Towards massive field-theory amplitudes from the cohomology of pure spinor superspace

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By analogy with the formula for the massless string disk amplitudes, we define massive field-theory tree amplitudes and conjecture that the BRST cohomology structure of pure spinor superspace fixes their form. We give evidence by deriving the pure spinor superspace expression of the massive field-theory n-point tree amplitude with one first-level massive and $n-1$ massless states in two ways: 1) from BRST cohomology arguments in pure spinor superspace and 2) from the α'^2 correction to the massless string amplitudes by inverting the unitarity constraint in superspace.

August 2024

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1. Statement of the problem

In a recent paper [1], the open superstring disk amplitude of $n{-}1$ massless states and one massive state $\underline{n}$ was decomposed in a similar fashion as the massless amplitudes of [2]. Namely, the full string amplitude is written as products of $(n{-}3)!$ worldsheet integrals F_Q^P independent of polarizations and partial subamplitudes $A(1,P,n{-}1|\underline{n})$:

$$\mathcal{A}(1,Q,n{-}1,\underline{n}) = \sum_{P\in S_{n-3}} F_Q^P A(1,P,n{-}1|\underline{n})\,. \tag{1.1}$$

The words P and Q encode the labels of the strings being scattered while the integrals F_Q^P have the same functional form as in the massless string scattering amplitude [2,3,4,5].

Since the amplitudes $A(1,P|\underline{n})$ play an analogous role as the massless field-theory amplitudes $A^{\rm YM}$ in the massless string disk amplitude counterpart of (1.1) given in [2], they will be called *massive field-theory amplitudes*. The same terminology will be used to expected generalizations of (1.1) with higher number of massive legs and/or higher mass levels [6] and should not be understood as taking the $\alpha'\to 0$ limit.

The focus of this paper will be to derive the pure spinor superspace expression for $A(1,P|\underline{n})$ whose component expansion following [7,8] reproduces the supersymmetric components found in [1]. The pure spinor superspace expression achieving this,

$$A(1,2,\ldots,n{-}1|\underline{n}) = \frac{i}{2\alpha'}\langle C^m_{1|2\ldots n-1}(\lambda H^m_n)\rangle\,, \tag{1.2}$$

will be derived in section 2 in two different ways:

1. Finding a BRST closed expression with the correct kinematic pole structure
2. Inverting the factorization of the massless amplitudes on their first massive pole

The first derivation uses the same BRST cohomology ideas [9] that were successfully used to obtain the pure spinor superspace expression of the massless SYM tree-level amplitudes [10]. The second derivation exploits the relation found in [11] between the massive field-theory amplitudes $A(1,P|\underline{n})$ and the ${\alpha'}^2$ correction of the disk massless amplitudes. Both derivations rely on BRST cohomology manipulations in pure spinor superspace.

Similarly to the massless case in [9], we conjecture that the BRST cohomology structure of pure spinor superspace fixes the field-theory massive amplitudes for higher mass levels and higher number of massive legs. The derivation of (1.2) here is the first step in this quest.

1.1. Preliminaries

For a review of the pure spinor formalism, we refer the reader to [12,13].

Equations of motion. The massless superfields $\big[A_\alpha, A_m, W^\alpha, F^{mn}\big]$ [14] and the massive superfields [15]

$$\lambda^\alpha B_{\alpha\beta} = (\lambda B)_\beta, \quad \lambda^\alpha H^m_\alpha = (\lambda H^m)\,, \quad C^\beta{}_\alpha \lambda^\alpha = (C\lambda)^\beta\,, \quad \lambda^\alpha F_{\alpha mn} = (\lambda F)_{mn}\,, \tag{1.3}$$

satisfy the following equations of motion [14,15,16] (with $A_{[m}B_{n]} = A_m B_n - A_n B_m$)

$$\begin{aligned}
QA_\beta + D_\beta V &= (\gamma^m\lambda)_\beta A_m\,, & QW^\alpha &= \frac{1}{4}(\lambda\gamma^{mn})^\alpha F_{mn},\\
QA_m &= \lambda\gamma^m W + \partial_m V\,, & QF_{mn} &= \partial_m(\lambda\gamma_n W) - \partial_n(\lambda\gamma_m W)\,,\\
Q(\lambda B)_\alpha &= (\lambda\gamma^m)_\alpha(\lambda H)_m\,, & Q(C\lambda)^\alpha &= \frac{1}{4}(\lambda\gamma^{mn})^\alpha(\lambda F)_{mn}\,,\\
Q(\lambda H^m) &= (\lambda\gamma^m C\lambda)\,, & Q(\lambda F)_{mn} &= \frac{1}{2}\partial_{[m}(\lambda\gamma_{n]}C\lambda) - \frac{1}{16}\partial^p(\lambda\gamma_{[m}C\gamma_{n]p}\lambda)\,,
\end{aligned} \tag{1.4}$$

where $Q = \lambda^\alpha D_\alpha$ is the pure spinor BRST operator acting on 10D superfields, and $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$ is the supersymmetric derivative satisfying $\{D_\alpha, D_\beta\} = \gamma^m_{\alpha\beta}\partial_m$. As factors of α' are being kept in all formulas, we find it convenient to list the length dimension of various quantities:

$$\begin{gathered}
[\alpha'] = 2, \quad [\lambda^\alpha] = [\theta^\alpha] = \frac{1}{2}, \quad [\partial_m] = -1, \quad [Q] = 0,\\
[A_\alpha] = \frac{1}{2}\,, \quad [A^m] = 0\,, \quad [W^\alpha] = -\frac{1}{2}\,, \quad [F^{mn}] = -1\,,\\
[B_{\alpha\beta}] = 1, \quad [H_{m\alpha}] = \frac{1}{2}\,, \quad [C^\beta{}_\alpha] = 0\,, \quad [F_{\alpha mn}] = -\frac{1}{2}\,.
\end{gathered} \tag{1.5}$$

Berends-Giele form of the massive field-theory amplitudes. Using the perturbiner approach, the supersymmetric partial amplitudes $A(P|\underline{n})$ with $|P|=n-1$ massless states represented by gluons e^m and gluinos χ^α and one first-level massive state represented by the bosons g^{mn}, b^{mnp} and fermions ψ^m_α (see appendix A) were found to be [1],

$$A(P|\underline{n}) = \phi^{mn}_P g_{\underline{n}\,mn} + \phi^{mnp}_P b_{\underline{n}\,mnp} + \phi^{m\alpha}_P \psi^m_{\underline{n}\alpha}\,, \tag{1.6}$$

where the massless SYM states (e^m, χ^α) are encoded in the deconcatenations

$$\begin{aligned}
\phi^{mn}_P &= \sum_{P=XY} \alpha'\big[\mathfrak{f}^{ma}_X \mathfrak{f}^{na}_Y + (\mathcal{X}_X\gamma^m\mathcal{X}_Y)k^n_Y\big] - \sum_{P=XYZ} 2\alpha'(\mathcal{X}_X\gamma^m\mathcal{X}_Z)\mathfrak{e}^n_Y + \mathrm{cyc}(P)\,, \qquad (1.7)\\
\phi^{mnp}_P &= \sum_{P=XY} \big[\mathfrak{e}^m_X\mathfrak{e}^n_Y k^p_Y - \frac{1}{12}(\mathcal{X}_X\gamma^{mnp}\mathcal{X}_Y)\big] + \sum_{P=XYZ} \frac{2}{3}\mathfrak{e}^m_X\mathfrak{e}^n_Y\mathfrak{e}^p_Z + \mathrm{cyc}(P)\,,\\
\phi^{m\alpha}_P &= \frac{2}{9}\alpha' k^n_P \sum_{P=XY} \mathfrak{f}^{mp}_X(\gamma_n\gamma_p\mathcal{X}_Y)^\alpha + \mathrm{cyc}(P)\,,
\end{aligned}$$

whose coefficients are adapted to the conventions of this paper and differ[1] from [1]. The currents $\mathfrak{e}^m_P$, $\mathfrak{f}^{mn}_P$ and $\mathcal{X}^\alpha_P$ are the Berends-Giele multiparticle polarizations of [17],

$$\begin{aligned}
\mathfrak{e}^m_P &= \frac{1}{k^2_P}\sum_{P=XY}\Big[\mathfrak{e}^m_Y(k_Y\cdot\mathfrak{e}_X) + \mathfrak{f}^{mn}_X\mathfrak{e}^n_Y + (\mathcal{X}_X\gamma^m\mathcal{X}_Y) - (X\leftrightarrow Y)\Big]\,, \qquad (1.8)\\
\mathfrak{f}^{mn}_P &= k^m_P\mathfrak{e}^n_P - k^n_P\mathfrak{e}^m_P - \sum_{XY=P}\big(\mathfrak{e}^m_X\mathfrak{e}^n_Y - \mathfrak{e}^n_X\mathfrak{e}^m_Y\big)\,,\\
\mathcal{X}^\alpha_P &= \frac{1}{k^2_P}\sum_{P=XY} k^n_P\big[\mathfrak{e}^m_X(\gamma_n\gamma_m\mathcal{X}_Y)^\alpha - \mathfrak{e}^m_Y(\gamma_n\gamma_m\mathcal{X}_X)^\alpha\big]\,,
\end{aligned}$$

starting with the single-letter gluon and gluino polarizations $\mathfrak{e}^m_i = e^m_i$, $\mathcal{X}^\alpha_i = \chi^\alpha_i$ and field strength $\mathfrak{f}^{mn}_i = k^m_i e^n_i - k^n_i e^m_i$.

The notation $+\mathrm{cyc}(P)$ instructs to add the cyclic permutations of the letters in P and $XY{=}P$ denotes the deconcatenations of P into non-empty words X and Y. In addition, the momentum k^m_P for a non-empty word $P=iQ$ is defined recursively by $k^m_{iQ} = k^m_i + k^m_Q$ with $k^m_\emptyset = 0$, where *letters* are indicated by lower case and *words* by upper case.

Massless representation of massive superfields. There are two distinct ways in which the massive superfields (labelled by k) appearing in the unintegrated pure spinor vertex operator can be represented [11,16,18] in terms of massless SYM superfields (labelled by i

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[1] I thank Oliver Schlotterer for pointing out the relation $\psi^{m\alpha} = k^n\gamma^{\alpha\beta}_n\psi^m_\beta$ between the Weyl fermions $\psi^{m\alpha}$ of [1] and the anti-Weyl fermions ψ^m_α from [15].

and j): in the *OPE* or *Berkovits-Chandia gauge* [16]. This rewriting is denoted by $\underline{k} \to i, j$. More precisely,

1. $\underline{k} \to i, j$ in the OPE gauge:

$$
\begin{aligned}
(\lambda B)_\alpha &= -2\alpha' \Big(ik_j^m (\gamma_m W_i)_\alpha V_j + ik_i^m (\lambda\gamma_m)_\alpha (W_i A_j) \\
&\qquad - \frac{1}{4} F_i^{mn} (\lambda\gamma^p\gamma^{mn})_\alpha A_j^p - \frac{1}{4} Q\big(F_{mn}^1 (\gamma^{mn} A_j)_\alpha\big)\Big) \\
(\lambda H^m) &= -2i\alpha' \Big(k_j^n F_i^{mn} V_j + k_i^m (\lambda\gamma^n W_i) A_j^n + k_i^m Q(W_i A_j)\Big) , \\
(C\lambda)^\alpha &= W_i^\alpha V_j , \\
(\lambda F)_{mn} &= F_i^{mn} V_j , \qquad\qquad 2\alpha' k_i \cdot k_j = -1 .
\end{aligned}
\tag{1.9}
$$

2. $\underline{k} \to i, j$ in the Berkovits-Chandia gauge:

$$
\begin{aligned}
(\lambda B)_\alpha &= (\gamma^{mnp}\lambda)_\alpha B_{mnp} , \\
(\lambda H_m) &= \frac{3}{7} (\lambda\gamma^{np} D) B_{mnp} , \\
(C\lambda)^\alpha &= \frac{1}{4} ik_q (\gamma^{qmnp}\lambda)^\alpha B_{mnp} , \\
(\lambda F)_{mn} &= \frac{1}{16}\Big(7ik_{[m}(\lambda H_{n]}) + ik_q(\lambda\gamma_{q[m} H_{n]})\Big) ,
\end{aligned}
\tag{1.10}
$$

with $2\alpha' k_i \cdot k_j = -1$ and

$$
B_{mnp} = \frac{1}{18}{\alpha'}^2 \Big[(W_i \gamma_{abmnp} W_j) k_i^a k_j^b + ik_j^q F^i_{q[m} F^j_{np]} + (i \leftrightarrow j)\Big] . \tag{1.11}
$$

It was shown in [11,16] that $\underline{k} \to k, k{+}1$ implies a relation between massive and massless amplitudes given by

$$
A(1, P|\underline{k})\big|_{\underline{k}\to k,k+1} = -\langle C_{1|P,k,k+1}\rangle , \qquad 2\alpha' k_k \cdot k_{k+1} = -1 , \tag{1.12}
$$

where the superfields $C_{1|P,Q,R}$ are the scalar BRST invariants encoding the ${\alpha'}^2$ terms of the massless string disk amplitudes, see (1.18).

Scalar BRST invariants. The superfield expansions of the scalar BRST invariants in terms of Berends-Giele currents follow from the recursion [19]

$$
\begin{aligned}
C_{i|j,k,l} &= M_i M_{j,k,l} , \\
C_{i|P,Q,R} &= M_i M_{P,Q,R} + M_i \otimes \big[C_{p_1|p_2\ldots p_{|P|},Q,R} - C_{p_{|P|}|p_1\ldots p_{|P|-1},Q,R} + (P \leftrightarrow Q, R)\big]
\end{aligned}
\tag{1.13}
$$

where $M_i \otimes M_A := M_{iA}$, M_P is the Berends-Giele current associated to the unintegrated massless vertex operator and

$$M_{A,B,C} \equiv \frac{1}{3}(\lambda\gamma_m \mathcal{W}_A)(\lambda\gamma_n \mathcal{W}_B)\mathcal{F}_C^{mn} + \text{cyc}(A,B,C)\,, \tag{1.14}$$

where $\mathcal{W}_P^\alpha$ and $\mathcal{F}_P^{mn}$ are Berends-Giele currents of the gluino and gluon field strengths, for more details see the review [13].

The first few outputs of the recursion (1.13) are given by

$$\begin{aligned} C_{1|2,3,4} &= M_1 M_{2,3,4}\,, \\ C_{1|23,4,5} &= M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}\,, \\ C_{1|234,5,6} &= M_1 M_{234,5,6} + M_{12} M_{34,5,6} + M_{123} M_{4,5,6} - M_{124} M_{3,5,6} \\ &\quad - M_{14} M_{23,5,6} - M_{142} M_{3,5,6} + M_{143} M_{2,5,6}\,, \end{aligned} \tag{1.15}$$

and can be checked to be BRST closed using [19]

$$\begin{aligned} QM_P &= \sum_{P=XY} M_X M_Y\,, \\ QM_{A,B,C} &= \sum_{XY=A} \big[M_X M_{Y,B,C} - (X \leftrightarrow Y)\big] + (A \leftrightarrow B,C)\,. \end{aligned} \tag{1.16}$$

The relation between the scalar BRST invariants and the ${\alpha'}^2$ correction to the massless disk amplitudes was discovered in [20]: writing the string disk amplitude as

$$A(P) = A^{\text{YM}}(P) + \zeta_2 {\alpha'}^2 A^{F^4}(P) + \mathcal{O}({\alpha'}^3) \tag{1.17}$$

it follows that A^{F^4} can be expanded as

$$A^{F^4}(1,P) = \sum_{XYZ=P} \langle C_{1|X,Y,Z}\rangle, \tag{1.18}$$

while the precise permutations in the inverse relation $\langle C_{1|P,Q,R}\rangle = \sum_S A^{F^4}(S)$ can be found in the algorithm of [21]. Note that these BRST invariants also capture parts of genus-one open-string amplitudes [22,23].

2. Massive field-theory amplitudes in superspace

Cohomology derivation of SYM amplitudes. When all external states are massless, the field-theory SYM amplitudes could be determined using the experimental observation that

> *two BRST-closed expressions with the same mass dimension and featuring the same kinematic poles have proportional component expansions.*

Since BRST closed expressions are gauge invariant and supersymmetric under the application of the pure spinor bracket [7], and the SYM tree amplitudes can be obtained from the $\alpha' \to 0$ limit of the tree-level open superstring amplitudes, SYM tree amplitudes must be represented by a BRST closed expression in pure spinor superspace. Using the observation above, any BRST closed expression of the same mass dimension and with the same kinematic poles must yield the components of the SYM tree amplitudes.

This observation led to the idea that SYM tree amplitudes could be fixed by the cohomology of pure spinor superspace [9], which eventually came into fruition with [10]. We now conjecture that the same idea applies equally well to the determination of *massive* field-theory amplitudes.

In this programme, there is an implicit assumption used to propose a pure spinor superspace expression reproducing the field-theory amplitude: as the starting point one uses superfields which are featured in the string amplitude prescription of [7]. This reasoning led to the development of the multiparticle SYM superfields inspired by OPEs [24], and to the Berends-Giele interpretation [17] of the cohomology method of [10]. We expect similar developments for the massive superfields.

We are now going to showcase these ideas to determine the pure spinor superspace expression for the massive field-theory amplitudes $A(P|\underline{n})$ involving one first-level massive and an arbitrary number of massless states.

2.1. From pure spinor superspace cohomology

We know that a single massive string state does not induce any kinematic poles, therefore we will start with the pure spinor superspace expression for the massive field-theory tree amplitudes with a single massive state, denoted $A(1, P|\underline{n})$. Fortunately, the component expansion of these amplitudes was determined in [1] as reviewed in section 1.1.

To find the pure spinor superspace expression that produces the Berends-Giele recursions of [1], the first step is to understand their kinematic pole structure.

The four-point amplitude $A(1,2,3|\underline{4})$ has the poles $1/s_{12}$, $1/s_{23}$ and $1/s_{13}$ – the same poles present in the scalar BRST invariant $C_{1|23,4,5}$ at multiplicity five [20]. The pattern repeats at higher multiplicities: the poles of $A(1,P|\underline{n})$ are the same as the poles in $C_{1|P,n,n+1}$ due to their dependence on Berends-Giele currents. Note that $A(1,P|\underline{k})$ and $C_{1|P,k,k+1}$ have been recently related in a different context [11,16].

The second step in deriving a pure spinor superspace expression is the proposal of a BRST-closed expression containing the same kinematic poles as outlined above. The disk amplitude computed in [25] between two massless and one first-level massive state was simplified in [16] using BRST cohomology manipulations to

$$A(1,2|\underline{3}) = \frac{i}{2\alpha'}\langle V_1(\lambda\gamma_m W_2)(\lambda H_3^m)\rangle\,. \tag{2.1}$$

The three-point disk amplitude (2.1) turns out to be, under the definition in section 1, proportional to the massive field-theory amplitude $A(1,2|\underline{3})$. Note that the expression in the right-hand side is BRST closed[2], as expected. To see this, one uses the equations of motion (1.4) together with the pure spinor constraint $(\lambda\gamma^m)_\alpha(\lambda\gamma_m)_\beta = 0$. Therefore, the simple pure spinor superspace expression (2.1) yields our starting BRST-closed expression and, by analogy with the massless case reviewed above, we expect the higher multiplicity expressions to closely follow the superspace structure of (2.1). To generalize the three-point expression to a BRST-closed expression of arbitrary multiplicity containing the same kinematic poles as the scalar BRST invariants, it will be convenient to define the following recursion:

Definition. *Pure spinor superfields $C^m_{1|P}$ for any non-empty word P are given by the following recursion*

$$\begin{aligned} C^m_{i|j} &= M_i(\lambda\gamma^m\mathcal{W}_j)\,, \\ C^m_{i|jk} &= M_i(\lambda\gamma^m\mathcal{W}_{jk}) + M_i\otimes\big[C^m_{j|k} - C^m_{k|j}\big] \\ C^m_{i|jPk} &= M_i(\lambda\gamma^m\mathcal{W}_{jPk}) + M_i\otimes\big[C^m_{j|Pk} - C^m_{k|jP}\big] \end{aligned} \tag{2.2}$$

where $M_i\otimes M_P = M_{iP}$.

The first few cases of the recursion (2.2) are given by

$$\begin{aligned} C^m_{1|2} &= M_1(\lambda\gamma^m\mathcal{W}_2)\,, \\ C^m_{1|23} &= M_{12}(\lambda\gamma^m\mathcal{W}_3) + M_1(\lambda\gamma^m\mathcal{W}_{23}) - M_{13}(\lambda\gamma^m\mathcal{W}_2) \\ C^m_{1|234} &= M_1(\lambda\gamma^m\mathcal{W}_{234}) + M_{12}(\lambda\gamma^m\mathcal{W}_{34}) + M_{123}(\lambda\gamma^m\mathcal{W}_4) - M_{124}(\lambda\gamma^m\mathcal{W}_3) \\ &\quad - M_{14}(\lambda\gamma^m\mathcal{W}_{23}) - M_{142}(\lambda\gamma^m\mathcal{W}_3) + M_{143}(\lambda\gamma^m\mathcal{W}_2) \end{aligned} \tag{2.3}$$

[2] By abuse of terminology, $\langle S\rangle$ is said to be BRST closed when $QS = 0$.

with similar expressions at higher multiplicities. Notice the similarity with the corresponding expansions of the scalar BRST invariants in (1.15); in fact $C^m_{1,P}$ can be obtained from those expansions by using the rule $M_{Q,k,k+1} \to (\lambda\gamma^m \mathcal{W}_Q)$. By analogy with the definition of the word recursion in [26], one infers that $C^m_{1|P}$ is annihilated by proper shuffles

$$C^m_{1|R\sqcup\mathchoice{}{}{}{}\sqcup S} = 0, \quad R, S \neq \emptyset, \tag{2.4}$$

where the shuffle product is recursively defined by $iA\sqcup\sqcup jB = i(A\sqcup\sqcup jB) + j(B\sqcup\sqcup iA)$ and $\emptyset\sqcup\sqcup A = A\sqcup\sqcup\emptyset = A$. Moreover, it is easy to see that the recursion (2.2) generates BRST closed superfields containing two pure spinors.

Using the BRST closed expressions $C^m_{1|P}$ given above, the massive amplitude $A(1, P|\underline{k})$ of arbitrary multiplicity is proposed to be

$$A(1, P|\underline{k}) = \frac{i}{2\alpha'}\langle C^m_{1|P}(\lambda H^m_k)\rangle\,. \tag{2.5}$$

Note that the shuffle symmetry (2.4) of $C^m_{1|P}$ implies, via the formula (2.5) that the massive amplitudes $A(1, P|\underline{k})$ satisfy the Kleiss-Kuijf [27] relations, in accordance with [1].

By construction, the right-hand side of (2.5) has the same kinematic poles as the left-hand side. In addition, one can easily show that (2.5) is BRST closed using the equation of motion $Q(\lambda H^m_k) = (\lambda\gamma^m C_k\lambda)$ and the pure spinor constraint $(\lambda\gamma_m)_\alpha(\lambda\gamma^m)_\beta = 0$. Therefore we conclude that (2.5) must yield component expansions in terms of polarizations and momenta proportional to the known components given by (1.6). Indeed, using the θ expansion of (λH^m) from the appendix A and the identities to extract component expansions automated in [28], we have explicitly verified (2.5) up to $k = 5$.

Therefore, the massive amplitudes (2.5) have been derived from the same pure spinor cohomology arguments as the SYM amplitudes of [9,10].

2.2. From massless ${\alpha'}^2$ amplitudes

The factorization of the massless $n+1$ amplitude on its first massive residue was shown to be equivalent to the statement [16]

$$A(1, P|\underline{k})\big|_{\underline{k}\to k,k+1} = -\langle C_{1|P,k,k+1}\rangle\,, \tag{2.6}$$

relating the massive amplitudes to the ${\alpha'}^2$ sector of the massless amplitudes, in agreement with the earlier observation in [11]. Equation (2.6) can be viewed as a consistency check due to unitarity, albeit written in a slightly unconventional form. This statement was explicitly verified [11] in terms of polarizations and momenta using the Berends-Giele construction of $A(1,P|\underline{k})$ given in (1.6) on the left-hand side, and the component expansion of the scalar BRST invariants available in [29]. In this case, the map $\underline{k} \to k, k{+}1$ is the component counterpart of the superfield prescription (1.10), see [16] for the precise details.

If one has the n-point massive amplitude, then the map $\underline{k} \to k, k{+}1$ relates it to the kinematic expression governing the ${\alpha'}^2$ expansion of the massless amplitude at $n{+}1$ points. The more interesting direction would be to derive the massive field-theory tree amplitudes starting from the massless string disk amplitudes; that is, to invert the factorization condition (2.6). We will demonstrate below that the cohomology structure of the pure spinor superspace allows us to do precisely that.

Inverting the factorization condition. Since the pure spinor superspace expressions for both sides of the factorization condition (2.6) as well as the superspace prescription of the map $\underline{k} \to k, k{+}1$ are known, we can exploit the simplicity of superspace to invert (2.6): That is, we want to arrive at the expression (2.5) by inverting the massless representation prescription $\underline{k} \to k, k{+}1$ given in (1.9) and (1.10), starting from the right-hand side given in (1.13).

In order to do this, it will be convenient to rewrite the scalar BRST invariants in an asymmetric manner. One can show using a combination of equations of motion, pure spinor constraint and gamma matrix identities that

$$C_{1|P,k,k+1} = C^m_{1|P}(\lambda\gamma^n \mathcal{W}_k)\mathcal{F}^{mn}_{k+1} - Q\widehat{M}_{1|P,k,k+1}\,, \tag{2.7}$$

where the labels k and $k{+}1$ are singled out to appear in different superfields. In this equation, $\widehat{M}_{1|P,k,k+1}$ is given by the ghost-number two expression obtained from the expansion of $C_{1|P,k,k+1}$ of (1.13) and replacing

$$M_{A,B,C} \to (\lambda\gamma^m\mathcal{W}_A)(\mathcal{W}_B\gamma_m\mathcal{W}_C) + (\lambda\gamma^m\mathcal{W}_B)(\mathcal{W}_A\gamma_m\mathcal{W}_C)\,. \tag{2.8}$$

For example,

$$\begin{aligned}
\widehat{M}_{1|2,3,4} &= M_1(\lambda\gamma^m\mathcal{W}_2)(\mathcal{W}_3\gamma_m\mathcal{W}_4) + M_1(\lambda\gamma^m\mathcal{W}_3)(\mathcal{W}_2\gamma_m\mathcal{W}_4)\,, \qquad (2.9)\\
\widehat{M}_{1|23,4,5} &= M_1(\lambda\gamma^m\mathcal{W}_{23})(\mathcal{W}_4\gamma_m\mathcal{W}_5) + M_1(\lambda\gamma^m\mathcal{W}_4)(\mathcal{W}_{23}\gamma_m\mathcal{W}_5)\\
&\quad + M_{12}(\lambda\gamma^m\mathcal{W}_3)(\mathcal{W}_4\gamma_m\mathcal{W}_5) + M_{12}(\lambda\gamma^m\mathcal{W}_4)(\mathcal{W}_3\gamma_m\mathcal{W}_5)\\
&\quad + M_{31}(\lambda\gamma^m\mathcal{W}_2)(\mathcal{W}_4\gamma_m\mathcal{W}_5) + M_{31}(\lambda\gamma^m\mathcal{W}_4)(\mathcal{W}_2\gamma_m\mathcal{W}_5)\,.
\end{aligned}$$

Since the pure spinor bracket annihilates BRST exact expressions we get

$$\langle C_{1|P,k,k+1}\rangle = \langle C^m_{1|P}(\lambda\gamma^n\mathcal{W}_k)\mathcal{F}^{mn}_{k+1}\rangle = \langle C^m_{1|P}(\lambda\gamma^n\mathcal{W}_{k+1})\mathcal{F}^{mn}_k\rangle\,, \tag{2.10}$$

where the second equality follows from the BRST cohomology identity

$$0 = \langle Q\big(C^m_{1,P}(\mathcal{W}_k\gamma^m\mathcal{W}_{k+1})\big)\rangle = \langle C^m_{1,P}(\lambda\gamma^n\mathcal{W}_k)\mathcal{F}^{mn}_{k+1}\rangle - \langle C^m_{1,P}(\lambda\gamma^n\mathcal{W}_{k+1})\mathcal{F}^{mn}_k\rangle\,. \tag{2.11}$$

The first equality represents the vanishing of BRST-exact expressions under the pure spinor bracket [7]. For the second equality, one uses $QC^m_{1|P} = 0$, the equation of motion for $\mathcal{W}^\alpha$ and the constraint $(\lambda\gamma_m)_\alpha(\lambda\gamma^m)_\beta = 0$.

In the OPE gauge, the prescription $\underline{k} \to k, k{+}1$ for the massless representation of the massive superfield (λH^m_k) is given by (1.9)

$$\begin{aligned}(\lambda H^m_k) &= -2\alpha'\Big(ik^n_{k+1}F^{mn}_kV_{k+1} + ik^m_k(\lambda\gamma^nW_k)A^n_{k+1} + ik^m_kQ(W_kA_{k+1})\Big) \\ &= -2i\alpha'\Big((\lambda\gamma^mW_k)(k_k\cdot A_{k+1}) - F^{mn}_k(\lambda\gamma^nW_{k+1}) + Q(F^{mn}_kA^n_{k+1}) + k^m_kQ(W_kA_{k+1})\Big)\,,\end{aligned} \tag{2.12}$$

where the second line follows from $Q(F^{mn}_kA^n_{k+1}) = ik^n_{k+1}F^{mn}_kV_{k+1} + ik^m_k(\lambda\gamma^nW_k)A^n_{k+1} - ik^n_k(\lambda\gamma^mW_k)A^n_{k+1} + F^{mn}_k(\lambda\gamma^nW_{k+1})$.

From (2.12), we can formally rewrite the factorization $\underline{k} \to k, k{+}1$ in the reverse direction to obtain

$$2i\alpha'F^{mn}_k(\lambda\gamma^nW_{k+1}) = (\lambda H^m_k) + 2i\alpha'(\lambda\gamma^mW_k)(k_k\cdot A_{k+1}) + Q\big((F^{mn}_kA^n_{k+1}) + k^m_k(W_kA_{k+1})\big)\,, \tag{2.13}$$

Finally, plugging (2.13) into the BRST-equivalent expression (2.7) of the scalar BRST invariant and using and that $C^m_{1,P}$ is BRST closed leads to

$$\begin{aligned}2i\alpha'\langle C_{1|P,k,k+1}\rangle &= 2i\alpha'\langle C^m_{1|P}F^{mn}_k(\lambda\gamma^nW_{k+1})\rangle \\ &= \langle C^m_{1,P}(\lambda H^m_k)\rangle + 2i\alpha'\langle C^m_{1|P}(\lambda\gamma^mW_k)(k_k\cdot A_{k+1})\rangle + \langle C^m_{1|P}Q(\ldots)\rangle \\ &= \langle C^m_{1|P}(\lambda H^m_k)\rangle \\ &= -2i\alpha' A(1,P|\underline{k})\,,\end{aligned} \tag{2.14}$$

where in the second line we used that $C^m_{1|P}(\lambda\gamma_m)_\alpha = 0$, integrated the BRST charge by parts and used that $C^m_{1|P}$ is BRST closed to obtain $\langle C^m_{1|P}Q(\ldots)\rangle = -\langle\big(QC^m_{1|P}\big)(\ldots)\rangle = 0$.

Therefore, inverting the massless representation map $\underline{k} \to k, k{+}1$ (which is equivalent to the factorization of the massless string amplitudes on their first massive pole [16]) yields the superspace expression of the massive field-theory amplitude:

$$\langle C_{1|P,k,k+1}\rangle \to -A(1,P|\underline{k}) \tag{2.15}$$

This completes the (formal) derivation of the massive field-theory amplitude $A(1,P|\underline{k})$ from the kinematics $\langle C_{1|P,k,k+1}\rangle$ of the ${\alpha'}^2$ correction to massless open-string disk amplitudes.

3. Conclusions

In this paper we derived, using BRST cohomology considerations, a compact pure spinor superspace expression for the massive field-theory amplitudes $A(1,2,\ldots|\underline{n})$. Furthermore, the same expression was also derived from the α'^2 correction to massless string amplitudes, as anticipated in [11]. The successful application, in the massive case, of the central idea in [9] for massless field-theory amplitudes leads us to conjecture that all massive field-theory amplitudes (as defined in section 1) can be obtained by BRST cohomology considerations.

Furthermore, BRST cohomology manipulations in pure spinor superspace are powerful enough to lead one to hope [11] that the expressions of massive field-theory amplitudes with higher number of massive legs and/or higher mass levels can be systematically obtained from the known massless disk amplitudes at higher α' orders. This paper gives evidence for the first step of this ladder, climbing the rest of the way is left for future work.

Acknowledgements: CRM thanks Oliver Schlotterer for useful comments on the draft, and Sitender Kashyap, Mritunjay Verma and Luis Ypanaqué for collaboration on related topics.

Appendix A. Theta expansion of massive superfields

The θ expansions of the massive superfields of the first massive level of the open superstring have been determined in [30]. In order to avoid problems due to different conventions – especially due to the convention $\partial_m \to k_m$ used in the component expansions via [28,31] – we will rederive the expansions here in a streamlined manner.

Equations of motion and recursion. With the definition

$$G^{mn} = -\frac{1}{144}\big[(D\gamma^m H^n) + (D\gamma^n H^m)\big]\,, \tag{A.1}$$

one can show that the massive superfields satisfy [30]

$$\begin{aligned}
D_\alpha G^{mn} &= -\frac{1}{18}\partial_p(\gamma^{pm}H^n)_\alpha - \frac{1}{18}\partial_p(\gamma^{pn}H^m)_\alpha\,, \qquad (A.2)\\
D_\alpha B_{mnp} &= -\frac{1}{18}(\gamma^{mn}H^p)_\alpha + \frac{\alpha'}{18}\partial_a\partial_m\Big((\gamma^{an}H^p)_\alpha - (\gamma^{ap}H^n)_\alpha\Big) + \mathrm{cyc}(mnp)\,,\\
D_\alpha H^m_\beta &= -\frac{9}{2}G_{mn}\gamma^n_{\alpha\beta} - \frac{3}{2}\partial_a B_{bcm}\gamma^{abc}_{\alpha\beta} + \frac{1}{4}\partial_a B_{bcd}\gamma^{mabcd}_{\alpha\beta}\,.
\end{aligned}$$

Denoting by $[K]_n$ the component of the superfield K of order $(\theta)^n$, the Euler operator (θD) satisfies $(\theta D)[K]_n = n[K]_n$. Therefore multiplying (A.2) by θ^α from the left gives rise to a recursion (note $\partial_m \to k_m$)

$$
\begin{aligned}
[G^{mn}]_k &= -\frac{1}{18k}\Big[k_p(\theta\gamma^{pm}[H^n]_{k-1}) + k_p(\theta\gamma^{pn}[H^m]_{k-1})\Big] \\
[B_{mnp}]_k &= -\frac{1}{18k}\Big[(\theta\gamma^{mn}[H^p]_{k-1}) - \alpha' k_a k_m\Big((\theta\gamma^{an}[H^p]_{k-1}) - (\theta\gamma^{ap}[H^n]_{k-1})\Big) + \mathrm{cyc}(mnp)\Big] \\
[H^m_\beta]_k &= \frac{1}{k}\Big[-\frac{9}{2}[G_{mn}]_{k-1}(\theta\gamma^n)_\beta - \frac{3}{2}k_a[B_{bcm}]_{k-1}(\theta\gamma^{abc})_\beta + \frac{1}{4}k_a[B_{bcd}]_{k-1}(\theta\gamma^{mabcd})_\beta\Big]\,.
\end{aligned}
\tag{A.3}
$$

starting with

$$
[G_{mn}]_0 = g_{mn} \quad [B_{mnp}]_0 = b_{mnp}\,, \quad [H^m_\alpha]_0 = \psi^m_\alpha \tag{A.4}
$$

of length dimensions $[g_{mn}] = 0$, $[b_{mnp}] = 1$ and $[\psi^m_\alpha] = \frac{1}{2}$. Using the recursion (A.3) yields the following θ expansion for $\lambda^\alpha H^m_\alpha$ in the Berkovits-Chandia gauge:

$$
\begin{aligned}
(\lambda H^m) = &\\
&(\lambda\psi^m) - \frac{1}{4}(\lambda\gamma^{kmpqr}\theta)b_{pqr} + \frac{3}{2}(\lambda\gamma^{kpq}\theta)b_{mpq} - \frac{9}{2}(\lambda\gamma^n\theta)g_{mn} \\
&+ \frac{1}{48}\Big[(\lambda\gamma^{mnpqr}\theta)(\theta\gamma^{np}\psi^q)k_i^r - 4(\lambda\gamma^{npq}\theta)(\theta\gamma^{mn}\psi^p)k_i^q - 2(\lambda\gamma^{npq}\theta)(\theta\gamma^{np}\psi^m)k_i^q\Big] \\
&- \frac{1}{12}\Big[(\lambda\gamma^{npq}\theta)(\theta\gamma^{nr}\psi^p)k_i^m k_i^q k_i^r \alpha' + \frac{3}{2}(\lambda\gamma^n\theta)(\theta\gamma^{mp}\psi^n)k_i^p + \frac{3}{2}(\lambda\gamma^n\theta)(\theta\gamma^{np}\psi^m)k_i^p\Big] \\
&- \frac{1}{576}(\lambda\gamma^{kmpqr}\theta)(\theta\gamma^{pqrtuvk}\theta)b_{tuv} - \frac{1}{32}(\lambda\gamma^{kmpqr}\theta)(\theta\gamma^{ptk}\theta)b_{qrt} \\
&+ \frac{1}{96}\Big[(\lambda\gamma^{kpq}\theta)(\theta\gamma^{mpqstuk}\theta)b_{stu} + 6(\lambda\gamma^{kpq}\theta)(\theta\gamma^{msk}\theta)b_{pqs} - 12(\lambda\gamma^{kpq}\theta)(\theta\gamma^{psk}\theta)b_{mqs}\Big] \\
&- \frac{1}{48}\Big[(\lambda\gamma^k\theta)(\theta\gamma^{pqr}\theta)b_{pqr}k^m - \frac{9}{2}(\lambda\gamma^k\theta)(\theta\gamma^{qrk}\theta)b_{mqr} - \frac{9}{2}(\lambda\gamma^n\theta)(\theta\gamma^{qrk}\theta)b_{nqr}k^m\Big] \\
&+ \frac{1}{48\alpha'}\Big[(\lambda\gamma^m\theta)(\theta\gamma^{npq}\theta)b_{npq} - \frac{9}{2}(\lambda\gamma^n\theta)(\theta\gamma^{mpq}\theta)b_{npq} - \frac{9}{2}(\lambda\gamma^n\theta)(\theta\gamma^{npq}\theta)b_{mpq}\Big] \\
&- \frac{1}{32}\Big[(\lambda\gamma^{kmpqr}\theta)(\theta\gamma^{pqs}\theta)g_{rs} - 4(\lambda\gamma^{kpq}\theta)(\theta\gamma^{mpr}\theta)g_{qr} + 4(\lambda\gamma^{kpq}\theta)(\theta\gamma^{psk}\theta)g_{qs}k^m\alpha'\Big] \\
&+ \frac{1}{16}\Big[(\lambda\gamma^{kpq}\theta)(\theta\gamma^{pqr}\theta)g_{mr} - 3(\lambda\gamma^n\theta)(\theta\gamma^{mqk}\theta)g_{nq} - 3(\lambda\gamma^n\theta)(\theta\gamma^{nqk}\theta)g_{mq}\Big] + \mathcal{O}(\theta^4)
\end{aligned}
\tag{A.5}
$$

where a vector k is written as an index if it is contracted with a gamma matrix: for example $(\lambda\gamma^k\theta)$ means $k_n(\lambda\gamma^n\theta)$. The θ^4 components are commented out in the TEX file. The θ expansion of the other superfields are not needed in this paper but can be easily generated from (A.3).

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