Probing Lambda-Gravity with Bose-Einstein Condensate

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We propose a precise test of two fundamental gravitational constants using a novel detector concept that exploits the dynamics of quantum phononic excitations in a trapped Bose-Einstein condensate (BEC), operable at the scale of table-top experiments. In this setup, the sensitivity is enhanced by approximately two orders of magnitude through the use of a tritter operation, which mixes phononic excitations with the BEC's ground state. The BEC exhibits unique sensitivity to the two key components of the gravitational potential in Λ-gravity: the Newtonian GM/r term and the cosmological constant Λr^2. Using state-of-the-art experimental design, we predict that the gravitational constant G could be measured with an accuracy up to 10^{-17} N m^2/kg^2, representing an improvement by two orders of magnitude over current measurements. Moreover, this experiment could establish the best Earth-based upper limit on Λ at $< 10^{-31}$ m^{-2}, marking the first laboratory-based probe of the cosmological constant. Additionally, the setup allows for the measurement of the distance-dependent behaviour of each term in the gravitational potential, providing a novel means to test modified gravity theories.

I. INTRODUCTION

The current standard model of cosmology, the ΛCDM model, is considered to be in remarkable agreement with the main cosmological data [1]. At the same time, along with the anomalies and tensions of the model and data, the nature of both dark matter and dark energy remains unexplained.

One of the main components of the ΛCDM model – dark energy – is often associated with the cosmological constant, Λ. The significance of Λ is not limited to its role as a parameter in the model but extends to its status as a fundamental constant, as demonstrated by Gurzadyan's theorem [2]. This theorem establishes that the most general function satisfying the equivalence of gravitational forces between a spherical mass and a point mass leads to the appearance of a term involving the cosmological constant Λ within the weak-field limit of General Relativity (GR) in the expression for the force [2]

$$F = -\frac{GMm}{r^2} + \frac{\Lambda r m c^2}{3} \ . \tag{1}$$

Besides its theoretical motivation, experimentally, the existence of the Λ term is well established as part of the standard ΛCDM cosmological model. Moreover, the Λ term is shown to fit observational data on the local Universe [3, 4], describes the dynamics of groups and clusters of galaxies [5, 6], and suggests a resolution for the Hubble tension [7–9] as a result of two flows, local and global ones [6, 10, 11].

It is important to note that Eq. 1 satisfies the first condition of the shell theorem (sphere-point equivalency) but not the second one, i.e., the condition of a force-free field inside a spherical shell. Namely, the second term in Eq. 1 predicts a force (non-force-free) field inside a shell, as indicated, for example, by the observational data on the determination of the structure of spiral galaxy disks by the spherical galactic halos, see e.g. [12]. So, the gravity is described not by one but two constants, G and Λ. Let us note that Einstein has denoted Λ as a universal constant in [13, 14]. The consequences of taking Λ into account as a universal constant were studied in [15]. The cosmological constant is the key element, along with the second law of thermodynamics, of Conformal Cyclic Cosmology and in the information transfer [16–18] and leads to the possibility of rescaling of physical constants from one aeon to another [15].

In this letter, we suggest using the potential of the table-top experiments with trapped Bose-Einstein condensates (BEC) to probe both fundamental constants of Λ-gravity and the dependence of both terms of Eq. 1 as a function of distance. The efforts to test modified gravity in table-top experiments have been very limited despite the great opportunities offered by the development of quantum sensors, which offer ultra-precise sensitivities [19]. In contrast, proposals to search for dark matter using quantum technologies have been more popular. For instance, recent proposals include searches using optomechanical systems [20], atom-interferometry [21] and Bose-Einstein Condensates [22].

Theoretical studies have shown that the collective excitations in trapped BECs are very sensitive to gravitational effects [22–27]. BECs can be cooled to nK temperatures, where the system exhibits distinct quantum behaviour. At

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very low temperatures and short time scales, BEC excitations behave as free quasiparticles (phonons) [28, 29] that respond to changes in the gravitational field. These excitations can be used to estimate physical parameters using methods from quantum metrology. Quantum phononic states involving entangled collective excitations provide sensitivities that, according to theoretical predictions, surpass those achievable through the quantum behaviour of small solid-state systems [30]. Additionally, resonances to periodic changes in the field can significantly amplify the sensitivity. For instance, resonant effects using phononic squeezed states were proposed to measure the gravitational potential and the respective accelerations produced by the oscillations of small masses near the BEC [26], by high-frequency gravitational waves [23, 25] and to devise a gravimeter and a gradiometer within the millimetre scale [31].

A recent theoretical study proposes employing phonon states in a frequency interferometric protocol to search for dark matter [22]. This approach involves using a tritter [32], an operation that mixes phonon states with the BEC's ground state. The tritter enhances sensitivity by producing a scaling proportional to $1/\sqrt{N_p N_0}$, where N_p represents the number of squeezed phonons and N_0 the number of atoms in the ground state, noting that generally $N_p \ll N_0$. The authors of [22] demonstrated that including the tritter significantly improved the sensitivity for gravitational-wave detection, compared to previous studies [23].

We propose a frequency interferometric method to measure the fundamental parameters G and Λ using a small oscillating mass near the BEC. Our results show that the tritter operation enables sensitivities approximately two orders of magnitude higher than previous proposals for measuring Newtonian accelerations [26]. If phonon squeezing of the order of 30.4 dB[1] can be successfully achieved in large BECs, this approach is anticipated to offer a highly sensitive method for testing gravitational potentials and associated accelerations at exceptionally low levels, potentially reaching sensitivities as low as 10^{-18} m/s^2. This exceptional sensitivity, in turn, would allow for the measurement of the gravitational constant G with a relative accuracy of about 10^{-7}, which is two orders of magnitude better than the current best measurement. Additionally, this experiment would establish the first Earth-based upper limit on $\Lambda < 10^{-31}\mathrm{m}^{-2}$. This direction is promising, as BEC spin-squeezing routinely reaches 6-8 dB in the laboratory [33–35], with best measurements yielding a factor of 25 dB in the number of squeezed atoms [36]. Additionally, theoretical studies indicate that the degree squeezing in the number of phonons required to achieve the aforementioned precision might be theoretically attainable [37–39].

[1] Decibels (dB) are given by the number of phonons (N_P) as #dB= $10 \times \log_{10}(N_p)$.

The Section II describes the proposed experimental setup, followed by Section III, which provides a quantum description of a BEC and discusses the dynamics of phonon excitations under the external gravitational potential. Section IV introduces the quantum metrology approach and presents the predicted sensitivity of the BEC to the gravitational potential. Section V presents results on the estimated precision of the experiment to test gravity, while Section VI draws the conclusions.

II. GRAVITY FROM AN OSCILLATING MASS

The gravitational force exerted on a mass m is defined by the potential ϕ.

$$\vec{F}_G = -m\vec{\nabla}\phi, \tag{2}$$

which is determined by Eq. 1

$$\phi = -\frac{MG}{r} - \frac{\Lambda r^2 c^2}{6}. \tag{3}$$

We consider the experiment where an oscillating sphere of mass M (of the order of 100 g) and frequency Ω is placed near a BEC of length L held by a uniform box trap potential [40], aligned with the direction of the oscillation. We assume the BEC (of the size of $L \simeq 100\mu m$) is placed at the distance $R_0 \gg L$ and that the size of the source mass is of the same order as the amplitude of its oscillation, $\delta_R(\simeq 1\mathrm{mm}) \ll R_0(\simeq 100\mathrm{mm})$.

A sketch of this setup is shown in Figure 1. The distance $r(x,t)$ between the center of the oscillating mass and point in BEC with coordinate $x \in [0, L]$ is given by

$$r(x,t) = R_0 + \delta_R \sin(\Omega t) + x \equiv R_0(1 + \Delta_{x,t}) \tag{4}$$

where $\Delta_{x,t} = [\delta_R \sin(\Omega t) + x]/R_0$ which will be used for the expansion, since $|\Delta_{x,t}| \ll 1$. The respective gravitational potential generated by the oscillating mass in BEC at point x is given by

$$\phi(x,t) = -\frac{MG}{r(x,t)} - \frac{\Lambda r(x,t)^2 c^2}{6}. \tag{5}$$

The expansion of $\phi(x,t)$ around R_0 up to the second order in $\Delta_{x,t}$ leads to the following expression for the $\phi(x,t)$:

$$\phi(x,t) = \phi_0^G(1 - \Delta_{x,t} + \Delta_{x,t}^2 + ...) + \phi_0^\Lambda(1 + 2\Delta_{x,t} + \Delta_{x,t}^2) \tag{6}$$

where $\phi_0^G = -\frac{MG}{R_0}$ and $\phi_0^\Lambda = -\frac{\Lambda R_0^2 c^2}{6}$. The part of $\phi(x,t)$ from Eq. 6 to which the BEC experiment has essential sensitivity (as we discuss below) is proportional to $[x\delta_R \sin(\Omega t)]$, originating from the $\Delta_{x,t}^2$ and higher-order $\mathcal{O}(\Delta_{x,t}^2)$ terms, and is given by

$$\phi_\Omega^{\mathrm{BEC}}(x,t) = \frac{2x\delta_R}{R_0^2}\left(\phi_0^G(1 + ...) + \phi_0^\Lambda\right)\sin(\Omega t) \equiv a_\Omega^{\mathrm{BEC}}(t)x, \tag{7}$$

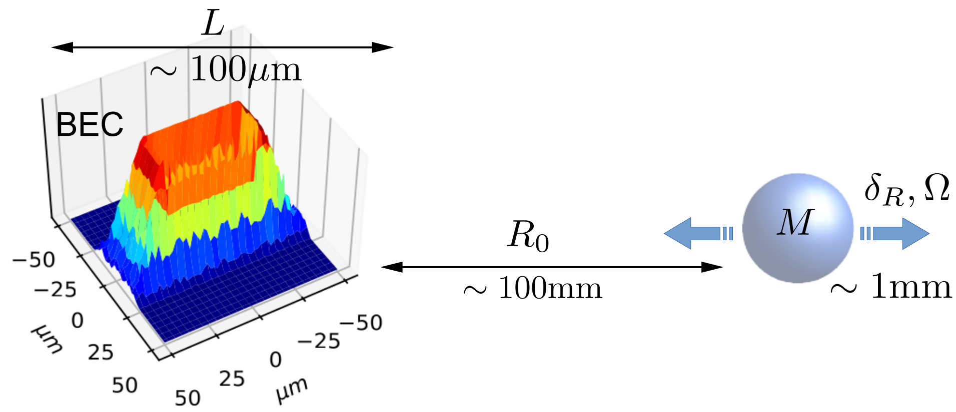

FIG. 1. A sketch of the experiment with oscillating sphere of mass M, frequency Ω and amplitude δ_R (on the right) placed at distance $R_0 \gg \delta_R$ to a BEC of length L (on the left).

where '...' denotes $\mathcal{O}\left(\frac{\delta R^2}{R_0^2}\right)$ terms, and

$$a_\Omega^{\rm BEC}(t) = \frac{2\delta_R}{R_0^2}\left(\phi_0^G(1+...) + \phi_0^\Lambda\right)\sin(\Omega t), \tag{8}$$

which is the time-dependent part of the acceleration exerted by the oscillating sphere on the BEC. The amplitude of this acceleration, $a^{\rm BEC}$, is the key observable in our study and is given by

$$a^{\rm BEC} \simeq \frac{2\delta_R}{R_0^2}\left(\frac{MG}{R_0} + \frac{\Lambda R_0^2 c^2}{6}\right), \tag{9}$$

where we have omitted the $\mathcal{O}(\frac{\delta R^2}{R_0^2})$ corrections to the first term with constant G. These corrections can be recovered if needed, depending on the required accuracy. To maintain simplicity in the following expressions, we will use the form of $a^{\rm BEC}$ as given here for the remainder of the paper.

III. QUANTUM FIELD TREATMENT OF BEC PHONONS

The standard description of a Bose gas with two-atom interactions [29], taking into account the gravitational potential of the oscillating sphere, is given by

$$\hat{H} = \int_\mathcal{V} \hat{\Psi}^\dagger\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm tr} - m\phi(x,t) + \frac{g}{2}\hat{\Psi}^\dagger\hat{\Psi}\right)\hat{\Psi}\, d^3x, \tag{10}$$

where $V_{\rm tr}$ is the trapping potential, $g = 4\pi\hbar^2 a_{sl}/m$ is the two-atom coupling constant, m the mass of the atoms in the BEC, a_{sl} the atomic scattering length and $\mathcal{V}$ is the confinement volume over which Eq. 10 is integrated. To produce a one-dimensional uniform density BEC in the x-direction, we set $V_{\rm tr} = 0$ inside the trap and impose von Neumann boundary conditions at the potential walls.

The stationary part of the gravitational potential $\phi(x,t)$, as well as the any x-independent terms, do not contribute to the system's dynamics. The only term contributing to the gravitational interaction is given by $\phi_\Omega^{\rm BEC}(x,t) = a_\Omega^{\rm BEC}(t)x$, with $a_\Omega^{\rm BEC}(t)$ given by Eq. 8 [26]. The field operator can be expanded as

$$\hat{\Psi}(\mathbf{r},t) = (\hat{\Psi}_0(\mathbf{r}) + \hat{\vartheta}(\mathbf{r},t))e^{-i\mu t/\hbar - i\int_0^t \delta\mu(t')dt'/\hbar}, \tag{11}$$

where $\hat{\Psi}_0(\mathbf{r})$ is the solution of the stationary Gross-Pitaevskii equation, $\hat{\vartheta}(\mathbf{r},t)$ is a small perturbation, μ is the chemical potential and $\delta\mu = \int h(t)d^3x$ is the time-dependent energy shift of the ground state.

Bose-Einstein condensation is achieved assuming that the temperature T of the Bose gas is much smaller than the condensation's critical temperature so that the ground state becomes macroscopically occupied. Making the *Bogoliubov approximation*, we can replace the field operator with a classical mean-field function $\hat{\Psi}_0(\mathbf{r}) = \hat{a}_0\,\psi_0(\mathbf{r}) \to \sqrt{N_0}\psi_0(\mathbf{r})$, where N_0 corresponds to the number of atoms in the ground state of the BEC. The perturbations are $\hat{\vartheta}(\mathbf{r},t) = \sum_{n\neq 0}\hat{a}_n(t)\,\psi_n(\mathbf{r})$, where $\hat{a}_n^\dagger$ and $\hat{a}_n$ are the creation and annihilation atom operators, satisfying the commutation relation $[\hat{a}_n, \hat{a}_l^\dagger] = \delta_{nl}$. To help solve the equations of motion, it is convenient to apply the *Bogoliubov transformation*

$$\hat{\vartheta}(\mathbf{r},t) = \sum_n\left(u_n(\mathbf{r})\hat{b}_n e^{-i\omega_n t} + v_n(\mathbf{r})\hat{b}_n^\dagger e^{i\omega_n t}\right), \tag{12}$$

where $\hat{b}_n^\dagger$ and $\hat{b}_n$ are the Bogoliubov mode creation and annihilation operators obeying $[\hat{b}_n, \hat{b}_l^\dagger] = \delta_{nl}$ and ω_n is the corresponding mode frequency. The mode functions $u_n(x)$, $v_n(x)$ follow the stationary Bogoliubov-de-Gennes equations and satisfy the orthogonality relation $\int(u_n^* u_l - v_n^* v_l)\,d^3x = \delta_{nl}$.

The energy spectrum is given by the dispersion relation

$$(\hbar\omega_n)^2 = (c_s\hbar k_n)^2 + (\hbar^2 k_n^2/2m)^2, \tag{13}$$

where

$$c_s := \sqrt{gn_0/m} = \sqrt{4\pi a_{sl} n_0}\hbar/m \tag{14}$$

is the speed of sound, n_0 is the BEC's density and $k_n = n\pi/L$ the mode number with $n \in \mathbb{Z}^+$. In the *low-energy limit* ($\hbar\omega_n \ll mc_s^2$), the dispersion relation is $\omega_n = c_s k_n$. The Bogoliubov modes in this limit correspond to phonons. In the interaction picture, we can rewrite the Hamiltonian as the sum of a diagonal part, $\hat{H}^{(0)} = \sum_n : \hbar\omega_n \hat{b}_n^\dagger \hat{b}_n :$ (where $::$ denotes normal ordering), plus an interaction term $\hat{H}^{(I)}$, that will be specified in what follows. We will consider resonant effects between the phononic modes and the oscillation frequency of the mass. In particular, we consider that the mode numbers satisfy the condition $n_\Omega := n + l = L\Omega/(\pi c_s)$, with n_Ω an odd integer, which corresponds to the resonance condition $\Omega \approx \omega_n + \omega_l$. In this case, the main contribution from the interaction Hamiltonian up to second order in the phonon operators under the rotating wave approximation [26]

$$\hat{H}^{(I)} = -\sum_{l<n_\Omega} i(-1)^{n_\Omega}|M_{nl}|a^{\mathrm{BEC}}\left(\hat{b}_n^\dagger\hat{b}_l^\dagger - \hat{b}_n\hat{b}_l\right), \tag{15}$$

for $n \neq l$, where the transition amplitude is given by

$$|M_{nl}| \approx \frac{mL^2(n^2+l^2)(1-(-1)^{n_\Omega})}{2\sqrt{2nl}(n^2-l^2)^2\pi^3\zeta}, \tag{16}$$

where $\zeta = \hbar/(\sqrt{2}mc_s)$ is the healing length.

The time evolution of the phonon modes is given by the operator $\hat{U}(t) = \exp(i\hat{H}^{(I)}t/\hbar)$, which explicitly reads

$$\hat{U}(t) = e^{\left[-\sum_{l<n_\Omega}(-1)^{n_\Omega}a^{\mathrm{BEC}}|M_{nl}|\left(\hat{b}_n^\dagger\hat{b}_l^\dagger - \hat{b}_n\hat{b}_l\right)t/\hbar\right]}. \tag{17}$$

This unitary operator corresponds to a two-mode squeezing transformation, parameterized by a^{BEC}. In the next section, we show how we can take advantage of this gravitational-induced evolution to estimate the value of a^{BEC}.

IV. QUANTUM METROLOGY

Quantum metrology provides strategies for optimizing the precision in estimating a physical parameter ϵ encoded by a unitary transformation $\hat{U}(\epsilon)$ in a system's quantum state [41–44]. Given the initial state of the system, called in this context the prove state, the optimal theoretical precision for measuring ϵ is obtained from the saturation of the quantum Cramér-Rao bound (QCRB)

$$\Delta\hat{\epsilon} \geq \frac{1}{\sqrt{N_m \mathcal{F}(\epsilon)}}, \tag{18}$$

where N_m is the number of measurements and $\mathcal{F}(\epsilon)$ is the quantum Fisher information (QFI) [45, 46]. The QCRB optimizes all positive-operator-valued measurement schemes and can be saturated for $N_m \to \infty$. When the measurement saturating the bound cannot be experimentally implemented, suboptimal viable measurements, such as heterodyne detection, can be carried out [43].

The QFI quantifies the distinguishability between two quantum states differing infinitesimally in the parameter ϵ. For Gaussian states, the QFI is easier to compute using the Covariance Matrix Formalism (CMF). This is a Phase space representation where a Gaussian state is completely determined by its first statistical moments given by the displacement vector $\boldsymbol{d}$, and its second moments, encoded in the covariance matrix $\mathbf{\Gamma}$. In the complex representation [47, 48], they are defined as:

$$\boldsymbol{d} \equiv \langle \hat{\mathbf{A}} \rangle, \tag{19a}$$

$$\Gamma_{ij} \equiv \langle \hat{\mathbf{A}}_i \hat{\mathbf{A}}_j^\dagger + \hat{\mathbf{A}}_j^\dagger \hat{\mathbf{A}}_i \rangle - 2\langle \hat{\mathbf{A}}_i \rangle \langle \hat{\mathbf{A}}_j^\dagger \rangle, \tag{19b}$$

where $\langle . \rangle$ denotes the expectation value of the state $\hat{\rho}$ and $\hat{\mathbf{A}} \equiv (\hat{A}_1, \ldots, \hat{A}_N; \hat{A}_1^\dagger, \ldots, \hat{A}_N^\dagger)^T$ is a $2N$ vector consisting of generic bosonic creation and annihilation operators in an N-dimensional Fock space. Using the CMF is very convenient here since Bogoliubov transformations are Gaussian. A Gaussian transformation $\hat{U}(\epsilon)$ preserves Gaussian states $\hat{\rho}'(\epsilon) = \hat{U}(\epsilon)\,\hat{\rho}(0)\,\hat{U}^\dagger(\epsilon)$. In the Phase space, the transformations correspond to symplectic matrices $\boldsymbol{S}(\epsilon)$ acting on displacement vector and the covariance matrix $\boldsymbol{d}'(\epsilon) = \boldsymbol{S}(\epsilon)\boldsymbol{d}(0)$ and $\mathbf{\Gamma}'(\epsilon) = \boldsymbol{S}(\epsilon)\mathbf{\Gamma}(0)\boldsymbol{S}^\dagger(\epsilon)$ [49].

In this formalism, the quantum Fisher information takes a simple form [50, 51]

$$\mathcal{F}(\epsilon) = \frac{1}{4}\mathrm{Tr}\left[\left(\mathbf{\Gamma}(\epsilon)^{-1}\dot{\mathbf{\Gamma}}(\epsilon)\right)^2\right] + 2\dot{\boldsymbol{d}}^\dagger(\epsilon)\mathbf{\Gamma}^{-1}(\epsilon)\dot{\boldsymbol{d}}(\epsilon). \tag{20}$$

where the dot represents the derivative with respect to ϵ and $\mathrm{Tr}[\cdot]$ is the trace of a matrix.

We implement a three-mode frequency interferometry scheme [22], depicted in Fig. 2, that involves the BEC ground state and two phonon modes. We select the modes n, l such that their sum resonates with the oscillation frequency of the mass.

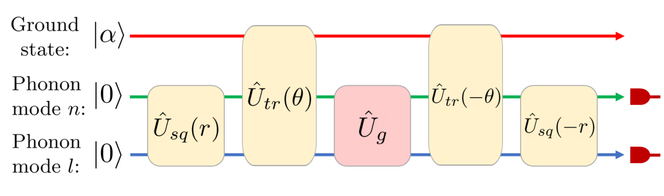

FIG. 2. The probe state is prepared by applying the transformation $\hat{U}_{\mathrm{sq}}(r)$ on two modes (represented by the green and blue lines) initially in the vacuum state and then mixing them with the BEC ground state (red line) through $\hat{U}_{\mathrm{tr}}(\theta)$. The gravitational parameters are encoded by $\hat{U}_g(a^{\mathrm{BEC}})$. To close the circuit, the inverse transformations are applied to the state. The number of phonons is counted at the output.

The ground state is well approximated by a coherent state $\hat{a}_0 |\alpha\rangle = \alpha_0 |\alpha\rangle$. For Bogoliubov transformations, squeezed states are known to be optimal probe states [52, 53]. Therefore, to prepare the probe state, we apply a two-mode squeezing transformation on the phonon modes, $\hat{U}_{\rm sq}(r) = e^{r(\hat{b}_n^\dagger \hat{b}_l^\dagger - \hat{b}_n \hat{b}_l)}$, where $r = |r|e^{i\vartheta_{\rm sq}}$ is the squeezing parameter and $\vartheta_{\rm sq} \in \mathbb{R}$ the squeezing phase. This operation parametrically populates the modes. Finally, to increase sensitivity, we mix the coherent state and the squeezed phonons with a tritter transformation $\hat{U}_{\rm tr}(\theta)$. The tritter transformation is generated from the Hamiltonian [32]

$$\hat{H}_{\rm tr}(\theta) = \frac{\hbar\theta}{\sqrt{2}}\left[e^{i\vartheta}\hat{a}_0^\dagger(\hat{b}_n + \hat{b}_l) + e^{-i\vartheta}\hat{a}_0(\hat{b}_n^\dagger + \hat{b}_l^\dagger)\right], \quad (21)$$

where $\theta, \vartheta \in \mathbb{R}$. In the Bogolibov approximation, $\hat{a}_0$ is replaced by $\sqrt{N_0}$. We assume that the number of condensed particles is fairly undepleted by the squeezing transformation and remains in a coherent state $|\alpha\rangle$, $N_0 = |\alpha_0^2|$. Also, we assume N_0 will remain reasonably undepleted after the tritter transformation. For this, θ cannot be too large, and in [22], it was shown that it has to fulfil relation (F3) in that work.

Once the oscillating mass is 'turned on', gravity acts on the phononic states via the two-mode squeezing transformation $\hat{U}_g$, given by (17). The change produced in the phonon states by gravity will be encoded by the parameter $a^{\rm BEC}$. To close the frequency interferometry scheme, we apply the inverse transformations $\hat{U}_{\rm sq}(-r)$ and $\hat{U}_{\rm tr}(-\theta)$. Finally, the trapping potential is released and single-atom detectors measure the velocities of the cloud of atoms, which then can be used to estimate the number of phonons in the final state [54].

Taking the optimum phase relationships $\vartheta_{\rm sq} = \pi/2$ and $\vartheta = \pi/4$, and assuming that $N \gg 0$, $\bar{N} \gg 0$ and $r \gg 1$, we find that the QFI is

$$\mathcal{F}(a^{\rm BEC}) \approx 8(|M_{nl}|t/\hbar)^2\theta^2 N_0 N, \quad (22)$$

which quantifies the change in the probe state produced by the gravitational force of the oscillating mass.

V. EXPERIMENTAL DETAILS AND RESULTS

To compute the sensitivity to the minimal acceleration Δa^{BEC}, we consider an experimental setup consisting of a one-dimensional ^{87}Rb BEC trapped by a uniform potential [40]. The mass of ^{87}Rb is $m = 1.44 \times 10^{-25}$ kg and the scattering length is $a_{sl} = 99\, r_B$ [55], where r_B ($\simeq 5.29 \times 10^{-11}$ m) is the Bohr's radius. The width-to-length ratio is given by $\alpha_{\rm WL} \leq 0.1$, computed from the ratio of axial to radial frequencies. This value is sufficient to ensure that phonons are constrained to move in one dimension [56]. The lengths reported for BECs range between ~ 50 μm up to 1000 μm [57, 58], while the number of condensed atoms N_0 can range between 1.6×10^3 and 1.1×10^9 atoms [59–61].

Previous work shows that the duration t of a single experiment is limited by the BEC and phonon half-lives, which are determined by two-body decay processes (Landau and Beliaev dampings) [62], three-body recombination [63, 64], and other sources of noise [22, 37]. These studies show that three-body recombination is the most limiting effect since it has the shortest time scale, $t_{\rm hl} = 3/(2Dn_0^2)$, where D is the decay constant [63]. For ^{87}Rb atoms, $D = 5.8 \times 10^{-30}$ cm^6 s^{-1} [65]. We consider small mode numbers since they enhance the sensitivity. Large mode numbers might also be difficult to populate and resolve due to the small phonon lifetimes [22].

We set the number of squeezed phonons to $N_p = 1100$ following the analysis done in [37]. This corresponds to a squeezing factor of 30.4 dB. While squeezing is well-established for photons in quantum optics [43, 66] and for number of phonons in quantum optomechanics [67, 68], the controlled creation and precise measurement of squeezing in the number of BEC phonons remains understudied. The phonons can be experimentally generated and squeezed by changing the atom-atom interactions or by periodically moving the trap boundaries, using an atomic version of the dynamical Casimir effect [37, 54]. While the experiment in [33] reports reaching 25 dB in the squeezing of the number of atoms in a BEC, further research is necessary to consistently reproduce these levels in the laboratory. High levels of phonon squeezing are theoretically possible [37–39]. However, this type of squeezing has been relatively underexplored in the laboratory [54, 69].

The following constraints must be taken into account to ensure Bose-Einstein condensation [29] and fulfil the experimental requirements. The BEC dilute regime requires that $n_0|a_{sl}|^3 \ll 1$ and the Bogoliubov approximation sets $N_{\rm exc} \ll N_0$, where the number of excited atoms in the n-mode is $N_{\rm exc} \approx (mc_s^2/\hbar\omega_n)N_p$. The modes l, n should fulfil the relation $\hbar\omega_{l,n} \ll mc_s^2$ to guarantee that the excitations are within the phonon regime, and the sum $l + n$ must be odd so that the phonon modes resonate with the oscillations driven by Eq. (15). The low-temperature regime requires $k_B T \ll \mu$, which is satisfied for regular experimental temperatures that can be as low as 0.5 nK for ^{87}Rb BECs [70]. The tritter angle must satisfy inequality (F3) from [22].

Considering that number of measurements is given by the integration (overall) time τ divided by the duration of a single experimental run t, $N_m = \tau/t$, and substituting Eq. (22) into Eq.(18) we obtain the sensitivity,

$$\Delta a^{\rm BEC} \approx \frac{\alpha_{WL}\hbar\pi^3\sqrt{2nl}(l^2 - n^2)^2}{16mN_0\theta\sqrt{La_{sl}\tau t N_p}(l^2 + n^2)}. \quad (23)$$

The values of the physical parameters that we consider in the numerical computation of the sensitivity are given in Table I. The table shows the sensitivity that is reached for three different values of the BEC length, where $\Delta a^{\rm BEC} = 4.8 \times 10^{-18}$ m/s^2 is the largest sensitivity obtained. The speed of sound (see Eq. (13)) is $c_s = 1.9$ mm/s for $n_0 = 10^{14}$ (cm^{-3}), and the respective frequency is given by, $\Omega = \pi c_s(n + l)/L = 17.7$Hz. The table also shows the

Parameter	Symbol	Values/range
Length	L (μm)	150, 500, 1000
BEC's width/L ratio	α_{WL}	0.3, 0.15, 0.05
Number of BEC atoms	N_0	10^9
Number of phonons	N_p	1100
Mode numbers	l, n	1,2
Single-experiment time	t (s)	1
Integration run time	τ(days)	60
^{87}Rb mass	m(kg)	1.44×10^{-25}
^{87}Rb scattering length	$a_{sl} = 99r_B$(m)	5.24×10^{-9}
Tritter angle	θ(rad)	0.31
Sensitivity $\times 10^{-18}$	$\Delta a^{\rm BEC}$(m/s^2)	74, 20, 4.8
Sensitivity $\times 10^{-17}$	ΔG(N m^2/kg^2)	37, 10, 2.3
Sensitivity $\times 10^{-31}$	$\Delta\Lambda$(1/m^2)	25, 6.7, 1.6

TABLE I. The set of parameters of BEC experiment (top part) which define the sensitivity (bottom part) to $\Delta a^{\rm BEC}$ (see Eq.(23)) for ^{87}Rb BEC. The bottom part of the table also provides the sensitivity to ΔG and $\Delta\Lambda$ derived from $\Delta a^{\rm BEC}$ and values of $R_0 = 100$mm, $\delta_R = 1$mm and M=100g using Eq.(24).

sensitivity to G and Λ derived from Eq. 9 and given by,

$$\Delta G = \Delta a^{BEC} \frac{R_0^3}{2M\delta_R}, \quad \Delta\Lambda = \Delta a^{\rm BEC} \frac{3}{\delta_R c^2} \ . \quad (24)$$

The relative accuracy $\Delta G/G$ obtained is of the order of 10^{-6}. This indicates that the BEC experiment has a unique potential to establish a new sensitivity to G, which is currently known with a relative uncertainty of $\Delta G/G \simeq 10^{-5}$ [71]. Moreover, as we demonstrate below, acceleration measurements at different values of R_0 will probe not only the functional dependence of the gravitational potential on distance but also further improve the accuracy of both G and Λ. It is important to note that in our derivations, we assume that the relative experimental uncertainties on the values of δ_R, R_0, and M are smaller than 10^{-6}, i.e. that the main uncertainty for ΔG is driven by $\Delta a^{\rm BEC}$.[2]

A special discussion regarding the sensitivity to Λ is warranted. As shown in Table I, the expected sensitivity to Λ is of the order of 10^{-30}m^{-2}, which would represent unprecedented accuracy for Earth-based experiments. However, it is important to note that this sensitivity should be regarded as an upper bound on Λ, still missing approximately 20 orders of magnitude compared to its actual value, as measured by the Planck experiment, $\Lambda = (1.09 \pm 0.028) \times 10^{-52}$m^{-2} [72].[3]

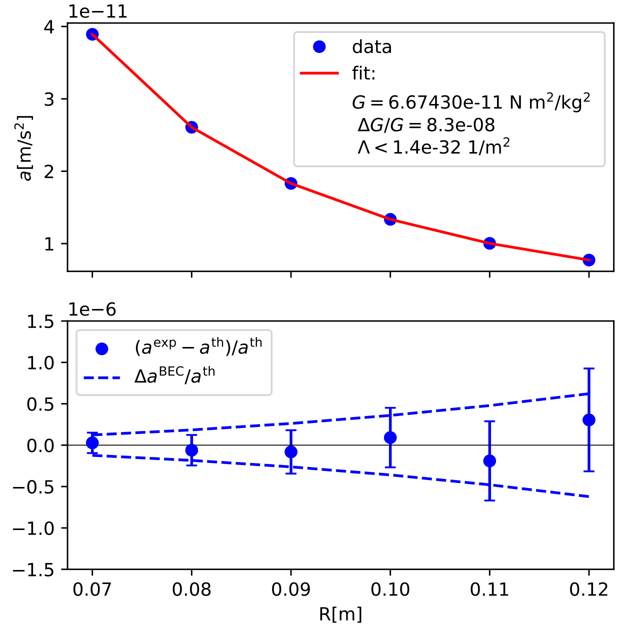

FIG. 3. Top: the example of the simulated experimental measurements of $a^{\rm BEC}$ for several R_0 values (blue circles) and their fit (red curve) to determine the value of G and the relative accuracy of its measurement as well as an upper limit on Λ. The value $\Delta a^{\rm BEC} = 4.8 \times 10^{-18}$ m/s^2 is assumed. Bottom: the relative deviation of the simulated experimental measurements of $a^{\rm exp}$ from its theoretical prediction $a^{\rm th}$ (blue circles). The blue dashed line presents $\Delta a^{\rm BEC}/a^{\rm th}$ ratio – the expected relative accuracy of $a^{\rm BEC}$ determination.

The accuracy of G and Λ measurements could be further improved when one studies the functional dependence of $a^{\rm BEC}$ versus R_0 and perform the respective fit. An example of such analysis is presented in Fig. 3 where we assume that measurements of $a^{\rm BEC}$ are performed for several values of R_0 and then fitted using Eq. 9, i.e. assuming the standard gravity model with G and Λ terms. The top panel of Fig. 3 presents simulated data (blue circles), the respective fit (red line), the value and uncertainty for G as well as the upper limit on Λ. The bottom panel shows deviations of "experimentally" measured values of $a^{\rm BEC}$, $a^{\rm exp}$, from the expected theory prediction, $a^{\rm th}$ (blue circles) with the errors consistent with the BEC sensitivity relative to $a^{\rm th}$ (blue dash line).[4] One can see the the fit further improves the uncertainty on G (reaching $\Delta G/G \simeq 10^{-7}$) and upper limit on Λ (reaching $\simeq 10^{-31}$m^{-2}).

[2] The uncertainties on δ_R, R_0, and M at $10^{-6} - 10^{-7}$ are achievable for our experimental setup, as suggested by CODATA [71] uncertainties for various experimental parameters. We thank Hendrik Ulbricht for valuable discussions and input on this point. In particular, to maintain the uncertainties for M at this level, it is necessary to place an oscillating mass in vacuum.

[3] We have converted [72] result from natural units to SI.

[4] The experimental errors on $a^{\rm exp}$ are not visible in the top panel since the relative error for $a^{\rm exp}$ is $\sim 10^{-6}$.

VI. CONCLUSIONS

We propose an Earth-based table-top experiment to precisely test the gravitational constant G and the cosmological constant Λ using a novel detector concept that exploits the dynamics of quantum phononic excitations in a trapped BEC, with sensitivity enhanced by approximately two orders of magnitude through a tritter operation.

As demonstrated by Gurzadyan's theorem, the fundamental constants we propose to test define the most general functional form of the gravitational potential that satisfies the equivalence of gravitational forces between a spherical mass and a point mass. The experiment we propose would probe G with an accuracy approximately two orders of magnitude better than current measurements. At the same time, this experiment would establish the best Earth-based upper limit on Λ, pioneering the first laboratory-based probe of the cosmological constant.

Our proposal explores the response of the collective modes of a BEC in the presence of an oscillating mass. Using quantum metrology, we derive an expression for the sensitivity in measuring the acceleration amplitude of the time-oscillating component of the gravitational potential. This assumes a probe state where the condensed atoms are in a coherent state, and two phonon modes are in a two-mode squeezed state.

For the experimental parameters given in Table I, we find that the proposed experiment is sensitive to accelerations on the order of $\simeq 10^{-17}$ m/s^2, which would improve the current accuracy of G measurements by about two orders of magnitude, achieving $\Delta G/G \simeq 10^{-7}$. This experiment would also establish the first Earth-based experimental upper limit on the cosmological constant Λ at approximately $10^{-31}\mathrm{m}^{-2}$. Moreover, the setup enables the measurement of the distance-dependent behaviour of each term in the gravitational potential, providing a new level of sensitivity for testing modified gravity theories.

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