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Costly learning under ambiguity

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ABSTRACT

In this paper, we propose a general model of information acquisition, Costly Bayesian Learning (CBL). Using menu choice framework, we provide an axiomatic characterization of the model, identify its parameters, apply a comparative statics, as well as study special cases of the model. Unlike many prominent models of information acquisition (e.g., the Rational Inattention model), a CBL agent can exhibit uncertainty averse behavior and prefer late resolution of uncertainty.

1. Introduction

This paper proposes a general model of information acquisition, Costly Bayesian Learning (CBL). In this model, the decision maker (DM) tries to learn about the *true mechanism* that governs uncertainty about the state, where optimal information depends on the DM's incentives that may change across different choice problems (i.e., menus). Using menu-choice data, we give an axiomatic characterization of the model, uniquely identify the model parameters, provide a comparative statics result, as well as study special cases of the general model including costly information acquisition with uncertainty aversion.

Many economic problems involve uncertainty such that the DM has to take an action without knowing the true state. For instance, a farmer may need to choose which crop to plant before knowing the weather conditions next season, a broker may have to decide which stock to purchase before learning about the future product sales, and a student may have to choose which program to complete before observing the job openings next year. Since a state that occurs in the future (e.g., weather conditions, product sales, or job openings) cannot be observed now, how can the DM improve her decision today? Is there a way to learn about the payoff relevant states without directly observing them even before they are realized? A natural way of obtaining information in these examples is to collect data about the weather conditions in the past; product sale history; or job openings in previous years. The CBL model, as we discuss in more detail below, does permit this way of statistical learning. By collecting information about the past outcomes a CBL agent can better understand which mechanism (i.e., distribution over states) explains the data, so as to make better predictions about the future realizations of the state and accordingly take more informed actions.

Since collecting data is costly, the DM's motives for information acquisition crucially depend on what alternatives are present for choice. As such, the DM would like to know in advance which choice problem is relevant for her payments. This suggests that the DM may be averse to randomizations over choice problems that resolve after the DM makes a contingent choice from each problem. In general, however, there can be other type of randomizations that are valuable for the DM. For instance, the DM can naturally exhibit preference for randomization due to hedging motives in addition to aversion to randomization due to information costs. A prominent model of information acquisition is the Rational Inattention (RI) model, which has been shown to have fruitful applications in economics. A rationally inattentive agent balances the benefits and costs by optimally choosing information about the *true state* of the world. While it does permit aversion to randomization, the RI model cannot accommodate preference for randomization. In particular, a rationally inattentive agent cannot value hedging at all. But as we discuss in Section 2 with a concrete example, there can be situations where the DM naturally prefers hedging opportunities. How can we resolve this issue?

The CBL model offers a viable solution; a CBL agent can have differing attitudes towards randomization over choice problems depending on the timing of resolution of uncertainty. If randomization is resolved after the DM makes a choice but before the state is revealed, the DM can become worse off just like an RI agent. But if randomization is resolved after the state is revealed, the DM can become better off

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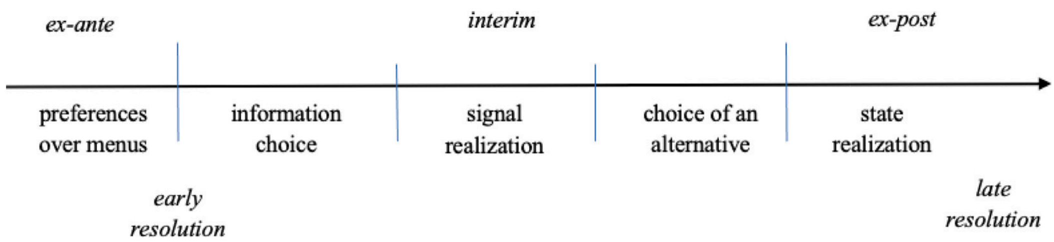

Fig. 1. Timeline.

unlike an RI agent. The RI model cannot capture this latter behavior because an RI agent is an expected utility maximizer. By contrast, we consider the agent (in the CBL model) as a second-order expected utility maximizer who collects information about the true distribution of the state. A second-order expected utility maximizer can strictly value late resolution of uncertainty (i.e., resolution after the state is revealed). Moreover, since information acquisition is costly, a CBL agent becomes averse to early resolution of uncertainty (i.e., resolution before the state is revealed but after the agent makes a choice). Thus, a CBL agent can exhibit both preference for late resolution and aversion to early resolution of uncertainty depending on the timing of the resolution relative to the realization of the state. In Section 2, we discuss some of these implications in more detail.

To differentiate between early and late resolution, we consider a menu-choice framework with lotteries that resolve either before or after the state is revealed. Our primitive is the DM's preferences over menus (i.e., choice problems). Each menu consists of lotteries that resolve before the state is revealed. Since the DM needs to make a choice from each given menu, early resolving lotteries allow us to model resolutions after the agent makes a choice but before the state is revealed. We call these lotteries *act-lotteries* since they yield acts. Each act maps states to lotteries that resolve after the state is revealed. Late resolving lotteries allow us to model resolutions after the state is revealed. We call these lotteries *prize-lotteries* since they yield prizes from which the DM eventually receives utility (see Fig. 1).

Our first main result provides an axiomatic characterization of the CBL model using the menu-choice framework (Theorem 1). We show that the CBL model has several implications for the DM's preferences over menus. Of these several implications, Posteriorwise Dominance (PD) is the key axiom differentiating the CBL model (see Section 4). Note that for a given distribution over states, each act can be reduced to a prize-lottery and each act-lottery can be reduced to a two-stage lottery (i.e., lottery over prize-lotteries). The PD axiom states that when comparing two act-lotteries, say P and Q, if P always (i.e., regardless of the distribution over states) induces a two-stage lottery that is preferred over its counterpart induced by Q, then Q will be seen as irrelevant whenever P is already present; that is, the DM would like to choose P instead of Q in any choice situation. As such, adding Q to a menu that already contains P can never increase the value of the menu. By contrast, the RI model satisfies the stronger Statewise Dominance (SD) axiom, which requires considering only degenerate distributions over states to deem an act-lottery irrelevant (see Section 6). Thus, it is relatively easier to observe indifferences with an RI agent than a CBL agent who can be, unlike an RI agent, averse to uncertainty and sensitive to the timing of resolution of uncertainty.[2]

In our discussions above, we have motivated the agent's choice behavior (in relation to uncertainty and information acquisition) by incorporating her behavior regarding compound lotteries. Halevy (2007) demonstrates with an experiment that there is close connection between people's attitudes towards uncertainty and compound lotteries. Seo (2009) axiomatizes Klibanoff et al. (2005)'s second-order expected utility model (i.e., the smooth ambiguity model) using the original Anscombe–Aumann setting of act-lotteries. Compound lotteries is part of this setting and non-reduction of compound lotteries is naturally tied to the attitudes towards uncertainty. This close connection can immediately extend to settings where information acquisition is possible. The general attitudes towards resolution of uncertainty that we discussed above (i.e., preference for late resolution) are well-documented in the literature and they have been utilized in explaining many puzzling behavior concerning uncertainty since the seminal work of Ellsberg (1961). In this regard, the CBL model does allow a richer set of attitudes towards resolution of uncertainty and reduction of compound lotteries since it requires a weaker dominance axiom. The weaker dominance axiom allows a CBL agent (unlike an RI agent) to be compatible with the behavior of uncertainty aversion in an information acquisition setting. Section 2 provides an illustration of the implications of the Posteriorwise Dominance for information acquisition.

Our work is related to de Oliveira et al. (2017) who give an axiomatic characterization of the RI model in a menu-choice framework with (Anscombe–Aumann) acts. We discuss the connection of the two models, CBL and RI, in more depth in Sections 4 and 6. Pennesi (2015) considers a choice setting with lotteries over menus of acts extending de Oliveira et al. (2017)'s framework, and provides an axiomatic characterization for the RI model. Ergin and Sarver (2015) consider a lotteries over menus of lotteries choice setting. Within this framework, they provide a rationale for the DM's seemingly intrinsic preferences for early resolution of uncertainty by the possibility of taking hidden actions at a later unmodeled stage. We show that when the DM is indifferent towards the timing of resolution of any uncertainty, the DM becomes a passive learner which extends Dillenberger et al. (2014)'s subjective-learning preferences in our choice setting.

In a recent working paper, Fabbri (2024) considers a related model of information acquisition to study rational inattention with ambiguity aversion. While, just like our CBL model, he considers uncertainty using smooth ambiguity (Klibanoff et al., 2005), there are several differences between the two information acquisition models. Firstly, while in Fabbri (2024)'s model the DM acquires information only about the realized state, the DM in the CBL model acquires information primarily about the true distribution of states. Secondly, while Fabbri (2024) focuses on the mutual-information function to represent the cost of information processing, we allow for a range of information cost functions. And thirdly, while Fabbri (2024) considers ex-post stochastic choice implications of his model (and therefore extends Matějka and McKay (2015)'s results) we consider ex-ante menu choice implications of the CBL model (and therefore extend de Oliveira et al. (2017)'s results). As such, we view our works complementary in the realm of rational inattention.

Finally, there are four parameters of the CBL model, a utility function, an increasing transformation, a second-order prior belief, and an information cost function. The first three parameters can be identified using standard techniques in the literature. Our second main result provides an elicitation procedure that can uniquely construct the cost parameter from menu-choice data (Theorem 2). In particular, our

[2] The RI model has been given axiomatic foundations with testable implications; see, e.g., Caplin and Dean (2015); de Oliveira et al. (2017); Ellis (2018); Hebert and Woodford (2023). See, e.g., Mackowiak et al. (2023) for a comprehensive review of the literature on economic applications of rational inattention.

axioms guarantee that for each menu there exists an equivalently good singleton menu. Using data on menus and their singleton equivalents, we show that the cost parameter can be approached from below in the limit. Moreover, we provide a comparative statics result, as well as characterize special cases of the CBL model including one with linear transformation (i.e., the RI model) and another one with concave transformation (i.e., the uncertainty averse CBL model).

The paper is organized as follows. In Section 2, we provide a simple example illustrating key implications of the CBL model differentiating it from the RI model. In Section 3, we present our menu-choice framework, introduce a general model of information acquisition, and define a set of canonical properties for the information cost function. Section 4 provides a set of testable axioms for menu choice data that are implied by the CBL model. Section 5.1 characterizes the CBL model by showing that these testable implications of the model are the only implications for menu choice data. In Section 5.2, we present our identification results for: a utility function, an increasing transformation, a second-order prior belief, and an information cost function. In particular, we show that the canonical properties of an information cost function are necessary and sufficient to uniquely elicit the information costs. Section 5.3 provides a comparative statics result. Section 6 characterizes special cases of the general CBL model, where the information acquisition model is either the RI model, or the "second-order" RI model, or the CBL model with a concave (or convex) transformation function capturing attitudes towards uncertainty, or a constrained Bayesian learning (ConBL) model capturing costless but constrained information acquisition, or a passive Bayesian learning (PBL) model capturing costless but fixed information acquisition. Section 7 concludes. Proofs are given in an Appendix.

2. Example

In this section, we provide a simple example to demonstrate differing choice implications of the CBL and RI models by considering a two-ball urn setting.

Two-ball urn

Suppose there is an urn which contains two balls, either black or white, such that the DM does not know how many black or white balls there are. She is given two bets, b and w, and asked to choose one of them. Specifically, one ball will be drawn from the urn, and depending on its color and the bet she chooses, she will be paid some money. Suppose that bet b pays \$100 (resp. \$0) when the drawn ball is black (resp. white) and likewise, bet w pays \$100 (resp. \$0) when it is white (resp. black). Since the DM has to make a choice in advance before a ball is drawn, she cannot observe the state (i.e., color of the drawn ball). In this case, like most individuals, the DM will be indifferent between choosing bet b and bet w since there is no information about the composition of the urn that can break the tie between the two bets.

Now suppose that the DM has the option of drawing a ball from the urn for inspection such that once she notes its color, the ball will be put back into the urn, she will be asked to choose between bet b and bet w afterwards, and depending on her choice of a bet and the color of a newly drawn ball from the urn, she will be paid. Intuitively, most individuals would like to use this option since they will acquire some information about the composition of the urn that can help improve their decision. For instance, if inspected ball is black (resp. white), then the DM can rule out the possibility that both balls are white (resp. black), which would make bet b (resp. w) more plausible. More precisely, suppose for instance the DM initially thinks that with $1/4$ probability the urn consists of two black balls, with $1/2$ probability it consists of one black and one white ball, and with $1/4$ probability it consists of two white balls. Thus, she is indifferent between bet b and bet w before drawing a ball since either choice of a bet will pay a positive amount with $1/2$ probability. However, once she draws a ball to inspect, then, as a Bayesian, her initial prior over the composition of the urn does change. In particular, if she draws a black (resp. white) ball, then she thinks that with $1/2$ probability the urn consists of two black (resp. white) balls, with $1/2$ probability it consists of one black and one white ball, and with 0 probability it consists of two white (resp. black) balls. Thus, choosing bet b (resp. w) after observing a black (resp. white) ball increases the DM's chances of receiving a positive amount from $1/2$ to $3/4$ probability. Hence, learning about the composition of the urn is beneficial for the DM and this remains true even when some cost is attached to it.

Since collecting data can be costly, the DM's motives for information acquisition crucially depend on what alternatives are present for choice. For instance, suppose the DM is told that with $1/2$ probability both bets, b and w, will be available, but with $1/2$ probability only bet b will be given. In this case, drawing a black (resp. white) ball for inspection will still induce the DM to choose bet b (resp. w) from menu $\{b, w\}$, but now the overall likelihood of obtaining a positive amount of payment drops from $3/4$ to $5/8$. Thus, the DM has less incentive to pay for the draw of a ball for inspection since she cannot fully utilize her information. As such, the DM would like to know in advance which choice problem is relevant for her payments. This suggests that the DM may be averse to randomizations over choice problems that resolve after the DM makes a contingent choice from each problem. In general, however, there can be other type of randomizations that are valuable for the DM. For instance, consider the bet, call it g, which yields (regardless of the color of the drawn ball) a lottery that pays \$100 with $1/2$ probability and \$0 with $1/2$ probability. Here, bet g can be seen as a randomization over bet b and bet w, which resolves after the color of the drawn ball is revealed. Introspection suggests that many individuals will prefer having bet g over having either bet b or bet w because bet g provides perfect hedging. Thus, the DM can naturally exhibit preference for randomization (due to hedging motives) in addition to aversion to randomization (due to information costs). While it does permit aversion to randomization, the RI model cannot accommodate preference for randomization. In particular, a rationally inattentive agent cannot value hedging at all suggesting that we may need to consider a more permissive model of information acquisition.

Menu-choice implications

To illustrate differing menu-choice implications of the CBL and RI models, consider the following choice situation. Suppose the DM faces a choice between menus $\{b, w\}$ and $\{b, g, w\}$. If the DM is an RI agent, she can never value adding alternative g to the menu $\{b, w\}$. By contrast, a CBL agent can strictly prefer the bigger menu $\{b, g, w\}$. The reason for this behavioral difference is that while an RI agent, as an expected utility maximizer, would like to choose only the "extreme alternatives", either b or w, a CBL agent, as a non-expected utility maximizer, may strictly prefer choosing the non-extreme alternative g since it provides hedging.

Notice that the bigger menu $\{b, g, w\}$ will be equivalent to menu $\{b, g_1, g_2, w\}$ where g_1 and g_2 are two identical copies of the bet g. Moreover, $\{b, g_1, g_2, w\}$ can be seen as the mixture of the menu $\{b, w\}$ by itself $\{b, w\}$ with equal weights, where the resolution of uncertainty happens after the state is realized. Here bet g_1 (resp., g_2) can be seen as the contingent plan of choosing bet b (resp., w) from the first-copy of the menu $\{b, w\}$ and bet w (resp., b) from the second-copy of the menu $\{b, w\}$. Similarly, the smaller menu $\{b, w\}$ can be seen as the mixture of it by itself with equal mixtures, where the resolution of uncertainty happens before the state is realized. As such, while an RI agent exhibits indifference between differing timings of resolution of uncertainty, a CBL agent can differentiate between them; in particular, a CBL agent can strictly prefer the late resolution of uncertainty.

Moreover, when the CBL agent chooses bet g from the menu $\{b, g, w\}$, she may not need to acquire any information in this case. The reason for this outcome is that since bet g provides hedging, choosing g only can be good enough to avoid the cost of information acquisition. In other words, for a CBL agent there could be choice situations where information acquisition incentives can be lowered with hedging possibilities. Such substitution opportunities between information acquisition and hedging bets cannot arise when the DM is an RI agent. Thus, the CBL model can allow for much richer behavior for both menu and act choice, as well as for information acquisition.

Numerical illustration

We now illustrate the above choice implications with a concrete model in the case of the two-ball urn example. Suppose that the DM is a CBL agent such that her preferences over acts can be represented by a second-order expected utility model. In particular, suppose that from any monetary outcome $x \in \mathbb{R}$, she receives a utility of $u(x) = x$, while she uses an increasing transformation function $v : \mathbb{R} \to \mathbb{R}$ to aggregate her expected utilities obtained under different first-order beliefs.

First suppose that the DM needs to make a choice from the set $\{b, w\}$. Note that when the DM does not draw a ball, she can choose either bet b or w with an equal expected payoff of $\frac{1}{4}v(100) + \frac{1}{2}v(50)$. When she draws a ball, however, her choice from the menu $\{b, w\}$ will be bet b (resp. w) if she draws a black ball (resp. white ball) with an expected payoff of $\frac{1}{2}v(100)+\frac{1}{2}v(50)$ in each case. As such, her net-benefit from the menu $\{b, w\}$ by drawing a ball will be $\frac{1}{2}v(100) + \frac{1}{2}v(50) - \frac{1}{4}v(100) - \frac{1}{2}v(50) = \frac{1}{4}v(100)$. Suppose that the DM prefers to draw a ball for inspection which costs her some $c > 0$ amount of payoff. As such, it must be that $c \leq \frac{1}{4}v(100)$.

Now suppose that bet g is added to her choice set and so the DM now needs to make a choice from the set $\{b, g, w\}$. Note that when the DM does not draw a ball, she can obtain an expected payoff of $v(50)$ if she chooses bet g (instead of b or w). Choosing bet g will be optimal for her in this case if it is true that v is a strictly concave function since then it will be that $v(50) > \frac{1}{4}v(100) + \frac{1}{2}v(50)$. Suppose that this is true. Moreover, if she draws a ball for inspection, then as before, her choice from the menu $\{b, g, w\}$ will be bet b (resp. w) if she draws a black (resp. white) ball with an expected payoff of $\frac{1}{2}v(100) + \frac{1}{2}v(50)$ in each case. As such, the DM will find it *not* optimal to draw a ball for inspection if it is true that $\frac{1}{2}v(100) + \frac{1}{2}v(50) - v(50) < c$, which is equivalent to $\frac{1}{2}v(100) - \frac{1}{2}v(50) < c$.

In other words, when $c > \frac{1}{2}v(100) - \frac{1}{2}v(50)$ and $c \leq \frac{1}{4}v(100)$, the DM will strictly prefer having the menu $\{b, g, w\}$ over the menu $\{b, w\}$. However, this is possible since we assumed that v is strictly concave implying that $\frac{1}{2}v(100) - \frac{1}{2}v(50) < \frac{1}{4}v(100)$. As such, the CBL agent can strictly prefer having the bigger menu $\{b, g, w\}$ instead of $\{b, w\}$. Moreover, as we just demonstrated, a CBL agent can acquire information when facing the menu $\{b, w\}$ while she does not acquire information when she faces the bigger menu $\{b, g, w\}$ with the hedging option g. By contrast, an RI agent cannot have strict preference between the menus $\{b, g, w\}$ and $\{b, w\}$. To see this, note that for an RI agent we must have $v(y) = y$ for all expected payoffs $y \in \mathbb{R}$. As such, under any scenario the value of bet g for an RI agent will be just equal to the half-half mixture of the value of bet b and bet w implying that g will never be chosen from the menu $\{b, g, w\}$ unless both b and w are equally good. Thus, an RI agent will always be indifferent between the menus $\{b, w\}$ and $\{b, g, w\}$. Moreover, adding a hedging option like g will never affect an RI agent's information acquisition behavior unlike a CBL agent; she will acquire the same information in either choice situation. This is true because an RI agent expects not to choose bet g under any posterior belief. As a result, an RI agent will only consider bet b and bet w for her information acquisition decision.

3. Preliminaries

In this section, we introduce our choice framework of menus of lotteries over state-contingent acts. We then describe a general information acquisition problem, and define the induced preference relation over menus.

3.1. Framework

Let $\Omega = \{\omega_1, \dots, \omega_n\}$ be a finite set of *states* and let Z be an arbitrary set of *prizes* with generic elements x, y, z.[3] A lottery is called *simple* if it has a finite support. Let $\Delta(Z)$ denote the set of simple lotteries over Z with typical elements p, q, r. We call a lottery in $\Delta(Z)$ a *prize-lottery*. An *act* is a map from the set of states Ω into the set of prize-lotteries $\Delta(Z)$. We denote by $\mathcal{F}$ the set of all acts with generic elements f, g, h. Let $\Delta(\mathcal{F})$ denote the set of simple lotteries over $\mathcal{F}$ with typical elements P, Q, R. We call a lottery in $\Delta(\mathcal{F})$ an *act-lottery*. A *menu* $F \subset \Delta(\mathcal{F})$ is a finite set of act-lotteries. Let $\mathbb{F}$ denote the collection of all menus with generic elements F, G, H.

For all $\alpha \in [0, 1]$, prize-lotteries $p, q \in \Delta(Z)$, acts $f, g \in \mathcal{F}$, act-lotteries $P, Q \in \Delta(\mathcal{F})$, and menus $F, G \in \mathbb{F}$, we denote (i) by $\alpha p+(1-\alpha)q$ the *mixed prize-lottery* $r \in \Delta(Z)$ such that $r(x) = \alpha p(x) + (1 - \alpha)q(x)$ for all $x \in Z$, (ii) by $\alpha f + (1 - \alpha)g$ the *mixed act* $h \in \mathcal{F}$ such that $h(\omega) = \alpha f(\omega) + (1 - \alpha)g(\omega)$ for all $\omega \in \Omega$, (iii) by $\alpha P + (1 - \alpha)Q$ the *mixed act-lottery* $R \in \Delta(\mathcal{F})$ such that $R(f) = \alpha P(f) + (1 - \alpha)Q(f)$ for all $f \in \mathcal{F}$, and (iv) by $\alpha F + (1 - \alpha)G$ the *mixed menu* $H \in \mathbb{F}$ such that $H = \{\alpha P + (1 - \alpha)Q \ : \ \forall P \in F \text{ and } \forall Q \in G\}$.

Note that while for the first two cases above, (i) and (ii), mixtures are resolved after a state is realized, for the last two cases, (iii) and (iv), mixtures are resolved before a state is realized. As such, we will consider two types of resolution of uncertainty in our setting. First type of resolution, which we call "early resolution", refers to resolution of which menu will be relevant for payment before the state is realized but after the DM makes a choice of an act-lottery from each menu. Second type of resolution, which we call "late resolution", refers to resolution of which act chosen from each menu will be relevant for payment after the state is realized.

With some abuse of notation, we identify a singleton menu $\{P\} \in \mathbb{F}$ with the act-lottery $P \in \Delta(\mathcal{F})$, and a constant act $f \in \mathcal{F}$ with the prize-lottery $p \in \Delta(Z)$ given that $f(\omega) = p$ for all $\omega \in \Omega$. An act $f \in \mathcal{F}$ can be identified with a degenerate act-lottery $\delta_f \in \Delta(\mathcal{F})$ and a prize $x \in Z$ can be identified with a degenerate prize-lottery $\delta_x \in \Delta(Z)$. For an act-lottery $P \in \Delta(\mathcal{F})$, let $f_P \in \mathcal{F}$ denote its induced act, where the resolution of the act-lottery is delayed by the determination of a state; that is, $f_P(\omega) = \sum P(f)f(\omega)$ for each $\omega \in \Omega$. Notice that any act-lottery $P \in \Delta(\mathcal{F})$ with only constant acts in its support can be identified as a (simple) two-stage compound lottery $P \in \Delta(\Delta(Z))$.

Let $\Delta(\Omega)$ denote the set of all possible beliefs about the likelihood of states in Ω with typical elements μ, ν, τ. Given a belief $\mu \in \Delta(\Omega)$, (i) an act $f \in \mathcal{F}$ induces a (simple) one-stage lottery $f_\mu \in \Delta(Z)$ such that $f_\mu = \sum_{\omega\in\Omega} \mu(\omega)f(\omega)$ and (ii) an act-lottery $P \in \Delta(\mathcal{F})$ induces a (simple) two-stage compound lottery $P_\mu \in \Delta(\Delta(Z))$ such that $P_\mu(p) = \sum_{f_\mu=p} P(f)$ for all $p \in \Delta(Z)$. Our primitive is a binary relation $\succsim$ over the set of menus, which represents the preferences of a decision-maker (henceforth, DM). The asymmetric and symmetric parts of $\succsim$ are denoted $\succ$ and $\sim$, respectively. We assume that both $\Delta(\Omega)$ and $\Delta(\Delta(\Omega))$ are measurable spaces, while both $\Delta(\Delta(\Omega))$ and $\Delta(\Delta(\Delta(\Omega)))$ are endowed with the weak* topology. In particular, since $\Delta(\Omega)$ is a subset of $\mathbb{R}^n$ it inherits the Borel $\sigma-$algebra generated by the (relative) box topology in $\Delta(\Omega)$, while $\Delta(\Delta(\Omega))$ has the Borel $\sigma-$algebra generated by its weak* topology. Finally, for any given measurable space $(X, \mathcal{A})$, we denote by $E_\theta[w(.)]$ the average value $\int_X w(x)\theta(dx)$ of a measurable function $w : X \to \mathbb{R}$ by a given measure $\theta : \mathcal{A} \to \mathbb{R}$.

3.2. The information acquisition problem

We consider a general information acquisition problem under uncertainty. Below we first describe how the DM receives utility from each act-lottery when information is not relevant. We then formalize the notion of information in our setting. Finally, we describe the general problem of information acquisition under uncertainty by discussing benefits and costs of information.

[3] We assume that Ω is finite to simplify the exposition. Our analysis can be extended to a general measurable space by straightforward modifications.

Second-order expected utility. The DM receives utility from any given act $f \in \mathcal{F}$ according to the second-order expected utility model

$$E_{\bar{m}}[v(E_{f_\mu}[u(x)])] = \int_{\Delta(\Omega)} v\left(\int_Z u(x) f_\mu(dx)\right) \bar{m}(d\mu), \tag{1}$$

where $u : Z \to \mathbb{R}$ is an unbounded utility function, $v : u(Z) \to \mathbb{R}$ is a strictly increasing transformation, and $\bar{m} \in \Delta(\Delta(\Omega))$ is a second-order prior.[4] The interpretation of this model is that the DM believes there is a true probability model in $\Delta(\Omega)$ which governs the likelihood of each state, but she is uncertain about which probability model $\mu \in \Delta(\Omega)$ is the correct one. However, she has enough information to form a prior belief $\bar{m} \in \Delta(\Delta(\Omega))$ on the likelihood of relevant probability models. Hence, using her second order prior belief $\bar{m}$, she aggregates each possible expected utility $v(E_{f_\mu}[u(x)])$ that she can gain under each probability model μ with her utility function u and a transformation v.

Furthermore, the DM evaluates an act-lottery by taking the expectation of second-order expected utility of each act in its support, and so

$$U_P^{u,v}(\bar{m}) = E_P[E_{\bar{m}}[v(E_{f_\mu}[u(x)])]] \tag{2}$$

gives the second-order expected utility of each act-lottery $P \in \Delta(\mathcal{F})$.

Information and blackwell ordering. We consider the possibility that the DM acquires information in order to improve her choices under uncertainty. In particular, the DM can acquire a noisy signal that conveys additional information about the true probability model. For instance, she can do this by sampling previous realizations of states. Each such sampling will result in a signal which would induce a posterior belief $m \in \Delta(\Delta(\Omega))$ from the prior $\bar{m}$ according to Bayes rule. Thus, a signal would lead to a *distribution over posteriors* $\pi \in \Delta(\Delta(\Delta(\Omega)))$ such that the expected posterior is equal to the prior. As a result, the collection of all possible signals can be given by the set

$$\Pi(\bar{m}) = \left\{ \pi \in \Delta(\Delta(\Delta(\Omega))) \ : \ \bar{m} = \int_{\Delta(\Delta(\Omega))} m\, \pi(dm) \right\}.$$

The set of signals $\Pi(\bar{m})$ is partially ordered in terms of their informativeness by the well-known Blackwell (1951) order, which in this context can be defined as follows:

Definition 1 (*Blackwell Order*)**.** Signal $\pi \in \Pi(\bar{m})$ is *more informative* than signal $\rho \in \Pi(\bar{m})$, denoted $\pi \trianglerighteq \rho$, if

$$\int_{\Delta(\Delta(\Omega))} \phi(m)\, \pi(dm) \geq \int_{\Delta(\Delta(\Omega))} \phi(m)\, \rho(dm)$$

for every convex continuous function $\phi : \Delta(\Delta(\Omega)) \to \mathbb{R}$.

Benefits and costs of information. Given a menu F, extracting a signal allows the DM to make a more informed choice from F because she can choose an act-lottery to maximize her second-order expected utility for each posterior $m \in \Delta(\Delta(\Omega))$. With a utility function $u : Z \to \mathbb{R}$ and a strictly increasing transformation function $v : u(Z) \to \mathbb{R}$, the *benefit of information* for a given signal $\pi \in \Pi(\bar{m})$ is therefore,

$$b_F^{u,v}(\pi) = \int_{\Delta(\Delta(\Omega))} \left[\max_{P \in F} U_P^{u,v}(m)\right] \pi(dm).$$

Since the integrand in square brackets is a convex continuous function on $\Delta(\Delta(\Omega))$, the benefits of information are increasing in the Blackwell order $\trianglerighteq$.

[4] Klibanoff et al. (2005) were the first to propose the smooth ambiguity model given in Eq. (1) to study attitudes towards ambiguity. They provide an axiomatic characterization of this model by using a rich choice setting of first-order and second-order acts. Seo (2009) provides an axiomatic characterization of this second-order expected utility model by using a lotteries of acts choice setting. Denti and Pomatto (2022) provide an axiomatic foundation for the smooth ambiguity model by considering Savage acts. For other related second-order expected utility modes, see, e.g., Neilson (1993), Nau (2006), and Ergin and Gul (2009).

A rational DM balances the benefit of information from a signal π against the cost for acquiring that signal. These costs are measured by an information cost function $c : \Pi(\bar{m}) \to [0, \infty]$, which associates a cost $c(\pi)$ to each signal $\pi \in \Pi(\bar{m})$. In the information acquisition problem we consider, the DM therefore chooses a signal π that maximizes the difference between benefits and costs of information $b_F^{u,v}(\pi) - c(\pi)$. We say a cost function is *proper* if it can assume a finite value and it is lower-semicontinuous.

3.3. Costly Bayesian learning preferences

In our framework, the DM chooses a menu with the expectation of acquiring information before she selects an act-lottery. We model information acquisition as illustrated above, and study the induced preference relation over menus.

Definition 2. A binary relation $\succsim$ over menus is a *costly Bayesian learning (CBL)* preference if it is represented by a functional $V : \mathbb{F} \to \mathbb{R}$, defined by

$$V(F) = \max_{\pi \in \Pi(\bar{m})} \left[b_F^{u,v}(\pi) - c(\pi)\right], \tag{3}$$

where $u : Z \to \mathbb{R}$ is an unbounded utility function, $v : u(Z) \to \mathbb{R}$ is a strictly increasing transformation, $\bar{m} \in \Delta(\Delta(\Omega))$ is a second-order prior, and $c : \Pi(\bar{m}) \to [0, \infty]$ is a proper information cost function. In this case, we also say $\succsim$ is represented by the parameters $(u, v, \bar{m}, c)$.

The assumptions on parameters $(u, v, \bar{m}, c)$ are standard. Unboundedness of u implies that the benefits of information are not bounded, which will be useful for unique identification of parameters. Monotonicity of v is natural property that ensures that more is better. Properness of c is a minimal assumption required to ensure that the maximization over costly signals is well-defined.

3.4. Canonical information costs

Properness is the only restriction that we impose on the cost function to define the information acquisition problem. On the other hand, there are a number of intuitive properties that, without loss of generality, can be imposed on an information cost function (see Corollary 1 in Section 5.2).

Definition 3. An information cost function $c : \Pi(\bar{m}) \to [0, \infty]$ is canonical if

(i) $c(\delta_{\bar{m}}) = 0$, where $\delta_{\bar{m}} \in \Pi(\bar{m})$ assigns probability 1 to the second-order prior $\bar{m}$ [groundedness: no information no cost],
(ii) $\pi \trianglerighteq \rho$ implies $c(\pi) \geq c(\rho)$ [monotonicity: more information more cost],
(iii) $\alpha c(\pi) + (1 - \alpha)c(\rho) \geq c(\alpha\pi + (1 - \alpha)\rho)$ [convexity: average cost of information exceeds cost of average information].

It is well known that mutual information satisfies properties (i)–(iii), and so the cost functions based on mutual information – which are frequently used in the literature – are canonical. As such, following Matějka and McKay (2015), a logit representation for ex-post choices can be obtained by using the CBL model with the mutual information as the cost of information.

4. Axioms

In the following, we consider eight axioms for the DM's preferences over menus. The first three axioms are standard in the menu-choice literature:

Axiom 1 (*Weak Order*)**.** For all menus F, G and H, (i) $F \succsim G$ or $G \succsim F$, and (ii) if $F \succsim G$ and $G \succsim H$, then $F \succsim H$.

Axiom 2 (*Continuity*)**.** For all menus F, G and H, the following sets are closed: $\{\alpha \in [0,1] : \alpha F + (1-\alpha)G \succsim H\}$ and $\{\alpha \in [0,1] : H \succsim \alpha F + (1-\alpha)G\}$.

Axiom 3 (*Unboundedness*)**.** There are prizes x and y, with $x \succ y$, such that for all $\alpha \in (0,1)$ there is a prize z satisfying either $y \succ \alpha z + (1-\alpha)x$ or $\alpha z + (1-\alpha)y \succ x$.

Axioms 1 and 2 ensure that preferences are complete, transitive and continuous. Axiom 3 implies that preferences over outcomes are unbounded (see, e.g., Maccheroni et al. (2006, Lemma 29)). The remaining axioms reflect more distinctive features of the information acquisition problem which defines the CBL preferences.

The DM chooses an act-lottery conditional on signal realizations by considering all possible alternatives in her menu. As such, when choosing a menu, the DM exhibits a *desire for flexibility* (Kreps, 1979); that is, adding an alternative to a menu can only make the DM better off since she can always ignore the added alternative if it does bring no added value.

Axiom 4 (*Preference for Flexibility*)**.** For all menus F and G, if $F \supset G$ then $F \succsim G$.

The DM solves an optimal signal extraction problem by balancing the benefits and costs of information that may differ from menu to menu. For instance, when the DM faces the menu F for sure, then she can focus her signal extraction on F, but if she faces a mixture of F with another (equally good) menu G, then she cannot focus her signal extraction on F or on G causing her potentially to lose value. This behavior reflects in our framework as a preference towards having one of the equally good menus F or G for sure rather than having a mixed menu $\alpha F + (1-\alpha)G$, a behavior which is called *aversion to contingent planning* (Ergin and Sarver, 2010).

Axiom 5 (*Aversion to Contingent Planning*)**.** For all menus F and G, if $F \sim G$ then $F \succsim \alpha F + (1-\alpha)G$ for all $\alpha \in (0,1)$.

Information is redundant for singleton menus since signal realizations do not help to choose a best alternative in a singleton menu and since the expected posterior belief is equal to the prior. Hence, the optimal information in a mixed menu $\alpha F + (1-\alpha)P$ depends only on α and F, and does not change if P is replaced by an alternative act-lottery Q. Thus, the DM's preferences exhibit an *independence of degenerate decisions* (Ergin and Sarver, 2010).

Axiom 6 (*Independence of Degenerate Decisions*)**.** For all menus F and G, act-lotteries P and Q, and $\alpha \in (0,1)$, if $\alpha F + (1-\alpha)P \succsim \alpha G + (1-\alpha)P$, then $\alpha F + (1-\alpha)Q \succsim \alpha G + (1-\alpha)Q$.

The DM acquires information only about the true probability model in $\Delta(\Omega)$. As a result, adding an act-lottery Q to a menu F can make the DM strictly better off only when there is some optimal information about the true probability model that would lead the DM to choose Q from $F \cup \{Q\}$. Thus, if F already contains an act-lottery that is preferred to Q posterior by posterior, adding Q to her opportunity set can make her neither better off nor worse off.

Axiom 7 (*Posteriorwise Dominance*)**.** For all menus F and act-lottery Q, if there exists $P \in F$ such that $P_\mu \succsim Q_\mu$ for all $\mu \in \Delta(\Omega)$, then $F \sim F \cup \{Q\}$.

In a menu of acts choice setting, de Oliveira et al. (2017) consider the first six axioms which we adapt to our menu of act-lotteries choice setting. Our seventh axiom is a dominance condition which, in our choice setting, weakens de Oliveira et al. (2017)'s counterpart, a statewise dominance (their seventh and last axiom). We argue in Section 6 that the (extended) statewise dominance axiom implies our posteriorwise dominance condition. Moreover, we show that an extension of de Oliveira et al. (2017)'s RI preferences (i.e., second-order RI preferences) can be characterized by this stronger dominance axiom in our setting while allowing for differing attitudes towards randomizations or the timing of resolution of uncertainty.

The DM is an expected utility maximizer. Thus, her preferences satisfy an independence axiom when they are reduced to the set of prize-lotteries, where neither uncertainty about states nor information acquisition is relevant.

Axiom 8 (*Independence of Prize-Lotteries*)**.** For all prize-lotteries p, q, r and $\alpha \in (0,1)$, $p \succsim q$ if and only if $\alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r$.

Our eighth and last axiom is a standard independence of prize-lotteries axiom adapted to our menu of act-lotteries choice setting. de Oliveira et al. (2017)'s Axioms 5 and 6 imply an independence of acts axiom. Moreover, by the definition of mixture acts in their setting, their RI preferences implicitly satisfy a reversal of order axiom (see Section 6). As a result, the independence of acts axiom they have together with their implicit reversal of order axiom imply independence of prize-lotteries axiom. However, we do not assume an independence of act-lotteries axiom nor a reversal of order axiom. Thus, our CBL preferences are more general than the RI preferences. In fact, we show in Section 6 that de Oliveira et al. (2017)'s RI model can be characterized in our setting by imposing on our CBL preferences either the independence of act-lotteries axiom or the reversal of order style axioms.

5. Analysis

In this section, we show that CBL preferences can be characterized by the set of axioms that we discussed in the previous section. Moreover, we show that the model parameters can be uniquely identified and compared across decision-makers using menu-choice data.

5.1. Characterization

The following result shows that Axioms 1–8 characterize all observable implications of CBL preferences.

Theorem 1. *A binary relation $\succsim$ over menus is a costly Bayesian learning preference if and only if it satisfies Axioms 1–8.*

Theorem 1 shows that the information acquisition problem in Section 3.2 implies a set of intuitive choice behavior that can be observed in our framework. It also establishes a formal connection between the literature on information acquisition about the distribution of the states and the decision-theory literature on menu-choice.

Proof sketch: Necessity part of the proof is straightforward to show. For sufficiency, Lemma 1 (Appendix A.2) shows that if a binary relation $\succsim$ satisfies Axioms 1–8, then there exist an unbounded utility function $u : Z \to \mathbb{R}$, a strictly increasing transformation $v : u(Z) \to \mathbb{R}$, a second-order prior $\bar{m}$, and an information cost function $c : \Pi(\bar{m}) \to [0,\infty]$ such that $(u, v, \bar{m}, c)$ represents $\succsim$. In particular, by considering the induced axioms over the set of act-lotteries, we obtain – as explained in more detail in the next section – a second-order expected utility representation for the restriction of the preferences on act-lotteries with parameters $(u, v, \bar{m})$. Using this fact, we then establish that every menu $F \in \mathbb{F}$ has a *singleton equivalent* $P_F \in \Delta(\mathcal{F})$ such that $P_F \sim F$. Using singleton equivalents and the second-order expected utility representation, (with a slight abuse of notation) a functional V over the set $\Phi_{\mathbb{F}} = \{\phi_F : \Delta(\Delta(\Omega)) \to \mathbb{R} \mid \phi_F(m) = \max_{P \in F} E_P E_m v(E_{f_\mu} u(x)),\ \forall m, \forall F\}$ can be defined such that $F \succsim G$ if and only if $V(\phi_F) \geq V(\phi_G)$ for all menus F and G. The remainder of the proof uses Axioms 1–8 to show that V is monotone, continuous and convex, and employs duality arguments to establish the desired representation.

5.2. Identification

In this section, we show how the parameters $(u, v, \bar{m}, c)$ in the information acquisition problem can be identified from menu-choice data.

Canonical information cost function

Given all other parameters $(u, v, \bar{m})$, we can obtain a unique canonical information cost function c by using menu-choice data. Note that for any given menu $F \in \mathbb{F}$, there exists an equivalent act-lottery $P_F \in \Delta(\mathcal{F})$ such that $F \sim P_F$.

Theorem 2. *Let $\succsim$ be a costly Bayesian learning preference such that the restriction of $\succsim$ to singleton menus is represented by $(u, v, \bar{m})$. Then the cost function $c : \Pi(\bar{m}) \to [0, \infty]$, defined by*

$$c(\pi) = \sup_{F \in \mathbb{F}} \left[b_F^{u,v}(\pi) - U_{P_F}^{u,v}(\bar{m}) \right], \tag{4}$$

is the unique canonical cost function such that $(u, v, \bar{m}, c)$ represents $\succsim$.

Theorem 2 shows that CBL preferences can be uniquely associated with a canonical information cost function c. In particular, the canonical cost function c can be constructed from choice data on singleton equivalents; since $c(\pi) \geq b_F^{u,v}(\pi) - U_{P_F}^{u,v}(\bar{m})$ for any information structure π, the singleton equivalent P_F can be used to determine a lower bound on the cost of π. Using other singleton equivalents then improves the lower bound. Theorem 2 shows that this procedure approximates $c(\pi)$ arbitrarily closely, thereby establishing a direct connection between the information cost function and menu-choice behavior.

Below we discuss how to uniquely identify other model parameters $(u, v, \bar{m})$ using standard techniques given in the literature.

Utility function, increasing transformation, and second-order prior

The restriction of the DM's preferences $\succsim$ over the set of first-order prize-lotteries $\Delta(Z)$ satisfies the axioms of expected utility (with unboundedness), and so they can be represented by the expected utility model with an unbounded utility function $u : Z \to \mathbb{R}$ such that $p \succsim q$ if and only if $E_p u(x) \geq E_q u(x)$. Moreover, u can be normalized such that $E_{p'} u(x) = 1$ and $E_{q'} u(x) = 0$ for some $p' \succ q'$.

The restriction of the DM's preferences $\succsim$ over the set of second-order prize-lotteries $\Delta(\Delta(Z))$ satisfies the axioms of expected utility (with unboundedness). Thus, there exists an unbounded function $\bar{U} : \Delta(Z) \to \mathbb{R}$ such that $\bar{P} \succsim \bar{Q}$ if and only if $E_{\bar{P}} \bar{U}(r) \geq E_{\bar{Q}} \bar{U}(r)$ for all $\bar{P}, \bar{Q} \in \Delta(\Delta(Z))$. Moreover, $\bar{U}$ can be normalized such that $E_{\bar{P}'} \bar{U}(r) = 1$ and $E_{\bar{Q}'} \bar{U}(r) = 0$ for some $\bar{P}' \succ \bar{Q}'$. Without loss of generality, let $\bar{P}' = \delta_{p'}$ and $\bar{Q}' = \delta_{q'}$. Since $\Delta(Z)$ can be embedded into $\Delta(\Delta(Z))$, we have $E_{\delta_p} \bar{U}(r) \geq E_{\delta_q} \bar{U}(r)$ if and only if $E_p u(x) \geq E_q u(x)$. This means that there exists a strictly increasing transformation $v : u(Z) \to \mathbb{R}$ such that $\bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$.

The restriction of the DM's preferences $\succsim$ over the set of act-lotteries $\Delta(\mathcal{F})$ satisfies the axioms of expected utility. Thus, there exists a function $U : \mathcal{F} \to \mathbb{R}$ such that $P \succsim Q$ if and only if $E_P[U(f)] \geq E_Q[U(f)]$ for all $P, Q \in \Delta(\mathcal{F})$. Note that we can let U such that $U(p) = \bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$. Since $\succsim$ satisfies Axiom 7, following Seo (2009, Lemma B.3-B10), there exists a second-order belief $\bar{m} \in \Delta(\Delta(\Omega))$ such that $U(f) = E_{\bar{m}}[v(E_{f_\mu} u(x))]$ yields the value of each act $f \in \mathcal{F}$.

Finally, as an implication of Theorem 2, we can identify all parameters of the costly information acquisition model up to standard positive affine transformations. For any two measures $m, m' \in \Delta(\Delta(\Omega))$, we say they are essentially equivalent (denoted by $m \approx m'$) if for all $P \in \Delta(\mathcal{F})$,

$$E_P[E_m[v(E_{f_\mu}[u(x)])]] = E_P[E_{m'}[v(E_{f_\mu}[u(x)])]].$$

Corollary 1. *If $(u, v, \bar{m}, c)$ and $(u', v', \bar{m}', c')$ represent the same CBL preferences $\succsim$ with canonical costs c and c', then there exists $\alpha, \lambda > 0$ and $\beta, \gamma \in \mathbb{R}$ such that $u' = \alpha u + \beta$, $v'(\alpha u + \beta) = \lambda v(u) + \gamma$, $\bar{m}' \approx \bar{m}$ and $c' = \lambda c$.*

5.3. Comparative statics

As an application of our identification results, we now consider a comparative measure of flexibility. Let DM1 and DM2 be two individuals who have CBL preferences $\succsim_1$ and $\succsim_2$, respectively, with canonical representations agreeing on the utilities, transformations, second-order priors, but the costs. We say that DM2 is less able to acquire information than DM1 when acquiring information is costlier for DM2 than DM1; that is, $c_2 \geq c_1$. Intuitively, when DM2 is less able to acquire information, she should find the option of committing to a singleton menu – which eliminates the need of acquiring information – more valuable than DM1. The following comparative defines when DM2 finds singleton menus more valuable than DM1.

Definition 4 (*Comparative Desire for Singletons*). Let $\succsim_1$ and $\succsim_2$ be two binary relations on the set of menus $\mathbb{F}$. Then $\succsim_2$ has a *stronger desire for singletons* than $\succsim_1$ if, for all F and P, whenever $P \succ_1 F$, then $P \succ_2 F$.

The following result shows that the comparative in Definition 4 characterizes when DM2 is less able to acquire information than DM1.

Theorem 3. *Let $\succsim_1$ and $\succsim_2$ be costly Bayesian learning preferences with canonical representations $(u_1, v_1, \bar{m}_1, c_1)$ and $(u_2, v_2, \bar{m}_2, c_2)$, respectively. Then, $\succsim_2$ has a stronger desire for singleton menus than $\succsim_1$ if and only if $(u_1, v_1, \bar{m}_1) = (u_2, v_2, \bar{m}_2)$ and $c_2 \geq c_1$.*

Theorem 3 provides a behavioral measure of comparative ability to acquire information. In particular, Theorem 3 implies that the utility difference between a menu and the singleton equivalent of the menu is higher for a DM who is less able to acquire information. As such, the DM will be willing to pay a higher premium for the option to have the singleton menu, thereby avoiding higher information costs.

6. Special cases

Special cases of CBL preferences can be characterized in terms of the additional restrictions they impose on menu-choice data.

6.1. Rationally inattentive preferences

As discussed earlier, de Oliveira et al. (2017)'s RI preferences can be seen as a special case of our CBL preferences.[5] We can establish this relation either by directly inspecting the representations or by establishing the underlying behaviors.

Structural relation. The RI model, which assigns a utility value to each menu of acts, can be given in our setting as

$$\hat{V}(F) = \max_{\hat{\pi} \in \Pi(\bar{\mu})} \left(E_{\hat{\pi}} \left[\max_{f \in F} \left(E_{f_\mu} w(x) \right) \right] - \hat{c}(\hat{\pi}) \right), \tag{5}$$

where $\bar{\mu}$ is a first-order prior, $\Pi(\bar{\mu}) = \{\pi \in \Delta(\Delta(\Omega)) : \bar{\mu} = \int \mu d\pi(\mu)\}$ is the set of viable information structures conveying information about the states, $w : Z \to \mathbb{R}$ is a utility function, and $\hat{c} : \Pi(\bar{\mu}) \to [0, \infty]$ is a proper information cost function.

Note that whenever the CBL model has an affine transformation function v, then the utility the DM obtains from each act f can be given by the first-order expected utility

$$E_{f_{\mu_{\bar{m}}}} v(u(x)) = \int_\Omega v[u(f(\omega))] d\mu_{\bar{m}}(\omega), \tag{6}$$

[5] de Oliveira et al. (2017) consider information acquisition via costly signal extraction only after a state is realized. In our setting, we allow for information acquisition even before a state is realized. As will be discussed below in more detail, when a CBL agent has linear transformation function, her menu choice behavior can be seen as an outcome of the RI model; that is, as if she has a unique prior over the state space which she updates by extracting signals about the realized state.

where $\mu_{\bar{m}} = \int_{\Delta(\Omega)} d\bar{m}(\mu)$ is the barycenter of the second-order probability measure $\bar{m}$. As such, according to the CBL model with an affine transformation function v, the value of a menu of acts F will be

$$V(F) = \max_{\pi \in \Pi(\bar{m})} \left(E_\pi \left[\max_{f \in F} E_m \left[v \left(E_{f_\mu} u(x) \right) \right] \right] - c(\pi) \right), \tag{7}$$

which is equivalent to

$$\max_{\pi \in \Pi(\bar{m})} \left(E_\pi \left[\max_{f \in F} E_{f_{\mu_m}} \left[v\left(u(x)\right) \right] \right] - c(\pi) \right), \tag{8}$$

where $\mu_m = \int_{\Delta(\Omega)} \mu \, m(d\mu)$ for all $m \in \Delta(\Delta(\Omega))$, which can be equivalently given as

$$\max_{\hat{\pi} \in \Pi(\mu_{\bar{m}})} \left(E_{\hat{\pi}} \left[\max_{f \in F} E_{f_{\mu_m}} \left[v\left(u(x)\right) \right] \right] - \hat{c}(\hat{\pi}) \right), \tag{9}$$

where $\Pi(\mu_{\bar{m}}) = \{\pi \in \Delta(\Delta(\Omega)) : \mu_{\bar{m}} = \int_{\Delta(\Omega)} \mu_m \pi(dm)\}$ is the set of viable information structures conveying information about the states and $\hat{c} : \Pi(\mu_{\bar{m}}) \to [0, \infty]$ is the suitably substituted for information cost function. Then, letting $\bar{\mu} = \mu_{\bar{m}}$ and $w(x) = v(u(x))$ for all $x \in Z$ above, we see that Eqs. (5) and (9) coincide showing that the RI model can be embedded into our setting as a CBL model representing preferences over menus of acts. Moreover, these RI preferences can be immediately extended to our general setting of menus of act-lotteries where the value of a menu of act-lotteries F will be

$$V(F) = \max_{\hat{\pi} \in \Pi(\mu_{\bar{m}})} \left(E_{\hat{\pi}} \left[\max_{P \in F} E_P \left(E_{f_{\mu_m}} \left[v\left(u(x)\right) \right] \right) \right] - \hat{c}(\hat{\pi}) \right). \tag{10}$$

Axiomatic relation. The RI model can be established as a special case of our CBL model by imposing either one of the four axioms below.

First, for the equivalence of Eqs. (7) and (8) above, we moved the expectation by the first-order beliefs out of the transformation function and reduced the second-order expectation to a first-order expectation. This possibility of reduction implies the following three axioms.[6]

Axiom 9 (*Indifference to the Timing of Resolution*)**.** For all act-lotteries P, we have $f_P \sim P$.

Axiom 9 states that the DM's preferences are not sensitive to the timing of resolution of uncertainty about which act among possible ones will be relevant for her payoffs once she makes a choice of an act-lottery.

Second, similar to the second observation, the DM must be indifferent between whether the mixture of acts is resolved before or after the state is determined.

Axiom 10 (*Reversal of Order*)**.** For all acts f, g, and mixture weights $\alpha \in (0, 1)$, we have $\alpha\delta_f + (1-\alpha)\delta_g \sim \alpha f + (1-\alpha)g$.

Axiom 10 states that the DM's preferences are not sensitive to the timing of resolution of uncertainty about which act of the possible two will be relevant for her payoffs once she makes a choice of an act-lottery.

Third, similar to the third observation, the DM must be indifferent between whether a second-order lottery is resolved at the first-stage or the second-stage.

Axiom 11 (*Reduction of Compound Lotteries*)**.** For all prize-lotteries p, q, and $\alpha \in (0, 1)$, we have $\alpha\delta_p + (1-\alpha)\delta_q \sim \alpha p + (1-\alpha)q$.

Axiom 11 states that the DM's preferences are not sensitive to the timing of resolution of uncertainty about which constant act of the possible two will be relevant for her payoffs once she makes a choice of an act-lottery.

Finally, the value of a single act f will be given by the first-order expected utility given in Eq. (6) whenever the CBL model has an affine transformation. This implies the following independence of acts axiom.

[6] In a similar vein, an Indifference to Convexification axiom can also be given when obtaining the RI model.

Axiom 12 (*Independence of Acts*)**.** For all acts f, g, h and $\alpha \in (0, 1)$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$.

Axiom 12 is a standard independence over acts axiom and typically imposed in order to obtain the subjective expected utility model in an Anscombe–Aumann choice setting.

Clearly, Axiom 9 implies Axiom 10, which implies Axiom 11.[7] In fact, the following results shows that for a given CBL preferences all of the four axioms stated above are equivalent to each other, and each one of them can characterize the extended RI preferences in our setting as a special case of the CBL preferences.

Proposition 1. *Let $\succsim$ be a costly Bayesian learning preference. Then the following are equivalent:*

(i) $\succsim$ satisfies either one of Axioms 9, 10, 11, or 12.
(ii) $\succsim$ is represented by the model given in Eq. (10).

Proposition 1 shows that the RI model can be identified as a special case of the CBL model by testing menu-choice preferences in several equivalent ways on the basis of either a timing of resolution type axioms or an independence of acts axiom.[8]

First, the timing of resolution type axioms reveal whether the DM is sensitive to the timing of resolution of randomizations or not. A DM following the RI model will be indifferent, though in general there can be viable reasons to expect that the DM will be not indifferent to the timing of resolutions; for instance, this is the case when the DM has multiple first-order priors and does not aggregate corresponding utilities with a linear transformation, as in the case of general CBL preferences. And second, verification (or violation) of the independence of acts axiom reveals if the DM is a subjective expected utility maximizer (or not). In general, the DM can violate the independence of acts axiom; for instance, this is the case when the DM exhibits uncertainty aversion, a phenomenon which is widely documented in the experimental literature.

6.2. Other special cases

In addition to the rationally inattentive preferences, there are other special cases of the CBL preferences.

Second-order rationally inattentive preferences

According to the RI model given in Eq. (10) in our framework, the DM can be seen as if she learns about the states, and not about the distribution of the states. This fact induces the following statewise dominance condition, which is clearly a strengthening of our posteriorwise dominance axiom.

Axiom 13 (*Statewise Dominance*)**.** For all menus F and act-lotteries Q, if there exists $P \in F$ such that $P_{\delta_\omega} \succsim Q_{\delta_\omega}$ for all $\omega \in \Omega$, then $F \sim F \cup \{Q\}$.

Axiom 13 states that the DM acquires information only about the states in Ω. As a result, adding an act-lottery Q to a menu F can make her strictly better off only when there is some information about the true state that would lead the DM to choose Q from $F \cup \{Q\}$. Thus, if F already contains an act-lottery that is preferred to Q state by state,

[7] Counterparts of Axioms 10 and 11 can be found in Seo (2009) who uses them in the original Anscombe–Aumann act choice setting to show that the first-order (subjective) expected utility representation can be obtained from his second-order expected utility representation. A counterpart of Axiom 9 was first given by Kreps (1988, p.107) in an Anscombe–Aumann choice setting by way of implicitly describing the condition within a discussion about the order of acts (i.e., horse-race lotteries) and prize-lotteries (i.e., roulette-wheel lotteries).

[8] In Proposition 1 above, a weaker Uncertainty Neutrality axiom can be imposed instead of the stronger Independence of Acts axiom.

adding Q to her opportunity set can make her neither better off nor worse off.

Although Axiom 13 is implied by the RI model given in Eq. (10), it does not restrict the transformation v to be an affine function. Instead, it only restricts the support of the second-order distribution m to contain Dirac measures on states. As such, Axiom 13 is also implied by the following model, which we call *second-order rationally inattentive preferences*:

$$V(F) = \max_{\pi \in \Pi(\bar{m})} \left(E_\pi \left[\max_{P \in F} E_P E_m \left[v \left(E_{f_{\delta_\omega}} u(x) \right) \right] \right] - c(\pi) \right), \tag{11}$$

where the support of the second-order prior $\bar{m}$ is a subset of the set $\cup_{\omega \in \Omega} \{\delta_\omega\}$. Unlike the RI model, the second-order RI model transforms the expected utilities received under each state $\omega \in \Omega$ by using the transformation function v while learning is kept about the realized state by using the second-order prior $\bar{m}$ which puts positive probability only on the set $\cup_{\omega \in \Omega} \{\delta_\omega\}$.[9] As a result, each second-order posterior m that the DM considers feasible puts positive probability only on the degenerate distributions δ_ω in the support of the second-order prior $\bar{m}$. The following result shows that Axiom 13 confines our general model of CBL preferences to the second-order RI model in Eq. (11).

Proposition 2. *Let $\succsim$ be a costly Bayesian learning preference. Then the following are equivalent:*

(i) $\succsim$ satisfies Axiom 13.
(ii) $\succsim$ is represented by the model given in Eq. (11).

The dominance axiom reveals directly what type of learning is undertaken by the DM. If the stronger axiom of dominance, Axiom 13, is satisfied, then the DM's information acquisition problem can be reduced to the second-order RI model as if she is learning about the realized state but with transforming state utilities with a function that may not be affine. If the DM violates this stronger axiom of dominance, but satisfies the relatively weaker axiom of dominance, Axiom 7, then her information acquisition problem will be on learning about the true distribution of states.

Uncertainty averse/seeking preferences

We have discussed in a previous subsection that any of the Axioms 9–12 can be violated by the DM with general CBL preferences, whenever she is not indifferent to the timing of resolution of randomizations. This is in particular a case for uncertainty averse decision-makers, who typically satisfy an aversion to uncertainty axiom, first given by Schmeidler (1989) in an Anscombe–Aumann choice setting.

Axiom 14 (*Uncertainty aversion*)**.** For all acts f, g and $\alpha \in (0,1)$, if $f \sim g$, then $\alpha f + (1-\alpha) g \succsim f$.

This axiom says that the DM can improve her payoff that she can obtain from singleton menus by way of mixing them with late resolution (i.e., resolution after the state is realized). As a result, a CBL agent who desires such late resolution of uncertainty will have a non-affine transformation function, and so she will not be indifferent to the timing of resolution of randomizations.[10]

Proposition 3. *Let $\succsim$ be a costly Bayesian learning preference with an increasing transformation v. Then the following are equivalent:*

(1) $\succsim$ satisfies Axioms 14.
(2) v is concave.

Note that a similar result, where the DM is uncertainty seeking, can be easily characterized with an uncertainty seeking axiom in which case the transformation function v will be convex.

In view of Propositions 2 and 3, we have the following corollary.

Corollary 2. *Let $\succsim$ be a costly Bayesian learning preference with an increasing transformation v. Then the following are equivalent:*

(i) $\succsim$ satisfies Axioms 13 and 14.
(ii) $\succsim$ is represented by the model given in Eq. (11) with a concave transformation v.

This result shows us that the RI model can be extended to the second-order RI model to accommodate uncertainty aversion while acquiring costly information within the perspective of rational inattention. In particular, the second-order RI model implies that a rationally inattentive agent can also be valuing hedging options, a natural behavior which has not been considered in the literature.

Constrained Bayesian learning

A general constrained Bayesian learning (ConBL) model can be given in our setting such that the value of a menu of act-lotteries will be

$$V(F) = \max_{\pi \in \Gamma(\bar{m})} \left(E_\pi \left[\max_{P \in F} U_P^{u,v}(m) \right] \right), \tag{12}$$

where $\Gamma(\bar{m})$ is a closed convex subset of the set of information structures $\Pi(\bar{m})$. Clearly, the ConBL model is a special case of our CBL preferences where the information cost function c satisfies $c(\pi) \in \{0, \infty\}$ with $c(\pi) = 0$ if and only if $\pi \in \Gamma(\bar{m})$. This model can be characterized by an axiom allowing for weak indifference to contingent plans with singleton menus (de Oliveira et al., 2017).

Axiom 15 (*Weak Indifference to Contingent Planning*)**.** For all menus F and act-lotteries P, if $F \sim P$, then $F \sim \alpha F + (1-\alpha) P$ for all $\alpha \in (0,1)$.

In the CBL model that we have characterized in Theorem 1, a DM is averse to mixing menus F and G unless there is a common information structure π that is optimal for both menus. For the constrained information acquisition problem in Eq. (12), any feasible information structure π in the set $\Gamma(\bar{m})$ is optimal for a singleton menu. Thus, the DM is indifferent towards mixtures with singleton menus. The following result shows that Axiom 15 is the only additional behavioral restriction of a ConBL model within the class of our general CBL model.

Proposition 4. *Let $\succsim$ be a costly Bayesian learning preference. Then the following are equivalent:*

(i) $\succsim$ satisfies Axiom 15,
(ii) $\succsim$ is represented by the model given in Eq. (12).

Passive Bayesian learning

There are many information acquisition models used in applied literature where the DM does not actively seek information, but rather is a passive recipient of it. A general passive Bayesian learning (PBL) model can be given in our setting such that the value of a menu of act-lotteries will be

$$V(F) = E_\pi \left[\max_{P \in F} U_P^{u,v}(m) \right] \tag{13}$$

where $\pi \in \Pi(\bar{m})$ is some fixed information structure that the DM expects to have for no cost. These types of models imply the following axiom in our choice setting, which de Oliveira et al. (2017) call indifference to contingent planning for arbitrary menus.

[9] This form of second-order expected utility is studied, among many others, by e.g., Neilson (1993) and Nau (2006).

[10] Recall that there are two types of resolution of uncertainty in our setting. First type of resolution, which we call "early resolution", refers to resolution of which menu will be relevant for payment after the DM makes a choice of an act-lottery from each menu but before the state is realized. Second type of resolution, which we call "late resolution", refers to resolution of which act chosen from a menu will be relevant for payment after the state is realized.

Axiom 16 (*Indifference to Contingent Planning*). *For all menus F and G, if $F \sim G$, then $F \sim \alpha F + (1-\alpha)G$ for all $\alpha \in (0,1)$.*

The following result shows that Axiom 16 is the only additional behavioral restriction of a PBL model within the class of our general CBL model.

Proposition 5. *Let $\succsim$ be a costly Bayesian learning preference. Then the following are equivalent:*

(i) $\succsim$ satisfies Axiom 16,
(ii) $\succsim$ is represented by the model given in Eq. (13).

By Proposition 1, it is immediate to see that the PBL model contains Dillenberger et al. (2014)'s subjective-learning preferences, and thus the subjective learning model can be characterized in our setting by adding one of the Axioms 9–12 to the PBL model.

7. Conclusion

In this paper, we have proposed a general model of information acquisition, Costly Bayesian Learning (CBL). By demonstrating with a simple two-ball urn example, we have argued that a CBL agent can have flexible attitudes towards the timing of resolution of uncertainty, and therefore can permit intuitive behavior for (i) choice between menus, (ii) choice of information, and (iii) choice of alternatives from menus which cannot be inferred by many prominent models of information acquisition. In particular, a CBL agent can utilize hedging opportunities to lower the need of costly information acquisition leading to better choices. A challenging issue for applied work on costly information acquisition problems is that the costs can be subjective and therefore not directly observable by the analyst. We show that the behavior of individuals with CBL preferences can be directly tested, and their hidden information costs can be identified and elicited with observable choice data in our rich choice setting.

A well-known model of information acquisition is the Rational Inattention (RI) model which has found a range of applications in the literature. We have shown that the RI model can be embedded into our setting as a special case of the CBL model. By imposing either (i) various forms of indifference to the timing of resolution of uncertainty axioms or (ii) an independence over acts axiom, the RI model can be characterized in our setting. Moreover, we have argued that a second-order RI model can be obtained in our setting by relaxing the DM's attitudes towards the timing of resolution of uncertainty while restricting the dominance axiom to the degenerate distributions over states. It can be easily verified that even when a second-order RI agent is incorrectly identified as a (first-order) RI agent, the information costs will still be correctly elicited by using the elicitation formula for information costs (given in Eq. (4)). This is, however, no longer true whenever the agent has general CBL preferences since then her attitudes towards the timing of resolution of uncertainty (via her transformation function v) affects the information costs. Thus, it is important to correctly identify whether the agent is learning about the true state or the true mechanism governing the likelihood of the states. As such, our analysis provides a plausible point of start for studying costly information acquisition with differing attitudes towards uncertainty.

CRediT authorship contribution statement

Kemal Ozbek: Writing – original draft, Formal analysis.

Declaration of competing interest

Author submit this paper for consideration for publication in Journal of Mathematical Economics. Author declare that he has no conflict of interest for manuscript.

Appendix

In this section, we prove the results given in Sections 5 and 6.

A.1. Preliminaries

We first introduce some notation and preliminary results required for the proofs. Let $\Delta^2(\Omega)$ denote the space $\Delta(\Delta(\Omega))$, and $\Delta^3(\Omega)$ denote $\Delta(\Delta(\Delta(\Omega)))$.

Niveloids. Denote by $C(\Delta^2(\Omega))$ the linear space of real-valued continuous functions defined on $\Delta^2(\Omega)$, and by $ca(\Delta^2(\Omega))$ the linear space of signed measures of bounded variation on $\Delta^2(\Omega)$ (Aliprantis and Border (2006, p. 399)). For each $\pi \in ca(\Delta^2(\Omega))$ and for each $\phi \in C(\Delta^2(\Omega))$, let

$$\langle \phi, \pi \rangle = \int_{\Delta^2(\Omega)} \phi(m)\, \pi(dm).$$

The linear space $C(\Delta^2(\Omega))$ is endowed with the supnorm and $ca(\Delta^2(\Omega))$ with the weak* topology. Therefore $ca(\Delta^2(\Omega))$ can be identified with the continuous dual space of $C(\Delta^2(\Omega))$ (Aliprantis and Border (2006, Corollary 14.15)), and $C(\Delta^2(\Omega))$ can be identified with the continuous dual space of $ca(\Delta^2(\Omega))$ (Aliprantis and Border (2006, Theorem 5.93)).

Let Ψ be a subset of $C(\Delta^2(\Omega))$, and consider a function $V : \Psi \to \mathbb{R}$. We say that V is *normalized* if $V(\alpha) = \alpha$ for each constant function $\alpha \in \Psi$; *monotone* if $V(\phi) \geq V(\psi)$ for all $\phi, \psi \in \Psi$ such that $\phi \geq \psi$; *translation invariant* if $V(\phi+\alpha) = V(\phi)+\alpha$ for each $\phi \in \Psi$ and $\alpha \in \mathbb{R}$ such that $\phi+\alpha \in \Psi$; a *niveloid* if $V(\phi)-V(\psi) \leq \sup\{\phi(m) - \psi(m) : m \in \Delta^2(\Omega)\}$ for each $\phi, \psi \in \Psi$. If V is a niveloid, then it is monotone and translation invariant, while the converse is true whenever $\Psi = \Psi + \mathbb{R}$. Moreover, if V is a niveloid, then V is Lipschitz continuous. If Ψ is a convex set and V is a convex niveloid, then there is a convex niveloid that extends V to $C(\Delta^2(\Omega))$.[11]

Notation and auxiliary results. Let Φ be the set of convex functions belonging to $C(\Delta^2(\Omega))$: Φ is a closed convex cone such that $0 \in \Phi$. Denote by Φ^* the dual cone of Φ, that is,

$$\Phi^* = \{\pi \in ca(\Delta^2(\Omega)) : \langle \phi, \pi \rangle \geq 0 \text{ for all } \phi \in \Phi\}.$$

The set Φ^* is also a closed convex cone such that $0 \in \Phi^*$. Moreover $\Phi = \Phi^{**}$ (see Aliprantis and Border (2006, Theorem 5.103)), that is,

$$\Phi = \{\phi \in C(\Delta^2(\Omega)) : \langle \phi, \pi \rangle \geq 0 \text{ for all } \pi \in \Phi^*\}.$$

Let $u : Z \to \mathbb{R}$ be a function and $v : u(Z) \to \mathbb{R}$ be an increasing transformation. Denote by $\Phi_{\mathbb{F}}$ the set of functions $\phi_F : \Delta^2(\Omega) \to \mathbb{R}$ for each menu F where,

$$\phi_F(m) = \max_{P \in F} E_P[E_m[v(E_{f_\mu}[u(x)])]],$$

for all $m \in \Delta^2(\Omega)$. Similarly, let $\Phi_{\Delta(\mathcal{F})}$ denote the set of functions $\phi_P : \Delta^2(\Omega) \to \mathbb{R}$ for each act-lottery P where $\phi_P(m) = E_P[E_m[v(E_{f_\mu}[u(x)])]]$ for each $m \in \Delta^2(\Omega)$, and let $\Phi_{\Delta(Z)}$ denote the set of functions $\phi_p : \Delta^2(\Omega) \to \mathbb{R}$ for each prize-lottery p where $\phi_p(m) = v(u(x))$ for each $m \in \Delta^2(\Omega)$. Observe that we have $\Phi_{\Delta(Z)} \subset \Phi_{\Delta(\mathcal{F})} \subset \Phi_{\mathbb{F}} \subset \Phi$. Moreover, $\alpha\phi_F + (1-\alpha)\phi_G = \phi_{\alpha F + (1-\alpha)G}$ for each pair of menus F and G, and $\alpha \in [0,1]$. Hence, in particular, $\Phi_{\mathbb{F}}$ is convex.

For any given menu F, let $\mathrm{co}(F)$ denote its convex hull. For any $P \in \Delta(\mathcal{F})$, let $supp(P)$ denote its support. For any $P \in \Delta(\mathcal{F})$, let $v(u(P))$ denote the vector $\lambda_P \in \mathbb{R}^{\Delta(\Omega)}$ such that $\lambda_P(\mu) = E_P[v(E_{f_\mu}[u(x)])]$ for each $\mu \in \Delta(\Omega)$ and let $v(u(F)) = \{\lambda_P : P \in F\}$ for each $F \in \mathbb{F}$. For any $f \in \mathcal{F}$, let $u_f \in u(Z)^{\Delta(\Omega)}$ denote the vector such that $u_f(\mu) = E_{f_\mu} u(x)$ for each $\mu \in \Delta(\Omega)$. Let $u(\mathcal{F}) = \{u_f : f \in \mathcal{F}\}$.

[11] See Cerreia-Vioglio et al. (2014) for the proofs of these results and a detailed analysis about niveloids in general.

A.2. Implications of Axioms 1–8

In this Section, we state and prove a lemma that provides a representation for a binary relation satisfying Axioms 1–8. We consider below utility functions which are unbounded above, while the case where they are unbounded below is analogous and therefore omitted.

Lemma 1. *Let $\succsim$ be a binary relation on $\mathbb{F}$ that satisfies Axioms 1–8. Then:*

(i) There exist an unbounded utility function $u : Z \to \mathbb{R}$, a strictly increasing transformation $v : u(Z) \to \mathbb{R}$, and a second-order prior $\bar{m} \in \Delta(\Delta(\Omega))$ such that the restriction of the preference order $\succsim$ over the set of singletons is represented by the second-order expected utility defined with parameters $(u, v, \bar{m})$.

(ii) Every menu $F \in \mathbb{F}$ has a singleton equivalent $P_F \in \Delta(\mathcal{F})$ such that $F \sim P_F$.

(iii) The function c^* such that $c^*(\pi) = \sup_{F\in\mathbb{F}} \left[b_F^{u,v}(\pi) - U_{P_F}^{u,v}(\bar{m})\right]$ for all $\pi \in \Pi(\bar{m})$ is proper.

(iv) The functional V defined by $V(F) = \max_{\pi\in\Pi(\bar{m})} \left[b_F^{u,v}(\pi) - c^*(\pi)\right]$ for all $F \in \mathbb{F}$ represents $\succsim$.

Proof. Let $\succsim$ be a binary relation on $\mathbb{F}$ that satisfies Axioms 1–8.

[Part (i)]: The restriction of the DM's preferences $\succsim$ over the set of first-order prize-lotteries $\Delta(Z)$ satisfies the vNM axioms of expected utility, and so they can be represented by the expected utility model with a utility function $u : Z \to \mathbb{R}$ such that $p \succsim q$ if and only if $E_p u(x) \geq E_q u(x)$. Note that by Axiom 3, u is unbounded. Moreover, u can be normalized such that $E_{p'} u(x) = 1$ and $E_{q'} u(x) = 0$ for some $p' \succ q'$.

The restriction of the DM's preferences $\succsim$ over the set of act-lotteries $\Delta(\mathcal{F})$ satisfies the axioms of expected utility. In particular, to see that the independence axiom holds, take any $\alpha \in (0,1)$ and $P, Q \in \Delta(\mathcal{F})$. Suppose that $P \sim Q$. By Axiom 5, we have $P = \alpha P + (1-\alpha)P \succsim \alpha Q + (1-\alpha)P$. By Axiom 6, $\alpha P + (1-\alpha)Q \succsim \alpha Q + (1-\alpha)Q = Q$. By Axiom 5 again, we have $Q = \alpha Q + (1-\alpha)Q \succsim \alpha P + (1-\alpha)Q$. So we conclude that $P \sim \alpha P + (1-\alpha)Q$ for any $\alpha \in (0,1)$ whenever $P \sim Q$. Now suppose that $P \succ Q$. We must have $P \succ 0.5P + 0.5Q \succ Q$. Otherwise, we have either (i) $0.5P + 0.5Q \succsim P$ or (ii) $Q \succsim 0.5P + 0.5Q$. By using Axiom 6, we obtain that $Q \succsim 0.5P + 0.5Q \succsim P$ in either case, a contradiction. Hence, following the arguments given in the proof of Herstein and Milnor (1953, Theorem 2, part c), we obtain that $P \succ \alpha P + (1-\alpha)Q \succ Q$ for any $\alpha \in (0,1)$. But then, by Ozbek (2023, Proposition 1), the preference order must satisfy independence over act-lotteries.

As a result, the restriction of the DM's preferences $\succsim$ over the set of second-order prize-lotteries $\Delta(\Delta(Z))$ satisfies the axioms of expected utility. Thus, there exists a function $\bar{U} : \Delta(Z) \to \mathbb{R}$ such that $\bar{P} \succsim \bar{Q}$ if and only if $E_{\bar{P}} \bar{U}(r) \geq E_{\bar{Q}} \bar{U}(r)$ for all $\bar{P}, \bar{Q} \in \Delta(\Delta(Z))$. Moreover, $\bar{U}$ can be normalized such that $E_{\bar{P}'} \bar{U}(r) = 1$ and $E_{\bar{Q}'} \bar{U}(r) = 0$ for some $\bar{P}' \succ \bar{Q}'$. Without loss of generality, let $\bar{P}' = \delta_{p'}$ and $\bar{Q}' = \delta_{q'}$. Since $\Delta(Z)$ can be embedded into $\Delta(\Delta(Z))$, we have $E_{\delta_p} \bar{U}(r) \geq E_{\delta_q} \bar{U}(r)$ if and only if $E_p u(x) \geq E_q u(x)$. This means that there exists a strictly increasing transformation $v : u(Z) \to \mathbb{R}$ such that $\bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$.

Finally, since the restriction of the DM's preferences $\succsim$ over the set of act-lotteries $\Delta(\mathcal{F})$ satisfies the vNM axioms of expected utility, there exists a function $U : \mathcal{F} \to \mathbb{R}$ such that $P \succsim Q$ if and only if $E_P[U(f)] \geq E_Q[U(f)]$ for all $P, Q \in \Delta(\mathcal{F})$. Note that we can let U such that $U(p) = \bar{U}(p) = v(E_p u(x))$ for all $p \in \Delta(Z)$. But then, since $\succsim$ satisfies Axiom 7, following Seo (2009, Lemma B.3-B10), there must exist a second-order belief $\bar{m} \in \Delta(\Delta(\Omega))$ such that $U(f) = E_{\bar{m}}[v(E_{f_\mu} u(x))]$ for all $f \in \mathcal{F}$. Combining all these, we conclude that the preference order $\succsim$ restricted to the set of singletons can be represented by the second-order expected utility model given in Eq. (2). □

[Part (ii)]: We establish this part in two claims.

Claim 1. *Let F and G be menus such that for each $Q \in G$ there is $P \in F$ such that $P_\mu \succsim Q_\mu$ for each $\mu \in \Delta(\Omega)$. Then $F \succsim G$.*

Proof. By Axiom 7, $F \sim F \cup \{Q_1\} \sim F \cup \{Q_1, Q_2\} \sim \ldots \sim F \cup G$. By Axiom 4, $F \cup G \succsim G$. Combining these, we obtain $F \succsim G$. □

Claim 2. *Every menu F has a singleton equivalent P_F such that $P_F \sim F$.*

Proof. Let $s(F) = \{f(w) : f \in supp(P), P \in F, w \in \Omega\}$ denote the set of prize-lotteries that are possible within menu F. Since F has finitely many act-lotteries, each act-lottery has finitely many acts in its support, and the state space has finitely many states, $s(F)$ must be finite. Therefore, there are some $p, q \in s(F)$ such that $p \succsim r \succsim q$ for all $r \in s(F)$. By part (i), the restriction of $\succsim$ over $\Delta(Z)$ has an expected utility representation for some $u : Z \to \mathbb{R}$, and so $E_p u(x) \geq E_r u(x) \geq E_q u(x)$ for all $r \in \text{co}(s(F))$.

By part (i) again, the restriction of $\succsim$ over $\Delta(\Delta(Z))$ has an expected utility representation. In particular, for any $P \in F$ and $\mu \in \Delta(\Omega)$, the value of P_μ can be given as $E_{P_\mu} v(E_r u(x))$ where $v : u(Z) \to \mathbb{R}$ is a strictly increasing transformation. As such, for all $r \in \text{co}(s(F))$, we have $v(E_p u(x)) \geq v(E_r u(x)) \geq v(E_q u(x))$ implying that $p \succsim P_\mu \succsim q$ for all $P \in F$ and $\mu \in \Delta(\Omega)$. By Claim 1, we must have $p \succsim F \succsim q$. By Axiom 2, the two sets

$$A = \{\alpha \in [0,1] : \alpha p + (1-\alpha)q \succsim F\} \quad \text{and}$$
$$B = \{\alpha \in [0,1] : F \succsim \alpha p + (1-\alpha)q\}.$$

are closed. Since $[0,1]$ is connected, there exists $\alpha \in A \cap B$ such that $\alpha p + (1-\alpha)q \sim F$. Let P_F be equal to the singleton $\alpha p + (1-\alpha)q$. □

[Part (iii)]: We need to show that (i) $c^*(\pi) \geq 0$ for all $\pi \in \Pi(\bar{m})$, (ii) $c^*(\pi) < \infty$ for some $\pi \in \Pi(\bar{m})$, and (iii) c^* is lower semi-continuous. Note that since u and v are normalized, and so we have $0 \in \Phi_{\mathbb{F}}$, it follows that $c^*(\pi) \geq E_\pi 0 - 0 = 0$ for all $\pi \in \Pi(\bar{m})$, showing (i). By Axiom 4, we have $F \succsim P$ for any $P \in F$, and so $P_F \succsim P$ for all $P \in F$ by Part (i). This means $c^*(\delta_{\bar{m}}) = \sup_{F\in\mathbb{F}} \left[b_F^{u,v}(\delta_{\bar{m}}) - U_{P_F}^{u,v}(\bar{m})\right] \leq 0$, and so $c^*(\delta_{\bar{m}}) = 0$, showing (ii). Finally, since c^* is a pointwise supremum of a family of continuous functions, it is lower semi-continuous. □

[Part (iv)]: We establish this part in several claims. Without loss of generality assume that $u(x) \geq 0$ for each $x \in Z$ whenever $u(Z)$ is lower bounded and closed.

Define the functional $V : \Phi_{\mathbb{F}} \to \mathbb{R}$ such that $V(\phi_F) = E_{P_F} U(f)$ where P_F is a singleton equivalent of F.[12] If P_F and Q_F are two certainty equivalents of F, then $P_F \sim Q_F$ and so $E_{P_F} U(f) = E_{Q_F} U(f)$. Next we show in two claims that V is monotone (i.e., $\phi_F \geq \phi_G$ implies $V(\phi_F) \geq V(\phi_G)$), and so V is well-defined; that is, whenever $\phi_F = \phi_G$, then $F \sim G$ for each pair of menus F and G.

Claim 3. *Consider a pair of menus F and G. If $\phi_F \geq \phi_G$, then for each $Q \in G$ there exists $P \in \text{co}(F)$ such that $P_\mu \succsim Q_\mu$ for each $\mu \in \Delta(\Omega)$.*

Proof. Assume, for contradiction, that there is some $Q \in G$ such that for all $P \in \text{co}(F)$ we have $Q_\mu \succ P_\mu$ for some $\mu \in \Delta(\Omega)$. Since the vector valued function $v(u(P))$ is linear in P, we have $\text{co}(v(u(F))) = v(u(\text{co}(F)))$, so that $v(u(\text{co}(F)))$ is convex, closed and bounded. Let $A = \{a \in \mathbb{R}^{\Delta(\Omega)} : a \geq v(u(Q))\}$, then A is a closed convex cone. Clearly, $v(u(\text{co}(F)))$ and A are disjoint sets. By a separating hyperplane theorem (Rockafellar (1970, Corollary 11.4.2)), there exists some $m \in \mathbb{R}^{\Delta(\Omega)}$ such that $E_P[E_m[v(E_{f_\mu} u(x))]] < E_m a(\mu)$ for all $a \in A$ and $P \in F$. Since $v(u(Q))$ belongs to A we have

[12] For convenience we use V to denote both the representation over menus and the induced representation over support functions.

$\max_{P\in F} E_P[E_m[v(E_{f_\mu}u(x))]] < E_Q[E_m[v(E_{f_\mu}u(x))]]$.

Since A is a cone, we can let $m \in \Delta^2(\Omega)$ implying $\phi_F(m) < \phi_G(m)$, a contradiction. □

Claim 4. *Consider a pair of menus F and G. If $G \subset \text{co}(F)$, then $F \succsim G$.*

Proof. Let $G = \{Q_1,\dots,Q_n\} \subset \text{co}(F)$. For all $i = 1,\dots,n$ we can write $Q_i = \sum_{j=1}^{m_i} \alpha^i_j P^i_j$ for $\alpha^i_1,\dots,\alpha^i_{m_i} \geq 0$ summing up to one, and $P^i_1,\dots,P^i_{m_i} \in F$. Hence

$$G \subset \sum_{j=1}^{m_1} \cdots \sum_{j'=1}^{m_n} \alpha^1_j \cdots \alpha^n_{j'} F = \sum_{k=1}^{l} \beta_k F.$$

By Axiom 4 we have that $\sum_{k=1}^{l} \beta_k F \succsim G$, so it is enough to check that $F \sim \sum_{k=1}^{l} \beta_k F$. We show this by induction on l. If $l = 1$, then $\sum_{k=1}^{l} \beta_k F = F \sim F$. Suppose now the claim is true for $l-1$. Observe that

$$\sum_{k=1}^{l} \beta_k F = \beta_l F + (1-\beta_l)\left(\sum_{k=1}^{l-1} \frac{\beta_k}{1-\beta_l} F\right).$$

Moreover, by inductive assumption $F \sim \sum_{k=1}^{l-1} \frac{\beta_k}{1-\beta_l} F$. Therefore $F \succsim \sum_{k=1}^{l} \beta_k F$ by Axiom 5. Since $F \subset \sum_{k=1}^{l} \beta_k F$, by Axiom 4 we obtain $\sum_{k=1}^{l} \beta_k F \succsim F$. Therefore $F \sim \sum_{k=1}^{l} \beta_k F$, as desired. □

By Claim 3, if $\phi_F \geq \phi_G$, then there exists a subset $H \subset \text{co}(F)$ such that for each $Q \in G$ there exists $R \in H$ such that $R_\mu \succsim Q_\mu$ for all $\mu \in \Delta(\Omega)$. By Claim 4, F is preferred to H, which, by Claim 1, is preferred to G. This shows that V is monotone, and so well-defined. Moreover, since $F \succsim G$ if and only if $P_F \succsim P_G$ by definition, we deduce that V represents $\succsim$ in the sense that $F \succsim G$ if and only if $V(\phi_F) \geq V(\phi_G)$.

Claim 5. *The functional V is a monotone, normalized, convex niveloid.*

Proof. We have already established that V is monotone. To see that it is normalized, notice that the set of constant functions in $\Phi_{\mathbb{F}}$ is identified with the set $\Phi_{\Delta(Z)}$, and for every prize-lottery p we have $V(\phi_p) = v(E_p u(x)) = \phi_p$, so that V is normalized.

We now show that V is a convex niveloid in several steps.

Step 1 (V is translation invariant): Using Axiom 6, the obvious adaptation of the argument in Maccheroni et al. (2006, Proof of Lemma 28) provides that whenever k belongs to $v(u(Z))$ we have for any $\phi_F \in \Phi_{\mathbb{F}}$,

$$V(\beta\phi_F + (1-\beta)k) = V(\beta\phi_F) + (1-\beta)k \quad \forall \beta \in (0,1).$$

Pick $\gamma > 1$, so that $\gamma\phi_F \in \Phi_{\mathbb{F}}$. Then,

$$V\left(\frac{1}{\gamma}(\gamma\phi_F) + \frac{\gamma-1}{\gamma}\left(\frac{\gamma}{\gamma-1}k\right)\right) = V\left(\frac{1}{\gamma}(\gamma\phi_F)\right) + \frac{\gamma-1}{\gamma}\left(\frac{\gamma}{\gamma-1}k\right) \quad \forall \alpha > 0.$$

This implies that $V(\phi_F + k) = V(\phi_F) + k$ whenever $k > 0$. Notice that we have just shown $V(\phi_F + k - k) = V(\phi_F + k) - k$ implying that $V(\phi_G + t) = V(\phi_G) + t$ for any $t < 0$ such that $\phi_G + t \in \Phi_{\mathbb{F}}$. Thus, V is translation invariant on $\Phi_{\mathbb{F}}$.

Step 2 (V is convex): To show that V is convex, suppose $V(\phi_F) = V(\phi_G)$. Then $F \sim G$ and, by Axiom 5, $F \succsim \alpha F + (1-\alpha)G$. Hence,

$$\alpha V(\phi_F) + (1-\alpha)V(\phi_G) = V(\phi_F) \geq V(\phi_{\alpha F+(1-\alpha)G}) = V(\alpha\phi_F + (1-\alpha)\phi_G).$$

Now suppose $V(\phi_G) > V(\phi_F)$, and define $\beta = V(\phi_G) - V(\phi_F) > 0$. Since $\phi_F + \beta \in \Phi_{\mathbb{F}}$,

$$V(\phi_F + \beta) = V(\phi_F) + \beta = V(\phi_F) + V(\phi_G) - V(\phi_F) = V(\phi_G),$$

where the first equality holds by translation invariance. Therefore,

$$\begin{aligned} V(\phi_G) &\geq V(\alpha(\phi_F + \beta) + (1-\alpha)\phi_G) = V(\alpha\phi_F + (1-\alpha)\phi_G) + \alpha\beta \\ &= V(\alpha\phi_F + (1-\alpha)\phi_G) + \alpha(V(\phi_G) - V(\phi_F)), \end{aligned}$$

so that $V(\alpha\phi_F + (1-\alpha)\phi_G) \leq \alpha V(\phi_F) + (1-\alpha)V(\phi_G)$ showing that V is convex.

Step 3 (V is a niveloid): Since V is translation invariant on $\Phi_{\mathbb{F}}$, we can extend V uniquely to $\Phi_{\mathbb{F}} + \mathbb{R}$ by defining $V(\phi) = V(\phi + k) - k$ for any $\phi \in \Phi_{\mathbb{F}} + \mathbb{R}$ and $k \in \mathbb{R}$ such that $\phi + k \in \Phi_{\mathbb{F}}$. This extension preserves not only translation invariance, but also monotonicity and convexity. Hence the extension of V is a convex niveloid on $\Phi_{\mathbb{F}} + \mathbb{R}$, and therefore on $\Phi_{\mathbb{F}}$. □

To complete to proof we apply the well-known Fenchel–Moreau theorem adapted to our framework.

Claim 6. *There exist a proper cost function $c : \Pi(\bar{m}) \to [0,\infty]$ such that*

$$V(\phi_F) = \max_{\pi\in\Pi(\bar{m})} \left(\langle\phi_F, \pi\rangle - c(\pi)\right), \quad \forall F \in \mathbb{F}.$$

Proof. Since $\Phi_{\mathbb{F}}$ is convex and V is a convex niveloid, there is a real-valued functional W defined on $C(\Delta^2(\Omega))$ which is a convex niveloid extending V (see Appendix A.1). Since W is a niveloid, it is continuous. Since W is continuous, convex and real-valued, by Rockafellar (1974, Theorem 11) the subdifferential of W is nonempty at each $\phi \in C(\Delta^2(\Omega))$, that is, for each ϕ there is $\pi \in ca(\Delta^2(\Omega))$ such that

$$\langle\phi, \pi\rangle - W(\phi) \geq \langle\psi, \pi\rangle - W(\psi) \quad \forall \psi \in C(\Delta^2(\Omega)). \tag{14}$$

Moreover, since W is a niveloid, it is monotone and translation invariant, so by Ruszczyński and Shapiro (2006, Theorem 2.2) we can let π be in $\Delta^3(\Omega)$. Define $V^* : \Delta^3(\Omega) \to (-\infty,\infty]$ such that

$$V^*(\pi) = \sup_{F\in\mathbb{F}} \langle\phi_F, \pi\rangle - V(\phi_F) \quad \forall \pi \in \Delta^3(\Omega).$$

Thus, for all ϕ_F and π, $V^*(\pi) \geqslant \langle\phi_F, \pi\rangle - V(\phi_F)$ and hence $V(\phi_F) \geqslant \langle\phi_F, \pi\rangle - V^*(\pi)$. Moreover, for any ϕ_F there exists a $\pi \in \Delta^3(\Omega)$ such that $\langle\phi_F, \pi\rangle - V(\phi_F) = V^*(\pi)$ by (14). As a result,

$$V(\phi_F) = \max_{\pi\in\Delta^3(\Omega)} \langle\phi_F, \pi\rangle - V^*(\pi) \quad \forall F \in \mathbb{F}.$$

Finally, we want to show that c is the restriction of V^* to $\Pi(\bar{m})$, in which case c coincides with c^* showing that it is a proper cost function by Part (iii). For this, we need to show that $V^*(\pi) < \infty$ implies $\pi \in \Pi(\bar{m})$ whenever v is not affine, and $V^*(\pi) < \infty$ implies $\pi \in \Pi(m)$ where $E_m\mu = E_{\bar{m}}\mu$ whenever v is affine.

First, suppose that v is not affine. In this case, suppose, for contradiction, that there exists some $\pi \in \Delta^3(\Omega) \setminus \Pi(\bar{m})$ such that $V^*(\pi) < \infty$. Let $m_\pi = E_\pi m \in \Delta^2(\Omega)$. By definition, we have $m_\pi \neq \bar{m}$. This means we can find some f such that $f(\omega), f(\omega') \in \{\delta_x, \delta_y\}$ for some $\delta_x \succ \delta_y$ and $\omega \neq \omega'$ satisfying

$$E_{m_\pi}[v(E_{f_\mu}u(x))] - E_{\bar{m}}[v(E_{f_\mu}u(x))] > 0. \tag{15}$$

But since u and v are unbounded, x and y above can be chosen such that the payoff difference in Eq. (15) becomes arbitrarily large. Thus $V^*(\pi) \geq \sup_{f\in F}\langle\phi_f, \pi\rangle - V(\phi_f) = \infty$, a contradiction.

Now suppose that v is affine. In this case, for each $n \in \mathbb{N}$, choose prizes x_n and y such that $v(u(x_n)) = n$ and $v(u(y)) = 0$. Fix some $\omega \in \Omega$ and consider an act f assuming prize x_n on ω and y otherwise. Then

$$\langle\phi_f, \pi\rangle - V^*(\pi) = nE_\pi[E_m[\mu(\omega)]] - V^*(\pi) \leq V(\phi_f) = nE_{\bar{m}}[\mu(\omega)].$$

Since the above inequality holds for each n, as long as $V^*(\pi) < \infty$, it follows that

$$E_\pi[E_m[\mu(\omega)]] \leq E_{\bar{m}}[\mu(\omega)] \quad \forall \omega \in \Omega,$$

and so, since $E_{\bar{m}}[\mu] \in \Delta(\Omega)$, it follows that $E_\pi[E_m[\mu(\omega)]] = E_{\bar{m}}[\mu(\omega)]$ for all $\omega \in \Omega$. Thus,

$$V(\phi_F) = \max_{\pi\in\Pi(m)} \langle\phi_F, \pi\rangle - V^*(\pi) \quad \forall F \in \mathbb{F},$$

where $m \in \Delta^2(\Omega)$ is such that $m = \bar{m}$ whenever v is not affine, and $E_m\mu = E_{\bar{m}}\mu$ whenever v is affine. □

With the demonstration of Part (iv), we complete the proof of Lemma 1. □

A.3. Proofs of the results in the text

Proof of Theorem 1. It is straightforward to show that a CBL preference satisfies Axioms 1–8. For the converse, let $\succsim$ be a binary relation that satisfies Axioms 1–8. Then by Lemma 1, the functional $V : \mathbb{F} \to \mathbb{R}$ defined by $V(F) = \max_{\pi \in \Pi(\bar{m})} \left[b_F^{u,v}(\pi) - c^*(\pi) \right]$ for all $F \in \mathbb{F}$ represents $\succsim$. □

For the following proofs of Propositions 1–5, let $\succsim$ be a CBL preference represented by $(u, v, \bar{m}, c)$ and suppose, without loss of generality, that c is canonical.

Proof of Proposition 1. It is clear that when $\succsim$ is represented by the RI model given in Eq. (10), then $\succsim$ satisfies Axioms 9–12.

For the converse, clearly Axiom 9 implies Axiom 10, which implies Axiom 11. Note that by Axiom 6, for all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, we have $f \succsim g$ if and only if $\alpha\delta_f + (1-\alpha)\delta_h \succsim \alpha\delta_g + (1-\alpha)\delta_h$. Moreover, if Axiom 12 holds, then we have $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$. In this case, for all $f, g, h \in \mathcal{F}$ and $\alpha \in (0,1)$, we have $f \succsim g$ if and only if $V_{\Delta(\mathcal{F})}(\alpha\delta_f + (1-\alpha)\delta_h) \geq V_{\Delta(\mathcal{F})}(\alpha\delta_g + (1-\alpha)\delta_h)$ and $V_{\mathcal{F}}(\alpha f + (1-\alpha)h) \geq V_{\mathcal{F}}(\alpha g + (1-\alpha)h)$, where $V_{\Delta(\mathcal{F})}$ and $V_{\mathcal{F}}$ are restrictions of V over $\Delta(\mathcal{F})$ and $\mathcal{F}$, respectively. This means there exists an increasing transformation $\lambda : \mathbb{R} \to \mathbb{R}$ such that $V_{\Delta(\mathcal{F})}(\delta_f) = \lambda(V_{\mathcal{F}}(f))$ for all $f \in \mathcal{F}$. But since $\delta_f \sim f$ for all $f \in \mathcal{F}$, the transformation λ must be the identify map. Hence, when Axiom 12 holds, we have $\alpha\delta_f + (1-\alpha)\delta_h \sim \alpha f + (1-\alpha)h$ for all $f, h \in \mathcal{F}$ and $\alpha \in (0,1)$ showing that Axiom 10 holds (which implies Axiom 11).

In sum, Axioms 9–10, and 12 all imply Axiom 11. Thus, if we show that Axiom 11 implies that v is affine, then we are done. To see this, let $p, q \in \Delta(Z)$ and $\alpha \in (0,1)$. Since $\succsim$ is represented by the CBL model given in Eq. (3) with a canonical cost function, we have $V(\alpha\delta_p + (1-\alpha)\delta_q) = \alpha\, v(E_p u(x)) + (1-\alpha)\, v(E_q u(x))$ and $V(\alpha p + (1-\alpha) q) = v(\alpha E_p u(x) + (1-\alpha) E_q u(x))$. Thus, if Axiom 11 holds, then we must have $V(\alpha\delta_p + (1-\alpha)\delta_q) = V(\alpha p + (1-\alpha) q)$. But this can happen only when v is affine, as desired. □

Proof of Proposition 2. It is straightforward to verify that when a CBL preference $\succsim$ is represented by the second-order RI model, then $\succsim$ satisfies Axiom 13. For the converse, suppose $\succsim$ satisfies Axiom 13. By Lemma 1 part (ii), restriction of $\succsim$ over the set of singleton menus is represented by the second-order expected utility defined with some parameters $(u, v, \bar{m})$. Let $P, Q \in \Delta(\mathcal{F})$ such that $P_{\delta_\omega} \sim Q_{\delta_\omega}$ for all $\omega \in \Omega$. By Axiom 13, we must have $P \sim Q$. But since P and Q are arbitrary act-lotteries, we must have $\bar{m}(E) = 0$ for any measurable set $E \subset \Delta(\Omega) \setminus (\cup \omega \in \Omega\{\delta_\omega\})$. □

Proof of Proposition 3. It is immediate to see that when $\succsim$ is represented by the CBL model with a concave transformation v, then $\succsim$ satisfies Axiom 14. For the converse, suppose that $\succsim$ satisfies Axiom 14. In this case, for any $f, g \in \mathcal{F}$ with $V(f) = V(g)$, we have $V(\alpha f + (1-\alpha) g) \geq V(f)$ for all $\alpha \in (0,1)$. Then, by a similar argument given in Claim 5, V is concave over $\mathcal{F}$; that is for all $f, g \in \mathcal{F}$, we have $V(\alpha f + (1-\alpha) g) \geq \alpha V(f) + (1-\alpha) V(g)$ for all $\alpha \in (0,1)$. Since $V(f) = E_{\bar{m}} v(E_{f_\mu} u(x))$ for all $f \in \mathcal{F}$, it follows that the function $E_{\bar{m}}[v] : u(\mathcal{F}) \to \mathbb{R}$ is concave, where for all $f \in \mathcal{F}$, we have $E_{\bar{m}}[v](u_f) = E_{\bar{m}} v(E_{f_\mu} u(x))$. Thus $v : u(Z) \to \mathbb{R}$ is concave. □

Proof of Proposition 4. If $\succsim$ is represented by the ConBL model given in Eq. (12), $\succsim$ satisfies Axiom 15. The converse of the proof follows from the proof of de Oliveira et al. (2017, Corollary 1) after making the obvious adaptations. □

Proof of Proposition 5. It is clear that if $\succsim$ is represented by the passive information acquisition model given in Eq. (13), then $\succsim$ satisfies Axiom 16. The converse direction of the proof follows immediately from the proof of de Oliveira et al. (2017, Corollary 2) after making the obvious adaptations. □

Proof of Theorem 2. Let $(u, v, \bar{m}, c)$ represents a CBL preference. By obvious adaptations of the proof for de Oliveira et al. (2017, Theorem 2), c can be taken as a cost function satisfying the canonical properties given in Definition 3. But then, the rest of the proof that c is the unique canonical cost function follows immediately from the proof of de Oliveira et al. (2017, Theorem 2). □

Proof of Corollary 1. Assume that the given CBL preference $\succsim$ is represented both by $(u, v, \bar{m}, c)$ and $(u', v', \bar{m}', c')$, where c and c' are canonical. Since the restriction of $\succsim$ to the set of prize-lotteries has an expected utility representation, there exist some $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u' = \alpha u + \beta$. Similarly, since the restriction of $\succsim$ to the set of second-order prize-lotteries has an expected utility representation, there exist some $\lambda > 0$ and $\gamma \in \mathbb{R}$ such that $v'(u') = \lambda v(u) + \gamma$. Since the restriction of $\succsim$ to the set of act-lotteries has a second-order expected utility representation, by Seo (2009, Lemma C1) for any $P \in \Delta(\mathcal{F})$, we have $E_P E_{\bar{m}'} v(E_{f_\mu} u(x)) = E_P E_{\bar{m}} v(E_{f_\mu} u(x))$ showing that $\bar{m}$ and $\bar{m}'$ are essentially equivalent. Finally, by Theorem 2, for all $\pi' \in \Pi(\bar{m}')$,

$$\begin{aligned}
c'(\pi') &= \sup_{F \in \mathbb{F}} \left[E_\pi \max_{P \in F} \left[E_P E_{m'} v'(E_{f_\mu} u'(x)) \right] - E_P E_{\bar{m}'} v'(E_{f_\mu} u'(x)) \right] \\
&= \sup_{F \in \mathbb{F}} \left[E_\pi \max_{P \in F} \left[E_P E_{m'} \lambda v(E_{f_\mu} \alpha u(x) + \beta) \right] - E_P E_{\bar{m}'} \lambda v(E_{f_\mu} \alpha u(x) + \beta) \right] \\
&= \lambda \sup_{F \in \mathbb{F}} \left[E_\pi \max_{P \in F} \left[E_P E_{m'} v(E_{f_\mu} \alpha u(x) + \beta) \right] - E_P E_{\bar{m}'} v(E_{f_\mu} \alpha u(x) + \beta) \right] \\
&= \lambda \sup_{F \in \mathbb{F}} \left[E_\pi \max_{P \in F} \left[E_P E_{m'} v(E_{f_\mu} u(x)) \right] - E_P E_{\bar{m}'} v(E_{f_\mu} u(x)) \right] \\
&= \lambda \sup_{F \in \mathbb{F}} \left[E_\pi \max_{P \in F} \left[E_P E_m v(E_{f_\mu} u(x)) \right] - E_P E_{\bar{m}} v(E_{f_\mu} u(x)) \right] = \lambda\, c(\pi)
\end{aligned}$$

where π' and π are induced by the same signal structure from second-order priors $\bar{m}'$ and $\bar{m}$, respectively. This completes the proof. □

Proof of Theorem 3. It is straightforward to show that whenever $\succsim_2$ has a stronger desire for singletons than $\succsim_1$, then the restriction of $\succsim_2$ and $\succsim_1$ over the set of act-lotteries coincide. Thus we can normalize the parameters such that $(u_1, v_1, \bar{m}_1) = (u_2, v_2, \bar{m}_2)$. In this case, we have $c_2 \geq c_1$ if and only if $V_1 \geq V_2$ by Theorem 2. Thus, given that we have $(u_1, v_1, \bar{m}_1) = (u_2, v_2, \bar{m}_2)$, to show the equivalence of the conditions in Theorem 3, we only need to show that (i) $P \succ_1 F$ implies $P \succ_2 F$ for all P and F if and only if (ii) $V_1(F) \geq V_2(F)$ for all F. First, suppose (i) holds. Take a menu F and an act-lottery P such that $F \sim_2 P$. By (i), we must have $F \succsim_1 P$. This means we have $V_1(F) \geq V_1(P) = V_2(P) = V_2(F)$, and so $V_1(F) \geq V_2(F)$ for all F showing that (ii) is satisfied. For the converse, suppose (ii) holds; that is, $V_1(G) \geq V_2(G)$ for all G. Take a menu F and an act-lottery P such that $V_2(F) \geq V_2(P)$. Since $V_1(F) \geq V_2(F)$ and $V_2(P) = V_1(P)$, we have $V_1(F) \geq V_1(P)$, as desired. □

Data availability

No data was used for the research described in the article.

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