Massless representation of massive superfields and tree amplitudes with the pure spinor formalism

Sitender Pratap Kashyap^{*}, Carlos R. Mafra^{α'}, Mritunjay Verma[†], Luis Ypanaqué[•]

 * Chennai Mathematical Institute, H1 SIPCOT IT Park, Kelambakkam, Tamil Nadu, India 603103
 ^{α'} Mathematical Sciences and STAG Research Centre, University of Southampton, Highfield, Southampton, SO17 1BJ, UK
 [†] Indian Institute of Technology Indore, Khandwa Road, Simrol, Indore 453552, India

We construct the unintegrated vertex operator at the first mass level of the open superstring from the OPE of massless vertices. Using BRST cohomology manipulations, the tree amplitude of two massless and one massive state is rewritten in terms of the pure spinor superspace kinematic expression of the massless four-point amplitude at the α'^2 level. A generalization relating the partial *n*-point tree amplitudes with one massive state and linear combinations of the α'^2 corrections to n+1 massless amplitudes is found and shown to be consistent with unitarity.

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^{*}email: sitender@cmi.ac.in

 $^{^{\}alpha'}$ email: c.r.mafra@soton.ac.uk

[†]email: mritunjay@iiti.ac.in

[•] email: luis.12yr@gmail.com

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1. Introduction

The successful calculation of the general massless open string tree amplitudes [1,2] with the pure spinor formalism [3] still does not have a counterpart involving massive states. One of the reasons for this situation is the added complexity in the description of massive superfields in ten dimensions and their use in constructing massive vertex operators.

However, the pure spinor formalism comes equipped with a powerful notion of a cohomological pure spinor superspace [4]. At the massless level, BRST cohomology manipulations give rise to many simplifications that have been exploited in several papers [5–9] culminating in the *n*-point tree amplitude of [1], see the review [10]. We wish to transfer some of the techniques and knowledge accumulated with pure spinor superspace expressions involving massless SYM superfields into the manipulation and simplification of massive amplitudes, with the hopes of advancing the knowledge of massive amplitudes beyond its current limited state. The unintegrated vertex operator at the first massive level and its superfield description was found in 2002 by Berkovits and Chandia [11], but it took many years until it was used in the calculation of the three point amplitude with two massless and one massive state [12].

In this paper, two major advantages of the pure spinor formalism in calculating massless tree amplitudes will start to be transferred to the study of massive amplitudes: the simplicity of the SYM superfield massless description and the BRST cohomology manipulations in pure spinor superspace. To accomplish the first goal we will construct the massive superfields for the unintegrated vertex operator using the OPEs between massless vertex operators¹. This will then give rise to a massless representation of the massive superfields.

We will make progress towards the second goal on a case by case basis, starting with the three point amplitude with one massive state computed in [12]. Firstly, it will be simplified using BRST cohomology manipulations in terms of the massive superfields. Subsequently, the massless SYM representation of the massive superfields will be plugged in, allowing several further BRST cohomology identities for massless expressions to be used. The end result expresses the three-point amplitude with one massive state in terms of the massless four-point pure spinor superspace expression capturing the α'^2 correction to the massless four-point open string tree amplitude. This result will be generalized using the component expression of the massive partial tree amplitudes found in [15] to linear combinations of α'^2 tree amplitudes. Finally, the generalization of this relation will be shown to follow from (or be compatible with) the factorization is slightly unusual as the sum over intermediate polarizations – which usually require two amplitudes connected via a propagator – will be used in a single amplitude with the rule $\underline{k} \rightarrow i, j$ defined in (3.25) and (3.51). This allows us to directly relate massive and massless string amplitudes.

Throughout this paper, repeated vector indices are summed irrespective of their downstairs/upstairs placement and we use the convention where the symmetrization or antisymmetrization over n indices does not contain the normalization $\frac{1}{n!}$.

¹ This construction with pure spinors was firstly announced in the companion of this paper [13] and later confirmed by similar calculations in [14].

2. Vertex operators

Physical states in the pure spinor formalism at the mass level n are defined as ghost number one vertex operators in the cohomology of the pure spinor BRST charge with conformal weight n at zero momentum [16].

2.1. Massless vertices

The unintegrated and integrated vertex operators describing the massless open string states are given by [3]

$$V = \lambda^{\alpha} A_{\alpha}, \qquad U = \partial \theta^{\alpha} A_{\alpha} + \Pi^{m} A_{m} + 2\alpha' d_{\alpha} W^{\alpha} + \alpha' N^{mn} F_{mn}.$$
(2.1)

The superfields $A_{\alpha}, A_m, W^{\alpha}$ and $F_{mn} = \partial_{[m} A_{n]}$ satisfy [17,10]

$$QA_{\beta} + D_{\beta}V = (\gamma^{m}\lambda)_{\beta}A_{m}, \qquad QW^{\beta} = -\frac{1}{4}(\gamma^{mn}\lambda)^{\beta}F_{mn},$$

$$QA_{m} = \lambda\gamma^{m}W + \partial_{m}V, \qquad QF_{mn} = \partial_{m}(\lambda\gamma_{n}W) - \partial_{n}(\lambda\gamma_{m}W),$$
(2.2)

where $Q = \lambda^{\alpha} D_{\alpha}$ is the pure spinor BRST operator acting on 10D superfields. The length dimensions are chosen such that

$$[\alpha'] = 2, \quad [V] = [U] = 1, \quad [A_{\alpha}] = \frac{1}{2}, \quad [A^m] = 0, \quad [W^{\alpha}] = -\frac{1}{2}, \quad [F^{mn}] = -1$$
(2.3)
$$[\lambda^{\alpha}] = [\theta^{\alpha}] = \frac{1}{2}, \quad [\partial_m] = -1, \quad [d_{\alpha}] = -\frac{1}{2}, \quad [\Pi^m] = 1, \quad [J] = 0, \quad [N^{mn}] = 0.$$

By stripping off λ^{α} from (2.2), one obtains the equations of motion written in terms of the covariant derivative D_{α} . We will use below both forms of these equations interchangeably. For convenience, when referring to a generic SYM superfield labelled by *i* we use the collective notation

$$K_i \in \{A^i_{\alpha}, A^i_m, W^{\alpha}_i, F^{mn}_i\}.$$
 (2.4)

2.2. Massive unintegrated vertex

The unintegrated vertex operator V(z) containing the open-string massive states with $(mass)^2 = 1/\alpha'$ was found in [11],

$$V(z) = [\lambda^{\alpha} [\partial \theta^{\beta} B_{\alpha\beta}]_0]_0 + [\lambda^{\alpha} [\Pi^m H^m_{\alpha}]_0]_0 + 2\alpha' [\lambda^{\alpha} [d_{\beta} C^{\beta}{}_{\alpha}]_0]_0 + \alpha' [\lambda^{\alpha} [N^{mn} F_{\alpha mn}]_0]_0, \quad (2.5)$$

where the normal-ordering bracket $[AB]_0$ is defined in (A.2). It was also shown in [11] that this vertex is BRST closed QV = 0 when the superfields obey the equations of motion

$$Q(\lambda B)_{\alpha} = (\lambda \gamma^{m})_{\alpha} (\lambda H)_{m}, \quad Q(\lambda H^{m}) = (\lambda \gamma^{m} C \lambda), \quad Q(C \lambda)^{\alpha} = \frac{1}{4} (\lambda \gamma^{mn})^{\alpha} (\lambda F)_{mn},$$
(2.6)

where we used the definitions (2.19) and omitted the slightly more complicated equation for $\lambda^{\alpha} F_{\alpha m n}$. The length dimension of the massive superfields in (2.5) is chosen to be

$$[V] = 2, \quad [B_{\alpha\beta}] = 1, \quad [H_{m\alpha}] = \frac{1}{2}, \quad [C^{\beta}{}_{\alpha}] = 0, \quad [F_{\alpha mn}] = -\frac{1}{2}.$$
 (2.7)

2.3. Massive vertex from the OPE of massless vertices

Massive vertex operators appear in the regular terms of OPEs of massless vertices [18]. This will allow us to construct the first-level massive unintegrated vertex operator in terms of the massless superfields [19,20] as

$$V(z) = \oint_{z} dw U_1(w) V_2(z), \qquad 2\alpha'(k_1 \cdot k_2) = -1, \qquad (2.8)$$

where the condition

$$2\alpha'(k_1 \cdot k_2) = -1, \qquad (2.9)$$

ensures the correct conformal weight one for the vertex V(z).

OPE of massless vertices. It is easy to see from the OPE expansion (A.1) that

$$V(w) = \oint dz U_1(z) V_2(w) = [U_1 V_2]_1, \qquad (2.10)$$

where the bracketed notation for the OPEs and normal ordering is reviewed in the appendix A. Using the normal ordered massless vertex operators²

$$U_1(z) = [\partial \theta^{\alpha} A^1_{\alpha}]_0(z) + [\Pi^m A^1_m]_0(z) + 2\alpha' [d_{\alpha} W^{\alpha}_1]_0(z) + \alpha' [N^{mn} F^{mn}_1]_0(z), \quad (2.11)$$

$$V_2(w) = [\lambda^{\alpha} A_{\alpha}^2]_0(w), \qquad (2.12)$$

 $^{^2}$ Note that there is no normal ordering ambiguity in the massless vertices due to the SYM equations of motion in the Lorenz gauge. Nevertheless, we write the normal ordering brackets in order to use the OPE formulas from the vertex operator algebra axioms of the appendix A.

we get (we write $[AB]_0 = [AB]$ when convenient to avoid cluttering)

$$V(w) = [\lambda^{\beta} [\partial \theta^{\alpha} (A_{\alpha}^{1} A_{\beta}^{2})]](w) + [\lambda^{\beta} [\Pi^{m} (A_{m}^{1} A_{\beta}^{2})]](w) - 2\alpha' i k_{2}^{n} [\lambda^{\beta} (\partial A_{n}^{1} A_{\beta}^{2})](w) + [\lambda^{\beta} [d_{\alpha} (W_{1}^{\alpha} A_{\beta}^{2})]](w) - [\lambda^{\beta} (\partial W_{1}^{\alpha} D_{\alpha} A_{\beta}^{2})](w) + [N^{mn} [\lambda^{\beta} (F_{1}^{mn} A_{\beta}^{2})]](w) + \frac{1}{2} (\gamma^{mn})^{\beta} \delta [\lambda^{\delta} (\partial F_{1}^{mn} A_{\beta}^{2})](w)$$
(2.13)

The appendix A.3 contains more details of this calculation.

Note that the factors of $(K_1K_2)(w)$ on the right-hand side are considered a single operator. For example, the term $[d_{\alpha}(W_1^{\alpha}A_{\beta}^2)]_0(w)$ is of the form $[AB]_0(w)$ with $A = d_{\alpha}(w)$ and $B = (W_1^{\alpha}A_{\beta}^2)(w) = W_1^{\alpha}(\theta)e^{ik_1\cdot X(w)}A_{\beta}^2(\theta)e^{ik_2\cdot X(w)} = W_1^{\alpha}(\theta)A_{\beta}^2(\theta)e^{ik\cdot X(w)}$, with $k = k_1 + k_2$.

2.4. Massive superfields in the OPE gauge

After expanding $\partial K_1 = \partial \theta^{\alpha} D_{\alpha} K_1 + \Pi^m i k_1^m K_1$ to rewrite factors like $(\partial A_1^n A_{\beta}^2)$ as $[\Pi^m i k_1^m (A_1^n A_{\beta}^2)]_0 + [\partial \theta^{\alpha} (D_{\alpha} A_1^n A_{\beta}^2)]_0$ and using (2.9) together with the SYM equations of motion we get (omitting the worldsheet position w from the right-hand side)

$$V(w) = [\lambda^{\alpha}[\partial\theta^{\beta}B_{\alpha\beta}]] + [\lambda^{\alpha}[\Pi^{m}H^{m}_{\alpha}]] + 2\alpha'[\lambda^{\alpha}[d_{\beta}C^{\beta}{}_{\alpha}]] + \alpha'[N^{mn}[\lambda^{\alpha}F_{\alpha mn}]]$$
(2.14)

where the massive superfields can be read off to be

$$B_{\alpha\beta} = -2\alpha' i k_2^m (\gamma^m W_1)_\beta A_\alpha^2 - \alpha' i k_1^m (\gamma^n W_1)_\beta (\gamma^{mn} A_2)_\alpha - \frac{\alpha'}{2} F_1^{mn} (\gamma^{mn} D)_\beta A_\alpha^2, \quad (2.15)$$

$$H^{m}_{\alpha} = A^{1}_{m}A^{2}_{\alpha} + 2\alpha'k^{1}_{m}(k^{2}\cdot A^{1})A^{2}_{\alpha} - 2\alpha'ik^{1}_{m}W^{\beta}_{1}D_{\beta}A^{2}_{\alpha} - \frac{\alpha'}{2}ik^{1}_{m}F^{1}_{np}(\gamma^{np}A_{2})_{\alpha}, \quad (2.16)$$

$$C^{\beta}{}_{\alpha} = W_1^{\beta} A_{\alpha}^2 \,, \tag{2.17}$$

$$F_{\alpha mn} = F_{mn}^1 A_{\alpha}^2 \,. \tag{2.18}$$

For reasons to become clear in section 2.5, this representation of the massive superfields in terms of massless SYM superfields will be called the *OPE gauge*. Their length dimensions are easily found to agree with (2.7).

Massive equations of motion. Defining the contraction of the massive superfields with a pure spinor

$$\lambda^{\alpha}B_{\alpha\beta} = (\lambda B)_{\beta}, \quad \lambda^{\alpha}H^{m}_{\alpha} = (\lambda H^{m}), \quad C^{\beta}{}_{\alpha}\lambda^{\alpha} = (C\lambda)^{\beta}, \quad \lambda^{\alpha}F_{\alpha m n} = (\lambda F)_{m n}, \quad (2.19)$$

and using the linearized SYM equations of motion (2.2) one readily finds the equations of motion of the massive superfields in terms of the BRST charge $Q = \lambda^{\alpha} D_{\alpha}$,

$$Q(\lambda B)_{\alpha} = (\lambda \gamma^m)_{\alpha} (\lambda H)_m \tag{2.20}$$

$$Q(\lambda H^m) = (\lambda \gamma^m C \lambda) \tag{2.21}$$

$$Q(C\lambda)^{\alpha} = \frac{1}{4} (\lambda \gamma^{mn})^{\alpha} (\lambda F)_{mn}$$
(2.22)

$$Q(\lambda F)_{mn} = ik_1^m (\lambda \gamma^n W_1) V_2 - ik_1^n (\lambda \gamma^m W_1) V_2$$
(2.23)

More details can be found in the appendix B.

2.4.1. BRST invariance of massive vertex

Recall that the BRST charge is

$$Q = \oint dz j(z) \tag{2.24}$$

where $j(z) = \lambda^{\alpha}(z)d_{\alpha}(z) = [\lambda^{\alpha}d_{\alpha}]_{0}(z)$ is the BRST current. We will evaluate the BRST variation of the massive unintegrated vertex in two different ways: directly from the definition (2.10) and using its explicit realization (2.14).

BRST variation from the definition. The massless vertices satisfy [3,10]

$$QU_{1}(w) = \oint dz j(z) U_{1}(w) = [jU_{1}]_{1}(w) = \partial V_{1}(w) , \qquad (2.25)$$
$$QV_{2}(w) = \oint dz j(z) V_{2}(w) = [jV_{2}]_{1}(w) = 0 .$$

Therefore, the BRST variation of the first massive unintegrated vertex operator (2.10) yields

$$QV(w) = [jV]_1(w) = [j[U_1V_2]_1]_1$$

$$= [U_1[jV_2]_1]_1 + [[jU_1]_1V_2]_1 = [\partial V_1V_2]_1 = 0,$$
(2.26)

where the second line follows from (A.15) and we used (A.11) in the last equality. So the massive unintegrated vertex (2.10) is BRST closed.

Evaluation using superfields. The explicit computation of $[jV]_1$ with V given by (2.14) is a bit tedious but straightforward. Using the identities (A.5) and (A.4) gives

$$QV = [jV]_{1} = [\partial\lambda^{\alpha}[\lambda^{\beta}B_{\beta\alpha}]] - [\partial\theta^{\beta}[\lambda^{\gamma}[\lambda^{\alpha}D_{\alpha}B_{\gamma\beta}]]] + [[\lambda^{\alpha}\partial\theta^{\gamma}][\lambda^{\beta}H_{\beta}^{m}]]\gamma_{\alpha\gamma}^{m} - [[\lambda^{\alpha}\Pi^{m}][\lambda^{\beta}C^{\gamma}{}_{\beta}]]\gamma_{\alpha\gamma}^{m} + [\Pi^{m}[\lambda^{\beta}[\lambda^{\alpha}D_{\alpha}H_{\beta}^{m}]]] - \frac{\alpha'}{2}[\lambda^{\beta}[[d_{\alpha}\lambda^{\gamma}]F_{\beta mn}]](\gamma^{mn})^{\alpha}{}_{\gamma} - 2\alpha'[d_{\gamma}[\lambda^{\beta}[\lambda^{\alpha}D_{\alpha}C^{\gamma}{}_{\beta}]]] + \alpha'[N^{mn}[\lambda^{\alpha}[\lambda^{\beta}D_{\beta}F_{\alpha mn}]]].$$

$$(2.27)$$

In order to compare terms we need to rewrite all nested brackets in a canonical order, say from right to left as in [A[B[CD]]]. After some work using the identities (A.10) to (A.12) one obtains

$$\begin{aligned} [\lambda^{\beta}[[d_{\alpha}\lambda^{\gamma}]F_{\beta mn}]] &= [d_{\alpha}[\lambda^{\beta}[\lambda^{\gamma}F_{\beta mn}]]] + [\partial\lambda^{\gamma}[\lambda^{\beta}(D_{\alpha}F_{\beta mn})]], \qquad (2.28) \\ [[\lambda^{\alpha}\partial\theta^{\gamma}][\lambda^{\beta}H_{\beta}^{m}]] &= [\partial\theta^{\gamma}[\lambda^{\alpha}[\lambda^{\beta}H_{\beta}^{m}]]], \\ [[\lambda^{\alpha}\Pi^{m}][\lambda^{\beta}C^{\gamma}{}_{\beta}]] &= [\Pi^{m}[\lambda^{\alpha}[\lambda^{\beta}C^{\gamma}{}_{\beta}]]] - 2\alpha'[\partial\lambda^{\alpha}[\lambda^{\beta}\partial^{m}C^{\gamma}{}_{\beta}]]. \end{aligned}$$

Plugging (2.28) into (2.27) and noticing that there are no normal ordering ambiguities among λ^{α} , $\partial \theta^{\alpha}$ and the massive superfields leads to

$$QV = [\partial \lambda^{\alpha} \lambda^{\beta} S^{1}_{\alpha\beta}] - [\partial \theta^{\alpha} [\lambda^{\beta} \lambda^{\gamma} S^{2}_{\alpha\beta\gamma}]] + [\Pi^{m} [\lambda^{\alpha} \lambda^{\beta} S^{3}_{m\alpha\beta}]]$$

$$+ 2\alpha' [d_{\alpha} [\lambda^{\beta} \lambda^{\gamma} S^{4\alpha}_{\beta\gamma}]] + \alpha' [N^{mn} [\lambda^{\alpha} \lambda^{\beta} S^{5}_{\alpha\beta mn}]],$$

$$(2.29)$$

where

$$\partial \lambda^{\alpha} \lambda^{\beta} S^{1}_{\alpha\beta} = (\lambda B \partial \lambda) + 2\alpha' (\partial \lambda \gamma^{m} \partial^{m} C \lambda) + \frac{\alpha'}{2} (\partial \lambda \gamma^{mn} D) (\lambda F)_{mn} \qquad (2.30)$$
$$= -\alpha' i k_{1}^{m} (\lambda \gamma^{n} W_{1}) (\partial \lambda \gamma^{mn} A_{2}) ,$$

$$\lambda^{\beta}\lambda^{\gamma}S^{2}_{\alpha\beta\gamma} = Q(\lambda B)_{\alpha} - (\lambda H^{m})(\lambda\gamma^{m})_{\alpha} = 0, \qquad (2.31)$$

$$\lambda^{\alpha}\lambda^{\beta}S^{3}_{m\alpha\beta} = Q(\lambda H^{m}) - (\lambda\gamma^{m}C\lambda) = 0, \qquad (2.32)$$

$$\lambda^{\beta}\lambda^{\gamma}S^{4\alpha}_{\beta\gamma} = Q(C\lambda)^{\alpha} - \frac{1}{4}(\lambda\gamma^{mn})^{\alpha}(\lambda F)_{mn} = 0, \qquad (2.33)$$

$$\lambda^{\alpha}\lambda^{\beta}S^{5}_{\alpha\beta mn} = Q(\lambda F)_{mn} = ik_{1}^{m}(\lambda\gamma^{n}W_{1})V_{2} - ik_{1}^{n}(\lambda\gamma^{m}W_{1})V_{2}.$$

$$(2.34)$$

The simplification in (2.30) follows from the linearized SYM equations of motion and the Dirac equation after plugging in the expression (2.15). The vanishing of the middle three lines follows from the BRST equations of motion (2.20), (2.21) and (2.22). Therefore,

$$QV = -\alpha' i k_1^m (\lambda \gamma^n W_1) (\partial \lambda \gamma^{mn} A_2) - 2\alpha' i [N^{mn} [\lambda^\alpha \lambda^\beta]] \gamma^m_{\beta\gamma} k_1^n W_1^\gamma A_\alpha^2 , \qquad (2.35)$$

where we pulled the superfields out of the normal ordering bracket as they do not have worldsheet singularities with the operators in $[N^{mn}[\lambda^{\alpha}\lambda^{\beta}]]$. The identity from [11]

$$[N^{mn}[\lambda^{\alpha}\lambda^{\beta}]]\gamma^{m}_{\beta\gamma} = \frac{1}{2}[J[\lambda^{\alpha}\lambda^{\beta}]]\gamma^{n}_{\beta\gamma} + \frac{5}{2}\lambda^{\alpha}(\gamma^{n}\partial\lambda)_{\gamma} + \frac{1}{2}(\lambda\gamma^{mn})^{\alpha}(\gamma^{m}\partial\lambda)_{\gamma}, \qquad (2.36)$$

rederived in (A.29), implies that

$$QV = -\alpha' i k_1^m (\lambda \gamma^n W_1) (\partial \lambda \gamma^{mn} A_2) - \alpha' i k_1^n (\partial \lambda \gamma^m W_1) (\lambda \gamma^{mn} A_2) = 0, \qquad (2.37)$$

where the Dirac equation eliminates the first two terms on the right-hand side of (2.36) when plugged into (2.35) and we used $(\lambda \gamma^m)_{\alpha} (\partial \lambda \gamma^m)_{\beta} + (\lambda \gamma^m)_{\beta} (\partial \lambda \gamma^m)_{\alpha} = 0$. Therefore the unintegrated massive vertex (2.14) is BRST closed, QV = 0.

2.5. Massive superfields in the Berkovits-Chandia gauge

We have seen above that the OPE calculation leads to an unintegrated massive vertex operator of the form

$$V(w) = \left[\lambda^{\alpha} \left[\partial \theta^{\beta} B_{\alpha\beta}\right]\right] + \left[\lambda^{\alpha} \left[\Pi^{m} H^{m}_{\alpha}\right]\right] + 2\alpha' \left[\lambda^{\alpha} \left[d_{\beta} C^{\beta}{}_{\alpha}\right]\right] + \alpha' \left[N^{mn} \left[\lambda^{\alpha} F_{\alpha mn}\right]\right], \quad (2.38)$$

with coefficients given in (2.15) to (2.18). It does not contain the fields $\partial \lambda^{\alpha}$ and J that would otherwise be present in the most general form of V of conformal weight one and ghost-number one. This parameterization in (2.15) to (2.18) was called the *OPE gauge*.

Berkovits-Chandia gauge. As shown in [11], the gauge invariance $\delta V = Q\Omega$ can be exploited to obtain a new parameterization for the superfields such that

$$B_{\alpha\beta} = \gamma^{mnp}_{\alpha\beta} B_{mnp} , \qquad \partial^m B_{mnp} = 0 , \qquad \gamma^{m\alpha\beta} H_{m\beta} = 0 , \qquad \partial^m H_{m\alpha} = 0 , \qquad (2.39)$$
$$C^{\alpha}{}_{\beta} = \frac{1}{4} (\gamma^{mpnq})^{\alpha}{}_{\beta} \partial_m B_{npq} , \qquad \gamma^{m\alpha\beta} F_{\alpha mn} = 0 ,$$

which we will call the *Berkovits-Chandia gauge*. As a side note, the normal ordering identity (A.6) yields $[N^{mn}[\lambda^{\alpha}F_{\alpha mn}]_0]_0 = [\lambda^{\alpha}[N^{mn}F_{\alpha mn}]_0]_0 + \frac{1}{2}[(\gamma^{mn}\partial\lambda)^{\alpha}F_{\alpha mn}]_0$ and therefore constraint $\gamma^{m\alpha\beta}F_{\alpha mn} = 0$ implies that

$$[N^{mn}[\lambda^{\alpha}F_{\alpha mn}]_{0}]_{0} = [\lambda^{\alpha}[N^{mn}F_{\alpha mn}]_{0}]_{0}, \qquad (2.40)$$

a relation that will be exploited later in (2.78).

This same gauge fixing will now be done starting from the vertex in the OPE gauge. Gauge-fixed massive superfields. The gauge invariance of the massive vertex $\delta V = Q\Omega$ with the most general superfield Ω of conformal weight one and ghost number zero,

$$\Omega = [\partial \theta^{\alpha} \Omega_{1\alpha}]_0 + [d_{\alpha} \Omega_2^{\alpha}]_0 + [\Pi^m \Omega_{3m}]_0 + [J \Omega_4]_0 + [N^{mn} \Omega_{5mn}]_0, \qquad (2.41)$$

will now be exploited to go from massive superfields in the OPE gauge (2.15)-(2.18) to massive superfields in the Berkovits-Chandia gauge satisfying (2.39).

The BRST variation $Q\Omega = [j\Omega]_1$, where $j = [\lambda^{\alpha} d_{\alpha}]_0$ is the BRST current, reads

$$Q\Omega = [\partial\theta^{\beta}\lambda^{\alpha}]_{0} \Big(-D_{\alpha}\Omega_{1\beta} + \gamma^{m}_{\alpha\beta}\Omega_{3m} \Big) + [\Pi^{m}\lambda^{\alpha}]_{0} \Big(D_{\alpha}\Omega_{3m} - \frac{1}{2\alpha'}\gamma^{m}_{\alpha\beta}\Omega_{2}^{\beta} \Big)$$

$$+ [d_{\beta}\lambda^{\alpha}]_{0} \Big(-D_{\alpha}\Omega_{2}^{\beta} - \frac{1}{2}(\gamma^{mn})^{\beta}_{\ \alpha}\Omega_{5mn} - \delta^{\beta}_{\alpha}\Omega_{4} \Big) + [J\lambda^{\alpha}]_{0}D_{\alpha}\Omega_{4}$$

$$+ [N^{mn}\lambda^{\alpha}]_{0}D_{\alpha}\Omega_{5mn} + \partial\lambda^{\alpha} \Big(\Omega_{1\alpha} + \gamma^{m}_{\alpha\beta}\partial_{m}\Omega_{2}^{\beta} - D_{\alpha}\Omega_{4} - \frac{1}{2}(\gamma^{mn})^{\beta}_{\ \alpha}D_{\beta}\Omega_{5mn} \Big).$$

$$(2.42)$$

However, the gauge variations of the massive superfields following from (2.42) need to be modified by a vector-spinor parameter Λ_n^{β} to account for the constraint [11]

$$[N^{mn}\lambda^{\alpha}]_{0}\gamma_{m\alpha\beta} - \frac{1}{2}[J\lambda^{\alpha}]_{0}\gamma^{n}_{\alpha\beta} = 2\partial\lambda^{\alpha}\gamma^{n}_{\alpha\beta}.$$
(2.43)

The resulting gauge transformations are given by

$$\delta B_{\alpha\beta} = -D_{\alpha}\Omega_{1\beta} + \gamma^{m}_{\alpha\beta}\Omega_{3m}, \qquad (2.44)$$

$$\delta H_{m\alpha} = D_{\alpha}\Omega_{3m} - \frac{1}{2\alpha'}\gamma^{m}_{\alpha\beta}\Omega_{2}^{\beta}, \qquad (2.44)$$

$$\delta C^{\beta}_{\ \alpha} = -\frac{1}{2\alpha'}D_{\alpha}\Omega_{2}^{\beta} - \frac{1}{4\alpha'}(\gamma^{mn})^{\beta}_{\ \alpha}\Omega_{5mn} - \frac{1}{2\alpha'}\delta^{\beta}_{\alpha}\Omega_{4}, \qquad (2.44)$$

$$\delta F_{\alpha m n} = \frac{1}{\alpha'}D_{\alpha}\Omega_{5mn} + \gamma_{m\alpha\beta}\Lambda^{\beta}_{n} - \gamma_{n\alpha\beta}\Lambda^{\beta}_{m}, \qquad (2.44)$$

and it is easy to determine the length dimensions of the gauge parameters

$$[\Omega_{\alpha}^{1}] = \frac{3}{2}, \quad [\Omega_{\alpha}^{2}] = \frac{5}{2}, \quad [\Omega_{3}^{m}] = 1, \quad [\Omega_{4}] = 2, \quad [\Omega_{5}^{mn}] = 2, \quad [\Lambda_{m}^{\alpha}] = -\frac{1}{2}.$$
(2.45)

Since there are no massive superfields proportional to J and $\partial \lambda^{\alpha}$ in the massive vertex V in the OPE gauge (2.38), the following constraints need to be satisfied as well

$$0 = D_{\alpha}\Omega_{4} - \alpha'\gamma^{m}_{\alpha\beta}\Lambda^{\beta}_{m}$$

$$0 = \Omega_{1\alpha} + \gamma^{m}_{\alpha\beta}\partial_{m}\Omega^{\beta}_{2} - D_{\alpha}\Omega_{4} - \frac{1}{2}(\gamma^{mn})^{\beta}_{\ \alpha}D_{\beta}\Omega_{5mn} - 4\alpha'\gamma^{m}_{\alpha\beta}\Lambda^{\beta}_{m}.$$

$$(2.46)$$

Note that we can eliminate the term involving Λ_m^{α} from the above two equations to arrive at a single condition

$$\Omega_{1\alpha} + \gamma^m_{\alpha\beta} \partial_m \Omega_2^\beta - 5D_\alpha \Omega_4 - \frac{1}{2} (\gamma^{mn})^\beta_{\ \alpha} D_\beta \Omega_{5mn} = 0.$$
 (2.47)

2.5.1. $B_{\alpha\beta}$ in the Berkovits-Chandia gauge

After the gauge transformation, the superfield $B_{\alpha\beta}$ takes the form

$$B'_{\alpha\beta} = B_{\alpha\beta} - D_{\alpha}\Omega_{1\beta} + \gamma^m_{\alpha\beta}\Omega_{3m}$$
(2.48)

Now, the bispinor $B'_{\alpha\beta}$ decomposes into a one-, three- and five-form parts

$$B'_{\alpha\beta} = \frac{1}{16} \gamma^m_{\alpha\beta} \gamma^{\sigma\tau}_m \left(B_{\sigma\tau} - D_{\sigma} \Omega_{1\tau} \right) + \gamma^m_{\alpha\beta} \Omega_{3m}$$

$$+ \frac{1}{96} \gamma^{mnp}_{\alpha\beta} \gamma^{\sigma\tau}_{mnp} \left(B_{\sigma\tau} - D_{\sigma} \Omega_{1\tau} \right)$$

$$+ \frac{1}{3840} \gamma^{mnpqr}_{\alpha\beta} \gamma^{\sigma\tau}_{mnpqr} \left(B_{\sigma\tau} - D_{\sigma} \Omega_{1\tau} \right)$$

$$(2.49)$$

However, the five-form part is BRST exact $\lambda^{\alpha}\lambda^{\beta} \left(B_{\alpha\beta} - D_{\alpha}\Lambda_{\beta}\right) = 0$ as shown in (B.9). So if we choose the gauge parameter $\Omega_{1\tau} = \Lambda_{\tau}$, where Λ_{τ} is defined in (B.10), we can eliminate the 5-form piece from (2.49). Further, choosing

$$\Omega_{3m} = -\frac{1}{16} \gamma_m^{\sigma\tau} \Big(B_{\sigma\tau} - D_{\sigma} \Lambda_\tau \Big)$$
(2.50)

eliminates the 1-form and $B'_{\alpha\beta}$ becomes

$$B'_{\alpha\beta} = \frac{1}{96} \gamma^{mnp}_{\alpha\beta} \gamma^{\sigma\tau}_{mnp} \Big(B_{\sigma\tau} - D_{\sigma} \Lambda_{\tau} \Big)$$
(2.51)

Simplifying the above expression using (B.10), we arrive at the result

$$B'_{\alpha\beta} = \gamma^{mnp}_{\alpha\beta} B'_{mnp} \tag{2.52}$$

where,

$$96B'_{mnp} = \left[(A_1\gamma_{mnp}A_2) + 8\alpha' A^1_{[m}F^2_{np]} - 4\alpha' (W_1\gamma^{mnp}W_2) + 2i\alpha' k^1_{[m}(A_1\gamma_{np]}W_2) + 4i\alpha' k^1_q(A_1\gamma_{mnpq}W_2) + (1\leftrightarrow 2) \right]$$
(2.53)

Note that the above expression of B'_{mnp} is invariant under the exchange of massless-particle labels. However, this is still not in the Berkovits-Chandia gauge since it does not satisfy the condition $k^m B'_{mnp} = 0$, where $k^m = k_1^m + k_2^m$. To satisfy this condition, we note that we are still allowed to change $\Omega_{1\alpha} = \Lambda_{\alpha}$ by shifting with any Φ_{α} which satisfies $Q(\lambda^{\alpha} \Phi_{\alpha}) = 0$. One can show that the following expressions are BRST closed (and also BRST exact)

$$\lambda^{\alpha} \Phi_{1\alpha} = ik_m^2 A_m^1 V_2 + ik_m^1 V_1 A_m^2 + A_m^1 (\lambda \gamma^m W_2) + (\lambda \gamma^m W_1) A_m^2 = Q(A_1 \cdot A_2)$$
(2.54)
$$\lambda^{\alpha} \Phi_{2\alpha} = ik_n^1 (\lambda \gamma^m W_1) F_{mn}^2 + ik_n^2 F_{mn}^1 (\lambda \gamma^m W_2) = -\frac{1}{2} Q(F_1^{mn} F_2^{mn}) .$$

Since the above expressions are BRST closed, we can modify the gauge parameters $\Omega_{1\alpha}$ and Ω_{3m} by terms involving $\Phi_{1\alpha}$ and $\Phi_{2\alpha}$. Choosing

$$\Omega_{1\alpha} = \Lambda_{\alpha} + 2\alpha' \Phi_{1\alpha} + \frac{4}{3} {\alpha'}^2 \Phi_{2\alpha} , \qquad (2.55)$$
$$\Omega_{3m} = -\frac{1}{16} \gamma_m^{\sigma\tau} \left(B_{\sigma\tau} - D_{\sigma} \left(\Lambda_{\tau} + 2\alpha' \Phi_{1\tau} + \frac{4}{3} {\alpha'}^2 \Phi_{2\tau} \right) \right) ,$$

we arrive at the following expression of B'_{mnp}

$$B'_{mnp} = \frac{1}{18} \alpha'(W_1 \gamma_{mnp} W_2) + \frac{1}{9} {\alpha'}^2 k^1_{[m} k^2_n (W_1 \gamma_{p]} W_2) + \frac{1}{18} i {\alpha'}^2 \Big[k^{2q} F^1_{q[m} F^2_{np]} + (1 \leftrightarrow 2) \Big] = \frac{1}{9} {\alpha'}^2 (W_1 \gamma^{k^1 k^2 mnp} W_2) + \frac{1}{18} {\alpha'}^2 \Big[i k^{2q} F^1_{q[m} F^2_{np]} + (1 \leftrightarrow 2) \Big], \qquad (2.56)$$

where we used the shorthand $(W_1 \gamma^{k^1 k^2 m n p} W_2) = k_a^1 k_b^2 (W_1 \gamma^{abmn p} W_2)$ in the second line. The equivalence between the first and second lines of (2.56) follows from the Dirac equation and $2\alpha' k_1 \cdot k_2 = -1$. It is straightforward to show that (2.56) indeed satisfies $k^m B'_{mnp} = 0$.

The explicit form of the gauge parameters defined in (2.55) in terms of the massless superfields is given by

$$\Omega_{1\alpha} = \frac{1}{2} \alpha' F_1^{mn} (\gamma_{mn} A_2)_{\alpha} + 2\alpha' (\gamma_m W_1)_{\alpha} A_2^m$$

$$+ \frac{4}{3} i \alpha'^2 k_n^1 (\gamma_m W_1)_{\alpha} F_2^{mn} + \frac{4}{3} i \alpha'^2 k_n^2 (\gamma_m W_2)_{\alpha} F_1^{mn} ,$$

$$\Omega_{3m} = 2i \alpha' k_m^1 (W_1 A_2) + \alpha' (W_1 \gamma_m W_2) + 2\alpha' F_{mn}^1 A_2^n$$

$$+ \frac{2}{3} i \alpha'^2 k_p^1 F_{mn}^1 F_2^{np} + \frac{2}{3} i \alpha'^2 k_p^2 F_{mn}^2 F_1^{np} .$$
(2.57)

2.5.2. $H_{m\alpha}$ in the Berkovits-Chandia gauge

We next consider the superfield $H_{m\alpha}$. After the gauge transformation, it becomes

$$H'_{m\alpha} = H_{m\alpha} + D_{\alpha}\Omega_{3m} - \frac{1}{2\alpha'}\gamma^m_{\alpha\beta}\Omega^\beta_2.$$
(2.58)

It is easy to see that the condition $\gamma^{m\alpha\beta}H'_{m\alpha} = 0$ is satisfied provided we choose

$$\Omega_{2}^{\alpha} = \frac{\alpha'}{5} \gamma^{m\alpha\beta} \Big(H_{m\beta} + D_{\beta} \Omega_{3m} \Big)$$

= $-4i {\alpha'}^{2} k_{m}^{1} W_{1}^{\alpha} A_{2}^{m} - \frac{4}{3} i {\alpha'}^{2} k_{n}^{1} A_{m}^{1} (\gamma^{mn} W_{2})^{\alpha} + \frac{2}{3} i {\alpha'}^{2} k_{n}^{2} (\gamma^{mn} W_{1})^{\alpha} A_{m}^{2} , \quad (2.59)$

which implies $(k_{12}^m = k_1^m + k_2^m)$

$$H_{\alpha}^{'m} = \frac{i\alpha'}{6} \Big(-5iF_1^{mn}(\gamma_n W_2)_{\alpha} - 2k_{12}^m A_n^1(\gamma^n W_2)_{\alpha} + k_p^1 A_n^1(\gamma^{mnp} W_2)_{\alpha} - 4\alpha' k_{12}^m (k^2 \cdot A^1) k_n^1(\gamma^n W_2)_{\alpha} + (1 \leftrightarrow 2) \Big).$$
(2.60)

It is straightforward to prove that (2.60) satisfies the transversality condition $k^m H'_{m\alpha} = 0$ where $k^m = k_1^m + k_2^m$. In addition, a long calculation using the massless equations of motion (2.2) (stripping off the pure spinor) and the constraint $2\alpha' k_1 \cdot k_2 = -1$ reveals that

$$H'_{m\alpha} = \frac{3}{7} (\gamma^{np})_{\alpha}{}^{\beta} D_{\beta} B'_{mnp} \,. \tag{2.61}$$

2.5.3. $C^{\beta}{}_{\alpha}$ in the Berkovits-Chandia gauge

Next, we consider the superfield $C^{\beta}_{\ \alpha}$. Its gauge transformation implies

$$C^{\prime\beta}_{\ \alpha} = C^{\beta}_{\ \alpha} - \frac{1}{2\alpha^{\prime}} D_{\alpha} \Omega^{\beta}_{2} - \frac{1}{4\alpha^{\prime}} (\gamma^{mn})^{\beta}_{\ \alpha} \Omega_{5mn} - \frac{1}{2\alpha^{\prime}} \delta^{\beta}_{\alpha} \Omega_{4} \,. \tag{2.62}$$

After applying the Fierz identity with respect to the indices α and β one can eliminate the zero- and two-form parts by choosing

$$\Omega_4 = \frac{\alpha'}{8} \left(C^{\alpha}_{\ \alpha} - \frac{1}{2\alpha'} D_{\alpha} \Omega_2^{\alpha} \right), \qquad (2.63)$$
$$\Omega_{5mn} = \frac{\alpha'}{8} (\gamma_{mn})^{\alpha}_{\ \beta} \left(-C^{\beta}_{\ \alpha} + \frac{1}{2\alpha'} D_{\alpha} \Omega_2^{\beta} \right).$$

In terms of the massless superfields, their explicit expressions are given by

$$\Omega_4 = -\frac{{\alpha'}^2}{6} F_1^{mn} F_{mn}^2 \,, \tag{2.64}$$

$$\Omega_{5mn} = {\alpha'}^2 \left(F_{a[m}^1 F_{n]a}^2 - 2F_{mn}^1 (ik^1 \cdot A^2) + \frac{3}{2} ik_{[m}^1 (W_1 \gamma_{n]} W_2) + \frac{1}{2} ik_{[m}^2 (W_1 \gamma_{n]} W_2) \right),$$

with $2\alpha' k_1 \cdot k_2 = -1$. Plugging these into (2.62), we find $(k_{12}^m = k_1^m + k_2^m)$,

$$C^{\prime\beta}{}_{\alpha} = \frac{\alpha^{\prime}}{6} (\gamma^{mnpq})^{\beta}{}_{\alpha} \left(\frac{1}{12} i k_m^{12} (W_1 \gamma_{npq} W_2) + k_m^1 k_n^2 A_p^1 A_q^2 \right).$$
(2.65)

It is easy to see that the above expression is equivalent to $(k^m = k_{12}^m)$

$$C^{\prime \beta}{}_{\alpha} = \frac{1}{4} i k_m (\gamma^{mnpq})^{\beta}{}_{\alpha} B^{\prime}_{npq} , \qquad (2.66)$$

with B'_{mnp} given in equation (2.56).

As a consistency check, we find that the gauge superfield parameters $\Omega_{1\alpha}$, Ω_2^{α} , Ω_4 and Ω_{5mn} fixed in (2.55), (2.59) and (2.64) satisfy the constraint (2.47).

2.5.4. $F_{\alpha mn}$ in the Berkovits-Chandia gauge

Finally, we consider the superfield $F_{\alpha mn}$. After the gauge transformation, it takes the form

$$F'_{\alpha mn} = F_{\alpha mn} + \frac{1}{\alpha'} D_{\alpha} \Omega_{5mn} + \gamma_{m\alpha\beta} \Lambda_n^{\beta} - \gamma_{n\alpha\beta} \Lambda_m^{\beta} \,. \tag{2.67}$$

Requiring the constraint $\gamma_m^{\alpha\beta}F'_{\beta mn} = 0$ and using (2.46) and $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$ implies that

$$\Lambda_n^{\alpha} = -\frac{1}{8\alpha'} \left(\alpha' \gamma_m^{\alpha\beta} F_{\beta mn} + (\gamma^m D)^{\alpha} \Omega_{mn}^5 + (\gamma^n D)^{\alpha} \Omega_4 \right).$$
(2.68)

Using the expressions of $F_{\alpha mn}$ from (2.18) and Ω_4 and Ω_{5mn} from (2.64), we get

$$\begin{split} \Lambda_{m}^{\alpha} &= -\frac{43}{96} A_{m}^{1} W_{2}^{\alpha} + \frac{53}{96} W_{1}^{\alpha} A_{m}^{2} - \frac{1}{96} A_{n}^{1} (\gamma^{mn} W_{2})^{\alpha} - \frac{1}{96} (\gamma^{mn} W_{1})^{\alpha} A_{n}^{2} \qquad (2.69) \\ &- \frac{43}{48} \alpha' k_{m}^{1} (k^{2} \cdot A^{1}) W_{2}^{\alpha} + 2\alpha' k_{m}^{1} W_{1}^{\alpha} (k^{1} \cdot A^{2}) + \frac{53}{48} \alpha' k_{m}^{2} W_{1}^{\alpha} (k^{1} \cdot A^{2}) \\ &- \frac{13}{16} \alpha' k_{m}^{1} k_{n}^{1} A_{p}^{1} (\gamma^{np} W_{2})^{\alpha} + \frac{3}{16} \alpha' k_{m}^{2} (\gamma^{np} W_{1})^{\alpha} k_{n}^{2} A_{p}^{2} - \frac{3}{8} \alpha' k_{m}^{2} k_{n}^{1} A_{p}^{1} (\gamma^{np} W_{2})^{\alpha} \\ &+ \frac{5}{8} \alpha' k_{m}^{1} (\gamma^{np} W_{1})^{\alpha} k_{n}^{2} A_{p}^{2} - \frac{1}{48} \alpha' (k^{2} \cdot A^{1}) k_{n}^{1} (\gamma^{mn} W_{2})^{\alpha} - \frac{1}{48} \alpha' k_{n}^{2} (\gamma^{mn} W_{1})^{\alpha} (k^{1} \cdot A^{2}) \,, \end{split}$$

which indeed satisfies the first equation in (2.46). Finally, we find $F'_{\alpha mn}$ in the Berkovits-Chandia gauge as

$$\begin{aligned} F'_{\alpha mn} &= -\frac{1}{24} A^{1}_{[m}(\gamma_{n]}W_{2})_{\alpha} - \frac{1}{48} A^{1}_{p}(\gamma^{mnp}W_{2})_{\alpha} + \frac{3}{8} \alpha' k^{1}_{[m}k^{2}_{n]}A^{1}_{p}(\gamma^{p}W_{2})_{\alpha} & (2.70) \\ &+ \frac{7}{16} \alpha' k^{1}_{[m}A^{1}_{n]}k^{1}_{p}(\gamma^{p}W_{2})_{\alpha} - \frac{1}{12} \alpha' (k^{2} \cdot A^{1}) k^{1}_{[m}(\gamma_{n]}W_{2})_{\alpha} \\ &+ \frac{3}{8} \alpha' k^{2}_{[m}A^{1}_{n]}k^{1}_{p}(\gamma^{p}W_{2})_{\alpha} + \frac{1}{16} \alpha' k^{1}_{p}A^{1}_{q}k^{1}_{[m}(\gamma_{n]pq}W_{2})_{\alpha} \\ &+ \frac{1}{8} \alpha' k^{1}_{p}A^{1}_{q}k^{2}_{[m}(\gamma_{n]pq}W_{2})_{\alpha} - \frac{1}{24} \alpha' (k^{2} \cdot A^{1}) k^{1}_{p}(\gamma^{mnp}W_{2})_{\alpha} \\ &+ (1 \leftrightarrow 2) \,, \end{aligned}$$

which satisfies $([mn] = mn - nm, k^m = k_1^m + k_2^m)$

$$F'_{\alpha m n} = \frac{1}{16} \Big(7ik_{[m} H'_{n]\alpha} + ik_q (\gamma_{q[m})_{\alpha}^{\ \beta} H'_{n]\beta} \Big) \,. \tag{2.71}$$

Furthermore, using the equation of motion for $H_{\alpha}^{\prime m}$ [21],

$$D_{\alpha}H_{\beta}^{\prime m} = -\frac{9}{4}G_{mn}^{\prime}\gamma_{\alpha\beta}^{n} - \frac{3}{2}\partial_{a}B_{bcm}^{\prime}\gamma_{\alpha\beta}^{abc} + \frac{1}{4}\partial_{a}B_{bcd}^{\prime}\gamma_{\alpha\beta}^{mabcd}, \qquad (2.72)$$

several gamma matrix identities and

$$\partial_p C^{\prime \alpha}{}_{\delta} \gamma_p^{\delta \beta} = -\frac{1}{4\alpha^{\prime}} (\gamma^{mnp})^{\alpha \beta} B^{\prime}_{mnp} , \qquad (2.73)$$
$$(\lambda \gamma^{mnabc} \lambda) B^{\prime}_{abc} = 2\alpha^{\prime} (\partial_p (\lambda \gamma^m C^{\prime} \gamma^n \gamma^p \lambda) - \partial_p (\lambda \gamma^n C^{\prime} \gamma^m \gamma^p \lambda)) ,$$

we obtain

$$Q(\lambda F')_{mn} = \frac{1}{2} \partial_{[m} (\lambda \gamma_{n]} C' \lambda) - \frac{1}{16} \partial^{p} (\lambda \gamma_{[m} C' \gamma_{n] p} \lambda), \qquad (2.74)$$

where $(\lambda \gamma^m C' \gamma^{np} \lambda) = (\lambda \gamma^m)_{\alpha} C'^{\alpha}{}_{\beta} (\gamma^{np} \lambda)^{\beta}$.

2.5.5. Massive superfields in the BC gauge: summary

In summary, the massive superfields in the Berkovits-Chandia gauge written in terms of massless superfields³ are given by the equations $B'_{\alpha\beta} = \gamma^{mnp}_{\alpha\beta}B'_{mnp}$ with B'_{mnp} in (2.56), $H'_{m\alpha}$ from (2.60), $C'^{\beta}{}_{\alpha}$ from (2.65) and $F'_{\alpha mn}$ from (2.70). Dropping the ' superscript from

 $^{^{3}}$ These results were firstly announced in November 2023 [13] and later confirmed with independent calculations by [14].

the notation and using $k^m = k_1^m + k_2^m$, one can show using the SYM equations of motion (2.2) that the above expression satisfies

$$B_{\alpha\beta} = \gamma_{\alpha\beta}^{mnp} B_{mnp} , \qquad (2.75)$$

$$H_{m\alpha} = \frac{3}{7} (\gamma^{np})_{\alpha}{}^{\beta} D_{\beta} B_{mnp} , \qquad (2.75)$$

$$C^{\beta}{}_{\alpha} = \frac{1}{4} i k_m (\gamma^{mnpq})^{\beta}{}_{\alpha} B_{npq} , \qquad (2.75)$$

$$F_{\alpha mn} = \frac{1}{16} \left(7 i k_{[m} H_{n]\alpha} + i k_q (\gamma_{q[m})_{\alpha}{}^{\beta} H_{n]\beta} \right) ,$$

where ([mn] = mn - nm)

$$B_{mnp} = \frac{1}{9} {\alpha'}^2 (W_1 \gamma^{k^1 k^2 m n p} W_2) + \frac{1}{18} {\alpha'}^2 \left[i k^{2q} F_{q[m}^1 F_{np]}^2 + (1 \leftrightarrow 2) \right], \quad 2\alpha' k_1 \cdot k_2 = -1,$$
(2.76)

in agreement with [11]. Moreover, the equations of motion of the superfields in the combination (2.19) are given by

$$Q(\lambda B)_{\alpha} = (\lambda \gamma^{m})_{\alpha} (\lambda H)_{m}, \qquad (2.77)$$

$$Q(\lambda H^{m}) = (\lambda \gamma^{m} C \lambda), \qquad (2.77)$$

$$Q(C \lambda)^{\alpha} = \frac{1}{4} (\lambda \gamma^{mn})^{\alpha} (\lambda F)_{mn}, \qquad (2.77)$$

$$Q(\lambda F')_{mn} = \frac{1}{2} \partial_{[m} (\lambda \gamma_{n]} C' \lambda) - \frac{1}{16} \partial^{p} (\lambda \gamma_{[m} C' \gamma_{n]p} \lambda), \qquad (2.77)$$

and resemble the massless SYM equations of motion (2.2). Note that the gauge transformations (2.44) preserve the first three equations but not the last. Finally, in the Berkovits-Chandia gauge the unintegrated vertex operator at mass level one becomes

$$V = [\lambda^{\alpha} [\partial \theta^{\beta} B_{\alpha\beta}]_{0}]_{0} + [\lambda^{\alpha} [\Pi^{m} H^{m}_{\alpha}]_{0}]_{0} + 2\alpha' [\lambda^{\alpha} [d_{\beta} C^{\beta}{}_{\alpha}]_{0}]_{0} + \alpha' [\lambda^{\alpha} [N^{mn} F_{\alpha mn}]_{0}]_{0}, \quad (2.78)$$

where the normal-ordering bracket $[AB]_0$ is defined in (A.2), and we used (2.40). Alternatively, it can also be suggestively rewritten in terms of the definition (2.19) as

$$V = [\partial \theta^{\alpha} (B\lambda)_{\alpha}]_0 + [\Pi^m (\lambda H^m)]_0 + 2\alpha' [d_{\alpha} (C\lambda)^{\alpha}]_0 + \alpha' [N^{mn} (\lambda F)_{mn}]_0, \qquad (2.79)$$

resembling the massless integrated vertex operator (2.11).

In the context of scattering amplitudes, the massless representation of massive superfields can be viewed as swapping a string label, say \underline{k} for the massive string, by a pair of massless string labels, say i, j (i=1 and j=2 in the example of (2.76)). At the level of superfields (both in the OPE or Berkovits-Chandia gauge), this swap will be denoted by

$$\underline{k} \to i, j, \qquad 2\alpha'(k_i \cdot k_j) = -1, \qquad (2.80)$$

where the constraint on the momenta must accompany the change of the superfields.

3. Relation between massive and massless string amplitudes

In this section we will show that using the massless parameterization of the massive superfields gives rise to an explicit relation between massive and massless amplitudes of the open string. The combinatorics of which can be described by an algorithm [22] closely related to the descent algebra [23,24]. To see this relation, we reinterpret the factorization condition to perform the equivalent of the sum over intermediate polarizations on a single amplitude rather than a quadratic expression connected via a propagator.

3.1. 3-point massive amplitude revisited

The tree-level scattering amplitude of two massless and one massive state was firstly computed using the pure spinor formalism in [12]. The result, despite correct, was obtained in a rather convoluted way using the OPEs of the pure spinor formalism after performing the θ expansions of the massive superfields obtained in [21]. Consequently, the simplicity of the result was lost. However, one can exploit the cohomological structure of the pure spinor formalism together with the equations of motion (2.20) to (2.23) to obtain a simple answer written in pure spinor superspace.

We start with the three-point amplitude prescription for massless strings labelled by 1 and 2 and one first-level massive string labelled by $\underline{3}$

$$4i{\alpha'}^2 A(1,2|\underline{3}) = \langle V_1^{(0)} V_2^{(0)} V_{\underline{3}}^{(1)} \rangle$$
(3.1)

where $V^{(0)}$ and $V^{(1)}$ denote the massless and first-level massive unintegrated vertices (2.12) and (2.78) (the superscript indicates the mass and has been added for clarity). The normalization on the left-hand side was chosen to make the amplitude dimensionless, $[A(1, 2|\underline{3})] = 0.$

We will use the techniques of [5,6,1] in which the OPEs among the vertices are evaluated up to the plane-wave factors. This results in a chiral CFT correlator multiplying an overall Koba-Nielsen factor [12]

$$\mathcal{I} = |z_{12}|^{2\alpha' k_1 \cdot k_2} |z_{13}|^{2\alpha' k_1 \cdot k_3} |z_{23}|^{2\alpha' k_2 \cdot k_3} = \frac{z_{13} z_{23}}{z_{12}}$$
(3.2)

which follows from momentum conservation and $k_1^2 = k_2^2 = 0$, $k_3^2 = -1/\alpha'$. In addition, the CFT correlator is evaluated up to BRST exact terms and this will be indicated by $A \sim B$. Defining

$$L_{\underline{3}1} \sim [V_{\underline{3}}^{(1)}V_{1}^{(0)}]_{1} \tag{3.3}$$

we get

$$4i{\alpha'}^2 A(1,2|\underline{3}) = \left(\frac{1}{z_{31}} \langle L_{\underline{3}1} V_2 \rangle - \frac{1}{z_{32}} \langle V_1 L_{\underline{3}2} \rangle \right) \mathcal{I} = -\frac{z_{23}}{z_{12}} \langle L_{\underline{3}1} V_2 \rangle + \frac{z_{13}}{z_{12}} \langle V_1 L_{\underline{3}2} \rangle .$$
(3.4)

A straightforward OPE calculation yields

$$L_{\underline{3}1} = -2\alpha'(\lambda H_3^m)\partial_m V_1 + 2\alpha'(C_3\lambda)^\beta D_\beta V_1 - \frac{\alpha'}{2}(\lambda F_3)_{mn}(\lambda\gamma^{mn}A_1).$$
(3.5)

The equation of motion (2.2) can be used to rewrite the second term of (3.5) as

$$2\alpha'(C_3\lambda)^{\beta}D_{\beta}V_1 = -2\alpha'(C_3\lambda)^{\beta}QA_{\beta}^1 + 2\alpha'(\lambda\gamma^m C_3\lambda)A_1^m$$

$$= -2\alpha'Q(A_1C_3\lambda) + \frac{\alpha'}{2}(\lambda F_3)_{mn}(\lambda\gamma^{mn}A_1) + 2\alpha'(\lambda\gamma^m C_3\lambda)A_1^m ,$$
(3.6)

leading to cancellations in (3.5). Furthermore, dropping the BRST exact term we arrive at

$$L_{\underline{3}1} \sim -2\alpha'(\lambda H_3^m)\partial_m V_1 + 2\alpha'(\lambda\gamma^m C_3\lambda)A_1^m$$

$$\sim -2\alpha'(\lambda H_3^m)QA_m^1 + 2\alpha'(\lambda H_3^m)(\lambda\gamma_m W_1) + 2\alpha'(\lambda\gamma^m C_3\lambda)A_1^m$$

$$\sim 2\alpha'Q((\lambda H_3^m)A_m^1) + 2\alpha'(\lambda H_3^m)(\lambda\gamma_m W_1)$$

$$\sim 2\alpha'(\lambda H_3^m)(\lambda\gamma_m W_1), \qquad (3.7)$$

where we used the equations of motion (2.21) and $QA_m^1 = \partial_m V_1 + (\lambda \gamma_m W_1)$. It is easy to see that $QL_{\underline{3}1} = 0$, as expected from the definition (3.3) and the identities (2.25) and (2.26). After plugging in $L_{\underline{i}j}$ from (3.3) into (3.4) the three-point amplitude becomes

$$4i\alpha'^{2}A(1,2|\underline{3}) = -2\alpha'\frac{z_{23}}{z_{12}}\langle(\lambda\gamma_{m}W_{1})V_{2}(\lambda H_{3}^{m})\rangle - 2\alpha'\frac{z_{13}}{z_{12}}\langle V_{1}(\lambda\gamma_{m}W_{2})(\lambda H_{3}^{m})\rangle$$
(3.8)

To simplify this answer further we will need the following:

Lemma. In pure spinor superspace, the following is true:

$$\langle V_2(\lambda \gamma_m W_1)(\lambda H_3^m) \rangle = \langle V_1(\lambda \gamma_m W_2)(\lambda H_3^m) \rangle.$$
(3.9)

Proof. Note that

$$Q(\lambda \gamma^m \mathcal{W}_{12}) = V_1(\lambda \gamma^m W_2) - V_2(\lambda \gamma^m W_1), \qquad (3.10)$$

where \mathcal{W}_{12} is the Berends-Giele current defined in [25]. Therefore,

$$\langle V_1(\lambda\gamma_m W_2)(\lambda H_3^m) \rangle = \langle Q(\lambda\gamma_m \mathcal{W}_{12})(\lambda H_3^m) \rangle + \langle V_2(\lambda\gamma_m W_1)(\lambda H_3^m) \rangle$$

$$= \langle (\lambda\gamma_m \mathcal{W}_{12})Q(\lambda H_3^m) \rangle + \langle V_2(\lambda\gamma_m W_1)(\lambda H_3^m) \rangle$$

$$= \langle (\lambda\gamma_m \mathcal{W}_{12})(\lambda\gamma^m C_3\lambda) \rangle + \langle V_2(\lambda\gamma_m W_1)(\lambda H_3^m) \rangle$$

$$= \langle V_2(\lambda\gamma_m W_1)(\lambda H_3^m) \rangle$$

$$(3.11)$$

where we used BRST integration by parts to arrive at the second line followed by the equation of motion (2.21) and the identity $(\lambda \gamma_m)_{\alpha} (\lambda \gamma^m)_{\beta} = 0$ in the last line, finishing the proof. \Box

Finally, using the Lemma (3.9) in (3.8), the three-point amplitude of two massless states and one first-level massive state becomes

$$A(1,2|\underline{3}) = \frac{i}{2\alpha'} \langle V_1(\lambda \gamma_m W_2)(\lambda H_3^m) \rangle .$$
(3.12)

This is independent of worldsheet positions as expected from Möbius invariance. Moreover, it is easy to show that the amplitude is BRST invariant.

3.2. Massless representation of the massive 3-point amplitude

Plugging in the massless representation of the massive superfield (λH_3^m) in the OPE gauge⁴ given in (B.2) (with the relabeling $1 \rightarrow 3, 2 \rightarrow 4$) into (3.12) one gets

$$A(1,2|\underline{3})|_{\underline{3}\to3,4} = \langle V_1(\lambda\gamma_m W_2) (F_3^{mn} k_4^n V_4 + k_3^m (\lambda\gamma^n W_3) A_4^n) \rangle, \qquad (3.13)$$

where $\underline{3} \rightarrow 3, 4$ defined in (2.80) represents the change to the massless representation of the massive superfield (λH_3^m) , and we dropped the BRST exact term $k_3^m Q(W_3 A_4)$ in (λH_3^m) from (B.2) because $V_1(\lambda \gamma^m W_2)$ is BRST closed. Using the equation of motion $k_4^n V_4 = Q A_4^n - (\lambda \gamma^n W_4)$ one rewrites the first term inside the parenthesis in (3.13) as

$$F_3^{mn}k_4^n V_4 = Q(F_3^{mn}A_4^n) - F_3^{mn}(\lambda\gamma^n W_4) - k_3^{[m}(\lambda\gamma^n]W_3)A_4^n.$$
(3.14)

Therefore, discarding the BRST exact term from (3.14), one obtains

$$F_3^{mn}k_4^n V_4 + k_3^m (\lambda \gamma^n W_3) A_4^n \sim -F_3^{mn} (\lambda \gamma^n W_4) + (\lambda \gamma^m W_3) (k_3 \cdot A_4) .$$
(3.15)

Plugging (3.15) into (3.13) and using $(\lambda \gamma_m)_{\alpha} (\lambda \gamma^m)_{\beta} = 0$ to drop the second term from (3.15) one finally arrives at

$$A(1,2|\underline{3})|_{\underline{3}\to3,4} = -\langle V_1(\lambda\gamma_m W_2) F_3^{mn}(\lambda\gamma_n W_4) \rangle.$$
(3.16)

⁴ Using BRST cohomology manipulations, one can show that the pure spinor superspace expression (3.13) is also obtained if (λH_3^m) in the Berkovits-Chandia gauge is used instead.

The superspace expression in the right-hand side of (3.16) is easily identified as the kinematic factor of the massless four-point open-string amplitude at one loop [26,5]

$$A(1,2,|\underline{3})|_{\underline{3}\to3,4} = -\langle C_{1|2,3,4} \rangle, \qquad 2\alpha' k_3 \cdot k_4 = -1, \qquad (3.17)$$

where $C_{1|2,3,4}$ is the four-point BRST invariant defined in [27,28] whose explicit component expansion [29] can be downloaded from [30]. When all states are bosonic, $\langle C_{1|2,3,4} \rangle$ is proportional to the famous t_8F^4 combination, where the t_8 tensor can be found in [31]. The constraint in the momenta is written explicitly for emphasis (as it is already implicitly required by (2.80)). Alternatively, the four-point BRST invariant was shown to be proportional to the α'^2 string correction of the massless four-point amplitude, denoted by $A^{F^4}(1,2,3,4)$ [27]

$$A(1,2|\underline{3})|_{\underline{3}\to3,4} = -A^{F^4}(1,2,3,4), \qquad 2\alpha' k_3 \cdot k_4 = -1.$$
(3.18)

Since the pure spinor superspace expression (3.16) contains four superfields and it is BRST invariant, one can expand it in components using regular four-point kinematics and apply the massive kinematics constraint $2\alpha'(k_3 \cdot k_4) = -1$ at the end of the calculations. Using the normalization $\langle (\lambda \gamma^m \theta) (\lambda \gamma^n \theta) (\lambda \gamma^p \theta) (\theta \gamma_{mnp} \theta) \rangle = 2880 {\alpha'}^2$ of the pure spinor bracket [3] we get

$$A(1,2|\underline{3})\big|_{\underline{3}\to3,4} = -\alpha'^2(k_1 \cdot k_2)(k_2 \cdot k_3)A^{\text{SYM}}(1,2,3,4), \qquad 2\alpha' k_3 \cdot k_4 = -1, \qquad (3.19)$$

where $A^{\text{SYM}}(1, 2, 3, 4)$ represents the four-point SYM field-theory amplitude.

Component expansion. On the one hand, the component expansion of the amplitude (3.12) when all external states are bosonic can be computed using the theta expansion of H^m_{α} found in [21],

$$A(1,2|\underline{3}) = \alpha' f_1^{mp} f_2^{pn} g_{\underline{3}mn} + 2i e_1^m k_2^n e_2^p b_{\underline{3}mnp} + (1\leftrightarrow 2), \qquad (3.20)$$

where g_{mn} is the symmetric traceless and b_{mnp} is the 3-form polarization subject to $k^m g_{mn} = k^m b_{mnp} = 0$. The length dimensions of the quantities in (3.20) are

$$[A(1,2|\underline{3})] = 0, \quad [g_{mn}] = 0, \quad [b_{mnp}] = 1.$$
(3.21)

In addition, $f_1^{mn} = k_1^m e_1^n - k_1^n e_1^m$ is the component field-strength and we rescalled the overall normalization of the amplitude given in [21] for convenience.

As shown in [21], the massive polarizations of the open string can be extracted from the massive superfields in the Berkovits-Chandia gauge as

$$g_{mn} = \frac{1}{64} (D\gamma_{(m}H_{n)}) \big|_{\theta=0}, \qquad b_{mnp} = \frac{9}{8} B_{mnp} \big|_{\theta=0}, \qquad (3.22)$$

where the overall normalizations were chosen for convenience. Using the massless representations of the massive superfields H^m_{α} and B_{mnp} from (2.75) in the Berkovits-Chandia gauge with labels 1 and 2 as in (2.76) yields [13]

$$g_{mn}(1,2) = \frac{1}{8} \left(e_1^m e_2^n + e_1^n e_2^m - \frac{1}{3} \delta_{mn}(e_1 \cdot e_2) \right)$$

$$+ \frac{1}{24} \alpha' \left(2(k_1^m k_1^n - 2k_1^m k_2^n)(e_1 \cdot e_2) + 6(k_2^m e_1^n + k_2^n e_1^m)(k_1 \cdot e_2) - \delta_{mn}(k_1 \cdot e_2)(k_2 \cdot e_1) + (1 \leftrightarrow 2) \right)$$

$$+ \frac{1}{6} \alpha'^2 k_{12}^m k_{12}^n (k_1 \cdot e_2)(k_2 \cdot e_1)$$

$$b_{mnp}(1,2) = \frac{i}{16} \alpha' \left(k_1^{[m} e_1^n e_2^{p]} + k_2^{[m} e_2^n e_1^{p]} \right)$$

$$+ \frac{i}{8} \alpha'^2 \left(k_1^{[m} k_2^n e_2^{p]}(k_2 \cdot e_1) + k_2^{[m} k_1^n e_1^{p]}(k_1 \cdot e_2) \right), \qquad 2\alpha'(k_1 \cdot k_2) = -1,$$
(3.23)

where the notation $g_{mn}(1,2)$ emphasizes the labels 1 and 2 of the massless polarizations on the right-hand side. Using the transversality $(k_i \cdot e_i) = 0$ and that the states 1 and 2 are massless, $k_1^2 = k_2^2 = 0$, together with the constraint $2\alpha'(k_1 \cdot k_2) = -1$, one can easily check that they are transverse, $k_{12}^m g_{mn}(1,2) = 0$ and $k_{12}^m b_{mnp}(1,2) = 0$ as well as traceless symmetric (g_{mn}) and antisymmetric (b_{mnp}) in their vectorial indices.

Finally, it follows from the above discussion that the two expressions (3.20) and (3.19) of the three-point amplitude with one massive and two massless states are related by

$$g_{\underline{3}mn} = g_{mn}(3,4), \quad b_{\underline{3}mnp} = b_{mnp}(3,4), \qquad 2\alpha'(k_3 \cdot k_4) = -1.$$
 (3.24)

More generally, we define in analogy with (2.80) the same notation $\underline{k} \to i, j$ to represent the swap to massless polarizations in place of the massive polarizations:

$$\underline{k} \to i, j: \quad g_{\underline{k}mn} = g_{mn}(i, j), \quad b_{\underline{k}mnp} = b_{mnp}(i, j), \qquad 2\alpha'(k_i \cdot k_j) = -1, \qquad (3.25)$$

where $g_{mn}(i,j)$ and $b_{mnp}(i,j)$ are given by (3.23).

3.3. Massive amplitudes as linear combinations of massless amplitudes

The observation (3.17) generalizes to higher multiplicities. To see this, we use the perturbiner construction of $A(P|\underline{n})$ recently found in [15]. In that paper, the superstring amplitude involving n-1 massless states and one massive state \underline{n} was packaged in terms of (n-3)! worldsheet integrals F_Q^P and partial subamplitudes $A(1, P, n-1|\underline{n})$ as

$$\mathcal{A}(1,Q,n-1,\underline{n}) = \sum_{P \in S_{n-3}} F_Q^P A(1,P,n-1|\underline{n}), \qquad (3.26)$$

where P and Q are words comprised of particle labels (letters) and F_Q^P have the same functional form as the string disk integrals in the massless string scattering amplitude [1,2,32,33]; the only difference stems from the massive constraint $k_{\underline{n}}^2 = -1/\alpha'$ affecting the relations among Mandelstam variables. When all external states are bosonic, the partial amplitudes $A(1, P|\underline{n})$ with |P| = n-1 massless states and one massive state at the first massive level are given by [15],

$$A(P|\underline{n}) = \phi_P^{mn} g_{\underline{n}\,mn} + \phi_P^{mnp} b_{\underline{n}\,mnp} \,, \tag{3.27}$$

where the |P| massless states are encoded in

$$\phi_P^{mn} = \alpha' \sum_{XY=P} f_X^{ma} f_Y^{an} + \operatorname{cyc}(P) , \qquad (3.28)$$

$$\phi_P^{mnp} = 2i \sum_{XY=P} e_X^m k_Y^n e_Y^p - \frac{4i}{3} \sum_{XYZ=P} e_X^m e_Y^n e_Z^p + \operatorname{cyc}(P) ,$$

where e_X^m and f_X^{mn} are the Berends-Giele multiparticle polarizations of [34],

$$e_P^m = \frac{1}{k_P^2} \sum_{XY=P} e_Y^m (k_Y \cdot e_X) + f_X^{mn} e_Y^n , \qquad (3.29)$$

$$f_P^{mn} = k_P^m e_P^n - k_P^n e_P^m - \sum_{XY=P} \left(e_X^m e_Y^n - e_X^n e_Y^m \right) .$$

In addition, the notation +cyc(P) instructs to add the cyclic permutations of the letters in P, XY=P denote the deconcatenations of P into non-empty words X and Y, and $k_{ij\ldots p}^m = k_i^m + k_j^m + \cdots + k_p^m$. Upon plugging the massless representations (3.23) of the massive polarizations into the amplitude (3.27), replacing the massive state \underline{n} by two massless states labelled n and n+1, straightforward but tedious calculations⁵ show that:

$$A(1,2|\underline{3})\big|_{\underline{3}\to3,4} = -\langle C_{1|2,3,4} \rangle, \qquad 2\alpha' k_3 \cdot k_4 = -1, \qquad (3.30)$$
$$A(1,2,3|\underline{4})\big|_{\underline{4}\to4,5} = -\langle C_{1|23,4,5} \rangle, \qquad 2\alpha' k_4 \cdot k_5 = -1, \\A(1,2,3,4|\underline{5})\big|_{\underline{5}\to5,6} = -\langle C_{1|234,5,6} \rangle, \qquad 2\alpha' k_5 \cdot k_6 = -1,$$

where the substitution rule in the left-hand side is given by (3.25), and the constraint in the momenta is written down for emphasis (in addition to featuring in (3.25)). The explicit components of the scalar BRST invariants are available to download from [30] and can be used to check the relations above. These results suggest the following generalization,

$$A(1, P|\underline{n})|_{\underline{n}\to n, n+1} = -\langle C_{1|P,n,n+1} \rangle, \qquad 2\alpha' k_n \cdot k_{n+1} = -1, \qquad (3.31)$$

relating the massive string amplitude with one massive external state to the α'^2 sector of the massless tree-level string amplitudes.

Massless string amplitudes at ${\alpha'}^2$ order. To make the connection to the ${\alpha'}^2$ correction of massless string tree amplitudes even clearer, we can explicitly rewrite (3.31) in terms of A^{F^4} , the ${\alpha'}^2$ corrections to massless string amplitudes defined by [27]

$$A(1, 2, \dots, n) = A^{\rm YM}(1, 2, \dots, n) + {\alpha'}^2 \zeta_2 A^{F^4}(1, 2, \dots, n) + \mathcal{O}({\alpha'}^3).$$
(3.32)

To see this, we note that the BRST invariants $\langle C_{1|X,Y,Z} \rangle$ can be expanded in terms of permutations of A^{F^4} , as argued in [27]. The precise permutations in this expansion turns out to be related to the Solomon descent algebra, as described in [22]. In particular, for each *n*-point BRST invariant $C_{1|X,Y,Z}$ characterized by words X, Y and Z, one can define precise permutations $\gamma_{1|X,Y,Z}$ of *n* labels (from the letters in 1, X, Y, Z) dubbed *BRSTinvariant permutations*. For example,

$$\gamma_{1|2,3,4} = W_{1234} + W_{1243} + W_{1324} + W_{1342} + W_{1423} + W_{1432}, \qquad (3.33)$$

where a permutation σ is written as W_{σ} for typographical reasons.

⁵ We acknowledge the use of FORM [35].

The relation found in [22] expands the BRST invariants as permutations of A^{F^4} ,

$$\langle C_{1|X,Y,Z} \rangle = \frac{1}{6} A^{F^4}(\gamma_{1|X,Y,Z}) \equiv \frac{1}{6} \sum_{\sigma \in \gamma_{1|X,Y,Z}} A^{F^4}(\sigma) \,.$$
(3.34)

For instance, using the permutations in (3.33) one gets

$$\langle C_{1|2,3,4} \rangle = \frac{1}{6} A^{F^4}(\gamma_{1|2,3,4})$$

$$= \frac{1}{6} \left(A^{F^4}(1234) + A^{F^4}(1243) + A^{F^4}(1324) + A^{F^4}(1342) + A^{F^4}(1423) + A^{F^4}(1432) \right)$$

$$= A^{F^4}(1234) ,$$

$$(3.35)$$

where in the last line we used the total symmetry of $A^{F^4}(1234)$.

In view of the identity (3.34), the general observation (3.31) yields a expansion in permutations of A^{F^4}

$$A(1,P|\underline{n})\big|_{\underline{n}\to n,n+1} = -\frac{1}{6}A^{F^4}(\gamma_{1|P,n,n+1}), \qquad 2\alpha' k_n \cdot k_{n+1} = -1.$$
(3.36)

For example, one gets

$$-6A(1,2,3|\underline{4})|_{\underline{4}\to4,5} = A_{12345}^{F^4} - A_{12354}^{F^4} - A_{12435}^{F^4} + A_{12453}^{F^4} + A_{12534}^{F^4} - A_{12543}^{F^4}$$
(3.37)
+ $A_{13245}^{F^4} - A_{13254}^{F^4} + A_{13425}^{F^4} - A_{13524}^{F^4} - A_{14235}^{F^4} - A_{14325}^{F^4},$

where the explicit permutations in $\gamma_{1|23,4,5}$ and the algorithm to generate them can be found in [22]. Note that we used the parity and cyclicity in the form of $A^{F^4}(1,2,\ldots,n) =$ $(-1)^n A^{F^4}(1,n,n-1,\ldots,2)$ to reduce the number of terms in (3.37).

3.4. Relating massive and massless string amplitudes via unitarity

In this section we will show that the result (3.31) can be explained from the factorization of the massless string amplitude in its first massive pole⁶. To see this, one computes the residue of the massless *n*-point tree-level amplitude when $s_{n-1,n} = -1$ (setting $\alpha' = \frac{1}{2}$).

 $^{^{6}}$ See [36] for similar considerations with the RNS formalism.

3.4.1. 4-point factorization

The massless 4-point tree amplitude is given by

$$\mathcal{A}_4 = \mathcal{A}(1, 2, 3, 4) = \langle C_{1|2, 3, 4} \rangle \frac{\Gamma(s_{34}) \Gamma(s_{23})}{\Gamma(1 + s_{34} + s_{23})}.$$
(3.38)

Using the well known result that $\operatorname{Res}_{x=-n}(\Gamma(x)) = (-1)^n/n!$ we get [37]

$$\operatorname{Res}_{s_{34}=-1}(\mathcal{A}_4) = -\langle C_{1|2,3,4} \rangle, \quad s_{34}=-1, \qquad (3.39)$$

which explains the first line of (3.30).

3.4.2. 5-point factorization

The five-point analysis proceeds similarly. But note that the Möbius symmetry gauge fixing $(z_1, z_4, z_5) \rightarrow (0, 1, \infty)$ in the usual formula [1,2]

$$\mathcal{A}_5 = \mathcal{A}(1, 2, 3, 4, 5) = A^{\rm YM}(12345)F_{23} + A^{\rm YM}(13245)F_{32}$$
(3.40)

with

$$F_{23} = \int_0^1 dz_3 \int_0^{z_3} dz_2 \, \frac{s_{12} s_{34}}{z_{12} z_{34}} \, \mathcal{I}_5, \quad F_{32} = \int_0^1 dz_3 \int_0^{z_3} dz_2 \, \frac{s_{13} s_{24}}{z_{13} z_{24}} \, \mathcal{I}_5 \tag{3.41}$$

is not well suited to obtain the residues with respect to s_{45} , since $z_5 \to \infty$ and the corresponding factor of $z_{54}^{s_{45}}$ is absent in the Koba-Nielsen factor $\mathcal{I}_5 = z_{21}^{s_{12}} z_{31}^{s_{13}} z_{32}^{s_{24}} z_{43}^{s_{34}}$. However, one can exploit the cyclicity of \mathcal{A}_5 to compute the residue as $s_{12} = -1$ and then apply three cyclic rotations in succession $s_{12} \to s_{23} \to s_{34} \to s_{45}$ to obtain the residue as $s_{45} = -1$. The calculations done in [38] show that

$$\operatorname{Res}_{s_{12}=-1}(\mathcal{A}_5) = s_{34}B(s_{13}+s_{23},s_{34}) \left(A^{\mathrm{YM}}(12345) \left(s_{23} - \frac{s_{24}(s_{13}+s_{23})}{s_{35}}\right) - A^{\mathrm{YM}}(13245) \frac{s_{13}s_{24}}{s_{35}} \right)$$
(3.42)

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Beta function. After noticing that

$$-\frac{1}{s_{12}}\langle C_{5|43,2,1}\rangle = A^{\rm YM}(12345)\left(s_{23} - \frac{s_{24}(s_{13} + s_{23})}{s_{35}}\right) - A^{\rm YM}(13245)\frac{s_{13}s_{24}}{s_{35}},\qquad(3.43)$$

which can be shown using the algorithm of [28] to rewrite

$$\langle C_{5|43,2,1} \rangle = A^{\rm YM}(52134)s_{12}s_{13} - A^{\rm YM}(52143)s_{12}s_{14}$$
 (3.44)

and expressing the result in the basis of $A^{\text{YM}}(12345)$ and $A^{\text{YM}}(13245)$, one gets

$$\operatorname{Res}_{s_{12}=-1}(\mathcal{A}_5) = s_{34}B(s_{13} + s_{23}, s_{34}) \langle C_{5|43,2,1} \rangle, \qquad s_{12} = -1.$$
(3.45)

The cyclic rotations of (3.45) give rise to

$$\operatorname{Res}_{s_{45}=-1} (\mathcal{A}_5) = s_{12} B(s_{14} + s_{15}, s_{12}) \langle C_{3|21,5,4} \rangle, \qquad s_{45} = -1$$
$$= -s_{12} B(-s_{12} - s_{13}, s_{12}) \langle C_{1|23,4,5} \rangle, \qquad s_{45} = -1, \qquad (3.46)$$

where in the last line we used the canonicalization identity [25] $\langle C_{3|21,5,4} \rangle = -\langle C_{1|23,4,5} \rangle$ and momentum conservation. This is compatible with the factorization

$$\operatorname{Res}_{s_{45}=-1}(\mathcal{A}_5) = \sum_{x} \mathcal{A}(1,2,3|\underline{x}) \mathcal{A}(4,5|\underline{x}), \qquad (3.47)$$

where \sum_{x} denotes a sum over the massive polarizations at the first mass level. To see this, note that the three- and four-point string amplitudes with one massive state x are [15] $(1 + s_{23} = -s_{12} - s_{23})$

$$\mathcal{A}(1,2,3|\underline{x}) = s_{12}B(-s_{12} - s_{13}, s_{12})A(1,2,3|\underline{x}), \qquad (3.48)$$
$$\mathcal{A}(4,5|\underline{x}) = A(4,5|\underline{x}),$$

where $A(P|\underline{x})$ is given by (3.27) (see also [39,40]). Compatibility of (3.46), (3.47) and (3.48) requires that

$$\sum_{x} A(1,2,3|\underline{x})A(\underline{x}|4,5) = -\langle C_{1|23,4,5} \rangle, \quad s_{45} = -1, \qquad (3.49)$$

which can be explicitly checked using (3.52) below.

3.4.3. Sum over intermediate massive polarizations

The justification given above for the first two lines of (3.30) was obtained by computing the first massive residue of the massless amplitudes. A more direct derivation follows from the interpretation that the left-hand sides of (3.30) are given by an sum over intermediate massive polarizations

$$A(1,2,\ldots,n-1|\underline{n})\big|_{\underline{n}\to n,n+1} = \sum_{\underline{x}} A(1,2,\ldots,n-1|\underline{x})A(n,n+1|\underline{x}).$$
(3.50)

To see this, note that the massless representation rules of (3.23) encapsulated in (3.25) follow from the factorization relations

$$\sum_{x} g_{mn}(k) g_{pq}(-k) \phi_{12}^{pq} = g_{mn}(1,2), \quad \sum_{x} b_{mnp}(k) b_{qrs}(-k) \phi_{12}^{qrs} = b_{mnp}(1,2), \quad (3.51)$$

where $\phi_P^{pq...}$ was defined in (3.28), and the momenta is $k = k_1 + k_2$. Moreover, the sum over the massive states x is performed by the completeness relations [41],

$$\sum_{x} g_{mn}(k)g_{pq}(-k) = \frac{1}{64} \Big((k_m k_p + 2\eta_{mp})(k_n k_q + 2\eta_{nq}) - \frac{1}{64} \Big((k_m k_p + 2\eta_{mp})(k_n k_q + 2\eta_{nq}) - \frac{1}{9} (k_m k_n + 2\eta_{mn})(k_p k_q + 2\eta_{pq}) + (m \leftrightarrow n) \Big) \\ \sum_{x} b_{mnp}(k)b_{qrs}(-k) = \frac{1}{256} (k_m k_q + 2\eta_{mq})(k_n k_r + 2\eta_{nr})(k_p k_s + 2\eta_{ps}) + [mnp] ,$$
(3.52)

where [mnp] instructs to antisymmetrize over the indices mnp and we set $\alpha' = \frac{1}{2}$.

With this interpretation, the relations (3.30) can be written as

$$\sum_{x} A(1,2|\underline{x})A(3,4|\underline{x}) = -\langle C_{1|2,3,4} \rangle, \quad 2\alpha'(k_3 \cdot k_4) = -1, \qquad (3.53)$$
$$\sum_{x} A(1,2,3|\underline{x})A(4,5|\underline{x}) = -\langle C_{1|23,4,5} \rangle, \quad 2\alpha'(k_4 \cdot k_5) = -1, \\\sum_{x} A(1,2,3,4|\underline{x})A(5,6|\underline{x}) = -\langle C_{1|234,5,6} \rangle, \quad 2\alpha'(k_5 \cdot k_6) = -1,$$

and have been explicitly verified. They suggest the generalization

$$\sum_{x} A(1, P|\underline{x}) A(n, n+1|\underline{x}) = -\langle C_{1|P,n,n+1} \rangle, \quad 2\alpha'(k_n \cdot k_{n+1}) = -1, \quad (3.54)$$

as the equivalent statement to (3.31). However, note the interpretation difference in how unitarity is actually implemented to arrive at the equivalent results (3.54) and (3.31).

4. Conclusions

In this paper we found an explicit realization of the massive superfields describing the open string states at the first mass level in terms of massless SYM fields. This was achieved through the calculation of OPEs between massless vertices, giving rise to a massless representation in the so-called OPE gauge (2.15)-(2.18). Additional manipulations were used to fix the gauge invariance of the unintegrated vertex operator due to BRST-exact pieces, with the end result being the massless representation in the Berkovits-Chandia gauge (2.75) and (2.76).

After simplifying the three-point amplitude of two massless and one massive state obtained in [12] to a single pure spinor superspace expression,

$$A(1,2|\underline{3}) = \frac{i}{2\alpha'} \langle V_1(\lambda \gamma_m W_2)(\lambda H_3^m) \rangle , \qquad (4.1)$$

the massless representation of the massive superfield H^m_{α} was then used through the superspace substitution (2.80). The resulting expression (3.16) related, at the superspace level, the massive amplitude (4.1) to the ${\alpha'}^2$ correction of the massless four-point open string amplitude as captured by the scalar BRST invariants [25]. The generalization of this relation was proposed as

$$A(1,P|\underline{n})\big|_{\underline{n}\to n,n+1} = -\langle C_{1|P,n,n+1}\rangle, \qquad 2\alpha' k_n \cdot k_{n+1} = -1, \qquad (4.2)$$

where the restriction $\underline{n} \to n, n+1$ on the left-hand side was defined in (3.25) from superfield considerations and later justified via factorization in (3.51). The proposal (4.2) was then explicitly checked in terms of polarizations and momenta up to $\underline{n} = 6$ and shown to be compatible with unitarity via the residue of the massless amplitudes at their first massive pole. The translation from the right-hand side of (4.2) to linear combinations of α'^2 massless amplitudes A^{F^4} (defined in [27]) follows from the descent algebra algorithm described in [22].

It would be interesting to invert the relation (4.2) to find the pure spinor superspace expression of the partial massive amplitudes (3.27). In addition, it may be possible to turn the observations in this paper into a constructive algorithm to compute massive amplitudes starting from the known expressions of the α'^2 massless open string amplitudes.

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Appendix A. OPEs of non-free fields

As explained in [42], a free field is defined as a field whose OPE with itself or its derivatives contain a single *constant* term. In the pure spinor formalism, the fields are not in general free as can be seen from the OPE of $d_{\alpha}(z)d_{\beta}(w)$ or $N^{mn}(z)N^{pq}(w)$. In this case, the definition of normal ordering of operators and the calculation of OPEs with normal ordered operators is done following a generalization of the conventional Wick theorem rules, see e.g. [43,42]. Let us briefly review the calculation of OPEs involving composite operators following the exposition of [44,45] (see also [46,47]).

A.1. Operator product expansion of composite operators

The OPE of A and B is defined as (N is a finite positive integer)

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{[AB]_n(w)}{(z-w)^n}$$
(A.1)

and the normal-ordered product of A and B, denoted (AB)(w), is given by

$$(AB)(w) = \oint \frac{dz}{z - w} A(z)B(w) = [AB]_0(w).$$
 (A.2)

Generalized Wick theorem. The calculation of nested OPEs of non-free fields can be done entirely at the level of the OPE brackets introduced above. The underlying techniques follow from the Borcherds identity

$$\sum_{j=0}^{\infty} \binom{p}{j} [[AB]_{r+j+1}C]_{p+q+1-j} =$$

$$\sum_{j=0}^{\infty} (-1)^{j} \binom{r}{j} \Big([A[BC]_{q+1+j}]_{p+r+1-j} - (-1)^{r+ab} [B[AC]_{p+1+j}]_{q+r+1-j} \Big), \quad p,q,r \in \mathbb{Z}$$
(A.3)

which plays a major role in vertex operator algebra [48]. In the above, a, b denote the Grassman parities of A and B, respectively. The two special cases of the Borcherds identity that are frequently used follow from (p+1=m, q+1=n, r=0) and (p=0, q+1=n, r+1=m); they give rise to identities for $[A[BC]_n]_m$ and $[[AB]_mC]_n$ [44,45]:

$$[A[BC]_n]_m = (-1)^{ab} [B[AC]_m]_n + \sum_{j=0}^{m-1} \binom{m-1}{j} [[AB]_{m-j}C]_{n+j}, \quad m \ge 1$$
(A.4)

$$[[AB]_mC]_n = \sum_{j=0}^{\infty} (-1)^j \binom{m-1}{j} \left([A[BC]_{n+j}]_{m-j} + (-1)^{m+ab} [B[AC]_{j+1}]_{m+n-j-1} \right)$$
(A.5)

where we used $\binom{m-1}{j} = \binom{m-1}{m-1-j}$ and relabeled $m-1-j \to j$ in (A.4). In particular, when the composite operators are normal ordered we get⁷ [43,47]

$$[A[BC]_0]_n = (-1)^{ab} [B[AC]_n]_0 + [[AB]_n C]_0 + \sum_{i=1}^{n-1} \binom{n-1}{i} [[AB]_{n-i} C]_i$$
(A.6)

$$= (-1)^{ab} \Big([B[AC]_n]_0 + \sum_{j=0}^{\infty} \frac{(-1)^{j+n}}{j!} [\partial^j [BA]_{j+n} C]_0 + \sum_{i=1}^{n-1} (-1)^i [[BA]_i C]_{n-i} \Big),$$

$$B_{i-1}^{\infty} \Big([A[BC]_n]_0 + (-1)^{ab} [B[AC]_n]_0 - (-1)^{ab} [BA]_{j+n} C]_0 + \sum_{i=1}^{n-1} (-1)^i [[BA]_i C]_{n-i} \Big),$$

$$(A.7)$$

$$[[AB]_{0}C]_{n} = \sum_{j=0}^{n-1} \left([A[BC]_{n+j}]_{-j} + (-1)^{ab} [B[AC]_{j+1}]_{n-j-1} \right)$$
(A.7)

$$= (-1)^{ab} \sum_{i=1}^{n-1} [B[AC]_{n-i}]_i + \sum_{j=0}^{n-1} \frac{1}{j!} \Big([\partial^j A[BC]_{n+j}]_0 + (-1)^{ab} [\partial^j B[AC]_{n+j}]_0 \Big)$$

Repeated application of these rules allow the computation of OPE brackets with arbitrary nesting. Some useful relations obeyed by the brackets are

$$[A[BC]_0]_0 = (-1)^{ab} [B[AC]_0]_0 + \sum_{i=1}^{\infty} (-1)^{1+i} \frac{1}{i!} [\partial^i [AB]_i C]_0$$
(A.8)

$$[[AB]_0C]_0 = [A[BC]_0]_0 + \sum_{i=1}^{\infty} \frac{1}{i!} \left([\partial^i A[BC]_i]_0 + (-1)^{ab} [\partial^i B[AC]_i]_0 \right)$$
(A.9)

and (with n a non-negative integer):

$$[BA]_n = (-1)^{n+ab} \left([AB]_n + \sum_{i=1}^{\infty} (-1)^i \frac{1}{i!} \partial^i [AB]_{n+i} \right)$$
(A.10)

$$[\partial AB]_n = (1-n)[AB]_{n-1}$$
(A.11)

$$[A\partial B]_{n} = \partial [AB]_{n} + (n-1)[AB]_{n-1}$$
(A.12)

$$[AB]_{-n} = \frac{1}{n!} [\partial^n AB]_0 \tag{A.13}$$

$$[AB]_{n-i} = \frac{(-1)^i}{\binom{n-1}{i}} \frac{1}{i!} [\partial^i AB]_n \tag{A.14}$$

Note that $\partial [AB]_n = [\partial AB]_n + [A\partial B]_n$. In addition, $[A,]_1$ is a graded derivation over all other brackets. This means that

$$[A[BC]_n]_1 = [[AB]_1C]_n + (-1)^{ab} [B[AC]_1]_n .$$
(A.15)

Furthermore, if the conformal weights of A and B are h_A and h_B then $[AB]_n$ has conformal weight $h_A + h_B - n$, i.e., the bracket $[]_n$ has conformal weight -n.

⁷ Note
$$\binom{n}{m} = (-1)^m \binom{-n+m-1}{m}$$
 when $n < 0$ implies $\binom{-1}{j} = (-1)^j$.

A.1.1. OPEs of superfields in a plane-wave basis

A superfield $K_i \in [A_\alpha, A_m, W^\alpha, F^{mn}]$ in a plane wave basis is expanded as

$$K_i(z) = K_i(\theta, X) = K_i(\theta(z))e^{ik_i \cdot X(z)}, \qquad (A.16)$$

for example $A_m^1(z) = A_m^1(\theta(z))e^{ik_1 \cdot X(z)}$. We are interested in the OPE of two such superfields. The definition of the OPE given in (A.1) needs to be generalized when the operators involve plane-wave factors $e^{ik \cdot X}$ as the behavior

$$:e^{ik_1X(z)}::=e^{ik_2X(w)}:=(z-w)^{2\alpha'k_1\cdot k_2}\left[1+(z-w)ik_1\cdot\partial X(z)+\mathcal{O}((z-w)^2)\right]:e^{ik_3\cdot X(w)}:$$

where $k_3 = k_1 + k_2$ is not of the form (A.1) unless $2\alpha' k_1 \cdot k_2$ is an integer. However, when $2\alpha' k_1 \cdot k_2 = -1$ the OPE can be written as

$$K_1(z)K_2(w) = \sum_{n=-\infty}^{N} \frac{[K_1 K_2]_n}{(z-w)^n}, \qquad 2\alpha' k_1 \cdot k_2 = -1$$
(A.17)

with

$$[K_1 K_2]_0(w) = \partial K_1(w) K_2(w) , \qquad (A.18)$$
$$[K_1 K_2]_1(w) = K_1(w) K_2(w) , \qquad [K_1 K_2]_{n \ge 2}(w) = 0 .$$

To see this, note that there is no worldsheet singularity between the factors $K_i(\theta)$; the OPE singularity comes entirely from the plane waves using $2\alpha' k_1 \cdot k_2 = -1$,

$$:e^{ik_1X(z)}:::e^{ik_2X(w)}:=\frac{e^{ik_1\cdot X(w)}e^{ik_2\cdot X(w)}}{(z-w)}\left[1+(z-w)ik_1\cdot\partial X(z)+\mathcal{O}((z-w)^2)\right] \quad (A.19)$$

The factors $K_i(\theta)$ contribute via the Taylor expansion

$$K_{1}(\theta(z))K_{2}(\theta(w)) = K_{1}(\theta(w))K_{2}(\theta(w)) + (z-w)\partial\theta^{\alpha}\partial_{\alpha}K_{1}(\theta(w))K_{2}(\theta(w)) + \mathcal{O}((z-w)^{2}).$$
(A.20)

From (A.19) and (A.20) it follows that

$$[K_1K_2]_0(w) = (ik_1 \cdot \partial X K_1(\theta(w)) + \partial \theta^{\alpha} \partial_{\alpha} K_1(\theta(w))) e^{ik_1 \cdot X(w)} K_2(\theta(w)) e^{ik_2 \cdot X(w)}$$
$$= \partial K_1(w) K_2(w) .$$
(A.21)

Similarly, $[K_1K_2]_1(w) = K_1(w)K_2(w)$ and $[K_1K_2]_{n \ge 2} = 0$.

A.2. OPEs in the pure spinor formalism

Using conventions for the open string, some of the basic OPEs of the pure spinor formalism used in this work are listed below (for brevity, the dependence on w is omitted on the right-hand side):

$$\begin{aligned} \partial \theta^{\alpha}(z) \Big\{ \partial \theta^{\beta}(w), \Pi^{m}(w), N^{mn}(w) \Big\} &\sim \text{regular}, \qquad d_{\alpha}(z) \partial \theta^{\beta}(w) \to \frac{\delta_{\alpha}^{\beta}}{(z-w)^{2}}, \qquad (A.22) \\ d_{\alpha}(z) K(w) \to \frac{D_{\alpha} K}{z-w}, \quad \Pi^{m}(z) K(w) \to -2\alpha' \frac{\partial^{m} K}{z-w}, \qquad d_{\alpha}(z) \Pi^{m}(w) \to \frac{(\gamma^{m} \partial \theta)_{\alpha}}{z-w} \\ d_{\alpha}(z) d_{\beta}(w) \to -\frac{1}{2\alpha'} \frac{\gamma_{\alpha\beta}^{m} \Pi_{m}}{z-w}, \quad \Pi^{m}(z) \Pi^{n}(w) \to -2\alpha' \frac{\eta^{mn}}{(z-w)^{2}}, \qquad d_{\alpha}(z) \theta^{\beta}(w) \to \frac{\delta_{\alpha}^{\beta}}{z-w} \\ J(z) J(w) \to -\frac{4}{(z-w)^{2}}, \quad J(z) \lambda^{\alpha}(w) \to \frac{\lambda^{\alpha}}{z-w}, \qquad N^{mn}(z) \lambda^{\alpha}(w) \to \frac{1}{2} \frac{(\gamma^{mn} \lambda)^{\alpha}}{z-w} \\ N^{mn}(z) N^{pq}(w) \to \frac{\delta^{p[m} N^{n]q} - \delta^{q[m} N^{n]p}}{z-w} - 3 \frac{\delta^{m[q} \delta^{p]n}}{(z-w)^{2}} \end{aligned}$$

where K(w) is a generic 10D superfield that does not depend on derivatives $\partial^k X^m$ and $\partial^k \theta^\alpha$ with $k \ge 1$.

A.3. Rearranging normal ordered brackets

The direct evaluation of the bracket $I_4 \equiv \alpha' [[N^{mn}F_1^{mn}]_0 [\lambda^{\beta}A_{\beta}^2]_0]_1$ using the rules in (A.5) and (A.4) gives

$$I_4 = \alpha' [\lambda^{\beta} [N^{mn} (F_1^{mn} A_{\beta}^2)]_0]_0 + \frac{\alpha'}{2} (\gamma^{mn})^{\beta} {}_{\gamma} [[F_1^{mn} \lambda^{\gamma}]_0 A_{\beta}^2]_0, \qquad (A.23)$$

which is not the result displayed in (the last line of) (2.13). We need to do further processing to obtain the last line in (2.13).

Notice that in the second term the SYM superfields F_1^{mn} and A_{β}^2 appear in different normal ordered brackets. Therefore an expression for a massive superfield cannot be identified as the singularity between F_1^{mn} and A_{β}^2 has not been taken into account. However, using the identity (A.9) followed by (A.8), the normal ordered bracket from (A.18) builds up and we get

$$\begin{split} [[F_1^{mn}\lambda^{\gamma}]_0 A_{\beta}^2]_0 &= [F_1^{mn}[\lambda^{\gamma}A_{\beta}^2]_0]_0 + [\partial\lambda^{\gamma}[F_1^{mn}A_{\beta}^2]_1]_0 \\ &= [\lambda^{\gamma}[F_1^{mn}A_{\beta}^2]_0]_0 + [\partial\lambda^{\gamma}(F_1^{mn}A_{\beta}^2)]_0 \\ &= [\lambda^{\gamma}(\partial F_1^{mn}A_{\beta}^2)]_0 + [\partial\lambda^{\gamma}(F_1^{mn}A_{\beta}^2)]_0 \,, \end{split}$$
(A.24)

and the result is proportional to the massive plane wave $e^{ik_3 \cdot X}$. Similarly, the first term in (A.23) can be rewritten using (A.8) and $[\lambda^{\beta} N^{mn}]_1 = -\frac{1}{2} (\gamma^{mn})^{\beta} {}_{\gamma} \lambda^{\gamma}$ as follows

$$\alpha'[\lambda^{\beta}[N^{mn}(F_1^{mn}A_{\beta}^2)]_0]_0 = \alpha'[N^{mn}[\lambda^{\beta}(F_1^{mn}A_{\beta}^2)]_0]_0 - \frac{\alpha'}{2}(\gamma^{mn})^{\beta}{}_{\gamma}[\partial\lambda^{\gamma}(F_1^{mn}A_{\beta}^2)]_0.$$
(A.25)

This leads to

$$I_{4} = \alpha' [\lambda^{\beta} [N^{mn} (F_{1}^{mn} A_{\beta}^{2})]_{0}]_{0} + \frac{\alpha'}{2} (\gamma^{mn})^{\beta} {}_{\gamma} \Big([\lambda^{\gamma} (\partial F_{1}^{mn} A_{\beta}^{2})]_{0} + [\partial \lambda^{\gamma} (F_{1}^{mn} A_{\beta}^{2})]_{0} \Big)$$

= $\alpha' [N^{mn} [\lambda^{\beta} (F_{1}^{mn} A_{\beta}^{2})]_{0}]_{0} + \frac{\alpha'}{2} (\gamma^{mn})^{\beta} {}_{\gamma} [\lambda^{\gamma} (\partial F_{1}^{mn} A_{\beta}^{2})]_{0} , \qquad (A.26)$

which is the result we used in the last line of (2.13).

A.3.1. Normal ordering identity

Using the pure spinor OPEs

$$[N^{mn}\lambda^{\alpha}]_{1} = \frac{1}{2}(\gamma^{mn}\lambda)^{\alpha}, \qquad [N^{mn}\lambda^{\alpha}]_{n\geq 2} = 0,$$

$$[\lambda^{\alpha}J]_{1} = -\lambda^{\alpha} \qquad \qquad [\lambda^{\alpha}J]_{n\geq 2} = 0,$$
 (A.27)

and the identities from the appendix A we can show a normal-ordering identity given in [11]

$$[N^{mn}[\lambda^{\alpha}\lambda^{\beta}]_{0}]_{0}\gamma^{m}_{\beta\gamma} = \frac{1}{2}[J[\lambda^{\alpha}\lambda^{\beta}]_{0}]_{0}\gamma^{n}_{\beta\gamma} + \frac{5}{2}\lambda^{\alpha}(\gamma^{n}\partial\lambda)_{\gamma} + \frac{1}{2}(\lambda\gamma^{mn})^{\alpha}(\gamma^{m}\partial\lambda)_{\gamma}$$
(A.28)

and used in the proof of QV = 0 in section 2. To see this note that (A.8) implies

$$[N^{mn}[\lambda^{\alpha}\lambda^{\beta}]_{0}]_{0}\gamma^{m}_{\beta\gamma} = [\lambda^{\alpha}[N^{mn}\lambda^{\beta}]_{0}]_{0}\gamma^{m}_{\beta\gamma} + [\partial[N^{mn}\lambda^{\alpha}]_{1}\lambda^{\beta}]_{0}\gamma^{m}_{\beta\gamma}$$

$$= \frac{1}{2}[\lambda^{\alpha}[J\lambda^{\beta}]_{0}]_{0}\gamma^{n}_{\beta\gamma} + 2\lambda^{\alpha}(\gamma^{n}\partial\lambda)_{\gamma} + \frac{1}{2}(\gamma^{mn}\partial\lambda)^{\alpha}(\lambda\gamma^{m})_{\gamma}$$

$$= \frac{1}{2}[J[\lambda^{\alpha}\lambda^{\beta}]_{0}]_{0}\gamma^{n}_{\beta\gamma} - \frac{1}{2}\partial\lambda^{\alpha}(\gamma^{n}\lambda)_{\gamma} + 2\lambda^{\alpha}(\gamma^{n}\partial\lambda)_{\gamma} + \frac{1}{2}(\gamma^{mn}\partial\lambda)^{\alpha}(\lambda\gamma^{m})_{\gamma}$$

$$= \frac{1}{2}[J[\lambda^{\alpha}\lambda^{\beta}]_{0}]_{0}\gamma^{n}_{\beta\gamma} + \frac{5}{2}\lambda^{\alpha}(\gamma^{n}\partial\lambda)_{\gamma} + \frac{1}{2}(\lambda\gamma^{mn})^{\alpha}(\gamma^{m}\partial\lambda)_{\gamma} \qquad (A.29)$$

where we used [11] (to show it, apply $[J, -]_2$ to both sides)

$$[N^{mn}\lambda^{\beta}]_{0}\gamma^{m}_{\beta\gamma} = \frac{1}{2}[J\lambda^{\beta}]_{0}\gamma^{n}_{\beta\gamma} + 2(\gamma^{n}\partial\lambda)_{\gamma}$$
(A.30)

to arrive at the second line while (A.8) has been used to arrive at the third line with $[\lambda^{\alpha}[J\lambda^{\beta}]_{0}]_{0} = [J[\lambda^{\alpha}\lambda^{\beta}]_{0}]_{0} - \partial\lambda^{\alpha}\lambda^{\beta}$. Finally, $(\partial\lambda\gamma^{m})_{\alpha}(\lambda\gamma^{m})_{\beta} + (\partial\lambda\gamma^{m})_{\beta}(\lambda\gamma^{m})_{\alpha} = 0$ leads to the fourth line and the identity (A.28) is demonstrated.

Appendix B. Equations of motion of massive superfields

From massless SYM in the OPE gauge. We will check that the equations of motion for the first-level massive superfields in our representation given in (2.20) to (2.23) are implied by the linearized SYM superfield equations of motion (2.2).

The massive equations of motion in a BRST language involve the combinations $(\lambda B)_{\alpha}$, (λH_m) , $(C\lambda)^{\alpha}$ and $(\lambda F)_{mn}$ defined in (2.19). Therefore it will be convenient to list these superfields after contracting the definitions (2.15) to (2.18) with the pure spinor λ :

$$\begin{aligned} (\lambda B)_{\alpha} &= -2i\alpha' k_m^2 (\gamma^m W_1)_{\alpha} V_2 - i\alpha' k_m^1 (\gamma_n W_1)_{\alpha} (\lambda \gamma^{mn} A_2) - \frac{\alpha'}{2} F_{mn}^1 (\gamma^{mn} D)_{\alpha} V_2 \\ &= -2i\alpha' k_m^2 (\gamma^m W_1)_{\alpha} V_2 + i\alpha' k_1^m (\gamma^n W_1)_{\alpha} (\lambda \gamma^n \gamma^m A_2) + i\alpha' k_1^m (\lambda \gamma^n W_1) (\gamma^n \gamma^m A_2)_{\alpha} \\ &+ \frac{\alpha'}{2} F_1^{mn} (\lambda \gamma^p \gamma^{mn})_{\alpha} A_2^p + \frac{\alpha'}{2} Q \Big(F_{mn}^1 (\gamma^{mn} A_2)_{\alpha} \Big) \end{aligned}$$
(B.1)

$$\begin{aligned} (\lambda H)_m &= A_m^1 V_2 + 2\alpha' k_m^1 (k^2 \cdot A^1) V_2 - 2i\alpha' k_m^1 W_1^\beta D_\beta V_2 - \frac{i\alpha'}{2} k_m^1 F_{np}^1 (\lambda \gamma^{np} A_2) \\ &= A_1^m V_2 + 2\alpha' k_m^1 (k^2 \cdot A^1) V_2 - 2i\alpha' k_m^1 (\lambda \gamma^n W_1) A_n^2 - 2i\alpha' k_m^1 Q(W_1 A_2) , \\ &= -2i\alpha' \Big(k_2^n F_1^{mn} V_2 + k_1^m (\lambda \gamma^n W_1) A_2^n + k_1^m Q(W_1 A_2) \Big) , \end{aligned}$$
(B.2)

$$(C\lambda)^{\alpha} = W_1^{\alpha} V_2 \tag{B.3}$$

$$(\lambda F)_{mn} = F_1^{mn} V_2 \tag{B.4}$$

In order to derive the above representations one uses the gamma matrix identity $\gamma^m_{\alpha(\beta}\gamma^m_{\gamma\delta)} = 0$, the Dirac equation, the linearized SYM equations of motion (2.2) as well as the pure spinor constraint. In particular,

$$-F_1^{mn}(\lambda\gamma^{mn}D)V_2 = Q(F_1^{mn}(\lambda\gamma^{mn}A_2)).$$
(B.5)

In addition, the first massive state condition $-2\alpha'(k_1 \cdot k_2) = 1$ implies that

$$-2i\alpha' k_2^n F_1^{mn} V_2 = A_1^m V_2 + 2\alpha' (k_2 \cdot A_1) k_1^m V_2$$
(B.6)

as easily seen after expanding the linearized field-strength $F_1^{mn} = ik_1^m A_1^n - ik_1^n A_1^n$.

A straightforward calculation using the usual set of identities leads to

$$Q(\lambda B)_{\alpha} = (\lambda \gamma^{m})_{\alpha} \Big[-2i\alpha' k_{2}^{m} F_{1}^{mn} V_{2} - 2i\alpha' k_{1}^{m} (\lambda \gamma^{n} W_{1}) A_{2}^{n} - i\alpha' k_{1}^{n} Q(W_{1} \gamma^{m} \gamma^{n} A_{2}) \Big]$$

$$= (\lambda \gamma^{m})_{\alpha} (\lambda H_{m}) + (\lambda \gamma^{m})_{\alpha} \Big[2i\alpha' k_{1}^{m} Q(W_{1} A_{2}) - i\alpha' k_{1}^{n} Q(W_{1} \gamma^{m} \gamma^{n} A_{2}) \Big]$$

$$= (\lambda \gamma^{m})_{\alpha} (\lambda H_{m}) .$$
(B.7)

To arrive at the last line, note that the two BRST-exact terms vanish after using $\gamma^m \gamma^n = -\gamma^n \gamma^m + 2\eta^{nm}$ and the Dirac equation.

Now, taking into account that $\lambda \gamma^m W$ is BRST closed and using the SYM equations of motion (2.2), the rewritten expression (B.2) leads to

$$Q(\lambda H)_m = -2\alpha'(k_1 \cdot k_2)(\lambda \gamma^m W_1)V_2 = (\lambda \gamma^m C\lambda)$$
(B.8)

where we used the first massive state condition $-2\alpha'(k_1 \cdot k_2) = 1$ from (2.9) and the definition (B.3). This proves the equation of motion (2.21).

The equation of motions (2.22) and (2.23) follow immediately from the linearized equations (2.2) and the definitions (B.3) and (B.4).

B.1. $(\lambda B \lambda)$ is BRST exact

It is easy to see from (B.7) and the pure spinor constraint that $(\lambda B \lambda)$ is BRST closed. We will now show that it is also BRST exact.

From (B.1) and the identities $(\lambda \gamma^m)_{\alpha} (\lambda \gamma^m)_{\beta} = 0$ and $(\lambda \gamma^{mnp} \lambda) = 0$ it follows that

$$(\lambda B\lambda) = -2i\alpha' k_m^2 (\lambda \gamma^m W_1) V_2 + \frac{\alpha'}{2} Q \left(F_{mn}^1 (\lambda \gamma^{mn} A_2) \right)$$
(B.9)
= $-2\alpha' Q \left(i(k_1 \cdot A_2) V_1 + A_1^m (\lambda \gamma^m W_2) + i(k_2 \cdot A_1) V_2 - \frac{1}{4} F_{mn}^1 (\lambda \gamma^{mn} A_2) \right),$

where we used the linearized equations (2.2). Therefore $\lambda^{\alpha}\lambda^{\beta}(B_{\alpha\beta}-D_{\alpha}\Lambda_{\beta})=0$ with

$$\Lambda_{\beta} = -2\alpha' \Big(i(k_1 \cdot A_2) A_{\beta}^1 + A_1^m (\gamma^m W_2)_{\beta} + i(k_2 \cdot A_1) A_{\beta}^2 - \frac{1}{4} F_{mn}^1 (\gamma^{mn} A_2)_{\beta} \Big) \,. \tag{B.10}$$

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