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T.M. Kulkarni and A. Agarwal

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by

Tejas M. Kulkarni and Anurag Agarwal

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Viscous Linear Instability of an Incompressible Round Jet

Tejas M. Kulkarni* **and Anurag Agarwal†**

Institute of Sound and Vibration Research, University of Southampton
Southampton SO17 1BJ

Abstract

Spatial viscous instability modes for an incompressible round jet have been computed. The incompressible linear stability equations are derived from the Navier Stokes equations in cylindrical polar coordinates. The instability modes are obtained by solving the two point boundary eigenvalue problem. The boundary conditions are obtained by using asymptotic analytical solutions for large and small (near the jet axis) r. In order to avoid the regular singularity at $r = 0$, power-series expansions for small values of r are derived. The governing equations are integrated by a fifth-order variable step Runge-Kutta method. Gram-Schmidt orthonormalisation is used to maintain the linear independence of solutions. The numerical method is validated against the results available in the literature.

*Visiting undergraduate student, Indian Institute of Technology Madras
†Rolls-Royce Lecturer, ISVR

1 Introduction

The main objective of the present summer project was to develop a C++ computer programme to obtain the viscous linear instability modes of an incompressible round jet. This required a numerical integration of the sixth order incompressible Orr-Sommerfeld equations in cylindrical polar coordinates. The starting point is the incompressible Navier-Stokes Equations. These equations are written in cylindrical coordinates and then linearized. Each of the linearized dependent variables (radial velocity - V, angular velocity - W, streamwise velocity - U) are separated into a mean, $\bar{\theta}$, and a fluctuating value, $\hat{\theta}$, where these fluctuations are modeled as: $\hat{\theta}(r, \phi, z, t) = \theta(r)e^{i(\alpha z - \omega t + n\phi)}$. This results in a system of six ordinary differential equations

$$\mathbf{Y}'(r) = \mathbf{A}\mathbf{Y}(r)$$

where $\mathbf{Y}$ is the vector of dependent variables, r is the independent variable, and $\mathbf{A}$ is a matrix that is a function of ω, α and r and Reynolds number R. Being a sixth order ordinary differential equation (ODE) we need six boundary conditions for the problem to be well posed. We specify three boundary conditions at a large value of r. These three boundary conditions lead to three solutions which satisfy the governing equations independently. The three independent solution vectors are then integrated integrated using a fifth-order variable step Runge-Kutta method from a large value of r to small value of r. But since this is a stiff ODE, the vectors become contaminated with computational round-off errors in the integration process. To preserve the linear independence of the solutions we need to orthonormalise the solution vectors at several steps in the integration. This is done by performing Gram-Schmidt orthonormalisation on the solutions. Thus, at the end of the integration process we obtain three independent solutions. The other three of the six boundary conditions are then specified using power-series for small values of r close to the axis of the jet. The power-series solution is necessary because the jet axis, $r = 0$, is a regular singular point of the governing equations in cylindrical polar coordinates. The linear combination of the integrated vectors is equated to the linear combination of the boundary conditions at small r, which gives a system of six algebraic equations. These equations represent an eigenvalue problem because, for any given value of ω only discrete α satisfy the equations for the spatial stability problem and vice-versa for the temporal stability problem. The eigenvalues are obtained by a shooting method for various values of frequency ω and Reynolds number R, for a fully developed velocity profile $\bar{U}$. Although only the spatial instability modes have been calculated in the present report, the same method applies to temporal modes as well.

The following sections detail each step of the derivation, including the numerical approach to solve the system and obtain the eigenvalues. A comparison for the critical Reynolds number obtained is also made with the values quoted in the literature.

In section 2 we derive the governing equations to be used in the numerical scheme. The numerical scheme is described in section 3. In section 4 we discuss some results and section 5 provides a summary.

2 Theory

In this section we develop the relevant equations to be used in the numerical scheme. To begin with, we start with the governing equations and then proceed to the relevant details of the derivation.

2.1 Governing Equations

To derive the linearized equations in polar coordinates we start with the incompressible Navier-Stokes Equations written in vector form:

$$\nabla \cdot \mathbf{u}^* = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{u}^*}{\partial t} + (\mathbf{u}^* \cdot \nabla)\mathbf{u}^* = -\frac{\nabla p^*}{\rho^*} + \nu \nabla^2 \mathbf{u}^*, \tag{2.2}$$

where * denotes a dimensional variable and $\mathbf{u}^* = \{U_z^*, V_r^*, W_\phi^*\}$. These equations are non-dimensionalised with respect to a length scale L_c^*, a velocity scale U_c^* and the ambient density ρ_o^* to give vector form of the incompressible dimensionless Navier-Stokes Equations:

$$\nabla \cdot \mathbf{u} = 0, \tag{2.3}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \frac{1}{R}\nabla^2 \mathbf{u} \tag{2.4}$$

where $R = L_c^* U_c^* / \nu_o^*$ is the Reynolds Number. In cylindrical polar coordinates, these equations become [1]:

[1]The details of the vector relations in cylindrical polar coordinates can be found in Appendix A

Continuity Equation

$$\frac{\partial V}{\partial r} + \frac{V}{r} + \frac{1}{r}\frac{\partial W}{\partial \phi} + \frac{\partial U}{\partial z} = 0 \tag{2.5}$$

r **- Momentum Equation**

$$\begin{aligned}\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial r} + \frac{W}{r}\frac{\partial V}{\partial \phi} + U\frac{\partial V}{\partial z} - \frac{W^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \\ \frac{1}{R}\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} - \frac{V}{r^2} - \frac{2}{r^2}\frac{\partial W}{\partial \phi}\Big]\end{aligned} \tag{2.6}$$

ϕ **- Momentum Equation**

$$\begin{aligned}\frac{\partial W}{\partial t} + V\frac{\partial W}{\partial r} + \frac{W}{r}\frac{\partial W}{\partial \phi} + U\frac{\partial W}{\partial z} + \frac{VW}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \\ \frac{1}{R}\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial W}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 W}{\partial \phi^2} + \frac{\partial^2 W}{\partial z^2} - \frac{W}{r^2} + \frac{2}{r^2}\frac{\partial V}{\partial \phi}\Big]\end{aligned} \tag{2.7}$$

z **- Momentum Equation**

$$\begin{aligned}\frac{\partial U}{\partial t} + V\frac{\partial U}{\partial r} + \frac{W}{r}\frac{\partial U}{\partial \phi} + U\frac{\partial U}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \\ \frac{1}{R}\Big[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial U}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2}\Big]\end{aligned} \tag{2.8}$$

where $V = V_r$, $U = U_z$ and $W = W_\phi$

2.2 Linearized Navier-Stokes Equations

In order to linearise the equations we have to make the following assumptions: 1) all variables have the form: $\theta = \bar{\theta} + \hat{\theta}$; 2) products of fluctuating quantities are negligible; 3) $\bar{V}_r = \bar{V}_\phi = 0$ (parallel mean flow assumption); 4)$\bar{\rho}$ and $\bar{U}$ are functions of r only. On making these substitutions for each variable, expanding and then subtracting the base flow equations from the new equations, we get the linearized equations:

Continuity Equation

$$\frac{\partial \hat{v}}{\partial r} + \frac{\hat{v}}{r} + \frac{1}{r}\frac{\partial \hat{w}}{\partial \phi} + \frac{\partial \hat{u}}{\partial z} = 0 \tag{2.9}$$

r - Momentum Equation

$$\frac{\partial \hat{v}}{\partial t} + \bar{U}\frac{\partial \hat{v}}{\partial z} = -\frac{1}{\bar{\rho}}\frac{\partial \hat{p}}{\partial r} + \frac{1}{R}\Big[\frac{1}{r}\frac{\partial \hat{v}}{\partial r} + \frac{\partial^2 \hat{v}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \hat{v}}{\partial \phi^2} + \frac{\partial^2 \hat{v}}{\partial z^2} - \frac{\hat{v}}{r^2} - 2\frac{\partial \hat{w}}{\partial \phi}\Big] \tag{2.10}$$

ϕ - Momentum Equation

$$\frac{\partial \hat{w}}{\partial t} + \bar{U}\frac{\partial \hat{w}}{\partial z} = \frac{1}{\bar{\rho}}\frac{1}{r}\frac{\partial \hat{p}}{\partial \phi} + \frac{1}{R}\Big[\frac{1}{r}\frac{\partial \hat{w}}{\partial r} + \frac{\partial^2 \hat{w}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \hat{w}}{\partial \phi^2} + \frac{\partial^2 \hat{w}}{\partial z^2} + \frac{2}{r^2}\frac{\partial \hat{w}}{\partial \phi} - \frac{\hat{w}}{r^2}\Big] \tag{2.11}$$

z - Momentum Equation

$$\frac{\partial \hat{u}}{\partial t} + \hat{v}\frac{\partial \bar{U}}{\partial r} + \bar{U}\frac{\partial u}{\partial z} = -\frac{1}{\bar{\rho}}\frac{\partial \hat{p}}{\partial z} + \frac{1}{R}\Big[\frac{\partial \hat{u}}{\partial r} + \frac{\partial^2 \hat{u}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \hat{u}}{\partial \phi^2} + \frac{\partial^2 \hat{u}}{\partial z^2}\Big] \tag{2.12}$$

Once the linearized equations are derived, the next task is to transform them into the stability equations by substituting for the variables with their normal mode forms, which will be detailed in the next subsection.

2.3 Stability Equations

The fluctuating variables are assumed to Fourier decompose into complex components typically of the form

$$\hat{\theta}(r, \phi, z, t) = Re[\theta(r)e^{i(\alpha z - \omega t + n\phi)}]$$

where α is the complex wavenumber, ω is the real radian frequency (spatial stability analysis) and n is the azimuthal mode number. Substitution for the variables $\hat{u}, \hat{v}, \hat{w}$ and $\hat{p}$ gives:

Continuity Equation

$$i\alpha u + v' + \frac{v}{r} + in\frac{w}{r} = 0, \tag{2.13}$$

r **- Momentum Equation**

$$i\alpha(\bar{U} - c)v + p' = \frac{1}{R}\Big[v'' + \frac{v'}{r} - \big\{\alpha^2 + \frac{(n^2+1)}{r^2}\big\}v - i\frac{2n}{r^2}w\Big], \tag{2.14}$$

ϕ **- Momentum Equation**

$$i\alpha(\bar{U} - c)w + \frac{in}{r}p = \frac{1}{R}\Big[w'' + \frac{w'}{r} - \big\{\alpha^2 + \frac{(n^2+1)}{r^2}\big\}w + i\frac{2n}{r^2}v\Big] \tag{2.15}$$

z **- Momentum Equation**

$$i\alpha(\bar{U} - c)u + \bar{U}'v + i\alpha p = \frac{1}{R}\Big[u'' + \frac{u'}{r} - \big(\alpha^2 + \frac{n^2}{r^2}\big)u\Big] \tag{2.16}$$

where c = ω/α and primes denote differentiation with respect to r.

In order to solve the linear system we must define the boundary conditions. According to Morris[3] these boundary conditions at the centreline of the jet depend on the mode n. Also, in the far field, the boundary conditions are that $\theta \to 0$ when $r \to \infty$. In particular for each mode we must implement the following boundary conditions once our system has been defined and before the eigenvalue problem is solved.

$$\left.\begin{aligned} u(0) = p(0) = 0, &\qquad n \neq 0, \\ v(0) = w(0) = 0, &\qquad n \neq 1, \\ v(0) + iw(0) = 0, &\qquad n = 1. \end{aligned}\right\} \tag{2.17}$$

2.4 Final form of the equations used in the program

It is now more convenient to write the set (2.13) - (2.16) in the form of a set of first-order equations. This is done simply by defining two new variables – u' and w'

$$\mathbf{Y}' = \mathbf{A}(r)\mathbf{Y} \tag{2.18}$$

where

$$\mathbf{Y} = \begin{bmatrix} u \\ v \\ w \\ p \\ u' \\ w' \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ (i\alpha) & (\frac{1}{r}) & (\frac{in}{r}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & (2\alpha^2 - \beta^2 + \frac{n^2}{r^2}) & (\frac{in}{r^2}) & 0 & (i\alpha) & (\frac{in}{r}) \\ (\beta^2 - 2\alpha^2 + \frac{n^2}{r^2}) & (\bar{U}'R) & 0 & (i\alpha R) & (-\frac{1}{r}) & 0 \\ 0 & (\frac{2in}{r^2}) & (\beta^2 + \frac{n^2+1}{r^2}) & (\frac{inR}{r}) & 0 & -(\frac{1}{r}) \end{bmatrix}$$

where $\beta = [\alpha^2 + i\alpha R(\bar{U} - c)]^{\frac{1}{2}}$. Because these equations are singular at $r = 0$, in order to develop a numerical technique to solve these equations it is necessary to find the form of the solution close to the axis of symmetry. Also, because the numerical integration cannot be carried out to infinity, boundary conditions are sought in the nearly undisturbed fluid surrounding the jet. These asymptotic solutions will be developed in the next subsection.

2.5 Asymptotic solutions of the stability equations

2.5.1 Near the axis of the jet ($r \to 0$)

For small values of r, Lessen and Singh[2] and Garg and Rouleau[1] obtained the solution to the disturbance equations by expanding the velocity and pressure fluctuations in a power-series in r. The recurrence relations for the coefficients in the power-series expansion are given by Garg and Rouleau[1]. These recurrence relations are used to derive the eigenfunctions near $r = 0$.

It is simpler to develop the series expansions of the eigenfunctions $v(r)$ and $w(r)$ in terms of the combinations

$$\begin{aligned} f(r) &= v(r) + iw(r), \\ g(r) &= v(r) - iw(r). \end{aligned} \tag{2.19}$$

Once $f(r)$ and $g(r)$ are known, it is simple to calculate $v(r)$ and $w(r)$. The transformed equations in terms of f, g, u, p are easily obtained by substituting for v and w in the stability equations. Also, the boundary conditions follow directly from (2.17).

The series expansions take the form:

$$S_j = r^{a_j}(S_{1j} + S_{2j}r^2 + S_{3j}r^4 + ... + S_{lj}r^{2(l-1)} + ...) \qquad (j = 1, 2, 3, 4). \tag{2.20}$$

where

$$\{a_1, a_2, a_3, a_4\} = \{(n+1), (n-1), n, n\}, \tag{2.21}$$

$S_j = (F, G, U, P)$, $S_{ij} = (F_i, G_i, U_i, P_i)$, F, G, U, P are the series expansions for f, g, u, p and F_i, G_i, U_i, P_i are the coefficients of powers of r in the series expansions. We use only the first 2 terms of the power series as the value of r chosen is sufficiently small. The coefficients used are:

$$F_1, G_1, U_1, P_1, F_2, G_2, U_2, P_2$$

Only three of the first four constants F_1, G_1, U_1 and P_1 are independent owing to the relation

$$F_1 = \frac{1}{n+1}\Big(\frac{B_1}{4n}G1 - \frac{R}{2}P1 - i\alpha U1\Big) \tag{2.22}$$

where $B_1 = -\alpha^2 - R(i\alpha C_1 - i\omega)$ and C_1 is the first coefficient in the series expansion of $\bar{U}$ when $\bar{U} = C_1 + C_2 r^2 + C_3 r^4 + \cdots$

Taking G_1, U_1 and P_1 to be independent, all of the other five coefficients can be expressed in terms of them. This enables any eigenfunction to be expressed as a sum of three terms; for example, the axial velocity eigenfunction $u(r)$ may be written as

$$u(r) = u_1(r)G_1 + u_2(r)U_1 + u_3(r)P_1.$$

This results in three independent sets of solutions $(u_1, v_1, w_1, p_1), (u_2, v_2, w_2, p_2)$ and (u_3, v_3, w_3, p_3). The set (2.13) - (2.16) is thus equivalent to three sets since each of the three sets of solutions must satisfy (2.13) - (2.16) independently. The power-series used for each of the variables u, v, w, p are given in Appendix B.

2.5.2 Far away from the axis of the jet (large r)

Far from the axis of the jet, most jets have a constant mean velocity. According to Morris[3], the form of solutions for the velocity and pressure fluctuations is conveniently derived if the vorticity equations are considered. By making use of the relationship between the vorticity and velocity components and using the z-momentum equation to obtain the pressure fluctuation, the asymptotic form, for large radii of the velocity and pressure disturbances can be shown to be

$$u = A_{1,2} H_n^{(1),(2)}(i\alpha r) + A_{3,4} H_n^{(1),(2)}(i\beta r) \tag{2.23}$$

$$v = -A_{1,2}\frac{i}{\alpha} H_n'^{(1),(2)}(i\alpha r) - A_{3,4}\frac{\alpha}{\beta^2} H_n'^{(1),(2)}(i\beta r) - A_{5,6}\frac{n}{r} H_n^{(1),(2)}(i\beta r) \tag{2.24}$$

$$w = A_{1,2}\frac{n}{\alpha r} H_n^{(1),(2)}(i\alpha r) + A_{3,4}\frac{n\alpha}{\beta^2 r} H_n^{(1),(2)}(i\beta r) + A_{5,6} H_n'^{(1),(2)}(i\beta r) \tag{2.25}$$

$$p = A_{1,2}(c - \bar{U}) H_n^{(1),(2)}(i\alpha r) \tag{2.26}$$

A Hankel function of the first or second kind is chosen, depending on the phase of the argument. The above solutions represent the form of the solutions in any region where the mean velocity is constant.

Note that the equations reported by Morris[3] are incorrect as they do not satisfy (2.13) - (2.16).

2.6 Mean velocity profile in the jet

Before performing the numerical integration of the stability equations, it is necessary to specify the mean velocity profile. Although the mean flow is diverging, the solutions presented here are based on the assumptions that the flow is locally parallel.

The streamwise velocity profile of the fully developed jet used in references[2], [3] and [4] is given by the following analytical equation:

$$\bar{U} = \frac{1}{(1+r^2)^2} \tag{2.27}$$

This particular choice for the mean velocity profile has been made because numerical results for it are available in literature.

The expansions and analytic solutions derived in the present section will form the basis of the numerical scheme described in the next section.

3 Numerical Scheme

The equations derived in the previous section represent a two-point boundary eigenvalue problem. However the equations need to be integrated first. We found that the numerical scheme was more robust if the numerical integration was carried out in the reverse direction from a large value, say $r = a$, to a small value $r = r_s$ near the jet axis. The form of the solutions far away from the axis (at $r = a$) being known from (2.5.2), is taken as the initial condition for the integration process. The vectors are integrated using a fifth order variable step Runge-Kutta integration method from $r = a$ to $r = r_s$. In order to preserve the linear independence of the three solutions, a Gram-Schmidt orthonormalisation process is performed at a number of steps within the range of the numerical integration.

The numerical solutions thus obtained after the integration process are of the form

$$u(r_s) = G_1 u_1(r_s) + U_1 u_2(r_s) + P_1 u_3(r_s) \tag{3.1}$$

Equating the integrated numerical solution at r_s with the power-series solution written at $r = r_s$ we get a set of six equations for the six variables involved $\{u, v, w, p, u', w'\}$. These equations can be written as

$$\mathbf{F}(\alpha, \omega, R)[A_{1,2}, A_{3,4}, A_{5,6}, G_1, U_1, P_1]^T = 0, \tag{3.2}$$

where $\mathbf{F}(\alpha, \omega, R)$ is a 6 x 6 matrix:

$$
\mathbf{F} = \begin{bmatrix}
H_n(i\alpha r) & H_n(i\beta r) & 0 & u_1 & u_2 & u_3 \\
-\frac{i}{\alpha}H_n'(i\alpha r) & -\frac{i\alpha}{\beta^2}H_n'(i\beta r) & -\frac{in}{r}H_n(i\beta r) & v_1 & v_2 & v_3 \\
\frac{n}{\alpha r}H_n(i\alpha r) & \frac{n\alpha}{\beta^2 r}H_n(i\beta r) & H_n'(i\beta r) & w_1 & w_2 & w_3 \\
(c-\bar{U})H_n(i\alpha r) & 0 & 0 & p_1 & p_2 & p_3 \\
H_n'(i\alpha r) & H_n(i\beta r) & 0 & u_1' & u_2' & u_3' \\
\frac{n}{\alpha}\left\{\frac{H_n(i\alpha r)}{r}\right\}' & \frac{n\alpha}{\beta^2}\left\{\frac{H_n(i\beta r)}{r}\right\}' & H_n''(i\beta r) & w_1' & w_2' & w_3'
\end{bmatrix} \tag{3.3}
$$

where the Hankel functions and the last three columns are evaluated at $r = r_s$

The eigenvalues can then be determined by satisfying the condition

$$
\det \mathbf{F} = 0. \tag{3.4}
$$

For a given ω, the eigenvalues α take on discrete values such that (3.4) is satisfied. To calculate the eigenvalues we need the values of the Hankel functions present in the determinant. The SLATEC[5] mathematical subroutine library has been used for this purpose. The values given by SLATEC show good agreement with those from Mathematica[6]. In order to calculate the eigenvalues using an iterative scheme it is helpful and important to find the approximate location of the eigenvalues because good initial guesses to start the iteration allow for fast convergence. For any given values of the Reynolds number and frequency there will be a discrete spectrum of eigenvalues. In order to locate the eigenvalues a shooting method is used. In the shooting method we guess the value for α to start with, then integrate from $r = a$

to $r = r_s$, compute det$F$ at r_s and finally iterate using Newton-Raphson method to find α that satisfies (3.4). For the root finding algorithm we need to specify an initial guess for the eigenvalue. The method is found to be extremely sensitive to the initial guess. In our calculations we are solving the spatial problem wherein, a value of α is sought for a given value of ω and Reynolds Number R.

We have also computed the value for the critical Reynolds number. The critical Reynolds number is the Reynolds number below which the jet is stable for every value of frequency ω. To find the critical Reynolds number several graphs were plotted covering the region near the maxima of the amplification factor. These plots give the value of critical Reynolds number as will be discussed in a subsequent section.

4 Results

4.1 Phase velocity and amplification factor

In stability analysis the most important eigenvalue is the one that is the most unstable (or least stable). For the present framework, this corresponds to the eigenvalue with the least imaginary part. In particular, as can be inferred from the expressions, the flow will be spatially unstable if the imaginary part of the complex wavenumber α is negative, that is if there are eigenvalues lying in the fourth quadrant of the complex-α plane. This region of the α plane is therefore explored for possible eigenvalues by the technique described in section 3. The results in the present section have been obtained by parametrically varying the Reynolds number and frequency ω for an azimuthal wave number n of 1.

The phase velocity and the amplification factor as a function of ω and R for the asymmetric $n = 1$ mode are shown in figures (4.1) and (4.2) respectively. The phase velocity gradually approaches the inviscid solution as R increases. For the $n = 1$ mode the phase velocity increases monotonically with frequency. The amplification factor does not behave in such a regular manner and an unusual phenomenon occurs. Increasing the value of R, the peak amplification factor increases at low frequencies. For values of R of order greater than 200 increasing the value of R decreases the peak amplification factor.

The graphs obtained for both, the amplification factor and the phase velocity, agree well with those given by Morris[3]. We expect that the values determined by our method are more accurate than previous calculations. The method converges well for low Reynolds number, while at higher Reynolds number the initial guess needs to be very close to the actual eigenvalue. This is because the problem becomes increasingly stiff as the Reynolds number increases.

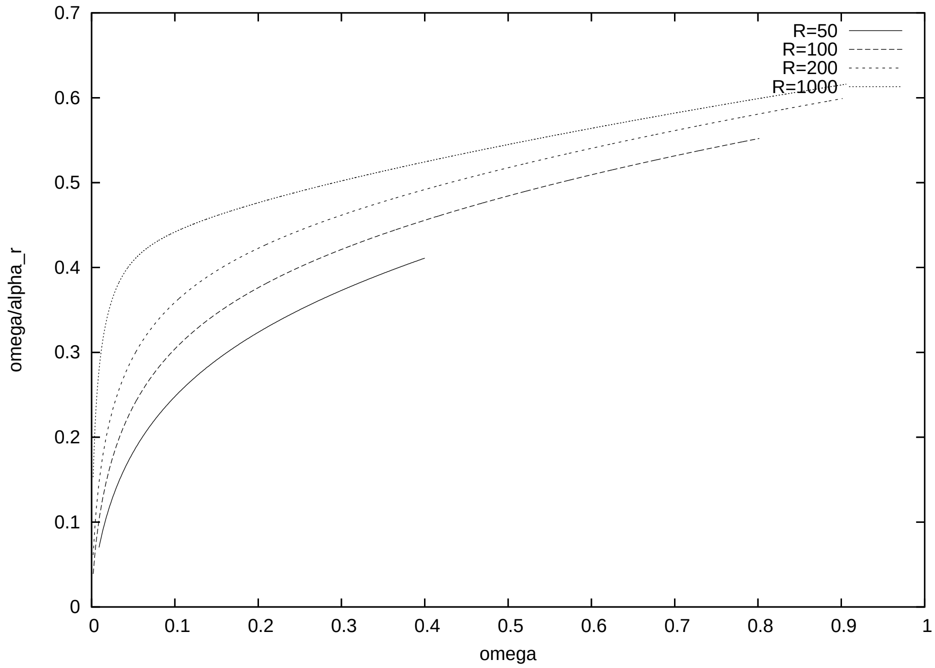

Figure 4.1: Phase Velocity as a function of ω and R for $n = 1$

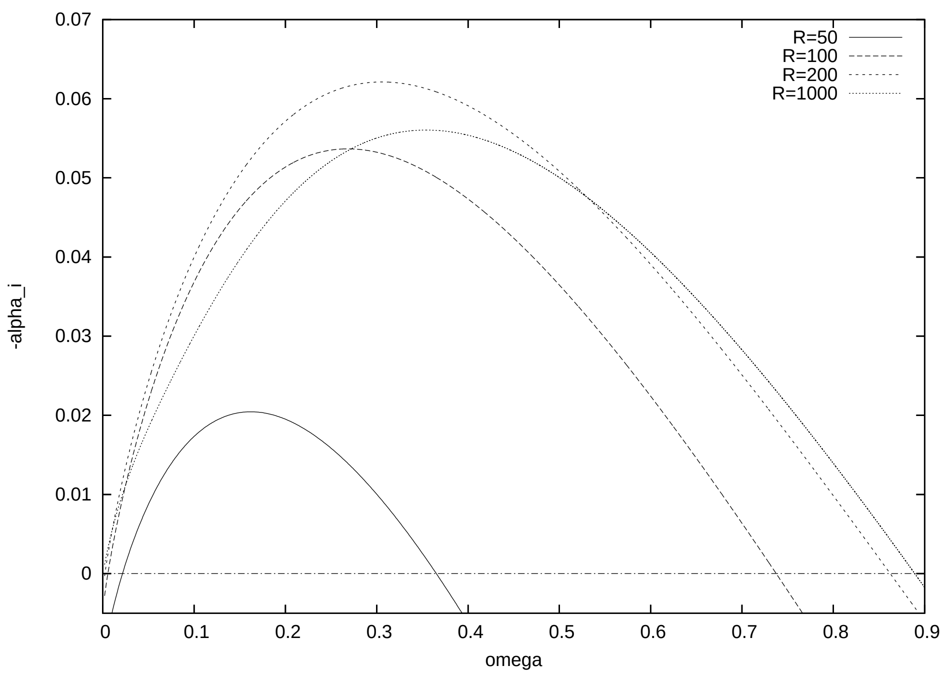

Figure 4.2: Amplification factor as a function of ω and R for $n = 1$

4.2 Critical Reynolds Number

To obtain the critical Reynolds number, we have plotted (figure (4.3)) the amplification factor for some values of R close to the critical Reynolds number predicted by other researchers. The critical Reynolds number is the point where the curve $-\alpha_i(\omega)$ becomes tangent to the $\alpha_i = 0$ line.

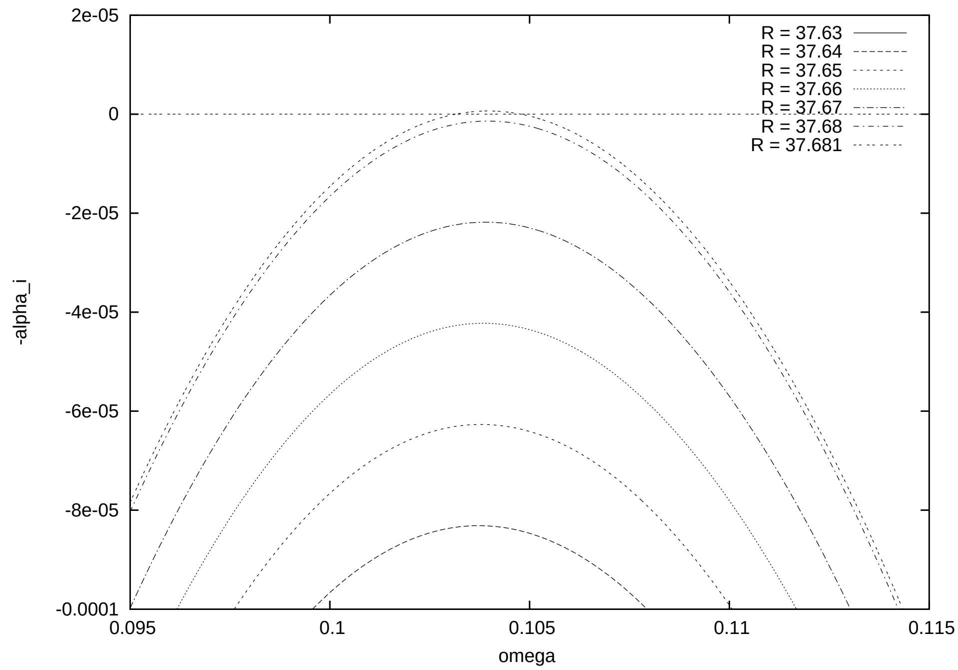

Figure 4.3: Amplification factor as a function of Reynolds number and ω

From this graph the critical Reynolds number was found to be 37.68. This is compared with some of the other values reported by researchers in table 1.

Reference	$R_{critical}$	$\alpha_{r-critical}$	$\omega_{critical}$
Morris[3]	37.64	0.44	0.1
Lessen and Singh[2]	37.9	0.3989	0.08
Adriana and Sandham[4]	37.8	0.417	0.09
Kulkarni and Agarwal	37.68	0.450481	0.104

Table 1: Comparison of critical Reynolds number for $n = 1$ mode

5 Summary

- The linear stability equations were derived in cylindrical polar coordinates.

- Asymptotic forms of the boundary conditions were derived for small and large r.

- The two point boundary eigenvalue problem was solved by a shooting method. This yields very accurate results. The average time for calculating a typical eigenvalue was 2.535 sec on a 1.9 GHz AMD 64 Athlon X2 machine.

- The numerical integration was performed by a fifth order variable step Runge-Kutta integration method.

- A computer program was developed in C++ to solve both the spatial and the temporal eigenvalue problem. Although the present report quotes the results of only the spatial problem, it works fine for the temporal problem as well.

- The critical Reynolds number was also computed and shown to be in good agreement with those reported in the literature.

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Appendix

A Vector Relations in Cylindrical Polar Coordinates

In order to obtain the Navier-Stokes equation in polar coordinates we need to define the following properties of vectors and scalars.

Gradient of a Scalar, $\nabla\phi$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\underline{e}_\theta + \frac{\partial\phi}{\partial z}\underline{e}_z \tag{A.1}$$

Gradient of a Vector, $\nabla\mathbf{V}$

$$\nabla\mathbf{V} = \begin{pmatrix} \frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_z}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r} & \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{V_r}{r} & \frac{1}{r}\frac{\partial V_z}{\partial\theta} \\ \frac{\partial V_r}{\partial z} & \frac{\partial V_\theta}{\partial z} & \frac{\partial V_z}{\partial z} \end{pmatrix} \tag{A.2}$$

Divergence of a Vector, $\nabla\cdot\mathbf{V}$

$$\nabla\cdot\mathbf{V} = \frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z} \tag{A.3}$$

where $\mathbf{V} = (V_r, V_\theta, V_z)$

Laplacian of a Scalar, $\nabla^2\phi$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} \tag{A.4}$$

Laplacian of a Vector, $\nabla^2 \mathbf{V}$

$$\nabla^2 \mathbf{V} \;=\; \begin{pmatrix} \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r}\frac{\partial V_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2} - \frac{V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial^2 V_r}{\partial z^2} \\ \frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial V_r}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2} - \frac{V_\theta}{r^2} + \frac{\partial^2 V_\theta}{\partial z^2} \\ \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r}\frac{\partial V_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2} + \frac{\partial^2 V_z}{\partial z^2} \end{pmatrix} \tag{A.5}$$

B Power-series expansions for small r

Recurrence Relations

Reference [1] gives the recurrence relations for the components of variables F, G, U, P. We use the first two terms in our power series because the value of r_s chosen is sufficiently small. The recurrence relations that we have used are:

$$F_1 = \frac{1}{n+1}\Big(\frac{B_1}{4n}G1 - \frac{R}{2}P1 - i\alpha U1\Big)$$

$$F_2 = \frac{1}{4(n+2)}F_1 B_1$$

$$G_2 = \frac{1}{2}(RP_1 - \frac{G_1 B_1}{2n})$$

$$U_2 = \frac{1}{4(n+1)}(i\alpha RP_1 - U_1 B_1 + RC_2 G_1)$$

$$P_2 = \frac{1}{R}\Big[-i\alpha U_2 + \frac{1}{4}\big\{F_1 B_1 + \frac{1}{(n+1)}(G_2 B_1 + G_1 B_2)\big\}\Big]$$

Definition of Vectors

Once the components of the variables are defined, the variables are formed:

$$F = r^{(n+1)}(F_1 + F_2 r^2)$$

$$G = r^{(n-1)}(G_1 + G_2 r^2)$$

$$U = r^n(U_1 + U_2 r^2)$$

$$P = r^n(P_1 + P_2 r^2)$$

From the above expressions for F and G we can derive the expressions for V and W using

$$V = (F+G)/2$$

$$W = (F+G)/2i$$

Thus we have the variables, U, V, W, P defined. Taking the partial derivative of U and W we get a set of 6 variables U, V, W, P, U' and W'. Note that the expressions for these variables are in terms of G_1, P_1 and U_1.

Components of the final vectors used in the computer programme

From these expressions we separate the coefficients of G_1, P_1 and U_1. Thus we get the three independent sets of vectors which satisfy the stability equations independently. The components of these three independent vectors are given by[2]:

Components u_i

$$u_1 = -\frac{r_3 R}{4},$$

$$u_2 = r\Big[1 - \frac{1}{8}r^2\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}\Big],$$

$$u_3 = \frac{1}{8}i\alpha r^3 R.$$

[2]All the expressions in this section have been derived with Mathematica[6].

Components v_i

$$v_1 = \frac{1}{2}\Big[1 - \frac{1}{8}r^2\big\{-\alpha^2 - (i\alpha - i\omega)R\big\} + \frac{1}{96}r^4\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}^2\Big],$$

$$v_2 = \frac{1}{2}\Big[-\frac{1}{2}i\alpha r^2 - \frac{1}{24}i\alpha r^4\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}\Big],$$

$$v_3 = \frac{1}{2}\Big[\frac{1}{4}r^2 R - \frac{1}{48}r^4 R\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}\Big].$$

Components w_i

$$w_1 = -\frac{1}{2}i\Big[-1 + \frac{3}{8}r^2\big\{-\alpha^2 - (i\alpha - i\omega)R\big\} + \frac{1}{96}r^4\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}^2\Big],$$

$$w_2 = -\frac{1}{2}i\Big[-\frac{1}{2}i\alpha r^2 - \frac{1}{24}i\alpha r^4\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}\Big],$$

$$w_3 = -\frac{1}{2}i\Big[-\frac{3}{4}r^2 R - \frac{1}{48}r^4 R\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}\Big].$$

Components p_i

$$p_1 = \frac{1}{2}i\alpha r^3,$$

$$p_2 = 0,$$

$$p_3 = r\big\{1 + \frac{1}{8}\alpha^2 r^2\big\}.$$

Components u_i'

$$u_1' = -\frac{3}{4}r^2 R,$$

$$u_2' = 1 - \frac{3}{8}r^2\big\{-\alpha^2 - (i\alpha - i\omega)R\big\},$$

$$u_3' = \frac{3}{8}i\alpha r^2 R.$$

Components w_i'

$$w_1' = -\frac{1}{2}i\Big[\frac{3}{4}r\big\{-\alpha^2 - (i\alpha - i\omega)R\big\} + \frac{1}{24}r^3\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}^2\Big],$$

$$w_2' = -\frac{1}{2}i\Big[-i\alpha r - \frac{1}{6}i\alpha r^3\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}\Big],$$

$$w_3' = -\frac{1}{2}i\Big[-\frac{3}{2}rR - \frac{1}{12}r^3R\big\{-\alpha^2 - (i\alpha - i\omega)R\big\}\Big].$$