $A_{\vec{\tau}}$ and $M_{\vec{\tau}}$ are related to three-point functions and anomalous dimensions of the $\mathcal{K}_{pq}$,

$$\mathbb{C}_{\vec{\tau}}\mathbb{C}^T_{\vec{\tau}} = A_{\vec{\tau}}, \qquad \mathbb{C}_{\vec{\tau}}\eta_{\vec{\tau}}\mathbb{C}^T_{\vec{\tau}} = M_{\vec{\tau}}. \tag{22}$$

Here, $\mathbb{C}_{(pq),(\tilde{p}\tilde{q})}$ is a $(d\times d)$ matrix of three-point functions $[\langle \mathcal{O}_p\mathcal{O}_q\mathcal{K}_{\tilde{p}\tilde{q}}\rangle]$, and η is a diagonal matrix encoding the anomalous dimensions of the eigenstates $\mathcal{K}_{pq}$:

$$\Delta_{pq} = \tau - l + \frac{2}{N^2}\eta_{pq} + O\left(\frac{1}{N^4}\right). \tag{23}$$

The η and $\mathbb{C}$ matrices are expanded for large λ as

$$\eta_{pq} = \eta^{(0)}_{pq} + \lambda^{-\frac{3}{2}}\eta^{(3)}_{pq} + \lambda^{-\frac{5}{2}}\eta^{(5)}_{pq} + \cdots,$$
$$\mathbb{C} = \mathbb{C}^{(0)} + \lambda^{-\frac{3}{2}}\mathbb{C}^{(3)} + \lambda^{-\frac{5}{2}}\mathbb{C}^{(5)} + \cdots. \tag{24}$$

The tree-level contributions $\eta^{(0)}$ induced by Eq. (A1) take an astonishingly simple form [7],

$$\eta^{(0)}_{pq} = -2M_t M_{t+l+1}/(\ell_{10}+1)_6, \tag{25}$$

where the numerator is given by

$$M_t = (t-1)(t+a)(t+a+b+1)(t+2a+b+2), \tag{26}$$

and the denominator is a Pochhammer of the effective ten-dimensional spin,

$$\ell_{10}(p) = l+a+2(i+r)-1-\frac{1+(-1)^{a+l}}{2}. \tag{27}$$

In Ref. [15], it was recognized that the appearance of ℓ_{10} signals the presence of a ten-dimensional conformal

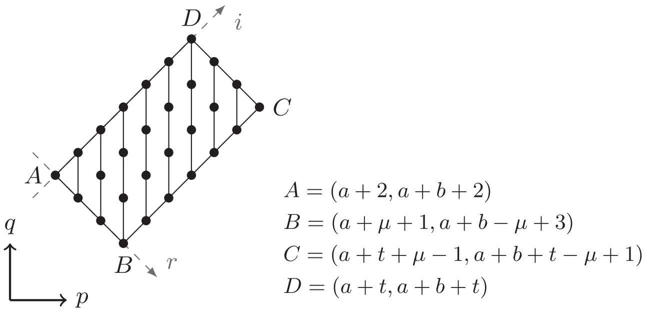

FIG. 1. Spectrum of anomalous dimensions $\eta^{(0)}_{pq}$ of the double-trace eigenstates $\mathcal{K}_{pq}$, represented by dots in the (p,q) plane. Anomalous dimensions which remain degenerate are connected by vertical lines of constant p.

symmetry. Note that ℓ_{10} only depends on the combination $i+r$ (or p, not q), so in general there are several states with the same anomalous dimension, and the resolution of the operator mixing in tree-level supergravity is only partial [7], as depicted in Fig. 1. This means that, although the eigenvalue problem is well posed, the leading-order three-point functions $\mathbb{C}^{(0)}$ are in general not fully determined by $\mathcal{M}^{(0)}$.

The first string corrections, $\eta^{(3)}$, are even simpler [17]. They are only nonzero for $l=a=0$ and $i=1$, $r=0$, (or $\ell_{10}=0$), where there is no partial degeneracy in the supergravity spectrum, corresponding to the leftmost corner labeled by A in Fig. 1. They take the form

$$\eta^{(3)} = -\frac{1}{840} M_t M_{t+l+1} \zeta_3 (t-1)_3 (t+b+1)_3. \tag{28}$$

Note that $\eta^{(0)}$ is a factor, and the total polynomial degree in t is 14. The fact that $\eta^{(3)}$ depends only on ℓ_{10}, instead of l, a, i, and r individually, suggests that the ten-dimensional conformal symmetry is respected also at order $\lambda^{-\frac{3}{2}}$. The corrections to the three-point functions are uniquely determined and vanish, $\mathbb{C}^{(3)}=0$.

The above result generalizes simply to states of the highest possible spin, $l=(n-3)$, at order $\lambda^{-\frac{n}{2}}$ with n odd. In this case, the only relevant terms in $\mathcal{M}^{(n)}$ are the highest powers in s, t, u, which are determined by the flat-space limit. Again, these terms are only nonzero for $a=0$ and $i=1$, $r=0$, and we find that $\mathbb{C}^{(n)}|_{l=n-3}=0$ with the anomalous dimension given by

$$\eta^{(n)}_{l=n-3} \propto -M_t M_{t+l+1} \zeta_n (t-1)_n (t+b+1)_n. \tag{29}$$

Note that the anomalous dimensions are invariant under

$$t \mapsto -t-b-2a-l-2. \tag{30}$$

As argued in Ref. [17], the ten-dimensional conformal symmetry, present in the supergravity anomalous dimensions $\eta^{(0)}_{pq}$, assigns an effective ten-dimensional spin ℓ_{10} to each eigenstate $\mathcal{K}_{pq}$ by means of Eq. (27). The above result [Eq. (29)] then suggests that this assignment is respected by the (tree-level) string corrections, to any order in $\lambda^{-\frac{1}{2}}$; i.e., the maximal exchanged spin ℓ_{10} at a given order $\lambda^{-\frac{n}{2}}$ in the flat-space Virasoro-Shapiro amplitude [Eq. (8)] determines which eigenstates $\mathcal{K}_{pq}$ develop an anomalous dimension, as well as which three-point functions are nonzero. We emphasize that this does not imply that the conformal symmetry is preserved by the string corrections—on the contrary, it will turn out that the $\lambda^{-\frac{5}{2}}$ corrections actually break it, albeit in a way consistent with the assignment [Eq. (27)].

Based on the above observations, we propose the following conditions on the double-trace data at order $\lambda^{-\frac{n}{2}}$:

1. $\eta^{(n)}_{pq} = 0 \quad \text{for } \ell_{10}(p) > n-3. \tag{31}$

2. $\mathbb{C}^{(n)}_{(pq),(\tilde{p}\tilde{q})} = 0 \quad \text{for } \ell_{10}(\tilde{p}) > n-3. \tag{32}$

3. $\eta^{(n)}_{i=1,r=0}$ is polynomial in t of degree $8+2n$. (33)

4. $\eta^{(n)}_{pq}$ only depends on $\ell_{10}(p)$ as $t\to\infty$. (34)

The constraint (31) says that ℓ_{10} dictates the nonzero contributions to η and generalizes the highest-spin $l=n-3$, $a=0$ result from Eq. (29). Similarly, the condition (32) says that the columns of $\mathbb{C}^{(n)}$ corresponding to operators with too high ten-dimensional spin vanish. In the $n=3$ case, it implies $\mathbb{C}^{(3)}=0$, since the first equation in Eq. (22) implies up to rescaling that $\mathbb{C}^{(0)}$ is an orthogonal matrix. Using the fact that $A_{\vec{\tau}}$ is independent of λ, its first correction $\mathbb{C}^{(3)}$ obeys

$$\mathbb{C}^{(3)}\mathbb{C}^{(0)T} + \mathbb{C}^{(0)}\mathbb{C}^{(3)T} = 0, \tag{35}$$

and therefore, after the change of basis, it is antisymmetric. If all but the first column vanishes, then the whole matrix vanishes. Importantly, for $n=5$, the same condition is weaker than the condition $\mathbb{C}^{(5)}=0$ examined in Ref. [17], since now there are generically three nonzero columns.

The condition (33) is an assumption on the anomalous dimension in the case of no partial degeneracy. The polynomial should obey the symmetry in Eq. (30) and is of the same order as in the maximal spin case [Eq. (29)]. The fourth condition (34) was also observed in Ref. [17], albeit under the (erroneously) stronger assumption $\mathbb{C}^{(5)}=0$. It relates to the restoration of ten-dimensional Lorentz symmetry in the flat-space limit (corresponding to $t\to\infty$).

IV. RESULTS

Imposing the conditions (31)–(34) in the case $n=5$, we find a unique consistent solution for the Mellin amplitude and the spectrum. We emphasize that the existence of a solution consistent with the ansatz for the Mellin amplitude, crossing symmetry, and the spectrum constraints is highly nontrivial. Actually, various computations in some channels have revealed that the constraints in Eqs. (31) and (32) are really a consequence of imposing Eq. (33) and an ansatz of the form [Eq. (14)] for the Mellin amplitude.

Here we summarize the form of the $\lambda^{-\frac{5}{2}}$ amplitude and the spectrum resulting from the above assumptions. First, we find that the $su(4)$ channels are constrained by $a \leq 2$, consistent with the ten-dimensional spin obeying $\ell_{10} \leq 2$ at this order. The resulting partial wave coefficients are

$$\begin{aligned}
&\tilde{\alpha}_{2,b} = \tilde{\beta}_{2,b} = 0, \qquad \frac{1}{2}\tilde{\gamma}_{2,b} = -\tilde{\alpha}_{1,b} = -\frac{1}{2}\tilde{\beta}_{1,b} = 1,\\
&\tilde{\gamma}_{1,b} = 2\left(\frac{mn}{4b_1}(\tilde{p}\tilde{q} + b_1) + (\Sigma^2 - 4)\right),\\
&\tilde{\alpha}_{0,b} = -\frac{1}{8}\left(3 + \frac{mn}{b_0}\right)(\tilde{p}\tilde{q} + b_0) + \frac{1}{2}(\Sigma^2 - 4),\\
&\tilde{\beta}_{0,b} = -\frac{nm}{4b_0}(\tilde{p}\tilde{q} + b_0),\\
&\tilde{\gamma}_{0,b} = -\frac{1}{128}\left[\frac{A}{b_0 - 5} + \frac{B}{b_0} + C\right]. \qquad (36)
\end{aligned}$$

Here, we define $\tilde{p} = (2p - m)$, $\tilde{q} = (2q - n)$ and $b_a = b(b + 4 + 2a)$, while for $\tilde{\gamma}_{0,b}$ we have (using $R = \tilde{p}\tilde{q} + b_0 + 8$)

$$\begin{aligned}
A &= -(m^2 - 1)(n^2 - 1)(\tilde{p}^2 - 1)(\tilde{q}^2 - 1),\\
B &= mn\tilde{p}\tilde{q}[mn(\tilde{p}\tilde{q} - 8) - 32(\Sigma^2 - 4)],\\
C &= 16(\Sigma^2 - 4)((2\Sigma + 1)^2 - 2mn) - 5m^2n^2 + 195\\
&\quad + 4b_0(\Sigma + 4)^2 + 4\Sigma(9\Sigma - 8R) - 13R^2 - 74R + 177b_0\\
&\quad + (m^2 + n^2)(2R - 4(\Sigma - 2)^2 - b_0 + 141). \qquad (37)
\end{aligned}$$

The anomalous dimensions are nonvanishing only for $\ell_{10} \leq 2$, constraining the possible values of (i, r, l, a). To write the anomalous dimensions $\eta^{(5)}_{i,r|l,a}$, we define the polynomial $\mathcal{T}$ as follows:

$$\begin{aligned}
N_t &= (t - 1)(t + a)(t + a + b + 1),\\
\mathcal{T}_{t,l,a,b} &= \frac{1}{166320}\zeta_5 M_t M_{t+l+1} N_t N_{-t-2a-b-l-2}. \qquad (38)
\end{aligned}$$

Note that $\mathcal{T}_{t,0,0,b} \propto \eta^{(3)}(t, b)$. For spin 2, we must have $i = 1$, $r = 0$, $a = 0$, and we find

$$\eta^{(5)}_{1,0|2,0} = \mathcal{T}_{t,2,0,b}(t+1)(t+2)(t+b+2)(t+b+3), \qquad (39)$$

which is just a particular case of Eq. (29).

For spin 1, we have $i = 1$, $r = 0$, and $a = 0, 1$:

$$\begin{aligned}
\eta^{(5)}_{1,0|1,0} &= \frac{1}{2}\mathcal{T}_{t,1,0,b}(t + 1)(t + b + 2)(2t(3 + b + t) + b),\\
\eta^{(5)}_{1,0|1,1} &= \mathcal{T}_{t,1,1,b}t(t + 2)(t + b + 3)(t + b + 5). \qquad (40)
\end{aligned}$$

The spin-zero anomalous dimensions have support on $a = 0, 1, 2$. For $a = 1, 2$, we have only $i = 1$, $r = 0$,

$$\begin{aligned}
\eta^{(5)}_{1,0|0,1} &= \frac{1}{2}\mathcal{T}_{t,0,1,b}t(t + b + 4)(2t^2 + 2(4 + b)t + b + 6),\\
\eta^{(5)}_{1,0|0,2} &= \mathcal{T}_{t,0,2,b}t(1 + t)(5 + b + t)(6 + b + t). \qquad (41)
\end{aligned}$$

In all the above cases, we have $\mathbb{C}^{(5)} = 0$. A pictorial representation of the spectrum for those cases is given in Fig. 2(a). On the other hand, the case $a = 0$ allows for generically three nonzero components, depending on the values of t and b. Using $\theta \equiv \tau + 2 = 2t + 2 + b$, the $i = 1$ component reads

$$\begin{aligned}
\eta^{(5)}_{1,0|0,0} &= \frac{77}{18}\mathcal{T}_{t,0,0,b}f_{b,t},\\
f_{b,t} &= \frac{9}{4}(\theta^2 - b_0)^2 - 35(\theta^2 - b_0) - 34b_0 + 639. \qquad (42)
\end{aligned}$$

Finally, the $(i, r) = (1, 1)$ and (2,0) components read

$$\begin{aligned}
\eta^{(5)}_{2,0|0,0} &= \frac{1}{9}\mathcal{T}_{t,0,0,b}\left(j_{b,t} - 10\sqrt{k_{b,t}}\right),\\
\eta^{(5)}_{1,1|0,0} &= \frac{1}{9}\mathcal{T}_{t,0,0,b}\left(j_{b,t} + 10\sqrt{k_{b,t}}\right),\\
j_{b,t} &= \frac{1}{4}f_{b,t} - \frac{15}{4}(\theta^2 + b_0 + 21),\\
k_{b,t} &= j_{b,t} + (\theta^2 + b_0)(\theta^2 + b_0 - 10). \qquad (43)
\end{aligned}$$

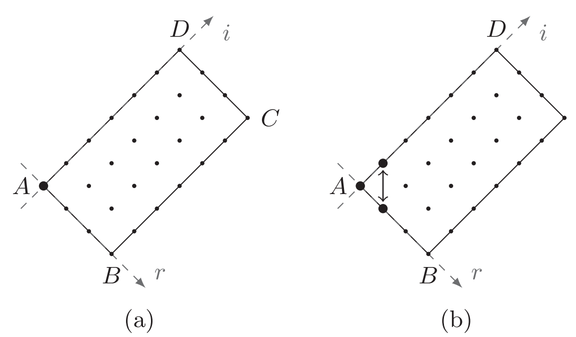

FIG. 2. Depiction of anomalous dimensions $\eta^{(5)}_{i,r|l,a}$: the nonvanishing ones are denoted by filled circles, while all others are zero. Diagram (a) describes the cases $(l, a) = (2, 0), (1, 1), (1, 0), (0, 2), (0, 1)$, where only one anomalous dimension is nonzero. Diagram (b) shows the case $(l, a) = (0, 0)$, where the arrow indicates the lifting of the residual degeneracy for $(i, r) = (1, 1)$ and $(2, 0)$.

Note that the residual partial degeneracy is lifted by the square root, as shown in Fig. 2(b). Moreover, in the $l = a = 0$ case, we have $\mathbb{C}^{(5)} \neq 0$.

V. DISCUSSION AND OUTLOOK

The results of the previous section provide a Mellin formula for all correlators at order $\lambda^{-\frac{5}{2}}$, as well as the corrections to the spectrum. The correlators are consistent with the results for $\langle 22qq\rangle$ [8,16] given above and $\langle 23q-1q\rangle$ derived in Ref. [17]. Note that the anomalous dimensions found here differ from those conjectured in Ref. [17], since we have found here that $\mathbb{C}^{(5)} \neq 0$ in general.

In the first case, where residual degeneracy is present in the supergravity spectrum, the $\lambda^{-\frac{5}{2}}$ corrections resolve it. Due to the residual twofold mixing problem, the appearance of square roots in the anomalous dimension is to be expected; this did not happen in supergravity due to the ten-dimensional conformal symmetry. In some cases, the square roots in Eq. (43) have to disappear:

(1) When $t = 2$, there is no degeneracy, and only two states acquire anomalous dimension. In fact, $k_{b,2} = j_{b,2}^2/100$, and $\eta^{(5)}_{1,1|0,0}$ becomes a rational function.

(2) When $b = 0$, $b = 1$, there is no degeneracy for any t [$\kappa = 1$ in Eq. (20)]: the square roots disappear again.

(3) In the flat-space limit $t \to \infty$, the square-root terms are suppressed and degeneracy is restored, respecting the ten-dimensional Lorentz symmetry.

The disappearance of the square roots in these cases is a strong check of the consistency of the solution. Finally, all the anomalous dimensions have some shared features:

(1) When expressed in terms of the twist τ (or $\theta = 2t + 2a + b + l + 2$) instead of t, they really depend on the $su(4)$ labels only through the Casimir combination $b_a = b(b + 4 + 2a)$.

(2) They enjoy the supergravity symmetry [Eq. (30)]: this in turn means that all the quartic polynomials f, j, k are actually quadratic in θ^2. We partly imposed this property in Eq. (33), but again in many examples it was found to follow from the other assumptions.

As mentioned earlier, the bootstrap constraints in Eqs. (31)–(34) were motivated by the result [Eq. (29)] for the highest-spin anomalous dimension and its agreement with the assignment of the ten-dimensional spin according to Eq. (27). The former statement is valid at any order $\lambda^{-\frac{n}{2}}$, with n odd. We thus believe that the methods developed here will continue to be effective at higher orders in $\lambda^{-\frac{1}{2}}$, the next case being λ^{-3}. It will be interesting to examine the first case of triple residual degeneracy at order $\lambda^{-\frac{7}{2}}$ to see if there is hope for an explicit formula for the spectrum. We hope that this may allow us to apply a bootstrap approach to the full classical string amplitude in AdS.

This in turn will provide valuable information on the λ dependence of the loop amplitudes. In fact, considering correlators of generic external charges at loop order and studying their mutual consistency under crossing might provide a way to further substantiate the validity of our bootstrap method.

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APPENDIX: SUPERGRAVITY MELLIN AMPLITUDE AND SUPERCONFORMAL BLOCKS

We give here the Mellin amplitude in supergravity,

$$\mathcal{M}^{(0)} = \sum_{i,j} \frac{\mathcal{N}_{ijk}\sigma^i\tau^j}{(s-\tilde{s}+2k)(t-\tilde{t}+2j)(u-\tilde{u}+2i)}, \qquad \text{(A1)}$$

with $i+j+k = p_3 + \min(0, \frac{p_{13}+p_{24}}{2}) - 2$ and the sum taken such that i, j, $k \geq 0$. Here, we have used

$$\tilde{s} = \min(p_1+p_2, p_3+p_4) - 2,$$
$$\tilde{t} = p_2 + p_3 - 2, \qquad \tilde{u} = p_1 + p_3 - 2. \qquad \text{(A2)}$$

Finally, the coefficients $\mathcal{N}_{ijk}$ are given by [13,17]

$$\mathcal{N}_{ijk} = \frac{8p_1p_2p_3p_4(i!j!k!)^{-1}}{\left[\frac{p_{43}+p_{21}+2i}{2}\right]!\left[\frac{p_{43}-p_{21}+2j}{2}\right]!\left[\frac{|p_{13}+p_{24}|+2k}{2}\right]!}. \qquad \text{(A3)}$$

The relevant superblocks for long multiplets were given in Refs. [26,27]. In our notation, they take the form

$$\mathbb{L}_{\vec{\tau}} = \mathcal{P}\mathcal{I}u^{\frac{p_{34}}{2}-2}Y_{[aba]}(y,\bar{y})\mathcal{B}^{2+\frac{\tau}{2}|l}(x,\bar{x}). \qquad \text{(A4)}$$

In Eq. (A4), we have

$$\mathcal{P} = N^{\frac{1}{2}\sum p_i} g_{12}^{\frac{p_1+p_2-p_{43}}{2}} g_{14}^{\frac{-p_{21}+p_{43}}{2}} g_{24}^{\frac{p_{21}+p_{43}}{2}} g_{34}^{p_3}, \qquad \text{(A5)}$$

where we introduce the propagators g_{ij}:

$$g_{ij} = y_{ij}^2/x_{ij}^2, \qquad x_{ij}^2 = (x_i - x_j)^2, \qquad y_{ij}^2 = y_i \cdot y_j. \qquad \text{(A6)}$$

The factor $\mathcal{I}$ in Eq. (A4) is given by

$$\mathcal{I}(x,\bar{x};y,\bar{y}) = (x-y)(x-\bar{y})(\bar{x}-y)(\bar{x}-\bar{y})/(y\bar{y})^2 \qquad \text{(A7)}$$

and is present due to superconformal symmetry [26,28]. The $su(4)$ blocks for $[a, b, a] = [\mu - \nu, 2\nu + p_{43}, \mu - \nu]$ are given in terms of Jacobi polynomials $J_\mu^{(\alpha,\beta)}$:

$$
\begin{aligned}
Y_{[aba]}(y,\bar{y}) &= (P_\nu(y)P_{\mu+1}(\bar{y}) - P_{\mu+1}(y)P_\nu(\bar{y}))/(y-\bar{y}),\\
P_\mu(y) &= \frac{\mu! y}{(\mu+1+p_{43})_\mu} J_\mu^{\left(\frac{p_{43}-p_{21}}{2},\frac{p_{21}+p_{43}}{2}\right)}\left(\frac{2}{y}-1\right).
\end{aligned} \tag{A8}
$$

Finally, the conformal blocks are given by

$$
\begin{aligned}
\mathcal{B}^{s|l}(x,\bar{x}) &= (-1)^l \frac{u^s x^{l+1}\mathbf{F}_{s+l}(x)\mathbf{F}_{s-1}(\bar{x}) - (x \leftrightarrow \bar{x})}{x-\bar{x}},\\
\mathbf{F}_s(x) &= {}_2F_1\left(s - \frac{p_{12}}{2}, s + \frac{p_{34}}{2}; 2s; x\right).
\end{aligned} \tag{A9}
$$

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