



An inertial extragradient method for solving strongly pseudomonotone equilibrium problems in Hilbert spaces

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Abstract

In this work, we propose an inertial extragradient method for solving strongly pseudomonotone equilibrium problems utilizing a novel self-adaptive stepsize approach. We establish the R -linear convergence rate of the proposed method without prior knowledge of the Lipschitz-type constants associated with the bifunction. We also discuss the application of the obtained results to variational inequality problems involving strongly pseudomonotone and Lipschitz continuous mapping. Numerical examples are presented to illustrate the efficiency of the proposed method.

Keywords Equilibrium problem · Inertial extragradient method · Strongly pseudomonotone bifunction · R -linear rate.

Mathematics Subject Classification 47J20 · 49J40 · 49M30

1 Introduction

Let C be a nonempty closed and convex subset of a real Hilbert space H . Let $f : H \times H \rightarrow \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. The equilibrium problem of the bifunction

Dedicate to Professor Le Dung Muu (Institute of Mathematics, VAST, Vietnam) on the occasion of his 75th birthday.

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f on C , denoted by $EP(f, C)$, is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \forall y \in C. \quad (1)$$

Equilibrium problem is also called the Ky Fan inequality due to his contribution to this field (Fan 1972). Mathematically, $EP(f, C)$ is a generalization of various important mathematical models including variational inequality problems, optimization problems and fixed point problems, see (Blum and Oettli 1994; Konnov 2000, 2007; Muu and Oettli 1992; Quoc and Muu 2012; Vuong and Strodiot 2020). EPs have been considered by many authors in recent years, e.g., see (Combettes and Hirstoaga 2005; Hieu et al. 2018; Iusem et al. 2009; Lyashko et al. 2011; Lyashko and Semenov 2016; Moudafi 2003; Nguyen et al. 2014; Strodiot et al. 2016; Tran et al. 2008; Vinh and Muu 2019; Vuong et al. 2015, 2012) and the references therein. Some notable methods for EPs have been proposed such proximal point methods (PPM) (Flam and Antipin 1997; Moudafi 1999), auxiliary problem principle methods (Mastroeni 2000) and gap function method (Mastroeni 2003). For an excellent collection of equilibrium modelling and applications, solutions existence as well as solution methods for EPs, we refer the readers to a recent monograph (Bigi et al. 2019).

The PPM is often used for solving monotone EPs, in which a regularized equilibrium problem is formed at each iteration. This sub-problem is strongly monotone, hence the solution is unique and can be computed more easily than solutions of the original problem. Solution approximations generated by the PPM will converge finitely or asymptotically to some solution of the original EPs (Flam and Antipin 1997; Moudafi 1999).

For solving pseudomonotone EPs, a powerful algorithm is the extragradient methods (EGM) studied originally by Korpelevich (1976) for variational inequalities and saddle point problems. This algorithm was extended to EPs in Tran et al. (2008) where the equilibrium bifunctions are pseudomonotone and satisfy a Lipschitz-type assumption. Under suitable conditions imposed on parameters and bifunctions, the iterative sequences generated by the EGM are proved to be convergent to some solution of $EP(f, C)$. In recent years, the extragradient methods have received great attention from the many authors, see, e.g., Censor et al. (2011); Nguyen et al. (2014); Vuong et al. (2015). The advantage of the extragradient method (Vinh and Muu 2019) is that the sub-problems are easier to solve than the PPM. Moreover, it can be applied to the more general class of bifunctions.

The main drawback of EGM is that the chosen step-size depends on the Lipschitz-type constants of the bifunctions (Dang 2017; Lyashko et al. 2011; Lyashko and Semenov 2016; Moudafi 1999). This fact can make some restrictions in applications because the Lipschitz-type constants are often unknown or difficult to estimate. In this work, we propose a new inertial extragradient method for solving strongly pseudo-monotone equilibrium problem. Note that, an extragradient method with inertial effect for solving EPs can be found in Vinh and Muu (2019) where the inertial parameters were chosen depending on the iteration gap and a summable sequence. In addition, no convergence rate was obtained in Vinh and Muu (2019).

The EGM variant proposed in this paper uses a new step-size rule which does not require the knowledge of the Lipschitz-type constants of the bifunction as in Vinh and Muu (2019). Moreover, the inertial parameter can be chosen independently to the iterations. Under suitable conditions on the parameters, we establish the linear convergence of the iterations to the unique solution of the EPs. As a consequence, we provide a linear convergence rate of a modified extragradient method for solving variational inequality problems in Hilbert spaces.

This paper is organized as follows: in Sect. 2, we collect some definitions and preliminary results for further use. Section 3 deals with analyzing the convergence of the proposed

algorithm. Finally, we discuss the applications to variational inequalities in Sect. 4, following with some numerical examples in Sect. 5.

2 Preliminaries

Let C be a nonempty closed convex subset of H . We begin with some concepts of monotonicity of a bifunction (Blum and Oettli 1994; Muu and Oettli 1992). A bifunction $f : H \times H \rightarrow \mathbb{R}$ is said to be:

(i) strongly monotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma \|x - y\|^2, \quad \forall x, y \in C.$$

(ii) strongly pseudomonotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) \geq 0 \implies f(y, x) \leq -\gamma \|x - y\|^2, \quad \forall x, y \in C.$$

(iii) satisfied Lipschitz-type condition on C if there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \|x - y\|^2 - c_2 \|y - z\|^2, \quad \forall x, y, z \in C.$$

From the definitions above, it is obvious that (i) \implies (ii).

The normal cone N_C to C at a point $x \in C$ is defined by

$$N_C(x) = \{w \in H : \langle w, x - y \rangle \geq 0, \forall y \in C\}.$$

For every $x \in H$, the metric projection $P_C(x)$ of x onto C is defined by

$$P_C(x) = \arg \min \{\|y - x\| : y \in C\}.$$

Since C is nonempty, closed and convex, $P_C(x)$ exists and is unique.

For each $x, z \in H$, by $\partial f(z, x)$, we denote the subdifferential of convex function $f(z, \cdot)$ at x , i.e.,

$$\partial f(z, x) := \{u \in H : f(z, y) \geq f(z, x) + \langle u, y - x \rangle, \forall y \in H\}.$$

In particular,

$$\partial f(z, z) = \{u \in H : f(z, y) \geq \langle u, y - z \rangle, \forall y \in H\}.$$

For proving the convergence of the algorithm, we need the following lemma.

Lemma 2.1 (Peypouquet 2015, Proposition 3.61) *Let C be a nonempty closed convex subset of H and $g : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function on H . Assume either that g is continuous at some point of C , or that there is an interior point of C where g is finite. Then, x^* is a solution to the following convex problem $\min \{g(x) : x \in C\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^* .*

Definition 2.1 (Ortega and Rheinboldt 1970) Let $\{x_n\}$ be a sequence in H .

(i) $\{x_n\}$ is said to converge R -linearly to x^* with rate $\rho \in [0, 1)$ if there is a constant $c > 0$ such that

$$\|x_n - x^*\| \leq c\rho^n \quad \forall n \in \mathbb{N}.$$

(ii) $\{x_n\}$ is said to converge Q -linearly to x^* with rate $\rho \in [0, 1)$ if

$$\|x_{n+1} - x^*\| \leq \rho \|x_n - x^*\| \quad \forall n \in \mathbb{N}.$$

3 Convergence analysis

Now, we are in a position to present a modified version of inertial extragradient method in Vinh and Muu (2019) for solving equilibrium problems.

Algorithm 3.1

Initialization. Let $u_0, u_1 \in H, \lambda_1 > 0, \rho \in [0, 1), \mu \in (0, 1)$. Let $\{\tau_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$.

Step 1. Given the current iterates u_{n-1} and u_n ($n \geq 1$), compute

$$\begin{cases} t_n = u_n + \rho(u_n - u_{n-1}), \\ v_n = \operatorname{argmin}_{y \in C} \{ \lambda_n f(t_n, y) + \frac{1}{2} \|y - t_n\|^2 \}. \end{cases}$$

If $v_n = t_n$ then stop and v_n is a solution. Otherwise, go to Step 2.

Step 2. Compute

$$u_{n+1} = \operatorname{argmin}_{y \in C} \{ \lambda_n f(v_n, y) + \frac{1}{2} \|y - t_n\|^2 \},$$

and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu}{2} \frac{\|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2}{f(t_n, u_{n+1}) - f(t_n, v_n) - f(v_n, u_{n+1})}, \lambda_n + \tau_n \right\} \\ \lambda_n + \tau_n \text{ otherwise.} \end{cases} \tag{2}$$

Set $n := n + 1$ and return to Step 1.

Remark 3.1 The self-adaptive stepsizes $\{\lambda_n\}$ chosen as in (2) is allowed to increase from iteration to iteration showing that the self-adaptive stepsize in this work is different from the studied self-adaptive stepsize in the literature (Hieu et al. 2018; Dang 2017; Muu and Quoc 2009; Tran et al. 2008; Vinh and Muu 2019). The initial stepsize λ_1 chosen in (2) could be arbitrary large allowing fast convergence from the beginning of the iterations. In particular, (Vinh and Muu 2019, Algorithm 1) proposed an inertial extragradient method for solving pseudo-monotone EPs with fixed stepsize and inertial parameters were chosen along the course of iterations depending on a summable series, which were different from Algorithm (3.1). In addition, the linear convergence rate was not investigated in (Vinh and Muu 2019).

In order to establish the convergence of Algorithm 3.1, we assume that bifunction $f : H \times H \rightarrow \mathbb{R}$ satisfies the following conditions.

Condition 1

- (A1) f is γ -strongly pseudomonotone on C .
- (A2) f satisfies Lipschitz-type condition on H with two constants c_1 and c_2 .
- (A3) $f(x, \cdot)$ is convex and lower semicontinuous on H for every fixed $x \in H$.
- (A4) Either $\operatorname{int} C \neq \emptyset$ or $f(x, \cdot)$ is continuous at some point in C for every $x \in H$.

Remark 3.2 From the condtions (A1) and (A2) we get $f(x, x) = 0$ for all $x \in C$. It is also known that under Condition 1, the problem $\operatorname{EP}(f, C)$ has unique solution (Muu and Quy 2015). It is easy to see that the uniqueness of solution is guaranteed by the strong pseudomonotonicity

of the bifunction f . For generic conditions of solution existence and their relaxations, we refer the readers to an excellent survey (Bigi et al. 2013, Section 2). In particular, the most simplest conditions for solution existence are: C is a bounded and convex set; $f(x, \cdot)$ is convex for each $x \in C$ and $f(\cdot, y)$ is continuous for each $y \in C$.

Next, we will establish the convergence rate of Algorithm 3.1. We start with the following lemmas which play an important role in proving the convergence of the proposed algorithm.

Lemma 3.1 (Yang 2020) *Let Condition 1 be satisfied. Let $\{\lambda_n\}$ be a sequence generated by Algorithm 3.1. Then*

$$\lim_{n \rightarrow \infty} \lambda_n = \lambda \in \left[\min \left\{ \frac{\mu}{2 \max\{c_1, c_2\}}, \lambda_1 \right\}, \lambda_1 + \tau \right],$$

where $\tau = \sum_{n=1}^{\infty} \tau_n$.

Lemma 3.2 *For any $\lambda > 0$ and $x, t \in H$, let*

$$z = \operatorname{argmin}_{y \in C} \left\{ \lambda f(x, y) + \frac{1}{2} \|y - t\|^2 \right\}, \tag{3}$$

then

$$\lambda (f(x, y) - f(x, z)) \geq \langle t - z, y - z \rangle \quad \forall y \in C.$$

Proof Since z is the unique solution of the strongly convex minimization problem (3). The optimality condition (Lemma 2.1) implies that there exists $s \in \partial f(x, z)$ such that

$$0 \in \lambda s + z - t + N_C(z),$$

Hence, by definition of this cone, we obtain that

$$\langle t - z - \lambda s, y - z \rangle \leq 0 \quad \forall y \in C. \tag{4}$$

On the other hand, since $s \in \partial f(x, z)$, we have

$$f(x, y) - f(x, z) \geq \langle s, y - z \rangle \quad \forall y \in C. \tag{5}$$

Combining (4) and (5), we obtain

$$\lambda (f(x, y) - f(x, z)) \geq \langle \lambda s, y - z \rangle \geq \langle t - z, y - z \rangle \quad \forall y \in C.$$

Lemma 3.3 *Let C be a nonempty, closed and convex subset of H and $f : H \times H \rightarrow \mathbb{R}$ be a bifunction satisfying Condition 1. Let u be the unique solution of $EP(f, C)$. Then the following inequality holds*

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|t_n - v_n\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|u_{n+1} - v_n\|^2 \\ &\quad - 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned} \tag{6}$$

Proof From

$$u_{n+1} = \operatorname{argmin}_{y \in C} \left\{ \lambda_n f(v_n, y) + \frac{1}{2} \|y - t_n\|^2 \right\},$$

by Lemma 3.2, we get

$$\lambda_n(f(v_n, y) - f(v_n, u_{n+1})) \geq \langle t_n - u_{n+1}, y - u_{n+1} \rangle \quad \forall y \in C.$$

Substituting $y := u \in C$, we obtain

$$\lambda_n(f(v_n, u) - f(v_n, u_{n+1})) \geq \langle t_n - u_{n+1}, u - u_{n+1} \rangle. \tag{7}$$

Since u is the unique solution of $EP(f, C)$ and $v_n \in C$, we have $f(u, v_n) \geq 0$. By the strong pseudomonotonicity assumption of f , we obtain $f(v_n, u) \leq -\gamma\|v_n - u\|^2$. It implies from (7) that

$$\begin{aligned} -\lambda_n f(v_n, u_{n+1}) &\geq \langle t_n - u_{n+1}, u - u_{n+1} \rangle - \lambda_n f(v_n, u) \\ &\geq \langle t_n - u_{n+1}, u - u_{n+1} \rangle + \lambda_n \gamma \|v_n - u\|^2. \end{aligned} \tag{8}$$

Again, since

$$v_n = \operatorname{argmin}_{y \in C} \left\{ \lambda_n f(t_n, y) + \frac{1}{2} \|y - t_n\|^2 \right\},$$

Lemma 3.2 implies

$$\lambda_n(f(t_n, u_{n+1}) - f(t_n, v_n)) \geq \langle t_n - v_n, u_{n+1} - v_n \rangle. \tag{9}$$

Adding (8) and (9) we get

$$\begin{aligned} &2\lambda_n(f(t_n, u_{n+1}) - f(t_n, v_n) - f(v_n, u_{n+1})) \\ &\geq 2\langle t_n - v_n, u_{n+1} - v_n \rangle + 2\langle t_n - u_{n+1}, u - u_{n+1} \rangle + 2\lambda_n \gamma \|v_n - u\|^2 \\ &= (\|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|u_{n+1} - t_n\|^2) + \\ &\quad + (\|t_n - u_{n+1}\|^2 + \|u_{n+1} - u\|^2 - \|t_n - u\|^2) + 2\lambda_n \gamma \|v_n - u\|^2 \\ &= \|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 + \|u_{n+1} - u\|^2 - \|t_n - u\|^2 + 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - \|t_n - v_n\|^2 - \|u_{n+1} - v_n\|^2 + 2\lambda_n(f(t_n, u_{n+1}) \\ &\quad - f(t_n, v_n) - f(v_n, u_{n+1})) - 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned} \tag{10}$$

On the other hand, from the definition of the sequence λ_n we get

$$2(f(t_n, u_{n+1}) - f(t_n, v_n) - f(v_n, u_{n+1})) \leq \frac{\mu}{\lambda_{n+1}} \left(\|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 \right). \tag{11}$$

Substituting (10) into (11) we obtain

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \|t_n - v_n\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \|u_{n+1} - v_n\|^2 \\ &\quad - 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned}$$

In the following theorem we will show that the sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution u with a R -linear rate.

Theorem 3.1 *Let C be a nonempty closed convex subset of H . Let $f : H \times H \rightarrow \mathbb{R}$ be a bifunction satisfying Condition 1. Let $\theta \in (0, 1)$ be arbitrary and ρ be a real number such that*

$$0 \leq \rho \leq \frac{\omega\epsilon}{\omega\epsilon + 2\omega + \epsilon}, \tag{12}$$

where $\omega := 1 - \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\}$ and $\epsilon := \frac{1}{2}(1 - \mu)(1 - \theta)\theta$. Then the sequence $\{u_n\}$ generated by Algorithm 3.1 converges in norm to the unique solution u of the problem $EP(f, C)$ with a R -linear rate.

Proof First, we show that there exists $N_1 \in \mathbb{N}$ such that

$$\|u_{n+1} - u\|^2 \leq \omega \|t_n - u\|^2 - \epsilon \|u_{n+1} - t_n\|^2 \quad \forall n \geq N_1. \tag{13}$$

Indeed, it follows from Lemma 3.1 that $\lim_{n \rightarrow \infty} \lambda_n = \lambda > 0$ and since $\mu < 1$, there exists $N > 0$ such that

$$\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) > 0 \quad \forall n \geq N.$$

Thanks to (6) and $\theta \in (0, 1)$, we have for all $n \geq N$ that

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|v_n - t_n\|^2 \\ &\quad - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) \|u_{n+1} - v_n\|^2 - 2\lambda_n\gamma \|v_n - u\|^2 \\ &= \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta \|v_n - t_n\|^2 \\ &\quad - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) \left[\|v_n - t_n\|^2 + \|u_{n+1} - v_n\|^2 \right] - 2\lambda_n\gamma \|v_n - u\|^2 \\ &\leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta \|v_n - t_n\|^2 \\ &\quad - \frac{1}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) \|u_{n+1} - t_n\|^2 - 2\lambda_n\gamma \|v_n - u\|^2, \end{aligned} \tag{14}$$

where we have used the Cauchy–Schwartz inequality in the last estimation. Moreover, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) &= \frac{1}{2}(1 - \mu)(1 - \theta) > \frac{1}{2}(1 - \mu)(1 - \theta)\theta, \\ \lim_{n \rightarrow \infty} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta &= (1 - \mu)\theta > 2 \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\}, \\ \lim_{n \rightarrow \infty} \lambda_n\gamma &= \lambda\gamma > \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\}. \end{aligned}$$

Using the definition of the limit, there exists $N_1 \in \mathbb{N}$ and $N_1 \geq N$, such that for all $n \geq N_1$

$$\frac{1}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) \geq \frac{1}{2}(1 - \mu)(1 - \theta)\theta,$$

$$\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta \geq 2 \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\},$$

and

$$\lambda_n \gamma \geq \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\}.$$

Using (14) we obtain for all $n \geq N_1$ that

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - 2 \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\} \|v_n - t_n\|^2 \\ &\quad - \frac{1}{2} (1 - \mu)(1 - \theta)\theta \|u_{n+1} - t_n\|^2 \\ &\quad - 2 \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\} \|v_n - u\|^2 \\ &= \|t_n - u\|^2 - \frac{1}{2} (1 - \mu)(1 - \theta)\theta \|u_{n+1} - t_n\|^2 \\ &\quad - 2 \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\} (\|v_n - t_n\|^2 + \|v_n - u\|^2) \\ &\leq \|t_n - u\|^2 - \frac{1}{2} (1 - \mu)(1 - \theta)\theta \|u_{n+1} - t_n\|^2 \\ &\quad - \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\} \|t_n - u\|^2 \\ &= \left(1 - \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\}\right) \|t_n - u\|^2 - \frac{1}{2} (1 - \mu)(1 - \theta)\theta \|u_{n+1} - t_n\|^2 \\ &\leq \left(1 - \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\}\right) \|t_n - u\|^2 - \frac{1}{2} (1 - \mu)(1 - \theta)\theta \|u_{n+1} - t_n\|^2 \\ &= \omega \|t_n - u\|^2 - \epsilon \|u_{n+1} - t_n\|^2. \end{aligned}$$

Next, we show that the sequence $\{u_n\}$ converges strongly to the unique solution u of the problem $EP(f, C)$. Indeed, we have

$$\begin{aligned} \|t_n - u\|^2 &= \|(1 + \rho)(u_n - u) - \rho(u_{n-1} - u)\|^2 \\ &= (1 + \rho)\|u_n - u\|^2 - \rho\|u_{n-1} - u\|^2 + \rho(1 + \rho)\|u_n - u_{n-1}\|^2 \end{aligned}$$

and

$$\begin{aligned} \|u_{n+1} - t_n\|^2 &= \|u_{n+1} - u_n - \rho(u_n - u_{n-1})\|^2 \\ &= \|u_{n+1} - u_n\|^2 + \rho^2\|u_n - u_{n-1}\|^2 - 2\rho \langle u_{n+1} - u_n, u_n - u_{n-1} \rangle \\ &\geq \|u_{n+1} - u_n\|^2 + \rho^2\|u_n - u_{n-1}\|^2 - 2\rho\|u_{n+1} - u_n\|\|u_n - u_{n-1}\| \\ &\geq \|u_{n+1} - u_n\|^2 + \rho^2\|u_n - u_{n-1}\|^2 - \rho\|u_{n+1} - u_n\|^2 - \rho\|u_n - u_{n-1}\|^2 \\ &= (1 - \rho)\|u_{n+1} - u_n\|^2 - \rho(1 - \rho)\|u_n - u_{n-1}\|^2. \end{aligned}$$

Combining these inequalities with (13) we obtain

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \omega(1 + \rho)\|u_n - u\|^2 - \omega\rho\|u_{n-1} - u\|^2 + \omega\rho(1 + \rho)\|u_n - u_{n-1}\|^2 \\ &\quad - \epsilon(1 - \rho)\|u_{n+1} - u_n\|^2 + \epsilon\rho(1 - \rho)\|u_n - u_{n-1}\|^2 \quad \forall n \geq N_1, \end{aligned}$$

or equivalently

$$\begin{aligned} & \|u_{n+1} - u\|^2 - \omega\rho\|u_n - u\|^2 + \epsilon(1 - \rho)\|u_{n+1} - u_n\|^2 \\ & \leq \omega \left[\|u_n - u\|^2 - \rho\|u_{n-1} - u\|^2 + \epsilon(1 - \rho)\|u_n - u_{n-1}\|^2 \right] \\ & \quad - (\omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho)) \|u_n - u_{n-1}\|^2 \quad \forall n \geq N_1. \end{aligned}$$

Setting

$$\Sigma_n := \|u_n - u\|^2 - \rho\|u_{n-1} - u\|^2 + \epsilon(1 - \rho)\|u_n - u_{n-1}\|^2,$$

since $\omega \in (0, 1)$, we can write

$$\begin{aligned} \Sigma_{n+1} & \leq \|u_{n+1} - u\|^2 - \omega\rho\|u_n - u\|^2 + \epsilon(1 - \rho)\|u_{n+1} - u_n\|^2 \\ & \leq \omega\Sigma_n - (\omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho)) \|u_n - u_{n-1}\|^2 \quad \forall n \geq N_1. \end{aligned}$$

We show that

$$\omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho) \geq 0.$$

Indeed, from (12) we get $\rho \in [0, 1)$, thus we obtain $1 + \rho \leq 2$ and $\rho(1 - \rho) \leq \rho$, hence

$$\begin{aligned} \omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho) & \geq \omega\epsilon(1 - \rho) - 2\omega\rho - \epsilon\rho \\ & = \omega\epsilon - \rho(\omega\epsilon + 2\omega + \epsilon) \geq 0. \end{aligned}$$

Therefore

$$\Sigma_{n+1} \leq \omega\Sigma_n \quad \forall n \geq N_1.$$

Next, we show that $\Sigma_n \geq 0$ for all n . From (12) we deduce

$$\rho \leq \frac{\omega\epsilon}{\omega\epsilon + 2\omega + \epsilon} \leq \frac{\omega\epsilon}{\omega\epsilon + 2\omega} = \frac{\epsilon}{2 + \epsilon},$$

which implies $\rho \leq \frac{\epsilon(1-\rho)}{2}$. Using this fact, we obtain

$$\begin{aligned} \Sigma_n & = (1 - \epsilon(1 - \rho))\|u_n - u\|^2 + \epsilon(1 - \rho) (\|u_n - u\|^2 + \|u_n - u_{n-1}\|^2) - \rho\|u_{n-1} - u\|^2 \\ & \geq (1 - \epsilon(1 - \rho))\|u_n - u\|^2 + \frac{\epsilon(1 - \rho)}{2}\|u_{n-1} - u\|^2 - \rho\|u_{n-1} - u\|^2 \\ & \geq (1 - \epsilon(1 - \rho))\|u_n - u\|^2 \geq 0. \end{aligned}$$

Hence

$$\Sigma_{n+1} \leq \omega\Sigma_n \leq \dots \leq \omega^{n-N_1+1}\Sigma_{N_1}.$$

$$\|u_n - u\|^2 \leq \frac{\Sigma_{N_1}}{\omega^{N_1-1}}\omega^n,$$

which means that $\{u_n\}$ converges R -linearly to u .

Remark 3.3 Using the similar technique in Vinh and Muu (2019); Yang (2020), one can obtain the weak convergence of Algorithm 3.1 under conditions: f is pseudomonotone on C ; $f(\cdot, y)$ is weakly upper semicontinuous on C , (A2), (A3), (A4) and the solution set $EP(f, C) \neq \emptyset$. We omit the proof here.

4 Application to variational inequalities

In this Section, we discuss the applications of the main result obtained in Sect. 3 for solving variational inequality problems in Hilbert spaces. Let $f(x, y) = \langle F(x), y - x \rangle, \forall x, y \in C$, where $F : H \rightarrow H$ is a continuous mapping. Then the equilibrium problems (1) becomes the variational inequality problem, i.e., find $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0 \quad \forall y \in C. \tag{15}$$

The solution set of (15) is denoted by $Sol(F, C)$. Moreover, we have

$$v_n = \operatorname{argmin}_{y \in C} \{ \lambda_n f(t_n, y) + \frac{1}{2} \|y - t_n\|^2 \} = P_C(t_n - \lambda_n F(t_n)).$$

We recall that the mapping F is δ -strongly pseudomonotone on C if there exists a constant $\delta > 0$ such that

$$\langle F(x), y - x \rangle \geq 0 \implies \langle F(y), y - x \rangle \geq \delta \|x - y\|^2 \quad \forall x, y \in C.$$

If F is L -Lipschitz continuous and strongly pseudomonotone, then the conditions (A1)-(A4) hold for f with $c_1 = c_2 = \frac{L}{2}$. Note that, under these assumption, $Sol(F, C)$ is nonempty and singleton (Kim et al. 2016). For solving variational inequality, we propose the following algorithm.

Algorithm 4.1

Initialization. Let $u_0, u_1 \in H, \lambda_1 > 0, \rho \in [0, 1), \mu \in (0, 1)$. Let $\{\tau_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$.

Step 1. Given the current iterates u_{n-1} and $u_n (n \geq 1)$, compute

$$\begin{cases} t_n = u_n + \rho(u_n - u_{n-1}), \\ v_n = P_C(t_n - \lambda_n F(t_n)). \end{cases}$$

If $v_n = t_n$ or $F(t_n) = 0$ then the stop and t_n is a solution of VI (15). Otherwise, go to Step 2.

Step 2. Compute

$$u_{n+1} = P_C(t_n - \lambda_n F(v_n)),$$

and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu}{2} \frac{\|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2}{\langle F(v_n) - F(t_n), u_{n+1} - v_n \rangle}, \lambda_n + \tau_n \right\} \\ \lambda_n + \tau_n \text{ otherwise.} \end{cases}$$

Set $n := n + 1$ and return to Step 1.

The following result is a direct consequence of Theorem 3.1.

Theorem 4.1 Assume that $F : H \rightarrow H$ is L -Lipschitz continuous on H and δ -strongly pseudomonotone on C . Let $\theta \in (0, 1)$ be arbitrary and ρ be a real number such that

$$0 \leq \rho \leq \frac{\omega \epsilon}{\omega \epsilon + 2\omega + \epsilon},$$

where $\omega := 1 - \min \left\{ \frac{(1 - \mu)\theta}{4}, \frac{\gamma\lambda}{2} \right\}$ and $\epsilon := \frac{1}{2}(1 - \mu)(1 - \theta)\theta$. Then the sequence $\{u_n\}$ generated by Algorithm 4.1 converges in norm to the unique solution $u \in \text{Sol}(F, C)$ with a R -linear rate.

Remark 4.1 As a consequence of Remark 3.3 we can also obtain the weak convergence of Algorithm 4.1 under conditions: F is pseudomonotone on H ; F is L -Lipschitz continuous on H ; F is sequentially weakly continuous on C and the solution set $\text{Sol}(F, C) \neq \emptyset$.

5 Numerical Illustrations

In this section, we consider some numerical results to illustrate the linear convergence of Algorithm 3.1 and compare its performance with the relaxed projection algorithm (Vuong and Strodiot 2020) and the inertial extragradient algorithm proposed by Vinh and Muu (2019). The codes are implemented in MATLAB.

Example 1: The bifunction f of the equilibrium problem comes from the Cournot-Nash equilibrium model considered by Quoc and Muu (2012). It is defined for each $x, y \in \mathbb{R}^5$, by

$$f(x, y) = \langle Px + Qy + r, y - x \rangle$$

where $r \in \mathbb{R}^5$, and P and Q are two square matrices of order 5 such that $P - Q$ is symmetric positive definite. It was proved by Quoc and Muu (2012) that the function f is strongly pseudo-monotone with modulus $\gamma = \lambda_{\min}(P - Q)$, the smallest eigenvalue of $P - Q$ and f satisfies the Lipschitz-type condition with modulus $c_1 = c_2 = \frac{1}{2}\|P - Q\|$. As in Quoc and Muu (2012), in our test, the vector r and the matrices P and Q are chosen as follows:

$$r = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{bmatrix}; \quad P = \begin{bmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}; \quad Q = \begin{bmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The constraint set of this problem is defined by

$$C = \left\{ x \in \mathbb{R}^5 \mid \sum_{i=1}^5 x_i \geq 0, \quad -5 \leq x_i \leq 5, \quad i = 1, 2, 3, 4, 5 \right\},$$

and its solution x^* is given by

$$x^* = (-0.725388, 0.803109, 0.72000, -0.866667, 0.200000)^T.$$

In this example, $\gamma = \lambda_{\min}(P - Q) = 0.7192$ and $L = \|P - Q\|$, $c_1 = c_2 = \frac{1}{2}\|P - Q\| = 1.45$. The stopping condition is $\text{ErrorBound} = \|u_{n+1} - u_n\| \leq 10^{-5}$.

For the projection type algorithms in Vuong and Strodiot (2020), we choose $\lambda = 1.9*\gamma/L^2$ with the non-relaxed parameter ($\alpha = 1$, red curve in Fig. 1) and $\lambda = \gamma/L^2$ with the over-relaxed parameter ($\alpha_{\max} = 1.014$, black curve in Fig. 1). Moreover, the blue curve in Fig. 1 presents the inertial extragradient algorithm in Vinh and Muu (2019) with the stepsize $\lambda = \frac{1}{2c_2}$, $\delta = 0.6$ and $\epsilon_n = \frac{1}{n^2}$, $n = 1, 2, \dots$. Figure 1 presents the comparisons of the performance of these methods with $u_0 = u_1 = (2, 1, 4, -1, -2)^T$ and the stopping condition 10^{-5} . It is clear that Algorithm 3.1 (green curve in Fig. 1) outperforms the others thanks to the very large initial stepsize ($\lambda_1 = 5000$) and inertial effect even though it is quite small ($\rho = 0.003$).

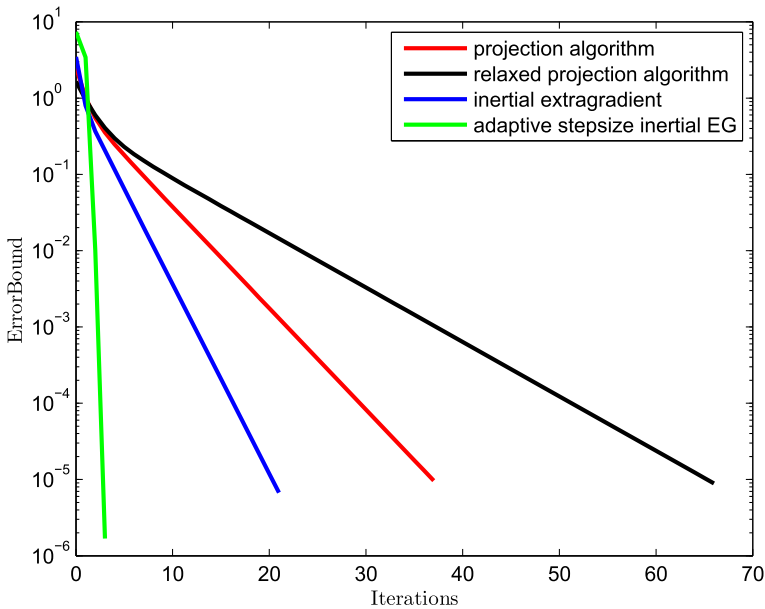


Fig. 1 Performance of different algorithms when $u_0 = u_1 = (2, 1, 4, -1, -2)^T$

The sequence $\{\tau_n\}$ is chosen the same as $\{\epsilon_n\}$. It is also noticed that the error estimates in Algorithm 3.1 has some oscillation phenomenon, which is well understood in algorithms with inertial effects for solving optimization problems. We believe that this is also the case for algorithms with inertial effects for solving EPs.

Example 2: Similar to Krawczyk and Uryasev (2000); Quoc and Muu (2012), in this example, we consider the problem of *River basin pollution game*. There are three players $j = 1, 2, 3$ located along a river. Each agent is engaged in an economic activity (paper pulp producing) at a chosen level x_j , but the players must meet the environmental condition set by a local authority. Pollutants may be expelled into the river, where they disperse. Two monitoring stations are located along the river, at which the local authority has set maximum pollutant concentration levels. The revenue and the expenditure for player j are

$$R_j(x) = [a_1 - a_2(x_1 + x_2 + x_3)]x_j,$$

and

$$F_j(x) = (b_{1j} + b_{2j}x_j)x_j,$$

respectively, where the parameters $a_1 = 3.0, a_2 = 0.01, b_{1j} = 0.1, 0.12, 0.15,$ and $c_{2j} = 0.01, 0.05, 0.01$ for $j = 1, 2, 3,$ respectively. The profit of player j is

$$K_j(x) = R_j(x) - F_j(x).$$

The constraint on emission imposed by the local authority at location is

$$q_i(x) = \sum_{j=1}^3 u_j v_{ij} x_j \leq 100, \quad i = 1, 2,$$

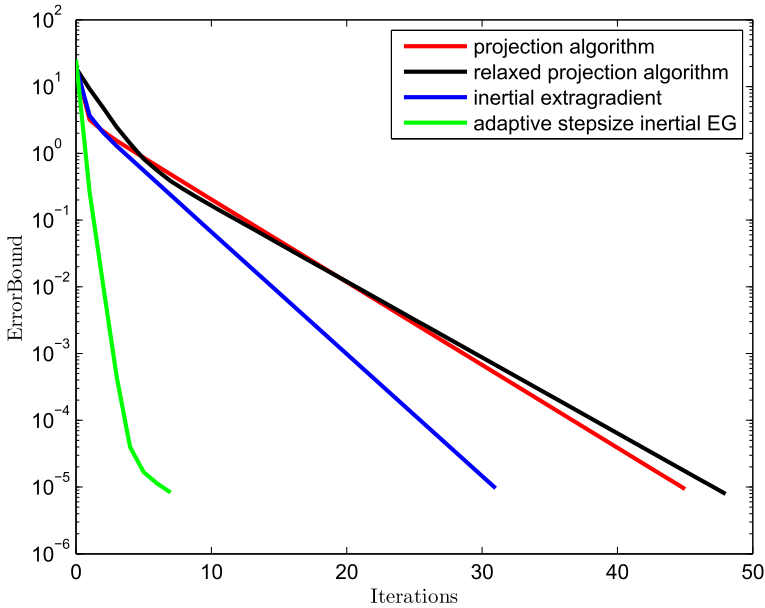


Fig. 2 Performance of different algorithms when $u_0 = u_1 = (0, 0, 0)^T$

where $u_j = 0.5, 0.25, 0.75, v_{1j} = 6.5, 5.0, 5.5$ and $v_{2j} = 4.583, 6.25, 3.75$ for $j = 1, 2, 3$, respectively. The level x_j is nonnegative for $j = 1, 2, 3$. Players try to maximize their profit $K_j(x)$ satisfying the condition $q_i(x) \leq 100, i = 1, 2$ and $x \geq 0$.

This problem can be reformulated as the equilibrium problem, where

$$P = \begin{bmatrix} a_2 + b_{21} & a_2 & a_2 \\ a_2 & a_2 + b_{22} & a_2 \\ a_2 & a_2 & a_2 + b_{23} \end{bmatrix}; \quad Q = \begin{bmatrix} a_2 + b_{21} & 0 & 0 \\ 0 & a_2 + b_{22} & 0 \\ 0 & 0 & a_2 + b_{23} \end{bmatrix}$$

and $r = \begin{bmatrix} b_{11} - a_1 \\ b_{12} - a_1 \\ b_{13} - a_1 \end{bmatrix}$. Note that since $P - Q$ is not positive definite, this linear equilibrium

problem is not strongly monotone. However, since $P + Q$ and Q are positive definite, we can choose matrix $T = 0.1 \times \text{diag}(b_{21}, b_{22}, b_{23})$ such that this problem is equivalent to a strongly monotone linear equilibrium problem as in Example 1 when $P_1 = P + T, Q_1 = Q - T$ and $r_1 = r$ (see Quoc and Muu (2012) for more details). For this choice, the corresponding parameters are $\gamma = \lambda_{\min}(P_1 - Q_1) = 0.019$ and $L = \frac{1}{2} \|P_1 - Q_1\| = 0.055$. The other parameters are similar to Example 1 except $\alpha_{\max} = 1.5$. We also compare the performance of four algorithms with $u_0 = u_1 = (0, 0, 0)^T$ and the stopping condition $\text{ErrorBound} = \|u_{n+1} - u_n\| \leq 10^{-5}$ in Fig. 2. Once again, Algorithm 3.1 demonstrates its effectiveness thanks to the large initial stepsize ($\lambda_1 = 5000$) and inertial effects.

Example 3: We continue with a very interesting model namely the Nash-Cournot oligopolistic equilibrium models of electricity markets introduced by Contreras et al. (2004). This model was reformulated as a strongly monotone equilibrium problem in Muu and Quy (2015); Quoc et al. (2012). Consider a Nash-Cournot oligopolistic equilibrium model arising in electricity markets with n^c ($n^c = 3$) generating companies and each company i ($i = 1, 2, 3$) (com. #) possesses several generating units n_i^c (gen. #). The quantities x and x^c

Table 1 The lower and upper bounds of the power generation of the generating unit and companies ($n^g = 6$)

| com. # | gen. # | x_{min}^g | x_{max}^g | x_{min}^c | x_{max}^c |
|--------|--------|-------------|-------------|-------------|-------------|
| 1 | 1 | 0 | 80 | 0 | 80 |
| 2 | 2 | 0 | 80 | 0 | 130 |
| 2 | 3 | 0 | 50 | 0 | 130 |
| 3 | 4 | 0 | 55 | 0 | 125 |
| 3 | 5 | 0 | 30 | 0 | 125 |
| 3 | 6 | 0 | 40 | 0 | 125 |

Table 2 The parameters of the generating unit cost functions c_j ($j = 1 \dots, 6$)

| gen. # | $\hat{\alpha}_j$ (\$Mw ² h) | $\hat{\beta}_j$ (\$/Mwh) | $\hat{\gamma}_j$ (\$/h) | $\tilde{\alpha}_j$ (\$/Mwh) | $\tilde{\beta}_j$ | $\tilde{\gamma}_j$ |
|--------|--|--------------------------|-------------------------|-----------------------------|-------------------|--------------------|
| 1 | 0.0400 | 2.00 | 0.00 | 2.0000 | 1.0000 | 25.0000 |
| 2 | 0.0300 | 1.75 | 0.00 | 1.7500 | 1.0000 | 28.5714 |
| 3 | 0.1250 | 1.00 | 0.00 | 1.0000 | 1.0000 | 8.0000 |
| 4 | 0.0116 | 3.25 | 0.00 | 3.2500 | 1.0000 | 86.2069 |
| 5 | 0.0500 | 3.00 | 0.00 | 3.0000 | 1.0000 | 20.0000 |
| 6 | 0.0500 | 3.00 | 0.00 | 3.0000 | 1.0000 | 20.0000 |

are the power generation of a unit and a company, respectively. The lower and upper bounds of the power generation of the generating units and companies are given in Table 1. The parameters of the cost function are pointed out in Table 2.

The cost of a generating unit j ($j = 1, \dots, 6$) is determined as follows

$$c_j(x_j) = \max\{\hat{c}_j(x_j), \tilde{c}_j(x_j)\}$$

where

$$\hat{c}_j(x_j) = \frac{\hat{\alpha}_j}{2}x_j^2 + \hat{\beta}_jx_j + \hat{\gamma}_j,$$

$$\tilde{c}_j(x_j) = \tilde{\alpha}_jx_j + \frac{\tilde{\beta}_j}{\tilde{\beta}_j + 1}\tilde{\gamma}_j^{\frac{-1}{\tilde{\beta}_j}}(x_j)^{\frac{\tilde{\beta}_j+1}{\tilde{\beta}_j}}.$$

It turns out that the cost of the generating unit j does not depend on other generating units. Let us denote n^g by the number of generating units of all companies (i.e. $n^g = \sum_{i=1}^{n^c} n_i^c$) and I_i by the index set of all generating units of the company i .

If the selling price of a unit of electricity is fixed at $378.4 - 2 \sum_{l=1}^{n^g} x_l$ then the profit of the company i that owns n_i^c generating units will be defined by

$$f_i(x) = \left(378.4 - 2 \sum_{l=1}^{n^g} x_l \right) \sum_{j \in I_i} x_j - \sum_{j \in I_i} c_j(x_j)$$

where $(x_{min}^g)_j \leq x_j \leq (x_{max}^g)_j$ ($j = 1, \dots, n^g$).

For each $i = 1, \dots, n^c$, let us define

$$\varphi_i(x, y) = \left[378.4 - 2 \left(\sum_{j \notin I_i} x_j + \sum_{j \in I_i} y_j \right) \right] \sum_{j \in I_i} y_j - \sum_{j \in I_i} c_j(y_j).$$

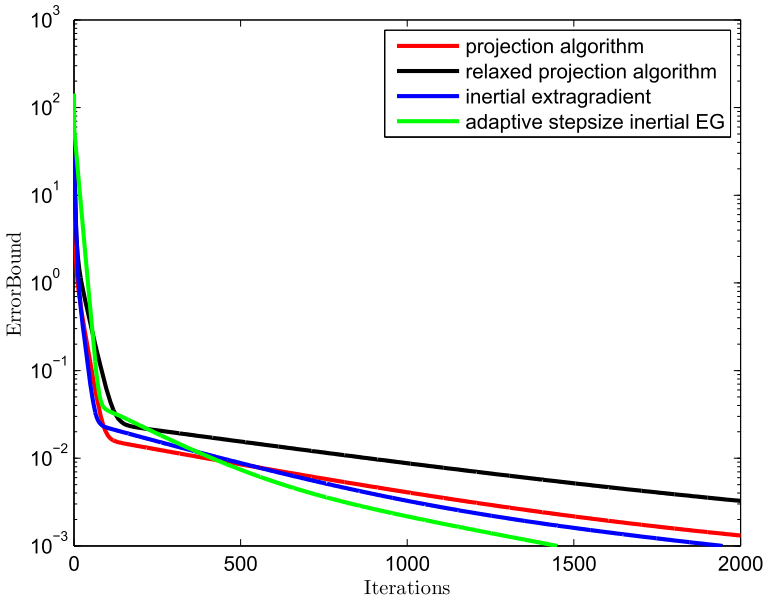


Fig. 3 Performance of different algorithms when $u_0 = u_1 = (0, 0, 0, 0, 0, 0)^T$

The Nikaido-Isoda function is defined as

$$f(x, y) = \sum_{i=1}^{n^c} [\varphi_i(x, x) - \varphi_i(x, y)].$$

Since each company intends to maximize their profit, the oligopolistic equilibrium model of electricity markets can be reformulated as an equilibrium problem of finding $x^* \in C^g$ such that

$$f(x^*, y) \geq 0, \forall y \in C^g$$

where C^g is the feasible set given by

$$C^g = \{x^g \in \mathbb{R}^{n^g} | (x_{min}^g)_j \leq x_j \leq (x_{max}^g)_j (j = 1, \dots, n^g)\}.$$

To figure out the properties of the bifunction f , first let us introduce two vectors $q^i = (q_1^i, q_2^i, \dots, q_{n^g}^i)^T$ and $\bar{q}^i = (\bar{q}_1^i, \dots, \bar{q}_{n^g}^i)^T$ in which

$$q_j^i = \begin{cases} 1 & \text{if } j \in I_i \\ 0 & \text{otherwise} \end{cases}$$

and $\bar{q}_j^i = 1 - q_j^i (j = 1, \dots, n^g)$. By (Quoc et al. 2012, Lemma 7), the equilibrium problem being solved can be formulated as

$$x^* \in C^g : f(x^*, y) = [(A + 1.5B)x + 0.5By + a]^T (y - x) + c(y) - c(x) \quad \forall y \in C^g,$$

where the two matrices A, B , the vector a , and the cost function are identified as follows

$$A = 2 \sum_{i=1}^{n^c} \bar{q}^i (q^i)^T, \quad B = 2 \sum_{i=1}^{n^c} q^i (q^i)^T$$

$$a = -378.4 \sum_{i=1}^{n^c} q^i, \quad c(x) = \sum_{i=1}^{n^c} \sum_{j \in I_i} c_j(x_j) = \sum_{j=1}^{n^g} c_j(x_j).$$

It is worth knowing that since c is a nonsmooth convex function and B is symmetric positive semidefinite, f is continuous on $C^g \times C^g$ and $f(x, \cdot)$ is convex but not smooth for all $x \in C^g$. In addition, the function f is only monotone but it can be transformed equivalently to a strongly monotone equilibrium problem by considering the following bifunction (see (Muu and Quy 2015, Lemma 4)).

$$f_1(x, y) = f(x, y) - \frac{1}{2}(y - x)^T B(y - x).$$

We test the Algorithm 3.1 with $u_0 = u_1 = (0, 0, 0, 0, 0, 0)^T$, the initial stepsize $\lambda_1 = 100$, the stopping condition $\text{ErrorBound} = \|u_{n+1} - u_n\| \leq 10^{-3}$ and small inertial effect ($\rho = 0.003$). For the other three algorithms, since the Lipschitz constants are not available, we select a positive stepsize so that the algorithms converge. We choose $\lambda = 0.02$ for the projection algorithm and $\lambda = 0.03, \alpha = 0.5$ for the relaxed projection algorithm. The inertial extragradient also uses the same stepsize with the relaxed projection. The sequences $\{\tau_n\}$ and $\{\epsilon_n\}$ are similar to Example 1. The results are reported in Fig. 3. We realize that the Algorithm 3.1 is still effective for this problem. We also notice that Algorithm 3.1 converges even faster with larger inertial effect, for example $\rho = 0.5$, but in this case condition (12) is not guaranteed.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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