# K-THEORY OF BERNOULLI SHIFTS OF FINITE GROUPS ON UHF-ALGEBRAS

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ABSTRACT. We show that the Bernoulli shift and the trivial action of a finite group G on a UHF-algebra of infinite type are KK<sup>G</sup>-equivalent and that the Bernoulli shift absorbs the trivial action up to conjugacy. As an application, we compute the K-theory of crossed products by approximately inner flips on classifiable C\*-algebras.

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#### 1. Introduction

In topological dynamics, a very fertile class of examples is given by Bernoulli shifts, that is, by the shift action of a group G on the product  $X^G := \prod_G X$  of G-many copies of a given compact space X. When the space X is moreover totally disconnected, the K-theory of the crossed product  $C(X^G) \rtimes_r G$  can be computed in many cases [CEL13]. These computations and the techniques appearing in them are not only of intrinsic interest, but they make possible the computation of the K-theory of C\*-algebras associated to large classes of (inverse) semigroups, wreath products, and many more examples [CEL13, Li19, Li22]. The simplest non-commutative analogue of a totally disconnected space is a UHF-algebra, that is, a (possibly infinite) tensor product of matrix algebras. The non-commutative version of the Bernoulli shift is the shift action of a group G on the tensor product  $A^{\otimes G} := \bigotimes_{q \in G} A$  for a given unital  $C^*$ -algebra A. Our main result computes

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the K-theory of the associated crossed product in the case that G is finite and that A is a UHF-algebra:

**Theorem A** (Theorem 2.8). Let G be a finite group, let Z be a G-set and let  $M_n$  be a UHF-algebra of infinite type. Then  $M_n$  is  $KK^G$ -equivalent to  $M_n^{\otimes Z}$  where we equip  $M_n$  with the trivial G-action and  $M_n^{\otimes Z}$  with the Bernoulli shift. In particular, we have

$$K_*\left(M_{\mathfrak{n}}^{\otimes Z} \rtimes G\right) \cong K_*(C^*(G) \otimes M_{\mathfrak{n}}) \cong K_*(C^*(G))[1/\mathfrak{n}].$$

The proof of Theorem A relies on a representation theoretic argument about invertibility of a certain element in the representation ring  $R_{\mathbb{C}}(G)$  after inverting sufficiently many primes (see Proposition 2.1). A byproduct of the proof is that the Bernoulli shift absorbs the trivial action not only in KK-theory, but up to conjugacy. We point out that this fact may alternatively be extracted from [HW08, Lemma 3.1].

**Theorem B** (Theorem 2.7). With the notation as in Theorem A, there is a G-equivariant isomorphism

$$M_n^{\otimes Z} \cong M_n \otimes M_n^{\otimes Z}$$
.

One immediate consequence of Theorem A and [Izu04, Theorem 3.13] is that the Bernoulli shift  $G \curvearrowright M_{\mathfrak{n}}^{\otimes Z}$  as above does not have the Rokhlin property (see Corollary 2.10). Beyond finite group actions, Theorem A also has consequences for infinite groups satisfying the Baum–Connes conjecture with coefficients [BCH94].

**Corollary C** (Corollary 2.11). Let G be a discrete group satisfying the Baum–Connes conjecture with coefficients, let Z be a G-set, let A be a G-C\*-algebra and let  $M_n$  be a UHF-algebra. Assume that Z is infinite or that  $\mathfrak n$  is of infinite type. Then the inclusion  $A \to A \otimes M_n$  induces an isomorphism

$$K_*\left(A \rtimes_r G\right)[1/\mathfrak{n}] \cong K_*\left(\left(A \otimes M_\mathfrak{n}^{\otimes Z}\right) \rtimes_r G\right).$$

*In particular, the right hand side is a*  $\mathbb{Z}[1/\mathfrak{n}]$ *-module.* 

Corollary C will be used in the follow-up paper [CEKN22] together with S. Chakraborty and S. Echterhoff to compute the K-theory of many more general Bernoulli shifts. Another consequence of Theorem A is that the Bernoulli shift of an amenable group G on a strongly self-absorbing (in the sense of [TW07]) C\*-algebra  $\mathcal D$  satisfying the UCT is KK<sup>G</sup>-equivalent to the trivial G-action on  $\mathcal D$  (see Corollary 2.12; for  $\mathcal D=\mathcal O_\infty$ , this is [Sza18, Corollary 6.9]).

In Section 3, we apply Theorem A and compute  $K_*$  ( $B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2$ ) whenever B is a C\*-algebra with approximately inner flip (in the sense of [ER78]) satisfying the assumptions of the Elliott classification programme<sup>1</sup>. Thanks to Tikuisis' classification of such C\*-algebras [Tik16], the computation reduces to the following special case (only the case  $\mathfrak{m}=1$  is relevant):

<sup>&</sup>lt;sup>1</sup>We refer to [Win18] and the references therein for an overview of the Elliott programme.

**Theorem D** (Theorem 3.2). For supernatural numbers  $\mathfrak n$  and  $\mathfrak m$  of infinite type, we have

$$\mathsf{K}_*\left(\mathfrak{F}_{\mathfrak{m},\mathfrak{n}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{r}}/\mathbb{Z}, & * = 0 \\ \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}, & * = 1 \end{cases}$$

where  $\mathfrak{r}$  is the greatest common divisor of  $\mathfrak{m}$  and  $\mathfrak{n}$ .

We refer to Section 3 for the definition of the notation appearing above. Our methods heavily build on Izumi's computation of K-theory of flip automorphisms on C\*-algebras with finitely generated K-theory [Izu19].

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#### 2. KK-THEORY OF BERNOULLI SHIFTS

For a finite group G, denote by  $R_{\mathbb{C}}(G)$  its representation ring, given by the free abelian group generated by all irreducible complex representations of G with the tensor product as multiplication. The character of a finite-dimensional complex representation  $\pi\colon G\to GL(V_\pi)$  is denoted by

$$\chi_{\pi} \colon \mathsf{G} o \mathbb{C}, \ \chi_{\pi}(g) \coloneqq \mathsf{tr}\left(V_{\pi} \xrightarrow{\pi(g)} V_{\pi}\right),$$

where tr denotes the (non-normalized) trace. Recall that the map

$$R_{\mathbb{C}}(G) \to \mathbb{C}_{class}(G), \ \pi \mapsto \chi_{\pi}$$

is an injective ring homomorphism with values in the algebra  $\mathbb{C}_{class}(G)$  of conjugation invariant functions on G with pointwise multiplication. There is a natural isomorphism  $R_{\mathbb{C}}(G) \cong KK^G(\mathbb{C},\mathbb{C})$ . We refer to [Ser77] for an introduction to representation theory of finite groups and to [Kas88] for the definition of equivariant KK-theory.

**Proposition 2.1.** Let G be a finite group, let  $k \ge 1$  and let Z be a finite G-set. Denote by  $\pi_k \colon G \to GL\left(\ell^2\left(\{1,\ldots,k\}^Z\right)\right)$  the permutation representation associated to the G-set  $\{1,\ldots,k\}^Z$ . Then the following hold.

- (1) There exist  $\alpha \in R_{\mathbb{C}}(G)$  and  $r \ge 1$  such that  $[\pi_k]^r = k\alpha$ .
- (2) There exist  $\beta \in R_{\mathbb{C}}(G)$  and  $l \geq 1$  such that  $[\pi_k] \cdot \beta = k^l$ .

*Proof.* By considering the standard basis in  $\ell^2(\{1,\ldots,k\}^Z)$ , it is easy to see that the trace of  $\pi_k(g)$  for  $g \in G$  is given by the number of g-fixed points in  $\{1,\ldots,k\}^Z$ . In other words, the character of  $\pi_k$  is given by

$$\chi_{\pi_k}(g) = k^{|Z/\langle g \rangle|}.$$

We therefore have

$$\prod_{g\in G}\left(\chi_{\pi_k}-k^{|Z/\langle g\rangle|}\right)=0 \text{ in } \mathbb{C}_{\text{class}}(G).$$

Since the map  $\pi \mapsto \chi_{\pi}$  is injective, we also have

$$\prod_{g \in G} \left( [\pi_k] - k^{|Z/\langle g \rangle|} \right) = 0 \text{ in } R_{\mathbb{C}}(G).$$

In particular, there are polynomials  $p, q \in \mathbb{Z}[t]$  satisfying

$$[\pi_k]^{|G|} = kp([\pi_k]), \quad [\pi_k] \cdot q([\pi_k]) = \prod_{g \in G} k^{|Z/\langle g \rangle|},$$

which proves the proposition.

**Definition 2.2.** Let Z be a set and let  $(A_z)_{z\in Z}$  be a collection of unital  $C^*$ -algebras. The infinite tensor product  $\bigotimes_{z\in Z} A_z$  is defined as

$$\bigotimes_{z\in\mathsf{Z}}\mathsf{A}_z\coloneqq\varinjlim_{\mathsf{F}}\bigotimes_{z\in\mathsf{F}}\mathsf{A}_z,$$

where the inductive limit is taken over all finite subsets  $F \subseteq Z$  with respect to the connecting maps  $a \mapsto a \otimes 1$ . Given a discrete group G, a unital  $C^*$ -algebra A and a G-set Z, the *Bernoulli shift* of G on  $A^{\otimes Z} := \bigotimes_Z A$  is the G-action induced by permuting the tensor factors according to the G-action on Z.

Countability of G and of a G-set Z and separability of A will be assumed only if the statement involves KK-theory. For example, we do not assume these when the statement is on K-theory of involved C\*-algebras.

**Definition 2.3.** A supernatural number is a formal product  $\mathfrak{n}=\prod_{\mathfrak{p}}\mathfrak{p}^{n_{\mathfrak{p}}}$  where  $\mathfrak{p}$  runs over all primes and  $n_{\mathfrak{p}}\in\{0,\ldots,\infty\}$ . The *UHF-algebra* associated to  $\mathfrak{n}$  is the infinite tensor product

$$M_{\mathfrak{n}} := \bigotimes_{\mathfrak{p}} M_{\mathfrak{p}^{\mathfrak{n}_{\mathfrak{p}}}},$$

with  $M_{p^\infty}:=M_p^{\otimes \mathbb{N}}$ . We call  $\mathfrak n$  or  $M_\mathfrak n$  of infinite type if  $n_p\in\{0,\infty\}$  for all  $\mathfrak p$ . We say that  $\mathfrak n=\prod \mathfrak p^{n_p}$  divides  $\mathfrak m=\prod \mathfrak p^{m_p}$  if  $n_p\leq m_p$  for all  $\mathfrak p$ .

*Remark* 2.4. Note that the above definition includes natural numbers and matrix algebras as a special case.

**Definition 2.5.** If M is an abelian group, we denote by M[1/n] the inductive limit of the system

$$M \xrightarrow{\cdot p_1} M \xrightarrow{\cdot p_2} M \xrightarrow{\cdot p_3} \cdots$$

where  $(p_1, p_2, ...)$  contains each prime dividing  $\mathfrak{n}$  infinitely many times.

*Remark* 2.6. If  $\mathfrak{q} = \prod_{\mathfrak{p}} \mathfrak{p}^{\mathfrak{n}_{\mathfrak{p}}}$  with  $\mathfrak{n}_{\mathfrak{p}} \geq 1$  for all  $\mathfrak{p}$ , then

$$M[1/\mathfrak{q}] \cong M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

If  $k \ge 1$  is a positive integer, then

$$\mathbb{Z}[1/k] \cong \left\{\frac{\mathfrak{m}}{k^{\mathfrak{n}}} \middle| \mathfrak{m} \in \mathbb{Z}, \mathfrak{n} \in \mathbb{Z}_{>0} \right\} \subseteq \mathbb{Q}.$$

In general, the group  $\mathbb{Z}[1/\mathfrak{n}]$  is different from the closely related group

$$\mathbb{Q}_{\mathfrak{n}} \coloneqq \left\{ \frac{\mathfrak{m}}{k} \middle| \mathfrak{m} \in \mathbb{Z}, k \in \mathbb{Z}_{>0} \text{ divides } \mathfrak{n} \right\},\,$$

unless n is of infinite type.

**Theorem 2.7** (cf. [HW08, Lemma 3.1]). Let G be a finite group, let  $M_n$  be a UHF-algebra and let Z be a G-set. Assume that Z is infinite or that  $\mathfrak n$  is of infinite type. Equip  $M_n$  with the trivial G-action and  $M_n^{\otimes Z}$  with the Bernoulli shift. Then there is an equivariant isomorphism

$$M_n^{\otimes Z} \otimes M_n \cong M_n^{\otimes Z}$$
.

If Z is infinite, and  $m < \infty$ , there is an equivariant isomorphism

$$M_{\mathfrak{m}}^{\otimes Z} \otimes M_{\mathfrak{m}^{\infty}} \cong M_{\mathfrak{m}}^{\otimes Z}.$$

*Proof.* Note that it suffices to prove the statement in the case that  $M_n = M_{p^k}$  (or  $M_m = M_{p^k}$ ) for a prime p and  $k \in \{0,1,\ldots,\infty\}$ . As before, if Z is finite, we denote by  $\pi_p$  the permutation representation of G on  $V_p := \ell^2(\{1,\ldots,p\}^Z)$ , so that  $M_p^{\otimes Z}$  is equivariantly isomorphic to  $End(V_p)$ .

Assume first that  $k=\infty$ . We only need to prove the theorem for (any) one G-orbit of Z so we may assume that Z is finite. Let  $\alpha \in R_{\mathbb{C}}(G)$  and  $r \geq 1$  be as in Proposition 2.1 so that  $[\pi_p]^r = p\alpha \in R_{\mathbb{C}}(G)$ . Since  $[\pi_p]$  is a non-negative linear combination of irreducible representations of G,  $\alpha$  has to be the class of a finite-dimensional representation  $\pi_\alpha \colon G \to GL(W_\alpha)$ . In particular, we have an equivariant isomorphism  $V_p^{\otimes r} \cong \mathbb{C}^p \otimes W_\alpha$ . Passing to endomorphisms, we obtain an equivariant isomorphism

$$\left(M_{\mathfrak{p}}^{\otimes Z}\right)^{\otimes r} \cong M_{\mathfrak{p}} \otimes End(W_{\alpha})$$

with the trivial G-action on  $M_p$ . By taking the infinite tensor product we obtain an equivariant isomorphism

$$M_{p^\infty}^{\otimes Z} \cong M_{p^\infty} \otimes End(W_\alpha)^{\otimes \mathbb{N}} \cong M_{p^\infty} \otimes M_{p^\infty} \otimes End(W_\alpha)^{\otimes \mathbb{N}} \cong M_{p^\infty} \otimes M_{p^\infty}^{\otimes Z}.$$

Assume now that  $k < \infty$  and that Z is infinite. Then Z contains infinitely many orbits of the same type. We may thus assume that Z is of the form  $Z = \bigsqcup_{\mathbb{N}} G/H$  for some subgroup  $H \subseteq G$ . Then there is an equivariant isomorphism  $M_{p^k}^{\otimes Z} \cong M_{p^\infty}^{\otimes G/H}$ . This reduces the proof to the case considered above.

**Theorem 2.8.** Let G be a finite group, let Z be a countable G-set and let  $M_n$  be a UHF-algebra of infinite type. Then the canonical inclusions

$$M_{\mathfrak{n}} \hookrightarrow M_{\mathfrak{n}} \otimes M_{\mathfrak{n}}^{\otimes Z} \hookleftarrow M_{\mathfrak{n}}^{\otimes Z}$$

are  $KK^G$ -equivalences, where  $M_n$  is endowed with the trivial action and where  $M_n^{\otimes Z}$  is endowed with the Bernoulli shift. If Z is infinite, and  $m < \infty$ , the same conclusion holds for the inclusions

$$M_{\mathfrak{m}^{\infty}} \hookrightarrow M_{\mathfrak{m}^{\infty}} \otimes M_{\mathfrak{m}}^{\otimes Z} \hookleftarrow M_{\mathfrak{m}}^{\otimes Z}.$$

*Proof.* It follows from Theorem 2.7 and the fact that  $M_n$  is strongly self-absorbing (in the sense of [TW07]) that the map

$$M_n^{\otimes Z} \hookrightarrow M_n \otimes M_n^{\otimes Z}$$

is a  $KK^G$ -equivalence. Similarly, if Z is infinite and  $\mathfrak{m} < \infty$ , the map

$$M_{\mathfrak{m}}^{\otimes Z} \hookrightarrow M_{\mathfrak{m}^{\infty}} \otimes M_{\mathfrak{m}}^{\otimes Z}$$

is a KK<sup>G</sup>-equivalence. We prove that the map

$$id_{\mathsf{M}_{\mathfrak{n}}} \otimes 1_{\mathsf{M}_{\mathfrak{n}}^{\otimes \mathsf{Z}}} \colon \mathsf{M}_{\mathfrak{n}} \to \mathsf{M}_{\mathfrak{n}} \otimes \mathsf{M}_{\mathfrak{n}}^{\otimes \mathsf{Z}}$$

is a KK<sup>G</sup>-equivalence. Note that this map is the inductive limit of the maps

$$id_{\mathsf{M}_{\mathfrak{n}}} \otimes 1_{\mathsf{M}_{k}^{\otimes Y}} \colon \mathsf{M}_{\mathfrak{n}} \to \mathsf{M}_{\mathfrak{n}} \otimes \mathsf{M}_{k}^{\otimes Y}$$

where k ranges over all positive integers that divide n and where Y ranges over all finite G-subsets of Z. It follows from the finiteness of G, the nuclearity of the involved algebras and [MN06, Proposition 2.6, Lemma 2.7] that the map in (2.1) is also the homotopy colimit (with respect to the triangulated structure of  $KK^G$ ) of the maps in (2.2). Since a homotopy colimit of  $KK^G$ -equivalences is a  $KK^G$ -equivalence  $^2$ , it suffices to show that the maps appearing in (2.2) are  $KK^G$ -equivalences. Note that the maps in (2.2) can be identified with the elements  $[id_{M_n}] \otimes_{\mathbb{C}} [\pi_k] \in KK^G(M_n, M_n)$ , where  $\pi_k \colon G \to GL\left(\ell^2\left(\{1,\ldots,k\}^Y\right)\right)$  is the permutation representation and  $[\pi_k]$  is its class in  $KK^G(\mathbb{C},\mathbb{C})$ . By Proposition 2.1, there is an element  $\beta \in KK^G(\mathbb{C},\mathbb{C})$  and  $l \geq 1$  such that  $[\pi_k]\beta = k^l$ . Thus  $[id_{M_n}] \otimes_{\mathbb{C}} [\pi_k]$  is invertible with inverse  $\frac{1}{k^l}[id_{M_n}] \otimes_{\mathbb{C}} \beta$ . The same proof shows that, if Z is infinite and  $m < \infty$ , the map

$$id_{M_{\mathfrak{m}^{\infty}}} \otimes 1_{M_{\mathfrak{m}}^{\otimes Z}} \colon M_{\mathfrak{m}^{\infty}} \to M_{\mathfrak{m}^{\infty}} \otimes M_{\mathfrak{m}}^{\otimes Z}$$

is a KK<sup>G</sup>-equivalence.

Remark 2.9. By [GL21, Theorem B] and [GHV22, Theorem B], a countable discrete group G is amenable if and only if for some (any) supernatural number  $\mathfrak{n}\neq 1$  of infinite type, the Bernoulli shift on  $M_\mathfrak{n}^{\otimes G}$  absorbs the trivial action on the Jiang-Su algebra  $\mathcal{Z}$  up to cocycle conjugacy. In particular (since  $M_\mathfrak{n}\cong M_\mathfrak{n}\otimes \mathcal{Z}$ ), the conclusion of Theorem 2.7 is false for

<sup>&</sup>lt;sup>2</sup>This follows from the axioms of a triangulated category. The fact that homotopy colimits of maps are not unique does not cause a problem here.

non-amenable groups. On the other hand, Theorem 2.8 together with the Higson-Kasparov Theorem [HK01] (applied in the form of [MN06, Theorem 8.5]) implies that if G is an amenable group, then  $M_n^{\otimes G}$  absorbs the trivial action on  $M_n$  up to KK<sup>G</sup>-equivalence. It is thus conceivable that a discrete group G is amenable if and only if the Bernoulli shift on  $M_n^{\otimes G}$  absorbs the trivial action on  $M_n$  up to cocycle conjugacy.

The following observation provides some evidence for this: Let G be an amenable group,  $n \neq 1$  a supernatural number of infinite type, and A a G-C\*-algebra. By the remarks above, the unital embedding

$$id\otimes 1\colon (A\otimes M_{\mathfrak{n}}^{\otimes G})\rtimes G\hookrightarrow (A\otimes M_{\mathfrak{n}}^{\otimes G})\rtimes G\otimes M_{\mathfrak{n}}$$

is a KK-equivalence between  $\mathcal{Z}$ -stable  $C^*$ -algebras that induces an isomorphism on the trace spaces, in particular it induces an isomorphism on the Elliott invariants. If we additionally assume that  $(A \otimes M_n^{\otimes G}) \rtimes G$  is simple, separable, nuclear, and satisfies the UCT (which happens in many cases of interest), then the classification of unital, simple, separable, nuclear,  $\mathcal{Z}$ -stable  $C^*$ -algebras satisfying the UCT [Phi00, EGLN15, TWW17, CET $^+$ 21] implies that  $(A \otimes M_n^{\otimes G}) \rtimes G \cong (A \otimes M_n) \rtimes G \otimes M_n$ . This condition is certainly necessary for  $M_n^{\otimes G}$  to absorb  $M_n$  up to cocycle conjugacy.

**Corollary 2.10.** Let  $G \neq \{e\}$  be a finite group, let Z be a G-set and let  $M_n$  be a UHF-algebra of infinite type. Then the Bernoulli shift of G on  $M_n^{\otimes Z}$  does not have the Rokhlin property.

*Proof.* Assume the contrary. Then [Izu04, Theorem 3.13] yields an isomorphism<sup>3</sup>

$$K_0\left(M_{\mathfrak{n}}^{\otimes Z} \rtimes G\right) \cong K_0\left(M_{\mathfrak{n}}^{\otimes Z}\right) = \mathbb{Z}[1/\mathfrak{n}].$$

On the other hand, Theorem 2.8 yields an isomorphism

$$\mathsf{K}_0\left(\mathsf{M}_\mathfrak{n}^{\otimes \mathsf{Z}} \rtimes \mathsf{G}\right) \cong \mathsf{K}_0(\mathsf{C}^*(\mathsf{G})) \otimes_{\mathbb{Z}} \mathbb{Z}[1/\mathfrak{n}] \cong \mathbb{Z}[1/\mathfrak{n}]^{\oplus \hat{\mathsf{G}}},$$

The following corollary will be used in the follow-up paper [CEKN22] with Sayan Chakraborty and Siegfried Echterhoff. We refer to [BCH94] for the formulation of the Baum–Connes conjecture with coefficients. Note that the Baum–Connes conjecture with coefficients holds for many groups, including a-T-menable groups [HK01] and hyperbolic groups [Laf12].

**Corollary 2.11.** Let G be a countable discrete group satisfying the Baum–Connes conjecture with coefficients, let Z be a G-set, let A be a G-C\*-algebra and let  $M_{\mathfrak{n}}$  be a UHF-algebra. Assume that Z is infinite or that  $\mathfrak{n}$  is of infinite type. Then the inclusion  $A \to A \otimes M_{\mathfrak{n}}$  induces an isomorphism

$$K_*\left(A\rtimes_r G\right)[1/\mathfrak{n}]\cong K_*\left(\left(A\otimes M_\mathfrak{n}^{\otimes Z}\right)\rtimes_r G\right).$$

<sup>&</sup>lt;sup>3</sup>Theorem 3.13 of [Izu04] is applicable by the combination of [Phi87, Proposition 7.1.3] and [Kis81, Theorem 3.1].

*In particular, the right hand side is a*  $\mathbb{Z}[1/\mathfrak{n}]$ *-module.* 

*Proof.* By an inductive limit argument, we may assume Z is countable and A is separable. If G is finite, the statement follows from Theorem 2.8 considering the commutative diagram

$$(2.3) \qquad A \longrightarrow A \otimes M_{\mathfrak{n}}^{\otimes Z} \xrightarrow{\varphi_1} A \otimes M_{\mathfrak{n}^{\infty}} \otimes M_{\mathfrak{n}^{\infty}}^{\otimes Z}$$

$$A \otimes M_{\mathfrak{n}^{\infty}}$$

where  $\phi_1, \phi_2$  are  $KK^G$ -equivalences. Assume now that G is infinite. Consider the diagram (2.3). We know that (the restrictions of)  $\phi_1, \phi_2$  are  $KK^H$ -equivalences for every finite subgroup  $H \subseteq G$ . Since G satisfies the Baum–Connes conjecture with coefficients, the results of [CEOO4] (see also [MN06]) imply that  $\phi_1$  and  $\phi_2$  induce isomorphisms of the K-theory groups of reduced crossed products by G. The statement follows from this by identifying  $K_*((A \otimes M_{n^\infty}) \rtimes_T G) \cong K_*(A \rtimes_T G)[1/n]$ .

We end this section with an application to Bernoulli shifts on strongly self-absorbing C\*-algebras. Recall that a separable, unital C\*-algebra  $\mathcal{D} \neq \mathbb{C}$  is strongly self-absorbing [TW07] if there is an isomorphism  $\mathcal{D} \cong \mathcal{D} \otimes \mathcal{D}$  which is approximately unitarily equivalent to the first factor inclusion  $\mathrm{id}_{\mathcal{D}} \otimes 1_{\mathcal{D}} \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ . Strongly self-absorbing C\*-algebras are automatically simple, nuclear [TW07] and  $\mathcal{Z}$ -stable [Win11]. By the combination of [TW07, Proposition 5.1] and the classification of unital, simple, separable, nuclear,  $\mathcal{Z}$ -stable C\*-algebras in the UCT class [Phi00, EGLN15, TWW17, CET+21], a complete list of strongly self-absorbing C\*-algebras satisfying the UCT is given by

$$(2.4) 2, M_{\mathfrak{n}}, \mathcal{O}_{\infty}, \mathcal{O}_{\infty} \otimes M_{\mathfrak{n}}, \mathcal{O}_{2},$$

where  $n \neq 1$  is a supernatural number of infinite type. The following corollary is a generalization of [Sza18, Corollary 6.9]:

**Corollary 2.12.** Let  $\mathfrak{D}$  be a strongly self-absorbing  $C^*$ -algebra satisfying the UCT and let G be a discrete group having a  $\gamma$ -element equal to  $I^4$ . Then, for any G-set Z, the G- $C^*$ -algebra  $\mathfrak{D}^{\otimes Z}$  equipped with the Bernoulli shift is  $KK^G$ -equivalent to  $\mathfrak{D}$  equipped with the trivial G-action.

For the proof, we need the following result of Izumi [Izu19] which we spell out here for later reference.

**Theorem 2.13** ([Izu19, Theorem 2.1], see also [Sza18, Lemma 6.8]). Let A, B be separable nuclear C\*-algebras, let H be a finite group and let Z be a finite H-set. Then, there is a map from KK(A,B) to  $KK^H(A^{\otimes Z},B^{\otimes Z})$  which in particular, sends a class of a \*-homomorphism  $\varphi$  to the class of  $\varphi^{\otimes Z}$ . Furthermore, this map

 $<sup>^4</sup>$ See [MN06, Section 7] for a definition of the  $\gamma$ -element. By the Higson–Kasparov theorem [HK01], this assumption is satisfied for all a-T-menable groups.

is compatible with the compositions and in particular sends a KK-equivalence to a KK<sup>H</sup>-equivalence. In particular, the Bernoulli shifts on  $A^{\otimes Z}$  and  $B^{\otimes Z}$  are KK<sup>H</sup>-equivalent if A and B are KK-equivalent.

Proof of Corollary 2.12. We claim that the unital embeddings

$$\mathfrak{D} \hookrightarrow \mathfrak{D} \otimes \mathfrak{D}^{\otimes \mathsf{Z}} \hookleftarrow \mathfrak{D}^{\otimes \mathsf{Z}}$$

are  $KK^G$ -equivalences. By the assumption on G, this amounts to showing that they are  $KK^H$ -equivalences for every finite subgroup  $H \subseteq G$ . (see [MN06, Theorem 7.3]). By the same homotopy co-limit argument as in the proof of Theorem 2.8, it is enough to show that the maps

$$\mathfrak{D} \hookrightarrow \mathfrak{D} \otimes \mathfrak{D}^{\otimes Y} \hookleftarrow \mathfrak{D}^{\otimes Y}$$

are  $KK^H$ -equivalences for all finite H-subsets Y of Z. Now Theorem 2.13 allows us to replace  $\mathcal D$  by a KK-equivalent C\*-algebra. Thanks to the list (2.4), this reduces the problem to the cases  $\mathcal D=\mathbb C$ ,  $\mathcal D=0$  and  $\mathcal D=M_n$ . The first two cases are trivial and the third one follows from Theorem 2.8.

#### 3. K-THEORY OF APPROXIMATELY INNER FLIPS

In this section we apply Theorem 2.8 to the K-theory of approximately inner flips. Recall that a C\*-algebra B is said to have approximately inner flip if the flip automorphism  $B\otimes B\to B\otimes B$ ,  $a\otimes b\mapsto b\otimes a$  is approximately inner, i.e. a point-norm limit of inner automorphisms. A C\*-algebra B with approximately inner flip must be simple, nuclear and have at most one trace [ER78]. An approximately inner flip necessarily induces the identity map on  $K_*(B\otimes B)$  and this largely restricts the class of C\*-algebras B with approximately inner flip. Effros and Rosenberg [ER78] showed that if B is AF, then B must be stably isomorphic to a UHF-algebra. Tikuisis [Tik16] determined a complete list of classifiable C\*-algebras with approximately inner flip. We would like to thank Dominic Enders, André Schemaitat and Aaron Tikuisis for informing us about a corrigendum stated below:

**Theorem 3.1.** ([EST22], Corrigendum to [Tik16, Theorem 2.2]) Let B be a separable, unital C\*-algebra with strict comparison, in the UCT class, which is either infinite or quasi-diagonal. The following are equivalent.

- (1) B has approximately inner flip;
- (2) B is Morita equivalent to one of the following C\*-algebras:
  - C;
  - $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$ ;
  - $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}\otimes \mathfrak{O}_{\infty}$ ;
  - $\bullet \mathcal{F}_{1,\mathfrak{m}}$

Here  $\mathfrak{m},\mathfrak{n}$  are supernatural numbers of infinite type such that  $\mathfrak{m}$  divides  $\mathfrak{n},\mathfrak{O}_{\infty}$  is the Cuntz algebra on infinitely many generators,  $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$  is the simple, separable, unital,  $\mathbb{Z}$ -stable, quasi-diagonal C\*-algebra in the UCT class with unique trace satisfying

$$K_0(\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}) \cong \mathbb{Q}_{\mathfrak{n}}, \ [1]_0 = 1, \ K_1(\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}) \cong \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z},$$

and  $\mathfrak{F}_{\mathfrak{m},\mathfrak{n}}$  is the unique unital Kirchberg algebra in the UCT class satisfying

$$K_0(\mathcal{F}_{\mathfrak{m},\mathfrak{n}}) \cong \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, \quad [1]_0 = 0, \quad K_1(\mathcal{F}_{\mathfrak{m},\mathfrak{n}}) \cong \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}.$$

Our strategy to compute the groups  $K_*(B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2)$  with B as in Theorem 3.1 builds on Izumi's remarkable computation of  $K_*(A^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2)$  for all separable nuclear C\*-algebras A in the UCT class and with finitely generated K-theory [Izu19]. Izumi's starting point is his Theorem 2.13 above. This allows him to replace the appearing C\*-algebras by finite direct sums of building blocks of the form  $\mathbb{C}$ ,  $C_0(\mathbb{R})$ ,  $\mathcal{O}_{n+1}$  and  $D_n$ , where  $\mathcal{O}_{n+1}$  denotes the Cuntz algebra on n+1 generators and where  $D_n$  denotes the dimension drop algebra. For B one of these building blocks, Izumi explicitly computes  $K_*(B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2)$ .

We follow the same strategy here. Thanks to Theorems 3.1 and 2.13 above, the following theorem and its corollary completely determine the K-theory groups  $K_*$  ( $B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2$ ) whenever B is a UCT C\*-algebra that is KK-equivalent to a classifiable C\*-algebra with an approximately inner flip.

**Theorem 3.2.** Let  $\mathfrak{m}$  and  $\mathfrak{n}$  be supernatural numbers of infinite type. Let  $F_{\mathfrak{m},\mathfrak{n}}$  be any  $C^*$ -algebra satisfying the UCT such that

$$K_*(F_{\mathfrak{m},\mathfrak{n}}) \cong \begin{cases} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, & *=0 \\ \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}, & *=1 \end{cases}.$$

Then we have

$$\mathsf{K}_*\left(\mathsf{F}_{\mathfrak{m},\mathfrak{n}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2
ight) \cong egin{cases} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{r}}/\mathbb{Z}, & *=0 \ \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{n}}/\mathbb{Z}, & *=1 \end{cases},$$

where  $\mathfrak{r}$  is the greatest common divisor of  $\mathfrak{m}$  and  $\mathfrak{n}$ .

**Corollary 3.3.** For any supernatural numbers m and n of infinite type, we have

$$K_*\left(\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Q}_\mathfrak{n} \oplus \mathbb{Q}_\mathfrak{n}, & *=0 \\ \mathbb{Q}_{\mathfrak{m}\mathfrak{n}}/\mathbb{Q}_\mathfrak{n} \oplus \mathbb{Q}_\mathfrak{m}/\mathbb{Z} \oplus \mathbb{Q}_\mathfrak{m}/\mathbb{Z}, & *=1 \end{cases}.$$

*Proof.* The algebra  $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}$  is KK-equivalent to  $M_{\mathfrak{n}}\oplus \mathcal{F}_{1,\mathfrak{m}}$ . Thus, using Theorem 2.13, we see that  $\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}^{\otimes \mathbb{Z}/2}\rtimes \mathbb{Z}/2$  is KK-equivalent to

$$(M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2) \oplus (\mathfrak{F}_{1,\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2) \oplus (M_{\mathfrak{n}} \otimes \mathfrak{F}_{1,\mathfrak{m}}).$$

Note that  $K_0(M_{\mathfrak{n}}\otimes \mathfrak{F}_{1,\mathfrak{m}})\cong 0$  and  $K_1(M_{\mathfrak{n}}\otimes \mathfrak{F}_{1,\mathfrak{m}})\cong \mathbb{Q}_{\mathfrak{n}}\otimes_{\mathbb{Z}}\mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}\cong \mathbb{Q}_{\mathfrak{mn}}/\mathbb{Q}_{\mathfrak{n}}$ . The assertion now follows from Theorem 3.2.

We break up the proof of Theorem 3.2 into two lemmas. We denote by  $[e_0], [e_1], [1] \in K_0(C^*(\mathbb{Z}/2))$  the classes of the trivial representation, the sign representation and the unit of  $C^*(\mathbb{Z}/2)$ . We will abuse notation and write KK-elements as arrows between  $C^*$ -algebras, well-aware that they might not be induced by \*-homomorphisms.

Lemma 3.4. We have

$$K_*\left(F_{\mathfrak{m},1}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, & *=0 \\ 0, & *=1 \end{cases}.$$

*Proof.* By the Kirchberg–Phillips classification theorem [Phi00], there is a unital \*-homomorphism

$$M_{\mathfrak{m}}\otimes \mathfrak{O}_{\infty} \to \mathfrak{F}_{\mathfrak{m},1}$$

such that the composition

$$\phi\colon M_{\mathfrak{m}}\xrightarrow{\mathrm{id}\,\otimes 1} M_{\mathfrak{m}}\otimes \mathfrak{O}_{\infty}\to \mathfrak{F}_{\mathfrak{m},1}$$

induces the canonical quotient map  $\mathbb{Q}_{\mathfrak{m}} \to \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}$  on  $K_0$ . Denote by

$$M_{\varphi} = \{(\alpha,f) \in M_{\mathfrak{m}} \oplus (C[0,1] \otimes \mathfrak{F}_{\mathfrak{m},1}) \mid \varphi(\alpha) = f(0)\}$$

the mapping cylinder of  $\varphi$  and by  $C_{\varphi} \coloneqq \ker(\mathrm{ev}_1 \colon M_{\varphi} \to \mathcal{F}_{\mathfrak{m},1})$  the mapping cone of  $\varphi$ . Note that the inclusion  $M_{\mathfrak{m}} \hookrightarrow M_{\varphi}$  is a KK-equivalence. The short exact sequence

$$(3.1) 0 \longrightarrow C_{\phi} \longrightarrow M_{\phi} \longrightarrow \mathcal{F}_{\mathfrak{m},1} \longrightarrow 0,$$

induces an exact sequence

$$(3.2) 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q}_{m} \longrightarrow \mathbb{Q}_{m}/\mathbb{Z} \longrightarrow 0$$

on  $K_0$  and 0 on  $K_1$ . We streamline the notations and re-write the exact sequence (3.1) as

$$0 \longrightarrow I \longrightarrow B \longrightarrow A_0 \longrightarrow 0$$
.

The only properties that we will use are the induced sequence (3.2) on  $K_0$  and that the map  $I \to B$  can be identified with the unital inclusion  $\mathbb{C} \hookrightarrow M_m$  in KK-theory. From now on, we follow the beautiful computations of [Izu19, Theorem 3.4]. Writing  $I_1 \coloneqq I \otimes B + B \otimes I \subseteq B^{\otimes \mathbb{Z}/2}$ , we have the following short exact sequences of  $\mathbb{Z}/2$ -C\*-algebras

$$0 \longrightarrow I_1 \longrightarrow B^{\otimes \mathbb{Z}/2} \longrightarrow A_0^{\otimes \mathbb{Z}/2} \longrightarrow 0,$$

$$(3.4) 0 \longrightarrow I^{\otimes \mathbb{Z}/2} \longrightarrow I_1 \longrightarrow (I \otimes A_0) \oplus (A_0 \otimes I) \longrightarrow 0.$$

Taking crossed-products and applying K-theory for (3.4) produces the 6-term exact sequence

$$(3.5) \qquad \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathsf{K}_0(\mathrm{I}_1 \rtimes \mathbb{Z}/2) \longrightarrow \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} .$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

Here the generators of  $\mathbb{Z} \oplus \mathbb{Z}$  are the image of [1] and  $[e_1]$  via the KK-equivalence  $\mathbb{C}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \to \mathrm{I}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2$  obtained from the KK-equivalence

 $\mathbb{C} \to I$  as in Theorem 2.13, and  $\mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}$  is identified with  $K_0(I \otimes A_0)$ . The canonical map from the exact sequence

$$(3.6) \qquad \underbrace{K_0(I \otimes I)}_{\cong \mathbb{Z}} \oplus \mathbb{Z} \to \underbrace{K_0(I \otimes B)}_{\cong \mathbb{Q}_m} \oplus \mathbb{Z} \to \underbrace{K_0(I \otimes A_0)}_{\cong \mathbb{Q}_m/\mathbb{Z}}$$

to the top row of (3.5) is an isomorphism (since it clearly is on the left and right hand terms). From this we see that  $K_0(I_1 \rtimes \mathbb{Z}/2) \cong \mathbb{Q}_{\mathfrak{m}} \oplus \mathbb{Z}$ , with the generator of  $\mathbb{Z}$  being the image of  $[e_1]$  via the KK-element  $\mathbb{C}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \to I^{\otimes \mathbb{Z}/2} \to I$ 

Taking crossed products and applying K-theory for (3.3) yields the 6-term exact sequence

$$(3.7) \qquad \mathbb{Q}_{\mathfrak{m}} \oplus \mathbb{Z} \longrightarrow \mathbb{Q}_{\mathfrak{m}} \oplus \mathbb{Q}_{\mathfrak{m}} \longrightarrow K_{0} \left( A_{0}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \right)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

where the generators  $\mathbb{Q}_{\mathfrak{m}} \oplus \mathbb{Q}_{\mathfrak{m}}$  over  $\mathbb{Q}_{\mathfrak{m}}$  are the images of [1] and [ $e_1$ ] by the KK-equivalence  $M_{\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2 \to B^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2$ . Here we have used Theorem 2.8. Thus the first arrow in the top row of (3.7) is the natural inclusion. We get  $K_1\left(A_0^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong 0$  and  $K_0\left(A_0^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}$ , generated by the image of  $\mathbb{Q}_{\mathfrak{m}}[e_1]$  in  $K_0\left(M_{\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right)$ .

## Lemma 3.5. We have

$$K_*\left(F_{1,\mathfrak{m}}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \begin{cases} 0, & *=0 \\ \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z} \oplus \mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}, & *=1 \end{cases}.$$

*Proof.* We write  $A_0 := F_{\mathfrak{m},1}$  as in the proof of Lemma 3.4 and use  $A := C_0(\mathbb{R}) \otimes A_0$  as a model for  $F_{1,\mathfrak{m}}$ . Note that the flip action on  $C_0(\mathbb{R})^{\otimes \mathbb{Z}/2}$  is conjugate to the action on  $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$  that is trivial on the first factor and reflects at the origin  $0 \in \mathbb{R}$  on the second factor. We thus have

$$(3.8) \hspace{1cm} K_*\left(A^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong K_{*+1}\left(\left(C_0(\mathbb{R}) \otimes A_0^{\otimes \mathbb{Z}/2}\right) \rtimes \mathbb{Z}/2\right).$$

We consider the short exact sequence

$$(3.9) \ 0 \to (C_0(-\infty,0) \oplus C_0(0,\infty)) \otimes A_0^{\otimes \mathbb{Z}/2} \to C_0(\mathbb{R}) \otimes A_0^{\otimes \mathbb{Z}/2} \to A_0^{\otimes \mathbb{Z}/2} \to 0$$
 of  $\mathbb{Z}/2$ -C\*-algebras. We have

$$\begin{split} K_*\left(\left((C_0(-\infty,0)\oplus C_0(0,\infty))\otimes A_0^{\otimes\mathbb{Z}/2}\right)\rtimes \mathbb{Z}/2\right) &\cong K_{*+1}\left(A_0^{\otimes\mathbb{Z}/2}\right) \\ &\cong \begin{cases} \mathbb{Q}_\mathfrak{m}/\mathbb{Z}, & *=0\\ 0, & *=1 \end{cases} \end{split}$$

by the Künneth theorem (since  $\operatorname{Tor}_{\mathbb{Z}}^{1}(\mathbb{Q}_{\mathfrak{m}}/\mathbb{Z},\mathbb{Q}_{\mathfrak{m}}/\mathbb{Z})\cong\mathbb{Q}_{\mathfrak{m}}/\mathbb{Z}$ ). In view of this and Lemma 3.4, taking crossed products and applying K-theory for (3.9) produces the 6-term exact sequence

By [Tik16, Lemma 1.1], the top row of (3.10) splits. Now the Lemma follows from (3.10) and (3.8).  $\Box$ 

By Theorem 3.1, the list of the classifiable C\*-algebras with approximately inner flip is up to KK-equivalences given by

$$\mathcal{E}_{\mathfrak{n},1,\mathfrak{m}}, \,\, \mathcal{F}_{1,\mathfrak{m}}$$

for supernatural numbers  $\mathfrak{m}$  and  $\mathfrak{n}$  where  $\mathfrak{m}$  divides  $\mathfrak{n}$ . Theorem 3.2 and Corollary 3.3 say that for any of these algebras A, we have an isomorphism

$$\mathsf{K}_*\left(\mathsf{A}^{\otimes \mathbb{Z}/2} \rtimes \mathbb{Z}/2\right) \cong \mathsf{K}_*(\mathsf{A} \otimes C^*_\mathsf{r}(\mathbb{Z}/2)).$$

The Künneth formula moreover implies  $K_*(A^{\otimes \mathbb{Z}/2}) \cong K_*(A)$ . This naturally raises the following question.

**Question 3.6.** Let A be any C\*-algebra satisfying the UCT with approximately inner flip. Is the  $\mathbb{Z}/2$ -C\*-algebra  $A^{\otimes \mathbb{Z}/2}$  equipped with the flip action  $KK^{\mathbb{Z}/2}$ -equivalent to A equipped with the trivial action?

We note that in order to answer this question positively, it would suffice to answer it positively for  $A = \mathcal{F}_{1,m}$ . Indeed,  $\mathcal{E}_{n,1,m}$  is KK-equivalent to  $M_n \oplus \mathcal{F}_{1,m}$  and  $M_n \otimes \mathcal{F}_{1,m}$  is KK-equivalent to zero if m divides n. In particular, the flip on  $\mathcal{E}_{n,1,m}^{\otimes \mathbb{Z}/2}$  is KK $\mathbb{Z}/2$ -equivalent to the sum of the flips on  $M_n^{\otimes \mathbb{Z}/2} \oplus \mathcal{F}_{1,m}^{\otimes \mathbb{Z}/2}$ . Since Theorem 2.8 provides a positive answer to Question 3.6 for  $A = M_n$ , a positive answer for  $A = \mathcal{F}_{1,m}$  would provide a positive answer for  $A = \mathcal{E}_{n,1,m}$ .

Unfortunately, the methods used to establish the  $KK^{\mathbb{Z}/2}$ -equivalence between  $M_n^{\otimes \mathbb{Z}/2}$  and  $M_n$  in Theorem 2.8 do not apply in the situation of Question 3.6. For once, there is no analogue of the representation theoretic argument in Proposition 2.1 for  $\mathcal{F}_{1,m}$ . Furthermore, the diagram

$$M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2} \to M_{\mathfrak{n}}^{\otimes \mathbb{Z}/2} \otimes M_{\mathfrak{n}} \leftarrow M_{\mathfrak{n}}$$

of  $KK^{\mathbb{Z}/2}$ -equivalences in Theorem 2.8 does not have an analogue for  $\mathcal{F}_{1,m}$  since the unit class  $[1]_0 \in K_0(\mathcal{F}_{1,m}) \cong 0$  is zero.

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