UNIVERSITY OF SOUTHAMPTON

Faculty of Social Sciences
School of Mathematics

Quasiconvexity and separability in relatively hyperbolic groups

by

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A thesis for the degree of Doctor of Philosophy

September 2024

University of Southampton

Abstract

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In this thesis, we aim to understand the behaviour of quasiconvexity under the basic operation of taking joins of subgroups. We will also study the relation between quasiconvexity and residual properties of relatively hyperbolic groups.

In particular, suppose that G is a relatively hyperbolic group, and let Q and R be relatively quasiconvex subgroups of G. We provide sufficient conditions for the join $\langle Q', R' \rangle$ of subgroups $Q' \leqslant Q$ and $R' \leqslant R$ to be relatively quasiconvex. Further, we determine the structure of the maximal parabolic subgroups of $\langle Q', R' \rangle$ in this setting.

We show that, given suitable assumptions on the profinite topology of G, these conditions can be arranged to hold for sufficiently deep finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$. As a consequence, we show that $\langle Q', R' \rangle$ decomposes as an amalgamated free product when the parabolic subgroups of Q and R are almost compatible.

Finally, we show that if G is hyperbolic relative to product separable subgroups, then the product of any finitely generated quasiconvex subgroups is separable in G. We record applications of this to various classes of nonpositively curved groups.

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Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as: A. Minasyan and L. Mineh. Quasiconvexity of virtual joins and separability of products in relatively hyperbolic groups. *Algebr. Geom. Topol.*, forthcoming
   L. Mineh. Structure of quasiconvex virtual joins, 2023

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Acknowledgements

Above all, thanks go to my supervisor Ashot Minasyan, without whose careful guidance this project would have been wholly infeasible. I am extremely grateful for all you taught me, and for the boundless patience you have shown me over the years.

I want to thank the many wonderful people I met along the way. With them, it was easy for the years to go by. Among them are Guy, Lewis, George, Jon, Abigail, Laura, and Emily. Thank you to my officemates for their help in developing a number of exciting new sports. Special thanks to Geraint for his friendship in difficult times, and to Sofiya for buying me pumpernickel.

Many thanks to the other friends in my life. To Pete for the continued repercussions of his actions, and to Jonathan for many interesting discussions and free dinners. To Sam, Harry, Shaked, and Marco for their company on a long walk. To Adam, Bartosz, Paul, Oliver, Ben: you know what you did.

Finally, a huge thank you to my family: to my parents for their consistent care and encouragement, and to Lana and John for their endless support and entertainment.

To Ciaran

Definitions and Abbreviations

$\leqslant_f$, $\lhd_f$	finite index (normal) subgroup
$V\Gamma, E\Gamma$	vertex set, edge set of graph Γ
$\lvert\cdot\rvert_X$, d_X	word length, word metric with respect to a subset X
$\min_X(Y)$	minimal word length $\min\{\lvert y\rvert_X \mid y \in Y\}$ of an element in Y
$\Gamma(G, X)$	Cayley graph of group G with respect to X
$N_X(Y, K)$	K-neighbourhood of subset Y with respect to d_X
$[x, y]$	geodesic starting at x and ending at y
p_-, p_+	initial and terminal endpoints of path p
p^{-1}	formal inverse of path p
$\ell(p)$	length of path p
$\mathrm{Lab}(p)$	label of combinatorial path p in $\Gamma(G, X)$
$\widetilde{p}$	element represented by $\mathrm{Lab}(p)$
$\lvert p\rvert_X$	word length $\lvert\widetilde{p}\rvert_X$ of $\widetilde{p}$
$\mathbb{N}, \mathbb{N}_0$	the sets $\{1, 2, 3, \ldots\}$ and $\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	the set of integers

Chapter 1

Introduction

The central theme of geometric group theory is to exhibit group actions on metric spaces, and from this extract information about the group. In his seminal essay, Gromov (1987) introduced the influential notion of a *hyperbolic group*, unifying the combinatorial and geometric methods that were being developed in group theory over the preceding decades. The key observation is that when a group admits a nice action on a space that has some nonpositive or negative curvature, many strong statements can be made about the algebraic properties of the group. *Hyperbolic metric spaces*, introduced in the same paper, serve as a robust model for negative curvature in arbitrary metric spaces. Indeed, hyperbolic groups are exactly the finitely generated groups that admit proper and cocompact actions by isometries on hyperbolic metric spaces. In this way, hyperbolic groups mimic the fundamental groups of compact hyperbolic manifolds.

The class of hyperbolic groups, though large, is somewhat restricted, and many natural and important examples of groups fall outside this class. For example, any group containing a higher rank free abelian subgroup cannot be hyperbolic. As such, it is often useful to relax the condition on the group action to allow for such examples. In this thesis, we will be focused primarily on the class of *relatively hyperbolic groups*, which admit cobounded actions on hyperbolic spaces that are in some sense proper away from a fixed collection of bounded subsets.

Relatively hyperbolic groups were suggested by Gromov (1987), and expanded on by various authors. The concept was more substantially developed by Bowditch (2012), Farb (1998), Druţu and Sapir (2005), Osin (2006b), and Groves and Manning (2008), whose varied definitions were shown equivalent by Hruska (2010). Relative hyperbolicity is a relative property of a group G in the sense that one must specify a collection of *peripheral subgroups* with respect to which G is relatively hyperbolic. Archetypal examples include small cancellation quotients of free products and fundamental groups of finite volume manifolds of pinched negative curvature, which are hyperbolic relative to the images of the free factors and to their cusp subgroups respectively (see, for example, Osin (2006b)).

We will also be interested in studying aspects of the *profinite topology* in groups. The profinite topology is an object that encodes information about the finite quotients of a group. Any group G can be equipped with the profinite topology by declaring (left) cosets of finite index subgroups of G to be a basis of open sets. This naturally makes G into a topological group, which is Hausdorff if and only if the trivial subgroup is closed: in this case G is called *residually finite*.

Let us introduce some terminology. We will say that a subset $U \subseteq G$ is *separable* in the profinite topology on G when U is closed. If each finitely generated subgroup of G is separable, we say that G is *LERF* (locally extended residually finite). Likewise, we say that G is *double coset separable* if for any two finitely generated subgroups $H, K \leqslant G$, the double coset HK is separable.

Knowing that certain subsets of groups are separable has important applications to geometric and algebraic problems. For instance, the membership problem is solvable for a finitely generated subgroup H of a finitely presented group G if H is separable in G (c.f. (Lyndon and Schupp, 1977, IV.4.6)). For fundamental groups of complexes, separability of subgroups and subsets corresponds to useful lifting properties in the complex. Double coset separability has recently proven to be instrumental in characterising the property of *virtual specialness* in relation to fundamental groups of nonpositively curved cube complexes (Haglund and Wise (2008)).

1.1 Quasiconvex subgroups and combination theorems

In trying to understand the structure of a group, it is essential to study the structure of its subgroups. We often restrict our attention to subgroups generated by finitely many elements of the group, as the infinitely generated subgroups can be especially wild. In the setting of hyperbolic groups, arbitrary finitely generated subgroups may still be quite poorly behaved. For instance, Rips (1982) developed a method to construct hyperbolic groups containing 2-generated normal subgroups exhibiting wild properties (e.g. with distortion greater than any computable function).

It is often fruitful, therefore, to further restrict one's attention to the class of *quasiconvex subgroups* of hyperbolic groups. Quasiconvex subgroups are exactly the finitely generated quasi-isometrically embedded subgroups of hyperbolic groups, and they play a central role in the theory of hyperbolic groups. It is a consequence of the above definition that quasiconvex subgroups are themselves hyperbolic.

If Q and R are quasiconvex subgroups of a hyperbolic group G, then the intersection $S = Q \cap R$ is also quasiconvex (Short (1991)). On the other hand, the join $\langle Q, R \rangle$ of Q

and R may fail to be quasiconvex. Indeed, cyclic subgroups of hyperbolic groups are quasiconvex, while infinite index normal subgroups are quasiconvex only when they are finite, so the construction of Rips mentioned above provides counterexamples. This failure can be remedied by considering instead *virtual joins*: subgroups of the form $\langle Q', R' \rangle$ for some finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$. Under the assumption that the intersection $S = Q \cap R$ is separable, Gitik (1999) showed that there exist finite index subgroups as above with $Q' \cap R' = S$ such that $\langle Q', R' \rangle$ is quasiconvex. Moreover, this virtual join $\langle Q', R' \rangle$ is naturally isomorphic to the amalgamated free product $Q' *_S R'$.

In the setting of relatively hyperbolic groups, the natural sub-objects are the *relatively quasiconvex subgroups*, which are themselves relatively hyperbolic in a way that is compatible with the ambient group. Basic examples of relatively quasiconvex subgroups are *maximal parabolic subgroups* (i.e. conjugates of the peripheral subgroups), *parabolic subgroups* (subgroups of maximal parabolic subgroups), and finitely generated quasi-isometrically embedded subgroups (Hruska (2010)).

Suppose that Q and R are relatively quasiconvex subgroups of relatively hyperbolic group G. In Hruska (2010), it was proven that the intersection $S = Q \cap R$ is again relatively quasiconvex. However, previously the existence of a relatively quasiconvex virtual join $\langle Q', R' \rangle$, for Q and R with $S = Q \cap R$ separable in G, was only known in a few special cases:

- Martínez-Pedroza (2009) proved it in the case when $R \leqslant P$, for some maximal parabolic subgroup P of G, such that $Q \cap P \subseteq R$;
- Martínez-Pedroza and Sisto (2012) proved it when Q and R have *compatible parabolics* (that is, for every maximal parabolic subgroup P of G either $Q \cap P \subseteq R \cap P$ or $R \cap P \subseteq Q \cap P$);
- Yang (2012) (unpublished; see also McClellan's thesis McClellan (2019)) proved it when R is a *full subgroup* of G (that is, for every maximal parabolic subgroup P in G, $R \cap P$ is either finite or has finite index in P).

Similarly to Gitik (1999), in all three cases above the authors establish an isomorphism between the virtual join $\langle Q', R' \rangle$ and the amalgamated free product $Q' *_{S'} R'$, where $S' = Q' \cap R' \leqslant_f S$ as an essential component of their proofs.

The extra assumptions on Q and R in each of the above results imply that Q and R have *almost compatible parabolics* (see Definition 1.3 below). Unfortunately this is still a significant restriction and a more general result is desirable. Moreover, in the absence of almost compatibility one cannot expect a virtual join to split as an amalgamated free product of Q' and R', for if both Q and R are subgroups of the same abelian peripheral subgroup of G then any virtual join $\langle Q', R' \rangle$ would again be abelian.

One of the goals of this thesis is to establish the quasiconvexity of virtual joins without making any compatibility assumptions on Q and R. However we need to impose stronger assumptions on the properties of the profinite topology on G than just separability of $S = Q \cap R$: we will require the finitely generated relatively quasiconvex subgroups to be separable and the peripheral subgroups are double coset separable.

We will say that a relatively hyperbolic group is *QCERF* if each of its finitely generated relatively quasiconvex subgroups are separable.

Theorem 1.1. *Let G be a finitely generated group that is QCERF hyperbolic relative to double coset separable subgroups. For any finitely generated relatively quasiconvex subgroups $Q, R \leqslant G$, there are finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ such that $\langle Q', R' \rangle$ is relatively quasiconvex.*

In fact, we establish the existence of many finite index subgroups of Q and R whose join is relatively quasiconvex rather than just a single pair, though the existential statement is a little technical: see Section 4.2 for details.

As mentioned above, a relatively quasiconvex subgroup Q of G is itself relatively hyperbolic in way that is compatible with the relative hyperbolicity of G. To be precise, Q is hyperbolic relative to infinite subgroups of the form $Q \cap P$ where $P \leqslant G$ is a maximal parabolic subgroup of G. As such, to understand the structure of the virtual joins obtained from Theorem 1.1 as relatively hyperbolic groups, we study these intersections. We find that the finite index subgroups Q' and R' may be chosen such that the intersection of the virtual join $\langle Q', R' \rangle$ with a maximal parabolic subgroup is itself, up to conjugacy, a join of intersections of Q and R with a maximal parabolic subgroup of G.

Theorem 1.2. *Let G be a finitely generated group that is QCERF hyperbolic relative to double coset separable subgroups, and let $Q, R \leqslant G$ be finitely generated relatively quasiconvex subgroups. Then there are finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ such that $\langle Q', R' \rangle$ is relatively quasiconvex and the following is true.*
Suppose that $P \leqslant G$ is a maximal parabolic subgroup with $\langle Q', R' \rangle \cap P$ infinite. Then there is $u \in \langle Q', R' \rangle$ such that

$$\langle Q', R' \rangle \cap P = u \langle Q' \cap K, R' \cap K \rangle u^{-1},$$

where $K = u^{-1} P u$.

Note that the conjugator u in the above statement is strictly necessary: suppose $K \leqslant G$ is a maximal parabolic subgroup of G such that either $Q' \cap K$ or $R' \cap K$ is infinite. Then for any $v \in \langle Q', R' \rangle$, the intersection $\langle Q', R' \rangle \cap vKv^{-1}$ contains $v(Q' \cap K)v^{-1}$ and $v(R' \cap K)v^{-1}$, and is therefore infinite. However, it may be that $u \in \langle Q', R' \rangle$ is such that the subgroups $Q' \cap P$ and $R' \cap P$ are both trivial, where $P = u^{-1}Ku$. This precludes the possibility that they generate $\langle Q', R' \rangle \cap P$.

We actually prove a more detailed characterisation of the subgroup $\langle Q', R' \rangle \cap P$: see Theorem 4.27. Using this stronger result, we generalise and unify the previous results of Martìnez-Pedroza, Sisto, McClellan, and Yang mentioned above. For this we will need to introduce some terminology and notation.

We will use a preorder $\preccurlyeq$ on the sets of subsets of a group G, introduced by Minasyan (2005b). Given subsets $U, V \subseteq G$, we will write $U \preccurlyeq V$ if there exists a finite subset $Y \subseteq G$ such that $U \subseteq VY$.

If d is a proper metric on G and U and V are subsets of G, then $U \preccurlyeq V$ if and only if U is contained in a finite d-neighbourhood of V. If U and V are subgroups of G then $U \preccurlyeq V$ is equivalent to $[U : U \cap V] < \infty$ (see (Minasyan, 2005b, Lemma 2.1)).

Definition 1.3. Let Q and R be subgroups of a relatively hyperbolic groups G, and let P be a maximal parabolic subgroup of G. We say that Q and R are *almost compatible at* P if $Q \cap P \preccurlyeq R \cap P$ or $Q \cap P \preccurlyeq R \cap P$. We will say that Q and R have *almost compatible parabolics* if Q and R are almost compatible at every maximal parabolic subgroup of G.

We note the condition of having almost compatible parabolics was introduced by Baker and Cooper (2008) in the context of discrete subgroups of $\mathrm{Isom}(\mathbb{H}^n)$. We are able to promote the condition of having almost parabolic subgroups to that of having compatible parabolics on the nose, after passing to finite index subgroups.

Theorem 1.4. *Let G be a finitely generated QCERF relatively hyperbolic group. Suppose that $Q, R \leqslant G$ are finitely generated relatively quasiconvex subgroups with almost compatible parabolics. There are finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ such that Q' and R' have compatible parabolics.*

Combining this with the combination theorem of Martínez-Pedroza and Sisto (2012), we obtain the following.

Corollary 1.5. *Let G be a finitely generated QCERF relatively hyperbolic group. Suppose that $Q, R \leqslant G$ are finitely generated relatively quasiconvex subgroups with almost compatible parabolics. Then there are finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ such that $\langle Q', R' \rangle$ is relatively quasiconvex and $\langle Q', R' \rangle \cong Q' *_{Q' \cap R'} R'$.*

Let us say a few words on the assumptions of the above theorems. The results apply to a wide class of relatively hyperbolic groups, including all limit groups, fundamental groups of many finite volume hyperbolic manifolds and many groups acting on CAT(0) cube complexes. Regarding QCERF-ness, Manning and Martínez-Pedroza (2010) proved that the following two statements are equivalent:

(a) every finitely generated group hyperbolic relative to a finite collection of LERF and slender subgroups is QCERF;
(b) all word hyperbolic groups are residually finite.

Recall that a group is called *slender* if every subgroup is finitely generated. The question of whether statement (b) is true is a well-known open problem. If the answer to it is positive then, for example, all finitely generated groups hyperbolic relative to virtually polycyclic subgroups will be QCERF.

Large classes of relatively hyperbolic groups have already been proved to be QCERF. One of the first results in this direction is due to Wilton (2008), who established QCERF-ness of limit groups. The ground-breaking work of Haglund and Wise (2008) and Agol (2013) implies that any word hyperbolic group acting geometrically on a CAT(0) cube complex is QCERF. One of the consequences of this result is that all geometrically finite Kleinian groups are QCERF. More recently, Groves and Manning (2022) extended this theory to relatively hyperbolic groups. They show that if a group is hyperbolic relative to a collection of LERF subgroups and admits a weakly relatively geometric action on a CAT(0) cube complex, then G is QCERF. Einstein and Ng (2021) showed that $C'(1/6)$-small cancellation quotients of free products exhibit such actions. It follows that a $C'(1/6)$-small cancellation quotient of a free product of LERF groups is QCERF, for example.

By a theorem of Lennox and Wilson (1979), all virtually polycyclic groups are double coset separable, hence the assumption about peripheral subgroups is automatically true in many relevant cases. However whether this assumption is actually necessary is less obvious. It is required in our approach, but it would be interesting to see whether Theorems 1.1 and 1.2 remain valid without it.

Metric conditions

Let G be a relatively hyperbolic group with finite generating set X, let $Q, R \leqslant G$ be relatively quasiconvex subgroups of G, and write $S = Q \cap R$. The overarching strategy we employ to prove the above results is to show that if $Q' \leqslant Q$ and $R' \leqslant R$ are subgroups satisfying a certain set of metric conditions, then the desired results on quasiconvexity and structure hold for their join. Then, we will use the assumptions of separability to find finite index subgroups of Q and R satisfying these conditions. First, therefore, we must introduce our collection of rather technical metric conditions.

Given a finite collection $\mathcal{P}$ of maximal parabolic subgroups of G, constants $B, C \geq 0$ and subgroups $Q' \leqslant Q$, $R' \leqslant R$, consider the following:

(C1) $Q' \cap R' = S$;

(C2) $\min_X\Big(Q\langle Q', R'\rangle Q \setminus Q\Big) \geq B$ and $\min_X\Big(R\langle Q', R'\rangle R \setminus R\Big) \geq B$;

(C3) $\min_X\Big((PQ' \cup PR') \setminus PS\Big) \geq C$, for each $P \in \mathcal{P}$.

Moreover, if not all of the subgroups in $\mathcal{P}$ are abelian then we will need two more conditions (here for subgroups $H \leqslant G$ and $P \in \mathcal{P}$, we use H_P to denote the intersection $H \cap P \leqslant P$):

(C4) $Q_P \cap \langle Q'_P, R'_P \rangle = Q'_P$ and $R_P \cap \langle Q'_P, R'_P \rangle = R'_P$, for every $P \in \mathcal{P}$;

(C5) $\min_X \left(q\langle Q'_P, R'_P \rangle R_P \setminus qQ'_P R_P \right) \geq C$, for each $P \in \mathcal{P}$ and all $q \in Q_P$.

Remark 1.6. If the peripheral subgroups of G are abelian then condition (C4) follows from (C1) and condition (C5) is trivially true. Indeed, if P is abelian, then, in the notation of (C4), $\langle Q'_P, R'_P \rangle = Q'_P R'_P$, hence

$$Q'_P \subseteq Q_P \cap \langle Q'_P, R'_P \rangle = Q_P \cap Q'_P R'_P = Q'_P (Q_P \cap R'_P) \subseteq Q'_P S_P = Q'_P,$$

where the last equality used that $S_P = S \cap P \subseteq Q'_P$ by (C1). The second equality of (C4) can be proved in the same fashion.

Similarly, if $q \in Q_P$ then $q\langle Q'_P, R'_P \rangle R_P = qQ'_P R'_P R_P = qQ'_P R_P$, so that

$$\min_X \left(q\langle Q'_P, R'_P \rangle R_P \setminus qQ'_P R_P \right) = \min_X(\emptyset) = +\infty,$$

thus (C5) holds.

Remark 1.7. As mentioned above, we are interested in the existence of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ satisfying the above metric conditions. Thus it may be easier to interpret the conditions when viewed through the lens of the profinite topology on G (see Section 4.2):

- conditions (C1) and (C4) can be ensured by choosing any finite index subgroup $M \leqslant_f G$ with $S \subseteq M$, and setting $Q' = Q \cap M$, $R' = R \cap M$;
- the existence of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ satisfying condition (C2) can be deduced from separability of Q and R in G;
- the existence of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ satisfying condition (C3) can be deduced from separability of the double coset PS in G;
- if $Q'_P \leqslant_f Q_P$ is already chosen then $R'_P \leqslant_f R_P$, satisfying (C5), can be constructed with the help of separability of the double coset $Q'_P R_P$ in P. Indeed, if $Q_P = \bigcup_{j=1}^{n} a_j Q'_P$, then the inequality in (C5) can be re-written as $\min_X \left(a_j \langle Q'_P, R'_P \rangle Q'_P R_P \setminus a_j Q'_P R_P \right) \geq C$, for every $j = 1, \ldots, n$. Thus our approach to establishing (C5) will be to choose $R' \leqslant_f R$ after $Q' \leqslant_f Q$ has already been constructed (in other words, R' will depend on Q').

The aim is to prove the following (see Theorem 3.26).

Theorem 1.8. *Let G be a finitely generated relatively hyperbolic group, and suppose that Q and R are relatively quasiconvex subgroups of G. There are constants $B, C \geq 0$ and finite family of maximal parabolic subgroups $\mathcal{P}$ such that the following is true. If $Q' \leqslant Q$ and $R' \leqslant R$ are relatively quasiconvex subgroups satisfying (C1)–(C5) with constants B, C, and family $\mathcal{P}$, then $\langle Q', R' \rangle$ is relatively quasiconvex.*

1.2 Product separability

It is often useful in group theory to know that products of certain subgroups are separable. For instance, double coset separability in limit groups and certain graphs of free groups was used to show that such groups satisfy the "geometric Hanna Neumann conjecture" concerning the rank of intersections of finitely generated subgroups in Fisher and Morales (2023). Recently, Abdenbi and Wise (2024) used separability of up to quintuple cosets of finitely generated subgroups of free groups to show that partial local isometries of special cube complexes extend to automorphisms of a larger special cube complex.

Definition 1.9. Let G be a group and let $s \in \mathbb{N}$. We say that P has property RZ_s if for arbitrary finitely generated subgroups $H_1, \ldots, H_s \leqslant P$, the product $H_1 \ldots H_s$ is separable in P. If G has property RZ_s for all $s \in \mathbb{N}$, we say that G is *product separable*.

Thus RZ_1 means that the group is LERF and RZ_2 is equivalent to double coset separability. The definition of RZ_s is due to Coulbois (2001); he named it after Ribes and Zalesskii, who proved in Ribes and Zalesskii (1993) that free groups are product separable, confirming a conjecture of Pin and Reutenauer (1991). Product separability was first considered in its connection to Rhodes' type II conjecture from semigroup theory (see Pin and Reutenauer (1991) and Pin (1989) for background). The property was further used to obtain a language-theoretic extension of Hrushovski's theorem on extending partial automorphisms of graphs by Herwig and Lascar (2000).

Let us recount what is known about separability of products in groups. Polycyclic groups are known to be double coset separable, though the integral Heisenberg group $\mathrm{Heis}_3(\mathbb{Z})$ (which is finitely generated nilpotent of class 2) contains a triple coset that is not separable (Lennox and Wilson (1979)). Double coset separability of free groups was first proved by Gitik and Rips (1995). Shortly after, Niblo (1992) came up with a new criterion for separability of double cosets and applied it to show that finitely generated Fuchsian groups and fundamental groups of Seifert-fibred 3-manifolds are double coset separable.

Previously, few examples of groups were known to be product separable: free abelian groups, free groups (Ribes and Zalesskii (1993)), groups of the form $F \times \mathbb{Z}$, where F is free (You (1997)), and locally quasiconvex LERF hyperbolic groups (Minasyan (2006)) (e.g., surface groups). Additionally, the class of product separable groups is closed under taking subgroups, finite index supergroups and free products (Coulbois (2001)). In his thesis, Coulbois (2000) also showed that groups of the form $G *_C F$ are product separable, where G is product separable, F is free, and C is a maximal cyclic subgroup of F. However, this class is not closed under direct products. Indeed, the direct product of non-abelian free groups is not even LERF (Allenby and Gregorac (1973)).

Generalising the result of Ribes and Zalesskii (1993), it was proven by Minasyan (2006) that the product of finitely many quasiconvex subgroups is separable in a QCERF word hyperbolic group. Moreover, Coulbois (2001) showed that, for every $s \in \mathbb{N}$, free products of groups with property RZ_s also have property RZ_s. Taken together, these facts motivate the following theorem.

Theorem 1.10. *Let G be a finitely generated group hyperbolic relative to a finite collection of subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$, and let $s \in \mathbb{N}$. Suppose that G is QCERF and H_ν has property RZ_s for each $\nu \in \mathcal{N}$. If $Q_1, \dots, Q_s \leqslant G$ are finitely generated relatively quasiconvex subgroups of G, then the product $Q_1 \dots Q_s$ is separable in G.*

The hypotheses in the above theorem are minimal, as parabolic subgroups are relatively quasiconvex. We note that separability of products of full relatively quasiconvex subgroups in a QCERF relatively hyperbolic group was proved by McClellan (2019). Using Theorem 1.10 we are able to expand the class of groups known to be product separable.

Theorem 1.11. *The following groups are product separable:*

- *limit groups;*
- *finitely generated Kleinian groups;*
- *balanced fundamental groups of graphs of free groups with cyclic edge groups.*

Recall that a group G is a *limit group* if it is finitely generated and fully residually free (i.e. for each finite subset $U \subseteq G$ there is a free group F and a homomorphism $\varphi \colon G \to F$ that is injective when restricted to U). Limit groups naturally arise in the study of algebraic geometry over groups, and played an important role in the solutions of Tarski's problems on the first order theory of free groups by Sela (2006) and Kharlampovich and Myasnikov (2006). Wilton (2008) proved they are LERF.

Kleinian groups are discrete subgroups of the isometry group of hyperbolic 3-space, which is isomorphic to $\mathrm{PSL}(2, \mathbb{C})$. Kleinian groups play a central role in hyperbolic geometry. Agol (2013) proved that finitely generated Kleinian groups are LERF.

Following Wise, we say that a group is *balanced* if for every infinite order element $g \in G$, g^n and g^m are conjugate only when $n = \pm m$. Wise (2000) proved that the fundamental group of a finite graph of free groups with cyclic edge groups is LERF if and only if it is balanced if and only if it does not contain any subgroups of the form $\langle a, t \mid ta^m t^{-1} = a^n \rangle$ with $n \neq \pm m$. The subgroups in this latter condition are exactly the *non-Euclidean Baumslag-Solitar groups*, which are in a sense the obvious obstructions to separability of subgroups in this context (as the cyclic subgroup $\langle a \rangle$ is not separable).

Parts of this thesis are based on joint work with Ashot Minasyan, and as such many sections include jointly written and edited material. Sections 3.3, 4.2–4.5, and 5.7 in particular contain material to which his contribution was large.

Chapter 2

Preliminaries

2.1 Basic notions

By a generating set X of G we will mean a set X together with a map $X \to G$ such that the image of X under this map generates G. The combinatorial Cayley graph $\Gamma(G, X)$ is the labelled directed graph whose vertex set is G, with an edge from an element g to an element h if $g^{-1}h \in X$.

We will identify the combinatorial Cayley graph with its geometric realisation. The latter is a geodesic metric space, though not necessarily uniquely geodesic. Thus, given $x, y \in \Gamma(G, X)$ there will usually be a choice for geodesic $[x, y]$, which will either be specified or will be clear from the context (e.g., if x and y already belong to some geodesic path under discussion, then $[x, y]$ will be chosen as the subpath of that path). Note that the metric d_X is proper if the generating set X is finite. In this thesis we work with metrics associated to both finite and infinite generating sets of groups, which may fail to be proper.

By a *combinatorial path* in a graph Γ, we will mean a sequence of edges $e_1, \ldots, e_n \in E\Gamma$ such that $(e_i)_+ = (e_{i+1})_-$ for each $i = 1, \ldots, n-1$. If $\gamma_1, \ldots, \gamma_n$ are combinatorial paths with $(\gamma_i)_+ = (\gamma_{i+1})_-$, for each $i \in \{1, \ldots, n-1\}$, we will denote their concatenation by $\gamma_1 \ldots \gamma_n$.

The following general fact will be used quite often.

Lemma 2.1. *Let G be a group and suppose that X is a finite generating set for G. If $A, B \leqslant G$ are subgroups of G then for every $K \geq 0$ there is a constant $K' = K'(A, B, K) \geq 0$ such that for any $x \in G$ we have*

$$N_X(xA, K) \cap N_X(xB, K) \subseteq N_X(x(A \cap B), K').$$

Proof. After applying the left translation by x^{-1}, which preserves the metric d_X, we can assume that $x = 1$. Now the statement follows, for example, from (Hruska, 2010, Proposition 9.4). □

We will also make use of the following elementary fact.

Lemma 2.2. *Let G be an infinite group and let $H, K \leqslant G$ be infinite subgroups. If all but finitely many elements of H are contained in K, then $H \subseteq K$.*

Proof. Suppose that $H \setminus K$ is finite, so that its complement (in H) $H \cap K$ is infinite. Let $g \in H \setminus K$. As $H \setminus K$ is finite and $H \cap K$ is infinite, there is some $h \in H \cap K$ such that $hg \notin H \setminus K$. That is to say, $hg \in H \cap K$. It follows that $g = (h^{-1})(hg) \in H \cap K$, a contradiction. Thus $H \setminus K$ must be empty and $H \subseteq K$ as required. □

2.1.1 Quasigeodesic paths

In this subsection we assume that Γ is a graph equipped with the standard path length metric $\mathrm{d}(\cdot, \cdot)$ giving edges unit length.

Definition 2.3 (Quasigeodesic)**.** Let $\lambda \geq 1$ and $c \geq 0$ and let p be an edge path in Γ. Recall that p is said to be (λ, c)*-quasigeodesic* if for every combinatorial subpath q of p we have

$$\ell(q) \leq \lambda\, \mathrm{d}(q_-, q_+) + c.$$

Note that in the literature, quasigeodesic paths may be defined with a lower bound on length as well as an upper bound. For us, all paths will be assumed continuous so the lower bound holds trivially. We will see in the next subsection that quasigeodesic paths are particularly well-behaved in hyperbolic spaces. First let us collect some general facts. In the following lemma we show that if we append short paths to the start and end of a quasigeodesic path, the result is quasigeodesic with only slightly worse constants.

Lemma 2.4. *Suppose that $s = rpt$ is a concatenation of three combinatorial paths r, p and t in Γ such that $\ell(r) \leq D$ and $\ell(t) \leq D$, for some $D \geq 0$, and p is (λ, c)-quasigeodesic, for some $\lambda \geq 1$ and $c \geq 0$. Then the path s is (λ, c')-quasigeodesic, where $c' = c + 2(\lambda + 1)D$.*

Proof. Consider an arbitrary combinatorial subpath q of s. We need to show that

$$\ell(q) \leq \lambda\, \mathrm{d}(q_-, q_+) + c + 2(\lambda + 1)D. \tag{2.1}$$

If q is contained in r or in t then the desired inequality follows from the assumptions that $\ell(r) \leq D$ and $\ell(t) \leq D$. Therefore we can further suppose that q_- is a vertex of rp

and q_+ is a vertex of pt. The bounds on the lengths of r and t imply that there is a combinatorial subpath a of p such that there are at most D edges of s between q_- and a_- and between a_+ and q_+. Thus $\mathrm{d}(q_-, a_-) \leq D$, $\mathrm{d}(q_+, a_+) \leq D$ and $\ell(q) \leq \ell(a) + 2D$

The assumption that p is (λ, c)-quasigeodesic implies that

$$\ell(q) \leq \ell(a) + 2D \leq \lambda \, \mathrm{d}(a_-, a_+) + c + 2D. \tag{2.2}$$

The triangle inequality gives $\mathrm{d}(a_-, a_+) \leq \mathrm{d}(q_-, q_+) + 2D$, which, combined with (2.2), shows that (2.1) holds, as required. □

We now show that the quasigeodesicity constants of a path obtained by replacing every edge of a quasigeodesic path with short paths are well-controlled.

Lemma 2.5. *Let $\lambda \geq 1, c \geq 0$ and $K \in \mathbb{N}$. Suppose that p is a combinatorial path in Γ and let p' be a path obtained by replacing some edges of p with combinatorial paths of length at most K. If p is (λ, c)-quasigeodesic then p' is $(K\lambda, 2K^2\lambda + Kc + 2K)$-quasigeodesic.*

Proof. Let q be any combinatorial subpath of p' and write $q_- = x$ and $q_+ = y$. We need to show that

$$\ell(q) \leq K\lambda \, \mathrm{d}(x, y) + 2K^2\lambda + Kc + 2K. \tag{2.3}$$

If q does not contain any vertices of p then $\ell(q) \leq K$ and (2.3) holds. Otherwise, let z and w be the first and the last vertices of q that lie on p respectively, and let r be the subpath of p starting at z and ending at w. The assumptions imply that $\mathrm{d}(x, z) \leq K$, $\mathrm{d}(y, w) \leq K$ and

$$\ell(q) \leq K\ell(r) + 2K. \tag{2.4}$$

Using the quasigeodesicity of p and the triangle inequality, we obtain

$$\ell(r) \leq \lambda \, \mathrm{d}(z, w) + c \leq \lambda \, \mathrm{d}(x, y) + 2K\lambda + c,$$

which, combined with (2.4), gives (2.3). □

2.1.2 Hyperbolic metric spaces

In this subsection we take (Γ, d) be a geodesic metric space.

Definition 2.6 (Gromov product). Let $x, y, z \in \Gamma$ be points. The *Gromov product* of x and y with respect to z is

$$\langle x, y \rangle_z = \frac{1}{2}\Big(\mathrm{d}(x, z) + \mathrm{d}(y, z) - \mathrm{d}(x, y) \Big).$$

It is easy to see that the Gromov products satisfy the following equations:

$$\mathrm{d}(x,y) = \langle y,z\rangle_x + \langle x,z\rangle_y, \ \mathrm{d}(y,z) = \langle x,z\rangle_y + \langle x,y\rangle_z \text{ and } \mathrm{d}(z,x) = \langle x,y\rangle_z + \langle y,z\rangle_x.$$

The following elementary property of Gromov products is an immediate consequence of the triangle inequality.

Remark 2.7. Suppose that x, y, z are points in Γ, u is a point on any geodesic segment from x to z, and v is a point on any geodesic segment from z to y. A straightforward application of the triangle inequality tells us that $\langle u,v\rangle_z \leq \langle x,y\rangle_z$.

Definition 2.8 (δ-thin triangle)**.** Let Δ be a geodesic triangle in Γ with vertices x, y, and z, and let $\delta \geq 0$. Denote by T_Δ the (possibly degenerate) tripod with edges of length $\langle x,y\rangle_z$, $\langle y,z\rangle_x$, and $\langle z,x\rangle_y$ respectively. There is a map from $\{x,y,z\}$ to the extremal vertices of T_Δ, which extends uniquely to a map $\phi\colon \Delta \to T_\Delta$, whose restriction to each side of Δ is an isometry. If the diameter in Γ of $\phi^{-1}(\{t\})$ is at most δ, for all $t \in T_\Delta$, then Δ is said to be *δ-thin*.

Definition 2.9 (Hyperbolic space)**.** The space Γ is said to be a *hyperbolic metric space* if there is a constant $\delta \geq 0$ such that every geodesic triangle in Γ is δ-thin.

The above definition of δ-hyperbolicity is not the most commonly used in the literature, though it is well-known to be equivalent to other definitions after possibly increasing δ: see, for example, (Bridson and Haefliger, 1999, III.H.1.17). For technical reasons we will always assume that δ is chosen to be sufficiently large so that all the definitions in this reference are satisfied.

In the remainder of this subsection we assume that Γ is a graph which, equipped with the standard path length metric $\mathrm{d}(\cdot,\cdot)$, is a δ-hyperbolic space for some $\delta \geq 0$.

Definition 2.10 (Broken line)**.** A *broken line* in Γ is a path p which comes with a fixed decomposition as a concatenation of combinatorial geodesic paths $p_1, \ldots, p_n$ in Γ, so that $p = p_1 p_2 \ldots p_n$. The paths $p_1, \ldots, p_n$ will be called the *segments* of the broken line p, and the vertices $p_- = (p_1)_-, (p_1)_+ = (p_2)_-, \ldots, (p_{n-1})_+ = (p_n)_-$ and $(p_{n+1})_+ = p_+$ will be called the *nodes* of p.

The following statement is a special case of Lemma 4.2 from Minasyan (2005a), applied to the situation when each p_i is geodesic (so, in the notation of that lemma, we can take $\overline{\lambda} = 1$, $\overline{c} = 0$ and $\nu = \delta$). Note that due to a slightly different definition of quasigeodesicity used in Minasyan (2005a), a (λ, c)-quasigeodesic in the sense of Minasyan (2005a) is $(1/\lambda, c/\lambda)$-quasigeodesic in the sense of Definition 2.3 above, and vice-versa. The statement gives sufficient conditions for a broken line to be quasigeodesic.

Lemma 2.11. *Let c_0, c_1 and c_2 be constants such that $c_0 \geq 14\delta$, $c_1 = 12(c_0 + \delta) + 1$ and $c_2 = 10(\delta + c_1)$. Suppose that $p = p_1 \ldots p_n$ is a broken line in Γ, where p_i is a geodesic with*

$(p_i)_- = x_{i-1}$, $(p_i)_+ = x_i$, $i = 1, \dots, n$. If $\mathrm{d}(x_{i-1}, x_i) \geq c_1$ for $i = 1, \dots, n$, and $\langle x_{i-1}, x_{i+1}\rangle_{x_i} \leq c_0$ for each $i = 1, \dots, n-1$, then the path p is $(4, c_2)$-quasigeodesic.

We will need an extension of the above lemma which allows the first and the last geodesic segments p_1 and p_n to be short.

Lemma 2.12. *For any constant c_0, satisfying $c_0 \geq 14\delta$, let $c_1 = c_1(c_0) = 12(c_0 + \delta) + 1$ and $c_3 = c_3(c_0) = 10(\delta + 2c_1)$.*
Suppose that $p = p_1 \dots p_n$ is a broken line in Γ, where p_i is a geodesic with $(p_i)_- = x_{i-1}$, $(p_i)_+ = x_i$, $i = 1, \dots, n$. If $\mathrm{d}(x_{i-1}, x_i) \geq c_1$ for $i = 2, \dots, n-1$, and $\langle x_{i-1}, x_{i+1}\rangle_{x_i} \leq c_0$ for each $i = 1, \dots, n-1$, then the path p is $(4, c_3)$-quasigeodesic.

Proof. This follows easily by combining Lemma 2.11 with Lemma 2.4. Indeed, there are four possibilities depending on whether or not $\mathrm{d}(x_0, x_1) \geq c_1$ and $\mathrm{d}(x_{n-1}, x_n) \geq c_1$. Since these cases are similar, we concentrate on the situation when $\mathrm{d}(x_0, x_1) < c_1$ and $\mathrm{d}(x_{n-1}, x_n) \geq c_1$. Then the path $q = p_2 p_3 \dots p_n$ is $(4, c_2)$-quasigeodesic by Lemma 2.11, where $c_2 = 10(\delta + c_1)$. Since $\ell(p_1) = \mathrm{d}(x_0, x_1) < c_1$, we can apply Lemma 2.4 to deduce that the path $p = p_1 \dots p_n = p_1 q$ is $(4, c_3)$-quasigeodesic, where $c_3 = c_2 + 10c_1 = 10(\delta + 2c_1)$ as required. □

2.2 Relatively hyperbolic groups

In this section we will define relatively hyperbolic groups and collect various properties that will be used throughout this work.

2.2.1 Definitions and basic properties

Hyperbolic groups are characterised by having *linear isoperimetric inequalities*: given a combinatorial loop in a Cayley complex corresponding to a finite presentation of such a group, the number of discs needed to fill that loop is bounded by a linear function of the length of the loop. This property can also be characterised algebraically in terms of the minimum number of relators needed to express a word representing the identity is a presentation.

We will define relatively hyperbolic groups following the approach of Osin, which builds on a notion of relative isoperimetric functions (for full details, see Osin (2006b)).

Definition 2.13 (Relative generating set, relative presentation)**.** Let G be a group, $X \subseteq G$ a subset and $\{H_\nu \mid \nu \in \mathcal{N}\}$ a collection of subgroups of G. The group G is said to be *generated by X relative to $\{H_\nu \mid \nu \in \mathcal{N}\}$* if it is generated by $X \cup \mathcal{H}$, where

$\mathcal{H} = \bigsqcup_{\nu \in \mathcal{N}} (H_\nu \setminus \{1\})$ (with the obvious map $X \cup \mathcal{H} \to G$). If this is the case, then there is a surjection

$$F = F(X) * (*_{\nu \in \mathcal{N}} H_\nu) \to G,$$

where $F(X)$ denotes the free group on X. Suppose that the kernel of this map is the normal closure of a subset $\mathcal{R} \subseteq F$. Then G can equipped with the *relative presentation*

$$\langle X, H_\nu, \nu \in \mathcal{N} \mid \mathcal{R} \rangle. \tag{2.5}$$

If X is a finite set, then G is said to be *finitely generated relative to* $\{H_\nu \mid \nu \in \mathcal{N}\}$. If $\mathcal{R}$ is also finite, G is said to be *finitely presented relative to* $\{H_\nu \mid \nu \in \mathcal{N}\}$ and the presentation above is a *finite relative presentation*.

With the above notation, we call the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ the *relative Cayley graph* of G with respect to X and $\{H_\nu \mid \nu \in \mathcal{N}\}$. Note that when X is itself a generating set of G, we have that $\mathrm{d}_{X \cup \mathcal{H}}(g, h) \leq \mathrm{d}_X(g, h)$, for all $g, h \in G$.

Definition 2.14 (Relative Dehn function)**.** Suppose that G has a finite relative presentation (2.5) with respect to a collection of subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$. If w is a word in the free group $F(X \cup \mathcal{H})$, representing the identity in G, then it is equal in F to a product of conjugates

$$w \stackrel{F}{=} \prod_{i=1}^{n} a_i r_i a_i^{-1},$$

where $a_i \in F$ and $r_i \in \mathcal{R}$, for each i. The *relative area* of the word w with respect to the relative presentation, $Area^{rel}(w)$, is the least number n among products of conjugates as above that are equal to w in F.

A *relative isoperimetric function* of the above presentation is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that $Area^{rel}(w)$ is at most $f(|w|)$, for every freely reduced word w in $F(X \cup \mathcal{H})$ representing the identity in G. If an isoperimetric function exists for the presentation, the smallest such function is called the *relative Dehn function* of the presentation.

Definition 2.15 (Relatively hyperbolic group)**.** Let G be a group and let $\{H_\nu \mid \nu \in \mathcal{N}\}$ be a collection of subgroups of G. If G admits a finite relative presentation with respect to this collection of subgroups which has a well-defined linear relative Dehn function, it is called *hyperbolic relative to* $\{H_\nu \mid \nu \in \mathcal{N}\}$. When it is clear what the relevant collection of subgroups is, we refer to G simply as a *relatively hyperbolic group*. The groups $\{H_\nu \mid \nu \in \mathcal{N}\}$ are called the *peripheral subgroups* of the relatively hyperbolic group G, and their conjugates in G are called *maximal parabolic subgroups*. Any subgroup of a maximal parabolic subgroup is said to be *parabolic*.

Lemma 2.16 (Osin (2006b), Corollary 2.54)**.** *Suppose that G is a group generated by a finite set X and hyperbolic relative to a collection of subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$, and let $\mathcal{H} = \bigsqcup_{\nu \in \mathcal{N}} (H_\nu \setminus \{1\})$. Then the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is δ-hyperbolic for some $\delta \geq 0$.*

Remark 2.17. If G is a finitely generated group that is finitely presented relative to a collection of nontrivial subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$, then this collection is necessarily finite (Osin, 2006b, Corollary 2.48).

Definition 2.18 (Path components)**.** Let p be a combinatorial path in $\Gamma(G, X \cup \mathcal{H})$. A non-trivial combinatorial subpath of p whose label consists entirely of elements of $H_\nu \setminus \{1\}$, for some $\nu \in \mathcal{N}$, is called an H_ν*-subpath* of p.
An H_ν-subpath is called an H_ν*-component* if it is not contained in any strictly longer H_ν-subpath. We will call a subpath of p an $\mathcal{H}$-subpath (respectively, an $\mathcal{H}$*-component*) if it is an H_ν-subpath (respectively, an H_ν-component), for some $\nu \in \mathcal{N}$.

Definition 2.19 (Connected and isolated components)**.** Let p and q be edge paths in $\Gamma(G, X \cup \mathcal{H})$ and suppose that s and t are H_ν-subpaths of p and q respectively, for some $\nu \in \mathcal{N}$. We say that s and t are *connected* if s_- and t_- belong to the same left coset of H_ν in G. The latter means that for all vertices u of s and v of t either $u = v$ or there is an edge e in $\Gamma(G, X \cup \mathcal{H})$ with $\mathrm{Lab}(e) \in H_\nu \setminus \{1\}$ and $e_- = u, e_+ = v$.
If s is an H_ν-component of a path p and s is not connected to any other H_ν-component of p then we say that s is *isolated* in p.

Definition 2.20 (Phase vertex)**.** A vertex v of a combinatorial path p in $\Gamma(G, X \cup \mathcal{H})$ is called *non-phase* if it is an interior vertex of an $\mathcal{H}$-component of p (that is, if it lies in an $\mathcal{H}$-component which it is not an endpoint of). Otherwise v is called *phase*.

Definition 2.21 (Backtracking)**.** If all $\mathcal{H}$-components of a combinatorial path p are isolated, then p is said to be *without backtracking*. Otherwise we say that p *has backtracking*.

Remark 2.22. If p is a geodesic edge path in $\Gamma(G, X \cup \mathcal{H})$ then every $\mathcal{H}$-component of p will consist of a single edge, labelled by an element from $\mathcal{H}$. Therefore every vertex of p will be phase. Moreover, it is easy to see that p will be without backtracking.

The following terminology is less standard than the above, but will be useful to us.

Definition 2.23 (Consecutive, adjacent and multiple backtracking)**.** Let $p = p_1 \dots p_n$ be a broken line in $\Gamma(G, X \cup \mathcal{H})$. Suppose that for some i, j, with $1 \leq i < j \leq n$, and $\nu \in \mathcal{N}$ there exist pairwise connected H_ν-components $h_i, h_{i+1}, \dots, h_j$ of the paths $p_i, p_{i+1}, \dots, p_j$, respectively. Then we will say that p has *consecutive backtracking* along the components $h_i, \dots, h_j$ of $p_i, \dots, p_j$. Moreover, if $j = i + 1$, we will call it an instance of *adjacent backtracking*, while if $j > i + 1$ will use the term *multiple backtracking*.

Quasigeodesic paths in Cayley graphs of hyperbolic groups are generally very well-behaved. We would like to say the same about quasigeodesics in the relative Cayley graphs of relatively hyperbolic groups, but it is a priori possible that such paths behave poorly due to the presence of $\mathcal{H}$-components. The following lemma is an essential tool in controlling $\mathcal{H}$-components, which moderates instances of such poor behaviour.

Lemma 2.24. *Let G be a group hyperbolic relative to subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$. Then there is a finite subset Ω of G and a constant $M \geq 1$ such that if $h_1, \ldots, h_n$ are isolated $\mathcal{H}$-components of a cycle q in $\Gamma(G, X \cup \mathcal{H})$, then $\widetilde{h_i} \in \langle\Omega\rangle$ for each $i = 1, \ldots, n$ and*

$$\sum_{i=1}^{n} |h_i|_\Omega \leq M\ell(q).$$

Proof. Since G is hyperbolic relative to $\{H_\nu \mid \nu \in \mathcal{N}\}$, it has a relative presentation satisfying a well-defined linear relative isoperimetric inequality. The statement then follows by applying (Osin, 2006b, Lemma 2.27). □

Remarkably, when the cycle in consideration is a polygon with geodesic sides, the bound on lengths of $\mathcal{H}$-components can be taken to depend only on the number of sides.

Proposition 2.25 (Osin (2007), Proposition 3.2)**.** *There is a constant $L \geq 0$ such that if P is a geodesic n-gon in $\Gamma(G, X \cup \mathcal{H})$ and some side p is an isolated $\mathcal{H}$-component of P then $|p|_\Omega \leq Ln$.*

A basic consequence of having a well-defined relative Dehn function is that the intersection of two distinct maximal parabolic subgroups is finite (Osin, 2006b, Theorem 1.4). We extend this result by showing that there are finitely many conjugacy classes of elements belonging to such intersections when the group is relatively hyperbolic.

Proposition 2.26. *Let $a, b \in G$ and $\lambda, \nu \in \mathcal{N}$ be such that $aH_\lambda a^{-1} \neq bH_\nu b^{-1}$. Then each element of $aH_\lambda a^{-1} \cap bH_\nu b^{-1}$ is conjugate to an element $h \in G$ with $|h|_\Omega \leq 4L$.*

Proof. Conjugating if necessary, we may assume that $a = 1$. Further, suppose that $b \in G$ is such that $|b|_{X \cup H}$ is minimal among elements in the coset $H_\lambda b$. Now let $g \in H_\lambda \cap bH_\nu b^{-1}$ be a nontrivial element, and let $h \in H_\nu$ be such that $g = bhb^{-1}$.

Let γ be a geodesic in $\Gamma(G, X \cup \mathcal{H})$ with $\gamma_- = 1$ and $\gamma_+ = b$. Further, let u be the H_λ-edge of $\Gamma(G, X \cup \mathcal{H})$ with $u_- = 1$ and $\widetilde{u} = g$, and let v be the H_ν-edge of $\Gamma(G, X \cup \mathcal{H})$ with $v_- = b$ and $\widetilde{v} = h$. Note that $v_+ = bh = gb$ by definition, so that $\gamma' = g \cdot \gamma$ (i.e. the translate of γ by g) has endpoints u_+ and v_+. Now consider the geodesic quadrilateral Q with sides u, γ, v, and γ'. We will show that u is isolated in Q.

If u and v are connected, then we must have $\lambda = \nu$ and both $u_- = 1$ and $v_- = b$ lie in the same H_λ-coset. However, this means that $H_\lambda = bH_\nu b^{-1}$, contrary to the assumption. Therefore u must be connected to an H_λ-component s of either γ or γ'. We suppose, without loss of generality, that s lies in γ. Since u and s are connected and $\gamma_- = u_- = 1$, the endpoints of s satisfy $d_{X\cup\mathcal{H}}(s_-, \gamma_-) \leq 1$ and $d_{X\cup\mathcal{H}}(s_+, \gamma_-) \leq 1$.

Therefore s must be the initial edge of γ, for otherwise the geodesicty of γ is contradicted. But then $\widetilde{s}^{-1}b \in H_\lambda b$ and

$$|\widetilde{s}^{-1}b|_{X\cup\mathcal{H}} = d_{X\cup\mathcal{H}}(\widetilde{s}, b) = d_{X\cup\mathcal{H}}(s_+, \gamma_+) < |\gamma|_{X\cup\mathcal{H}} = |b|_{X\cup\mathcal{H}}$$

contradicting the minimality of b.

As s cannot contain v or be an H_λ-component of γ or γ', u is isolated in Q. Proposition 2.25 then tells us that $|g|_\Omega = |u|_\Omega \leq 4L$, as required. □

As the set Ω is finite, there are only finitely many elements of G whose length with respect to Ω is less than any given number. The result then immediate.

Corollary 2.27. *There are finitely many conjugacy classes of elements in G belonging to more than one maximal parabolic subgroup.*

2.2.2 Geodesics and quasigeodesics in relatively hyperbolic groups

Convention 2.1. Unless explicitly stated otherwise, for the remainder of this thesis we will assume that G is a finitely generated group with finite generating set X. We will suppose that G is hyperbolic relative to infinite subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$, and that X is also a finite relative generating set corresponding to a finite presentation relative to $\{H_\nu \mid \nu \in \mathcal{N}\}$ which satisfies a well-defined linear relative isoperimetric inequality. Moreover, in this case we may assume without loss of generality that X contains the set Ω obtained from Lemma 2.24, so that $|g|_X \leq |g|_\Omega$ for all $g \in G$ (for example, see (Osin, 2006b, §3.1)).

The following is a basic observation about the lengths of paths in the relative Cayley graph whose $\mathcal{H}$-components are uniformly short.

Lemma 2.28. *Let p be a path in $\Gamma(G, X \cup \mathcal{H})$ and suppose there is a constant $\Theta \geq 1$ that for any $\mathcal{H}$-component h of p, we have $|h|_X \leq \Theta$. Then $|p|_X \leq \Theta\ell(p)$.*

Proof. We can write p as a concatenation $p = a_0h_1a_1 \ldots a_{n-1}h_na_n$, where $h_1, \ldots, h_n$ are the $\mathcal{H}$-components of p and $a_0, \ldots, a_n$ are subpaths of p all whose edges are labelled by elements of $X^{\pm 1}$.

It follows from the triangle inequality that

$$|p|_X = \mathrm{d}_X(p_-, p_+) \leq \sum_{i=0}^{n} \mathrm{d}_X((a_i)_-, (a_i)_+) + \sum_{i=1}^{n} \mathrm{d}_X((h_i)_-, (h_i)_+).$$

Since each edge of a_i is labelled by an element of $X^{\pm 1}$, we have that $\mathrm{d}_X((a_i)_-, (a_i)_+) \leq \ell(a_i)$, for all $i = 0, \ldots, n$. Moreover for each $i = 1, \ldots, n$, $\mathrm{d}_X((h_i)_-, (h_i)_+) = |h_i|_X \leq \Theta\ell(h_i)$ by the hypothesis of the lemma, as $\ell(h_i) \geq 1$.

Combining the above three inequalities with the fact that $\Theta \geq 1$, we obtain

$$|p|_X \leq \sum_{i=0}^{n} \ell(a_i) + \sum_{i=1}^{n} \Theta\ell(h_i) \leq \Theta\Big(\sum_{i=0}^{n} \ell(a_i) + \sum_{i=1}^{n} \ell(h_i)\Big) = \Theta\ell(p). \qquad \square$$

Lemma 2.29. *For any $\lambda \geq 1$, $c \geq 0$ and $A \geq 0$ there is a constant $\eta = \eta(\lambda, c, A) \geq 0$ such that the following is true.*
Suppose that p is a (λ, c)-quasigeodesic path in $\Gamma(G, X \cup \mathcal{H})$ possessing an isolated $\mathcal{H}$-component h such that $|h|_X \geq \eta$. Then $|p|_X \geq A$.

Proof. Let $M \geq 1$ be the constant from Lemma 2.24, and set

$$\eta = M(1+\lambda)A + Mc. \tag{2.6}$$

Let q be a path in $\Gamma(G, X \cup \mathcal{H})$, labelled by a word over $X^{\pm 1}$, with endpoints $q_- = p_-$ and $q_+ = p_+$, such that $\ell(q) = |p|_X$.

Consider the cycle $r = pq^{-1}$ in $\Gamma(G, X \cup \mathcal{H})$, formed by concatenating p and the inverse of q. By the quasigeodesicity of p, $\ell(p) \leq \lambda|p|_{X\cup\mathcal{H}} + c \leq \lambda|p|_X + c$. Now $\ell(r) = \ell(p) + \ell(q)$, therefore

$$\ell(r) \leq (1+\lambda)|p|_X + c. \tag{2.7}$$

Since h is isolated in p it must also be an isolated $\mathcal{H}$-component of the cycle r (because all edges of q are labelled by letters from $X^{\pm 1}$). Hence $|h|_X \leq M\ell(r)$ by Lemma 2.24, so (2.7) implies that

$$|p|_X \geq \frac{1}{1+\lambda}(\ell(r) - c) \geq \frac{1}{M(1+\lambda)}(|h|_X - Mc). \tag{2.8}$$

Combining the above inequality with (2.6) and the assumption that $|h|_X \geq \eta$, we obtain the desired bound $|p|_X \geq A$. $\square$

Lemma 2.30. *There is a constant $\xi \geq 0$ such that if v is a vertex of a geodesic p in $\Gamma(G, X \cup \mathcal{H})$, then $\mathrm{d}_X(p_-, v) \leq \xi|p|_X^2$.*

Proof. For $\lambda \geq 0$ and $A \geq 0$, the constant $\eta(\lambda, 0, A)$ of Lemma 2.29 is a multiple of A that depends only on λ: see (2.6). Thus there is $\xi \geq 0$ such that $\eta(1, 0, |p|_X) = \xi|p|_X$. Now an application of Lemma 2.29 tells us that if h is an $\mathcal{H}$-component of p, then $|h|_X \leq \eta(1, 0, |p|_X) = \xi|p|_X$. Finally, noting that there are at most $|p|_{X\cup\mathcal{H}} \leq |p|_X$ edges of p between p_- and v gives that $\mathrm{d}_X(p_-, v) \leq \xi|p|_X^2$ as required. $\square$

Lemma 2.31. *Let $L \geq 0$ be the constant provided by Proposition 2.25. If p_1 and p_2 are geodesic paths in $\Gamma(G, X \cup \mathcal{H})$ with $(p_1)_+ = (p_2)_-$, and s and t are connected H_ν-components of p_1, p_2 respectively, for some $\nu \in \mathcal{N}$, then $\mathrm{d}_X(s_+, t_-) \leq 3L$.*

Proof. Since the component s of p_1 is connected to the component t of p_2, we know that $h = (s_+)^{-1} t_- \in H_\nu$. If $h = 1$ then $s_+ = t_-$ and there is nothing to prove, otherwise s_+ and t_- are endpoints of an edge e labelled by h in $\Gamma(G, X \cup \mathcal{H})$.

Consider the geodesic triangle Δ with vertices s_+, $(p_1)_+$ and t_-, where the sides $[s_+, (p_1)_+]$ and $[(p_1)_+, t_-]$ are chosen to be subpaths of p_1 and p_2 respectively, and the side $[s_+, t_-]$ is the edge e.

If $v \in [s_+, (p_1)_+]$ is a vertex belonging to the left coset $s_+ H_\nu$ then $\mathrm{d}_{X \cup \mathcal{H}}(s_-, v) = 1$ and $s_+ \in [s_-, v]$ in p_1. Since $\mathrm{d}_{X \cup \mathcal{H}}(s_-, s_+) = 1$ and p_1 is geodesic, we can conclude that $v = s_+$. Similarly, the only vertex of $[(p_1)_+, t_-]$ which belongs to the left coset $t_- H_\nu = s_+ H_\nu$ is t_-. It follows that the edge e is an isolated H_ν-component of Δ. Hence $\mathrm{d}_X(s_+, t_-) \leq 3L$ by Proposition 2.25. □

Yet another characterisation of hyperbolicity in metric spaces goes as follows: a geodesic triangle Δ is said to be *δ-slim* if each of its sides is contained in a δ-neighbourhood of the other two, and a space is δ-hyperbolic if all geodesic triangles are δ-slim. There is a strong analogue for this property in the setting of relative Cayley graphs.

Proposition 2.32 (Osin (2006b), Theorem 3.26)**.** *Let Δ be a combinatorial geodesic triangle in $\Gamma(G, X \cup \mathcal{H})$ with sides p, q and r. There is a constant $\sigma \in \mathbb{N}_0$ such that for any vertex $u \in p$, there is a vertex $v \in q \cup r$ with $\mathrm{d}_X(u, v) \leq \sigma$.*

Likewise, the *Morse property* of quasigeodesics in hyperbolic spaces, which states that quasigeodesics whose endpoints are close stay in uniform neighbourhoods of one another, has an analogue for relative Cayley graphs.

Definition 2.33 (k-similar paths)**.** Let p and q be paths in $\Gamma(G, X \cup \mathcal{H})$, and let $k \geq 0$. The paths p and q are said to be *k-similar* if $\mathrm{d}_X(p_-, q_-) \leq k$ and $\mathrm{d}_X(p_+, q_+) \leq k$.

Proposition 2.34 (Osin (2006b), Proposition 3.15, Lemma 3.21 and Theorem 3.23)**.** *For any $\lambda \geq 1$, $c, k \geq 0$ there is a constant $\kappa = \kappa(\lambda, c, k) \geq 0$ such that if p and q are k-similar (λ, c)-quasigeodesics in $\Gamma(G, X \cup \mathcal{H})$ and p is without backtracking, then*

1. *for every phase vertex u of p, there is a phase vertex v of q with $\mathrm{d}_X(u, v) \leq \kappa$;*
2. *every $\mathcal{H}$-component s of p, with $|s|_X \geq \kappa$, is connected to an $\mathcal{H}$-component of q.*

Moreover, if q is also without backtracking then

3. *if s and t are connected $\mathcal{H}$-components of p and q respectively, then we have $\mathrm{d}_X(s_-, t_-) \leq \kappa$ and $\mathrm{d}_X(s_+, t_+) \leq \kappa$*

One of the tools for proving Theorem 1.8 will be a result of Martínez-Pedroza (2009).

Proposition 2.35 (Martínez-Pedroza (2009), Proposition 3.1)**.** *There are constants $\zeta_0 \geq 0$ and $\lambda_0 \geq 1$ such that the following holds. If $q = r_0 s_1 \dots r_n s_{n+1}$ is a concatenation of geodesic paths $r_0, s_1, \dots, r_n, s_{n+1}$ in $\Gamma(G, X \cup \mathcal{H})$ such that*

1. *s_i is an $\mathcal{H}$-component of q, for each $i = 1, \dots, n+1$,*
2. *$|s_i|_X \geq \zeta_0$, for every $i = 1, \dots, n+1$,*
3. *s_i is not connected to s_{i+1}, for every $i = 1, \dots, n$,*

then q is $(\lambda_0, 0)$-quasigeodesic in $\Gamma(G, X \cup \mathcal{H})$ without backtracking.

We will actually need a slightly more general version of Proposition 2.35, as follows.

Proposition 2.36. *There exist constants $\lambda \geq 1$ and $c \geq 0$ such that for every $\rho \geq 0$ there is $\zeta_1 > 0$ such that the following holds. Suppose that $p = a_0 b_1 a_1 \dots b_n a_n$ is a concatenation of geodesic paths $a_0, b_1, \dots, b_n, a_n$ in $\Gamma(G, X \cup \mathcal{H})$ such that*

1. *b_i is an $\mathcal{H}$-subpath of p, for each $i = 1, \dots, n$,*
2. *$|b_i|_X \geq \zeta_1$, for each $i = 1, \dots, n$;*
3. *b_i is not connected to b_{i+1}, for every $i = 1, \dots, n-1$;*
4. *if b_i is connected to a component h of a_i or a_{i-1} then $|h|_X \leq \rho$, $i = 1, \dots, n$.*

Then p is a (λ, c)-quasigeodesic without backtracking.

Proof. The argument below employs the following trick: for each $i = 1, \dots, n$, we replace the $\mathcal{H}$-component of p containing b_i by a single edge s_i, and then embed the resulting path p' into a larger path q to which Proposition 2.35 can be applied. Since a subpath of a (λ, c)-quasigeodesic path without backtracking is again (λ, c)-quasigeodesic and without backtracking, this will complete the proof. In order to construct the path q we add an extra infinite peripheral subgroup Z by embedding G into a larger relatively hyperbolic group G'.

Let us consider the free product $G' = G * Z$, where $Z = \langle z \rangle$ is an infinite cyclic group. Since G is hyperbolic relative to the family $\{H_\nu \mid \nu \in \mathcal{N}\}$, the group G' is hyperbolic relative to the union $\{H_\nu \mid \nu \in \mathcal{N}\} \cup \{Z\}$ (this can be fairly easily deduced from the definition or from many existing combination theorems for relatively hyperbolic groups, e.g. (Osin, 2006a, Corollary 1.5)).

Note that G embeds in G' and G' is generated by the finite set $X' = X \sqcup \{z\}$. Let $\mathcal{H}' = \mathcal{H} \sqcup Z \setminus \{1\}$, so that the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is naturally a subgraph of the Cayley graph $\Gamma(G', X' \cup \mathcal{H}')$. Therefore we can think of p as a path in $\Gamma(G', X' \cup \mathcal{H}')$.

The normal form theorem for free products (Lyndon and Schupp, 1977, Theorem IV.1.2) implies that the embedding of G into G' is isometric with respect to both proper and relative metrics, more precisely

$$\mathrm{d}_X(g, h) = \mathrm{d}_{X'}(g, h) \text{ and } \mathrm{d}_{X \cup \mathcal{H}}(g, h) = \mathrm{d}_{X' \cup \mathcal{H}'}(g, h), \text{ for all } g, h \in G. \tag{2.9}$$

An alternative way to see this is to use the retraction $r : G' \to G$, such that $r(x) = x$ for all $x \in X$ and $r(z) = 1$. Then $r(X') = X \cup \{1\}$, $r(H_\nu) = H_\nu$, for all $\nu \in \mathcal{N}$, and $r(Z) = \{1\}$.

Let $\zeta_0 \geq 0$ and $\lambda_0 \geq 1$ be the constants provided by Proposition 2.35 applied to the group G', its finite generating set X' and its Cayley graph $\Gamma(G', X' \cup \mathcal{H}')$. Set $\zeta_1 = \zeta_0 + 2\rho + 1 > 0$.

For each $i = 1, \dots, n$, let t_i denote the H_{ν_i}-component of p containing the edge b_i, $\nu_i \in \mathcal{N}$. Note that $t_1, \dots, t_n$ are pairwise distinct by condition (3), in particular no two of them share a common edge. In view of Remark 2.22, for every $i = 1, \dots, n$ we can represent t_i as a concatenation $t_i = h_{i-1} b_i f_i$, where

- h_{i-1} is either the last edge and an H_{ν_i}-component of a_{i-1} if a_{i-1} ends with an H_{ν_i}-component, or h_{i-1} is the trivial path, consisting of the vertex $(a_{i-1})_+$, if a_{i-1} does not end with an H_{ν_i}-component;
- f_i is the first edge and an H_{ν_i}-component of a_i if a_i starts with an H_{ν_i}-component, or f_i is the trivial path, consisting of the vertex $(a_i)_-$, if a_i does not start with an H_{ν_i}-component.

Note that for each $i = 1, \dots, n$ we have $|h_{i-1}|_X \leq \rho$ and $|f_i|_X \leq \rho$, by condition (4). By (2) and the triangle inequality we get

$$|t_i|_X \geq |b_i|_X - 2\rho \geq \zeta_0 + 1, \text{ for } i = 1, \dots, n. \tag{2.10}$$

Therefore p decomposes as a concatenation

$$p = r_0 t_1 r_1 \dots t_n r_n,$$

where r_i is a subpath of a_i, $i = 0, \dots, n$, so that $a_0 = r_0 h_0$, $a_1 = f_1 r_1 h_1, \dots, a_n = f_n r_n$.

By (2.10) the endpoints of the H_{ν_i}-component t_i of p must be distinct, hence there is an edge s_i joining them in $\Gamma(G, X \cup \mathcal{H})$, such that $\mathrm{Lab}(s_i) \in H_{\nu_i} \setminus \{1\}$, $i = 1, \dots, n$. Now, (2.10) and (2.9) imply that

$$|s_i|_{X'} = |t_i|_{X'} = |t_i|_X \geq \zeta_0, \text{ for } i = 1, \dots, n.$$

Choose $k \in \mathbb{N}$ so that $|z^k|_{X'} \geq \zeta_0$ and let s_{n+1} be the edge in $\Gamma(G', X' \cup \mathcal{H}')$, starting at $p_+ = (r_n)_+$ and labelled by z^k. Observe that $|s_{n+1}|_{X'} = |z^k|_{X'} \geq \zeta_0$.

Consider the path q in $\Gamma(G', X' \cup \mathcal{H}')$, defined as the concatenation $q = r_0 s_1 \dots r_n s_{n+1}$. By (2.9) the paths $r_0, \dots, r_n$ are still geodesic in $\Gamma(G', X' \cup \mathcal{H}')$, and $s_1, \dots, s_{n+1}$ are $\mathcal{H}'$-components of q, by construction. Finally, s_i is not connected to s_{i+1}, for $i = 1, \dots, n-1$, because elements of G that belong to different H_ν-cosets continue to do so in G', and s_n is not connected to s_{n+1} because H_{ν_n} and Z are distinct peripheral subgroups of G'. Therefore all of the assumptions of Proposition 2.35 are satisfied, which allows us to conclude that the path q is $(\lambda_0, 0)$-quasigeodesic without backtracking in $\Gamma(G', X' \cup \mathcal{H}')$.

Consequently, the path $p' = r_0 s_1 r_1 \ldots s_n r_n$ is $(\lambda_0, 0)$-quasigeodesic without backtracking in $\Gamma(G', X' \cup \mathcal{H}')$, as a subpath of q. Since p' only contains vertices and edges from $\Gamma(G, X \cup \mathcal{H})$, we see that p' is also $(\lambda_0, 0)$-quasigeodesic without backtracking in $\Gamma(G, X \cup \mathcal{H})$.

Now, the original path p can be obtained by replacing the edges $s_1, \ldots, s_n$ of p' by paths $t_1, \ldots, t_n$, each of which has length at most 3. Hence, by Lemma 2.5, p is $(3\lambda_0, 18\lambda_0 + 6)$-quasigeodesic. Since p' is without backtracking and every $\mathcal{H}$-component of p is connected to an $\mathcal{H}$-component of p' (and vice-versa), by construction, the path p must also be without backtracking.

Thus we have shown that the path p is (λ, c)-quasigeodesic without backtracking in $\Gamma(G, X \cup \mathcal{H})$, where $\lambda = 3\lambda_0$ and $c = 18\lambda_0 + 6$. □

2.2.3 Quasiconvex subsets in relatively hyperbolic groups

In this thesis we shall use the definition of a relatively quasiconvex subgroup given by Osin (2006b). For convenience we state it in the case of arbitrary subsets rather than just subgroups.

Definition 2.37 (Relatively quasiconvex subset)**.** A subset $Q \subseteq G$ is said to be *relatively quasiconvex* (with respect to $\{H_\nu \mid \nu \in \mathcal{N}\}$) if there exists $\varepsilon \geq 0$ such that for every geodesic path q in $\Gamma(G, X \cup \mathcal{H})$, with $q_-, q_+ \in Q$, and every vertex v of q we have $\mathrm{d}_X(v, Q) \leq \varepsilon$. Any such number $\varepsilon \geq 0$ will be called a *quasiconvexity constant* of Q.

Osin proved that relative quasiconvexity of a subset is independent of the choice of a finite generating set X of G: see (Osin, 2006b, Proposition 4.10) – the proof there is stated for relatively quasiconvex subgroups but actually works more generally for relatively quasiconvex subsets.

We outline some basic properties of quasiconvex subsets and subgroups of G in the next two lemmas.

Lemma 2.38. *Let Q be a relatively quasiconvex subset of G. Then*

(a) the subset gQ is relatively quasiconvex, for every $g \in G$;

(b) if $T \subseteq G$ lies at a finite d_X-Hausdorff distance from Q then T is relatively quasiconvex.

Proof. Claim (a) follows immediately from the fact that left multiplication by g induces an isometry of G with respect to both the proper metric d_X and the relative metric $\mathrm{d}_{X \cup \mathcal{H}}$.

To prove claim (b), suppose that $\varepsilon \geq 0$ is a quasiconvexity constant of Q and the d_X-Hausdorff distance between Q and T is less than $k \in \mathbb{N}$. Consider any geodesic path t in $\Gamma(G, X \cup \mathcal{H})$ with $t_-, t_+ \in T$, and take any vertex v of t. Then there are $x, y \in Q$ such that $\mathrm{d}_X(x, t_-) \leq k$ and $\mathrm{d}_X(y, t_+) \leq k$. Let q be any geodesic connecting x with y. Then q is k-similar to t, hence there is a vertex u of q such that $\mathrm{d}_X(v, u) \leq \kappa$, where $\kappa = \kappa(1, 0, k) \geq 0$ is the global constant given by Proposition 2.34 applied to k-similar geodesics. By the relative quasiconvexity of Q, there exists $w \in Q$ such that $\mathrm{d}_X(u, w) \leq \varepsilon$. Moreover, $\mathrm{d}_X(w, T) \leq k$ by assumption. Therefore $\mathrm{d}_X(v, T) \leq \kappa + \varepsilon + k$, thus T is relatively quasiconvex in G. □

Lemma 2.39. *Suppose that $Q \leqslant G$ is a relatively quasiconvex subgroup. Then for all $g \in G$ and $Q' \leqslant_f Q$ the subgroups gQg^{-1} and Q' are relatively quasiconvex in G.*

Proof. By claim (a) of Lemma 2.38, the coset gQ is relatively quasiconvex and the d_X-Hausdorff distance between this coset and gQg^{-1} is at most $|g|_X$, hence gQg^{-1} is relatively quasiconvex in G by claim (b) of the same lemma.

Suppose that $Q = \bigcup_{i=1}^{m} Q'h_i$, where $h_i \in Q$, $i = 1, \ldots, m$. Then the d_X-Hausdorff distance between Q and Q' is bounded above by $\max\{|h_i|_X \mid 1 \leq i \leq m\}$, so Q' is relatively quasiconvex by Lemma 2.38(b). □

Corollary 2.40. *Any parabolic subgroup of G is relatively quasiconvex.*

Proof. Let $H = gQg^{-1}$ be a parabolic subgroup, where $g \in G$ and $Q \leqslant H_\nu$, for some $\nu \in \mathcal{N}$. The subgroup Q is relatively quasiconvex in G (with quasiconvexity constant 0), because any geodesic connecting two elements of Q consists of a single edge in $\Gamma(G, X \cup \mathcal{H})$. Therefore H is relatively quasiconvex by Lemma 2.39. □

Lemma 2.41. *Let P be a maximal parabolic subgroup of G and let Q be a finitely generated relatively quasiconvex subgroup of G. Then the subgroups P and $Q \cap P$ are finitely generated.*

Proof. The fact that each H_ν is finitely generated, provided G is finitely generated, was proved by Osin in (Osin, 2006b, Theorem 1.1).

Now, Hruska proved in (Hruska, 2010, Theorem 9.1) that every quasiconvex subgroup Q of G is itself relatively hyperbolic and maximal parabolic subgroups of Q are precisely the infinite intersections of Q with maximal parabolic subgroups of G. In other words, if $P \leqslant G$ is maximal parabolic, then $Q \cap P$ is either finite or a maximal parabolic subgroup of Q. Combined with Osin's result (Osin, 2006b, Theorem 1.1) mentioned above we can conclude that if Q is finitely generated then so is $Q \cap P$, as required. □

Lemma 2.42. *Let $Q \leqslant G$ be a hyperbolic relatively quasiconvex subgroup of G. Then for any maximal parabolic subgroup $P \leqslant G$, the intersection $Q \cap P$ is quasiconvex in Q. In particular, $Q \cap P$ is hyperbolic.*

Proof. Again Q is hyperbolic relative to a collection of infinite subgroups of the form $Q \cap H$, where $H \leqslant G$ is a maximal parabolic subgroup of G (Hruska, 2010, Theorem 9.1). Thus if $Q \cap P$ is infinite, it is a maximal parabolic subgroup of Q and is undistorted in Q by (Osin, 2006b, Lemma 5.4). It follows that $Q \cap P$ is quasiconvex in Q and hence hyperbolic (Bridson and Haefliger, 1999, Proposition III.Γ.3.7). On the other hand, if $Q \cap P$ is finite then it is trivially hyperbolic. □

The following property of quasiconvex subgroups will be useful.

Lemma 2.43. *Let $Q, R \leqslant G$ be relatively quasiconvex subgroups of G. For every $\zeta \geq 0$ there exists a constant $\mu = \mu(\zeta) \geq 0$ such that the following holds.*
Suppose $x \in G$, $a \in Q$, $b \in R$ are some elements, $[x, xa]$ and $[x, xb]$ are geodesic paths in $\Gamma(G, X \cup \mathcal{H})$, and $u \in [x, xa]$, $v \in [x, xb]$ are vertices such that $\mathrm{d}_X(u, v) \leq \zeta$. Then there is an element $z \in x(Q \cap R)$ such that $\mathrm{d}_X(u, z) \leq \mu$ and $\mathrm{d}_X(v, z) \leq \mu$.

Proof. Denote by $\varepsilon \geq 0$ a quasiconvexity constant of the subgroups Q and R. After applying the left translation by x^{-1}, which is an isometry with respect to both metrics d_X and $\mathrm{d}_{X \cup \mathcal{H}}$, we can assume that $x = 1$. Let $K' = K'(Q, R, \varepsilon + \zeta)$ be the constant given by Lemma 2.1.

Since $x = 1 \in Q \cap R$, $xa = a \in Q$ and $xb = b \in R$, by the relative quasiconvexity of Q and R we know that $u \in N_X(Q, \varepsilon)$ and $v \in N_X(R, \varepsilon)$. By the assumptions $\mathrm{d}_X(u, v) \leq \zeta$, it follows that $u \in N_X(Q, \varepsilon + \zeta) \cap N_X(R, \varepsilon + \zeta)$, hence $u \in N_X(Q \cap R, K')$ by Lemma 2.1.

Thus there exists $z \in Q \cap R$ such that $\mathrm{d}_X(u, z) \leq K'$, and, hence, $\mathrm{d}_X(v, z) \leq K' + \zeta$ by the triangle inequality. Therefore the statement of the lemma holds for $\mu = K' + \zeta$. □

We record two existing combination theorems for relatively quasiconvex subgroups of relatively hyperbolic groups.

Theorem 2.44 (Martínez-Pedroza (2009), Theorem 1.1)**.** *Suppose that Q is a relatively quasiconvex subgroup of G, P is a maximal parabolic subgroup of G and $D = Q \cap P$. There is a constant $C \geq 0$ such that the following holds. If $H \leqslant P$ is any subgroup satisfying*

1. *$H \cap Q = D$, and*
2. *$\min_X(H \setminus D) \geq C$,*

then the subgroup $A = \langle H, Q \rangle$ is relatively quasiconvex in G and is naturally isomorphic to the amalgamated free product $H *_D Q$.

Moreover, for every maximal parabolic subgroup T of G, there exists $u \in A$ such that

$$\text{either } A \cap T \subseteq uQu^{-1} \text{ or } A \cap T \subseteq uHu^{-1}.$$

Theorem 2.45 ((Martínez-Pedroza and Sisto, 2012, Theorem 2))**.** *Let Q and R be relatively quasiconvex subgroups with compatible parabolics, and let $S' \leqslant_f S = Q \cap R$ be a finite index subgroup of their intersection. There is a constant $M = M(Q, R, S') \geq 0$ such that the following is true.*

If $Q' \leqslant Q$ and $R' \leqslant R$ satisfy $Q' \cap R' = S'$ and $|g|_X \geq M$ for all $g \in (Q' \cup R') \setminus S'$, then $\langle Q', R' \rangle$ is relatively quasiconvex and $\langle Q', R' \rangle \cong Q' *_{S'} R'$.

Chapter 3

Joins of quasiconvex subgroups

This chapter of the thesis is devoted to the proof of Theorem 1.8 and a metric version of Theorem 1.2. Let us start by giving brief outlines of the arguments. We continue to follow Convention 2.1 and take two finitely generated relatively quasiconvex subgroups $Q, R \leqslant G$. Set $S = Q \cap R$ and suppose that $Q' \leqslant Q$ and $R' \leqslant R$ are subgroups satisfying conditions (C1)-(C5), with a suitable finite collection of maximal parabolic subgroups $\mathcal{P}$ and parameters B and C that are sufficiently large.

Every element $g \in \langle Q', R' \rangle$ can be written as a product of elements of Q' and R', which gives rise to a broken geodesic line in $\Gamma(G, X \cup \mathcal{H})$ (not necessarily uniquely), whose label represents g in G. We choose a path p from the collection of such broken lines, representing g, that is minimal in a certain sense. The path p may fail to be uniformly quasigeodesic, as it may travel through H_ν-cosets for an arbitrarily long time. We do, however, have some metric control over such instances of backtracking, using the fact that Q' and R' satisfy conditions (C1)-(C5) and the minimality of p.

We construct a new path from p, which we call the *shortcutting* of p, that turns out to be uniformly quasigeodesic. Informally speaking, the shortcutting of p is obtained by replacing each maximal instance of backtracking in consecutive geodesic segments of p with a single edge, then connecting these edges in sequence by geodesics. The resulting path can be seen to satisfy the hypotheses of Proposition 2.36. It follows that the shortcutting of p is uniformly quasigeodesic, and hence $\langle Q', R' \rangle$ is relatively quasiconvex. This proves Theorem 1.8.

To establish the result on the structure of maximal parabolic subgroups of $\langle Q', R' \rangle$, we must further analyse shortcuttings of broken lines representing parabolic elements of this subgroup. Up to conjugacy in $\langle Q', R' \rangle$, the problem reduces to studying elements whose associated paths essentially constitute one large instance of backtracking through a single H_ν-coset. We leverage this to construct a broken line approximating the original, each of whose segments represents an element of $Q' \cap K$ or $R' \cap K$ for a particular maximal parabolic subgroup $K \leqslant G$, giving the desired result.

3.1 Constructing quasigeodesics from broken lines

In this section we detail a procedure that takes as input a broken line and a natural number, and outputs another broken line together with some additional vertex data. We show that if a broken line satisfies certain metric conditions, then the new path constructed through this procedure is uniformly quasigeodesic.

The outline of the construction is as follows: we begin with a broken line $p = p_1 \dots p_n$ in $\Gamma(G, X \cup \mathcal{H})$. Starting from the initial vertex p_-, we note in sequence (along the vertices of p) the vertices marking the start and end of maximal instances of consecutive backtracking in p involving sufficiently long $\mathcal{H}$-components. Once we have done this, we construct the new path by connecting (in the same sequence) the marked vertices with geodesics.

Procedure 3.1 (Θ-shortcutting)**.** Fix a natural number $\Theta \in \mathbb{N}$ and let $p = p_1 \dots p_n$ be a broken line in $\Gamma(G, X \cup \mathcal{H})$. Let $v_0, \dots, v_d$ be the enumeration of all vertices of p in the order they occur along the path (possibly with repetition), so that $v_0 = p_-$, $v_d = p_+$ and $d = \ell(p)$.

We construct broken lines $\Sigma(p, \Theta)$ and $\Sigma_0(p, \Theta)$, which we call Θ-*shortcuttings* of p, which come with a finite set $V(p, \Theta) \subset \{0, \dots, d\} \times \{0, \dots, d\}$ corresponding to indices of vertices of p that we shortcut along.

In the algorithm below we will refer to numbers $s, t, N \in \{0, \dots, d\}$ and a subset $V \subseteq \{0, \dots, d\} \times \{0, \dots, d\}$. To avoid excessive indexing these will change value throughout the procedure. The parameters s and t will indicate the starting and terminal vertices of subpaths of p in which all $\mathcal{H}$-components have lengths less than Θ. The parameter N will keep track of how far along the path p we have proceeded. The set V will collect all pairs of indices (s, t) obtained during the procedure. We initially take $s = 0$, $N = 0$ and $V = \emptyset$.

Step 1: If there are no edges of p between v_N and v_d that are labelled by elements of $\mathcal{H}$, then add the pair (s, d) to the set V and skip ahead to Step 4. Otherwise, continue to Step 2.

Step 2: Let $t \in \{0, \dots, d\}$ be the least natural number with $t \geq N$ for which the edge of p with endpoints v_t and v_{t+1} is an $\mathcal{H}$-component h_i of a geodesic segment p_i of p, for some $i \in \{1, \dots, n\}$.

If $i = n$ or if h_i is not connected to a component of p_{i+1} then set $j = i$. Otherwise, let $j \in \{i + 1, \dots, n\}$ be the maximal integer such that p has consecutive backtracking along $\mathcal{H}$-components $h_i, \dots, h_j$ of segments $p_i, \dots, p_j$. Proceed to Step 3.

Step 3: If

$$\max \left\{ |h_k|_X \;\middle|\; k = i, \dots, j \right\} \geq \Theta,$$

then add the pair (s, t) to the set V and redefine $s = N$ in $\{1, \dots, d\}$ to be the index of the vertex $(h_j)_+$ in the above enumeration $v_0, \dots, v_d$ of the vertices of p. Otherwise let N be the index of $(h_i)_+$, and leave s and V unchanged.

Return to Step 1 with the new values of s, N and V.

Step 4: Set $V(p, \Theta) = V$. The above constructions gives a natural ordering of $V(p, \Theta)$:

$$V(p, \Theta) = \{(s_0, t_0), \dots, (s_m, t_m)\},$$

where $s_k \leq t_k < s_{k+1}$, for all $k = 0, \dots, m-1$. Note that $s_0 = 0$ and $t_m = d$. Proceed to Step 5.

Step 5: For each $k = 0, \dots, m$, let f_k be a geodesic segment (possibly trivial) connecting v_{s_k} with v_{t_k}. Similarly for each $k = 0, \dots, m$ let p'_k be the (possibly trivial) subpath of p with endpoints v_{s_k} and v_{t_k}. Note that when $k < m$, v_{t_k} and $v_{s_{k+1}}$ are in the same left coset of H_ν, for some $\nu \in \mathcal{N}$. If $v_{t_k} = v_{s_{k+1}}$ then let e_k be the trivial path at v_{t_k}, otherwise let e_k be an edge of $\Gamma(G, X \cup \mathcal{H})$ starting at v_{t_k}, ending at $v_{s_{k+1}}$ and labelled by an element of $H_\nu \setminus \{1\}$.

We define the broken line $\Sigma(p, \Theta)$ to be the concatenation $f_0 e_1 f_1 e_2 \dots f_{m-1} e_m f_m$. We also define the broken line $\Sigma_0(p, \Theta)$ to be the concatenation $p'_0 e_1 p'_1 e_2 \dots p'_{m-1} e_m p'_m$.

Remark 3.1. Let us collect some observations about Procedure 3.1.

(a) Since p has only finitely many vertices and N increases at each iteration of Step 3 above, the procedure will always terminate after finitely many steps.

(b) The newly constructed broken lines $\Sigma(p, \Theta)$ and $\Sigma_0(p, \Theta)$ have the same endpoints as p, and each of their nodes of is a vertex of p.

(c) By construction, for any $k \in \{0, \dots, m\}$ the subpath of p between v_{s_k} and v_{t_k} contains no edge labelled by an element $h \in \mathcal{H}$ satisfying $|h|_X \geq \Theta$. In particular, if $\Theta = 1$ then the paths $p'_0, \dots, p'_m$ contain no edges labelled by elements of $\mathcal{H}$.

Figure 3.1 below sketches an example of the output of Procedure 3.1.

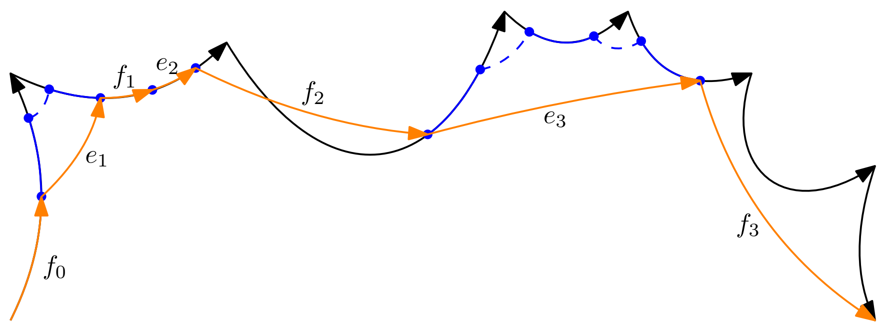

FIGURE 3.1: An example of a shortcutting of a path p in $\Gamma(G, X \cup \mathcal{H})$.

We begin with the following observation.

Lemma 3.2. *Let $\lambda \geq 1$ and $c \geq 0$. Let $p = p_1 \dots p_n$ be a (λ, c)-quasigeodesic broken line in $\Gamma(G, X \cup \mathcal{H})$ with $|p_i|_{X\cup\mathcal{H}} > \lambda + c$ for each $i = 2, \dots, n-1$. Then the path $\Sigma_0(p, 1)$ obtained from Procedure 3.1 is a (λ, c)-quasigeodesic without backtracking.*

Proof. Let q be a subpath of $\Sigma_0 = \Sigma_0(p, 1) = p'_0 e_1 p'_1 \dots p'_{m-1} e_m p'_m$. Since for each i, p'_i is a subpath of p and e_i consists of at most a single edge, q_- and q_+ are vertices of p. Let p' be the subpath of p with $p'_- = q_-$ and $p'_+ = q_+$. The path q can be obtained by replacing subpaths of p' with single edges, so that the length of q is bounded by the length of p'. Then by the quasigeodescity of p we have

$$\ell(q) \leq \ell(p') \leq \lambda \, d_{X\cup\mathcal{H}}(p'_-, p'_+) + c = \lambda \, d_{X\cup\mathcal{H}}(q_-, q_+) + c,$$

so Σ_0 is (λ, c)-quasigeodesic.

We must now show that Σ_0 is without backtracking, so suppose for a contradiction that it does have backtracking. As noted in Remark 3.1(c) the subpaths $p'_0, \dots, p'_m$ contain no $\mathcal{H}$-subpaths. That is, if h is an $\mathcal{H}$-subpath of Σ_0, it must be one of the paths $e_1, \dots, e_m$. Therefore it must be that there are integers $1 \leq k < l \leq m$ such that e_k and e_l are nontrivial connected $\mathcal{H}$-subpaths of Σ_0. Thus,

$$d_{X\cup\mathcal{H}}((e_k)_+, (e_l)_-) \leq 1. \tag{3.1}$$

Let h_1 be the $\mathcal{H}$-component of a segment of p with $(h_1)_+ = (e_k)_+$ and let h_2 be the $\mathcal{H}$-component of a segment of p with $(h_2)_- = (e_l)_-$. Since e_k and e_l are connected, so are h_1 and h_2. Following Remark 2.22, h_1 and h_2 cannot lie in the same segment of p. If h_1 and h_2 lie in adjacent segments of p, then they are part of the same instance of consecutive backtracking and the construction of Σ_0 is contradicted. Otherwise, the path p'_k contains a full segment of p, say p_s, with $1 < s < n$. Then

$$\ell(p_s) \leq \ell(p'_k) \leq \lambda \, d_{X\cup\mathcal{H}}((e_k)_+, (e_l)_-) + c \leq \lambda + c, \tag{3.2}$$

by quasigeodesicity of p and (3.1). However, since p_s is a geodesic, $|p_s|_{X\cup\mathcal{H}} = \ell(p_s)$. Therefore (3.2) contradicts the lemma hypothesis that $\ell(p_s) > \lambda + c$. □

In the next definition we describe paths that will serve as the prototypical input for Procedure 3.1. Essentially, such paths will be broken lines with long segments with instances of consecutive backtracking well-controlled.

Definition 3.3 (Tamable broken line)**.** Let $p = p_1 \dots p_n$ be a broken line in $\Gamma(G, X \cup \mathcal{H})$, and let $B, C, \zeta \geq 0, \Theta \in \mathbb{N}$. We say that p is (B, C, ζ, Θ)-*tamable* if all of the following conditions hold:

(i) $|p_i|_X \geq B$, for $i = 2, \dots, n-1$;

(ii) $\langle (p_i)_-, (p_{i+1})_+ \rangle^{rel}_{(p_i)_+} \leq C$, for each $i = 1, \dots, n-1$;

(iii) whenever p has consecutive backtracking along $\mathcal{H}$-components $h_i, \dots, h_j$, of segments $p_i, \dots, p_j$, such that
$$\max\left\{ |h_k|_X \,\middle|\, k = i, \dots, j \right\} \geq \Theta,$$
it must be that $\mathrm{d}_X\Big((h_i)_-, (h_j)_+\Big) \geq \zeta$.

The main goal of this section is to prove the following result about quasigeodesicity of shortcuttings for tamable paths with appropriate constants.

Proposition 3.4. *Given arbitrary $C \geq 14\delta$ and $\eta \geq 0$ there are constants $\lambda = \lambda(C) \geq 1$, $c = c(C) \geq 0$ and $\zeta = \zeta(\eta, C) \geq 1$ such that for any natural number $\Theta \geq \zeta$ there is $B_0 = B_0(\Theta, C) \geq 0$ satisfying the following.*
Let $p = p_1 \dots p_n$ be a (B_0, C, ζ, Θ)-tamable broken line in $\Gamma(G, X \cup \mathcal{H})$ and let $\Sigma(p, \Theta)$ be the Θ-shortcutting obtained by applying Procedure 3.1 to p, so that $\Sigma(p, \Theta) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$. Then e_k is non-trivial, for each $k = 1, \dots, m$, and $\Sigma(p, \Theta)$ is (λ, c)-quasigeodesic without backtracking.
Moreover, for any $k \in \{1, \dots, m\}$, if we denote by e'_k the $\mathcal{H}$-component of $\Sigma(p, \Theta)$ containing e_k, then $|e'_k|_X \geq \eta$.

The idea of the proof will be to show that under the above assumptions the broken line $\Sigma(p, \Theta)$ satisfies the hypotheses of Proposition 2.36.

Notation 3.2. For the remainder of this section we fix arbitrary constants $C \geq 14\delta$ and $\eta \geq 0$. We let $\rho = \kappa(4, c_3, 0)$, where $c_3 = c_3(C) \geq 0$ is the constant from Lemma 2.12 and $\kappa(4, c_3, 0)$ is the constant obtained by applying Proposition 2.34 to $(4, c_3)$-quasigeodesics. Let $\zeta_1 > 0$, $\lambda \geq 1$ and $c \geq 0$ be the constants given by Proposition 2.36, applied with constant ρ. Note that the constants λ and c only depend on C and do not depend on η.

We now define the constant ζ by
$$\zeta = \max\Big\{\zeta_1, \eta\Big\} + 2\rho + 1. \tag{3.3}$$

Finally we take any natural number $\Theta \geq \zeta$ and
$$B_0 = \max\Big\{(12C + 12\delta + 1)\Theta,\ 4(1 + c_3)\Theta + 1\Big\}. \tag{3.4}$$

The proof of Proposition 3.4 will consist of the following four lemmas. Throughout these lemmas we use the constants defined above and assume that $p = p_1 \dots p_n$ is a (B_0, C, ζ, Θ)-tamable broken line in $\Gamma(G, X \cup \mathcal{H})$. As before, we write $v_0, \dots, v_d$ for the set of vertices of p in the order of their appearance. We let the Θ-shortcutting of p, $\Sigma(p, \Theta) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$, and the set $V(p, \Theta) = \{(s_0, t_0), \dots, (s_m, t_m)\}$ be those obtained by applying Procedure 3.1 to p.

Lemma 3.5. *For each $k = 1, \dots, m$, we have $|e_k|_X \geq \zeta > 0$.*

Proof. By the construction in Procedure 3.1, there are pairwise connected $\mathcal{H}$-components $h_i, \dots, h_j$ of consecutive segments of p, such that $j \geq i$, $(h_i)_- = (e_k)_-$, $(h_j)_+ = (e_k)_+$ and $\max\{|h_l|_X \mid l = i, \dots, j\} \geq \Theta$.
If $j = i$ we see that $|e_k|_X = |h_i|_X \geq \Theta \geq \zeta$, and if $j > i$ then we know that $|e_k|_X \geq \zeta$ by property (iii) from Definition 3.3. □

Lemma 3.6. *The subpaths of p between v_{s_k} and v_{t_k} are $(4, c_3)$-quasigeodesic for each $k = 0, \dots, m$.*

Proof. We write $c_1 = c_1(C) = 12C + 12\delta + 1$, as in Lemma 2.12.

Choose any $k \in \{0, \dots, m\}$ and denote by p' be the subpath of p starting at v_{s_k} and terminating at v_{t_k}. If v_{s_k} and v_{t_k} are both vertices of p_i, for some $i \in \{1, \dots, n\}$, then p' is geodesic and we are done. Otherwise $p' = p'_i p_{i+1} \dots p_{j-1} p'_j$, for some $i, j \in \{1, \dots, n\}$, with $i < j$, where p'_i is a terminal segment of p_i and p'_j is an initial segment of p_j.

By Remark 3.1(c), the paths $p_{i+1}, \dots, p_{j-1}$ contain no $\mathcal{H}$-components h with $|h|_X \geq \Theta$. Since p is (B_0, C, ζ, Θ)-tamable, $|p_l|_X \geq B_0$ for each $l = i+1, \dots, j-1$ by condition (i). Thus we can combine Lemma 2.28 with (3.4) to obtain

$$\mathrm{d}_{X\cup\mathcal{H}}\Big((p_l)_-, (p_l)_+\Big) = \ell(p_l) \geq \frac{1}{\Theta}|p_l|_X \geq \frac{B_0}{\Theta} \geq c_1, \text{ for each } l \in \{i+1, \dots, j-1\}.$$

Again, from the assumption that p is (B_0, C, ζ, Θ)-tamable, we have that

$$\langle (p_l)_-, (p_{l+1})_+ \rangle^{rel}_{(p_l)_+} \leq C, \text{ for all } l = i, \dots, j-1,$$

using condition (ii). In view of Remark 2.7,

$$\langle (p'_i)_-, (p_{i+1})_+ \rangle^{rel}_{(p'_i)_+} \leq C \text{ and } \langle (p_{j-1})_-, (p'_j)_+ \rangle^{rel}_{(p_{j-1})_+} \leq C.$$

Therefore we can use Lemma 2.12 to conclude that p' is $(4, c_3)$-quasigeodesic, as required. □

Lemma 3.7. *If $k \in \{0, \dots, m-1\}$ and h is an $\mathcal{H}$-component of f_k or f_{k+1} that is connected to e_{k+1}, then $|h|_X \leq \rho$.*

Proof. Arguing by contradiction, suppose that h is an $\mathcal{H}$-component of f_k connected to e_{k+1} and satisfying $|h|_X > \rho$ (the other case when h is an $\mathcal{H}$-component of f_{k+1} is similar). Remark 2.22 tells us that h is a single edge of f_k. Moreover, since h and e_{k+1} are connected and $(f_k)_+ = (e_{k+1})_-$, we have $d_{X\cup\mathcal{H}}(h_-, (f_k)_+) \leq 1$. The geodesicity of

f_k in $\Gamma(G, X \cup \mathcal{H})$ now implies that h must in fact be the last edge of f_k, so that $h_+ = (f_k)_+ = v_{t_k}$.

Let $p' = p'_i p_{i+1} \dots p_{j-1} p'_j$ be the subpath of p with $p'_- = v_{s_k}$ and $p'_+ = v_{t_k}$, where p'_i and p'_j are non-trivial subpaths of p_i and p_j respectively. By Lemma 3.6, p' is $(4, c_3)$-quasigeodesic.

Since $|h|_X > \rho = \kappa(4, c_3, 0)$ we may apply Proposition 2.34 to find that h is connected to an $\mathcal{H}$-component of p' (which may consist of multiple edges, each of which is an $\mathcal{H}$-component of a segment of p). We write h' for the final edge of this $\mathcal{H}$-component and denote by u the edge of p with endpoints v_{t_k} and v_{t_k+1} (see Figure 3.2). Procedure 3.1 and the assumption that h is connected to e_{k+1} imply that u is an $\mathcal{H}$-component of a segment of p and h' and u are connected as $\mathcal{H}$-subpaths of p.

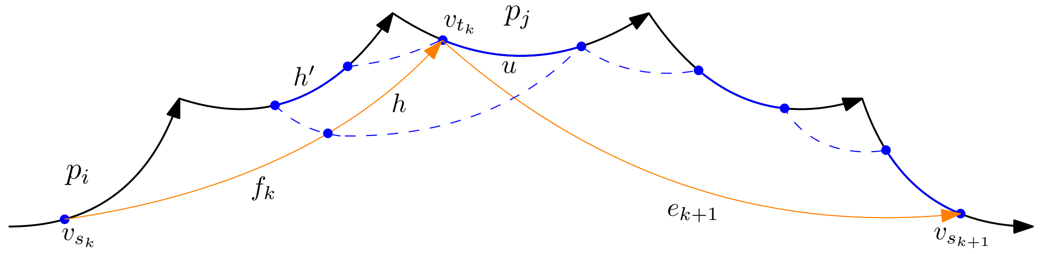

FIGURE 3.2: Illustration of Lemma 3.7.

Suppose, first, that p'_j is a proper subpath of p_j, so that u belongs to the segment p_j, as shown on Figure 3.2. Then there are the following possibilities.

<u>Case 1</u>: h' is an edge of p_j.

In this case h' and u are connected distinct $\mathcal{H}$-subpaths of p_j, which is a geodesic. This contradicts the observation of Remark 2.22, that geodesics are without backtracking and $\mathcal{H}$-components of geodesics are single edges.

<u>Case 2</u>: h' is an $\mathcal{H}$-component of p_{j-1}.

Let $t \in \{0, \dots, d\}$ be such that $v_t = h'_-$, and note that

$$s_k \leq t < t_k. \tag{3.5}$$

By the construction from Procedure 3.1, there are pairwise connected $\mathcal{H}$-components $h_j, \dots, h_{j+l}$, of segments $p_j, \dots, p_{j+l}$, with $(e_{k+1})_- = (h_j)_- = v_{t_k}$ and $(e_{k+1})_+ = (h_{j+l})_+ = v_{s_{k+1}}$, such that

$$\max\{|h_j|_X, \dots, |h_{j+l}|_X\} \geq \Theta$$

and $l \in \{0, \dots, n-j\}$ is chosen to be maximal with this property. Then the components $h', h_j, \dots, h_{j+l}$ constitute a larger instance of consecutive backtracking, starting at $h'_- = v_t$, with

$$\max\left\{|h'|_X, |h_j|_X, \dots, |h_{j+l}|_X\right\} \geq \Theta.$$

In view of (3.5), this contradicts the choice of t_k and the inclusion of (s_k, t_k) in the set $V(p, \Theta)$ at Steps 2 and 3 of Procedure 3.1.

Case 3: h' is an $\mathcal{H}$-component of one of the paths $p'_i, p_{i+1}, \dots, p_{j-2}$.

Then the subpath q of p' from h'_+ to $p'_+ = v_{t_k}$ contains all of p_{j-1}. By Remark 3.1(c), p_{j-1} contains no $\mathcal{H}$-components r satisfying $|r|_X \geq \Theta$. Therefore, in view of Lemma 2.28 and the assumption that p is (B_0, C, ζ, Θ)-tamable, we can deduce that $\Theta\ell(p_{j-1}) \geq |p_{j-1}|_X \geq B_0$. Combining this with the $(4, c_3)$-quasigeodesicity of p', we obtain

$$\mathrm{d}_{X\cup\mathcal{H}}(h'_+, p'_+) \geq \frac{1}{4}\Big(\ell(q) - c_3\Big) \geq \frac{1}{4}\Big(\ell(p_{j-1}) - c_3\Big) \geq \frac{B_0}{4\Theta} - \frac{c_3}{4} > 1,$$

where the last inequality follows from (3.4). On the other hand, the fact that h' and h are connected gives $d_{X\cup\mathcal{H}}(h'_+, p'_+) = \mathrm{d}_{X\cup\mathcal{H}}(h'_+, h_+) \leq 1$, contradicting the above.

In each case we arrive at a contradiction, so it is impossible that $|h|_X > \rho$ if p'_j is a proper subpath of p_j. If p'_j is instead the whole subpath p_j, we may carry out a similar analysis. In this situation it must be that u is an $\mathcal{H}$-component of the segment p_{j+1}. We now have only two relevant cases to consider: h' is an $\mathcal{H}$-component of p_j or h' is an $\mathcal{H}$-component of one of the paths $p'_i, p_{i+1}, \dots, p_{j-1}$. Both of them will lead to contradictions similarly to Cases 2 and 3 above.

Therefore it must be that $|h|_X \leq \rho$, as required. □

Lemma 3.8. *For each $k \in \{1, \dots, m-1\}$, the $\mathcal{H}$-subpaths e_k and e_{k+1} of $\Sigma(p, \Theta)$ are not connected.*

Proof. Suppose that e_k is connected to e_{k+1} for some $k \in \{1, \dots, m-1\}$. As before, according to Procedure 3.1, there exist two sets of pairwise connected $\mathcal{H}$-components of consecutive segments of p, $h_1, \dots, h_i$ and $q_1, \dots, q_j$, such that $(h_1)_- = (e_k)_-$, $(h_i)_+ = (e_k)_+$, $(q_1)_- = (e_{k+1})_-$, $(q_j)_+ = (e_{k+1})_+$ and

$$\max\left\{|h_1|_X, \dots, |h_i|_X\right\} \geq \Theta, \quad \max\left\{|q_1|_X, \dots, |q_j|_X\right\} \geq \Theta.$$

Since e_k and e_{k+1} are connected, h_i and q_1 will be connected $\mathcal{H}$-subpaths of p, in particular they cannot be contained in the same segment of the broken line p by Remark 2.22. If h_i and q_1 are $\mathcal{H}$-components of adjacent segments of p, then the components $h_1, \dots, h_i, q_1, \dots, q_j$ constitute a longer instance of consecutive backtracking in p, which contradicts the construction of e_k in Procedure 3.1.

Therefore it must be the case that the subpath p' of p between $(e_k)_+ = (h_i)_+ = v_{s_k}$ and $(e_{k+1})_- = (q_1)_- = v_{t_k}$ contains at least one full segment p_l (with $1 < l < n$). By Remark 3.1(c) the path p' has no $\mathcal{H}$-components h satisfying $|h|_X \geq \Theta$. Therefore we can combine Lemma 2.28 with the fact that p is (B_0, C, ζ, Θ)-tamable to deduce that

$$\ell(p') \geq \ell(p_l) \geq \frac{|p_l|_X}{\Theta} \geq \frac{B_0}{\Theta}. \tag{3.6}$$

Moreover, by Lemma 3.6 the path p' is $(4, c_3)$-quasigeodesic so that

$$\ell(p') \leq 4d_{X\cup\mathcal{H}}((e_k)_+, (e_{k+1})_-) + c_3 \leq 4 + c_3,$$

where the last inequality is true because e_k and e_{k+1} are connected. Combined with (3.6), the above inequality gives $B_0 \leq (4 + c_3)\Theta$, which contradicts the choice of B_0 in (3.4).

Therefore e_k and e_{k+1} cannot be connected, for any $k \in \{1, \ldots, m-1\}$. □

Proof of Proposition 3.4. The construction, together with Lemmas 3.5, 3.7 and 3.8, show that the Θ-shortcutting $\Sigma(p, \Theta) = f_0 e_1 f_1 \ldots f_{m-1} e_m f_m$ satisfies the hypotheses of Proposition 2.36 and e_k is non-trivial, for each $k = 1, \ldots, m$. Therefore $\Sigma(p, \Theta)$ is (λ, c)-quasigeodesic without backtracking.

For the final claim of the proposition, consider any $k \in \{1, \ldots, m\}$ and denote by e'_k the H_ν-component of $\Sigma(p, \Theta)$ containing e_k, for some $\nu \in \mathcal{N}$. Lemma 3.8 implies that e'_k is the concatenation $h_1 e_k h_2$, where h_1 is either trivial or it is an H_ν-component of f_{k-1}, and h_2 is either trivial or it is an H_ν-component of f_k. Combining the triangle inequality with Lemmas 3.5, 3.7 and equation (3.3), we obtain

$$|e'_k|_X \geq |e_k|_X - |h_1|_X - |h_2|_X \geq \zeta - 2\rho \geq \eta,$$

as required. □

We will also need the following result, which asserts some control over the subpaths of a tamable broken line between the shortcuts. Recall that given a broken line p and $\Theta \in \mathbb{N}$, Procedure 3.1 gives us a collection of subpaths $p'_0, \ldots, p'_m$ of p.

Lemma 3.9. *Under the same hypotheses as Proposition 3.4, the following is true. For each $i = 0, \ldots, m$, the shortcutting $\Sigma_0(p'_i, 1)$ is a $(4, c_3)$-quasigeodesic without backtracking, and each of its $\mathcal{H}$-components h satisfies $|h|_X \leq 3L + 2\Theta$, where $L \geq 0$ is the constant of Proposition 2.25.*

Proof. Let t be a segment of p_i', which is a geodesic. By Remark 3.1, any $\mathcal{H}$-component h of t has $|h|_X \leq \Theta$, so by Lemma 2.28 and the assumption that p is tamable

$$|t|_{X\cup\mathcal{H}} = \ell(t) \geq \frac{B_0}{\Theta} > 4 + c_3$$

whenever t is not the first or last segment of p_i'. Moreover, Lemma 3.6 gives that p_i' is $(4, c_3)$-quasigeodesic. Therefore by Lemma 3.2, $\Sigma_0(p_i', 1)$ is a $(4, c_3)$-quasigeodesic without backtracking.

Now let h be an $\mathcal{H}$-component of $\Sigma_0(p_i', 1)$. Then h is either an $\mathcal{H}$-component of a segment of p_i' or shares its endpoints with two connected $\mathcal{H}$-components q and r of segments of p_i'. In the former case, $|h|_X \leq \Theta$ and we are done, so suppose the latter.

The path p_i' is a broken line with $p_i' = p_j' p_{j+1} \dots p_{k-1} p_k'$, where p_j' (respectively, p_k') is a subpath of p_j with $(p_j')_- = v_{s_i}$ and $(p_j')_+ = (p_j)_+$ (respectively, of p_k with $(p_k')_- = (p_k)_-$ and $(p_k')_+ = v_{t_i}$). As in Remark 3.1(c), each $\mathcal{H}$-component h of the paths $p_j', p_{j+1}, \dots, p_{k-1}, p_k'$ satisfies $|h|_X \leq \Theta$. This implies

$$|q|_X + |r|_X \leq 2\Theta. \tag{3.7}$$

Since each segment of p is geodesic, q and r must be connected $\mathcal{H}$-components of distinct segments of p, say p_a and p_b. Without loss of generality, we assume $a < b$. If $b > a+1$ then the subpath of p_i' between q_- and r_+ contains the entire segment p_{a+1}. By Lemma 2.28,

$$\ell(p_{a+1}) \geq \frac{1}{\Theta}|p_{a+1}| \geq \frac{B_0}{\Theta}, \tag{3.8}$$

where the last inequality is given by condition (i) of tamability.

Lemma 3.6 tells us that p_i' is $(4, c_3)$-quasigeodesic. Combining this fact with (3.8) and the choice of B_0, we have

$$\mathrm{d}_{X\cup\mathcal{H}}(q_-, r_+) \geq \frac{1}{4}\ell(p_{a+1}) - \frac{c_3}{4} \geq \frac{B_0 - \Theta c_3}{4\Theta} > 1.$$

On the other hand, q and r are connected, so that $\mathrm{d}_{X\cup\mathcal{H}}(q_-, r_+) \leq 1$, a contradiction. Therefore q and r must lie in adjacent segments p_a and p_{a+1} of p.

If $q_+ \neq r_-$, then there is an $\mathcal{H}$-edge h' in $\Gamma(G, X\cup\mathcal{H})$ with $h'_- = q+$ and $h'_+ = r_-$. The edge h' must be isolated in the geodesic triangle $h \cup [q_+, (p_a)_+] \cup [(p_a)_+, r_-]$. Thus by Proposition 2.25, we have $|h'|_X = \mathrm{d}_X(q_+, r_-) \leq 3L$. Otherwise $q_+ = r_-$, in which case $\mathrm{d}_x(q_+, r_-) = 0$. Together with (3.7), we obtain

$$|h|_X = \mathrm{d}_X(q_-, r_+) \leq \mathrm{d}_X(q_-, q_+) + \mathrm{d}_X(q_+, r_-) + \mathrm{d}_X(r_-, r_+) \leq 3L + 2\Theta,$$

as required. □

3.2 Path representatives

Let us set the notation that will be used in the next few sections.

Convention 3.3. As G is relatively hyperbolic, the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ is δ-hyperbolic, for some $\delta \in \mathbb{N}$ (see Lemma 2.16). Furthermore, we assume that $Q, R \leqslant G$ are fixed relatively quasiconvex subgroups of G, with a quasiconvexity constant $\varepsilon \geq 0$, and denote $S = Q \cap R$. We fix U and V nonempty subsets of either Q or R (independently of one another).

Let $Q' \leqslant Q$ and $R' \leqslant R$ be arbitrary subgroups. Elements of $U\langle Q', R'\rangle V$ are labels of certain broken lines in $\Gamma(G, X \cup \mathcal{H})$ which can be assigned a numerical invariant. When this numerical invariant is minimal and Q' and R' satisfy certain conditions, these broken lines are will satisfy useful geometric properties. In this section we define path representatives of elements of $U\langle Q', R'\rangle V$.

Definition 3.10 (Path representative)**.** Consider an element $g \in U\langle Q', R'\rangle V$. Let $p = up_1 \dots p_n v$ be a broken line in $\Gamma(G, X \cup \mathcal{H})$ with geodesic segments $u, p_1, \dots, p_n$, and v such that

- $\widetilde{p} = g$;
- $\widetilde{p_i} \in Q' \cup R'$ for each $i \in \{1, \dots, n\}$;
- $\widetilde{u} \in U$ and $\widetilde{v} \in V$.

We will call p a *path representative* of g.

To choose an optimal path representative we define their types.

Definition 3.11 (Type of a path representative)**.** Suppose that $p = up_1 \dots p_n v$ is a path representative of an element of $U\langle Q', R'\rangle V$. Let T denote the set of all $\mathcal{H}$-components of the segments of p. We define the *type* $\tau(p)$ of p to be the triple

$$\tau(p) = \Big(n, \ell(p), \sum_{t \in T} |t|_X\Big) \in {\mathbb{N}_0}^3.$$

Definition 3.12 (Minimal type)**.** Given $g \in U\langle Q', R'\rangle V$, the set $\mathcal{S}$ of all path representatives of g is non-empty. Therefore the subset $\tau(\mathcal{S}) = \{\tau(p) \mid p \in \mathcal{S}\} \subseteq {\mathbb{N}_0}^3$, where ${\mathbb{N}_0}^3$ is equipped with the lexicographic order, will have a unique minimal element. We will say that $p = up_1 \dots p_n v$ is a *path representative of g of minimal type* if $\tau(p)$ is the minimal element of $\tau(\mathcal{S})$.

The minimality of the type of a path representative is thus a numerical condition on the segments of a path representative and the total lengths of their components. In the next few sections we will study local properties induced by this global condition.

Convention 3.4. We call the sets U and V *trivial* when $U = V = \{1\}$. In this case the segments of a path representative of $U\langle Q', R'\rangle V$ corresponding to U and V are necessarily constant paths on a vertex. We may omit mention of these segments and of the sets U and V in this setting (i.e. we simply refer to path representatives of $\langle Q', R'\rangle$).

Definition 3.13. We say that U and V are *Q'/R'-absorbing* if both of the following conditions hold:

- $UQ' = U$ or $UR' = U$;
- $Q'V = V$ or $R'V = V$.

Remark 3.14. Suppose that U and V are trivial or Q'/R'-absorbing and note that if p_1 and p_2 are paths with $(p_1)_+ = (p_2)_-$ whose labels both represent elements of Q' (or, respectively, both R'), then the label of any geodesic $[(p_1)_-, (p_2)_+]$ also represents an element of Q' (respectively, R'). Hence in a path representative of $g \in U\langle Q', R'\rangle V$ of minimal type, the labels of consecutive segments alternate between representing elements of $Q \setminus S$ and $R \setminus S$, whenever g is not an element of $UQ'V$ or $UR'V$.

Notation 3.5. Let $x, y, z \in G$. We will write

$$\langle x, y\rangle_z^{rel} = \frac{1}{2}\Big(\mathrm{d}_{X\cup\mathcal{H}}(x,z) + \mathrm{d}_{X\cup\mathcal{H}}(y,z) - \mathrm{d}_{X\cup\mathcal{H}}(x,y)\Big)$$

to denote the Gromov product of x and y with respect to z in the relative metric $\mathrm{d}_{X\cup\mathcal{H}}$.

Lemma 3.15 (Gromov products are bounded)**.** *There is a constant $C_0 \geq 0$ such that the following holds.*
Suppose U and V are either trivial or Q'/R'-absorbing. Let $Q' \leqslant Q$ and $R' \leqslant R$ be subgroups satisfying condition (C1). If $p = up_1 \dots p_n v$ is a minimal type path representative of an element $g \in U\langle Q', R'\rangle V$ and $f_0, \dots, f_{n+2} \in G$ are the nodes of p then $\langle f_{i-1}, f_{i+1}\rangle_{f_i}^{rel} \leq C_0$ for each $i = 1, \dots, n+1$.

Proof. Let $\sigma \in \mathbb{N}_0$ be the constant from Proposition 2.32 and let $\mu = \mu(\sigma) \geq 0$ be given by Lemma 2.43. Set $C_0 = \mu + \delta + 2\sigma + 2$, and assume that $p = up_1 \dots p_n v$ is a path representative of $g \in U\langle Q', R'\rangle V$ of minimal type.

Take any $i \in \{1, \dots, n+1\}$ and let t_1 and t_2 be the consecutive segments of p with $f_i = (t_1)_+ = (t_2)_-$. If either t_1 or t_2 are the constant path then the statement is trivial, so suppose otherwise. Choose vertices $x \in t_1$ and $y \in t_2$ so that

$$\mathrm{d}_{X\cup\mathcal{H}}(f_i, x) = \mathrm{d}_{X\cup\mathcal{H}}(f_i, y) = \lfloor \langle f_{i-1}, f_{i+1}\rangle_{f_i}^{rel} \rfloor.$$

As $\Gamma(G, X\cup\mathcal{H})$ is δ-hyperbolic, we must have $\mathrm{d}_{X\cup\mathcal{H}}(x,y) \leq \delta$.

If $\langle f_{i-1}, f_{i+1}\rangle_{f_i}^{rel} < C_0$ then we are done, so suppose otherwise. Then we have $\mathrm{d}_{X\cup\mathcal{H}}(x, f_i) \geq \delta + \sigma + 1 \in \mathbb{N}$, so there is a vertex x_1 on the subpath $[x, f_i]$ of t_1 such that

$$\mathrm{d}_{X\cup\mathcal{H}}(x_1, x) = \delta + \sigma + 1.$$

Applying Proposition 2.32 to the geodesic triangle Δ with sides $[x, f_i]$, $[f_i, y]$ and $[x, y]$ (here we choose $[f_i, y]$ to be a subpath of t_2), we can find some vertex $y_1 \in [x, y] \cup [f_i, y]$ with $d_X(x_1, y_1) \leq \sigma$. If $y_1 \in [x, y]$, then, by the triangle inequality,

$$d_{X\cup\mathcal{H}}(x_1, x) \leq d_{X\cup\mathcal{H}}(x_1, y_1) + d_{X\cup\mathcal{H}}(x, y) \leq \sigma + \delta,$$

which would contradict the choice of x_1. Therefore it must be that $y_1 \in [f_i, y]$ (see Figure 3.3).

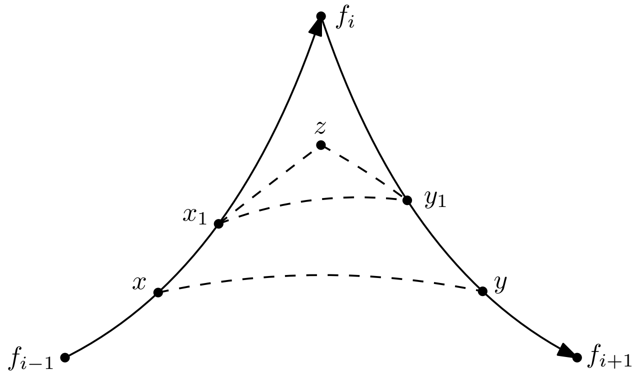

FIGURE 3.3: Illustration of Lemma 3.15.

Since the path representative p has minimal type, in view of Remark 3.14 we must have either $\widetilde{t_1} \in Q$ and $\widetilde{t_2} \in R$ or $\widetilde{t_1} \in R$ and $\widetilde{t_2} \in Q$. Without loss of generality let us assume the former. We can apply Lemma 2.43 to find $z \in f_i(Q \cap R)$ with $d_X(x_1, z) \leq \mu$ and $d_X(y_1, z) \leq \mu$. Let t'_1 be a geodesic path in $\Gamma(G, X \cup \mathcal{H})$ joining $f_{i-1} = (t_1)_-$ with z and let t'_2 be a geodesic path joining z with $f_{i+1} = (t_2)_+$. Observe that $Q' \cap R' = Q \cap R$ by (C1), whence

$$\widetilde{t'_1} = f_{i-1}^{-1} z = f_{i-1}^{-1} f_i f_i^{-1} z \in \widetilde{t_1}(Q' \cap R').$$

Similarly, $\widetilde{t'_2} \in (Q' \cap R')\widetilde{t_1}$. When t_1 is one of the segments $p_1, \ldots, p_{n-1}$, we have that $\widetilde{t'_1} \in Q'(Q' \cap R') = Q'$ and $\widetilde{t'_2} \in (Q' \cap R')R' = R'$. In the case that t_1 is the segment u (respectively, p_n), we have $\widetilde{t'_1} \in U(Q' \cap R') \subseteq U$ and $\widetilde{t'_2} \in (Q' \cap R')R' = R'$ (respectively, $\widetilde{t'_1} \in Q'(Q' \cap R') = Q'$ and $\widetilde{t'_2} \in (Q' \cap R')V \subseteq V$) as U and V are Q'/R'-absorbing. In any case the broken line p' obtained by replacing the segments t_1 and t_2 of p with t'_1 and t'_2 is also a path representative of the same element $g \in U\langle Q', R'\rangle V$.

Since p has minimal type, by the assumption, it must be the case that $\ell(t_1) + \ell(t_2) \leq \ell(t'_1) + \ell(t'_2)$, which can be re-written as

$$d_{X\cup\mathcal{H}}(f_{i-1}, f_i) + d_{X\cup\mathcal{H}}(f_i, f_{i+1}) \leq d_{X\cup\mathcal{H}}(f_{i-1}, z) + d_{X\cup\mathcal{H}}(z, f_{i+1}). \tag{3.9}$$

Since $x_1 \in t_1$, we have $\mathrm{d}_{X\cup\mathcal{H}}(f_{i-1}, f_i) = \mathrm{d}_{X\cup\mathcal{H}}(f_{i-1}, x_1) + \mathrm{d}_{X\cup\mathcal{H}}(x_1, f_i)$. On the other hand, we have

$$\mathrm{d}_{X\cup\mathcal{H}}(f_{i-1}, z) \leq \mathrm{d}_{X\cup\mathcal{H}}(f_{i-1}, x_1) + \mathrm{d}_{X\cup\mathcal{H}}(x_1, z) \leq \mathrm{d}_{X\cup\mathcal{H}}(f_{i-1}, x_1) + \mu,$$

by the triangle inequality. Similarly,

$$\mathrm{d}_{X\cup\mathcal{H}}(f_i, f_{i+1}) = \mathrm{d}_{X\cup\mathcal{H}}(f_i, y_1) + \mathrm{d}_{X\cup\mathcal{H}}(y_1, f_{i+1})$$

and

$$\mathrm{d}_{X\cup\mathcal{H}}(z, f_{i+1}) \leq \mathrm{d}_{X\cup\mathcal{H}}(y_1, f_{i+1}) + \mu.$$

Combining the above inequalities with (3.9), we obtain

$$\mathrm{d}_{X\cup\mathcal{H}}(x_1, f_i) + \mathrm{d}_{X\cup\mathcal{H}}(f_i, y_1) \leq 2\mu. \tag{3.10}$$

Now, by construction, we have

$$\mathrm{d}_{X\cup\mathcal{H}}(x_1, f_i) = \mathrm{d}_{X\cup\mathcal{H}}(x, f_i) - \mathrm{d}_{X\cup\mathcal{H}}(x_1, x) = \lfloor \langle f_{i-1}, f_{i+1} \rangle^{rel}_{f_i} \rfloor - (\delta + \sigma + 1). \tag{3.11}$$

On the other hand, since $\mathrm{d}_{X\cup\mathcal{H}}(x_1, y_1) \leq \sigma$, we achieve

$$\mathrm{d}_{X\cup\mathcal{H}}(f_i, y_1) \geq \mathrm{d}_{X\cup\mathcal{H}}(x_1, f_i) - \mathrm{d}_{X\cup\mathcal{H}}(x_1, y_1) \geq \lfloor \langle f_{i-1}, f_{i+1} \rangle^{rel}_{f_i} \rfloor - (\delta + 2\sigma + 1). \tag{3.12}$$

After combining (3.11), (3.12) and (3.10), we obtain

$$2\lfloor \langle f_{i-1}, f_{i+1} \rangle^{rel}_{f_i} \rfloor - (2\delta + 3\sigma + 2) \leq 2\mu.$$

Therefore, we can conclude that $\langle f_{i-1}, f_{i+1} \rangle^{rel}_{f_i} \leq \mu + \delta + 2\sigma + 2 = C_0$, as required. □

3.3 Adjacent backtracking in path representatives

In this section we continue working under Convention 3.3. Our goal here is to study the possible backtracking within two adjacent segments in a minimal type path representative.

Lemma 3.16. *For all non-negative numbers ζ and ξ there exists $\tau = \tau(\zeta, \xi) \geq 0$ such that the following holds.*
Suppose U and V are either trivial or Q'/R'-absorbing. Let $Q' \leqslant Q$ and $R' \leqslant R$ be subgroups satisfying (C1), $g \in U\langle Q', R'\rangle V$ and p is a path representative of g of minimal type. If s_1 and s_2 are connected $\mathcal{H}$-components of consecutive segments t_1 and t_2 of p respectively, such that $\mathrm{d}_X((s_1)_-, (s_2)_+) \leq \zeta$ and $\mathrm{d}_X((s_1)_+, (t_1)_+) \leq \xi$, then $|s_1|_X \leq \tau$ and $|s_2|_X \leq \tau$.

Proof. Let $\mu = \mu(\zeta) \geq 0$ be the constant from Lemma 2.43. Since $|X| < \infty$ and $|\mathcal{N}| < \infty$ we can define the constant $k \geq 0$ as follows:

$$k = \max\{K'(Q \cap R, cH_\nu c^{-1}, \xi + \mu) \mid \nu \in \mathcal{N},\ c \in G,\ |c|_X \leq \xi\}, \tag{3.13}$$

where for each $c \in G$ and $\nu \in \mathcal{N}$ the constant $K'(Q \cap R, cH_\nu c^{-1}, \xi + \mu)$ is given by Lemma 2.1. Let $L \geq 0$ be the constant from Proposition 2.25 and set $\tau = 2k + 2\xi + \zeta + 3L \geq 0$.

Let $p = up_1 \dots p_n v$ be a path representative of some $g \in U\langle Q', R'\rangle V$ of minimal type. Suppose that s_1 and s_2 are connected H_ν-components of consecutive segments t_1 and t_2 of p respectively, for some $\nu \in \mathcal{N}$, such that $d_X((s_1)_-, (s_2)_+) \leq \zeta$ and that $d_X((s_1)_+, (t_1)_+) \leq \xi$.

Note that, by Lemma 2.31,

$$d_X((s_1)_+, (s_2)_-) \leq 3L. \tag{3.14}$$

Denote $x = (t_1)_+ = (t_2)_- \in G$, $a = x^{-1}s_+ \in G$ and $b = x^{-1}t_- \in G$: see Figure 3.4.

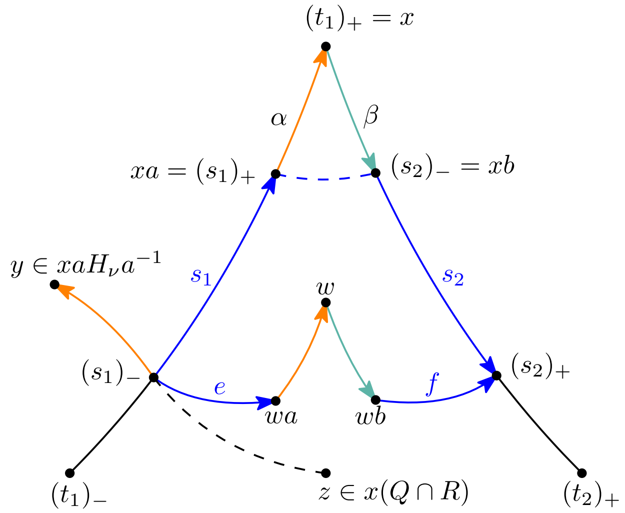

FIGURE 3.4: Illustration of Lemma 3.16.

Note that

$$aH_\nu = bH_\nu, \text{ hence } aH_\nu a^{-1} = bH_\nu b^{-1}, \tag{3.15}$$

because the H_ν-components s_1 and s_2 are connected. Using the lemma hypotheses and (3.14) we also have

$$|a|_X = d_X(x, (s_1)_+) \leq \xi \text{ and } |b|_X \leq d_X(x, (s_1)_+) + d_X((s_1)_+, (s_2)_-) \leq \xi + 3L. \tag{3.16}$$

In view of Remark 3.14, without loss of generality we can assume that $\mathrm{Lab}(t_1)$ represents an element of Q and $\mathrm{Lab}(t_2)$ represents an element of R in G (the other case can be treated similarly). Applying Lemma 2.43, we can find $z \in x(Q \cap R)$ such that $d_X((s_1)_-, z) \leq \mu$.

Consider the element $y = (s_1)_- a^{-1} = xa\widetilde{s_1}^{-1}a^{-1} \in xaH_\nu a^{-1}$, and observe that $d_X((s_1)_-, y) = |a^{-1}|_X \le \xi$. Moreover, $d_X((s_1)_-, x(Q \cap R)) \le d_X((s_1)_-, z) \le \mu$, whence

$$(s_1)_- \in N_X\Big(x(Q \cap R), \xi + \mu\Big) \cap N_X\Big(xaH_\nu a^{-1}, \xi + \mu\Big).$$

Therefore, according to Lemma 2.1, there exists $w \in x(Q \cap R \cap aH_\nu a^{-1})$ such that

$$d_X((s_1)_-, w) \le k, \tag{3.17}$$

where $k \ge 0$ is the constant defined in (3.13).

Let α be the subpath of t_1 from $(s_1)_+ = xa$ to $(t_1)_+ = x$. Choose the geodesic path $[wa, w]$ as the translate $wx^{-1}\alpha$. Observe that $(s_1)_- \in xaH_\nu$ and $wa \in xaH_\nu a^{-1}a = xaH_\nu$ lie in the same H_ν-coset. Thus $d_{X\cup\mathcal{H}}((s_1)_-, wa) \le 1$; if $(s_1)_- = wa$ we let e be the trivial path in $\Gamma(G, X \cup \mathcal{H})$ consisting of the single vertex s_-, and otherwise we let e be the edge of $\Gamma(G, X \cup \mathcal{H})$ labelled by an element of $H_\nu \setminus \{1\}$ that joins $(s_1)_-$ to wa. Define the path q in $\Gamma(G, X \cup \mathcal{H})$ as the concatenation

$$q = [(t_1)_-, (s_1)_-]\, e\, [wa, w], \tag{3.18}$$

where $[(t_1)_-, (s_1)_-]$ is chosen as the initial segment of t_1.

Since $\ell(e) \le 1 = d_{X\cup\mathcal{H}}((s_1)_-, (s_2)_+)$, we can bound the length of the path q from above as follows:

$$\begin{aligned}\ell(q) &= d_{X\cup\mathcal{H}}((t_1)_-, (s_1)_-) + \ell(e) + d_{X\cup\mathcal{H}}(wa, w)\\ &\le d_{X\cup\mathcal{H}}((t_1)_-, (s_1)_-) + d_{X\cup\mathcal{H}}((s_1)_-, (s_1)_+) + d_{X\cup\mathcal{H}}(xa, x) = \ell(t_1).\end{aligned} \tag{3.19}$$

Now we construct a similar path from w to $(t_2)_+$. Let β be the subpath of t_2 from $(t_2)_- = x$ to $(s_2)_- = xb$. Choose the geodesic path $[w, wb]$ as the translate $wx^{-1}\beta$. Recall that $(s_2)_+ \in xbH_\nu$ and note that the inclusion $w \in xaH_\nu a^{-1}$, together with (3.15), imply that $wb \in xbH_\nu$ also. If $(s_2)_+ = wb$ then let f be the trivial path in $\Gamma(G, X \cup \mathcal{H})$ consisting of the single vertex $(s_2)_+$, otherwise let f be the edge in $\Gamma(G, X \cup \mathcal{H})$ joining the vertices wb and $(s_2)_+$ with $\mathrm{Lab}(f) \in H_\nu \setminus \{1\}$. We now define the path r in $\Gamma(G, X \cup \mathcal{H})$ as the concatenation

$$r = [w, wb]\, f\, [t_+, (t_2)_+], \tag{3.20}$$

where $[(s_2)_+, (t_2)_+]$ is chosen as the ending segment of t_2. Similarly to the case of q we can estimate that

$$\ell(r) \le \ell(t_2). \tag{3.21}$$

Note that since $q_- = (t_1)_- = x\widetilde{t_1}^{-1}$, $q_+ = w \in x(Q \cap R)$ and $Q \cap R = Q' \cap R'$ by (C1), we have $\widetilde{q} \in \widetilde{t_1}(Q' \cap R')$. Similarly, $\widetilde{r} \in (Q' \cap R')\widetilde{t_2}$.

Let t_1' be a geodesic path from $q_- = (t_1)_-$ to $q_+ = w$, and let t_2' be a geodesic path from $w = r_-$ to $(t_2)_+ = r_+$. When t_1 is one of the segments $p_1, \ldots, p_{n-1}$, we have that $\widetilde{t_1'} \in Q'(Q' \cap R') = Q'$ and $\widetilde{t_2'} \in (Q' \cap R')R' = R'$. In the case that t_1 is the segment u (respectively, p_n), we have $\widetilde{t_1'} \in U(Q' \cap R') \subseteq U$ and $\widetilde{t_2'} \in (Q' \cap R')R' = R'$ (respectively, $\widetilde{t_1'} \in Q'(Q' \cap R') = Q'$ and $\widetilde{t_2'} \in (Q' \cap R')V \subseteq V$) as U and V are Q'/R'-absorbing. Hence, the broken line p' obtained by replacing t_1 and t_2 in p with t_1' and t_2' respectively is a path representative of the same element $g \in G$.

If at least one of the paths q, r is not geodesic in $\Gamma(G, X \cup \mathcal{H})$, then, in view of (3.19) and (3.21) we have

$$\ell(t_1') + \ell(t_2') < \ell(q) + \ell(r) \leq \ell(t_1) + \ell(t_2),$$

hence $\ell(p) = \ell(u) + \ell(v) + \sum_{i=1}^{n} \ell(p_i) > \ell(p')$, contradicting the minimality of the type of p.

Hence both q and r must be geodesic in $\Gamma(G, X \cup \mathcal{H})$, so we can further assume that $t_1' = q$ and $t_2' = r$. Moreover, the inequality $\ell(p) \leq \ell(p')$ must hold by the minimality of the type of p. Therefore $\ell(t_1) + \ell(t_2) \leq \ell(q) + \ell(r)$, which, in view of (3.19) and (3.21), implies that $\ell(q) = \ell(t_1)$, $\ell(r) = \ell(t_2)$ and $\ell(p) = \ell(p')$. In particular, e and f are actual edges of $\Gamma(G, X \cup \mathcal{H})$ (and not trivial paths).

The definition (3.18) of q implies that $\mathrm{Lab}(q)$ can differ from $\mathrm{Lab}(t_1)$ in at most one letter, which is the label of the H_ν-component e in $\mathrm{Lab}(q)$ and the label of the H_ν-component s_1 in $\mathrm{Lab}(t_1)$. Indeed,

$$\begin{aligned} \mathrm{Lab}(t_1) &= \mathrm{Lab}([(t_1)_-, (s_1)_-])\,\mathrm{Lab}(s_1)\,\mathrm{Lab}(\alpha), \quad \text{and} \\ \mathrm{Lab}(q) &= \mathrm{Lab}([(t_1)_-, (s_1)_-])\,\mathrm{Lab}(e)\,\mathrm{Lab}(\alpha), \end{aligned}$$

where we used the fact that $[wa, w]$ is the left translate of α, by definition, and hence its label is the same $\mathrm{Lab}(\alpha)$. Similarly, (3.20) implies $\mathrm{Lab}(r)$ can differ from $\mathrm{Lab}(t_1)$ in at most one letter which is the label of f in r and the label of s_2 in t_2. The minimality of the type of p therefore implies that

$$|s_1|_X + |s_2|_X \leq |e|_X + |f|_X. \tag{3.22}$$

Now, using the triangle inequality, (3.17) and (3.16) we obtain

$$|e|_X = \mathrm{d}_X((s_1)_-, wa) \leq \mathrm{d}_X((s_1)_-, w) + \mathrm{d}_X(w, wa) \leq k + |a|_X \leq k + \xi. \tag{3.23}$$

To estimate $|f|_X$ we also use the inequality $\mathrm{d}_X((s_1)_-, (s_2)_+) \leq \zeta$:

$$\begin{aligned} |f|_X = \mathrm{d}_X((s_2)_+, wb) &\leq \mathrm{d}_X((s_2)_+, w) + |b|_X \\ &\leq \mathrm{d}_X((s_2)_+, (s_1)_-) + \mathrm{d}_X((s_1)_-, w) + \xi + 3L \\ &\leq \zeta + k + \xi + 3L. \end{aligned} \tag{3.24}$$

Combining (3.22)–(3.24) together, we achieve

$$\max\{|s_1|_X, |s_2|_X\} \le |e|_X + |f|_X \le 2k + 2\xi + \zeta + 3L = \tau.$$

This inequality completes the proof of the lemma. □

The following auxiliary definition will only be used in the remainder of this section.

Definition 3.17. Let $C_0 \ge 0$ be the constant provided by Lemma 3.15, let $L \ge 0$ be the constant given by Proposition 2.25 and let $\kappa = \kappa(1, 0, 3L) \ge 0$ be the constant from Proposition 2.34.
Define the sequences $(\zeta_j)_{j\in\mathbb{N}}$, $(\xi_j)_{j\in\mathbb{N}}$ and $(\tau_j)_{j\in\mathbb{N}}$ of non-negative real numbers as follows. Set $\zeta_1 = \kappa$, $\xi_1 = C_0 + 1$ and $\tau_1 = \max\{\kappa, \tau(\zeta_1, \xi_1)\}$, where $\tau(\zeta_1, \xi_1)$ is given by Lemma 3.16.
Now suppose that $j > 1$ and the first $j - 1$ members of the three sequences have already been defined. Then we set

$$\zeta_j = \kappa,\ \xi_j = C_0 + 1 + \sum_{k=1}^{j-1} \tau_k \text{ and } \tau_j = \max\{\kappa, \tau(\zeta_j, \xi_j)\},$$

where where $\tau(\zeta_j, \xi_j)$ is given by Lemma 3.16.

Lemma 3.18. *There exists a constant $C_1 \ge 0$ such that the following is true.
Suppose U and V are either trivial or Q'/R'-absorbing. Let $Q' \leqslant Q$ and $R' \leqslant R$ be subgroups satisfying (C1) and let $p = up_1 \dots p_n v$ be a minimal type path representative for an element $g \in U\langle Q', R'\rangle V$. Suppose that q and r are connected $\mathcal{H}$-components of adjacent segments t_1 and t_2 of p. Then $\mathrm{d}_X(q_+, (t_1)_+) \le C_1$ and $\mathrm{d}_X((t_1)_+, r_-) \le C_1$.*

Proof. Denote $x = (t_1)_+ = (t_2)_- \in G$. First, let us show that

$$\mathrm{d}_{X\cup\mathcal{H}}(q_+, x) \le C_0 + 1, \tag{3.25}$$

where $C_0 \ge 0$ is the global constant provided by Lemma 3.15. Indeed, the latter lemma states that $\langle (t_1)_-, (t_2)_+\rangle_x^{rel} \le C_0$. Since q_+ and r_- are points on the geodesics t_1 and t_2, Remark 2.7 implies that

$$\langle q_+, r_-\rangle_x^{rel} \le \langle (t_1)_-, (t_2)_+\rangle_x^{rel} \le C_0.$$

Consequently,

$$\begin{aligned} C_0 \ge \langle q_+, r_-\rangle_x^{rel} &= \frac{1}{2}\Big(\mathrm{d}_{X\cup\mathcal{H}}(x, q_+) + \mathrm{d}_{X\cup\mathcal{H}}(x, r_-) - \mathrm{d}_{X\cup\mathcal{H}}(q_+, r_-)\Big) \\ &\ge \frac{1}{2}\Big(2\,\mathrm{d}_{X\cup\mathcal{H}}(x, q_+) - 2\,\mathrm{d}_{X\cup\mathcal{H}}(q_+, r_-)\Big) \ge \mathrm{d}_{X\cup\mathcal{H}}(x, q_+) - 1, \end{aligned}$$

where the last inequality used the fact that $\mathrm{d}_{X\cup\mathcal{H}}(q_+, r_-) \le 1$, which is true because q and r are connected $\mathcal{H}$-components. This establishes the inequality (3.25).

Let α denote the subpath of t_1 starting at q_+ and ending at x, and let β denote the subpath of t_2 starting at x and ending at r_-. Let $s_1, \dots, s_l$, $l \in \mathbb{N}_0$, be the set of all $\mathcal{H}$-components of α listed in the reverse order of their occurrence. That is, s_1 is the last $\mathcal{H}$-component of α (closest to $\alpha_+ = x$) and s_l is the first $\mathcal{H}$-component of α (closest to $\alpha_- = q_+$). Note that, by (3.25),

$$l \leq \ell(\alpha) = d_{X \cup \mathcal{H}}(x, q_+) \leq C_0 + 1. \tag{3.26}$$

Let $L \geq 0$ be the constant given by Proposition 2.25. Then by Lemma 2.31,

$$\mathrm{d}_X(\alpha_-, \beta_+) = \mathrm{d}_X(q_+, r_-) \leq 3L. \tag{3.27}$$

It follows that the geodesic paths α and β^{-1} are $3L$-similar in $\Gamma(G, X \cup \mathcal{H})$. Let $\kappa = \kappa(1, 0, 3L) \geq 0$ be the constant provided by Proposition 2.34.

We will now prove the following.

Claim 3.6. For each $j = 1, \dots, l$ we have

$$|s_j|_X \leq \tau_j, \tag{3.28}$$

where $\tau_j \geq 0$ is given by Definition 3.17.

We will establish the claim by induction on j. For the base of induction, $j = 1$, note that if $|s_1|_X < \kappa$ then the inequality $|s_1|_X \leq \tau_1$ will be true by definition of τ_1. Thus we can suppose that $|s_1|_X \geq \kappa$. In this case, by Proposition 2.34, s_1 must be connected to some $\mathcal{H}$-component of β^{-1}. Claim (3) of the same proposition implies that there is an $\mathcal{H}$-component s_1' of β, such that s_1 is connected to s_1' and $\mathrm{d}_X((s_1)_-, (s_1')_+) \leq \kappa$. Note that, by construction, s_1 and s_1' are also connected $\mathcal{H}$-components of t_1 and t_2 respectively.

Observe that the subpath of α from $(s_1)_+$ to x is labelled by letters from $X^{\pm 1}$ because it has no $\mathcal{H}$-components. Therefore $\mathrm{d}_X((s_1)_+, x) \leq \ell(\alpha) \leq C_0 + 1$. Consequently, we can apply Lemma 3.16 to deduce that $|s_1|_X \leq \tau(\zeta_1, \xi_1)$, where $\zeta_1 = \kappa$ and $\xi_1 = C_0 + 1$.

Thus we have shown that $|s_1|_X \leq \tau_1$, where $\tau_1 = \max\{\kappa, \tau(\zeta_1, \xi_1)\}$, and the base of induction has been established.

Now, suppose that $j > 1$ and inequality (3.28) has been proved for all strictly smaller values of j. If $|s_j|_X < \kappa$ then are done, because $\tau_j \geq \kappa$ by definition. So we can assume that $|s_j|_X \geq \kappa$. As before, we can use Proposition 2.34, to find an $\mathcal{H}$-component s_j' of β such that s_j is connected to s_j' and $\mathrm{d}_X((s_j)_-, (s_j')_+) \leq \kappa$.

By construction, $s_1, \dots, s_{j-1}$ is the list of all $\mathcal{H}$-components of the subpath $[(s_j)_+, x]$ of α, hence

$$d_X((s_j)_+, x) \leq \ell(\alpha) + \sum_{k=1}^{j-1} |s_k|_X \leq C_0 + 1 + \sum_{k=1}^{j-1} \tau_k,$$

where the second inequality used (3.26) and the induction hypothesis. This allows us to apply Lemma 3.16 again, and conclude that $|s_j|_X \leq \tau(\zeta_j, \xi_j)$, where $\zeta_j = \kappa$ and $\xi_j = C_0 + 1 + \sum_{k=1}^{j-1} \tau_k$.

Thus, $|s_j|_X \leq \max\{\kappa, \tau(\zeta_j, \xi_j)\} = \tau_j$, as required. Hence the claim has been proved by induction on j.

We are finally ready to prove the main statement of the lemma. Since $s_1, \dots, s_l$ is the list of all $\mathcal{H}$-components of α, we can combine the inequalities (3.26) and (3.28) to achieve

$$d_X(q_+, (t_1)_+) = |\alpha|_X \leq \ell(\alpha) + \sum_{j=1}^{l} |s_j|_X \leq C_0 + 1 + \sum_{j=1}^{l} \tau_j \leq C_0 + 1 + \sum_{j=1}^{\lfloor C_0+1 \rfloor} \tau_j.$$

On the other hand, by the triangle inequality and (3.27), we have

$$d_X((t_1)_+, r_-) \leq 3L + d_X(q_+, (t_1)_+) \leq 3L + C_0 + 1 + \sum_{j=1}^{\lfloor C_0+1 \rfloor} \tau_j.$$

We have shown that the constant $C_1 = 3L + C_0 + 1 + \sum_{j=1}^{\lfloor C_0+1 \rfloor} \tau_j > 0$ is an upper bound for $d_X(q_+, (t_1)_+)$ and $d_X((t_1)_+, r_-)$, thus the lemma is proved. □

The next lemma shows that, among path representatives of minimal type, instances of adjacent backtracking where at least one of the components is sufficiently long with respect to the proper metric d_X must have initial and terminal vertices far apart in d_X.

Lemma 3.19 (Adjacent backtracking is long)**.** *For any $\zeta \geq 0$ there is $\Theta_0 = \Theta_0(\zeta) \in \mathbb{N}$ such that the following holds.*
Suppose U and V are either trivial or Q'/R'-absorbing. Let $Q' \leqslant Q$ and $R' \leqslant R$ be subgroups satisfying (C1) and let $p = p_1 \dots p_n$ be a minimal type path representative for an element $g \in U\langle Q', R' \rangle V$. Suppose that adjacent segments t_1 and t_2 of p have connected $\mathcal{H}$-components s_1 and s_2 respectively, satisfying

$$\max\{|s_1|_X, |s_2|_X\} \geq \Theta_0.$$

Then $d_X((s_1)_-, (s_2)_+) \geq \zeta$.

Proof. For any $\zeta \geq 0$ we can define $\Theta_0 = \lfloor \tau(\zeta, C_1) \rfloor + 1$, where C_1 is the constant from Lemma 3.18 and $\tau(\zeta, C_1)$ is provided by Lemma 3.16. It follows that whenever $d_X((s_1)_-, (s_2)_+) < \zeta$, we have $|s_1|_X < \Theta_0$ and $|s_2|_X < \Theta_0$, which is the contrapositive of the required statement. □

3.4 Multiple backtracking in path representatives

We will keep working under Convention 3.3. In this section we deal with multiple backtracking in path representatives of elements from $U\langle Q', R'\rangle V$. Proposition 3.22 below uses condition (C3) to show that any instance of multiple backtracking essentially takes place inside a single parabolic subgroup. In order to achieve this we first prove two auxiliary statements.

Notation 3.7. Let $C_1 \geq 0$ be the constant given by Lemma 3.18 and $M \geq 0$. We will denote by $\mathcal{P}_M$ the finite collection of parabolic subgroups of G defined by

$$\mathcal{P}_M = \{tH_\nu t^{-1} \mid \nu \in \mathcal{N}, |t|_X \leq C_1 + M\}.$$

Consider the subset $O = \{o \in PS \mid P \in \mathcal{P}_1,\ |o|_X \leq 2C_1\}$ of G. Since $|O| < \infty$, we can choose and fix a finite subset $\Xi \subseteq S$ such that every element $o \in O$ can be written as $o = fh$, where $f \in P$, for some $P \in \mathcal{P}_1$, and $h \in \Xi$. We define a constant E by

$$E = \max\{|h|_X \mid h \in \Xi\} \geq 0. \tag{3.29}$$

Lemma 3.20. *There exists a constant $D \geq 0$ such that the following holds.*
Let $\nu \in \mathcal{N}$ and $b \in G$ be an element with $|b|_X \leq C_1$, so that $P = bH_\nu b^{-1} \in \mathcal{P}_0$, and let p be a geodesic path in $\Gamma(G, X \cup \mathcal{H})$ with $\widetilde{p} \in Q \cup R$. Suppose that there is a vertex v of p and an element $u \in P$ such that $v \in Pb = bH_\nu$ and $u^{-1}p_- \in S = Q \cap R$. Then there exists a geodesic path p' in $\Gamma(G, X \cup \mathcal{H})$ such that

- *$p'_- = u$ and $\mathrm{d}_X(p'_+, v) \leq D$;*
- *if $\widetilde{p} \in Q$ then $\widetilde{p'} \in Q \cap P$, otherwise $\widetilde{p'} \in R \cap P$.*

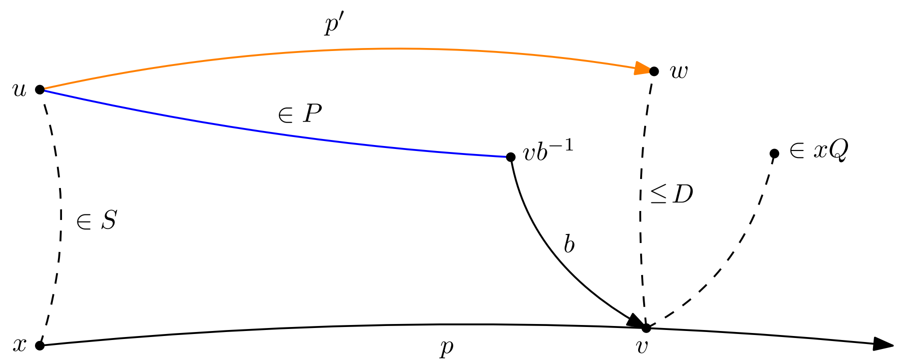

FIGURE 3.5: Illustration of Lemma 3.20.

Proof. Let $K = \max\{C_1, \varepsilon\} \geq 0$, where ε is the quasiconvexity constant of Q and R, and let

$$D = \max\{K'(Q, P, K), K'(R, P, K) \mid P \in \mathcal{P}_0\}, \tag{3.30}$$

where $K'(Q, P, K)$ and $K'(R, P, K)$ are obtained from Lemma 2.1.

Denote $x = p_- \in G$ and assume, without loss of generality, that $\widetilde{p} \in Q$ (the case $\widetilde{p} \in R$ can be treated similarly). By the quasiconvexity of Q, we have that $\mathrm{d}_X(v, xQ) \leq \varepsilon$. Moreover, $xQ = uQ$ as $u^{-1}x \in S \subseteq Q$.

By the assumptions, $vb^{-1} \in P$, hence $\mathrm{d}_X(v, P) \leq |b|_X \leq C_1$. Since $uP = P$ we see that

$$v \in N_X(uQ, \varepsilon) \cap N_X(uP, C_1).$$

Applying Lemma 2.1, we find $w \in u(Q \cap P)$ such that $\mathrm{d}_X(v, w) \leq D$ (see Figure 3.5). Let p' be any geodesic in $\Gamma(G, X \cup \mathcal{H})$ starting at u and ending at w. It is easy to see that p' satisfies all of the required properties, so the lemma is proved. □

The next lemma describes how condition (C3) is used.

Lemma 3.21. *Let $M \geq 0$ and suppose that subgroups $Q' \leqslant Q$ and $R' \leqslant R$ satisfy conditions (C1) and (C3) with constant C and family $\mathcal{P}$ such that $C \geq M + C_1 + 1$ and $\mathcal{P} \supseteq \mathcal{P}_M$. Let $P = bH_\nu b^{-1} \in \mathcal{P}_M$, for some $\nu \in \mathcal{N}$ and $b \in G$, with $|b|_X \leq M$, and let p be a path in $\Gamma(G, X \cup \mathcal{H})$ with $\widetilde{p} \in Q' \cup R'$.*
Suppose that there is a vertex v of p and an element $u \in P$ satisfying $u^{-1}p_- \in S$, $v \in Pb$, and $d_X(v, p_+) \leq C_1$. Then there exists a geodesic path p' such that $(p')_- = u$, $\widetilde{p'} \in P$, $(p')_+^{-1}p_+ \in S$, and $\mathrm{d}_X((p')_+, p_+) \leq E$, where E is the constant from (3.29). *In particular, if $\widetilde{p} \in Q'$ (respectively, $\widetilde{p} \in R'$) then $\widetilde{p'} \in Q' \cap P$ (respectively, $\widetilde{p'} \in R' \cap P$).*

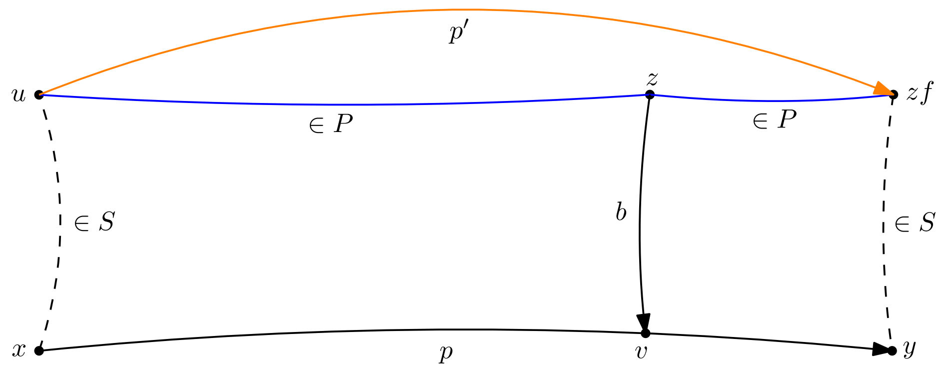

FIGURE 3.6: Illustration of Lemma 3.21.

Proof. Denote $x = p_-$, $y = p_+$ and $z = vb^{-1} \in P$ (see Figure 3.6). Then $u^{-1}z \in P$ and $x^{-1}y = \widetilde{p} \in Q' \cup R'$. Since $u^{-1}x \in S = Q' \cap R'$, we obtain

$$u^{-1}y = (u^{-1}x)(x^{-1}y) \in Q' \cup R',$$

whence $z^{-1}y = (z^{-1}u)(u^{-1}y) \in P(Q' \cup R')$. Now, observe that

$$|z^{-1}y|_X = \mathrm{d}_X(z, y) \leq \mathrm{d}_X(z, v) + \mathrm{d}_X(v, y) \leq |b|_X + C_1 \leq M + C_1 < C.$$

Condition (C3) now implies that $z^{-1}y \in PS$. That is, $z^{-1}y = fh$ for some $f \in P$ and $h \in \Xi$, where Ξ is the finite subset of S defined above the statement of the lemma. Let p' be a geodesic path starting at u and ending at $zf \in P$. Then $\widetilde{p'} = u^{-1}zf \in P$,

$$(p')_+^{-1} p_+ = f^{-1}z^{-1}y = h \in S \text{ and } \mathrm{d}_X((p')_+, p_+) = |h|_X \le E.$$

The last statement of the lemma follows from (C1) and the observation that

$$\widetilde{p'} = u^{-1}(p')_+ = u^{-1}p_- \,\widetilde{p}\,(p_+)^{-1}(p')_+ \in S\,\widetilde{p}\,S. \qquad \square$$

Proposition 3.22. *Let $D \ge 0$ is the constant provided by Lemma 3.20, and let E be given by* (3.29). *Suppose that $Q' \leqslant Q$ and $R' \leqslant R$ are subgroups satisfying (C1) and (C3), with constant $C \ge 2C_1 + 1$ and family $\mathcal{P} \supseteq \mathcal{P}_0$. Further, suppose that U and V are either trivial or Q'/R'-absorbing.*

Let p be a path representative for an element $g \in U\langle Q', R'\rangle V$ with minimal type. If p has consecutive backtracking along $\mathcal{H}$-components $h_i, \dots, h_j$ of consecutive segments $p_i, \dots, p_j$ respectively, then there is a subgroup $P \in \mathcal{P}_0$ and a path $p' = p'_i \dots p'_j$ satisfying the following properties:

- *(i) p'_k is geodesic with $\widetilde{p'_k} \in P$ for all $k = i, \dots, j$;*
- *(ii) $(p'_i)_+ = (p_i)_+$, $(p'_k)_+^{-1}(p_k)_+ \in S$ and $\mathrm{d}_X((p'_k)_+, (p_k)_+) \le E$, for all $k = i+1, \dots, j-1$;*
- *(iii) $\mathrm{d}_X(p'_-, (h_i)_-) \le D$ and $\mathrm{d}_X(p'_+, (h_j)_+) \le D$;*
- *(iv) $\widetilde{p'_i} \in Q \cap P$ if $\widetilde{p}_i \in Q$, and $\widetilde{p'_i} \in R \cap P$ if $\widetilde{p}_i \in R$; similarly, $\widetilde{p'_j} \in Q \cap P$ if $\widetilde{p}_j \in Q$, and $\widetilde{p'_j} \in R \cap P$ if $\widetilde{p}_j \in R$;*
- *(v) for each $k \in \{i+1, \dots, j-1\}$, $\mathrm{Lab}(p'_k)$ either represents an element of $Q' \cap P$ or an element of $R' \cap P$.*

Proof. Let $p = up_1 \dots p_n v$ be a path representative of $g \in U\langle Q', R'\rangle V$ of minimal type. For simplicity, we assume that $h_i, \dots, h_j$ are connected $\mathcal{H}$-components of the segments $p_i, \dots, p_j$. The case when the consecutive backtracking begins in the segment u or ends in the segment v may be dealt with identically.

Figure 3.7 below is a sketch of the path p' above the subpath $p_i p_{i+1} \dots p_{j-1} p_j$ of p.

Note that claim (v) follows from claim (ii) and condition (C1), so we only need to establish claims (i)–(iv).

By the assumptions, there is $\nu \in \mathcal{N}$ such that for each $k \in \{i, \dots, j\}$, the path p_k is a concatenation $p_k = a_k h_k b_k$, where h_k is an H_ν-component of p_k and a_k, b_k are subpaths of p_k.

According to Lemma 3.18, we have

$$|b_k|_X \le C_1, \text{ for } k = i, \dots, j-1. \tag{3.31}$$

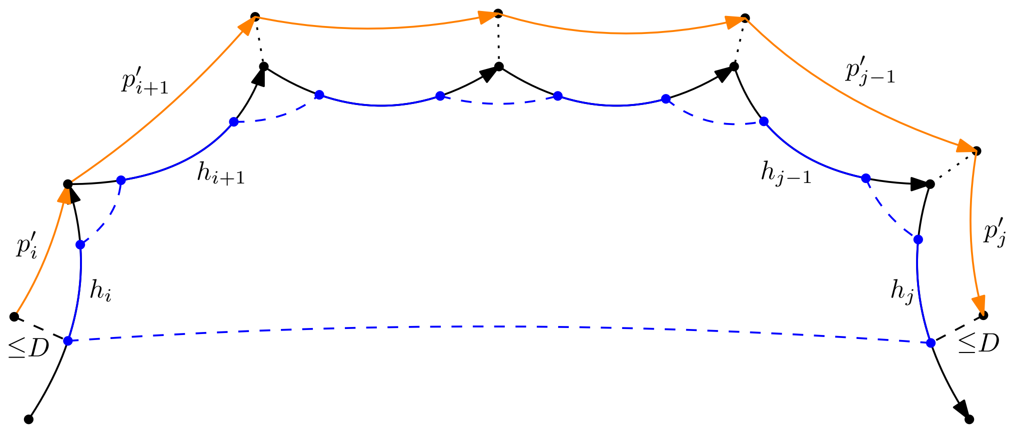

FIGURE 3.7: Illustration of Proposition 3.22.

After translating everything by $(p_i)_+^{-1}$ we can assume that $(p_i)_+ = 1$. From here on, we let $b = \widetilde{b_i}^{-1} \in G$ and $P = bH_v b^{-1}$. As noted in (3.31), $|b|_X = |b_i|_X \leq C_1$, so $P \in \mathcal{P}_0$.

Since the components h_i and h_k are connected, for every $k = i+1, \ldots, j$, the elements $(h_i)_+ = (b_i)_- = b$ and $(h_k)_+$ all belong to the same left coset $bH_v = Pb$, thus

$$(h_k)_+ \in Pb, \text{ for all } k = i+1, \ldots, j. \tag{3.32}$$

The rest of the argument will be divided into three steps.

Step 1: construction of the path p'_i.

Set $u_i = (p_i)_+ = 1$ and $v_i = (h_i)_-$. Then $v_i = \widetilde{b_i}^{-1}\widetilde{h_i}^{-1} \in bH_v = Pb$, so the path p_i^{-1}, its vertex v_i and the element $u_i = 1 \in P$ satisfy the assumptions of Lemma 3.20. Therefore there exists a path t with $t_- = u_i$, $d_X(t_+, v_i) \leq D$ and such that $\widetilde{t} \in Q \cap P$ if $\widetilde{p_i} \in Q$ and $\widetilde{t} \in R \cap P$ if $\widetilde{p_i} \in R$.

It is easy to check that the path $p'_i = t^{-1}$ enjoys the required properties.

Step 2: construction of the paths p'_k, for $k = i+1, \ldots, j-1$.

We will define the paths p'_k by induction on k. For $k = i+1$ we consider the path p_{i+1}, its vertex $v_{i+1} = (h_{i+1})_+$ and the element $u_i = 1 = (p_{i+1})_-$. Since $v_{i+1} \in Pb$ by (3.32) and $d_X(v_{i+1}, (p_{i+1})_+) = |b_{i+1}|_X \leq C_1$ by (3.31), we can apply Lemma 3.21 to find a geodesic path p'_{i+1} starting at u_i and satisfying the required conditions.

Now suppose that the required paths $p'_{i+1}, \ldots, p'_m$ have already been constructed for some $m \in \{i+1, \ldots, j-2\}$. To construct the path p'_{m+1}, let v_{m+1} be the vertex $(h_{m+1})_+$ of p_{m+1} and set $u_m = (p'_m)_+$. Then $u_m \in P$ and $u_m^{-1}(p_{m+1})_- = (p'_m)_+^{-1}(p_m)_+ \in S$ by the induction hypothesis. In view of (3.32) and (3.31), $v_{m+1} \in Pb$ and $d_X(v_{m+1}, (p_{m+1})_+) \leq C_1$, therefore we can find a geodesic path p'_{m+1} with the desired properties by using Lemma 3.21.

Thus we have described an inductive procedure for constructing the paths p'_k, for $k = i+1, \dots, j-1$.

<u>*Step 3:*</u> construction of the path p'_j.

This step is similar to Step 1: the path p'_j will start at $u_{j-1} = (p'_{j-1})_+ \in P$ and can be constructed by applying Lemma 3.20 to the path p_j and the elements $v_j = (h_j)_+ \in Pb$, $u_{j-1} \in P$.

We have thus constructed a sequence of geodesic paths $p'_i, \dots, p'_j$ whose concatenation p' satisfies all the properties from the proposition. □

We will now prove the main result of this section, which states that the initial and terminal vertices of an instance of multiple backtracking in a minimal type path representative must lie far apart in the proper metric d_X, provided $Q' \leqslant Q$ and $R' \leqslant R$ satisfy (C1)–(C5) with sufficiently large constants.

Proposition 3.23 (Multiple backtracking is long)**.** *For any $\zeta \geq 0$ there is $C_2 = C_2(\zeta) \geq 0$ such that if $Q' \leqslant Q$ and $R' \leqslant R$ are subgroups satisfying conditions (C1)-(C5) with constants $B \geq C_2$ and $C \geq C_2$ and a family $\mathcal{P} \supseteq \mathcal{P}_0$, then the following is true.*
Suppose U and V are trivial or Q'/R'-absorbing and let $g \in U\langle Q', R'\rangle V$ with $g \notin UQ'V$ and $g \notin UR'V$. Let p be a minimal type path representative of g in $\Gamma(G, X \cup \mathcal{H})$. If p has multiple backtracking along $\mathcal{H}$-components $h_i, \dots, h_j$ of consecutive segments of p, then $\mathrm{d}_X((h_i)_-, (h_j)_+) \geq \zeta$.

Proof. Let $\zeta \geq 0$ and define $C_2(\zeta) = \max\{2C_1, \zeta + 2D\} + 1$, where $D \geq 0$ is the constant obtained from Lemma 3.20. Suppose that $p_i, \dots, p_j$ are consecutive segments of p such that h_k is an H_ν-component of p_k for each $k = i, \dots, j$.

In view of the assumptions we can apply Proposition 3.22 to find a path $p' = p'_i \dots p'_j$ and $P \in \mathcal{P}_0$ satisfying properties (i)–(v) from its statement. Let α be a geodesic with $\alpha_- = (p'_j)_-$ and $\alpha_+ = (p_j)_-$, and let $\beta = p'_{i+1} \dots p'_{j-1}$. We will denote $x_k = \widetilde{p_k}$ and $x'_k = \widetilde{p'_k}$, for each $k \in \{i, \dots, j\}$, and $z = \widetilde{\alpha}$. Condition (C1), together with property (ii) of Proposition 3.22, tell us that $z \in S = Q' \cap R'$, and property (v) yields that

$$\widetilde{\beta} = x'_{i+1} \dots x'_{j-1} \in \langle Q'_P, R'_P \rangle \tag{3.33}$$

(as before, for a subgroup $H \leqslant G$ we denote by $H_P \leqslant G$ the intersection $H \cap P$).

Now suppose, for a contradiction, that $\mathrm{d}_X((h_i)_-, (h_j)_+) < \zeta$. Then

$$|p'|_X = \mathrm{d}_X(p'_-, p'_+) < \zeta + 2D < C_2 \leq \min\{B, C\}, \tag{3.34}$$

by claim (iii) of Proposition 3.22. There are four cases to consider depending on whether $\widetilde{p_i}$ and $\widetilde{p_j}$ are elements of Q or R.

Case 1: $x_i = \widetilde{p_i} \in Q$ and $x_j = \widetilde{p_j} \in Q$. Then, by claim (iv) of Proposition 3.22, both x'_i and x'_j are elements of Q_P . It follows that $\widetilde{p'} \in Q_P \langle Q'_P, R'_P \rangle Q_P \subseteq Q\langle Q', R'\rangle Q$. By (3.34) and (C2), there is $q \in Q$ such that $\widetilde{p'} = q$. Therefore

$$\widetilde{\beta} = x_i'^{-1}\, \widetilde{p'}\, x_j'^{-1} = x_i'^{-1}\, q\, x_j'^{-1} \in Q. \tag{3.35}$$

Combining (3.35) with (3.33) and using condition (C4), we get

$$\widetilde{\beta} \in Q \cap \langle Q'_P, R'_P\rangle = Q_P \cap \langle Q'_P, R'_P\rangle = Q'_P.$$

Let γ be any geodesic path in $\Gamma(G, X \cup \mathcal{H})$ starting at $(p_i)_-$ and ending at $(p_j)_+$. Then γ shares the same endpoints with the path $p_i\beta\alpha p_j$, therefore their labels represent the same element of G:

$$\widetilde{\gamma} = x_i\, \widetilde{\beta}\, z\, x_j \in x_i Q'_P\, S x_j = x_i Q' x_j.$$

When neither p_i and p_j are segments corresponding to U and V respectively, $x_i, x_j \in Q'$ and so $\widetilde{\gamma} \in Q'$. Thus we can use γ to obtain another path representative for g by replacing the subpath $p_i \dots p_j$ in p with γ, which consists of strictly fewer geodesic subpaths than $p = p_1 \dots p_n$. This contradicts the minimality of the type of p. If U and V are trivial we are now done, so suppose otherwise.

For the remaining possibilities, the assumption that U and V are Q'/R'-absorbing gives that either $\widetilde{\gamma} \in UQ'$, $\widetilde{\gamma} = Q'V$, or $g = \widetilde{\gamma} \in UQ'V$. In the former two cases, take geodesics γ_1 and γ_2 with $(\gamma_1)_- = \gamma_-$, $(\gamma_1)_+ = (\gamma_2)_-$, and $(\gamma_2)_+ = \gamma_+$ such that $\widetilde{\gamma_1} \in U$ and $\widetilde{\gamma_2} \in Q'$ (respectively, $\widetilde{\gamma_1} \in Q'$ and $\widetilde{\gamma_2} \in V$). We may then replace $p_i \dots p_j$ with $\gamma_1\gamma_2$ as before to obtain a path representative of lesser type. In the final case, the hypothesis that $g \notin UQ'V$ is contradicted. Hence Case 1 is considered.

Case 2: $\widetilde{p_i}$ and $\widetilde{p_j}$ are elements of R. This case can be dealt with identically to Case 1.

Case 3: $x_i = \widetilde{p_i} \in Q$ and $x_j = \widetilde{p_j} \in R$. Then $x'_i \in Q_P$ and $x'_j \in R_P$ by claim (iv) of Proposition 3.22. Hence $\mathrm{Lab}(p')$ represents an element of $x'_i\langle Q'_P, R'_P\rangle R_P$ with $x'_i \in Q_P$. In view of (3.34), we can use condition (C5) to deduce that $\widetilde{p'} \in x'_i Q'_P R_P$. It follows that

$$\widetilde{\beta} = (x'_i)^{-1}\, \widetilde{p'}\, (x'_j)^{-1} \in Q'_P\, R_P,$$

so there exist $q \in Q'_P$ and $r \in R_P$ such that $\widetilde{\beta} = qr$. Combining this with (3.33) we conclude that $r = q^{-1}\widetilde{\beta} \in R_P \cap \langle Q'_P, R'_P\rangle$, so $r \in R'_P$ by condition (C4), whence

$$\widetilde{\beta} = qr \in Q'_P\, R'_P. \tag{3.36}$$

Observe that the paths $\gamma = p_i \dots p_j$ and $p_i \beta \alpha p_j$ have the same endpoints, hence their labels represent the same element of G:

$$\widetilde{\gamma} = x_i \widetilde{\beta} z x_j \in x_i Q'_P \, R'_P \, S x_j \subseteq x_i Q' R' x_j.$$

When neither p_i and p_j are segments corresponding to U and V respectively, $x_i \in Q'$ and $x_j \in R'$ and so $\widetilde{\gamma} \in Q'R'$. Therefore there are elements $q_1 \in Q'$ and $r_1 \in R'$ such that $\widetilde{\gamma} = q_1 r_1$.

Let γ_1 be a geodesic path in $\Gamma(G, X \cup \mathcal{H})$ starting at $\gamma_- = (p_i)_-$ and ending at $\gamma_- q_1$ and let γ_2 be a geodesic path starting at $(\gamma_1)_+$ and ending at $(\gamma_1)_+ r_1 = \gamma_+ = (p_j)_+$. Since $\widetilde{\gamma_1} = q_1 \in Q'$ and $\widetilde{\gamma_2} = r_1 \in R'$ the path obtained from p by replacing the subpath $p_i \dots p_j$ with $\gamma_1 \gamma_2$ is a path representative of g. Moreover, it consists of fewer than n geodesic segments because $j > i + 1$ (by the definition of multiple backtracking), contradicting the minimality of the type of p. If U and V are trivial, we are now done, so suppose otherwise.

For the remaining possibilities, the assumption that U and V are Q'/R'-absorbing gives that either $\widetilde{\gamma} \in UR'$ or $\widetilde{\gamma} = Q'V$, or $g = \widetilde{\gamma} \in UV$. In the former two cases, take geodesics γ_1 and γ_2 with $(\gamma_1)_- = \gamma_-$, $(\gamma_1)_+ = (\gamma_2)_-$, and $(\gamma_2)_+ = \gamma_+$ such that $\widetilde{\gamma_1} \in U$ and $\widetilde{\gamma_2} \in Q'$ (respectively, $\widetilde{\gamma_1} \in Q'$ and $\widetilde{\gamma_2} \in V$). We may then replace $p_i \dots p_j$ with $\gamma_1 \gamma_2$ as before to obtain a path representative of lesser type. In the final case, the fact that $g \in UV$ contradicts the hypothesis that g is not an element of $UQ'V$ or $UR'V$.

<u>*Case 4:*</u> $x_i = \widetilde{p_i} \in R$ and $x_j = \widetilde{p_j} \in Q$. Then $x'_i \in R_P$ while $x'_j \in Q_P$, which implies that $\widetilde{p'} \in R_P \langle Q'_P, R'_P \rangle x'_j$, hence $\widetilde{p'}^{-1} \in (x'_j)^{-1} \langle Q'_P, R'_P \rangle R_P$.

By (3.34), we can use (C5) to conclude that $\widetilde{p'}^{-1} \in (x'_j)^{-1} Q'_P R_P$, thus $\widetilde{p'} \in R_P Q'_P x'_j$. The rest of the argument proceeds similarly to the previous case, leading to a contradiction with the minimality of the type of p. Hence Case 4 is also impossible.

We have arrived at a contradiction in each of the four cases, so $d_X((h_i)_-, (h_j)_+) \geq \zeta$, as required. □

3.5 Metric quasiconvexity theorem

In this section we prove Theorem 1.8. As usual, we work under Convention 3.3. First we will show that if some subgroups $Q' \leqslant Q$ and $R' \leqslant R$ satisfy conditions (C1)-(C5) with appropriately large constants, then minimal type path representatives of $\langle Q', R' \rangle$ meet the conditions of Proposition 3.4. We will then use the quasigeodesicity of shortcuttings of these path representatives to obtain quasiconvexity of $\langle Q', R' \rangle$.

Lemma 3.24. *Suppose that $Q' \leqslant Q$ and $R' \leqslant R$ satisfy (C2) with constant $B \geq 0$. Then*

$$\min_X \Big((Q' \cup R') \setminus S \Big) \geq B.$$

Proof. Let $g \in (Q' \cup R') \setminus S$. If $g \in Q'$ then $g \notin R$ as $g \notin S$. Therefore $g \in Q' \setminus R \subseteq R\langle Q', R' \rangle R \setminus R$, whence $|g|_X \geq B$ by (C2). Similarly, if $g \in R'$ then $g \in Q\langle Q', R' \rangle Q \setminus Q$, and (C2) again implies that $|g|_X \geq B$. □

Notation 3.8. For the remainder of this section we fix the following notation:

- C_0 is the constant provided by Lemma 3.15;
- $C_0' = \max\{C_0, 14\delta\}$ and $c_3 = c_3(C_0')$ is the constant obtained by applying Lemma 2.12;
- $\lambda = \lambda(C_0')$ and $c = c(C_0')$ are the first two constants from Proposition 3.4;
- $C_1 \geq 0$ is the constant from Lemma 3.18;
- $\mathcal{P}_0$ is the finite family of parabolic subgroups of G defined by

  $$\mathcal{P}_0 = \{ tH_\nu t^{-1} \mid \nu \in \mathcal{N}, |t|_X \leq C_1 \}$$

  as in Notation 3.7 (with $M = 0$).

Lemma 3.25. *For each $\eta \geq 0$ there are constants $B_1 = B_1(\eta) \geq 0$, $C_3 = C_3(\eta) \geq 0$, $\zeta = \zeta(\eta) \geq 1$, and $\Theta_1 = \Theta_1(\eta) \in \mathbb{N}$ such that the following is true.*
Suppose that $Q' \leqslant Q$ and $R' \leqslant R$ are subgroups satisfying conditions (C1)-(C5) with constants $B \geq B_1$ and $C \geq C_3$ and family $\mathcal{P} \supseteq \mathcal{P}_0$. If $p = p_1 \dots p_n$ is a minimal type path representative for an element $g \in \langle Q', R' \rangle$ then p is $(B, C_0', \zeta, \Theta_1)$-tamable.
Moreover, let $\Sigma(p, \Theta_1) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$ be the Θ_1-shortcutting of p obtained from Procedure 3.1, and let e_k' be the $\mathcal{H}$-component of $\Sigma(p, \Theta_1)$ containing e_k, $k = 1, \dots, m$. Then $\Sigma(p, \Theta_1)$ is a (λ, c)-quasigeodesic without backtracking and $|e_k'|_X \geq \eta$, for each $k = 1, \dots, m$.

Proof. We define the following constants:

- $\zeta = \zeta(\eta, C_0') \geq 0$, the constant provided by Proposition 3.4;
- $C_3 = C_2(\zeta) \geq 0$, where $C_2(\zeta)$ is given by Proposition 3.23;
- $\Theta_1 = \max\{\Theta_0(\zeta), \zeta\}$, where Θ_0 is the constant of Lemma 3.19;
- $B_1 = \max\{B_0(\Theta_1, C_0'), C_2(\zeta)\} \geq 0$, where B_0 is the remaining constant of Proposition 3.4.

Let $B \geq B_1$ and $C \geq C_3$. Suppose that Q', R', g and p are as in the statement of the lemma. In view of Remark 3.14, $\widetilde{p}_i \in (Q' \cup R') \setminus S$, for every $i = 2, \dots, n-1$. Therefore, by Lemma 3.24, we have

$$|p_i|_X \geq B, \text{ for each } i = 2, \dots, n-1. \tag{3.37}$$

On the other hand, Lemma 3.15 tells us that

$$\langle (p_i)_-, (p_{i+1})_+ \rangle^{rel}_{(p_i)_+} \leq C_0 \leq C_0', \text{ for each } i = 1, \dots, n-1. \tag{3.38}$$

Now suppose that p has consecutive backtracking along $\mathcal{H}$-components $h_i, \ldots, h_j$ of segments $p_i, \ldots, p_j$ satisfying

$$\max\left\{|h_i|_X, \ldots, |h_j|_X\right\} \geq \Theta_1.$$

If $j = i + 1$ then Lemma 3.19 and the choice of Θ_1 give that $\mathrm{d}_X((h_i)_-, (h_j)_+) \geq \zeta$. Otherwise Proposition 3.23 gives the same inequality. The above together with (3.37) and (3.38) show that p is $(B, C_0', \zeta, \Theta_1)$-tamable.

The remaining claims of the lemma follow from Proposition 3.4. □

We can now deduce the relative quasiconvexity of $\langle Q', R'\rangle$ by applying Lemma 3.25 with $\eta = 0$.

Theorem 3.26. *Suppose that $Q' \leqslant Q$ and $R' \leqslant R$ are relatively quasiconvex subgroups of G satisfying conditions (C1)-(C5) with family $\mathcal{P} \supseteq \mathcal{P}_0$ and constants $B \geq B_1(0)$, $C \geq C_3(0)$, where $B_1(0)$ and $C_3(0)$ are the constants provided by Lemma 3.25 applied to the case when $\eta = 0$. Then the subgroup $\langle Q', R'\rangle$ is relatively quasiconvex in G.*

Proof. By assumption the subgroups Q' and R' are relatively quasiconvex, with some quasiconvexity constant $\varepsilon' \geq 0$. For any element $g \in \langle Q', R'\rangle$ consider a geodesic τ in $\Gamma(G, X \cup \mathcal{H})$ with $\tau_- = 1$ and $\tau_+ = g$. Let u be any vertex of τ.

Since $g \in \langle Q', R'\rangle$, it has a path representative $p = p_1 \ldots p_n$ of minimal type, with $p_- = 1$. Let $\Sigma(p, \Theta) = f_0 e_1 f_1 \ldots f_{m-1} e_m f_m$ be the Θ-shortcutting of p obtained from Procedure 3.1, where $\Theta = \Theta_1(0)$ is provided by Lemma 3.25. This lemma implies that p is (B, C_0', ζ, Θ)-tamable and $\Sigma(p, \Theta)$ is a (λ, c)-quasigeodesic without backtracking, where $\lambda \geq 1$ and $c \geq 0$ are the constants fixed in Notation 3.8. Therefore, by Proposition 2.34, there is a phase vertex v of $\Sigma(p, \Theta)$ with $\mathrm{d}_X(u, v) \leq \kappa(\lambda, c, 0)$.

Since each e_i is a single edge, the vertex v lies on the geodesic subpath f_i of $\Sigma(p, \Theta)$, for some $i \in \{0, \ldots, m\}$. The subpath of p sharing endpoints with f_i is $(4, c_3)$-quasigeodesic by Lemma 3.6. Hence there is a vertex w of p such that $\mathrm{d}_X(v, w) \leq \kappa(4, c_3, 0)$, by Proposition 2.34.

Now w is a vertex of a subpath p_j of p, for some $j \in \{1, \ldots, n\}$. Let $x = (p_j)_-$, and note that $x \in \langle Q', R'\rangle$. Without loss of generality, suppose that $\widetilde{p_j} \in Q'$ (the case when $\widetilde{p_j} \in R'$ can be treated similarly). Then by the relative quasiconvexity of Q', $\mathrm{d}_X(w, xQ') \leq \varepsilon'$, whence $\mathrm{d}_X(w, \langle Q', R'\rangle) \leq \varepsilon'$. Therefore

$$\begin{aligned}\mathrm{d}_X(u, \langle Q', R'\rangle) &\leq \mathrm{d}_X(u, v) + \mathrm{d}_X(v, w) + \mathrm{d}_X(w, \langle Q', R'\rangle) \\ &\leq \kappa(\lambda, c, 0) + \kappa(4, c_3, 0) + \varepsilon',\end{aligned}$$

so that $\langle Q', R'\rangle$ is a relatively quasiconvex subgroup of G, with the quasiconvexity constant $\kappa(\lambda, c, 0) + \kappa(4, c_3, 0) + \varepsilon'$. □

3.6 Shortcuttings for paths representing parabolic elements

In this section we study the behaviour of shortcuttings of tamable broken lines that represent elements from parabolic subgroups of G. The aim is to show that tamable broken lines representing elements of some $bH_\nu b^{-1}$ consist of essentially a single instance of consecutive backtracking that involves all its segments, given that the element is sufficiently long in comparison to the conjugator b.

As a simplifying assumption, throughout this section we will often assume that b is such that $|b|_{X\cup\mathcal{H}}$ is minimal among elements in its left H_ν-coset. We observe that it does not cost us a lot to make such an assumption.

Remark 3.27. Let $b \in G$ and $\nu \in \mathcal{N}$. Suppose $|b|_{X\cup\mathcal{H}}$ is not minimal among elements of bH_ν. Let $b_1 = bh \in bH_\nu$ be such a minimal element, so that $|b_1|_{X\cup\mathcal{H}} < |b|_{X\cup\mathcal{H}}$. Since $b_1 \in bH_\nu$, it must be that $|b|_{X\cup\mathcal{H}} \le |b_1|_{X\cup\mathcal{H}} + 1$. Combining these inequalities, we in fact have that $|b|_{X\cup\mathcal{H}} = |b_1|_{X\cup\mathcal{H}} + 1$. Therefore the path $p = [1, b_1]e$, where e is an H_ν-edge labelled by h^{-1}, is a geodesic in $\Gamma(G, X \cup \mathcal{H})$. Moreover, if $|b|_X \le M$ then by Lemma 2.30, $|b_1|_X = d_X(1, e_-) \le \xi M^2$ where ξ is the constant of that lemma.

Lemma 3.28. *For any $M \ge 0$ there is $N_0 = N_0(M) \ge 1$ such that the following is true. Let $b \in G$ with $|b|_X \le M$, and let p be a geodesic in $\Gamma(G, X \cup \mathcal{H})$ with $\widetilde{p} \in bH_\nu b^{-1}$ for some $\nu \in \mathcal{N}$. Suppose that $|b|_{X\cup\mathcal{H}}$ is minimal among elements of bH_ν and denote by h the H_ν-edge with $h_- = p_- b$ and $\widetilde{p} = b\widetilde{h}b^{-1}$. If $|p|_X \ge N_0$, then h is connected to an H_ν-component h' of p with*

$$d_X(h_-, h'_-) \le 3L \quad \text{and} \quad d_X(h_+, h'_+) \le 3L,$$

where L is the constant from Proposition 2.25.

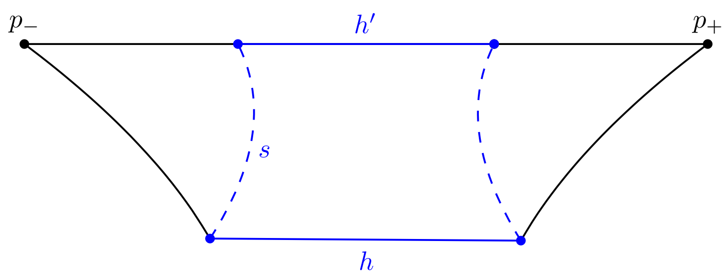

FIGURE 3.8: Illustration of Lemma 3.28.

Proof. Take $N_0 = 2M + \kappa$, where $\kappa = \kappa(1, 0, M)$ is the constant from Proposition 2.34 applied to M-similar geodesics. Suppose that $|p|_X \ge N_0$, so that $|h|_X \ge |p|_X - 2M \ge \kappa$ by the triangle inequality. Now we apply Proposition 2.34 to the M-similar geodesics h and p, which shows that h is connected to an H_ν-component h' of p. If $h'_- = h_-$, then we are done, so suppose otherwise. Take $s = [h_-, h'_-]$ to be the H_ν-edge in $\Gamma(G, X \cup \mathcal{H})$ labelled by the element $h_-^{-1}h'_- \in H_\nu$.

We will show that s is isolated in the geodesic triangle $[p_-, h_-] \cup [p_-, h'_-] \cup s$, whence we can conclude that $|s|_X = \mathrm{d}_X(h_-, h'_-) \leq 3L$ by applying Proposition 2.25. Suppose for a contradiction that s is connected to an H_ν-component t of $[p_-, h_-]$. Since s is connected to h, h is also connected to t. That is, the vertices t_- and h_- lie in the same H_ν-coset which implies that $\mathrm{d}_{X\cup\mathcal{H}}(t_-, h_-) \leq 1$. However, by minimality of $|b|_{X\cup\mathcal{H}}$ among elements of bH_ν, t cannot be the final edge of $[p_-, h_-]$. This means that $\mathrm{d}_{X\cup\mathcal{H}}(t_-, h_-) \geq 2$ by geodesicity of $[p_-, h_-]$, a contradiction. Similarly, if s is connected to an H_ν-component t of $[p_-, h'_-]$, then t is in turn connected to h', this time contradicting geodesicity of p (via Remark 2.22).

Thus $\mathrm{d}_X(h_-, h'_-) \leq 3L$ by Proposition 2.25. The same argument, by symmetry, shows that $\mathrm{d}_X(h_+, h'_+) \leq 3L$. □

For the remainder of the section, we fix a constant $C \geq 14\delta$, let $\lambda = \lambda(C)$ and $c = c(C)$ be the constants obtained from Proposition 3.4. Given any $\eta \geq 0$ we will write $\zeta(\eta, C)$ and, given any $\Theta \geq \zeta(\eta, C)$, we write $B_0 = B_0(\Theta, C)$ for the constants obtained from the same proposition. Finally, let $c_3 = c_3(C)$ be the constant of Lemma 2.12.

Lemma 3.29. *There is a constant $\kappa_0 = \kappa_0(C) \geq 0$ such that for any $M \geq 0, \eta \geq 0, \Theta \geq \zeta = \zeta(\eta, C)$ there is $N_1 = N_1(\Theta, M) \geq 1$ such that the following holds. Let $b \in G$ with $|b|_X \leq M$. Let $p = p_1 \dots p_n$ be a (B_0, C, ζ, Θ)-tamable broken line, and suppose that $\widetilde{p} \in bH_\nu b^{-1}$ for some $\nu \in \mathcal{N}$. Suppose that $|b|_{X\cup\mathcal{H}}$ is minimal among elements of bH_ν and denote by $\Sigma(p, \Theta) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$ its Θ-shortcutting. Let h be the H_ν-edge in $\Gamma(G, X \cup \mathcal{H})$ with $h_- = p_- b$ such that $\widetilde{p} = b\widetilde{h}b^{-1}$.*
If $|p|_X \geq N_1$, then h is connected to e_k for some $k = 1, \dots, m$ and

$$\mathrm{d}_X(h_-, (e_k)_-) \leq \kappa_0 \quad \text{and} \quad \mathrm{d}_X(h_+, (e_k)_+) \leq \kappa_0.$$

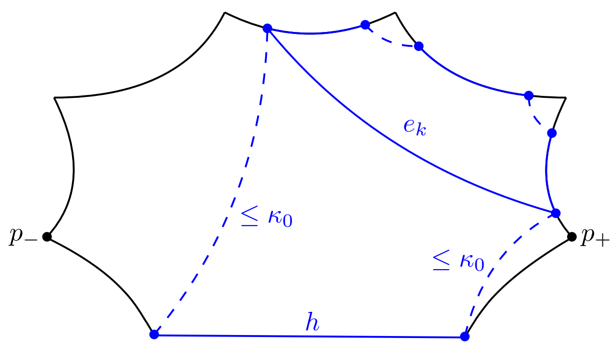

FIGURE 3.9: Illustration of Lemma 3.29.

Proof. We take the constants

- $\kappa_1 = \kappa(\lambda, c, 0)$ and $\kappa_2 = \kappa(4, c_3, 0)$, obtained by applying Proposition 2.34 to (λ, c)- and $(4, c_3)$-quasigeodesics with the same endpoints respectively;
- $N_1 = \max\{N_0, 2M + 9L + 2\kappa_1 + 2\kappa_2 + 2\Theta\} + 1$, where $N_0 = N_0(M)$ is the constant of Lemma 3.28 and L is the constant of Proposition 2.25;

- $\kappa_0 = \kappa_1 + \rho + 3L$, where $\rho = \rho(C)$ is the constant of Lemma 3.7.

Suppose that $|p|_X \geq N_1$. First observe that N_1 is greater than N_0, so that by Lemma 3.28 h is connected to an H_v-component h' of a geodesic $[p_-, p_+]$ (see Figure 3.10) with

$$\mathrm{d}_X(h_-, h'_-) \leq 3L \quad \text{and} \quad \mathrm{d}_X(h_+, h'_+) \leq 3L. \tag{3.39}$$

As $\widetilde{p} = bhb^{-1}$, the triangle inequality gives us that

$$\begin{aligned} |h|_X &\geq |p|_X - 2|b|_X \\ &\geq N_1 - 2M \\ &\geq 9L + 2\kappa_1 + 2\kappa_2 + 2\Theta + 1. \end{aligned} \tag{3.40}$$

Combining (3.39) and (3.40) yields that $|h'|_X \geq |h|_X - 6L \geq \kappa_1$. Moreover, by Proposition 3.4, $\Sigma(p, \Theta)$ is (λ, c)-quasigeodesic without backtracking. Therefore Proposition 2.34 tells us that there is an H_v-component h'' of $\Sigma(p, \Theta)$ connected to h' such that

$$\mathrm{d}_X(h'_-, h''_-) \leq \kappa_1 \quad \text{and} \quad \mathrm{d}_X(h'_+, h''_+) \leq \kappa_1. \tag{3.41}$$

Suppose for a contradiction that h'' is an H_v-component of f_k for some $k = 0, \dots, m$. Let p' be the subpath of p with $p'_- = (f_k)_-$ and $p'_+ = (f_k)_+$. Lemma 3.9 tells us that $\Sigma_0(p', 1)$ is $(4, c_3)$-quasigeodesic without backtracking. We have that $|h''|_X \geq |h|_X - 6L - 2\kappa_1 \geq \kappa_2$ by combining equations (3.39), (3.40), and (3.41). Hence by Proposition 2.34, h'' is connected to an H_v-component h''' of $\Sigma_0(p', 1)$ with

$$\mathrm{d}_X(h''_-, h'''_-) \leq \kappa_2 \quad \text{and} \quad \mathrm{d}_X(h''_+, h'''_+) \leq \kappa_2 \tag{3.42}$$

as in Figure 3.10.

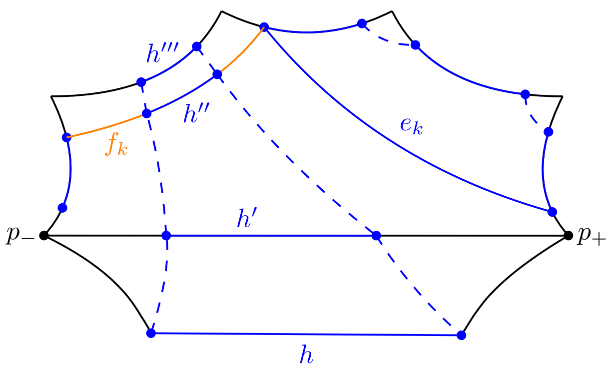

FIGURE 3.10: Illustration of proof of Lemma 3.29.

By the triangle inequality and equations (3.39)-(3.42), we have

$$|h'''|_X \geq |h|_X - 6L - 2\kappa_1 - 2\kappa_2 \geq 3L + 2\Theta_1 + 1,$$

whereas by Lemma 3.9, $|h'''|_X \leq 3L + 2\Theta$, a contradiction. Therefore h'' cannot be an H_ν-component of f_k. It follows that h'' is a component of $\Sigma(p, \Theta)$ containing e_k for some $k = 1, \ldots, m$ and thus that h is connected to e_k (as in Figure 3.9).

It remains to show the inequality in the lemma statement. Following Remark 2.22, h'' consists of at most three edges, one being e_k and the (possible) other two being edges, respectively the last and the first, of the geodesics f_{k-1} and f_k. Lemma 3.7 then implies that

$$d_X(h''_-, (e_k)_-) \leq \rho \quad \text{and} \quad d_X(h''_+, (e_k)_+) \leq \rho. \tag{3.43}$$

Finally, (3.39), (3.41), and (3.43) together with the choice of κ_0 give the inequalities

$$d_X(h_-, (e_k)_-) \leq \kappa_0 \quad \text{and} \quad d_X(h_+, (e_k)_+) \leq \kappa_0$$

as required. □

Lemma 3.30. *For any $M \geq 0$, there is a constant $\eta_0 = \eta_0(M) \geq 0$ such that for any $\Theta \geq \zeta = \zeta(\eta_0, C)$ the following is true.*
Let $b \in G$ with $|b|_X \leq M$. Let $p = p_1 \ldots p_n$ a be a (B_0, C, ζ, Θ)-tamable broken line, and suppose that $\widetilde{p} \in bH_\nu b^{-1}$ for some $\nu \in \mathcal{N}$. Suppose that $|b|_{X \cup \mathcal{H}}$ is minimal among elements of bH_ν and denote by $\Sigma(p, \Theta) = f_0 e_1 f_1 \ldots f_{m-1} e_m f_m$ the Θ-shortcutting of the path p. If $|p|_X \geq N_1$ (where $N_1 = N_1(\Theta, M)$ is the constant of Lemma 3.29), then $m = 1$.

Proof. We fix the following constants:

- $\kappa_1 = \kappa(\lambda, c, 0)$ and $\kappa_2 = \kappa(1, 0, 3L)$, the constants obtained by applying Proposition 2.34 to (λ, c)-quasigeodesics with the same endpoints and $3L$-similar geodesics respectively;
- $\eta = \eta(1, 0, M + 1)$ is provided by Lemma 2.29;
- $\eta_0 = \eta + 2\kappa_1 + 2\kappa_2 \geq 0$;

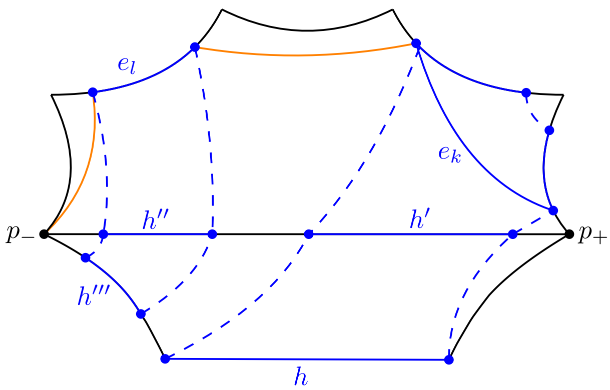

FIGURE 3.11: Illustration of Lemma 3.30.

Since $\widetilde{p} \in bH_\nu b^{-1}$, denote by h the H_ν-edge with $h_- = p_- b$ and $\widetilde{p} = b\widetilde{h}b^{-1}$. Lemma 3.29 tells us that h is connected to e_k for some $k = 1, \ldots, m$, so $m \geq 1$. Moreover, by

Lemma 3.28, h is connected to an H_ν-component h' of a geodesic $[p_-, p_+]$ and

$$d_X(h_-, h'_-) \leq 3L \quad \text{and} \quad d_X(h_+, h'_+) \leq 3L.$$

In particular, this implies that $[p_-, h_-]$ and $[p_-, h'_-]$ are $3L$-similar, and as are $[h_+, p_+]$ and $[h'_+, p_+]$.

Suppose for a contradiction that $m > 1$, so that there is $l \neq k$ with $1 \leq l \leq m$. By Proposition 3.4, the shortcutting $\Sigma(p, \Theta)$ is (λ, c)-quasigeodesic without backtracking, and further the $\mathcal{H}$-component e'_l of $\Sigma(p, \Theta)$ containing e_l satisfies the inequality

$$|e'_l|_X \geq \eta_0. \tag{3.44}$$

Now by Proposition 2.34, e'_l is connected to an $\mathcal{H}$-component h'' of the geodesic $[p_-, p_+]$ with

$$d_X(h''_-, (e'_l)_-) \leq \kappa_1 \quad \text{and} \quad d_X(h''_+, (e'_l)_+) \leq \kappa_1. \tag{3.45}$$

Since $\Sigma(p, \Theta)$ is without backtracking, h'' must be distinct from h': if not, then e'_l and e'_k would be connected $\mathcal{H}$-components of $\Sigma(p, \Theta)$.

We consider only the case that h'' is an $\mathcal{H}$-component of the subpath $[p_-, h'_-]$ of $[p_-, p_+]$, with the other case being dealt with identically. It follows from (3.44), (3.45), and the definition of η_0 that $|h''|_X \geq \kappa_2$. Since $[p_-, h'_-]$ and $[p_-, h_-]$ are $3L$-similar geodesics, Proposition 2.34 tells us that h'' is connected to an $\mathcal{H}$-component h''' of $[p_-, h_-]$ (respectively $[h_+, p_+]$) and h'' and h''' satisfy

$$d_X(h''_-, h'''_-) \leq \kappa_2 \quad \text{and} \quad d_X(h''_+, h'''_+) \leq \kappa_2. \tag{3.46}$$

Combining (3.45), (3.46), and (3.44), we see that

$$\begin{aligned} |h'''|_X &\geq |e'_l|_X - d_X(h'''_-, (e'_l)_-) - d_X(h'''_+, (e'_l)_+) \\ &\geq \eta_0 - 2(\kappa_1 + \kappa_2) \geq \eta, \end{aligned}$$

where the last inequality comes from the definition of η_0. Now we may apply Lemma 2.29 to see that

$$|b|_X = |[p_-, h_-]|_X \geq M + 1 > M$$

contradicting the fact that $|b|_X \leq M$. □

Lemma 3.31. *For any $M \geq 0$ and $\Theta \geq \zeta = \zeta(\eta_0, C)$ (where $\eta_0 = \eta_0(M)$ is the constant of Lemma 3.30) there is $B_2 = B_2(M, \Theta, C) \geq 0$ such that the following is true.*
Let $b \in G$ with $|b|_X \leq M$. Let $p = p_1 \dots p_n$ be a (B_2, C, ζ, Θ)-tamable broken line, and suppose that $\widetilde{p} \in bH_\nu b^{-1}$ for some $\nu \in \mathcal{N}$. Suppose that $|b|_{X \cup \mathcal{H}}$ is minimal among elements of bH_ν and denote by $\Sigma(p, \Theta) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$ the Θ-shortcutting of the path p.

If $|p|_X \geq N_1$ (where $N_1 = N_1(\Theta, M)$ is the constant of Lemma 3.29) then $(e_1)_-$ is a non-terminal vertex of p_1p_2, and $(e_m)_+$ is a non-initial vertex of $p_{n-1}p_n$. Moreover, if $|p_1|_X \geq B_2$ (respectively, $|p_n|_X \geq B_2$), then $(e_1)_-$ is a non-terminal vertex of p_1 (respectively, $(e_m)_+$ is a non-initial vertex of p_n).

Proof. Define $B_2 = \max\{B_0, (4M + 8 + c_0)\Theta\}$. Denote by h the H_ν-edge with $h_- = p_-b$ and $\widetilde{p} = b\widetilde{h}b^{-1}$. By Lemma 3.30 we have $m = 1$, and so by Lemma 3.29, h is connected to e_1.

We prove only the statement involving $(e_1)_-$, for a symmetrical argument shows the corresponding statement for $(e_1)_+$. Suppose to the contrary, so that $(e_1)_-$ is a vertex of p_i for $i > 2$. The subpath p' of p with endpoints $p'_- = p_-$ and $p'_+ = (e_1)_-$ is a $(4, c_3)$-quasigeodesic broken line in $\Gamma(G, X \cup \mathcal{H})$ by Lemma 3.6. Each $\mathcal{H}$-component h of the segments of p' satisfies $|h|_X \leq \Theta$ by Remark 3.1(c). Moreover, p_2 is a subpath of p' and $|p_2|_X \geq B_1$ by tamability condition (i). Then by Lemma 2.28,

$$\ell(p') \geq \ell(p_2) \geq \frac{B_2}{\Theta} \geq 4M + 8 + c_0,$$

whence by quasigeodesicity of p' we have

$$|p'|_{X\cup\mathcal{H}} \geq \frac{1}{4}\ell(p') - \frac{c_3}{4} \geq M + 2. \tag{3.47}$$

On the other hand, we have $d_{X\cup\mathcal{H}}(p_-, h_-) = |b|_{X\cup\mathcal{H}} \leq |b|_X \leq M$ and that $d_{X\cup\mathcal{H}}(h_-, (e_1)_-) \leq 1$ since h and e_1 are connected. It follows, then, that:

$$|p'|_{X\cup\mathcal{H}} \leq M + 1,$$

contradicting (3.47). This means that p' cannot contain the entire subpath p_2. Hence $p'_- = (e_1)_-$ must be a non-terminal vertex of p_1p_2. If, in addition, $|p_1|_X \geq B_2$ the same argument shows $(e_1)_-$ is a non-terminal vertex of p_1, as p_1 is also a subpath of p'. □

3.7 Reduction to short conjugators

In this section we again follow the notation of Convention 3.3. Our aim now is to obtain a metric version Theorem 1.2. We will first prove the special case for conjugates of the peripheral subgroups by uniformly short elements. In this case, taking Q' and R' with sufficiently deep index, the conjugator $u \in \langle Q', R' \rangle$ in the statement of theorem will be trivial. In particular, we will prove the following:

Proposition 3.32. *For any $M \geq 0$ there exist constants $B_3 = B_3(M) \geq 0$ and $C_4 = C_4(M) \geq 0$ such that the following is true.*

Suppose $Q' \leqslant Q$ and $R' \leqslant R$ satisfy conditions (C1)-(C5) with constants B_3, C_4, and family $\mathcal{P}_M$ (see Notation 3.7). If $P \in \mathcal{P}_M$ is such that $\langle Q', R' \rangle \cap P$ is infinite, then

$$\langle Q', R' \rangle \cap P = \langle Q' \cap P, R' \cap P \rangle.$$

Proof. Let $P \in \mathcal{P}_M$ and suppose that $\langle Q', R' \rangle \cap P$ is infinite. We will fix the following notation for the proof:

- $P = bH_\nu b^{-1}$ where $\nu \in \mathcal{N}$ and $b \in G$ with $|b|_X \leq M$;
- $b_1 \in bH_\nu$ which has minimal length with respect to $d_{X \cup \mathcal{H}}$ and $|b_1|_X \leq \xi M^2$ (as in Remark 3.27), where ξ is the constant of Lemma 2.30;
- $C_0' = \max\{C_0, 14\delta\}$, where C_0 is the constant of Lemma 3.15;
- $\eta_0 = \eta_0(\xi M^2)$ is the constant of Lemma 3.30;
- $B_1 = B_1(\eta_0), C_3 = C_3(\eta_0)$, and $\Theta_1 = \Theta_1(\eta_0)$ are the constants obtained from Lemma 3.25 applied with η_0;
- $N_1 = N_1(\Theta_1, \xi_0 M^2)$ and $\kappa_0 = \kappa_0(C_0')$ are the constants of Lemma 3.29;
- $B_3 = \max\{B_1, B_2(\xi M^2, \Theta_1, C_0')\}$, where $B_2(\xi M^2, \Theta_1, C_0')$ is the constant of Lemma 3.31 and $C_4 = \max\{C_3, M + C_1 + 1\}$, where C_1 is the constant of Lemma 3.18.

By assumption, $\langle Q', R' \rangle \cap P$ is infinite, so there is an element $g \in \langle Q', R' \rangle \cap P$ with $|g|_X \geq N_1$. Let $p = p_1 \dots p_n$ be a path representative of minimal type for g (as an element of $U\langle Q', R' \rangle V$, with $U = V = \{1\}$) with $p_- = 1$. If $n = 1$ then $g = \widetilde{p} \in (Q' \cup R') \cap P$ and we are done, so suppose that $n > 1$. We write h for the H_ν-edge of $\Gamma(G, X \cup \mathcal{H})$ with $h_- = b_1$ and $g = b_1 \widetilde{h} b_1^{-1}$.

We consider the shortcutting $\Sigma(p, \Theta_1) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$ of p obtained from Procedure 3.1. Lemma 3.24, together with the fact that p is minimal and $n > 1$, gives us that $|p_i|_X \geq B_3$ for each $i = 1, \dots, n$. Moreover, Lemma 3.25 gives that p is $(B_3, C_0', \zeta, \Theta_1)$-tamable. Lemmas 3.30 and 3.31 tell us that $m = 1$ and that $(e_1)_-$ and $(e_1)_+$ are non-terminal and non-initial vertices of p_1 and p_n respectively. As such, we may suppose that f_0 and f_1 are chosen to be subpaths of the geodesics p_1 and p_n respectively, so that e_1 is an $\mathcal{H}$-component of $\Sigma(p, \Theta_1)$. Moreover, Lemma 3.29 implies that e_1 is connected to h with

$$d_X(h_-, (e_1)_-) \leq \kappa_0 \quad \text{and} \quad d_X(h_+, (e_1)_+) \leq \kappa_0. \tag{3.48}$$

It follows that $(e_1)_- H_\nu = b_1 H_\nu = bH_\nu$. Denote by $h_1, \dots, h_n$ the pairwise connected H_ν-components of the segments $p_1, \dots, p_n$ that constitute the instance of consecutive backtracking associated to e_1.

We will inductively construct a sequence of paths $p_1', \dots, p_{n-1}'$ (cf. Proposition 3.22) with the following properties:

- $(p_1')_- = 1$;

- $\widetilde{p_i'} \in (Q' \cup R') \cap P$ for each $i = 1, \ldots, n-1$;
- $(p_i')^{-1} p_i \in S$ for each $i = 1, \ldots, n-1$.

It is straightforward to verify that p_1 satisfies the hypotheses of Lemma 3.21 with $u = 1, v = (e_1)_-$, and subgroup $bH_\nu b^{-1}$. Thus there is p_1' with $(p_1')_- = 1, \widetilde{p_1'} \in (Q' \cup R') \cap P$, and $(p_1')_+^{-1}(p_1)_+ \in S$. Similarly, for any $1 < i \leq n-1$, we can use Lemma 3.18 to verify that we can apply Lemma 3.21 to the path p_i with $u = (p_{i-1}')_+, v = (h_i)_+$, and $P = bH_\nu b^{-1}$. We thus obtain a path p_i' with $(p_i')_- = (p_{i-1}')_+, \widetilde{p_i'} \in (Q' \cup R') \cap P$, and $(p_i')^{-1} p_i \in S$.

We will write $z = (p_{n-1}')_+ = \widetilde{p_1'} \ldots \widetilde{p_{n-1}'} \in \langle Q' \cap P, R' \cap P \rangle$. Since $g \in P$ and $z \in P$, it is also true that $z^{-1}g \in P$. Moreover,

$$\begin{aligned} z^{-1}g &= z^{-1}(p_{n-1})_+(p_{n-1})_+^{-1} g \\ &= ((p_{n-1}')_+^{-1}(p_{n-1})_+)\widetilde{p_n} \in S(Q' \cup R') = Q' \cup R', \end{aligned}$$

so that $z^{-1}g \in (Q' \cap P) \cup (R' \cap P)$. Thus $g = zz^{-1}g \in \langle Q' \cap P, R' \cap P \rangle$.

Since g was an arbitrary element of $\langle Q', R' \rangle \cap P$ with $|g|_X \geq N_1$, we have shown that all but finitely many elements of $\langle Q', R' \rangle \cap P$ lie in $\langle Q' \cap P, R' \cap P \rangle$. Now applying Lemma 2.2 shows that the former subgroup is contained in the latter. The reverse inclusion is immediate. □

To complete the proof of the theorem, we reduce computation of the subgroup $\langle Q', R' \rangle \cap P$, where $P = bH_\nu b^{-1}$ is an arbitrary maximal parabolic subgroup, to the case when P belongs to a fixed finite set of maximal parabolic subgroups. An application of Proposition 3.32 will then yield the general result.

To do this, we observe that when $\langle Q', R' \rangle \cap P$ is infinite, the conjugator b has a decomposition as an element of $\langle Q', R' \rangle Qx$ or $\langle Q', R' \rangle Rx$ where $x \in G$ has uniformly bounded length with respect to d_X. Thus, up to conjugation by an element in $\langle Q', R' \rangle$, we need only consider intersections of the form $\langle Q', R' \rangle \cap sxH_\nu x^{-1}s^{-1}$, where $s \in Q \cup R$ and $\nu \in \mathcal{N}$.

Lemma 3.33. *There are constants $B_4 \geq 0$ and $\sigma \geq 0$ such that the following is true. Suppose $Q' \leqslant Q$ and $R' \leqslant R$ satisfy conditions (C1)-(C5) with constants $B_4, C_3(1)$ (obtained from Lemma 3.25) and family $\mathcal{P}_0$ (as in Notation 3.7, with $M = 0$). Let $P = bH_\nu b^{-1}$ be a maximal parabolic subgroup, with $|b|_{X \cup \mathcal{H}}$ minimal among elements of bH_ν.*
Suppose that $\langle Q', R' \rangle \cap P$ is infinite. Then there are elements $s \in Q \cup R, u \in \langle Q', R' \rangle$, and $x \in G$ such that $b = usx$ and $|x|_X \leq \sigma$. In particular,

$$\langle Q', R' \rangle \cap P = u\Big(\langle Q', R' \rangle \cap sxH_\nu x^{-1}s^{-1}\Big)u^{-1},$$

and $usxH_\nu x^{-1}s^{-1}u^{-1} = P$. Moreover, if $Q' \cap P$ or $R' \cap P$ is infinite, then we may take $u = 1$ in the above.

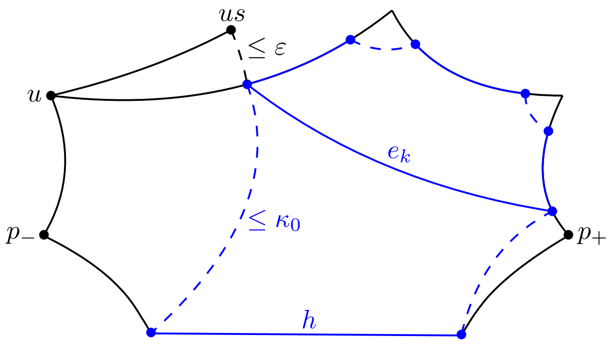

FIGURE 3.12: Illustration of Lemma 3.33.

Proof. We define the following notation for this proof:

- $C_0' = \max\{C_0, 14\delta\}$, where C_0 is the constant of Lemma 3.15;
- $\zeta = \zeta(1), \Theta_1 = \Theta_1(1), B_1 = B_1(1)$, and $C_3 = C_3(1)$ are the constants of Lemma 3.25;
- $B_4 = B_0(\Theta_1, C_0')$ is the constant of Proposition 3.4;
- $N_1 = N_1(\Theta_1, |b|_X)$ is obtained from Lemma 3.29;
- $\sigma = \kappa_0 + \varepsilon$, where $\kappa_0 = \kappa_0(C_0')$ is the constant of Lemma 3.29.

Since $\langle Q', R'\rangle \cap P$ is infinite, there is an element $g \in \langle Q', R'\rangle$ with $|g|_X \geq N_1$. Let $p = p_1 \dots p_n$ be a minimal type path representative of g (as an element of $U\langle Q', R'\rangle V$, with U and V trivial) with $p_- = 1$, and let h be the H_ν-edge of $\Gamma(G, X \cup \mathcal{H})$ such that $h_- = b$ and $g = \widetilde{p} = b\widetilde{h}b^{-1}$.

By Proposition 3.25, p is $(B_4, C_0', \zeta, \Theta_1)$-tamable. We will denote by $\Sigma(p, \Theta_1) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$ the Θ_1-shortcutting of p obtained from Procedure 3.1. Then by Lemma 3.29, h is connected to e_k for some $k = 1, \dots, m$ with

$$d_X(b, (e_k)_-) = d_X(h_-, (e_k)_-) \leq \kappa_0. \tag{3.49}$$

Take $u = (p_i)_- \in \langle Q', R'\rangle$. If $\widetilde{p_i} \in Q'$ then by quasiconvexity of Q, there is an element $s \in Q$ such that

$$d_X(us, (e_k)_-) \leq \varepsilon. \tag{3.50}$$

Otherwise $\widetilde{p_i} \in R'$, whence by the quasiconvexity of R, there is an element $s \in R$ satisfying the same inequality. In either case, take $x = s^{-1}u^{-1}b$ and observe that combining (3.49) with (3.50) gives

$$|x|_X = d_X(b, us) \leq d_X(b, (e_k)_-) + d_X(us, (e_k)_-) \leq \kappa_0 + \varepsilon = \sigma.$$

It is immediate from the definition of x that $b = usx$, whence $us^{-1}x^{-1}H_v xsu^{-1} = bH_v b^{-1} = P$. It follows that

$$u\Big(\langle Q', R'\rangle \cap sxH_v x^{-1}s^{-1}\Big)u^{-1} = u\langle Q', R'\rangle u^{-1} \cap usxH_v x^{-1}s^{-1}u^{-1}$$
$$= \langle Q', R'\rangle \cap P,$$

as required.

Finally, note that when $Q' \cap P$ is infinite we may take $g \in Q' \cap P$ with $|g|_X \geq N_1$, in which case p consists of a single geodesic segment. Following the above argument in this case gives that e_k is an H_v-component of this single segment and $u = p_- = 1$. The case with $R' \cap P$ infinite is identical. □

When s is not an element of Q' or R', the element sx obtained above cannot be further decomposed in a useful way, but it does fit into a sort of dichotomy. We find that either $\langle Q', R'\rangle$ intersects $sxH_v x^{-1}s^{-1}$ in an elementary way, or that sx is an element of $Q'yH_v$ or $R'yH_v$, where y has uniformly bounded length with respect to d_X. This completes the reduction (up to $\langle Q', R'\rangle$-conjugacy) of computing $\langle Q', R'\rangle \cap P$ from arbitrary maximal parabolic $P \leqslant G$ to finitely many conjugates of H_v, for $v \in \mathcal{N}$.

Proposition 3.34. *There are constants $B_5, C_5, \tau \geq 0$ such that if $Q' \leqslant Q$ and $R' \leqslant R$ satisfy (C1)-(C5) with B_5, C_5 and family $\mathcal{P}_0$ (as in Notation 3.7) then the following is true.*
Let $s \in Q \cup R$, $x \in G$ with $|x|_X \leq \sigma$ (where σ is the constant of Lemma 3.33), and $v \in \mathcal{N}$. If $\langle Q', R'\rangle \cap sxH_v x^{-1}s^{-1}$ is infinite then one of the following holds:

- *$s \in Q' \cup R'$ and $\langle Q', R'\rangle \cap sxH_v x^{-1}s^{-1} = s\langle Q' \cap xH_v x^{-1}, R' \cap xH_v x^{-1}\rangle s^{-1}$, or*
- *$s \in Q$ and $\langle Q', R'\rangle \cap sxH_v x^{-1}s^{-1} = Q' \cap sxH_v x^{-1}s^{-1}$, or*
- *$s \in R$ and $\langle Q', R'\rangle \cap sxH_v x^{-1}s^{-1} = R' \cap sxH_v x^{-1}s^{-1}$, or*
- *there are elements $t \in Q' \cup R'$ and $y \in G$, with $|y|_X \leq \tau$, such that $sx \in tyH_v$. In particular,*
$$\langle Q', R'\rangle \cap sxH_v x^{-1}s^{-1} = t\Big(\langle Q', R'\rangle \cap yH_v y^{-1}\Big)t^{-1}$$
and $tyH_v y^{-1}t^{-1} = sxH_v x^{-1}s^{-1}$.

Proof. In this proof we use the following notation:

- C_0 and C_1 are the constants of Lemmas 3.15 and 3.18 respectively, and $C_0' = \max\{C_0, 14\delta\}$;
- $x_1 \in xH_v$ has minimal length with respect to $\mathrm{d}_{X\cup\mathcal{H}}$ and $|x_1|_X \leq \xi\sigma^2$ (as in Remark 3.27), where ξ is the constant of Lemma 2.30;
- $\eta_0 = \eta_0(\xi\sigma^2)$ is the constant of Lemma 3.30;
- $\Theta_1 = \Theta_1(\eta_0)$ is the constant obtained from Lemma 3.25;
- $B_5 = \max\{B_1(\eta_0), B_2(\xi\sigma^2, \Theta_1, C_0'), B_3(\sigma)\}$ and $C_5 = \max\{C_3(\eta_0), C_4(\sigma)\}$, where $B_1(\eta_0)$ and $C_3(\eta_0)$ are the constants of Lemma 3.25 $B_2(\xi_0\sigma^2, \Theta_1, C_0')$ is the constant of Lemma 3.31, and $B_3(\sigma)$ and $C_4(\sigma)$ are those of Proposition 3.32;

- $\kappa_0 = \kappa_0(C_0')$ and $N_1 = N_1(\Theta_1, \xi_0\sigma^2)$ are the constants of Lemma 3.29;
- $\tau = \max\{C_1, B_5 + \xi_0\sigma^2 + \kappa_0\}$.

If $s \in Q' \cup R'$, then $s^{-1}\langle Q', R'\rangle s = \langle Q', R'\rangle$ so that

$$\langle Q', R'\rangle \cap sxH_\nu x^{-1}s^{-1} = s\Big(\langle Q', R'\rangle \cap xH_\nu x^{-1}\Big)s^{-1}.$$

Applying Theorem 3.32 gives us that $\langle Q', R'\rangle \cap xH_\nu x^{-1} = \langle Q' \cap xH_\nu x^{-1}, R' \cap xH_\nu x^{-1}\rangle$. Combining these two equalities gives the first case of the proposition. Thus we may assume $s \notin Q' \cup R'$ for the remainder of the proof.

If $s \in Q$, we define $U = s^{-1}Q'$, and otherwise set $U = s^{-1}R'$. In either case let $V = U^{-1}$. The sets U and V are Q'/R'-absorbing in both cases. Throughout this proof we will assume that $s \in Q$, with the case that $s \in R$ being identical. Note that these two cases are mutually exclusive, for otherwise we would have $s \in Q \cap R = Q' \cap R'$ by (C1), contradicting our assumption.

If $g \in Q'$ for all $g \in \langle Q', R'\rangle \cap sxH_\nu x^{-1}s^{-1}$ with $|g|_X \geq N_1 + 2|s|_X$, then by Lemma 2.2 we have $\langle Q', R'\rangle \cap sxH_\nu x^{-1}s^{-1} = Q' \cap sxH_\nu x^{-1}s^{-1}$ and we are done. Suppose to the contrary, then, that there exists some element $g \in \langle Q', R'\rangle \cap sxH_\nu x^{-1}s^{-1}$ with $|g|_X \geq N_1 + 2|s|_X$ such that $g \notin Q'$. Then $s^{-1}gs \notin s^{-1}Q's$, and so $s^{-1}gs$ (as an element of $U\langle Q', R'\rangle V$) has a minimal type path representative $p = up_1 \ldots p_n v$ with $n > 0$ and $p_- = 1$. Moreover, we have $\left|s^{-1}gs\right|_X \geq N_1$.

Since $x_1 H_\nu = xH_\nu$ and $\widetilde{p} \in xH_\nu x^{-1}$, we have $\widetilde{p} \in x_1 H_\nu x_1^{-1}$ also. Let h be the H_ν-edge of $\Gamma(G, X \cup \mathcal{H})$ with $h_- = x_1$ and $s^{-1}gs = \widetilde{p} = x_1\widetilde{h}x_1^{-1}$. Denote the Θ_1-shortcutting of p by $\Sigma(p, \Theta_1) = f_0e_1f_1 \ldots f_{m-1}e_mf_m$.

By Lemma 3.25 the path p is $(B_5, C_0', \zeta, \Theta_1)$-tamable. Lemma 3.29 tells us that h is connected to e_k for some $k = 1, \ldots, m$ and $\mathrm{d}_X(h_-, (e_k)_-) \leq \kappa_0$. Moreover, by Lemma 3.30, $k = m = 1$, so that $\Sigma(p, \Theta_1) = f_0e_1f_1$ and

$$\mathrm{d}_X(h_-, (e_1)_-) \leq \kappa_0. \tag{3.51}$$

Applying Lemma 3.31, we see that $(e_1)_-$ is a non-terminal vertex of up_1 and $(e_1)_+$ is a non-initial vertex of $p_n v$. In any of the cases, p_1 contains an H_ν-component h' that is connected to e_1 (and is thus, in turn, connected to h). We will bound $\mathrm{d}_X(\widetilde{u}, h'_-)$.

Case 1: Suppose first that $(e_1)_-$ is a vertex of p_1. As h' is the H_ν-component of p_1 connected to e_1, it must be that $(e_1)_- = h'_-$. By Lemma 3.31, we must have $\mathrm{d}_X(1, \widetilde{u}) = |u|_X \leq B_5$. Further, $\mathrm{d}_X(1, h_-) = |x_1|_X \leq \xi_0\sigma^2$. Combining these two inequalities with (3.51), we obtain

$$\begin{aligned}\mathrm{d}_X(\widetilde{u}, h'_-) &= \mathrm{d}_X(\widetilde{u}, 1) + \mathrm{d}_X(1, h_-) + \mathrm{d}_X(h_-, h'_-)\\ &\leq B_5 + \xi_0\sigma^2 + \kappa_0.\end{aligned} \tag{3.52}$$

Case 2: Suppose now that $(e_1)_-$ is a non-terminal vertex of u. Since $(e_1)_+$ is a vertex of either p_n or v, e_1 comes from an instance of consecutive backtracking along the segments $u, p_1, \ldots, p_n$ and possibly v. In particular, u contains an H_ν-component connected to h'. By Lemma 3.18,

$$\mathrm{d}_X(\widetilde{u}, h'_-) = \mathrm{d}_X(u_+, h'_-) \leq C_1 \tag{3.53}$$

concluding the second case.

Since $\widetilde{u} \in s^{-1}Q'$, there is some $t \in Q'$ such that $\widetilde{u} = s^{-1}t$. Take $y = t^{-1}sh'_-$. Following (3.52) and (3.53), we have

$$|y|_X = \mathrm{d}_X(s^{-1}t, h'_-) = \mathrm{d}_X(\widetilde{u}, h'_-) \leq \tau.$$

Moreover, $s^{-1}ty = h'_- \in x_1 H_\nu = xH_\nu$ since h' and h are connected, and so $sx \in tyH_\nu$. It follows that $tyH_\nu y^{-1}t^{-1} = sxH_\nu x^{-1}s^{-1}$ and

$$\begin{aligned} t\Big(\langle Q', R'\rangle \cap yH_\nu y^{-1}\Big)t^{-1} &= t\langle Q', R'\rangle t^{-1} \cap tyH_\nu y^{-1}t^{-1} \\ &= \langle Q', R'\rangle \cap sxH_\nu x^{-1}s^{-1} \end{aligned}$$

as required. □

3.8 Structure of maximal parabolic subgroups

This section is dedicated to proving the following theorem.

Theorem 3.35. *There is s finite set $\mathcal{K}$ of maximal parabolic subgroups of G and constants $B_6, C_6 \geq 0$ such that if $Q' \leqslant Q$ and $R' \leqslant R$ satisfy conditions (C1)-(C5) with constants $B \geq B_6, C \geq C_6$, and family $\mathcal{P} \supseteq \mathcal{P}_\tau$ (as in Notation 3.7), where τ is the constant obtained from Proposition 3.34, then the following is true.*
Suppose that P is such that $\langle Q', R'\rangle \cap P$ infinite. Then there is an element $u \in \langle Q', R'\rangle$ such that either

(i) $\langle Q', R'\rangle \cap P = uQ'u^{-1} \cap P$ or,
(ii) $\langle Q', R'\rangle \cap P = uQ'u^{-1} \cap P$ or,
(iii) $\langle Q', R'\rangle \cap P = u\langle Q' \cap K, R' \cap K\rangle u^{-1}$, where $K = u^{-1}Pu$ is an element of $\mathcal{K}$.

Moreover, if either $Q' \cap P$ or $R' \cap P$ is infinite, then we may take $u = 1$ in cases (i) and (ii), and $u \in Q' \cup R'$ in case (iii).

Proof. We define $B_6 = \max\{B_3(\tau), B_4, B_5\}$ and $C_6 = \max\{C_3(1), C_4(\tau), C_5\}$, where $B_3(\tau)$ and $C_4(\tau)$ are the constants of Theorem 3.32, B_4 is the constant of Lemma 3.33, $C_3(1)$ is the constant from Lemma 3.25, and B_5 and C_5 are the constants of Proposition 3.34. Take $\mathcal{K}$ to be the set $\{yH_\nu y^{-1} \in G \mid \nu \in \mathcal{N}, |y|_X \leq \tau\}$.

Let $Q' \leqslant Q$ and $R' \leqslant R$ be subgroups satisfying conditions (C1)-(C5) with constants $B \geq B_6, C \geq C_6$, and finite family $\mathcal{P} \supseteq \mathcal{P}_\tau$. Let $P = bH_\nu b^{-1}$ be a maximal parabolic subgroup of G such that $\langle Q', R' \rangle \cap P$ is infinite and $|b|_{X \cup \mathcal{H}}$ minimal among elements of bH_ν.

By Lemma 3.33, there is $v \in \langle Q', R' \rangle$ and $s \in Q \cup R$ such that

$$\langle Q', R' \rangle \cap P = v\Big(\langle Q', R' \rangle \cap sxH_\nu x^{-1}s^{-1}\Big)v^{-1}, \tag{3.54}$$

where $\nu \in \mathcal{N}$ and $x \in G$ with $|x|_X \leq \sigma$ and $b = vsx$. It follows that

$$vsxH_\nu x^{-1}s^{-1}v^{-1} = bH_\nu b^{-1} = P. \tag{3.55}$$

Moreover, when $Q' \cap P$ or $R' \cap P$ is infinite, v may be taken to be trivial.

Applying Proposition 3.34, we have either that

$$\langle Q', R' \rangle \cap sxH_\nu x^{-1}s^{-1} = s\langle Q' \cap xH_\nu x^{-1}, R' \cap xH_\nu x^{-1}\rangle s^{-1} \quad \text{with } s \in Q' \cup R', \tag{3.56}$$

or that one of the following equations holds

$$\begin{aligned} \langle Q', R' \rangle \cap sxH_\nu x^{-1}s^{-1} &= Q' \cap sxH_\nu x^{-1}s^{-1} \quad \text{with } s \in Q, \\ \langle Q', R' \rangle \cap sxH_\nu x^{-1}s^{-1} &= R' \cap sxH_\nu x^{-1}s^{-1} \quad \text{with } s \in R, \end{aligned} \tag{3.57}$$

or finally that

$$\langle Q', R' \rangle \cap sxH_\nu x^{-1}s^{-1} = t\Big(\langle Q', R' \rangle \cap yH_\nu y^{-1}\Big)t^{-1} \tag{3.58}$$

where $t \in (Q' \cup R')$, $y \in G$ with $|y|_X \leq \tau$, and $sx \in tyH_\nu$ so that

$$tyH_\nu y^{-1}t^{-1} = sxH_\nu x^{-1}s^{-1}. \tag{3.59}$$

If (3.56) holds, then we set $u = vs$ and $K = xH_\nu x^{-1}$. The equality

$$\langle Q', R' \rangle \cap P = u\langle Q' \cap K, R' \cap K\rangle u^{-1}$$

then follows immediately from (3.54). Observe that (3.55) tells us that $K = u^{-1}Pu$. Moreover, noting that $|x|_X \leq \sigma \leq \tau$, we have that $xH_\nu x^{-1} \in \mathcal{K}$, as required.

If instead one of the equations of (3.57) holds, then from (3.54) we obtain

$$\langle Q', R' \rangle \cap P = v\Big(Q' \cap s^{-1}x^{-1}H_\nu xs\Big)v^{-1}$$

or

$$\langle Q', R' \rangle \cap P = v\Big(R' \cap s^{-1}x^{-1}H_\nu xs\Big)v^{-1},$$

where in either case setting $u = v$ gives the desired conclusion by (3.55).

Lastly, if (3.58) holds, then (3.54) gives that

$$\langle Q', R' \rangle \cap P = vt\Big(\langle Q', R' \rangle \cap yH_v y^{-1}\Big)t^{-1}v^{-1}. \tag{3.60}$$

By the choice of B and C, and the fact that $|y|_X \leq \tau$, we can apply Theorem 3.32 to obtain

$$\langle Q', R' \rangle \cap yH_v y^{-1} = \langle Q' \cap yH_v y^{-1}, R' \cap yH_v y^{-1} \rangle. \tag{3.61}$$

Combining (3.60) and (3.61) we conclude that

$$\langle Q', R' \rangle \cap P = vt\langle Q' \cap K, R' \cap K \rangle t^{-1}v^{-1},$$

where $K = yH_v y^{-1} \in \mathcal{K}$. We set $u = vt$ and note that $u \in \langle Q', R' \rangle$, since $t \in Q' \cup R'$. Since $v = 1$ when $Q' \cap P$ or $R' \cap P$ is infinite, we have $u \in Q' \cup R'$ in these cases. Finally, observing that (3.55) and (3.59) give $K = t^{-1}sxH_v x^{-1}s^{-1}t = u^{-1}Pu$ completes the proof. □

We note that in this setting nothing can yet be said about the intersections $Q' \cap K$ and $R' \cap K$ appearing in case (iii) of the above theorem. A priori, it may be the case that $\langle Q', R' \rangle \cap P$ is infinite, while both of these intersections are finite. In Section 4.6, we are able to rule out (in a strong way) such exceptional possibilities by passing to appropriately deep finite index subgroups.

Chapter 4

Separability and virtual combination theorems

We continue to work under Conventions 2.1 and 3.3. Suppose now that G is QCERF and its peripheral subgroups are double coset separable. In Theorem 4.12 we use the separability assumptions on G and $\{H_\nu \mid \nu \in \mathcal{N}\}$ to deduce the existence of a finite index subgroup $M \leqslant_f G$ such that $Q' = Q \cap M \leqslant_f Q$, $R' = R \cap M \leqslant_f R$ satisfy conditions (C1)-(C5) with constants B and C large enough to apply Theorem 1.8 (as suggested in Remark 1.7). Conditions (C1) and (C4) are essentially automatic. Conditions (C2), (C3) and (C5) can be assured to hold for the subgroups Q' and R' using Lemma 4.5 by the QCERF condition on G, separability of double cosets PS (where P is one of finitely many maximal parabolic subgroups) and double coset separability of the peripheral subgroups, respectively.

The remaining technical difficulty is in showing that the double cosets of the form PS as above are separable in G. To this end, we prove a general result about lifting separability of certain double cosets in amalgamated free products. This is then combined with a result of Martínez-Pedroza (Theorem 2.44), allowing us to deduce Theorem 1.1 from Theorem 1.8. We note that simplifications are possible in the special cases that the peripheral subgroups of G are virtually abelian, or when Q and R have almost compatible parabolics.

Finally, we deduce the full-strength version of Theorem 1.2 on the structure of maximal parabolic subgroups of $\langle Q', R' \rangle$. To do this, we use the QCERF condition, together with the fact that the set $\mathcal{K}$ obtained from Theorem 3.35 is finite and independent of Q' and R', to find appropriate finite index subgroups that avoid all the possible pathologies. As an consequence, we obtain Theorem 1.4 and hence Corollary 1.5. We finish by noting a number of other applications of this result.

4.1 The profinite topology

Let G be a group. The *profinite topology* on G is the topology $\mathcal{PT}(G)$ whose basis consists of left cosets to finite index subgroups of G.

A subset $Z \subseteq G$ is called *separable* (in G) if it is closed in $\mathcal{PT}(G)$. Evidently finite unions and arbitrary intersections of separable subsets are separable. It is easy to see that a subset $Z \subseteq G$ is separable if and only if for every $g \in G \setminus Z$, there is a finite group Q and a homomorphism $\varphi\colon G \to Q$ such that $\varphi(g) \notin \varphi(Z)$ in Q. A subgroup $H \leq G$ is separable if and only if it is the intersection of the finite index subgroups of G containing it.

We recall that a group G is called *residually finite* if the trivial subgroup is separable in G, and it is called *LERF* if each of its finitely generated subgroups are separable. If G is a relatively hyperbolic group, it is called *QCERF* if each of its finitely generated relatiely quasiconvex subgroups are separable.

The following observation stems from the fact that the group operations of taking an inverse and multiplying by a fixed element are homeomorphisms with respect to the profinite topology.

Remark 4.1. Let Z be a separable subset of a group G. Then for every $g \in G$ the subsets Z^{-1}, gZ and Zg are also separable.

Lemma 4.2. *Suppose that A is a subgroup of a group G.*

- *(a) Every subset of A which is closed in $\mathcal{PT}(G)$ is also closed in $\mathcal{PT}(A)$.*
- *(b) If every finite index subgroup of A is separable in G then every closed subset of $\mathcal{PT}(A)$ is closed in $\mathcal{PT}(G)$.*

Proof. Claim (a) immediately follows from the observation that the intersection of A with any basic closed subset from $\mathcal{PT}(G)$ is either empty or is a basic closed subset of $\mathcal{PT}(A)$.

If each finite index subgroup of A is separable in G then, in view of Remark 4.1, every basic closed set in $\mathcal{PT}(A)$ is closed in the profinite topology of G. Claim (b) of the lemma now follows from the fact that any closed subset of A is the intersection of basic closed sets. □

Lemma 4.3. *Let G be a group with subgroups A, B. Suppose that $A' \leqslant_f A$, $B' \leqslant_f B$ and $A'B'$ is separable in G. Then AB is separable in G.*

Proof. Let $A = \bigsqcup_{i=1}^{m} a_i A'$ and $B = \bigsqcup_{j=1}^{n} B' b_j$. Then $AB = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} a_i A' B' b_j$, which is separable in G by Remark 4.1. □

We recall the following preorder on subsets of G introduced at the beginning of this thesis. Given subsets $U, V \subseteq G$, we will write $U \preccurlyeq V$ if there exists a finite subset $Y \subseteq G$ such that $U \subseteq VY$.

Lemma 4.4. *Let A, B be subgroups of a group G such that $A \preccurlyeq B$. If B is separable in G then so are the double cosets AB and BA.*

Proof. By (Minasyan, 2005b, Lemma 2.1) $A \cap B$ has finite index in A, so $A = \bigsqcup_{i=1}^{m} a_i(A \cap B)$, for some $a_1, \ldots, a_m \in A$. It follows that $AB = \bigcup_{i=1}^{m} a_i B$, so it is separable by Remark 4.1. The same remark also implies that $BA = (AB)^{-1}$ is separable in G. □

The main use of the profinite topology in this thesis stems from the following elementary facts.

Lemma 4.5. *Let G be a group generated by a finite set X, and let $P \leqslant G$ be a subgroup. Suppose that Z is a separable subset of P.*

(a) If a finite subset $U \subseteq P$ is disjoint from Z then there is a normal finite index subgroup $N \lhd_f P$ such that $U \cap ZN = \emptyset$. Thus the image of U in the quotient P/N will be disjoint from the image of Z.

(b) For every constant $C \geq 0$ there is a finite index normal subgroup $N \lhd_f P$ such that

$$\min_X(ZN \setminus Z) \geq C.$$

(c) For any finite subset $A \subseteq P$ and any $C \geq 0$ there exists $N \lhd_f P$ such that

$$\min_X(aZN \setminus aZ) \geq C, \quad \textit{for all } a \in A.$$

Proof. (a) Let $U = \{u_1, \ldots, u_m\} \subseteq P$. Since $u_i \notin Z$ and Z is separable in P, there exists $N_i \lhd_f P$ such that $u_i N_i \cap Z = \emptyset$, for each $i = 1, \ldots, m$. We set $N = \bigcap_{i=1}^{m} N_i \lhd_f P$, so that $u_i N \cap Z = \emptyset$. That is, $u_i \notin ZN$ for all $i = 1, \ldots m$. Therefore $U \cap ZN = \emptyset$ and (a) has been proved.

Claim (b) follows by applying claim (a) to the finite subset $U = \{g \in P \setminus Z \mid |g|_X < C\}$ of P.

To prove (c), suppose that $A = \{a_1, \ldots, a_k\} \subseteq P$. By Remark 4.1, $a_j Z$ is separable in P, for every $j = 1, \ldots, k$, so, according to part (b), there exists $N_j \lhd_f P$ such that

$$\min_X(a_j Z N_j \setminus a_j Z) \geq C, \text{ for each } j = 1, \ldots, k.$$

It is straightforward to verify that the normal subgroup $N = \bigcap_{j=1}^{k} N_j \lhd_f P$ has the required property. □

We can use separability to lift finite index subgroups of a subgroup to finite index subgroups of the entire group.

Lemma 4.6. *Let G be a group with subgroups $K \leqslant_f H \leqslant G$. If K is separable in G, then there is $L \leqslant_f G$ such that $L \cap H = K$*

Proof. Since K is of finite index in H, we can write $H = K \cup Kh_1 \cup \cdots \cup Kh_m$ for some $h_1, \ldots h_m \in H \setminus K$. The subgroup K is separable in G, meaning that it is closed in $\mathcal{PT}(G)$. Following Remark 4.1, the union $Kh_1 \cup \cdots \cup Kh_m$ is also closed in $\mathcal{PT}(G)$. Thus the subset $(G \setminus H) \cup K = G \setminus (Kh_1 \cup \cdots \cup Kh_m)$ is open in $\mathcal{PT}(G)$ and contains the identity. It follows from the definition of the profinite topology that there is a finite index normal subgroup $N \lhd_f G$ with $N \subseteq (G \setminus H) \cup K$. Observe that $Kh_i \cap N = \emptyset$, for every $i = 1, \ldots, m$, so $N \cap H \leqslant K$. Now set $L = KN \leqslant_f G$. Then $L \cap H = KN \cap H = K(N \cap H) = K$, as required. □

Separability of certain double cosets may also be used for similar purposes.

Lemma 4.7. *Let G be a group, $H, Q \leqslant G$ be subgroups of G and let $K \leqslant_f H$ be a finite index subgroup of H, with $Q \cap H \subseteq K$. If KQ is separable in G, then there is a finite index subgroup $M \leqslant_f G$ such that $Q \subseteq M$ and $M \cap H \subseteq K$.*

Proof. Let $H = K \cup Kh_1 \cup \cdots \cup Kh_m$, where $h_1, \ldots, h_m \in H \setminus K$. Note that $KQ \cap H = K(Q \cap H) = K$, so $h_1, \ldots, h_m \notin KQ$. The double coset KQ is profinitely closed, so, by Lemma 4.5(a), there exists $N \lhd_f G$ such that

$$\{h_1, \ldots, h_m\} \cap KQN = \emptyset.$$

Let $M = QN \leqslant_f G$, so that the above implies $Kh_i \cap M = \emptyset$, for each $i = 1, \ldots, m$. We then have $Q \subseteq M$ and $M \cap H \subseteq K$, as required. □

For the remainder of this section we take G to be a relatively hyperbolic group.

Lemma 4.8. *Suppose that G is QCERF. If Q is a finitely generated relatively quasiconvex subgroup of G then every subset of Q which is closed in $\mathcal{PT}(Q)$ is also closed in $\mathcal{PT}(G)$.*

Proof. By Lemma 2.39 every subgroup of finite index in Q is finitely generated and relatively quasiconvex, hence it is separable in G as G is QCERF. The claim of the lemma now follows from Lemma 4.2(b). □

When G is residually finite, we have control over finite parabolic subgroups of relatively quasiconvex subgroups, up to passing to finite index subgroups. Note that we do not require G or Q to be finitely generated in the following.

Proposition 4.9. *Suppose that G is residually finite, and let $Q \leqslant G$ be a relatively quasiconvex subgroup. Then there is a finite index subgroup $Q' \leqslant_f Q$ such that for any maximal parabolic subgroup $P \leqslant G$, the subgroup $Q' \cap P$ is either infinite or trivial. Moreover, if Q is finitely generated, Q' may be taken to be normal in Q.*

Proof. Let S be a set of representatives of conjugacy classes of nontrivial elements of G belonging to more than one maximal parabolic subgroups of G. Corollary 2.27 tells us that the set S is finite. That G is residually finite means exactly that the trivial subgroup $\{1\}$ is separable, and $\{1\} \cap S = \emptyset$ so Lemma 4.5(a) gives us a finite index normal subgroup $G_1 \lhd_f G$ with $G_1 \cap S = \emptyset$. As G_1 is normal, it thus contains no nontrivial elements that belong to more than one maximal parabolic subgroup of G.

Let $Q_1 = G_1 \cap Q \lhd_f Q$. Now by (Osin, 2006b, Theorem 4.2), there are only finitely many conjugacy classes of finite order hyperbolic elements in Q_1 (an element of G is called *hyperbolic* if it is not conjugate to an element of H_ν for any $\nu \in \mathcal{N}$). Similarly to before, by residual finiteness there is $Q' \lhd_f Q_1$ excluding each of these elements by Lemma 4.5(a). When Q is finitely generated, we may replace Q' by a finite index subgroup that is characteristic in Q_1 (Lyndon and Schupp, 1977, Theorem IV.4.7), and is hence normal in Q.

We will show that the subgroup $Q' \leqslant_f Q$ has the desired property. Let $\mathcal{P}$ be a set of maximal parabolic subgroups of G such that Q_1 is hyperbolic relative to the collection of infinite subgroups $\{Q_1 \cap H \mid H \in \mathcal{P}\}$ (see (Hruska, 2010, Theorem 9.4)). Let $P \leqslant G$ be a maximal parabolic subgroup of G, and suppose that $Q' \cap P$ is nontrivial. If $Q' \cap P$ contains an element of infinite order then we are done, so suppose $x \in Q' \cap P$ is a nontrivial element of finite order. By construction, Q' contains no elements of finite order that are hyperbolic in Q_1, so x must be parabolic in Q_1. That is, there is $q \in Q_1$ such that $qxq^{-1} \in Q_1 \cap H$ for some $H \in \mathcal{P}$. It follows that

$$x \in Q_1 \cap P \cap q^{-1}Hq \subseteq G_1 \cap P \cap q^{-1}Hq,$$

whence we must have $P = q^{-1}Hq$ by the definition of G_1. This implies that

$$Q_1 \cap P = Q_1 \cap q^{-1}Hq = q(Q_1 \cap H)q^{-1}$$

and since $Q_1 \cap H$ is infinite, $Q_1 \cap P$ is infinite as well. The result then follows by noting that $Q' \cap P$ has finite index in $Q_1 \cap P$. □

4.2 Using separability to establish metric conditions

For this section, let G be a group generated by finite set X, fix subgroups $Q, R \leqslant G$ and a finite collection $\mathcal{P}$ of subgroups of G. As usual, we write $S = Q \cap R$. We will exhibit the existence of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ which satisfy the metric conditions (C1)–(C5), using certain separability assumptions. The existential statement we are primarily interested in quantifies the existence of finite index subgroups satisfying a property as follows:

(E) there exists $L \leqslant_f G$ with $S \subseteq L$ such that for any $L' \leqslant_f L$ satisfying $S \subseteq L'$, there exists $M \leqslant_f L'$ with $Q \cap L' \subseteq M$ such that for any $M' \leqslant_f M$ satisfying $Q \cap L' \subseteq M'$, we can choose $Q' = Q \cap M'$ and $R' = R \cap M' \leqslant_f R$.

Remark 4.10. The statement (E) above is stable under intersections. That is, suppose that for $i = 1, 2$, there are subgroups $L_i \leqslant_f G$ with $S \subseteq L_i$ such that for any $L_i' \leqslant_f L_i$ satisfying $S \subseteq L_i'$, there exists $M_i \leqslant_f L_i'$ with $Q \cap L_i' \subseteq M_i$ such that for any $M_i' \leqslant_f M_i$ satisfying $Q \cap L_i' \subseteq M_i'$, the subgroups $Q \cap M_i'$ and $R \cap M_i' \leqslant_f R$ satisfy some properties P_i. Take $L = L_1 \cap L_2 \leqslant_f G$ and note that $S \subseteq L$, and let $L' \leqslant_f L$ be such that $S \subseteq L'$. Now $L' \leqslant_f L_1 \cap L_2 \leqslant_f L_1, L_2$, so there are subgroups $M_i \leqslant_f L'$ with $Q \cap L' \subseteq M_i$ for $i = 1, 2$. Now take $M = M_1 \cap M_2$, and note that $Q \cap L' \subseteq M$. Let $M' \leqslant_f M$ be any finite index subgroup with $Q \cap L' \subseteq M'$. Then $M' \leqslant_f M_1, M_2$ so the subgroups $Q' = Q \cap M'$ and $R' = R \cap M'$ satisfy both properties P_1 and P_2 by the statement of (E).

We start with finding assumptions for establishing (C2) and (C3).

Proposition 4.11. *Suppose that Q and R are separable in G and PS is separable in G, for each $P \in \mathcal{P}$. Then for any constants $B, C \geq 0$ there exists a finite index subgroup $L \leqslant_f G$, with $S \subseteq L$, such that conditions (C2) and (C3) are satisfied by arbitrary subgroups $Q' \leqslant Q \cap L$ and $R' \leqslant R \cap L$.*

Proof. Combining the separability of Q and R in G with Lemma 4.5, we can find $E_1, E_2 \lhd_f G$ such that $\min_X(QE_1 \setminus Q) \geq B$ and $\min_X(RE_2 \setminus R) \geq B$. Set $N_0 = E_1 \cap E_2 \lhd_f G$ and observe that

$$QSN_0Q = QN_0Q = QQN_0 = QN_0 \subseteq QE_1,$$

as Q is a subgroup containing S and normalising N_0 in G. Similarly, $RSN_0R = RN_0 \subseteq RE_2$, therefore

$$\min_X(QSN_0Q \setminus Q) \geq B \quad \text{and} \quad \min_X(RSN_0R \setminus R) \geq B. \tag{4.1}$$

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$. The assumptions imply that for every $i \in \{1, \ldots, k\}$ the double coset P_iS is separable in G, hence we can apply Lemma 4.5 again to find finite index normal subgroups $N_i \lhd_f G$ satisfying

$$\min_X(P_iSN_i \setminus P_iS) \geq C, \quad \text{for each } i = 1, \ldots, k. \tag{4.2}$$

Now set $L = \bigcap_{i=0}^k SN_i \leqslant_f G$, and choose arbitrary subgroups $Q' \leqslant Q \cap L$ and $R' \leqslant R \cap L$. Then $S \subseteq L$ and $\langle Q', R' \rangle \subseteq L \subseteq SN_i$, for all $i = 0, \ldots, k$, by construction, hence (C2) holds by (4.1) and (C3) holds by (4.2), as desired. □

We are now in position to prove the main result of this section.

Theorem 4.12. *Suppose that that for every $P \in \mathcal{P}$ all of the following hold:*

(S1) *Q and R are separable in G;*

(S2) *the double coset PS is separable in G;*

(S3) *for all $K \leqslant_f P$ and $T \leqslant_f Q$, satisfying $S \subseteq T$ and $T \cap P \subseteq K$, the double coset KT is separable in G;*

(S4) *for all $U \leqslant_f Q \cap P$, the double coset $U(R \cap P)$ is separable in P.*

Then, given arbitrary constants $B, C \geq 0$, there exist a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that conditions (C1)–(C5) are all satisfied.

Proof. The idea is that assumption (S1) will take care of condition (C2), (S2) will take care of (C3), and that (S3) and (S4) will take care of (C5). The subgroups Q' and R' will satisfy $Q' = Q \cap M'$ and $R = R \cap M'$, for some $M' \leqslant_f G$, with $S \subseteq M'$, which will immediately imply (C1) and (C4).

Let $\mathcal{P} = \{P_1, \dots, P_k\}$. Arguing just like in the proof of Proposition 4.11 (using the assumptions (S1) and (S2)), we can finite finite index normal subgroups $N_i \lhd_f G$, $i = 0, \dots, k$, such that

$$\min_X(QSN_0Q \setminus Q) \geq B, \ \min_X(RSN_0R \setminus R) \geq B \ \text{ and}$$

$$\min_X(P_iSN_i \setminus P_iS) \geq C, \ \text{ for each } i = 1, \dots, k.$$

We can now define a finite index subgroup $L \leqslant_f G$ by $L = \bigcap_{i=0}^k SN_i$. Note that $S \subseteq L$ by construction, and for each $i \in \{1, \dots, k\}$ we have

$$\min_X(QLQ \setminus Q) \geq B, \ \min_X(RLR \setminus R) \geq B \ \text{ and } \ \min_X(P_iL \setminus P_iS) \geq C. \tag{4.3}$$

Choose an arbitrary finite index subgroup $L' \leqslant_f L$, with $S \subseteq L'$, and define $Q' = Q \cap L'$, so that $S \leqslant Q' \leqslant_f Q$.

To construct $R' \leqslant_f R$, consider any $i \in \{1, \dots, k\}$ and denote $Q_i = Q \cap P_i$, $R_i = R \cap P_i$ and $Q'_i = Q' \cap P_i \leqslant_f Q_i$. Choose some elements $a_{i1}, \dots, a_{in_i} \in Q_i$ such that $Q_i = \bigsqcup_{j=1}^{n_i} a_{ij} Q'_i$. Assumption (S4) implies that the subset $Q'_i R_i$ is separable in P_i and hence, by Lemma 4.5(c), there exists $F_i \lhd_f P_i$ such that

$$\min_X\left(a_{ij}Q'_iR_iF_i \setminus a_{ij}Q'_iR_i\right) \geq C, \ \text{ for } j = 1, \dots, n_i. \tag{4.4}$$

Define $K_i = Q'_iF_i \leqslant_f P_i$. Then $Q' \cap P_i = Q'_i \subseteq K_i$ and $a_{ij}K_iR_i = a_{ij}Q'_iR_iF_i$, for each $j = 1, \dots, n_i$. Therefore, from (4.4) we can deduce that

$$\min_X\left(a_{ij}K_iR_i \setminus a_{ij}Q'_iR_i\right) \geq C, \ \text{ for all } j = 1, \dots, n_i. \tag{4.5}$$

By assumption (S3), the double coset K_iQ' is separable in G, so we can apply Lemma 4.7 to find $M_i \leqslant_f G$ such that $Q' \subseteq M_i$ and $M_i \cap P_i \subseteq K_i$.

We now let $M = \bigcap_{i=1}^{k} M_i \cap L'$ and observe that $Q' \leqslant M \leqslant_f L'$ and $M \cap P_i \subseteq K_i$ for each $i \in \{1, \dots, k\}$. Inequality (4.5) yields

$$\min_X \Big(a_{ij}(M \cap P_i)R_i \setminus a_{ij}Q_i'R_i \Big) \geq C, \;\text{ for all } i = 1, \dots, k \text{ and } j = 1, \dots, n_i. \tag{4.6}$$

We can now choose an arbitrary finite index subgroup $M' \leqslant_f M$, with $Q' \subseteq M'$, and define $R' = R \cap M'$. Observe that $M' \leqslant_f G$, by construction, hence $R' \leqslant_f R$.

Let us check that the subgroups Q' and R' obtained above satisfy conditions (C1)–(C5). Indeed, by construction, $S = Q \cap R \subseteq Q'$, so $S \subseteq R \cap M' = R'$, hence

$$S \subseteq Q' \cap R' \subseteq Q \cap R = S,$$

thus (C1) holds. We also have $Q' = Q \cap L' = Q \cap M'$, as $Q' \subseteq M' \subseteq L'$, hence

$$Q' \subseteq Q \cap \langle Q', R' \rangle \subseteq Q \cap M' = Q',$$

thus $Q \cap \langle Q', R' \rangle = Q'$. After intersecting both sides of the latter equation with an arbitrary $P \in \mathcal{P}$, we get $Q_P \cap \langle Q', R' \rangle = Q'_P$, hence

$$Q'_P \subseteq Q_P \cap \langle Q'_P, R'_P \rangle \subseteq Q_P \cap \langle Q', R' \rangle = Q'_P,$$

thus $Q_P \cap \langle Q'_P, R'_P \rangle = Q'_P$. Similarly, $R_P \cap \langle Q'_P, R'_P \rangle = R'_P$, so condition (C4) is satisfied.

Conditions (C2) and (C3) hold by (4.3), because $Q', R' \subseteq L$ by construction.

To prove (C5), take $P_i \in \mathcal{P}$ for any $i \in \{1, \dots, k\}$, and denote $Q_i = Q \cap P_i$, $Q'_i = Q' \cap P_i$, $R_i = R \cap P_i$ and $R'_i = R' \cap P_i$, as before. For any $q \in Q_i$ there exists $j \in \{1, \dots, n_i\}$ such that $q \in a_{ij}Q'_i$. It follows that

$$q\langle Q'_i, R'_i \rangle R_i = a_{ij}\langle Q'_i, R'_i \rangle R_i \;\text{ and }\; qQ'_iR_i = a_{ij}Q'_iR_i. \tag{4.7}$$

Since $\langle Q'_i, R'_i \rangle \leqslant M \cap P_i$, we can combine (4.7) with (4.6) to deduce that

$$\min_X \Big(q\langle Q'_i, R'_i \rangle R_i \setminus qQ'_iR_i \Big) \geq C,$$

which establishes condition (C5). Thus the proof is complete. □

4.3 Double coset separability in amalgamated free products

In this section we develop a method for establishing the separability assumptions (S2) and (S3) of Theorem 4.12 using amalgamated products. The idea is that when G is a relatively hyperbolic group, P is a maximal parabolic subgroup and Q is a relatively quasiconvex subgroup of G, we can apply the combination theorem of Martínez-Pedroza (Theorem 2.44) to find a finite index subgroup $H \leqslant_f P$ such that $A = \langle H, Q \rangle \cong H *_{H \cap Q} Q$, so proving the separability of PQ in G can be reduced to proving the separability of HQ in the amalgamated free product A.

The next proposition gives a criterion for showing separability of double cosets in amalgamated free products.

Proposition 4.13. *Let $A = B *_D C$ be an amalgamated free product, where we consider B, C and D as subgroups of A with $B \cap C = D$. Suppose that D is separable in A, and $U \subseteq B$, $V \subseteq C$ are arbitrary subsets.*
If the product UD (respectively, DV) is separable in A then the product UC (respectively, BV) is separable in A.

Proof. We will prove the statement in the case of UC, as the other case is similar. If $U = \emptyset$ then $UC = \emptyset$, so we can suppose that U is non-empty. Take any $u \in U$. According to Remark 4.1, without loss of generality we can replace U with $u^{-1}U$ to assume that $1 \in U$.

Consider any element $g \in A \setminus UC$; since $1 \in U$, we deduce that $g \notin C$. We will construct a homomorphism from A to a finite group L which separates the image of g from the image of UC.

Since $g \notin D$, it has a reduced form $g = x_1 x_2 \dots x_k$, where x_i belongs to one of the factors B, C, for each i, consecutive elements x_i, x_{i+1} belong to different factors, and $x_i \notin D$ for all $i = 1, \dots, k$ (see (Lyndon and Schupp, 1977, p. 187)).

Since D is separable in A, by Lemma 4.5(a) there is a finite group M and a homomorphism $\varphi : A \to M$ such that

$$\varphi(x_i) \notin \varphi(D) \text{ in } M, \text{ for every } i = 1, \dots, k. \tag{4.8}$$

Denote by $\overline{B}$, $\overline{C}$ and $\overline{D}$ the φ-images if B, C and D in M respectively. We can then consider the amalgamated free product $\overline{A} = \overline{B} *_{\overline{D}} \overline{C}$, together with the natural homomorphism $\psi : A \to \overline{A}$, which is compatible with φ on B and C (in other words, $\psi|_B = \varphi|_B$ and $\psi|_C = \varphi|_C$). It follows that φ factors through ψ. That is, $\varphi = \overline{\varphi} \circ \psi$, where $\overline{\varphi} : \overline{A} \to M$ is the natural homomorphism extending the embeddings of $\overline{B}$ and $\overline{C}$ in M. Equation (4.8) now implies that

$$\psi(x_i) \notin \overline{D} \text{ in } \overline{A}, \text{ for every } i = 1, \dots, k. \tag{4.9}$$

Denote $\overline{x}_i = \psi(x_i) \in \overline{A}$, $i = 1, \dots, k$. In view of (4.9), $\psi(g) = \overline{x}_1 \dots \overline{x}_k$ is a reduced form in the amalgamated free product $\overline{A}$. We will now consider several cases.

Case 1: assume that $k \geq 3$. Then the above reduced form for $\psi(g)$ has length $k \geq 3$, so by the normal form theorem for amalgamated free products (Lyndon and Schupp, 1977, Theorem IV.2.6), it cannot be equal to an element from $\psi(UC) = \psi(U)\overline{C} \subseteq \overline{B}\,\overline{C}$, which would necessarily have a reduced form of length at most 2 in $\overline{A}$. Therefore $\psi(g) \notin \psi(UC)$ in $\overline{A}$.

Since $\overline{B}$ and $\overline{C}$ are finite groups, their amalgamated free product $\overline{A}$ is residually finite (in fact, $\overline{A}$ is a virtually free group – see (Serre, 1980, Proposition 2.6.11)), so the finite subset $\psi(UC)$ is closed in the profinite topology on $\overline{A}$. Hence there is a finite group L and a homomorphism $\eta : \overline{A} \to L$ such that $\eta(\psi(g)) \notin \eta(\psi(UC))$ in L. The composition $\eta \circ \psi : A \to L$ is the required homomorphism separating the image of g from the image of UC, and the consideration of Case 1 is complete.

Case 2: suppose that $k = 2$, $x_1 \in C \setminus D$ and $x_2 \in B \setminus D$. Then $\overline{x}_1 \in \overline{C} \setminus \overline{D}$ and $\overline{x}_2 \in \overline{B} \setminus \overline{D}$ by (4.9), so that $\psi(g) = \overline{x}_1\overline{x}_2$ is a reduced form of length 2 in $\overline{A}$. Again, the normal form theorem for amalgamated free products implies that $\psi(g) \notin \overline{BC}$ in $\overline{A}$, hence $\psi(g) \notin \psi(UC)$ and we can find the required finite quotient L of A as in Case 1.

Case 3: $g = bc$, where $b \in B \setminus UD$ and $c \in C$ (here we allow $c \in D$, so this case also covers the situation when $k = 1$).

This is the only case where we need to use the assumption that UD is separable in A. This assumption implies that we can find a finite group M and a homomorphism $\varphi : A \to M$ satisfying

$$\varphi(b) \notin \varphi(UD) \text{ in } M.$$

As above, we can construct the amalgamated free product $\overline{A} = \overline{B} *_{\overline{D}} \overline{C}$, together with the natural homomorphism $\psi : A \to \overline{A}$, such that φ factors through ψ. It follows that

$$\psi(b) \notin \psi(UD) = \psi(U)\overline{D} \text{ in } \overline{A}. \tag{4.10}$$

Observe that $\psi(g) \notin \psi(UC) = \psi(U)\overline{C}$ in $\overline{A}$. Indeed, otherwise we would have

$$\psi(b) = \psi(g)\psi(c^{-1}) \in \psi(U)\overline{C} \cap \overline{B} = \psi(U)(\overline{C} \cap \overline{B}) = \psi(U)\overline{D},$$

which would contradict (4.10) (in the first equality we used the fact that $\overline{B}$ is a subgroup of $\overline{A}$ containing the subset $\psi(U)$). We can now argue as in Case 1 above to find a homomorphism from A to a finite group L separating the image of g from the image of UC.

It is not hard to see that since $g \notin UC$ in A, the above three cases cover all possibilities, hence the proof is complete. □

In the next two corollaries we assume that $A = B *_D C$ is the amalgamated free product of its subgroups B, C, with $B \cap C = D$.

Corollary 4.14. *Suppose that D is a separable subgroup in A. Then B, C and BC are all separable in A.*

Proof. The separability of C and B in A follows from Proposition 4.13, after choosing $U = \{1\}$ and $V = \{1\}$. The separability of BC is also a consequence of Proposition 4.13, where we take $U = B$ (so that $UD = BD = B$). □

We will not need the next corollary in the following, but it may be of independent interest and can be used to strengthen some of the statements proved in Section 4.4.

Corollary 4.15. *Suppose that $U \subseteq B$, $V \subseteq C$ are subsets such that UD and DV are separable in A. Then the subset UDV is separable in A.*

Proof. If either U or V are empty then UDV is empty, and, hence, separable in A. Thus we can suppose that there exist some elements $u \in U$ and $v \in V$. By Remark 4.1. the subsets $u^{-1}UD \subseteq B$ and $DVv^{-1} \subseteq C$ are separable in A. Since both of them contain D, we see that $D = u^{-1}UD \cap DVv^{-1}$, thus D is separable in A.

By Proposition 4.13, the products UC and BV are separable in A, so the statement follows from the observation that

$$UC \cap BV = UDV \text{ in } A. \qquad \square$$

When U and V are subgroups, the above corollary shows that we can use separability of double cosets UD and DV to deduce separability of the triple coset UDV. Moreover, if both U and V are subgroups containing D, Corollary 4.15 implies that the double coset $UV = UDV$ is separable in A, as long as U and V are separable in A.

4.4 Separability of double cosets when one factor is parabolic

Throughout this section we will assume that G is group generated by a finite subset X and hyperbolic relative to a collection of peripheral subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$.

Our goal in this section will be to establish separability of double cosets required by conditions (S2) and (S3) of Theorem 4.12. All statements in this section will assume that finitely generated relatively quasiconvex subgroups of G are separable – that is, G is QCERF.

The next statement is essentially a corollary of the combination theorem of Martínez-Pedroza (Theorem 2.44).

Proposition 4.16. *Suppose that G is QCERF. Let P be a maximal parabolic subgroup of G, let $Q \leqslant G$ be a finitely generated relatively quasiconvex subgroup and let $D = Q \cap P$. Then there exists a finite index subgroup $H \leqslant_f P$ such that all of the following properties hold:*

- *$Q \cap H = D$;*
- *the subgroup $A = \langle H, Q\rangle$ is relatively quasiconvex in G;*
- *A is naturally isomorphic to $H *_D Q$;*
- *D is separable in A;*
- *every subset of A which is closed in $\mathcal{PT}(A)$ is also closed in $\mathcal{PT}(G)$.*

Proof. Let $C \geq 0$ be the constant provided by Theorem 2.44, applied to the maximal parabolic subgroup P and the relatively quasiconvex subgroup Q. By QCERF-ness, Q is separable in G, so by Lemma 4.5 there exists $N \lhd_f G$ such that $\min_X(QN \setminus Q) \geq C$. Therefore, after setting $H = P \cap QN \leqslant_f P$, we get $\min_X(H \setminus D) = \min_X(H \setminus Q) \geq C$.

Note that since $D = P \cap Q \subseteq H \subseteq P$, we have $H \cap Q = D$. Hence we can apply Theorem 2.44 to conclude that $A = \langle H, Q\rangle$ is relatively quasiconvex in G and is naturally isomorphic to the amalgamated free product $H *_D Q$.

Recall, from Lemma 2.41 and Corollary 2.40, that P is finitely generated and relatively quasiconvex in G, hence it is separable in G by QCERF-ness. It follows that $D = P \cap Q$ is separable in G, which implies that it is separable in A by Lemma 4.2.

Observe that H and Q are both finitely generated, hence A is finitely generated and relatively quasiconvex in G. Therefore Lemma 4.8 yields the last assertion of the proposition, that every subset of A which is closed in $\mathcal{PT}(A)$ is also closed in $\mathcal{PT}(G)$. □

By combining Proposition 4.16 with Proposition 4.13 we obtain the first double coset separability result when one of the factors is parabolic and the other one is finitely generated and relatively quasiconvex.

Proposition 4.17. *Assume that G is QCERF. Let P be a maximal parabolic subgroup of G, let $R \leqslant G$ be a finitely generated relatively quasiconvex subgroup of G. Suppose that $D \leqslant P$ is a subgroup satisfying the following condition:*

$$\text{for each } U \leqslant_f D, \text{ the double coset } U(R \cap P) \text{ is separable in } P. \tag{4.11}$$

Then the double coset DR is separable in G.

Proof. According to Proposition 4.16, there exists $H \leqslant_f P$ such that the subgroup $A = \langle H, R\rangle$ is naturally isomorphic to the amalgamated free product $H *_E R$, where $E = R \cap P = R \cap H$ is separable in A, and every closed subset from $\mathcal{PT}(A)$ is separable in G.

Denote $U = D \cap H \leqslant_f D$. By assumption (4.11), UE is separable in P. Since P is finitely generated and relatively quasiconvex in G, we can conclude that UE is separable in G by Lemma 4.8. As $UE \subseteq A \leqslant G$, UE will also be closed in $\mathcal{PT}(A)$, so we can apply Proposition 4.13 to deduce that the double coset UR is closed in $\mathcal{PT}(A)$. It follows that this double coset is separable in G and, since $U \leqslant_f D$, Lemma 4.3 implies that DR is separable in G, as desired. □

Note that when $D = Q \cap P$, the condition (4.11) is exactly (S4). We can now prove that (S3) of Theorem 4.12 holds as long as the relatively hyperbolic group G is QCERF.

Corollary 4.18. *Suppose that G is QCERF, P is a maximal parabolic subgroup of G and $Q \leqslant G$ is a finitely generated relatively quasiconvex subgroup. Then for all finite index subgroups $K \leqslant_f P$ and $T \leqslant_f Q$ the double coset KT is separable in G.*

Proof. Note that T is finitely generated and relatively quasiconvex in G by Lemma 2.39. Hence, to apply Proposition 4.17 we simply need to check that for any $U \leqslant_f K$ the double coset $U(T \cap P)$ is separable in P. The latter is true because $U(T \cap P)$ is a basic closed set in $\mathcal{PT}(P)$, being a finite union of right cosets to $U \leqslant_f P$. Therefore KT is separable in G by Proposition 4.17. □

The proof of assumption (S2) of Theorem 4.12 is slightly more involved because the intersection of two finitely generated relatively quasiconvex subgroups need not be finitely generated. It turns out to follow from (S4) when G is QCERF.

Proposition 4.19. *Let P be a maximal parabolic subgroup of G, let $Q, R \leqslant G$ be finitely generated relatively quasiconvex subgroups, let $S = Q \cap R$. Suppose that G is QCERF and condition* (S4) *is satisfied. Then the double coset PS is separable in G.*

Proof. Let $D = Q \cap P$. Since (S4) is satisfied, Proposition 4.17 tells us that the double coset DR is separable in G. Moreover, G is QCERF so Q is separable in G. Now, observe that $DR \cap Q = D(R \cap Q) = DS$, since $D \leqslant Q$. Thus the double coset DS is separable in G.

According to Proposition 4.16, there exists a finite index subgroup $H \leqslant_f P$ such that $Q \cap H = D$, $A = \langle H, Q \rangle \cong H *_D Q$, D is separable in A, and every closed subset in $\mathcal{PT}(A)$ is closed in $\mathcal{PT}(G)$. The double coset DS is separable in A by Lemma 4.2, so HS is closed in $\mathcal{PT}(A)$ by Proposition 4.13. It follows that HS is closed in $\mathcal{PT}(G)$, which implies that the double coset PS is separable in G by Lemma 4.3. Thus the proof is complete. □

4.5 Quasiconvexity of virtual joins

We will follow Conventions 2.1 and 3.3. Further, we will assume that both Q and R are finitely generated subgroups throughout the section. In this section we will prove Theorem 1.1 from the introduction. First we deal with the special case that G is hyperbolic relative to a collection of virtually abelian groups. As mentioned at the beginning of this chapter, we obtain a simpler existential statement in this setting.

Theorem 4.20. *Suppose that G is QCERF with abelian peripheral subgroups. There exists a finite index subgroup $L \leqslant_f G$, with $S \subseteq L$, such that if $Q' \leqslant Q \cap L$ and $R' \leqslant R \cap L$ are relatively quasiconvex subgroups of G satisfying (C1), then $\langle Q', R' \rangle$ is relatively quasiconvex.*

Proof. By combining the assumptions with Lemma 2.41, we know that maximal parabolic subgroups of G are finitely generated abelian groups. Since such groups are slender, all relatively quasiconvex subgroups of G are finitely generated (Hruska, 2010, Corollary 9.2). Moreover, finitely generated abelian groups are LERF, and hence, they are double coset separable (because the product of two subgroups is again a subgroup). Therefore the double coset PS is separable in G for any maximal parabolic subgroup $P \leqslant G$ by Proposition 4.19.

In view of Proposition 4.11, for any finite collection $\mathcal{P}$, of maximal parabolic subgroups of G, and any $B, C \geq 0$ there exists $L \leqslant_f G$, with $S \subseteq L$, such that any subgroups $Q' \leqslant Q \cap L$ and $R' \leqslant R \cap L$ satisfy conditions (C1)–(C3), as long as $Q' \cap R' = S$. Remark 1.6 tells us that these subgroups automatically satisfy conditions (C4) and (C5). Thus we can obtain the statement by applying Theorem 3.26. □

Corollary 4.21. *Suppose that G is QCERF with virtually abelian peripheral subgroups. There exists $L \leqslant_f G$ such that if $Q' \leqslant Q \cap L$ and $R' \leqslant R \cap L$ are relatively quasiconvex subgroups of G satisfying $Q' \cap R' = S \cap L$ then the subgroup $\langle Q', R' \rangle$ is also relatively quasiconvex in G.*

Proof. By the assumptions for each $\nu \in \mathcal{N}$ there exists a finite index abelian subgroup $K_\nu \leqslant_f H_\nu$. Since G is QCERF, each K_ν is separable in G (it is finitely generated by Lemma 2.41 and it is relatively quasiconvex by Corollary 2.40). Thus, in view of Lemma 4.6, for every $\nu \in \mathcal{N}$ there exists $L_\nu \leqslant_f G$ such that $L_\nu \cap H_\nu = K_\nu$.

Since $|\mathcal{N}| < \infty$, the intersection $\bigcap_{\nu \in \mathcal{N}} L_\nu$ has finite index in G, hence it contains a finite index normal subgroup $G_1 \lhd_f G$. Note that for any $g \in G$ and any $\nu \in \mathcal{N}$ we have

$$G_1 \cap g H_\nu g^{-1} = g(G_1 \cap H_\nu) g^{-1} \subseteq g(L_\nu \cap H_\nu) g^{-1} = g K_\nu g^{-1}, \tag{4.12}$$

where the first equality follows from the normality of G_1, the middle inclusion follows from the fact that $G_1 \subseteq L_\nu$, and the last equality is due to the fact that $L_\nu \cap H_\nu = K_\nu$. By Lemma 2.39, G_1 is finitely generated and relatively quasiconvex in G, hence, by

(Hruska, 2010, Theorem 9.1) it is hyperbolic relative to representatives of G_1-conjugacy classes of the intersections $G_1 \cap gH_vg^{-1}$, $g \in G$. Thus, in view of (4.12), all peripheral subgroups in G_1 are abelian.

By (Hruska, 2010, Corollary 9.3), a subgroup of G_1 is relatively quasiconvex in G_1 (with respect to the above family of peripheral subgroups) if and only if it is relatively quasiconvex in G. Therefore G_1 is QCERF and $Q_1 = Q \cap G_1 \leqslant_f Q$, $R_1 = R \cap G_1 \leqslant_f R$ are finitely generated relatively quasiconvex subgroups of G_1 by Lemma 2.39. After denoting $S_1 = S \cap G_1 = Q_1 \cap R_1$, we can apply Theorem 4.20 to find a finite index subgroup $L \leqslant_f G_1$ such that $S_1 \subseteq L$ (thus, $S_1 = S \cap L$) and the subgroup $\langle Q', R' \rangle$ is relatively quasiconvex in G_1, for arbitrary $Q' \leqslant Q_1 \cap L = Q \cap L$ and $R' \leqslant R_1 \cap L = R \cap L$ satisfying $Q' \cap R' = Q_1 \cap R_1 = S_1$. We can use (Hruska, 2010, Corollary 9.3) again to deduce that $\langle Q', R' \rangle$ is relatively quasiconvex in G. □

The following collects the results of the previous sections, allowing us to find subgroups Q' and R' to which Theorem 3.26 can be applied.

Proposition 4.22. *Suppose G is QCERF with double coset separable peripheral subgroups. Then for any $B \geq 0, C \geq 0$, and finite family $\mathcal{P}$ of maximal parabolic subgroups of G, there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) satisfying (C1)-(C5) with constants B and C and family $\mathcal{P}$.*

Proof. We check that all the assumptions of Theorem 4.12 are satisfied for every $P \in \mathcal{P}$. Indeed, assumption (S1) holds because because G is QCERF and assumption (S3) is true by Corollary 4.18.

Note that the subgroups $Q \cap P$ and $R \cap P$ are finitely generated by Lemma 2.41, hence condition (S4) follows from the double coset separability of P. Finally, assumption (S2) holds by Proposition 4.19. The statement now follows by applying Theorem 4.12. □

Theorem 4.23. *Suppose G is QCERF with double coset separable peripheral subgroups. There exists a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that $\langle Q', R' \rangle$ is relatively quasiconvex.*

Proof. This follows immediately from Theorem 3.26 and Proposition 4.22. □

Recall that Q and R are said to have almost compatible parabolics if for every maximal parabolic subgroup $P \leqslant G$, either $Q \cap P \preccurlyeq R \cap P$ or $R \cap P \preccurlyeq Q \cap P$. We find that in the case when Q and R have almost compatible parabolics, it is actually not necessary to assume that the peripheral subgroups are double coset separable:

Proposition 4.24. *Suppose that G is QCERF and that Q and R have almost compatible parabolics. Then for any $B \geq 0, C \geq 0$, and finite family $\mathcal{P}$ of maximal parabolic subgroups of*

G, there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) satisfying (C1)-(C5) with constants B and C and family $\mathcal{P}$.

Proof. As before, we will be verifying the assumptions of Theorem 4.12. Let P be an arbitrary maximal parabolic subgroup of G. Assumption (S1) follows from the QCERF-ness of G and assumption (S3) follows from Corollary 4.18.

Let $U \leqslant_f Q \cap P$. Since Q and R have almost compatible parabolics and $Q \cap P \preccurlyeq U$, we know that either $U \preccurlyeq R \cap P$ or $R \cap P \preccurlyeq U$. Note that both U and $R \cap P$ are finitely generated by Lemma 2.41 and relatively quasiconvex by Corollary 2.40, so they are separable because G is QCERF. Lemma 4.4 now implies that the double coset $U(R \cap P)$ is separable in G, thus condition (S4) is satisfied by Lemma 4.2. Finally, assumption (S2) holds by Proposition 4.19. This concludes the proof. □

Again, applying Theorem 4.12 yields the following.

Theorem 4.25. *Suppose that G is QCERF and that Q and R have almost compatible parabolics. There exists a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that $\langle Q', R' \rangle$ is relatively quasiconvex.*

4.6 Structure of virtual joins and combination theorems

For this section we will assume G is a finitely generated QCERF relatively hyperbolic group, with Q and R finitely generated relatively quasiconvex subgroups of G. We will prove a more detailed version of Theorem 1.2 from the introduction.

Proposition 4.26. *Suppose that either Q and R have almost compatible parabolic subgroups or that each peripheral subgroup of G is double coset separable. Then there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) satisfying the hypotheses of Theorem 3.35 such that the following is true.*
Suppose that $P \leqslant G$ is a maximal parabolic subgroup of G with $\langle Q', R' \rangle \cap P$ infinite and $u \in \langle Q', R' \rangle$ is the element obtained from Theorem 3.35. If Q' and R' are almost compatible at $u^{-1}Pu$, then either $\langle Q', R' \rangle \cap P = uQ'u^{-1} \cap P$ or $\langle Q', R' \rangle \cap P = uR'u^{-1} \cap P$. In particular, at least one of $Q' \cap u^{-1}Pu$ or $R' \cap u^{-1}Pu$ is infinite.

Proof. Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be the finite set of maximal parabolic subgroups of G provided by Theorem 3.35. If each H_ν is double coset separable, then by Proposition 4.22, there are subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) satisfying (C1)-(C5) with constants B_6, C_6 (provided by Theorem 3.35) and finite family $\mathcal{P}_\tau$, where τ is the constant of Proposition 3.34. Otherwise, the same conclusion holds in the case that Q and R have almost compatible parabolics, by applying

Proposition 4.24. More precisely, there exists $L \leqslant_f G$ with $S \subseteq L$ such that for any $L' \leqslant_f L$ with $S \subseteq L'$, there is $M \leqslant_f L'$ with $Q \cap L' \subseteq M$ such that for any $M' \leqslant_f M$ with $Q \cap L' \subseteq M'$, the subgroups $Q' = Q \cap M'$ and $R' = R \cap M'$ satisfy these conditions. All such Q' and R' meet the hypotheses of Theorem 3.35.

We will show that the subgroup L can be modified so that the desired conclusion holds. Fix some $i = 1, \dots, n$ and note that since G is QCERF, Q and R are separable. Thus their intersection S is also separable. Whenever $S \cap K_i \leqslant_f Q \cap K_i$, let U_i be a finite set of coset representatives of $S \cap K_i$ in $Q \cap K_i$, and otherwise take U_i to be the empty set. Similarly, whenever $S \cap K_i \leqslant_f R \cap K_i$, let V_i be a finite set of coset representatives of $S \cap K_i$ in $R \cap K_i$, and otherwise take V_i to be the empty set. Take $U = \bigcup_{i=1}^{n} (U_i \cup V_i)$, and note that U is a finite set disjoint from S.

Since S is separable, Lemma 4.5(a) gives us $N \lhd_f G$ such that $SN \leqslant_f G$ is a subgroup containing S with $U \cap SN = \emptyset$. We take $L_0 = L \cap SN \leqslant_f G$, noting that again $S \subseteq L_0$ and $L_0 \cap U = \emptyset$. For any $L' \leqslant_f L_0$ with $S \subseteq L'$, we have that $L' \leqslant_f L$. Now there is $M \leqslant_f L'$ with $Q \cap L' \subseteq M$ as in (E). Let $M' \leqslant_f M$ be any finite index subgroup with $Q \cap L' \subseteq M'$ and write $Q' = Q \cap M'$, $R' = R \cap M'$. By Proposition 4.22, Q' and R' also satisfy (C1)-(C5), so Theorem 3.35 holds.

Let $P \leqslant G$ be a maximal parabolic subgroup of G such that $\langle Q', R' \rangle \cap P$ is infinite, and let $u \in \langle Q', R' \rangle$ be the element provided by Theorem 3.35. If either of the first two cases of the theorem hold, then we are done. Otherwise there is $i = 1, \dots, n$ such that

$$\langle Q', R' \rangle \cap P = u \langle Q' \cap K_i, R' \cap K_i \rangle u^{-1},$$

with $K_i = u^{-1} P u$. By assumption, Q' and R' are almost compatible at K_i. In other words, either $S \cap K_i \leqslant_f Q' \cap K_i$ or $S \cap K_i \leqslant_f R' \cap K_i$. In the former case, by the constructions of N and Q' we have

$$S \cap K_i \subseteq Q' \cap K_i = Q \cap M' \cap K_i \subseteq Q \cap SN \cap K_i = S \cap K_i,$$

so that $Q' \cap K_i = S \cap K_i$. It follows that $\langle Q' \cap K_i, R' \cap K_i \rangle = R' \cap K_i$. Thus

$$\langle Q', R' \rangle \cap P = u \langle Q' \cap K_i, R' \cap K_i \rangle u^{-1} = u (R' \cap K_i) u^{-1} = u R' u^{-1} \cap P,$$

as required. An identical argument (with the roles of Q' and R' swapped) gives us that $\langle Q', R' \rangle \cap P = u Q' u^{-1} \cap P$ when $S \cap K_i \leqslant_f R' \cap K_i$.

To conclude, note that if $Q' \cap u^{-1} P u$ is finite, Q' and R' are almost compatible at $u^{-1} P u$, whence $u R' u^{-1} \cap P = \langle Q', R' \rangle \cap P$ is infinite by the hypotheses. □

We are ready to prove the main result of this section.

Theorem 4.27. *Suppose that either Q and R have almost compatible parabolics or that each peripheral subgroup of G is double coset separable. Then there is a finite set $\mathcal{K}$ of maximal parabolic subgroups of G and a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that the following is true.*
Suppose that $P \leqslant G$ is a maximal parabolic subgroup with $\langle Q', R' \rangle \cap P$ infinite. Then there is an element $u \in \langle Q', R' \rangle$ such that either

(i) $\langle Q', R' \rangle \cap P = uQ'u^{-1} \cap P$ *or,*
(ii) $\langle Q', R' \rangle \cap P = uR'u^{-1} \cap P$ *or,*
(iii) $\langle Q', R' \rangle \cap P = u\langle Q' \cap K, R' \cap K \rangle u^{-1}$ *where $K = u^{-1}Pu$ is an element of $\mathcal{K}$, and Q' and R' are not almost compatible at K.*

Moreover, if either $Q' \cap P$ or $R' \cap P$ is infinite, then we may take $u = 1$ in cases (i) and (ii), and $u \in Q' \cup R'$ in case (iii).

Proof. Let $\mathcal{K}$ be the finite set of maximal parabolic subgroups provided by Theorem 3.35. There is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) satisfying Proposition 4.26.

Let $P \leqslant G$ be a maximal parabolic subgroup of G such that $\langle Q', R' \rangle \cap P$ is infinite, and suppose that $\langle Q', R' \rangle \cap P$ is not equal to $uQ'u^{-1} \cap P$ or $uR'u^{-1} \cap P$ for any $u \in \langle Q', R' \rangle$. Then Theorem 3.35 gives us $u \in \langle Q', R' \rangle$ such that $\langle Q', R' \rangle \cap P = u\langle Q' \cap K, R' \cap K \rangle u^{-1}$, where $K = u^{-1}Pu$ is an element of $\mathcal{K}$. Suppose that Q' and R' are almost compatible at K. But then Proposition 4.26, gives that either $\langle Q', R' \rangle \cap P = uR'u^{-1} \cap P$ or $\langle Q', R' \rangle \cap P = uQ'u^{-1} \cap P$ respectively. In either case we obtain a contradiction, completing the proof. □

When Q and R have almost compatible parabolics, then so do any pair of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$. It follows that the third case of Theorem 4.27 cannot occur for such Q and R.

Corollary 4.28. *Suppose that Q and R have almost compatible parabolics. There is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) with the following property. Let $P \leqslant G$ be a maximal parabolic subgroup of G with $\langle Q', R' \rangle \cap P$ infinite. Then there is $u \in \langle Q', R' \rangle$ such that $\langle Q', R' \rangle \cap P$ is equal to either $uQ'u^{-1} \cap P$ or $uR'u^{-1} \cap P$. In particular, either $uQ'u^{-1} \cap P$ or $uR'u^{-1} \cap P$ is infinite. Moreover, if either $Q' \cap P$ or $R' \cap P$ is infinite, we may take $u = 1$ in the above.*

We now prove Theorem 1.4 and Corollary 1.5, with more precise existential statements than given in the introduction. In particular, we find a finite index subgroup $Q_1 \leqslant_f Q$ that takes over the role of Q in (E) in the following.

Theorem 4.29. *Suppose that Q and R have almost compatible parabolics. There is a finite index subgroup $Q_1 \leqslant_f Q$ and a family of pairs of finite index subgroups $Q' \leqslant_f Q_1$ and $R' \leqslant_f R$ as in (E) such that Q' and R' have compatible parabolics.*

Proof. By Proposition 4.9, there is a finite index subgroup $Q_1 \leqslant_f Q$ such that if $P \leqslant G$ is a maximal parabolic subgroup of G, then $Q_1 \cap P$ is either infinite or trivial. Let $Q' \leqslant_f Q_1$ and $R' \leqslant_f R$ be finite index subgroups as in (E) satisfying Corollary 4.28. Since Q and R have almost compatible parabolics, so do Q' and R'.

Let $P \leqslant G$ be a maximal parabolic subgroup of G. If $Q' \cap P$ is finite, then $Q_1 \cap P$ is finite and thus trivial by Proposition 4.9. In this case $Q' \cap P = \{1\} \leqslant R' \cap P$. On the other hand, if $Q' \cap P$ is infinite then so is $\langle Q', R' \rangle \cap P$. Now applying Corollary 4.28, we obtain that $\langle Q', R' \rangle \cap P = Q' \cap P$ or $\langle Q', R' \rangle \cap P = R' \cap P$. It follows that either $R' \cap P \leqslant Q' \cap P$ or $Q' \cap P \leqslant R' \cap P$ as required. □

Theorem 4.30. *Suppose that Q and R have almost compatible parabolics. There is a finite index subgroup $Q_1 \leqslant_f Q$ and a family of pairs of finite index subgroups $Q' \leqslant_f Q_1$ and $R' \leqslant_f R$ as in (E) such that $\langle Q', R' \rangle$ is relatively quasiconvex and $\langle Q', R' \rangle \cong Q' *_{Q' \cap R'} R'$.*

Proof. Suppose Q and R have almost compatible parabolics and let $Q_1 \leqslant_f Q$ be the finite index subgroup provided by Theorem 4.29. Note that $S' = Q_1 \cap R$ is a fixed finite index subgroup of $Q \cap R$ depending only on Q. Take $M = M(Q, R, S') \geq 0$ to be the constant of Theorem 2.45.

Following Remark 4.10, we may combine Proposition 4.24 and Theorem 4.29 to obtain a family of pairs of finite index subgroups $Q' \leqslant_f Q_1$ and $R' \leqslant_f R$ as in (E) that have compatible parabolics and satisfy condition (C2) with parameter M. By Lemma 3.24, $\min_X (Q' \cup R') \setminus S' \geq M$. Note that (E) ensures that $Q' \cap R' = S'$. Now applying Theorem 2.45, we see that $\langle Q', R' \rangle$ is relatively quasiconvex and $\langle Q', R' \rangle \cong Q' *_{Q' \cap R'} R'$ as required. □

A relatively quasiconvex subgroup of G is said to be *strongly relatively quasiconvex* if its intersection with each maximal parabolic subgroup of G is finite, and *full* if its intersection with each maximal parabolic subgroup of G is either finite or has finite index in that parabolic. Strongly relatively quasiconvex subgroups are necessarily hyperbolic (Osin, 2006b, Theorem 4.16). Note that if either of Q and R are strongly quasiconvex or full, then they have almost compatible parabolics. As a consequence of Theorem 4.27 one obtains the analogue of Theorem 1.1 for these classes of subgroups.

Corollary 4.31. *If Q and R are strongly (respectively, full) relatively quasiconvex subgroups, then there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that $\langle Q', R' \rangle$ is also strongly (respectively, full) relatively quasiconvex.*

Proof. Recall that if Q and R are strongly quasiconvex or full, they have almost compatible parabolics. Let Q and R be strongly relatively quasiconvex subgroups of G. By Remark 4.10, Theorem 4.23, and Corollary 4.28, there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that $\langle Q', R' \rangle$ is relatively quasiconvex and the conclusion of Corollary 4.28 holds. Let $P \leqslant G$ be a maximal parabolic subgroup of G. Since Q and R have finite intersections with maximal parabolic subgroups of G, so do their subgroups Q' and R'. In particular, $uQ'u^{-1} \cap P$ and $uR'u^{-1} \cap P$ are finite for all $u \in \langle Q', R' \rangle$. Corollary 4.28 now directly implies that $\langle Q', R' \rangle \cap P$ is finite. Therefore $\langle Q', R' \rangle$ is strongly relatively quasiconvex.

Now suppose that Q and R are full relatively quasiconvex subgroups, and again let $Q' \leqslant_f Q$ and $R' \leqslant_f R$ be subgroups as in (E) for which Theorem 4.23 and Corollary 4.28 hold. If $P \leqslant G$ is a maximal parabolic subgroup of G such that $\langle Q', R' \rangle \cap P$ is infinite, then by Corollary 4.28, there is $u \in \langle Q', R' \rangle$ such that at least one of $Q' \cap u^{-1}Pu$ or $R' \cap u^{-1}Pu$ is infinite. Without loss of generality, say that $R' \cap u^{-1}Pu$ is infinite. Now $R' \cap u^{-1}Pu$ has finite index in $R \cap u^{-1}Pu$, which has finite index in $u^{-1}Pu$ since R is fully relatively quasiconvex. Conjugating by u, we see that $uR'u^{-1} \cap P$ has finite index in P. Observing that $\langle Q', R' \rangle \cap P$ contains $uR'u^{-1} \cap P$ completes the proof. □

As an immediate consequence of the above, the virtual joins $\langle Q', R' \rangle$ are hyperbolic when both Q and R are strongly relatively quasiconvex. It may be of interest that this conclusion in fact holds the under slightly weaker hypotheses.

Corollary 4.32. *If Q is hyperbolic and R is strongly relatively quasiconvex, then there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that $\langle Q', R' \rangle$ is relatively quasiconvex and hyperbolic.*

Proof. Let Q be a hyperbolic relatively quasiconvex subgroup of G, and R a strongly relatively quasiconvex subgroup of G. Since R is strongly quasiconvex, Q and R have almost compatible parabolics, we can apply Corollary 4.28. Let $Q' \leqslant_f Q$ and $R' \leqslant_f R$ be subgroups as in (E) for which Corollary 4.28 holds, and let $P \leqslant G$ be a maximal parabolic subgroup of G with $\langle Q', R' \rangle \cap P$ infinite.

Since R is strongly relatively quasiconvex, $uR'u^{-1} \cap P$ is finite for all $u \in \langle Q', R' \rangle$. Hence Corollary 4.28 implies that $\langle Q', R' \rangle \cap P = uQ'u^{-1} \cap P$ for some $u \in \langle Q', R' \rangle$. By Lemma 2.39, $uQ'u^{-1}$ is relatively quasiconvex. Now applying Lemma 2.42 gives that $uQ'u^{-1} \cap P$ is hyperbolic. By Hruska (Hruska, 2010, Theorem 9.1), $\langle Q', R' \rangle$ is hyperbolic relative to a collection of hyperbolic groups. Finally, (Osin, 2006b, Corollary 2.41) yields that $\langle Q', R' \rangle$ is hyperbolic. □

Chapter 5

Product separability in nonpositively curved groups

This chapter of the thesis is dedicated to proving Theorem 1.10 from the introduction. In order to do this we must generalise the discussion of path representatives from Sections 3.2, 3.3, and 3.4, adapting the proofs there to deal with additional technicalities. Let us give a summary of the argument.

Let G be a QCERF finitely generated relatively hyperbolic group with a finite collection of peripheral subgroups $\{H_\nu \mid \nu \in \mathcal{N}\}$. Suppose that, for each $\nu \in \mathcal{N}$, the subgroup H_ν has property RZ_s. Let $F_1, \dots, F_s \leqslant G$ be finitely generated relatively quasiconvex subgroups. In order to show that the product $F_1 \dots F_s$ is separable, we proceed by induction on s. Note that the case that $s = 1$ is exactly the QCERF condition, so we may assume $s > 1$. For ease of reading we relabel the subgroups $F_1 = Q, F_2 = R, F_3 = T_1, \dots, F_s = T_m$, where $m = s - 2 \geq 0$.

We approximate the product $QRT_1 \dots T_m$ with sets of the form $Q\langle Q', R'\rangle RT_1 \dots T_m$, where $Q' \leqslant_f Q$ and $R' \leqslant_f R$ are finite index subgroups of Q and R respectively. Observe that we can write these sets as finite unions

$$Q\langle Q', R'\rangle RT_1 \dots T_m = \bigcup_{i,j} a_i \langle Q', R'\rangle b_j T_1 \dots T_m, \tag{5.1}$$

where the elements a_i and b_j are coset representatives of Q' and R' in Q and R respectively. Note that the products on the right-hand side of (5.1) now involve only $s - 1$ subgroups. By Theorem 1.1, the subgroups Q' and R' can be chosen so that $\langle Q', R'\rangle$ is relatively quasiconvex, hence we can apply the induction hypothesis to show that such products are separable in G.

It then remains to prove that the product $QRT_1 \dots T_m$ is, in fact, an intersection of subsets of the form $Q\langle Q', R'\rangle RT_1 \dots T_m$ as above. To this end, we study path representatives $qp_1 \dots p_n rt_1 \dots t_m$ of elements of $Q\langle Q', R'\rangle RT_1 \dots T_m$ in a similar manner to Chapter 3. The main additional difficulty comes from controlling instances of multiple backtracking that involve segments in the $t_1 \dots t_m$ part of the path. We introduce new metric conditions (C2-m) and (C5-m) to deal with these technicalities.

Finally, we will conclude by collecting the applications of Theorem 1.10 to product separability. In particular, we prove Theorem 1.11, giving new examples of product separable groups. For limit groups and the fundamental groups of graphs of free groups, our proofs are predicated on showing that the groups are LERF and *locally quasiconvex* (i.e. all finitely generated subgroups are relatively quasiconvex), while for Kleinian groups we apply some deep theorems coming from the theory of 3-manifolds to obtain the result.

5.1 Auxiliary definitions

Convention 5.1. In addition to Conventions 2.1 and 3.3, we will assume that $T_1, \dots, T_m \leqslant G$ are fixed relatively quasiconvex subgroups of G, with quasiconvexity constant $\varepsilon \geq 0$, where $m \in \mathbb{N}_0$. Moreover, we use Q' and R' to denote subgroups of Q and R respectively and assume that $Q' \cap R' = S$ (that is, Q' and R' satisfy (C1)).

5.1.1 New metric conditions

Suppose $B, C \geq 0$ are some constants, $\mathcal{P}$ is a finite collection of maximal parabolic subgroups of G, and $\mathcal{U}$ is a finite family of finitely generated relatively quasiconvex subgroups of G. We will be interested in the following generalisations of conditions (C2) and (C5) to the multiple coset setting:

(C2-m) $\min_X\Big(R\langle Q', R'\rangle R T_1 \dots T_j \setminus R T_1 \dots T_j\Big) \geq B$, for each $j = 0, \dots, m$;

(C5-m) $\min_X\Big(q\langle Q'_P, R'_P\rangle R_P (U_1)_P \dots (U_j)_P \setminus q Q'_P R_P (U_1)_P \dots (U_j)_P\Big) \geq C$, for each $P \in \mathcal{P}$, all $q \in Q_P$, any $j \in \{0, \dots, m\}$ and arbitrary $U_1, \dots, U_j \in \mathcal{U}$, where $(U_i)_P = U_i \cap P \leqslant P$.

Remark 5.1. Let us make the following observations.

- When $j = 0$, the inequality from condition (C2-m) reduces to $\min_X(R\langle Q', R'\rangle R \setminus R) \geq B$, which is a part of (C2); on the other hand, the inequality from condition (C5-m) simply becomes (C5). In particular, for each $m \geq 0$, (C5-m) implies (C5).
- In our usage of (C5-m), the set $\mathcal{U}$ will consists of finitely many conjugates of $T_1, \dots, T_m$; in fact, $U_i = T_i^{a_i}$, for some $a_i \in G$, $i = 1, \dots, m$.

Remark 5.2. Similarly to conditions (C1)-(C5), the above metric conditions are best understood with a view towards the profinite topology.

- To prove separability of products of relatively quasiconvex subgroups we argue by induction on the number of factors. That is, we assume that the product of $m+1$ relatively quasiconvex subgroups is separable and then deduce the separability of the product of $m+2$ relatively quasiconvex subgroups. The existence of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ realising condition (C2-m) will be deduced from this inductive assumption.
- The existence of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ realising condition (C5-m), given a finite family $\mathcal{U}$, will be deduced from the assumption that the peripheral subgroups of G each satisfy the property RZ_{m+2}.

5.1.2 Path representatives for products of subgroups

In this subsection we define path representatives for elements of $Q\langle Q', R'\rangle RT_1 \dots T_m$ similarly to the path representatives for elements of $U\langle Q', R'\rangle V$ from Definition 3.10 and discuss their properties.

Definition 5.3 (Product path representative)**.** Let g be an element of the set $Q\langle Q', R'\rangle RT_1 \dots T_m$. Suppose that $p = qp_1 \dots p_n r t_1 \dots t_m$ is a broken line in $\Gamma(G, X \cup \mathcal{H})$ satisfying the following properties:

- $\widetilde{p} = g$;
- $\widetilde{q} \in Q$ and $\widetilde{r} \in R$;
- $\widetilde{p_i} \in Q' \cup R'$ for each $i \in \{1, \dots n\}$;
- $\widetilde{t_i} \in T_i$ for each $i \in \{1, \dots m\}$.

We say that p is a *product path representative* of g in the product $Q\langle Q', R'\rangle RT_1 \dots T_m$.

The type of a product path representative is defined similarly to Definition 3.11.

Definition 5.4 (Type and width of a product path representative)**.** Let $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$ and let $p = qp_1 \dots p_n r t_1 \dots t_m$ be a product path representative of g. Denote by Y the set of all $\mathcal{H}$-components of the segments of p. We define the *width* of p as the integer n and the *type* of p as the triple

$$\tau(p) = \Big(n, \ell(p), \sum_{y \in Y} |y|_X\Big) \in \mathbb{N}_0{}^3.$$

The following observation will be useful.

Remark 5.5. Suppose $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$ can be written as a product

$$g = xy_1 \dots y_n z u_1 \dots u_m,$$

where $x \in Q$, $y_1, \dots y_n \in Q' \cup R'$, $z \in R$ and $u_i \in T_i$, for each $i = 1, \dots, m$. Then g has a product path representative of width n.

Similarly to path representatives of elements of $U\langle Q', R'\rangle V$, we will be interested in product path representatives whose type is minimal (as an element of ${\mathbb{N}_0}^3$ under the lexicographic ordering). Given an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$, such a product path representative is always guaranteed to exist. Let us make the following observation (c.f. Remark 3.14).

Remark 5.6. Suppose that $p = qp_1 \dots p_n rt_1 \dots t_m$ is a minimal type product path representative of an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$ such that $g \notin QRT_1 \dots T_m$. Then $n > 0$, $\widetilde{p_1} \in R' \setminus S$, $\widetilde{p_n} \in Q' \setminus S$ and the labels of $p_1, \dots, p_n$ alternate between representing elements of $R' \setminus S$ and $Q' \setminus S$. In particular, the integer n must be even.

Note that in Definition 5.3 the geodesic paths q, r and $t_1, \dots, t_m$ are always counted as segments of the path p, even if they end up being trivial paths. For example a minimal type product path representative of an element $g \in R'Q'T_1 \dots T_m \setminus QRT_1 \dots T_m$ will be a broken line $p = qp_1p_2rt_1 \dots t_m$ with $m + 4$ segments, where q and r are trivial paths.

The main results from Sections 3.2 and 3.3 can be adapted to apply to minimal type product path representatives of elements of $Q\langle Q', R'\rangle RT_1 \dots T_m \setminus QRT_1 \dots T_m$ with only superficial differences. As such, the proofs of the following generalisations of Lemmas 3.15, 3.18 and 3.19, respectively, will be omitted.

Lemma 5.7. *There is a constant $C_0 \geq 0$ such that the following holds.*
Consider any element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$ with $g \notin QRT_1 \dots T_m$. Let $p = qp_1 \dots p_n rt_1 \dots t_m$ be a product path representative of g of minimal type, with nodes $f_0, \dots, f_{n+m+2}$ (that is, $f_0 = q_-$, $f_i = (p_i)_-$, for each $i \in \{1, \dots, n\}$, $f_{n+1} = r_-$, $f_{n+1+j} = (t_j)_-$, for each $j \in \{1, \dots, m\}$, and $f_{n+m+2} = (t_m)_+$). Then $\langle f_{i-1}, f_{i+1}\rangle^{rel}_{f_i} \leq C_0$, for all $i \in \{1, \dots, n + m + 1\}$.

Lemma 5.8. *There is a constant $C_1 \geq 0$ such that the following is true.*
Consider a minimal type product path representative $p = qp_1 \dots p_n rt_1 \dots t_m$ for an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m \setminus QRT_1 \dots T_m$. If a and b are adjacent segments of p, with $a_+ = b_-$, and h and k are connected $\mathcal{H}$-components of a and b respectively, then $\mathrm{d}_X(h_+, a_+) \leq C_1$ and $\mathrm{d}_X(a_+, k_-) \leq C_1$.

Lemma 5.9. *For any $\zeta \geq 0$ there is $\Theta_0 = \Theta_0(\zeta) \in \mathbb{N}$ such that the following is true.*
Consider a minimal type product path representative $p = qp_1 \dots p_n rt_1 \dots t_m$ for an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m \setminus QRT_1 \dots T_m$. Suppose that a and b are adjacent segments of p, with $a_+ = b_-$, and h and k are connected $\mathcal{H}$-components of a and b respectively, such that

$$\max\{|h|_X, |k|_X\} \geq \Theta_0.$$

Then $\mathrm{d}_X(h_-, k_+) \geq \zeta$.

5.2 Multiple backtracking in product path representatives: two special cases

As in Chapter 3, the main difficulty lies in dealing with multiple backtracking in our chosen path representatives. In this section we will consider two of the possible cases. We will be working under Convention 5.1.

Throughout the rest of the thesis we fix the following notation.

Notation 5.2. let C_1 be the larger of the constants provided by Lemmas 3.18 and 5.8, and write $\mathcal{P}_0$ for the finite collection of maximal parabolic subgroups of G given by

$$\mathcal{P}_0 = \{{H_\nu}^b \mid \nu \in \mathcal{N}, |b|_X \leq C_1\}.$$

The following lemma is roughly analogous to Lemma 3.20.

Lemma 5.10. *For any $L \geq 0$ and any relatively quasiconvex subgroup $T \leqslant G$ there is a constant $L' = L'(L,T) \geq 0$ such that the following is true.*
Let $P = {H_\nu}^b \in \mathcal{P}_0$, for some $\nu \in \mathcal{N}$ and $b \in G$, with $|b|_X \leq C_1$, and let t be a geodesic path in $\Gamma(G, X \cup \mathcal{H})$, with $\widetilde{t} \in T$. Suppose that $v \in Pb = bH_\nu$ is a vertex of t and $u \in P$ is an element satisfying $\mathrm{d}_X(u, t_-) \leq L$. Denote $a = u^{-1}t_- \in G$. Then there is a geodesic path t' in $\Gamma(G, X \cup \mathcal{H})$ such that

- *$t'_- = u$ and $\mathrm{d}_X(t'_+, v) \leq L'$;*
- *$\widetilde{t'} \in T^a \cap P$;*
- *$(t'_+)^{-1} t_+ \in aT$.*

Proof. Let $K = \max\{C_1, \sigma + L\}$, where $\sigma \geq 0$ is a quasiconvexity constant for T. Denote

$$L' = \max\{K'(P, T^a, K) \mid P \in \mathcal{P}_0,\ a \in G,\ |a|_X \leq L\}, \tag{5.2}$$

where $K'(P, T^a, K)$ is obtained from Lemma 2.1.

The hypotheses that $v \in Pb$ and $|b|_X \leq C_1$ imply that $\mathrm{d}_X(v, P) \leq |b|_X \leq C_1$. As $u \in P$, we have $P = uP$ and so

$$\mathrm{d}_X(v, uP) \leq C_1. \tag{5.3}$$

Set $x = t_- = ua$. Since $\widetilde{t} \in T$, we have $\mathrm{d}_X(v, xT) \leq \sigma$, as T is σ-quasiconvex. Hence

$$\mathrm{d}_X(v, uT^a) = \mathrm{d}_X(v, xTa^{-1}) \leq \mathrm{d}_X(v, xT) + |a|_X \leq \sigma + L.$$

Combining the latter inequality with (5.3) allows us to apply Lemma 2.1 to find an element $z \in u(T^a \cap P)$ such that $\mathrm{d}_X(v, z) \leq L'$, where $L' \geq 0$ is the constant from (5.2). Now take t' to be any geodesic in $\Gamma(G, X \cup \mathcal{H})$ with $t'_- = u$ and $t'_+ = z$. It is straightforward to verify that t' satisfies the first two of the required properties. For the last property, observe that

$$(t'_+)^{-1}t_+ = \left((t'_+)^{-1}u\right)\left(u^{-1}t_-\right)\left(t_-^{-1}t_+\right) = \widetilde{t'}^{-1}a\widetilde{t} \in T^a aT = aT. \qquad \square$$

The following notation will be fixed for the remainder of the section.

Notation 5.3. Let D be the constant from Lemma 3.20, corresponding to C_1 and $\mathcal{P}_0$ (from Notation 5.2) and subgroups Q, R. We define constants $L_1, \dots . L_{m+1}$ as follows:

$$L_1 = D + C_1 \quad \text{and} \quad L_{i+1} = L'(L_i, T_i) + C_1, \text{ for each } i = 1, \dots, m,$$

where L' is obtained from Lemma 5.10.
We also define the family of subgroups

$$\mathcal{U}_0 = \bigcup_{i=1}^{m} \left\{ T_i^g \,\middle|\, i \in \{1, \dots, m\}, g \in G, |g|_X \le L_i \right\},$$

consisting of finitely many conjugates of the subgroups $T_1, \dots, T_m$. Note that, by Lemma 2.39, each $U \in \mathcal{U}_0$ is a relatively quasiconvex subgroup of G.

The next proposition describes how we approximate an instance of consecutive backtracking that involves the $t_1 \dots t_m$-part of a product path representative of an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m \setminus QRT_1 \dots T_m$; it complements Proposition 3.22 which takes care of backtracking within the $qp_1 \dots p_n r$-part.

Proposition 5.11. *Suppose that $p = qp_1 \dots p_n r t_1 \dots t_m$ is a product path representative of minimal type for an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m \setminus QRT_1 \dots T_m$. Let $P = H_\nu{}^b \in \mathcal{P}_0$, for some $\nu \in \mathcal{N}$ and $b \in G$, with $|b|_X \le C_1$.*
Suppose that $h_1, \dots, h_j$ are connected H_ν-components of the segments $t_1, \dots, t_j$, respectively, with $j \in \{1, \dots, m\}$, such that $(h_1)_- \in Pb = bH_\nu$. If $u_1 \in P$ is an element satisfying $\mathrm{d}_X(u_1, (t_1)_-) \le L_1$ then there exist elements $a_1, \dots, a_j \in G$ and a broken line $t'_1 \dots t'_j$ in $\Gamma(G, X \cup \mathcal{H})$ such that the following conditions hold:

- *(i)* $(t'_1)_- = u_1$ *and* $\mathrm{d}_X((t'_j)_+, (h_j)_+) \le L_{j+1}$*;*
- *(ii)* $a_{i+1} \in a_i T_i$*, for* $i = 1, \dots, j-1$*;*
- *(iii)* $a_i = (t'_i)_-^{-1}(t_i)_-$ *and* $|a_i|_X \le L_i$*, for each* $i = 1, \dots, j$*;*
- *(iv)* $\widetilde{t'_i} \in T_i^{a_i} \cap P$*, for all* $i = 1, \dots, j$*.*

Proof. We start by setting $a_1 = u_1^{-1}(t_1)_-$, so that $|a_1|_X = \mathrm{d}_X(u_1, (t_1)_-) \le L_1$. Note that $(h_1)_+ = (h_1)_-\widetilde{h_1} \in bH_\nu = Pb$. Therefore we can apply Lemma 5.10 to find geodesic t'_1 in $\Gamma(G, X \cup \mathcal{H})$ such that $(t'_1)_- = u_1$, $\mathrm{d}_X((t'_1)_+, (h_1)_+) \le L'(L_1, T_1)$, $\widetilde{t'_1} \in T_1^{a_1} \cap P$ and

$$(t'_1)_+^{-1}(t_1)_+ \in a_1 T_1. \tag{5.4}$$

It follows that properties (ii)–(iv) are satisfied for $i = 1$, while property (i) holds because $L_2 \ge L'(L_1, T_1)$ by definition. If $j = 1$ then property (ii) is vacuously true.

We can now suppose that $j > 1$. Then h_1 is connected to the component h_2 of t_2, so, according to Lemma 5.8, $\mathrm{d}_X((h_1)_+, (t_1)_+) \leq C_1$. Set $u_2 = (t_1')_+$ and $a_2 = u_2^{-1}(t_1)_+$. Note that $a_2 \in a_1 T_1$ by (5.4) and

$$
\begin{aligned}
|a_2|_X = \mathrm{d}_X((t_1)'_+, (t_1)_+) &\leq \mathrm{d}_X((t_1')_+, (h_1)_+) + \mathrm{d}_X((h_1)_+, (t_1)_+) \\
&\leq L'(L_1, T_1) + C_1 = L_2.
\end{aligned}
$$

Since $(t_2)_- = (t_1)_+$, we see that $a_2 = u_2^{-1}(t_2)_-$ and $\mathrm{d}_X(u_2, (t_2)_-) = |a_2|_X \leq L_2$.

Now, observe that $u_2 = u_1 \widetilde{t_1'} \in P$ and $(h_2)_+ \in bH_\nu = Pb$, as h_2 is connected to h_1. This allows us to use Lemma 5.10 to find a geodesic path t_2' in $\Gamma(G, X \cup \mathcal{H})$ such that $(t_2')_- = u_2 = (t_1')_+$, $\mathrm{d}_X((t_2')_+, (h_2)_+) \leq L'(L_2, T_2)$, $\widetilde{t_2'} \in T_2^{a_2} \cap P$ and $(t_2')_+^{-1} t_+ \in a_2 T_2$ (see Figure 5.1).

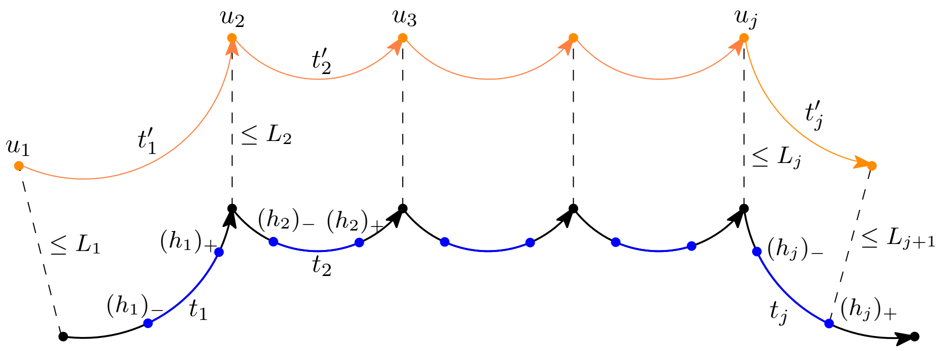

FIGURE 5.1: The new path $t_1' \dots t_j'$ constructed in Proposition 5.11.

If $j = 2$ then we are done, otherwise we construct the remaining elements $a_3, \dots, a_j$ and the paths $t_3', \dots, t_j'$ inductively, similarly to the construction of a_2 and t_2' above. □

The next two propositions prove that, under certain conditions, instances of multiple backtracking are long. Essentially, they generalise Proposition 3.23. The first of these shows how we can use condition (C5-m) to deal with particular instances of multiple backtracking.

Proposition 5.12. *For each $\zeta \geq 0$ there is a constant $C_2 = C_2(\zeta) \geq 0$ such that if $Q' \leqslant Q$ and $R' \leqslant R$ satisfy conditions (C1), (C3) and (C5-m) with constant $C \geq C_2$ and finite families $\mathcal{P}$ and $\mathcal{U}$, such that $\mathcal{P}_0 \subseteq \mathcal{P}$ and $\mathcal{U}_0 \subseteq \mathcal{U}$, then the following is true.*

Let $p = qp_1 \dots p_n r t_1 \dots t_m$ be a minimal type product path representative for some element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$, with $g \notin QRT_1 \dots T_m$. Suppose that p has multiple backtracking along H_ν-components $h_1, \dots, h_k$ of its segments, for some $\nu \in \mathcal{N}$, such that

- *h_1 is an H_ν-component of either q or p_i, for some $i \in \{1, \dots, n-1\}$, with $\widetilde{p_i} \in Q'$;*
- *h_k is an H_ν-component of a segment t_j, for some $j \in \{1, \dots, m\}$.*

Then $\mathrm{d}_X((h_1)_-, (h_k)_+) \geq \zeta$.

Proof. Take
$$C_2 = \max\{2C_1, D + \zeta + L_j \mid j = 1, \dots, m+1\} + 1,$$
where D and L_j are defined in Notation 5.3, and suppose that $C \geq C_2$.

The proof employs the same strategy as Proposition 3.23: we first construct a path whose endpoints are close to $(h_1)_-$ and $(h_k)_+$ and whose label represents an element of a parabolic subgroup. We will then obtain a contradiction with the minimality of the type of p, using condition (C5-m).

We will focus on the case when h_1 is an H_ν-component of p_i, for some index $i \in \{1, \dots, n-1\}$ with $\widetilde{p_i} \in Q'$, with the case when h_1 is an H_ν-component of q being similar. Note that since $g \notin QRT_1 \dots T_m$, it must be that $n \geq 2$ by Remark 5.6. After translating by $(p_i)_+^{-1}$, we may assume that $(p_i)_+ = 1$. We write $b = (h_1)_+$ and note that, according to Lemma 5.8,
$$|b|_X = \mathrm{d}_X((h_1)_+, (p_i)_+) \leq C_1. \tag{5.5}$$

Let $P = bH_\nu b^{-1} \in \mathcal{P}_0 \subseteq \mathcal{P}$. Since $h_1, \dots, h_k$ are pairwise connected, the vertices $(h_l)_+$ lie in the same left coset bH_ν, for all $l = 1, \dots, k$, thus
$$(h_l)_+ \in Pb, \text{ for all } l = 1, \dots, k. \tag{5.6}$$

We construct a new broken line $p' = p'_i \dots p'_n r' t'_1 \dots t'_j$ in two steps. It will be used in conjunction with condition (C5-m) to obtain a product path representative of g with lesser type than p.

<u>Step 1:</u> we start by constructing geodesic paths $p'_i, p'_{i+1}, \dots, p'_n$ and r' by using condition (C3) and applying Lemmas 3.20 and 3.21, in exactly the same way as in the proof of Proposition 3.22. The newly constructed paths will have the following properties:

- $\widetilde{p'_i} \in Q_P$, $\widetilde{p'_l} \in Q'_P \cup R'_P$, for each $l = i+1, \dots, n$, and $\widetilde{r'} \in R_P$;
- $\mathrm{d}_X((p'_i)_-, (h_1)_-) \leq D$ and $(p'_i)_+ = (p_i)_+ = 1$;
- $(p'_l)_+ = (p'_{l+1})_-$, for $l = i, \dots, n-1$;
- $r'_- = (p'_n)_+$ and $\mathrm{d}_X(r'_+, (h_{k-j})_+) \leq D$;
- $(p'_l)_+^{-1}(p_l)_+ \in S$, for $l = i+1, \dots, n$.

<u>Step 2:</u> we now construct geodesic paths $t'_1, \dots, t'_j$ as follows. Set $u_1 = (r')_+$ and observe that since $(p'_{i+1})_- = (p'_i)_+ = 1$, we have
$$u_1 = \widetilde{p'_{i+1}} \dots \widetilde{p'_n} \widetilde{r'} \in P.$$

By Lemma 5.8, we have $d_X((h_{k-j})_+, (t_1)_-) = d_X((h_{k-j})_+, r_+) \le C_1$. Moreover, by Step 1 above, $d_X(u_1, (h_{k-j})_+) \le D$. Therefore

$$d_X(u_1, (t_1)_-) \le C_1 + D = L_1.$$

Together with (5.6) this allows us to apply Proposition 5.11 to find elements $a_1, \dots, a_j \in G$ and a broken line $t'_1 t'_2 \dots t'_j$ in $\Gamma(G, X \cup \mathcal{H})$ such that

- $(t'_1)_- = u_1$ and $d_X((t'_j)_+, (h_k)_+) \le L_{j+1}$;
- $a_{l+1} \in a_l T_l$, for $l = 1, \dots, j-1$;
- $a_l = (t'_l)_-^{-1}(t_l)_-$ and $|a_l|_X \le L_l$, for each $l = 1, \dots, j$;
- $\widetilde{t'_l} \in T_l^{a_l} \cap P$, for all $l = 1, \dots, j$.

Observe that

$$\begin{aligned} a_1 = (t'_1)_-^{-1}(t_1)_- = u_1^{-1} r_+ &= (r'^{-1}_+ r'_-)(r'^{-1}_- r_-)(r_-^{-1} r_+) \\ &= \widetilde{r'}^{-1} (p'_n)_+^{-1} (p_n)_+ \widetilde{r} \in R_P S R \subseteq R. \end{aligned} \tag{5.7}$$

We now define a new broken line p' in $\Gamma(G, X \cup \mathcal{H})$ by

$$p' = p'_i \dots p'_n r' t'_1 \dots t'_j.$$

Note that $\widetilde{p'} \in \widetilde{p'_i} \langle Q'_P, R'_P \rangle R_P (T_1^{a_1})_P \dots (T_j^{a_j})_P$ (where $\widetilde{p'_i} \in Q_P$), $d_X(p'_-, (h_1)_-) \le D$, and $d_X(p'_+, (h_k)_+) \le L_{j+1}$. Moreover, $T_l^{a_l} \in \mathcal{U}_0 \subseteq \mathcal{U}$, for each $l = 1, \dots, j$.

Now, suppose, for a contradiction, that $d_X((h_1)_-, (h_k)_+) < \zeta$. Then, by the triangle inequality,

$$|p'|_X \le D + \zeta + L_{j+1} < C_2.$$

Thus, as $C \ge C_2$, we can apply (C5-m) to deduce that $\widetilde{p'} \in \widetilde{p'_i} Q'_P R_P (T_1^{a_1})_P \dots (T_j^{a_j})_P$. Therefore, there exist elements $z \in \widetilde{p'_i} Q'_P$, $x \in R$ and $y_l \in T_l$, $l = 1, \dots, j$, such that $\widetilde{p'} = z x y_1^{a_1} \dots y_j^{a_j}$. By construction, for each $l = 1, \dots, j-1$ there is $b_l \in T_l$ such that $a_{l+1} = a_l b_l$, and so $a_l^{-1} a_{l+1} = b_l \in T_l$. Recalling that $(p'_i)_+ = (p_i)_+ = 1$, the above yields

$$\widetilde{p'} = z x y_1^{a_1} \dots y_j^{a_j} = z x a_1 y_1 b_1 y_2 b_2 \dots b_{j-1} y_j a_j^{-1}. \tag{5.8}$$

Let α and β be geodesic segments in $\Gamma(G, X \cup \mathcal{H})$ connecting $(p_i)_-$ with $(p'_i)_-$ and $(t'_j)_+$ with $(t_j)_+$ respectively. Since $(p_i)_+ = (p'_i)_+$, we have

$$\widetilde{\alpha} = (p_i)_-^{-1}(p'_i)_- = (p_i)_-^{-1}(p_i)_+ (p'_i)_+^{-1}(p'_i)_- = \widetilde{p_i}\, \widetilde{p'_i}^{-1}. \tag{5.9}$$

On the other hand, it follows from the construction that

$$\widetilde{\beta} = (t'_j)_+^{-1}(t_j)_+ = \widetilde{t'_j}^{-1} (t'_j)_-^{-1}(t_j)_- \widetilde{t_j} = \widetilde{t'_j}^{-1} a_j \widetilde{t_j} \in T_j^{a_j} a_j T_j = a_j T_j. \tag{5.10}$$

The broken lines p and $\gamma = qp_1 \dots p_{i-1}\alpha p'\beta t_{j+1} \dots t_m$ have the same endpoints in $\Gamma(G, X \cup \mathcal{H})$. Hence, in view of (5.9) and (5.8), we obtain

$$\begin{aligned} g = \widetilde{p} = \widetilde{\gamma} &= \widetilde{q}\,\widetilde{p_1} \dots \widetilde{p_{i-1}}\,\widetilde{\alpha}\,\widetilde{p'}\,\widetilde{\beta}\,\widetilde{t_{j+1}} \dots \widetilde{t_m} \\ &= \widetilde{q}\,\widetilde{p_1} \dots \widetilde{p_{i-1}}(\widetilde{p_i}\,\widetilde{p_i'}^{-1})(zxa_1y_1b_1y_2b_2 \dots b_{j-1}y_ja_j^{-1})\widetilde{\beta}\,\widetilde{t_{j+1}} \dots \widetilde{t_m} \\ &= \widetilde{q}\,\widetilde{p_1} \dots \widetilde{p_{i-1}}(\widetilde{p_i}\,\widetilde{p_i'}^{-1}z)(xa_1)(y_1b_1) \dots (y_{j-1}b_{j-1})(y_ja_j^{-1}\widetilde{\beta})\widetilde{t_{j+1}} \dots \widetilde{t_m}. \end{aligned} \tag{5.11}$$

Recall that $\widetilde{q} \in Q$, $\widetilde{p_1}, \dots, \widetilde{p_{i-1}} \in Q' \cup R'$ and $\widetilde{t_l} \in T_l$, for $l = j+1, \dots, m$, by definition. On the other hand, $\widetilde{p_i}\,\widetilde{p_i'}^{-1} z \in Q'\,\widetilde{p_i'}^{-1}\,\widetilde{p_i'}\,Q'_P = Q'$, $xa_1 \in R$ by (5.7) and $y_lb_l \in T_l$, for each $l = 1, \dots, j-1$, by construction. Finally, $y_ja_j^{-1}\,\widetilde{\beta} \in T_ja_j^{-1}a_jT_j = T_j$ by (5.10). Thus, following Remark 5.5, the product decomposition (5.11) for g gives us a product path representative of g with width $i < n$. This contradicts the minimality of the type of p, so the proposition is proved. □

Condition (C2-m) can be used deal with another case of multiple backtracking.

Proposition 5.13. *For every $\zeta \geq 0$ there is a constant $B_1 = B_1(\zeta) \geq 0$ such that if $Q' \leqslant Q$ and $R' \leqslant R$ satisfy condition (C2-m) with constant $B \geq B_1$ then the following is true.*

Let $p = qp_1 \dots p_nrt_1 \dots t_m$ be a minimal type product path representative for some element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$, with $g \notin QRT_1 \dots T_m$, and let $\nu \in \mathcal{N}$. Suppose that p has multiple backtracking along H_ν-components $h_1, \dots, h_k$ of its segments such that

- *h_1 is an H_ν-component of p_i, for some $i \in \{1, \dots, n-1\}$, with $\widetilde{p_i} \in R'$;*
- *h_k is an H_ν-component of t_j for some $j \in \{1, \dots, m\}$.*

Then $d_X((h_1)_-, (h_k)_+) \geq \zeta$.

Proof. Take $B_1 = \zeta + 2\varepsilon + 1$, where $\varepsilon \geq 0$ is a quasiconvexity constant for the subgroups R and $T_1, \dots, T_m$ (as in Convention 5.1), and let $B \geq B_1$. Suppose, for a contradiction, that $d_X((h_1)_-, (h_k)_+) < \zeta$.

Since $\widetilde{p_i} \in R'$, we have $d_X((h_1)_-, (p_i)_+ R) \leq \varepsilon$, by the quasiconvexity of R. Therefore there is a geodesic path p_i' in $\Gamma(G, X \cup \mathcal{H})$, such that $\widetilde{p_i'} \in R$, $d_X((p_i')_-, (h_1)_-) \leq \varepsilon$ and $(p_i')_+ = (p_i)_+$. Similarly, using the quasiconvexity of T_j, we can find a geodesic path t_j' in $\Gamma(G, X \cup \mathcal{H})$, such that $\widetilde{t_j'} \in T_j$, $(t_j')_- = (t_j)_-$ and $d_X((t_j')_+, (h_k)_+) \leq \varepsilon$. Let p' be the broken line $p_i'p_{i+1} \dots p_nrt_1 \dots t_{j-1}t_j'$.

Observe that $\widetilde{p'} \in R\langle Q', R'\rangle RT_1 \dots T_j$ and, by the triangle inequality, $|p'|_X \leq \zeta + 2\varepsilon$. Therefore we can apply condition (C2-m) to $\widetilde{p'}$ to find that $\widetilde{p'} = xy_1 \dots y_j$, where $x \in R$ and $y_l \in T_l$, for each $l = 1, \dots, j$.

The broken lines p and $\gamma = qp_1 \dots p_i p_i'^{-1} p' t_j'^{-1} t_j \dots t_m$ have the same endpoints, hence

$$\begin{aligned} g = \widetilde{p} = \widetilde{\gamma} &= \widetilde{q}\, \widetilde{p_1} \dots \widetilde{p_i}\, \widetilde{p_i'}^{-1}\, \widetilde{p'}\, \widetilde{t_j'}^{-1}\, \widetilde{t_j} \dots \widetilde{t_m} \\ &= \widetilde{q}\, \widetilde{p_1} \dots \widetilde{p_{i-1}} (\widetilde{p_i}\, \widetilde{p_i'}^{-1} x) y_1 \dots y_{j-1} (y_j\, \widetilde{t_j'}^{-1}\, \widetilde{t_j}) \widetilde{t_{j+1}} \dots \widetilde{t_m}. \end{aligned} \tag{5.12}$$

Note that $\widetilde{p_i}\, \widetilde{p_i'}^{-1} x \in R$ and $y_j\, \widetilde{t_j'}^{-1}\, \widetilde{t_j} \in T_j$. In view of Remark 5.5, the product decomposition of g from (5.12) can be used to obtain a product path representative p'' of g with width $i - 1 < n$. Thus the type of p'' is strictly less than the type of p, which yields the desired contradiction. □

5.3 Multiple backtracking in product path representatives: the general case

The statements of Propositions 3.23, 5.12 and 5.13 show that for elements of the set $Q\langle Q', R'\rangle RT_1 \dots T_m \setminus QRT_1 \dots T_m$, instances of multiple backtracking in a minimal type product path representative $p = qp_1 \dots p_n r t_1 \dots t_m$, that start at a component of q, or $p_1, \dots, p_{n-1}$, are long. We cannot draw the same conclusion in all cases since we have no control over the elements $\widetilde{r}, \widetilde{t_1}, \dots, \widetilde{t_m}$. Therefore in this section we use a different approach. Proposition 5.16 below shows that in the remaining cases we can find a product path representative with one of the segments from the tail section $rt_1 \dots t_m$ being short with respect to the proper metric d_X. Note that the main constant $\xi_0 = \xi_0(Q', \zeta)$, produced in this proposition, will depend on Q' (unlike the constants C_1, D, $C_2(\zeta)$, $B_1(\zeta)$, …, defined previously) but will be independent of R'.

As before, we work under Convention 5.1. We will also keep using Notation 5.2 and 5.3. Let us start with the following elementary observation.

Lemma 5.14. *For any $\zeta \geq 0$ and any given subsets $A_1, \dots, A_k \subseteq G$, $k \geq 1$, there is a constant $\xi = \xi(\zeta, A_1, \dots, A_k) \geq 0$ such that if $g \in A_1 \dots A_k$ and $|g|_X \leq \zeta$, then there exist $a_1 \in A_1$,…, $a_k \in A_k$ such that $g = a_1 \dots a_k$ and $|a_i|_X \leq \xi$, for all $i \in \{1, \dots, k\}$.*

Proof. For each $g \in A_1 \dots A_k$ fix some elements $a_{1,g} \in A_1, \dots, a_{k,g} \in A_k$ such that $g = a_{1,g} \dots a_{k,g}$. Now we can define

$$\xi = \max\left\{ |a_{1,g}|_X, \dots, |a_{k,g}|_X \,\middle|\, g \in A_1 \dots A_k,\ |g|_X \leq \zeta \right\} < \infty.$$

Clearly ξ has the required property. □

Definition 5.15 (Tail thickness)**.** Suppose that $Q' \leqslant Q$, $R' \leqslant R$ and $p = qp_1 \dots p_n r t_1 \dots t_m$ is a path representative of an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$. The *tail thickness* of p is defined as $\omega_X(p) = \min\{|r|_X, |t_1|_X, \dots, |t_{m-1}|_X\}$.

Proposition 5.16. *For each $\zeta \geq 0$, let $C_2 = C_2(\zeta)$ be the larger of the two constants provided by Propositions 3.23 and 5.12, and let $B_1 = B_1(\zeta)$ be given by Proposition 5.13. Set $B_2 = B_2(\zeta) = \max\{C_2(\zeta), B_1(\zeta)\}$.*
Suppose that $Q' \leqslant Q$ is a relatively quasiconvex subgroup of G containing $S = Q \cap R$. Then there exists a constant $\xi_0 = \xi_0(Q', \zeta) \geq 0$ such that if $R' \leqslant R$ and Q', R' satisfy conditions (C1)-(C4), (C2-m) and (C5-m), with constants $B \geq B_2$ and $C \geq C_2$ and collections of subgroups $\mathcal{P} \supseteq \mathcal{P}_0$ and $\mathcal{U} \supseteq \mathcal{U}_0$, then the following is true.
Let $p = qp_1 \dots p_n r t_1 \dots t_m$ be a minimal type product path representative for some element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m$, with $g \notin QRT_1 \dots T_m$. Suppose that p has multiple backtracking along $\mathcal{H}$-components $h_1, \dots, h_k$ of its segments, with $k \geq 3$ and $\mathrm{d}_X((h_1)_-, (h_k)_+) \leq \zeta$. Then $m \geq 1$ and there is a product path representative p' for g (not necessarily of minimal type) such that $\omega_X(p') \leq \xi_0$.

Proof. Let $\varepsilon' \geq 0$ be a quasiconvexity constant for Q'. Take $\xi_0 = \xi_0(Q', \zeta) \geq 0$ to be the maximum, taken over all indices i and j satisfying $1 \leq i \leq j \leq m$, of the constants

$$\xi(\zeta + \varepsilon + \varepsilon', Q', R, T_1, \dots, T_j), \;\; \xi(\zeta + 2\varepsilon, R, T_1, \dots, T_j) \text{ and } \xi(\zeta + 2\varepsilon, T_i, \dots, T_j),$$

obtained from Lemma 5.14.

Suppose that $h_1, \dots, h_k$ are as in the statement, with $\mathrm{d}_X((h_1)_-, (h_k)_+) \leq \zeta$. There are four possible cases to consider, depending on the segments of p to which the $\mathcal{H}$-components h_1 and h_k belong to. If h_k is an $\mathcal{H}$-component of one of the segments $p_2, \dots, p_n$ or r, then one obtains a contradiction to the minimality of type of p by following the same argument as in Proposition 3.23 (recall that (C5-m) implies (C5) by Remark 5.1).

If h_1 is an $\mathcal{H}$-component of one of the segments $q, p_1, \dots, p_{n-1}$ and h_k is an $\mathcal{H}$-component of one of the segments $t_1, \dots, t_m$, we obtain a contradiction by applying either Proposition 5.12 or 5.13 (depending on whether h_1 is a component of a segment of p representing an element of Q or R, respectively).

It remains to consider the possibility when h_1 is an $\mathcal{H}$-component of one of the segments $p_n, r, t_1, \dots, t_m$. It follows that h_k is an $\mathcal{H}$-component of t_j, for some $j \in \{1, \dots, m\}$, in particular $m \geq 1$. For simplicity we treat only the case when h_1 is an $\mathcal{H}$-component of p_n; the remaining cases can be dealt with similarly.

Note that $\widetilde{p_n} \in Q'$ by Remark 5.6. By the relative quasiconvexity of Q' and T_j there are geodesic paths α and β in $\Gamma(G, X \cup \mathcal{H})$ satisfying

$$\mathrm{d}_X(\alpha_-, (h_1)_-) \leq \varepsilon', \; \alpha_+ = (p_n)_+ \text{ and } \widetilde{\alpha} \in Q',$$
$$\beta_- = (t_j)_-, \; \mathrm{d}_X(\beta_+, (h_k)_+) \leq \varepsilon \; \text{ and } \widetilde{\beta} \in T_j.$$

Let $\gamma = \alpha r t_1 \dots t_{j-1}\beta$. Observe that $\widetilde{\gamma} \in Q'RT_1 \dots T_j$ and, by the triangle inequality,

$$|\gamma|_X = \mathrm{d}_X(\alpha_-, \beta_+) \leq \varepsilon' + \zeta + \varepsilon.$$

Thus, applying Lemma 5.14, we can find elements $x \in Q'$, $y \in R$, $z_1 \in T_1, \dots, z_j \in T_j$ such that $\widetilde{\gamma} = xyz_1 \dots z_j$ and

$$|y|_X \leq \xi_0. \tag{5.13}$$

Therefore

$$\begin{aligned} g = \widetilde{p} &= \widetilde{q}\,\widetilde{p_1} \dots \widetilde{p_n}(\widetilde{\alpha}^{-1}\widetilde{\alpha})\widetilde{r}\,\widetilde{t_1} \dots \widetilde{t_{j-1}}(\widetilde{\beta}\,\widetilde{\beta}^{-1})\widetilde{t_j} \dots \widetilde{t_m} \\ &= \widetilde{q}\,\widetilde{p_1} \dots \widetilde{p_n}\,\widetilde{\alpha}^{-1}\,\widetilde{\gamma}\widetilde{\beta}^{-1}\,\widetilde{t_j} \dots \widetilde{t_m} \\ &= \widetilde{q}\,\widetilde{p_1} \dots \widetilde{p_{n-1}}(\widetilde{p_n}\,\widetilde{\alpha}^{-1}\,x)yz_1 \dots z_{j-1}(z_j\,\widetilde{\beta}^{-1}\,\widetilde{t_j})\widetilde{t_{j+1}} \dots \widetilde{t_m}. \end{aligned} \tag{5.14}$$

Following Remark 5.5, the product decomposition (5.14) gives rise to a product path representative $p' = q'p'_1 \dots p'_n r' t'_1 \dots t'_m$ for g, where $\widetilde{q'} = \widetilde{q} \in Q$, $\widetilde{p'_i} = \widetilde{p_i} \in Q' \cup R'$, for $i = 1, \dots, n-1$, $\widetilde{p'_n} = \widetilde{p_n}\,\widetilde{\alpha}^{-1}\,x \in Q'$, $\widetilde{r'} = y \in R$, $\widetilde{t'_l} = z_l \in T_l$, for $l = 1, \dots, j-1$, $\widetilde{t'_j} = z_j\,\widetilde{\beta}^{-1}\,\widetilde{t_j} \in T_j$ and $\widetilde{t'_s} = \widetilde{t_s} \in T_s$, for $s = j+1, \dots, m$. In view of (5.13), we see that $\omega_X(p') \leq |y|_X \leq \xi_0$, so the proof is complete. □

The following proposition is an analogue of Lemma 3.25. It employs the constant $C'_0 = \max\{C_0, 14\delta\}$, where C_0 is provided by Lemma 5.7, and the constants $\lambda = \lambda(C'_0) \geq 1$ and $c = c(C'_0) \geq 0$, given by Proposition 3.4.

Proposition 5.17. *For any $\eta \geq 0$ there are constants $\zeta = \zeta(\eta) \geq 0$, $C_3 = C_3(\eta) \geq 0$, $\Theta_1 = \Theta_1(\eta) \in \mathbb{N}$ and $B_3 = B_3(\eta) \geq 0$ such that if $B \geq B_3$, $C \geq C_3$ then there exists $E = E(\eta, Q', B) \geq 0$ such that the following holds.*
Suppose Q' and R' satisfy conditions (C1)-(C4), (C2-m) and (C5-m), with constants B and C, and families $\mathcal{P} \supseteq \mathcal{P}_0$ and $\mathcal{U} \supseteq \mathcal{U}_0$. Let p be a minimal type product path representative for an element $g \in Q\langle Q', R'\rangle RT_1 \dots T_m \setminus QRT_1 \dots T_m$. Assume that for any product path representative p' for g we have $\omega_X(p') \geq E$. Then p is $(B, C'_0, \zeta, \Theta_1)$-tamable.
Let $\Sigma(p, \Theta_1) = f_0 e_1 f_1 \dots e_l f_l$ denote the Θ_1-shortcutting of p, obtained by applying Procedure 3.1, and let e'_j be the $\mathcal{H}$-component of $\Sigma(p, \Theta_1)$ containing e_j, $j = 1, \dots, l$. Then $\Sigma(p, \Theta_1)$ is a (λ, c)-quasigeodesic without backtracking and $|e'_j|_X \geq \eta$, for each $j = 1, \dots, l$.

Proof. The proof is similar to the argument in Lemma 3.25. Let us define the necessary constants:

- $\zeta = \zeta(\eta, C'_0)$ is the constant from Proposition 3.4;
- $\Theta_1 = \max\{\Theta_0(\zeta), \zeta\}$, where Θ_0 is the constant from Lemma 5.9;
- $B_2(\zeta)$ and $C_3 = C_2(\zeta)$ are the constants provided by Proposition 5.16;
- $B_3 = \max\{B_0(\Theta_1, C'_0), B_2(\zeta)\}$, where $B_0(\Theta_1, C'_0)$ is the constant from Proposition 3.4;

and, finally, for any given $B \geq B_3, C \geq C_3$, we set

- $E = \max\{B, \xi_0(\eta, Q') + 1\}$, where $\xi_0(\eta, Q')$ is the constant from Proposition 5.16.

Suppose that Q', R', g and $p = qp_1 \dots p_n r t_1 \dots t_m$ are as in the statement of the proposition. We will now show that p is $(B, C_0', \zeta, \Theta_1)$-tamable.

Since Q' and R' satisfy (C2), Lemma 3.24 together with Remark 5.6 imply that $|p_i|_X \geq B$, for each $i = 1, \dots, n$. Moreover, by assumption, $|r|_X, |t_1|_X, \dots, |t_{m-1}|_X \geq E \geq B$, so condition (i) of Definition 3.3 is satisfied. On the other hand, condition (ii) is satisfied by Lemma 5.7.

If condition (iii) of Definition 3.3 is not satisfied then p must have consecutive backtracking along $\mathcal{H}$-components $h_1, \dots, h_k$ of its segments, such that

$$\max\Big\{|h_i|_X \,|\, i = 1, \dots, k\Big\} \geq \Theta_1 \text{ and } \mathrm{d}_X((h_1)_-, (h_k)_+) < \zeta.$$

Lemma 5.9 rules out the case of adjacent backtracking ($k = 2$), so it must be that $k \geq 3$. That is, $h_1, \dots, h_k$ is an instance of multiple backtracking in p. Proposition 5.16 now applies, giving a product path representative p' for g with $\omega_X(p') \leq \xi_0(\eta, Q') < E$. This contradicts a hypothesis of the proposition, so p must also satisfy condition (iii).

Therefore p is $(B, C_0', \zeta, \Theta_1)$-tamable, and we can apply Proposition 3.4 to achieve the desired conclusion. □

5.4 Establishing conditions (C2-m) and (C5-m) using separability

For this section, we will take G to be a group generated by finite set X.

We will exhibit, under suitable assumptions on the profinite topology on G, the existence of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ satisfying conditions (C1)-(C4), (C2-m) and (C5-m). We begin with (C2-m).

Lemma 5.18. *Let $Q, R, T_1, \dots, T_m \leqslant G$ be subgroups, and let $S = Q \cap R$. Suppose that $RT_1 \dots T_l$ is separable in G, for each $l = 0, \dots, m$. Then for any $B \geq 0$ there is a finite index subgroup $N \leqslant_f G$, with $S \subseteq N$, such that arbitrary subgroups $Q' \leqslant Q \cap N$ and $R' \leqslant R \cap N$ satisfy condition (C2-m) with constant B.*

Proof. For each $l \in \{0, \dots, m\}$ the product $RT_1 \dots T_l$ is separable, so, by Lemma 4.5(b), there is a finite index normal subgroup $M_l \lhd_f G$ such that

$$\min_X(RT_1 \dots T_l M_l \setminus RT_1 \dots T_l) \geq B, \text{ for all } l = 0, \dots, m. \tag{5.15}$$

Define the subgroup $M = \bigcap_{l=0}^{m} M_l \lhd_f G$, and take $N = SM \leqslant_f G$. Observe that

$$RNRT_1 \dots T_l = RSMRT_1 \dots T_l = RSRT_1 \dots T_l M = RT_1 \dots T_l M, \text{ for all } l = 0, \dots, m. \tag{5.16}$$

Now choose arbitrary subgroups $Q' \leqslant Q \cap N$ and $R' \leqslant R \cap N$, so that $\langle Q', R' \rangle \subseteq N$. Since $M \subseteq M_l$ for all l, we can combine (5.15) with (5.16) to draw the desired conclusion. □

We now tackle (C5-m). The next statement is similar to Theorem 4.12.

Lemma 5.19. *Suppose that G is a group generated by finite set X and $m \in \mathbb{N}_0$. Let $Q, R \leqslant G$ be some subgroups, and let $\mathcal{P}$ and $\mathcal{U}$ be finite collections of subgroups of G such that*

1. *each $P \in \mathcal{P}$ has property RZ_{m+2};*
2. *the subgroups $Q \cap P$, $R \cap P$ and $U \cap P$ are finitely generated, for all $P \in \mathcal{P}$ and all $U \in \mathcal{U}$;*
3. *if $P \in \mathcal{P}$, $K \leqslant_f P$ and $L \leqslant_f Q$ then KL is separable in G.*

Then for any $C \geq 0$ and any finite index subgroup $Q' \leqslant_f Q$, there is a finite index subgroup $O \leqslant_f G$, with $Q' \subseteq O$, such for any $R' \leqslant R \cap O$ the subgroups Q' and R' satisfy (C5-m) with constant C and collections $\mathcal{P}$ and $\mathcal{U}$.

Proof. As usual, for subgroups $H \leqslant G$ and $P \in \mathcal{P}$ we denote $H \cap P$ by H_P.

Fix an enumeration $\mathcal{P} = \{P_1, \dots, P_k\}$ and let $Q' \leqslant_f Q$ be a finite index subgroup of Q. Given any $i \in \{1, \dots, k\}$, we choose some coset representatives $a_{i1}, \dots, a_{in_i} \in Q_{P_i}$ of Q'_{P_i}, so that $Q_{P_i} = \bigsqcup_{j=1}^{n_i} a_{ij} Q'_{P_i}$. Let $\mathbb{U}$ be the finite set consisting of all l-tuples $(U_1, \dots, U_l)$, where $l \in \{0, \dots, m\}$ and $U_1, \dots, U_l \in \mathcal{U}$.

Consider any $i \in \{1, \dots, k\}$ and $\underline{u} = (U_1, \dots, U_l) \in \mathbb{U}$, where $l \in \{0 \dots, m\}$. Note that $Q'_{P_i} \leqslant_f Q_{P_i}$ is finitely generated, for each $i = 1, \dots, k$, since Q_{P_i} is itself finitely generated by assumption (2). Combining assumptions (1) and (2), the subset $Q'_{P_i} R_{P_i} (U_1)_{P_i} \dots (U_l)_{P_i}$ is separable in P_i. Therefore, by Lemma 4.5(c), for any $C \geq 0$ there is $F_{i,\underline{u}} \lhd_f P_i$ such that

$$\min_X \left(a_{ij} Q'_{P_i} F_{i,\underline{u}} R_{P_i} (U_1)_{P_i} \dots (U_l)_{P_i} \setminus a_{ij} Q'_{P_i} R_{P_i} (U_1)_{P_i} \dots (U_l)_{P_i} \right) \geq C, \tag{5.17}$$

for all $j = 1, \dots, n_i$.

Define $K_{i,\underline{u}} = Q'_{P_i} F_{i,\underline{u}} \leqslant_f P_i$. Then (5.17) implies that for every $j = 1, \dots, n_i$ we have

$$\min_X \left(a_{ij} K_{i,\underline{u}} R_{P_i} (U_1)_{P_i} \dots (U_l)_{P_i} \setminus a_{ij} Q'_{P_i} R_{P_i} (U_1)_{P_i} \dots (U_l)_{P_i} \right) \geq C. \tag{5.18}$$

Assumption (3) tells us that the double coset $K_{i,\underline{u}}Q'$ is separable in G, and since $Q' \cap P_i = Q'_{P_i} \subseteq K_{i,\underline{u}}$, we can apply Lemma 4.7 to find a finite index subgroup $O_{i,\underline{u}} \leqslant_f G$ such that $Q' \subseteq O_{i,\underline{u}}$ and $O_{i,\underline{u}} \cap P_i \subseteq K_{i,\underline{u}}$.

We can now define a finite index subgroup O of G by

$$O = \bigcap_{i=1}^{k} \bigcap_{\underline{u} \in \mathbb{U}} O_{i,\underline{u}} \leqslant_f G.$$

Observe that $Q' \subseteq O$ and $O \cap P_i \subseteq K_{i,\underline{u}}$, for each $i = 1, \ldots, k$ and all $\underline{u} \in \mathbb{U}$. Consider any subgroup $R' \leqslant R \cap O$. Then $Q'_{P_i} \cup R'_{P_i} \subseteq O \cap P_i$, so (5.18) yields that

$$\min_X \left(a_{ij} \langle Q'_{P_i}, R'_{P_i} \rangle R_{P_i} (U_1)_{P_i} \ldots (U_l)_{P_i} \setminus a_{ij} Q'_{P_i} R_{P_i} (U_1)_{P_i} \ldots (U_l)_{P_i} \right) \geq C, \tag{5.19}$$

for arbitrary $i = 1, \ldots, k$, $l = 0, \ldots, m$, $U_1, \ldots, U_l \in \mathcal{U}$ and any $j = 1, \ldots, n_i$.

Given any $i \in \{1, \ldots, k\}$ and any $q \in Q_{P_i}$, there is $j \in \{1, \ldots, n_i\}$ such that $qQ'_{P_i} = a_{ij}Q'_{P_i}$. It follows that $q\langle Q'_{P_i}, R'_{P_i} \rangle = a_{ij} \langle Q'_{P_i}, R'_{P_i} \rangle$, which, combined with (5.19), shows that Q' and R' satisfy condition (C5-m), as required. □

For the next result we will follow the notation of Convention 5.1.

Proposition 5.20. *Suppose that G is QCERF, the product $RT_1 \ldots T_l$ is separable in G, for every $l = 0, \ldots, m$, and each peripheral subgroup of G has property RZ_{m+2}. Let $\mathcal{P}$ be a finite collection of maximal parabolic subgroups and let $\mathcal{U}$ be a finite collection of finitely generated relatively quasiconvex subgroups in G.*
Then for any $B, C \geq 0$ there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) such that Q' and R' satisfy conditions (C1)-(C4), (C2-m) and (C5-m) with constants B and C and collections $\mathcal{P}$ and $\mathcal{U}$.

Proof. Fix some constants $B, C \geq 0$. By Proposition 4.22, there is a family of pairs of finite index subgroups $Q' \leqslant_f Q$ and $R' \leqslant_f R$ as in (E) satisfying (C1)-(C5) with constants B, C, and family $\mathcal{P}$. Let $L \leqslant_f G$ with $S \subseteq L$ be the finite index subgroup of G provided by the existential statement (E) here.

By the hypothesis on G, the subsets $RT_1 \ldots T_l$ are separable in G, for each $l = 0, \ldots, m$. We can therefore apply Lemma 5.18 to obtain a finite index subgroup $N \leqslant_f G$ from its statement (in particular, $S \subseteq N$). Now we define the finite index subgroup $L_1 \leqslant_f G$, from the statement of the proposition, by setting $L_1 = L \cap N$. Clearly L_1 contains S. Take any $L' \leqslant_f L_1$, with $S \subseteq L'$, and set $Q' = Q \cap L' \leqslant_f Q$. Let $M \leqslant_f L'$ be the subgroup provided by (E) above.

Lemma 2.41 and Corollary 4.18 imply that all the assumptions of Lemma 5.19 are satisfied, so let $O \leqslant_f G$ be the subgroup given by this lemma, with $Q' \subseteq O$. We now define the finite index subgroup $M_1 \leqslant_f L'$, from the statement of the proposition, by $M_1 = M \cap O$.

Evidently, M_1 contains Q'. Choose an arbitrary finite index subgroup $M' \leqslant_f M_1$, with $Q' \subseteq M'$, and set $R' = R \cap M'$. Observe that $M' \leqslant_f G$, by construction, hence $R' \leqslant_f R$. By Proposition 4.22, such Q' and R' satisfy (C1)-(C4) with constants B, C and family $\mathcal{P}$, while by Lemmas 5.18 and Lemma 5.19 they satisfy (C2-m) and (C5-m) with constants B, C and families $\mathcal{P}$ and $\mathcal{U}$. □

5.5 Separability of double cosets in relatively hyperbolic groups

Let us work under Conventions 2.1 and 3.3. In this section we show that if G is QCERF with double coset separable peripheral subgroups, and $Q, R \leqslant G$ are finitely generated relatively quasiconvex subgroups, then the double coset QR is separable in G. Conceptually, the idea for proving the more general Theorem 1.10 is similar to the double coset case presented below, but with many additional technicalities. Indeed, to prove the separability of double cosets requires only the simpler machinery developed in Chapter 3. As such, we present it separately from the proof of Theorem 1.10.

Let us recall some notation required for the following. By $\mathcal{P}_0$ we mean the finite set of maximal parabolic subgroups defined in Notation 3.7 (with $M = 0$). Let C_0 be the constant obtained from Lemma 3.15, and write $C_0' = \max\{C_0, 14\delta\}$. The constant $c_3 = c_3(C_0')$ is obtained from Lemma 2.12, and the constants $\lambda = \lambda(C_0')$ and $c = c(C_0')$ are from Proposition 3.4. Finally, given $\eta \geq 0$, we will write $\Theta_1(\eta)$, $B_1(\eta)$, and $C_3(\eta)$ for the constants obtained from Lemma 3.25.

Lemma 5.21. *For any $A \geq 0$ there exist $\beta = \beta(A) \geq 0$ and $\gamma = \gamma(A) \geq 0$ such that if $Q' \leqslant Q$ and $R' \leqslant R$ satisfy conditions (C1)-(C5) with constants β, γ, and family $\mathcal{P}_0$, then*

$$\min_X \left(Q\langle Q', R'\rangle R \setminus QR \right) \geq A.$$

Proof. Given any $A \geq 0$ let $\eta = \eta(\lambda, c, A)$ be the constant provided by Lemma 2.29. Using Lemma 3.25, set

$$\Theta = \Theta_1(\eta),\ \beta = \max\{B_1(\eta), (4A + c_3)\Theta\},\ \text{and}\ \gamma = C_3(\eta).$$

Suppose that Q' and R' satisfy conditions (C1)-(C5) with constants β, γ, and family $\mathcal{P}_0$, and let $g \in Q\langle Q', R'\rangle R$ be any element with $|g|_X < A$. Let $p = qp_1 \dots p_n r$ be a path representative of g with minimal type (as in Definition 3.10 with $U = Q$ and $V = R$). By Lemma 3.25, the broken line p is (B, C_0', ζ, Θ)-tamable, the Θ-shortcutting $\Sigma(p, \Theta) = f_0 e_1 f_1 \dots f_{m-1} e_m f_m$ is (λ, c)-quasigeodesic without backtracking, and, for each $k = 1, \dots, m$, e'_k, the $\mathcal{H}$-component of $\Sigma(p, \Theta)$ containing e_k, is isolated and satisfies $|e'_k|_X \geq \eta$.

If $m \geq 1$, then, according to Lemma 2.29, $|g|_X = |\Sigma(p, \Theta)|_X \geq A$, contradicting our assumption. Therefore it must be the case that $m = 0$ and $\Sigma(p, \Theta) = f_0$. Since $p_- = (f_0)_-$ and $p_+ = (f_0)_+$, Lemma 3.6 tells us that p is $(4, c_3)$-quasigeodesic. Moreover, following Remark 3.1(c), we see that p_i has no $\mathcal{H}$-component h with $|h|_X \geq \Theta$, for each $i = 1, \dots, n$.

Now, arguing by contradiction, suppose that $g \notin QR$. Then $\widetilde{p_1} \in R' \setminus S$ (by Remark 3.14), so $|p_1|_X \geq \beta$, by Lemma 3.24. Lemma 2.28 now implies that

$$\ell(p_1) \geq \beta/\Theta \geq 4A + c_3.$$

Since $\ell(p) \geq \ell(p_1)$, the $(4, c_3)$-quasigeodesicity of p yields

$$A > |g|_X \geq |g|_{X \cup \mathcal{H}} = |p|_{X \cup \mathcal{H}} \geq \frac{1}{4}\left(\ell(p) - c_3\right) \geq A,$$

which is a contradiction. Therefore $g \in QR$ and the lemma is proved. □

Theorem 5.22. *The double coset QR is separable in G.*

Proof. Consider any $g \in G \setminus QR$, and set $A = |g|_X + 1$. Let $\beta = \beta(A)$ and $\gamma = \gamma(A)$ be the constants obtained from Lemma 5.21. By Remark 4.10, we may combine Theorems 4.22 and 4.23 there are subgroups $Q' \leqslant_f Q$, $R' \leqslant_f R$ satisfying (C1)-(C5) with constants β, γ and finite family $\mathcal{P}_0$ such that $\langle Q', R' \rangle$. By Lemma 5.21, we have $g \notin Q\langle Q', R' \rangle R$.

Now Q' and R' are finitely generated, $\langle Q', R' \rangle$ is finitely generated and relatively quasiconvex. Hence it is separable in G by the assumption that G is QCERF. Observe that since Q' and R' are finite index subgroups in Q and R respectively,

$$Q\langle Q', R' \rangle R = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} a_i \langle Q', R' \rangle b_j,$$

where $a_1, \dots, a_n$ are left coset representatives of Q' in Q, and $b_1, \dots, b_m$ are right coset representatives of R' in R. Recalling Remark 4.1, we see that the subset $Q\langle Q', R' \rangle R$ is separable in G, thus it is a closed set containing QR but not containing g. Since we found such a set for an arbitrary $g \in G \setminus QR$, we can conclude that QR is closed in $\mathcal{PT}(G)$, as required. □

5.6 Separability of quasiconvex products in relatively hyperbolic groups

The goal of this section is to prove Theorem 1.10.

Remark 5.23. Let G be a relatively hyperbolic group. Suppose that $s \in \mathbb{N}$ and the product of any s finitely generated relatively quasiconvex subgroups is separable in G. If $Q_1, \dots, Q_s$ are finitely generated quasiconvex subgroups of G and $a_0, \dots, a_s \in G$ are arbitrary elements, then the subset $a_0 Q_1 a_1 \dots Q_s a_s$ is separable in G.
Indeed, observe that the subset

$$a_0 Q_1 a_1 \dots Q_s a_s = Q_1^{a_0} Q_2^{a_0 a_1} \dots Q_s^{a_0 \dots a_{s-1}} a_0 \dots a_s$$

is a translate of a product of conjugates of the subgroups $Q_1, \dots, Q_s$. Combining Lemma 2.39 with Remark 4.1 and the assumption on G yields the desired conclusion.

Proof of Theorem 1.10. We induct on s. The case $s = 1$ is equivalent to the QCERF property of G, while the case $s = 2$ is Theorem 5.22. Thus we can assume that $s > 2$ and the product of any $s - 1$ finitely generated relatively quasiconvex subgroups is separable in G.

Let $Q_1, \dots, Q_s$ be finitely generated relatively quasiconvex subgroups of G. For ease of notation we write $m = s - 2$, $Q = Q_1$, $R = Q_2$ and $T_i = Q_{i+2}$, for $i \in \{1, \dots, m\}$. We work under Convention 5.1.

Arguing by contradiction, suppose that the subset $QRT_1 \dots T_m = Q_1 \dots Q_s$ is not separable in G. Then there exists $g \in G \setminus QRT_1 \dots T_m$ such that g belongs to the profinite closure of $QRT_1 \dots T_m$ in G. Let us fix the following notation for the remainder of the proof:

- $C_0' = \max\{C_0, 14\delta\} \geq 0$, where C_0 is the constant obtained from Lemma 5.7;
- $c_3 = c_3(C_0') \geq 0$ is the constant obtained from Lemma 2.12;
- $\lambda = \lambda(C_0') \geq 1$ and $c = c(C_0') \geq 0$ are obtained from Proposition 3.4, applied with the constant C_0';
- $\mathcal{P} = \mathcal{P}_0$ is the finite family of maximal parabolic subgroups of G from Notation 5.2;
- $\mathcal{U} = \mathcal{U}_0$ is the finite collection of finitely generated relatively quasiconvex subgroups of G from Notation 5.3;
- $A = |g|_X + 1$ and $\eta = \eta(\lambda, c, A) \geq 0$ is obtained from Lemma 2.29;
- $\zeta = \zeta(\eta) \geq 0$, $\Theta = \Theta_1(\eta) \geq 0$, $C_3 = C_3(\eta) \geq 0$ and $B_3 = B_3(\eta) \geq 0$ are the constants obtained from Proposition 5.17;
- $B = \max\{B_3(\eta), (4A + c_3)\Theta_1\}$ and $C = C_3(\eta)$.

Observe that, by the induction hypothesis, the product $RT_1 \dots T_l$ is separable in G, for every $l = 0, \dots, m$. Moreover, by assumption the peripheral subgroups of G satisfy RZ_s, so Proposition 5.20 applies. Now by Remark 4.10, we may combine Proposition 5.20 and Theorem 4.23 to obtain a family of pairs of finite index subgroups $Q' \leqslant_f Q$ amd $R' \leqslant R$ as in (E) that satisfy (C1)-(C4), (C2-m), and (C5-m) with constants B, C and families $\mathcal{P}$ and $\mathcal{U}$ and with $\langle Q', R' \rangle$ relatively quasiconvex.

Let $L \leqslant_f G$ with $S \subseteq L$ be the finite index subgroup provided by (E) for the above, and fix $Q' = Q \cap L$. Let $M \leqslant_f L$ with $Q' \subseteq M$ be the finite index subgroup provided by (E) corresponding to $L' = L$. Now take $E = E(\eta, Q', B) \geq 0$ to be the constant provided by Proposition 5.17. Let $\{M_j \mid j \in \mathbb{N}\}$ be an enumeration of the finite index subgroups of M containing Q', and define the subgroups

$$M'_i = \bigcap_{j=1}^{i} M_j \leqslant_f L \text{ and } R'_i = M'_i \cap R \leqslant_f R, \; i \in \mathbb{N}. \tag{5.20}$$

Note that for every $i \in \mathbb{N}$, $Q' \subseteq M'_i$, so the condition (E) ensures that the subgroups Q' and R'_i satisfy conditions (C1)-(C4), (C2-m) and (C5-m) with constants B, C and families $\mathcal{P}, \mathcal{U}$, defined above, and $\langle Q', R'_i \rangle$ is relatively quasiconvex. For each $i \in \mathbb{N}$, we consider the subset

$$K_i = Q\langle Q', R'_i \rangle R T_1 \dots T_m.$$

Choose coset representatives $x_1, \dots, x_a \in Q$ and $y_{i,1}, \dots, y_{i,b_i} \in R$ such that $Q = \bigcup_{j=1}^{a} x_j Q'$ and $R = \bigcup_{k=1}^{b_i} R'_i y_{i,k}$. Then

$$Q\langle Q', R'_i \rangle R = \bigcup_{j=1}^{a} \bigcup_{k=1}^{b_i} x_j \langle Q', R'_i \rangle y_{i,k},$$

hence K_i may be written as the finite union

$$K_i = \bigcup_{j=1}^{a} \bigcup_{k=1}^{b_i} x_j \langle Q', R'_i \rangle y_{i,k} T_1 \dots T_m.$$

Therefore, for every $i \in \mathbb{N}$, the subset K_i is separable in G by Remark 5.23 and the induction hypothesis. Since each K_i contains the product $QRT_1 \dots T_m$ and g is in the profinite closure of $QRT_1 \dots T_m$, it must be the case that $g \in K_i$, for every $i \in \mathbb{N}$. The remainder of the proof will be dedicated to showing that we obtain a contradiction by considering sufficiently large values of i.

For each $i \in \mathbb{N}$, let $\mathcal{S}_i$ be the set of product path representatives of g in the set $K_i = Q\langle Q', R'_i \rangle R T_1 \dots T_m$ (see Definition 5.3, where R' is replaced by R'_i). We will now consider two cases depending on the tail thickness of the representatives in $\mathcal{S}_i$.

Case 1: there exists $i \in \mathbb{N}$ such that $\inf_{p' \in \mathcal{S}_i} \omega_X(p') \geq E$.

Choose a product path representative of minimal type $p = qp_1 \ldots p_n r t_1 \ldots t_m$ for g in K_i. Note that $n \geq 1$ and $\widetilde{p_1} \in R'_i \setminus S$ because $g \notin QRT_1 \ldots T_m$ (see Remark 5.6). By the assumption and the above construction, we can apply Proposition 5.17 to conclude that p is (B, C'_0, ζ, Θ)-tamable and the shortcutting $\Sigma(p, \Theta) = f_0 e_1 f_1 \ldots f_{l-1} e_l f_l$, obtained from Procedure 3.1, is (λ, c)-quasigeodesic without backtracking, with $|e'_k|_X \geq \eta$ for each $k = 1, \ldots, l$ (where e'_k denotes the $\mathcal{H}$-component of $\Sigma(p, \Theta)$ containing e_k).

If $l > 0$, then applying Lemma 2.29 to the path $\Sigma(p, \Theta)$ gives

$$|g|_X = |p|_X = |\Sigma(p, \Theta)|_X \geq A > |g|_X,$$

by the choice of η, which gives a contradiction.

Therefore it must be that $l = 0$. Then p is $(4, c_3)$-quasigeodesic by Lemma 3.6 and, according to Remark 3.1(c), no segment of p contains an $\mathcal{H}$-component h with $|h|_X \geq \Theta$. By the quasigeodesicity of p and the fact that p_1 is a subpath of p, we have

$$|g|_{X \cup \mathcal{H}} = |p|_{X \cup \mathcal{H}} \geq \frac{1}{4}(\ell(p) - c_3) \geq \frac{1}{4}(\ell(p_1) - c_3). \tag{5.21}$$

Applying Lemma 2.28 to the geodesic p_1 in $\Gamma(G, X \cup \mathcal{H})$ we obtain

$$\ell(p_1) \geq \frac{1}{\Theta}|p_1|_X \geq \frac{B}{\Theta} \geq 4A + c_3, \tag{5.22}$$

where the second inequality follows from the fact that $\widetilde{p_1} \in R'_i \setminus S$ and Lemma 3.24. Combining (5.21) and (5.22), we get

$$|g|_X \geq |g|_{X \cup \mathcal{H}} \geq \frac{1}{4}(4A + c_3 - c_3) = A > |g|_X,$$

which is a contradiction.

Case 2: for all $i \in \mathbb{N}$ we have $\inf_{p' \in \mathcal{S}_i} \omega_X(p') < E$.

For each $i \in \mathbb{N}$ there is a product path representative $p_i = q_i p_{1,i} \ldots p_{n_i,i} r_i t_{1,i} \ldots t_{m,i} \in \mathcal{S}_i$ for g such that $th(p_i) \leq E$. It must either be the case that $\liminf_{i \to \infty} |r_i|_X \leq E$ or that $\liminf_{i \to \infty} |t_{j,i}|_X \leq E$, for some $j \in \{1, \ldots, m\}$. We will consider the former case, as the latter is very similar.

Since there are only finitely many elements $x \in G$ with $|x|_X \leq E$, we may pass to a subsequence $(p_{i_k})_{k \in \mathbb{N}}$ such that $\widetilde{r_{i_k}} = r \in R$ is some fixed element, for all $k \in \mathbb{N}$. It follows that

$$g = \widetilde{p_{i_k}} \in Q\langle Q', R'_{i_k} \rangle r T_1 \ldots T_m, \quad \text{for each } k \in \mathbb{N}. \tag{5.23}$$

Now, $g \notin QrT_1 \dots T_m$ (as $y \in R$), and the subset $QrT_1 \dots T_m$ is separable in G by the induction hypothesis and Remark 5.23. By Lemma 4.5(a), there is a finite index normal subgroup $N \lhd_f G$ such that $g \notin QNrT_1 \dots T_m$. The subgroup $M \cap QN$ has finite index in M and contains Q', therefore $M \cap QN = M_{j_0}$, for some $j_0 \in \mathbb{N}$.

Choose $k \in \mathbb{N}$ such that $i_k \geq j_0$, so that $M'_{i_k} \subseteq M_{j_0} \subseteq QN$ (see the definition (5.20)). Then $R'_{i_k} = M'_{i_k} \cap R \subseteq QN$, hence

$$Q\langle Q', R'_{i_k}\rangle rT_1 \dots T_m \subseteq QNyT_1 \dots T_m. \tag{5.24}$$

Since $g \notin QNrT_1 \dots T_m$, inclusions (5.23) and (5.24) contradict each other.

We have arrived to a contradiction at each of the two cases, hence the proof is complete. □

5.7 New examples of product separable groups

In this section we prove Theorem 1.11. We proceed with the examples in the order they are listed in the theorem statement. Limit groups are the easiest to treat.

Proposition 5.24. *Limit groups are product separable.*

Proof. Dahmani (2003) and, independently, Alibegović (2005) proved that every limit group is hyperbolic relative to a collection of conjugacy class representatives of its maximal non-cyclic finitely generated abelian subgroups.

Moreover, Wilton (2008) showed that limit groups are LERF and Dahmani (2003) showed they are *locally quasiconvex* (that is, each of their finitely generated subgroups is relatively quasiconvex with respect to the given peripheral structure). Therefore our Theorem 1.10 yields that limit groups are product separable. □

For Kleinian groups, we require the following two lemmas to deal with the case when one of the factors is not relatively quasiconvex.

Lemma 5.25. *Let N be a group and $n \geq 2$ be an integer. Suppose that $H_1, \dots, H_n$ are subgroups of N such that $H_i \lhd N$, for some $i \in \{1, \dots, n\}$, and the image of the product $H_1 \dots H_{i-1}H_{i+1} \dots H_n$ is separable in N/H_i. Then $H_1 \dots H_n$ is separable in N.*

Proof. Let $\varphi : N \to N/H_i$ denote the natural epimorphism. By the assumptions, the subset $S = \varphi(H_1 \dots H_{i-1}H_{i+1} \dots H_n)$ is separable in N/H_i. Observe that

$$H_1 \dots H_n = (H_1 \dots H_{i-1}H_{i+1} \dots H_n)H_i = \varphi^{-1}(S),$$

as $H_i \lhd N$, whence $H_1 \dots H_n$ is closed in the profinite topology on N because group homomorphisms are continuous with respect to profinite topologies. □

Lemma 5.26. *Let G be a group with finitely generated subgroups $F_1, \dots, F_n \leqslant G$, $n \geq 2$. Suppose there exists a finite index subgroup $G' \leqslant_f G$ and an index $i \in \{1, \dots, n\}$ such that $F_i' = F_i \cap G' \lhd G'$ and G'/F_i' has property RZ_{n-1}. Then $F_1 \dots F_n$ is separable in G.*

Proof. Let $N \lhd_f G$ be a finite index normal subgroup contained in G', and set $H_j = F_j \cap N$, for $j = 1, \dots, n$.

Since $[F_j : H_j] < \infty$, for each $j = 1, \dots, n$, the product $F_1 \dots F_n$ can be written as a finite union of subsets of the form $h_1 H_1 h_2 H_2 \dots h_n H_n$, where $h_1, \dots, h_n \in G$. Observe that

$$h_1 H_1 h_2 H_2 \dots h_n H_n = H_1^{g_1} H_2^{g_2} \dots H_n^{g_n} g_n,$$

where $g_j = h_1 \dots h_j \in G$, $j = 1, \dots, n$. Thus, in view of Remark 4.1, in order to prove the separability of $F_1 \dots F_n$ in G it is enough to show that the product $H_1^{g_1} H_2^{g_2} \dots H_n^{g_n}$ is separable, for arbitrary $g_1, \dots, g_n \in G$.

Given any elements $g_1, \dots, g_n \in G$, the subgroups $H_1^{g_1}, H_2^{g_2}, \dots, H_n^{g_n} \leqslant G$ are finitely generated and are contained in N. Moreover, since the subgroup $H_i = F_i \cap N = F_i' \cap N$ is normal in N and $N \leqslant G'$ is normal in G, we see that $H_i^{g_i} \lhd N$ and

$$N/H_i^{g_i} = N^{g_i}/H_i^{g_i} \cong N/H_i \leqslant G'/F_i'.$$

Therefore the group $N/H_i^{g_i}$ has RZ_{n-1}, as a subgroup of G'/F_i', so the image of the product $H_1^{g_1} \dots H_{i-1}^{g_{i-1}} H_{i+1}^{g_{i+1}} \dots H_n^{g_n}$ is separable in $N/H_i^{g_i}$. Lemma 5.25 now implies that $H_1^{g_1} H_2^{g_2} \dots H_n^{g_n}$ is separable in N, hence it is also separable in G by Lemma 4.2(b). As we observed above, the latter yields the separability of $F_1 \dots F_n$ in G, as required. □

Proposition 5.27. *Finitely generated Kleinian groups are product separable.*

Proof. Let G be a finitely generated discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$. We will first reduce the proof to the case when $\mathbb{H}^3/G$ is a finite volume manifold. This idea is inspired by the argument of Manning and Martínez-Pedroza used in the proof of (Manning and Martínez-Pedroza, 2010, Corollary 1.5).

Using Selberg's lemma, we can find a torsion-free finite index subgroup $K \leqslant G$. Since product separability of K implies that of G (Ribes, 2017, Lemma 11.3.5), without loss of generality we can assume that G is torsion-free. It follows that G acts freely and properly discontinuously on $\mathbb{H}^3$, so that $M = \mathbb{H}^3/G$ is a complete hyperbolic 3-manifold.

If M has infinite volume then, by (Matsuzaki and Taniguchi, 1998, Theorem 4.10), G is isomorphic to a geometrically finite Kleinian group. Thus we can further assume that

G is geometrically finite, which allows us to apply (Brooks, 1986, Theorem 2) to find an embedding of G into a torsion-free Kleinian group G' such that $\mathbb{H}^3/G'$ is a finite volume manifold. If G' is product separable, then so is any subgroup of it, hence we have made the promised reduction.

Thus we can suppose that $G = \pi_1(M)$, for a hyperbolic 3-manifold M of finite volume. The tameness conjecture, proved by Agol (2004) and Calegari and Gabai (2006), combined with (Canary, 1996, Corollary 8.3), implies that any finitely generated subgroup $F \leqslant G$ is either geometrically finite or is a virtual fibre subgroup. The latter means that there is a finite index subgroup $G' \leqslant_f G$ such that $F' = F \cap G' \lhd G'$ and $G'/F' \cong \mathbb{Z}$.

By (Matsuzaki and Taniguchi, 1998, Theorem 3.7), G is a geometrically finite subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, hence it is finitely generated and hyperbolic relative to a finite collection of finitely generated virtually abelian subgroups, each of which is product separable by (Ribes, 2017, Lemma 11.3.5). Moreover, by (Hruska, 2010, Corollary 1.6), a subgroup of G is relatively quasiconvex if and only if it is geometrically finite. Finally, G is LERF (and, hence, QCERF) by (Agol, 2013, Corollary 9.4).

Let $F_1, \dots, F_n$ be finitely generated subgroups of G, $n \geq 2$. If F_j is geometrically finite, for all $j = 1, \dots, n$, then the product $F_1 \dots F_n$ is separable in G by Theorem 1.10. Thus we can suppose that F_i is not geometrically finite, for some $i \in \{1, \dots, n\}$. By the above discussion, in this case F_i must be a virtual fibre subgroup of G. Since $\mathbb{Z}$ is product separable, we can apply Lemma 5.26 to conclude that $F_1 \dots F_n$ is separable in G, completing the proof. □

Limit groups and Kleinian groups are hyperbolic relative to virtually abelian subgroups. The peripheral subgroups from relatively hyperbolic structures on groups in Proposition 5.29 will be fundamental groups of graphs of cyclic groups, which motivates the next auxiliary lemma.

Lemma 5.28. *Suppose that G is the fundamental group of a finite graph of infinite cyclic groups. If G is balanced then it is product separable.*

Proof. Suppose that $G = \pi_1(G_-, \Gamma)$, where (G_-, Γ) is a graph of groups, associated to a finite connected graph Γ with vertex set $V\Gamma$ and edge set $E\Gamma$. According to the assumptions, each vertex group G_v, $v \in V\Gamma$, is infinite cyclic. W use G_e to denote the edge group corresponding to $e \in E\Gamma$ (see (Dicks and Dunwoody, 1989, Section I.3) for the definition and general theory of graphs of groups).

If $|E\Gamma| = 0$ then G is cyclic, thus product separable. We proceed by induction on $|E\Gamma|$.

Assume first that one of the edge groups G_e is trivial. If removing e disconnects Γ then G splits as a free product $G_1 * G_2$, where G_1, G_2 are the fundamental groups of finite graphs of infinite cyclic groups corresponding to the two connected components of $\Gamma \setminus \{e\}$. Otherwise, $G \cong G_1 * G_2$, where G_1 the fundamental group of a finite graph of infinite cyclic groups corresponding to the graph $\Gamma \setminus \{e\}$ and G_2 is infinite cyclic. Moreover, G_1 and G_2 will be balanced as subgroups of a balanced group G. Hence G_1 and G_2 will be product separable by induction, so $G \cong G_1 * G_2$ will be product separable by (Coulbois, 2001, Theorem 1).

Therefore we can assume that every edge group G_e is infinite cyclic. This means that G is a *generalised Baumslag-Solitar group*. The assumption that G is balanced now translates into the assumption that G is *unimodular*, using Levitt's terminology from Levitt (2007). We can now apply (Levitt, 2007, Proposition 2.6) to deduce that G has a finite index subgroup K isomorphic to the direct product $F \times \mathbb{Z}$, where F is a free group.

Now, $K \cong F \times \mathbb{Z}$ is product separable by You's result (You, 1997, Theorem 5.1), hence G is product separable as a finite index supergroup of K (see (Ribes, 2017, Lemma 11.3.5)). Thus the lemma is proved. □

Proposition 5.29. *Let G be the fundamental group of a finite graph of free groups with cyclic edge groups. If G is balanced then it is product separable.*

Proof. Without loss of generality we can assume that each vertex group is a finitely generated free group (in particular, G is finitely generated). Indeed, otherwise $G \cong G_1 * F$, where G_1 is the fundamental group of a finite graph of finitely generated free groups with cyclic edge groups and F is free (this follows from the fact that any element of a free group is the product of only finitely many free generators). In this case we can deduce the product separability of G from the product separability of G_1 and F by (Coulbois, 2001, Theorem 1) (recall that F is product separable by (Ribes and Zalesskii, 1993, Theorem 2.1)).

Now, for each vertex group G_v, choose and fix a finite family of maximal infinite cyclic subgroups $\mathbb{P}_v$ such that

(a) no two subgroups from $\mathbb{P}_v$ are conjugate in G_v;

(b) for every edge e incident to v in Γ, the image of the cyclic group G_e in G_v is conjugate into one of the subgroups from $\mathbb{P}_v$.

Condition (a) means that each G_v is hyperbolic relative to the finite family $\mathbb{P}_v$ (for example, by (Bowditch, 2012, Theorem 7.11)), and condition (b) means that each edge group of the given splitting of G is parabolic in the corresponding vertex groups. Therefore we can apply (Bigdely and Wise, 2013, Theorem 1.4) to conclude that G is hyperbolic relative to a finite collection of subgroups $\mathbb{Q}$, where each $Q \in \mathbb{Q}$ acts cocompactly on a *parabolic tree* (see (Bigdely and Wise, 2013, Definition 1.3)) with vertex stabilisers conjugate to elements of $\bigcup_{v \in V\Gamma} \mathbb{P}_v$ and edge stabilisers conjugate to elements of $\{G_e \mid e \in \Gamma\}$. The structure theorem for groups acting on trees (Dicks and Dunwoody, 1989, Section I.4.1) implies that every $Q \in \mathbb{Q}$ is isomorphic to the fundamental group of a finite graph of infinite cyclic groups. Since Q is balanced, being a subgroup of G, we can apply Lemma 5.28 to conclude that each $Q \in \mathbb{Q}$ is product separable. By Wise's result (Wise, 2000, Theorem 5.1) G is LERF, hence we can apply our Theorem 1.10 to deduce that the product of a finite number of finitely generated relatively quasiconvex subgroups is separable in G.

To establish the product separability of G it remains to show that it is locally quasiconvex. To achieve this we will again use the results of Bigdely and Wise. More precisely, according to (Bigdely and Wise, 2013, Theorem 2.6), a subgroup of G is relatively quasiconvex if it is *tamely generated*.

Let $H \leqslant G$ be a finitely generated subgroup. The splitting of G as the fundamental group of the graph of groups (G_-, Γ) induces a splitting of H as the fundamental group of a graph of groups (H_-, Δ), where for each vertex $u \in V\Delta$ the stabiliser H_u is equal to $H \cap G_v{}^g$, for some $v \in V\Gamma$ and some $g \in G$. Moreover, the graph Δ is finite, because H is finitely generated (see (Dicks and Dunwoody, 1989, Proposition I.4.13)). Note that every edge group from (H_-, Δ) is cyclic, hence each vertex group H_u, $u \in V\Delta$, must be finitely generated as H is finitely generated (see (Bigdely and Wise, 2013, Lemma 2.5)).

According to (Bigdely and Wise, 2013, Definition 0.1), H is tamely generated if for every $u \in V\Delta$ the subgroup $H_u = H \cap G_v{}^g$ is relatively quasiconvex in $G_v{}^g$, equipped with the peripheral structure $\mathbb{P}_v{}^g$. But the latter is true because $G_v{}^g$ is a finitely generated free group, so any finitely generated subgroup is undistorted, and hence it is relatively quasiconvex with respect to any peripheral structure on $G_v{}^g$, by (Hruska, 2010, Theorem 1.5). Thus every finitely generated subgroup $H \leqslant G$ is tamely generated, and so it is relatively quasiconvex in G by (Bigdely and Wise, 2013, Theorem 2.6). □

Remark 5.30. If G is finitely generated and is the fundamental group of a finite graph of virtually free groups with virtually cyclic edge groups, G has a torsion-free finite index subgroup K (Shepherd and Woodhouse, 2022, Proposition 3.13). As K has finite index in G, K is balanced and is isomorphic to the fundamental group of a finite graph of free groups with cyclic edge groups. Now the product separability of G follows from the product separability of K by (Ribes, 2017, Lemma 11.3.5). Therefore in the restricted case that G is finitely generated, Theorem 1.11 extends to graphs of virtually free groups with virtually cyclic edge groups.

Remark 5.31. In the case when the graph of groups has two vertices and one edge (so that G is a free amalgamated product of two free groups over a cyclic subgroup), Proposition 5.29 was originally proved by Coulbois in his thesis: see (Coulbois, 2000, Theorem 5.18). We can use similar methods to recover another result of Coulbois: if

$G = H *_C F$, where H is product separable, F is free and C is a maximal cyclic subgroup in F then G is product separable (Coulbois, 2000, Theorem 5.4). Indeed, in this case G will be hyperbolic relative to $\mathbb{Q} = \{H\}$ and will be LERF by Gitik's theorem (Gitik, 1997, Theorem 4.4). As in the proof of Proposition 5.29, the results from Bigdely and Wise (2013) imply that G is locally quasiconvex. Therefore G is product separable by Theorem 1.10.

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