UNIVERSITY OF SOUTHAMPTON
SCHOOL OF ENGINEERING SCIENCES
AERODYNAMICS & FLIGHT MECHANICS GROUP

Governing equations for a new compressible Navier-Stokes solver in general cylindrical coordinates

by

Richard D. Sandberg

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The aim of this report is to derive the governing equations for a new compressible Navier-Stokes solver in general cylindrical coordinates, i.e. the streamwise and radial directions are mapped to general coordinates. A literature review revealed that simulations of complex cylindrical geometries have mostly been conducted using general curvilinear coordinates (e.g. DeBonis & Scott, 2002). The method presented in this report is chosen over three-dimensional general curvilinear coordinates as it requires a smaller number of terms to be computed. Furthermore, only two-dimensional metric terms need to be stored, reducing the required allocated memory and therefore resulting in a more efficient code.

1 Governing equations in cylindrical coordinates

The compressible Navier-Stokes equations consist of conservation of mass, momentum and total energy. The flow is assumed to be an ideal gas with constant specific heat coefficients. All quantities are made dimensionless using the flow quantities at a reference location in the flow; here the free-stream/inflow location is used. The radius of the body is chosen as the reference length. The non-dimensionalization results in the following dimensionless parameters:

$$Re = \frac{\rho^*_\infty u^*_\infty r^*}{\mu^*_\infty} \quad , \quad M = \frac{u^*_\infty}{a^*_\infty} \quad , \quad Pr = \frac{\mu^*_\infty c^*_p}{\kappa^*_\infty} \quad .$$

With z, r, θ denoting the streamwise, radial and azimuthal directions, respectively, and u, v, w denoting the velocity components in z, r, θ directions, respectively, the non-dimensional compressible Navier–Stokes equations in cylindrical coordinates are

$$\frac{\partial U}{\partial t} + \frac{\partial A}{\partial z} + \frac{\partial B}{\partial r} + \frac{1}{r}\frac{\partial C}{\partial \theta} + \frac{1}{r} D = 0 \tag{1}$$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{pmatrix} \tag{2}$$

$$A = \begin{pmatrix} \rho u \\ \rho u u + p - \tau_{zz} \\ \rho u v - \tau_{rz} \\ \rho u w - \tau_{\theta z} \\ \rho u H + q_z - u\tau_{zz} - v\tau_{rz} - w\tau_{\theta z} \end{pmatrix} \quad B = \begin{pmatrix} \rho v \\ \rho u v - \tau_{rz} \\ \rho v v + p - \tau_{rr} \\ \rho v w - \tau_{\theta r} \\ \rho v H + q_r - u\tau_{rz} - v\tau_{rr} - w\tau_{\theta r} \end{pmatrix}$$

$$C = \begin{pmatrix} \rho w \\ \rho u w - \tau_{\theta z} \\ \rho v w - \tau_{\theta r} \\ \rho w w + p - \tau_{\theta\theta} \\ \rho w H + q_\theta - u\tau_{\theta z} - v\tau_{\theta r} - w\tau_{\theta\theta} \end{pmatrix} \quad D = \begin{pmatrix} \rho v \\ \rho u v - \tau_{rz} \\ \rho v v - \rho w w - \tau_{rr} + \tau_{\theta\theta} \\ 2\rho v w - 2\tau_{\theta r} \\ \rho v H + q_r - u\tau_{rz} - v\tau_{rr} - w\tau_{\theta r} \end{pmatrix} ,$$

where the total energy is defined as $E = T/[\gamma(\gamma-1)M^2] + 1/2 u_i u_i$ with $\gamma = 1.4$, and the total enthalpy is $H = E + p/\rho$.

The molecular stress tensor components are

$$\tau_{zz} = \frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} - \frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right)\right] \quad (3)$$

$$\tau_{rr} = \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r} - \frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right)\right] \quad (4)$$

$$\tau_{\theta\theta} = \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r} + 2\frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right)\right] \quad (5)$$

$$\tau_{rz} = \frac{\mu}{Re}\left[\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right] \quad (6)$$

$$\tau_{\theta z} = \frac{\mu}{Re}\left[\frac{\partial w}{\partial z} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right] \quad (7)$$

$$\tau_{\theta r} = \frac{\mu}{Re}\left[\frac{1}{r}\left(\frac{\partial v}{\partial \theta} - w\right) + \frac{\partial w}{\partial r}\right] . \quad (8)$$

The heat-flux vector components are

$$q_z = \frac{-\mu}{Pr(\gamma-1)M^2 Re}\frac{\partial T}{\partial z} \quad (9)$$

$$q_r = \frac{-\mu}{Pr(\gamma-1)M^2 Re}\frac{\partial T}{\partial r} \quad (10)$$

$$q_\theta = \frac{-\mu}{Pr(\gamma-1)M^2 Re}\frac{1}{r}\frac{\partial T}{\partial \theta} , \quad (11)$$

where the Prandtl number is assumed to be constant at $Pr = 0.72$. The molecular viscosity μ is computed using Sutherland's law (c.f. White, 1991), setting the ratio of the Sutherland constant over freestream temperature to 0.36867. To close the system of equations, the pressure is obtained from the non-dimensional equation of state $p = (\rho T)/(\gamma M^2)$.

2 Transformation to general cylindrical coordinates

In order to allow for complex geometries, the (r, z) plane is mapped to general coordinates (ξ, η). Hence, streamwise and radial derivatives need to be expressed in terms of the new variables. By using the chain rule, the following expressions can be derived

$$\frac{\partial}{\partial z} = \frac{1}{J}\left[\frac{\partial r}{\partial \eta}\frac{\partial}{\partial \xi} - \frac{\partial r}{\partial \xi}\frac{\partial}{\partial \eta}\right] = r^*_\eta\frac{\partial}{\partial \xi} - r^*_\xi\frac{\partial}{\partial \eta} , \quad (12)$$

$$\frac{\partial}{\partial r} = \frac{1}{J}\left[-\frac{\partial z}{\partial \eta}\frac{\partial}{\partial \xi} + \frac{\partial z}{\partial \xi}\frac{\partial}{\partial \eta}\right] = z^*_\xi\frac{\partial}{\partial \eta} - z^*_\eta\frac{\partial}{\partial \xi} , \quad (13)$$

where J is the determinant of the coordinate transformation. For conciseness, the metric terms are abbreviated as, e.g., $r_\eta = \frac{\partial r}{\partial \eta}$ so that $J = z_\xi r_\eta - z_\eta r_\xi$, and the asterisk denotes a metric term already divided by J.

Following Anderson (1995), the r and z derivatives of equation (1) can be transformed as follows (provided that the grid transformation does not vary in time)

$$\frac{\partial U}{\partial t} = -\frac{1}{J}\left[\frac{\partial}{\partial \xi}\left(Ar_\eta - Bz_\eta\right) + \frac{\partial}{\partial \eta}\left(-Ar_\xi + Bz_\xi\right)\right] - \frac{1}{r}\frac{\partial C}{\partial \theta} - \frac{1}{r}D \tag{14}$$

Unfortunately, it is not straightforward to plug in the vectors A and B into (14), as they contain terms that are pre-multiplied by $1/r$. As $r = r(\xi, \eta)$, when taking ξ and η derivatives of A and B the chain rule needs to be applied. To illustrate this procedure, the derivation of the radial momentum equation is chosen as example. Here, we only focus on the derivatives in the ξ and η directions,

$$\begin{aligned} &\frac{\partial}{\partial \xi}\Big[\left(\rho uv - \tau_{rz}\right) r_\eta - \left(\rho vv + p - \tau_{rr}\right) z_\eta\Big] \\ -&\frac{\partial}{\partial \eta}\Big[\left(\rho uv - \tau_{rz}\right) r_\xi - \left(\rho vv + p - \tau_{rr}\right) z_\xi\Big] \,. \end{aligned} \tag{15}$$

The only term containing $1/r$ is the stress tensor component τ_{rr}. Therefore, the equation is separated into

$$\begin{aligned} &\frac{\partial}{\partial \xi}\left[\left(\rho uv - \tau_{rz}\right) r_\eta - \left(\rho vv + p - \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right]\right) z_\eta\right] - \frac{\partial}{\partial \xi}\left[\frac{1}{r}\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\eta\right] \\ -&\frac{\partial}{\partial \eta}\left[\left(\rho uv - \tau_{rz}\right) r_\xi - \left(\rho vv + p - \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right]\right) z_\xi\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{r}\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\xi\right] \,. \end{aligned} \tag{16}$$

Applying the chain rule to the last term of each row results in

$$\begin{aligned} \frac{\partial}{\partial \xi}\left[\frac{1}{r}\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\eta\right] &= \frac{1}{r}\frac{\partial}{\partial \xi}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\eta\right] - \frac{1}{r^2} r_\xi z_\eta \frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) \\ \frac{\partial}{\partial \eta}\left[\frac{1}{r}\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\xi\right] &= \frac{1}{r}\frac{\partial}{\partial \eta}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\xi\right] - \frac{1}{r^2} r_\eta z_\xi \frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) \end{aligned}$$

Substituting these two expressions into (16) yields

$$\begin{aligned} &\frac{\partial}{\partial \xi}\left[\left(\rho uv - \tau_{rz}\right) r_\eta - \left(\rho vv + p - \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right]\right) z_\eta\right] \\ -&\frac{\partial}{\partial \eta}\left[\left(\rho uv - \tau_{rz}\right) r_\xi - \left(\rho vv + p - \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right]\right) z_\xi\right] \\ -&\frac{1}{r}\frac{\partial}{\partial \xi}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\eta\right] + \frac{1}{r}\frac{\partial}{\partial \eta}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\xi\right] \\ +&\frac{1}{r^2}\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right)\underbrace{\left(r_\xi z_\eta - r_\eta z_\xi\right)}_{-J} \,. \end{aligned} \tag{17}$$

The last term of (17) occurs because of the partial derivative with respect to r of τ_{rr}. In the case of streamwise derivatives $\frac{\partial}{\partial z}$, we get $(r_\xi r_\eta - r_\eta r_\xi) = 0$, hence $1/r^2$ terms cancel. The resulting governing equations are now summarized. For brevity, the inner derivatives are not written in terms of ξ, η, however the streamwise and radial derivatives need to be computed with equations (12) and (13).

2.1 Continuity equation

$$\frac{\partial \rho}{\partial t} = -\frac{1}{J}\left[\frac{\partial}{\partial \xi}\left(\rho u r_\eta - \rho v z_\eta\right) + \frac{\partial}{\partial \eta}\left(-\rho u r_\xi + \rho v z_\xi\right)\right] - \frac{1}{r}\frac{\partial(\rho w)}{\partial \theta} - \frac{(\rho v)}{r} \tag{18}$$

2.2 Streamwise momentum equation

$$\begin{aligned}\frac{\partial(\rho u)}{\partial t} = -\frac{1}{J}\Bigg\{ & \frac{\partial}{\partial \xi}\left[\left(\rho u u + p - \frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right]\right) r_\eta - \left(\rho u v - \tau_{rz}\right) z_\eta\right] \\ & -\frac{\partial}{\partial \eta}\left[\left(\rho u u + p - \frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right]\right) r_\xi - \left(\rho u v - \tau_{rz}\right) z_\xi\right] \\ & +\frac{1}{r}\frac{\partial}{\partial \xi}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) r_\eta\right] - \frac{1}{r}\frac{\partial}{\partial \eta}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) r_\xi\right] \Bigg\} \\ & -\frac{1}{r}\frac{\partial}{\partial \theta}\left[\rho u w - \tau_{\theta z}\right] - \frac{1}{r}\left[\rho u v - \tau_{rz}\right]\end{aligned} \tag{19}$$

2.3 Radial momentum equation

$$\begin{aligned}\frac{\partial(\rho v)}{\partial t} = -\frac{1}{J}\Bigg\{ & \frac{\partial}{\partial \xi}\left[\left(\rho u v - \tau_{rz}\right) r_\eta - \left(\rho v v + p - \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right]\right) z_\eta\right] \\ & -\frac{\partial}{\partial \eta}\left[\left(\rho u v - \tau_{rz}\right) r_\xi - \left(\rho v v + p - \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right]\right) z_\xi\right] \\ & -\frac{1}{r}\frac{\partial}{\partial \xi}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\eta\right] + \frac{1}{r}\frac{\partial}{\partial \eta}\left[\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) z_\xi\right] \Bigg\} \\ & +\frac{1}{r^2}\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) \\ & -\frac{1}{r}\frac{\partial}{\partial \theta}\left[\rho v w - \tau_{\theta r}\right] - \frac{1}{r}\left[\rho(vv - ww) - \frac{2\mu}{Re}\left(\frac{\partial v}{\partial r} - \frac{1}{r}\left(\frac{\partial w}{\partial \theta} + v\right)\right)\right]\end{aligned} \tag{20}$$

2.4 Azimuthal momentum equation

$$
\begin{aligned}
\frac{\partial(\rho w)}{\partial t} = -\frac{1}{J}\Bigg\{ & \frac{\partial}{\partial \xi}\left[\left(\rho u w - \frac{\mu}{Re}\frac{\partial w}{\partial z}\right) r_\eta - \left(\rho v w - \frac{\mu}{Re}\frac{\partial w}{\partial r}\right) z_\eta\right] \\
& -\frac{\partial}{\partial \eta}\left[\left(\rho u w - \frac{\mu}{Re}\frac{\partial w}{\partial z}\right) r_\xi - \left(\rho v w - \frac{\mu}{Re}\frac{\partial w}{\partial r}\right) z_\xi\right] \\
& -\frac{1}{r}\frac{\partial}{\partial \xi}\left[\frac{\mu}{Re}\left(\frac{\partial u}{\partial \theta} r_\eta - \left(\frac{\partial v}{\partial \theta} - w\right) z_\eta\right)\right] + \frac{1}{r}\frac{\partial}{\partial \eta}\left[\frac{\mu}{Re}\left(\frac{\partial u}{\partial \theta} r_\xi - \left(\frac{\partial v}{\partial \theta} - w\right) z_\xi\right)\right]\Bigg\} \\
& -\frac{1}{r^2}\frac{\mu}{Re}\left(\frac{\partial v}{\partial \theta} - w\right) \\
& -\frac{1}{r}\frac{\partial}{\partial \theta}\Big[\rho w w + p - \tau_{\theta\theta}\Big] - \frac{1}{r}\Big[2\rho v w - 2\tau_{\theta r}\Big]
\end{aligned}
\tag{21}
$$

2.5 Energy equation

$$
\begin{aligned}
\frac{\partial(\rho E)}{\partial t} = -\frac{1}{J}\Bigg\{ & \frac{\partial}{\partial \xi}\Bigg[\left(\rho u H + q_z - u\frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right] - v\tau_{rz} - w\frac{\mu}{Re}\frac{\partial w}{\partial z}\right) r_\eta \\
& \quad - \left(\rho v H + q_r - u\tau_{rz} - v\frac{2\mu}{3Re}\left[2\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z}\right] - w\frac{\mu}{Re}\frac{\partial w}{\partial r}\right) z_\eta\Bigg] \\
& -\frac{\partial}{\partial \eta}\Bigg[\left(\rho u H + q_z - u\frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right] - v\tau_{rz} - w\frac{\mu}{Re}\frac{\partial w}{\partial z}\right) r_\xi \\
& \quad - \left(\rho v H + q_r - u\tau_{rz} - v\frac{2\mu}{3Re}\left[2\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z}\right] - w\frac{\mu}{Re}\frac{\partial w}{\partial r}\right) z_\xi\Bigg] \\
& +\frac{1}{r}\frac{\partial}{\partial \xi}\Bigg[\frac{\mu}{Re}\left[u\frac{2}{3}\left(\frac{\partial w}{\partial \theta} + v\right) - w\frac{\partial u}{\partial \theta}\right] r_\eta \\
& \quad - \frac{\mu}{Re}\left[\frac{2}{3} v\left(\frac{\partial w}{\partial \theta} + v\right) - w\left(\frac{\partial v}{\partial \theta} - w\right)\right] z_\eta\Bigg] \\
& -\frac{1}{r}\frac{\partial}{\partial \eta}\Bigg[\frac{\mu}{Re}\left[u\frac{2}{3}\left(\frac{\partial w}{\partial \theta} + v\right) - w\frac{\partial u}{\partial \theta}\right] r_\xi \\
& \quad - \frac{\mu}{Re}\left[\frac{2}{3} v\left(\frac{\partial w}{\partial \theta} + v\right) - w\left(\frac{\partial v}{\partial \theta} - w\right)\right] z_\xi\Bigg] \Bigg\} \\
& +\frac{1}{r^2}\frac{\mu}{Re}\left[\frac{2}{3} v\left(\frac{\partial w}{\partial \theta} + v\right) - w\left(\frac{\partial v}{\partial \theta} - w\right)\right] \\
& -\frac{1}{r}\frac{\partial}{\partial \theta}\Big[\rho w H + q_\theta - u\tau_{\theta z} - v\tau_{\theta r} - w\tau_{\theta\theta}\Big] - \frac{1}{r}\Big[\rho v H + q_r - u\tau_{rz} - v\tau_{rr} - w\tau_{\theta r}\Big]
\end{aligned}
\tag{22}
$$

3 Vector form of general cylindrical coordinates

The equations in general cylindrical coordinates can be written as

$$
\begin{aligned}
\frac{\partial U}{\partial t} = -\frac{1}{J}\{ \quad & \left[\frac{\partial}{\partial \xi}\left(\hat{A} r_\eta - \hat{B} z_\eta\right) + \frac{\partial}{\partial \eta}\left(-\hat{A} r_\xi + \hat{B} z_\xi\right)\right] \\
+\frac{1}{r} \quad & \left[\frac{\partial}{\partial \xi}\left(\hat{A}_r r_\eta - \hat{B}_r z_\eta\right) + \frac{\partial}{\partial \eta}\left(-\hat{A}_r r_\xi + \hat{B}_r z_\xi\right)\right]\Bigg\} \\
-\frac{1}{r}\frac{\partial C}{\partial \theta} - \frac{1}{r} D - \frac{1}{r^2}\hat{B}_{rr}\ , &
\end{aligned}
\tag{23}
$$

where

$$
\hat{A} = \begin{pmatrix} \rho u \\ \rho u u + p - \frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right] \\ \rho u v - \tau_{rz} \\ \rho u w - \frac{\mu}{Re}\frac{\partial w}{\partial z} \\ \rho u H + q_z - u\frac{2\mu}{3Re}\left[2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right] - v\tau_{rz} - w\frac{\mu}{Re}\frac{\partial w}{\partial z} \end{pmatrix},
$$

$$
\hat{B} = \begin{pmatrix} \rho v \\ \rho u v - \tau_{rz} \\ \rho v v + p - \frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right] \\ \rho v w - \frac{\mu}{Re}\frac{\partial w}{\partial r} \\ \rho v H + q_r - u\tau_{rz} - v\frac{2\mu}{3Re}\left[-\frac{\partial u}{\partial z} + 2\frac{\partial v}{\partial r}\right] - w\frac{\mu}{Re}\frac{\partial w}{\partial r} \end{pmatrix},
$$

$$
\hat{A}_r = \begin{pmatrix} 0 \\ \frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) \\ 0 \\ -\frac{\mu}{Re}\frac{\partial u}{\partial \theta} \\ \frac{\mu}{Re}\left[u\frac{2}{3}\left(\frac{\partial w}{\partial \theta} + v\right) - w\frac{\partial u}{\partial \theta}\right] \end{pmatrix}, \quad
\hat{B}_r = \begin{pmatrix} 0 \\ 0 \\ \frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) \\ -\frac{\mu}{Re}\left(\frac{\partial v}{\partial \theta} - w\right) \\ \frac{\mu}{Re}\left[\frac{2}{3}v\left(\frac{\partial w}{\partial \theta} + v\right) - w\left(\frac{\partial v}{\partial \theta} - w\right)\right] \end{pmatrix},
$$

$$
\hat{B}_{rr} = \begin{pmatrix} 0 \\ 0 \\ -\frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) \\ \frac{\mu}{Re}\left(\frac{\partial v}{\partial \theta} - w\right) \\ \frac{\mu}{Re}\left[w\left(\frac{\partial v}{\partial \theta} - w\right) - \frac{2}{3}v\left(\frac{\partial w}{\partial \theta} + v\right)\right] \end{pmatrix}.
$$

In order to make the new code most efficient, a good compromise between the number of arithmetic operations and the number of three dimensional arrays stored needs to be found. The emphasis here lies on reducing the number of stored arrays to a minimum while minimizing the number of repeated operations. One choice is to simply store all nine velocity derivatives. However, for this option, the number of operations is high as the stress tensor components constantly need to be reassembled. The most efficient choice of arrays to store appears to be the following set

$$
\begin{array}{lll}
1)\ \tilde{\tau}_{zz} = \frac{2\mu}{3Re}\left(2\frac{\partial u}{\partial z} - \frac{\partial v}{\partial r}\right) & 2)\ \tau_{rz} = \frac{\mu}{Re}\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z}\right) & 3)\ \tilde{\tau}_{\theta z} = \frac{\mu}{Re}\frac{\partial w}{\partial z} \\
4)\ \tilde{\tau}_{rr} = \frac{2\mu}{3Re}\left(2\frac{\partial v}{\partial r} - \frac{\partial u}{\partial z}\right) & 5)\ \tilde{\tau}_{\theta r} = \frac{\mu}{Re}\frac{\partial w}{\partial r} & 6)\ \tau_1 = \frac{2\mu}{3Re}\left(\frac{\partial w}{\partial \theta} + v\right) \\
7)\ \tau_2 = \frac{\mu}{Re}\frac{\partial u}{\partial \theta} & 8)\ \tau_3 = \frac{\mu}{Re}\left(\frac{\partial v}{\partial \theta} - w\right) . &
\end{array}
$$

Using these eight arrays in addition to the primitive variables and the heat-flux vector, $\hat{A}, \hat{B}, \hat{A}_r, \hat{B}_r$ and $\hat{B}_{rr}$ can be assembled. In order to evaluate C and D, the following relations can be used

$$
\tau_{\theta z} = \tilde{\tau}_{\theta z} + \frac{1}{r}\tau_2 \quad , \quad \tau_{\theta r} = \tilde{\tau}_{\theta r} + \frac{1}{r}\tau_3 \quad , \quad \tau_{rr} = \tilde{\tau}_{rr} - \frac{1}{r}\tau_1 \quad , \quad \tau_{\theta\theta} = -\tilde{\tau}_{zz} - \tilde{\tau}_{rr} + \frac{2}{r}\tau_1 \ . \qquad (24)
$$

Finally, all vectors required for (23) can be written as

$$
\hat{A} = \begin{pmatrix} \rho u \\ \rho uu + p - \tilde{\tau}_{zz} \\ \rho uv - \tau_{rz} \\ \rho uw - \tilde{\tau}_{\theta z} \\ \rho uH + q_z - u\tilde{\tau}_{zz} - v\tau_{rz} - w\tilde{\tau}_{\theta z} \end{pmatrix} , \quad \hat{B} = \begin{pmatrix} \rho v \\ \rho uv - \tau_{rz} \\ \rho vv + p - \tilde{\tau}_{rr} \\ \rho vw - \tilde{\tau}_{\theta r} \\ \rho vH + q_r - u\tau_{rz} - v\tilde{\tau}_{rr} - w\tilde{\tau}_{\theta r} \end{pmatrix} ,
$$

$$
\hat{A}_r = \begin{pmatrix} 0 \\ \tau_1 \\ 0 \\ -\tau_2 \\ u\tau_1 - w\tau_2 \end{pmatrix} , \quad \hat{B}_r = \begin{pmatrix} 0 \\ 0 \\ \tau_1 \\ -\tau_3 \\ v\tau_1 - w\tau_3 \end{pmatrix} , \quad \hat{B}_{rr} = \begin{pmatrix} 0 \\ 0 \\ -\tau_1 \\ \tau_3 \\ w\tau_3 - v\tau_1 \end{pmatrix} ,
$$

$$
C = \begin{pmatrix} \rho w \\ \rho uw - (\tilde{\tau}_{\theta z} + \frac{1}{r}\tau_2) \\ \rho vw - (\tilde{\tau}_{\theta r} + \frac{1}{r}\tau_3) \\ \rho ww + p - (-\tilde{\tau}_{zz} - \tilde{\tau}_{rr} + \frac{2}{r}\tau_1) \\ \rho wH + q_\theta - u(\tilde{\tau}_{\theta z} + \frac{1}{r}\tau_2) - v(\tilde{\tau}_{\theta r} + \frac{1}{r}\tau_3) - w(-\tilde{\tau}_{zz} - \tilde{\tau}_{rr} + \frac{2}{r}\tau_1) \end{pmatrix} ,
$$

$$
D = \begin{pmatrix} \rho v \\ \rho uv - \tau_{rz} \\ \rho vv - \rho ww - (\tilde{\tau}_{rr} - \frac{1}{r}\tau_1) + (-\tilde{\tau}_{zz} - \tilde{\tau}_{rr} + \frac{2}{r}\tau_1) \\ 2\rho vw - 2(\tilde{\tau}_{\theta r} + \frac{1}{r}\tau_3) \\ \rho vH + q_r - u\tau_{rz} - v(\tilde{\tau}_{rr} - \frac{1}{r}\tau_1) - w(\tilde{\tau}_{\theta r} + \frac{1}{r}\tau_3) \end{pmatrix} .
$$

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