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RESEARCH ARTICLE

The difference between the weak core and the strong core from the design point of view

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Abstract

From a normative viewpoint, there is no compelling reason for preferring the weak over the strong core, and vice versa. However, the situation changes significantly from a mechanism design perspective. We work in a rights structures environment, where the role of the social planner is to allocate rights to individuals or coalitions which allow them to change the status-quo state. While coalitions are irrelevant for implementation in weak core (Koray and Yildiz in J. Econ. Theory 176:479-502, 2018; Korpela et al. in J. Econ. Theory 185:104953, 2020), our results show that they are fundamental for implementation in strong core. We fully characterize the implementation of social choice rules in strong core to outline this distinction. For robustness, we also characterize double implementation in weak and strong core which we show to be equivalent to implementation in weak core. Finally, we show that this equivalence breaks down in the more realistic case of implementation by codes of rights, where the set of states coincides with the set of outcomes.

Keywords Implementation · Rights structures · Weak core · Strong core

JEL Classification C79 · D47 · D71 · D82

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1 Introduction

Implementation theory studies the realization of collective goals, when the social planner does not have the relevant information and must rely on the agents to reveal it. Although this is usually achieved through designing mechanisms as noncooperative game forms,[1] a new approach to the implementation problem involves the design of "cooperative game forms" instead. Koray and Yildiz (2018) introduced the notion of a *rights structure*, where the role of the planner is to design the "power distribution" in the society by assigning rights to agents or coalitions that allow them to change the status quo state. This departure has been especially fruitful in overcoming well-known weaknesses of traditional implementation.[2]

A rights structure consists of (i) a state space,[3] (ii) an effectivity correspondence that, for each pair of states, assigns the family of coalitions that are effective for such change and, finally, (iii) an outcome function, which maps states into outcomes. This is the object of design for the social planner. The challenge is to design a rights structure such that the outcomes corresponding to the equilibrium (stable) states coincide with the social planner's goal, for any possible preference profile of the agents.

However, the choice of the equilibrium concept is non-trivial, as it reflects the strategic concerns of the agents and depends on the problem at hand. This paper focuses on the weak and strong core as equilibrium concepts. A state s is in the weak (strong) core if, for any state t, no coalition K is effective in changing the state from s to t, where all its members strictly prefer the outcome in t to the outcome in s (weakly prefer the outcome in t to the outcome in s, with strict preference for at least one member respectively).

We fully characterize implementation in strong core. Regarding the weak core, Korpela et al. (2018) show that Maskin-monotonicity and unanimity are necessary and sufficient. In the case of the strong core, we identify a new condition, *weak SC-monotonicity*, which, again alongside unanimity, is necessary and sufficient for implementation in strong core.

This result, when combined with the characterization of rules implementable in weak core, illuminates an interesting aspect related to the two equilibrium concepts: While any social choice rule that can be implemented in weak core by assigning rights to coalitions can be implemented by assigning rights only to individuals (Koray and Yildiz 2018), this equivalence breaks down in the case of strong core. In other words, we show that coalitions are fundamental for implementation in strong core, which is not the case for the weak core. Moreover, as Maskin-monotonicity implies weak SC-monotonicity and the converse is not true, we show that, in general, there are rules that are implementable in strong but not in weak core. We thus implicitly outline the importance of coalitional rights in mechanism design.

This result sharply contrasts conventional wisdom in policy-making. This is because, in many policy settings, coalition formation is considered undesirable and

[1] As pioneered by Maskin (1999).

[2] Such as integer and modulo games, as well as the sensitivity of the equilibrium outcome to the timing according to which the strategic interaction will take place.

[3] Not to be confused with the notion of state space in traditional mechanism design, as the set of possible type profiles.

something the institution design exercise should rule out. Moreover, as a matter of fact, Koray and Yildiz (2018) show that coalition formation can be excluded without loss of generality in the case of the weak core. Instead, we show that the social planner can utilize coalition formation as a powerful tool to implement strictly more social goals in the strong core compared to the weak core. To more transparently outline this point, we apply our results in a many-to-one matching environment with indifferences (as found in Erdil and Ergin 2017). Specifically, we show that the set of (strongly) Pareto efficient and stable matchings is implementable by rights structures in strong but not in weak core. Given the impossibility of achieving stability in a strategy-proof manner when both sides of the market are strategic agents, our result recovers some optimism for its implementation through decentralized protocols. Most importantly, our application shows that the strong core is indispensable in environments where indifferences matter.

While very crucial, we understand nevertheless that the choice of the appropriate equilibrium notion can be quite arbitrary from an ex-ante perspective: Indeed, it is hardly plausible to know in advance whether agents will join a coalition when they do not lose (as in the strong core) or when they strictly gain (as in the weak core). Most importantly, it is not necessarily the case that all agents follow the same behavioral protocol. To address these concerns, we explore *double implementation* in weak and strong core. This implementation exercise is robust to the coalition formation protocol the agents utilize (i.e. weak vs strong core). Surprisingly, it turns out that double implementation is equivalent to weak core implementation. As a last note, we show, however that when we restrict attention to the more realistic setting of *codes of rights*,[4] the latter equivalence breaks down. We fully characterize double implementation by codes of rights to illuminate this interesting asymmetry and show that this robustness comes at a cost compared to weak and strong core implementation.

Our paper is structured as follows: Section 2 discusses the relevant literature, and Section 3 contains our main results. Section 4 presents the application in a matching environment, and Section 5 presents our results on double implementation. Finally, Section 6 concludes. Appendix I contains longer proofs of our results, and Appendix II contains further results.

2 Relevant literature

Since the pioneering work of Maskin (1999), who studies the implementation of a social choice rule in Nash Equilibrium, the theory of implementation has explored numerous avenues, outlining the possibilities (and impossibilities) that arise in various environments. To name a few, implementation in subgame perfect Nash Equilibrium (Moore and Repullo 1988; Abreu and Sen 1990; Vartiainen 2007), repeated implementation in Nash Equilibrium (Mezzetti and Renou 2017), implementation in undominated strategies (Jackson 1992) or implementation with behavioral agents

[4] A rights structure where the set of states coincides with the set of outcomes and the outcome function is the identity map.

(Korpela 2012; de Clippel 2014; Lombardi and Yoshihara 2020; Hayashi et al. 2023).[5] Nevertheless, the theory has been criticized for its reliance on unnatural mechanisms.

The introduction of *rights structures* by Koray and Yildiz (2018) addresses the above concerns.[6] As this environment admits cooperative game-theoretic solution concepts, it can naturally incorporate coalition formation processes, and the results do not rely on the exact way the strategic interaction will take place. This is an interesting exercise, allowing one to penetrate the heart of the incentive problem without using unnatural features.

Since then, the literature on implementation by rights structures has been expanding exploring different equilibrium notions, including myopic ones such as the core (Koray and Yildiz 2018; Korpela et al. 2020; Savva 2021a; Korpela et al. 2024; Lombardi et al. 2023), farsighted ones (Korpela et al. 2021), stable sets (Korpela et al. 2022, 2021), and can also incorporate behavioral considerations (Savva 2021; Korpela et al. 2024).

The paper most closely related to our study is Lombardi et al. (2023), who study strong core implementation by codes of rights. They show that the intersection of strong Pareto with the no-envy correspondence is implementable in strong core by codes of rights in a rationing environment. Moreover, they show that weak and strong core implementation by codes of rights are logically independent, which is not true in our setting.

3 Main results

We first define our setting. Our environment consists of a set of agents $N = \{1, ..., n\}$, with $n \geq 2$ and a set of social outcomes X. The set of all nonempty coalitions is denoted by $\mathcal{N}_0 = 2^N \setminus \{\emptyset\}$. Each agent i has a complete and transitive preference relation over X that is denoted by R_i, and $(R_1, ..., R_n) \in \mathcal{R}$ is called a preference profile. As usual, by P_i we denote the asymmetric and by I_i we denote the symmetric part of R_i. For all $i \in N$, $x \in X$ and $R \in \mathcal{R}$, let $L_i(x, R) \equiv \{y \in X | x R_i y\}$ be the *lower contour set* of i with respect to x in preference profile R. In a similar manner, we define the *strict lower contour set* of i with respect to x in preference profile R as $SL_i(x, R) \equiv \{y \in X | x P_i y\}$. Finally, given a coalition K outcome x and preference profile R, let $I_K(x, R) = \cap_{i \in K}\{y \in X | x I_i y\}$, and $SL_K(x, R) = \cup_{i \in K} SL_i(x, R)$.

A *social choice rule* (SCR) $\phi : \mathcal{R} \rightrightarrows X$ is a correspondence that assigns a nonempty set of outcomes to each possible preference profile. By $\phi(\mathcal{R})$ we denote the image of ϕ.

A *rights structure* is a tuple $\Gamma = (S, h, \gamma)$, such that S is a state space, $h : S \to X$ is an outcome function, and $\gamma : S \times S \rightrightarrows \mathcal{N}_0 \cup \{\emptyset\}$ is a code of rights. For each $(s, t) \in S \times S$, $\gamma(s, t)$ is interpreted as the family of coalitions that are effective in moving from s to t. An individual-based rights structure is a rights structure such that,

[5] This is by no means an exhaustive list. A slightly old but excellent review of the topic can be found in Jackson (2001).

[6] The concept was originally introduced by Sertel (2002). Although not in an implementation framework, a similar notion can be found in McQuillin and Sugden (2011). Some earlier results that utilize effectivity functions can be found in Moulin and Peleg (1982) and Peleg and Winter (2002).

for all $(s, t) \in S \times S$, $S \in \gamma(s, t)$ implies that $S = \{i\}$ for some $i \in N$ that is, γ contains only singletons.

We now proceed with the definitions of our implementation notions.

Definition 3.1 An SCR ϕ *is implementable in weak core* by the rights structure $\Gamma = (S, h, \gamma)$ provided that, for all $R \in \mathcal{R}$, it holds:

(i) For all $x \in \phi(R)$, there exists $s \in S$ with $h(s) = x$ and, moreover, for all $t \in S$ and all $K \in \mathcal{N}_0$ such that $K \in \gamma(s, t)$, $x R_i h(t)$ for some $i \in K$.
(ii) For all $s \in S$, if for all $t \in S$ and all $K \in \mathcal{N}_0$ with $K \in \gamma(s, t)$, $h(s) R_i h(t)$ for some $i \in K$, then $h(s) \in \phi(R)$.[7]

Definition 3.2 An SCR ϕ *is implementable in strong core* by the rights structure $\Gamma = (S, h, \gamma)$ provided that, for all $R \in \mathcal{R}$, it holds:

(i) For all $x \in \phi(R)$, there exists $s \in S$ such that $h(s) = x$ and, moreover, for all $t \in S$ and all $K \in \mathcal{N}_0$ such that $K \in \gamma(s, t)$, $x P_i h(t)$ for some $i \in K$ or $x R_j h(t)$ for all $j \in K$.
(ii) For all $s \in S$, if for all $t \in S$ and all $K \in \mathcal{N}_0$ such that $K \in \gamma(s, t)$, $h(s) P_i h(t)$ for some $i \in K$ or $h(s) R_j h(t)$ for all $j \in K$, then $h(s) \in \phi(R)$.[8]

The above definitions uncover the essence of implementation in weak vs implementation in strong core.[9] The idea is that in the definition of weak core implementation, agent i can be thought of as a "whistle-blower" in that, in any preference profile R where $x \in \phi(R)$, he will prohibit a move from s where $h(s) = x$ to some other state t. At the same time, if at some preference profile R, there is no whistle-blower for some state s, then the outcome $h(s)$ has to be socially optimal at R. A crucial point to note is that, while the presence of the whistle-blower in the definition of weak core implementation is essential, it is not in the definition of the strong core. The reason is that the absence of the whistle-blower can be compensated with coalitional considerations. Everyone in the coalition can be indifferent between $h(s)$ and $h(t)$.

A natural restriction on a rights structure is to define the set of states S to be the set of outcomes X. Under this assumption, a *code of rights* is a rights structure such that $S \equiv X$ and $h : S \to X$ is the identity map. Then, an individual-based code of rights is a code of rights such that for all $x, y \in X$, $\gamma(x, y)$ contains only singletons when it is non-empty. As the state space in a rights structure can be quite abstract, implementation by codes of rights admits an intuitive interpretation and gives a more applied scope to the implementation problem.

The equilibrium and implementation notions for codes of rights (weak and strong core) are defined in a similar manner. To avoid unnecessary repetitions and since the majority of our results concern implementation by rights structures, we will refer to

[7] Equivalently, $\Gamma = (S, h, \gamma)$ implements ϕ in weak core if, for all $R \in \mathcal{R}$, $\phi(R) = h(C(\Gamma, R))$, where $C(\Gamma, R) = \{s \in S|$ there is no $t \in S$ and $K \in \gamma(s, t)$ such that for all $i \in K$, $h(t) P_i h(s)\}$.

[8] Equivalently, $\Gamma = (S, h, \gamma)$ implements ϕ in strong core provided that for all $R \in \mathcal{R}$, $\phi(R) = h(SC(\Gamma, R))$, where $SC(\Gamma, R) = \{s \in S|$ there is no $t \in S$ and $K \in \gamma(s, t)$ such that for all $i \in K$, $h(t) R_i h(s)$, with strict preference for at least one $j \in K\}$.

[9] We thank an anonymous referee for suggesting this formulation.

the latter as simply "implementation" and, only when we consider implementation by codes of rights will we refer to it as "implementation by codes of rights".

The first condition that we present, which is necessary for implementation in weak core, is Maskin-monotonicity:

Definition 3.3 An SCR ϕ satisfies Maskin-monotonicity with respect to $Y \supseteq \phi(\mathcal{R})$, if for all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $i \in N$

$$L_i(x, R) \cap Y \subseteq L_i(x, R'),$$

we have $x \in \phi(R')$.

Maskin-monotonicity is a well-known condition in mechanism design and social choice. It says that whenever an outcome x is socially optimal at preference profile R and weakly moves up in every agent's preference ranking when moving from R to R', then it must remain socially optimal in R' as well. Our condition is slightly different than the original formulation (Maskin 1999), as it concerns the intersection of the agents' lower contour sets with a set Y, thus taking the set of feasible outcomes into account, which is crucial in order to obtain a full characterization. As the flavor though is essentially the same, we chose to keep the same name.[10]

In their characterization of weak core implementation, Koray and Yildiz (2018) use a similar condition, called *image-monotonicity*. Its formulation is identical to Maskin-monotonicity with the exception that, instead of $L_i(x, R) \cap Y$, image-monotonicity concerns the intersection $L_i(x, R) \cap \phi(\mathcal{R})$ that is, the intersection of the lower contour sets with the image of ϕ. Image-monotonicity implies Maskin-monotonicity. To see this, it suffices to observe that since $Y \supseteq \phi(\mathcal{R})$, it follows that $L_i(x, R) \cap \phi(\mathcal{R}) \subseteq L_i(x, R')$ provided that $L_i(x, R) \cap Y \subseteq L_i(x, R')$. The converse implication is not generally true; therefore, Maskin-monotonicity is weaker than image-monotonicity.

We now proceed to the definition of our main condition weak SC-monotonicity:

Definition 3.4 An SCR ϕ satisfies weak SC-monotonicity with respect to $Y \supseteq \phi(\mathcal{R})$ if for all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $K \in \mathcal{N}_0$

$$[I_K(x, R) \bigcup SL_K(x, R)] \cap Y \subseteq I_K(x, R') \bigcup SL_K(x, R'),$$

we have $x \in \phi(R')$.

The interpretation of weak SC-monotonicity is intuitive: we can imagine the union $I_K(x, R) \cup SL_K(x, R)$ as coalition K's *collective lower contour set* for outcome x at R for the case of the strong core. Indeed, one can notice that if an outcome y is not in K's collective lower contour set and K can move from x to y, then it will do so. Then, one interpretation is that if x is socially optimal at R and, for any coalition K, the outcomes in its collective lower contour set for x at R are contained in its collective lower contour set for x at R', then x must be socially optimal at R' as well.[11]

Our final condition is a variant of unanimity:

Definition 3.5 An SCR ϕ satisfies unanimity with respect to $Y \supseteq \phi(\mathcal{R})$, if for all $R \in \mathcal{R}$ and $x \in Y$, whenever for all $i \in N$, $Y \subseteq L_i(x, R)$, then we have $x \in \phi(R)$.

[10] A similar condition appears in Moore and Repullo (1990).

[11] Again, taking into account the intersection with the set Y.

The interpretation is straightforward: If x is weakly on top of everyone's ranking at profile R (again considering the restriction to a set Y), then it must be selected as socially optimal. We are now ready to state our main result:

Theorem 1 *(A) (Korpela et al. (2018)) The following statements are equivalent:*

(i) An SCR ϕ is implementable in weak core.
(ii) An SCR ϕ satisfies Maskin-monotonicity and unanimity with respect to some $Y \supseteq \phi(\mathcal{R})$.
(iii) An SCR ϕ is implementable in weak core by an individual-based rights structure.

(B) The following statements are equivalent:

(i) An SCR ϕ is implementable in strong core.
(ii) An SCR ϕ satisfies weak SC-monotonicity and unanimity with respect to some $Y \supseteq \phi(\mathcal{R})$.

Proof See Appendix I. □

Theorem 1(A) generalizes the results of Koray and Yildiz (2018) who partially characterize implementation in weak core with rights structures in the universal domain of strict preferences. As they point out, an interesting corollary emerges: If an SCR is implementable in weak core by some rights structure, then it is also implementable by individual-based rights structures.[12] This implies that, in the case of the weak core, coalitional power is inconsequential and *practically of no value* to the social planner.

Theorem 1(B) extends the result of Lombardi et al. (2023) from codes of rights to the general case of rights structures. In this case, the equivalence with implementation by individual-based rights structures breaks down. While this peculiarity has been shown in a codes of rights environment for the case of the weak core (Korpela et al. 2021), we show that *coalitions are fundamental in a rights structure environment* as well, when one considers the strong core to be the appropriate solution concept. To illustrate this, we use the following example[13]:

Example 3.1 Let $\mathcal{R} = \{R, R'\}$, $X = \{x, y, z\}$ and $N = \{1, 2\}$:

R		R'	
R_1	R_2	R'_1	R'_2
y	xy	y	x
x	z	x	y
z		z	z

[12] However, given that their assumption of universal domain of strict preferences restricts the set of implementable SCRs, their result does not imply that this is true in general. In fact, the rights structure that they use cannot be used to derive the general result.

[13] As usual, $\frac{a}{b}$ means that aPb, and ab means that aIb.

Consider the rule that assigns the set of strongly Pareto optimal outcomes to each preference profile, that is, $\phi(R) = \{y\}$ and $\phi(R') = \{x, y\}$. First, the reader can verify that it satisfies weak SC-monotonicity and unanimity with respect to $Y = \{x, y\}$ and is thus implementable in strong core. However, we will show that it is not implementable in strong core by an individual-based rights structure.

To see this, consider an individual-based rights structure $\Gamma = (S, h, \gamma)$ and assume it implements ϕ in strong core. Since $x \in \phi(R')$, by implementability, there must exist $\bar{s} \in S$ where $h(\bar{s}) = x$, and, for all $t \in S$ and $i \in N$ such that $\{i\} \in \gamma(\bar{s}, t)$, we have $x \equiv h(\bar{s}) R'_i h(t)$.

Additionally, we have $x \notin \phi(R)$ so, for all $s \in S$ where $h(s) = x$, there must exist $t_s \in S$ and $i_s \in N$ such that $\{i_s\} \in \gamma(s, t_s)$ and $h(t_s) P_{i_s} h(s) \equiv x$. Specifically, as for agent 2 x is ranked at the top at both R and R', the only eligible candidate for i_s is 1, where $h(t_s) = y$. So, we established that, for all $s \in S$ where $h(s) = x$, $\{1\} \in \gamma(s, t)$ and $h(t) = y$. In such case though, $\bar{s}$ would fail to be a strong core equilibrium state at preference profile R', as agent 1's preferences are the same across R and R', a contradiction.

Now, as the following proposition shows, Maskin-monotonicity implies weak SC-monotonicity.

Proposition 1 *If an SCR ϕ satisfies Maskin-monotonicity with respect to $Y \supseteq \phi(\mathcal{R})$, then it satisfies weak SC-monotonicity with respect to Y.*

Proof Suppose that ϕ satisfies Maskin-monotonicity with respect to Y and assume that, for some $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, we have for all $K \in \mathcal{N}_0$, $[I_K(x, R) \cup SL_K(x, R)] \cap Y \subseteq I_K(x, R') \cup SL_K(x, R')$. We need to show that $x \in \phi(R')$ as weak SC-monotonicity requires. Now, since the above assumption holds for all K, let $K = \{i\}$. Then, we have that for all $i \in N$, $[I_i(x, R) \cup SL_i(x, R)] \cap Y = L_i(x, R) \cap Y \subseteq I_i(x, R') \cup SL_i(x, R') = L_i(x, R')$ and, by Maskin-monotonicity, we have $x \in \phi(R')$ as required. □

Using Proposition 1, we are able to logically connect the two related implementation notions:

Corollary 1 *If an SCR ϕ is implementable in weak core, then it is also implementable in strong core.*

Proof Direct implication of Theorem 1 and Proposition 1. □

While Maskin-monotonicity implies weak SC-monotonicity, the converse is not true, as shown in the example below:

Example 3.2 Let $N = \{1, 2\}$, $X = \{x, y\}$, $\mathcal{R} = \{R, R'\}$, $\phi(R) = \{x, y\}$, and $\phi(R') = \{y\}$.

R		R'	
R_1	R_2	R'_1	R'_2
x	y	xy	y
y	x		x

Let $Y = X$ and note that the premise of weak SC-monotonicity is not satisfied as $I_{\{1,2\}}(x, R) \cup SL_{\{1,2\}}(x, R) = \{x, y\} \nsubseteq I_{\{1,2\}}(x, R') \cup SL_{\{1,2\}}(x, R') = \{x\}$ and it thus holds vacuously. Also, notice that since $x \in \phi(R) \setminus \phi(R')$, $L_1(x, R) = \{x, y\} = L_1(x, R')$, and $L_2(x, R) = \{x\} = L_2(x, R')$, Maskin-monotonicity is violated.

Therefore, ϕ satisfies weak SC-monotonicity while it does not satisfy Maskin-monotonicity, which (given that it satisfies unanimity) makes it implementable in strong, but not in weak core.

The most important takeaway point from the above implementation exercise is that a social planner can implement strictly more SCRs in strong core than in weak core, as far as rights structures are concerned. However, this expansion comes only through the employment of coalitions. In other words, the social planner can implement an SCR that is not implementable in weak core (but is weakly SC-monotonic) in strong core, only if they are confident that coalitions will form (under the strong core behavioral protocol). And, indeed, in many environments it is natural to assume that no coalition formation will take place. In such cases, the weak and the strong core are no different from the design point of view.

Another interesting observation is in order when we compare our results with Lombardi et al. (2023): When we restrict attention to codes of rights, weak core implementation is logically independent from strong core implementation. However, as we have shown for the more general case of rights structures, weak core implementation implies strong core implementation. The channel for this peculiarity goes through Theorem 1: if an SCR is implementable in weak core, then it is implementable in weak core by some individual-based rights structures. However, for individual-based rights structures, it is easy to see that the weak core coincides with the strong core. Therefore, the SCR is implementable in strong core (by individual-based rights structures) as well. In a codes of rights environment though, as the equivalence between weak core implementation and individual-based weak core implementation does not hold in general, this logical implication breaks down. This implies that, in a codes of rights environment, whether a social planner designs an institution to implement in weak or strong core carries certain trade-offs. In a rights structures environment on the other hand, the choice of the strong core should be a "no-brainer": it allows for strictly more SCRs to be implemented.

We close this section with another useful remark: Lombardi et al. (2023) show that a condition named SC-monotonicity is necessary and almost sufficient (alongside unanimity) for implementation in strong core by codes of rights. We present it below:

Definition 3.6 An SCR ϕ satisfies SC-monotonicity if for all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $K \in \mathcal{N}_0$

$$\bigcap_{R \in \phi^{-1}(x)} [I_K(x, R) \bigcup SL_K(x, R)] \cap Y \subseteq I_K(x, R') \bigcup SL_K(x, R'),$$

we have $x \in \phi(R')$.

The intuition for this condition derives from the nature of the codes of rights environment. As the reader will notice, it bears some similarity with weak SC-monotonicity. However, now we have to take the intersection of the collective lower contour set of coalition K with respect to $x \in \phi(R)$, *for all the preference profiles where x is socially optimal*, that is, for all $R \in \phi^{-1}(x)$. This restriction comes from the fact that, in a codes of rights environment, the planner cannot infer any information regarding the true preference profile (and the respective lower contour sets) from the current state. Therefore, they have to consider all possible preference profiles for which x is socially optimal.

It is easy to see that SC-monotonicity implies weak SC-monotonicity,[14] while the converse is not true, as shown in the example below:

Example 3.3 Let $N = \{1, 2\}$, $X = \{x, y\}$, $\mathcal{R} = \{R, R', R''\}$ and $\phi(R) = \phi(R') = \{x, y\}$, while $\phi(R'') = \{y\}$. The preference profiles are given below:

R		R'		R''	
R_1	R_2	R'_1	R'_2	R''_1	R''_2
xy	y	y	xy	y	y
	x	x		x	x

First, we check that weak SC-monotonicity holds when moving from R to R'' and from R' to R''. This is easy to verify as $L_1(x, R) = \{x, y\} \nsubseteq \{x\} = L_1(x, R'')$ and $L_2(x, R') = \{x, y\} \nsubseteq \{x\} = L_2(x, R'')$. Therefore, the premises of weak SC-monotonicity do not hold and it holds vacuously. Now, we show that SC-monotonicity fails in this example. To see this, note that:

- $\bigcap_{R\in\phi^{-1}(x)} L_1(x, R) = \{x\} = L_1(x, R'')$.
- $\bigcap_{R\in\phi^{-1}(x)} L_2(x, R) = \{x\} = L_2(x, R'')$.
- $\bigcap_{R\in\phi^{-1}(x)}[I_{\{1,2\}}(x, R) \cup SL_{\{1,2\}}(x, R)] = \{x\} = I_{\{1,2\}}(x, R'') \cup SL_{\{1,2\}}(x, R'')$.

Therefore, the premises of SC-monotonicity are satisfied, yet $x \notin \phi(R'')$, which implies that SC-monotonicity fails. This makes the above SCR implementable in strong core by rights structures but not by codes of rights. This outlines the natural expansion of the implementable SCRs when the planner can enrich the state space with more information.

4 Application

In this section, we provide an application of our implementation results in a many-to-one matching environment with indifferences as in Erdil and Ergin (2017). This is a natural environment for an application of our previous results for two main reasons.

[14] It suffices to notice that, for any outcome x, preference profiles R, R' and coalition K, the set $\bigcap_{R\in\phi^{-1}(x)}[I_K(x, R) \cup SL_K(x, R)]$ is a subset of $I_K(x, R) \cup SL_K(x, R)$.

First, indifferences matter. Indeed, after the introduction of the one-to-one matching problem with strict preferences,[15] there has been a growing consensus[16] that indifferences cannot be ignored in a matching environment. This is especially due to the increasing use of online platforms, where candidates evaluate large numbers of possible matches. In such cases, preferences are naturally more coarse and indifferences are more likely to occur.

And, second, as is well-known in the matching literature, strategy-proofness is incompatible with stability in any type of matching algorithms, when both sides of the market are considered (Roth and Sotomayor 1992). Therefore, a market designer might be interested in knowing whether it is possible to decentralize the implementation of Pareto stability by relaxing the strategy-proofness condition. Our framework provides a fruitful path to explore this dimension.

We proceed with the formal definitions following closely the setting of Erdil and Ergin (2017). We have the set of agents $N = W \cup F$, with $W \cap F = \emptyset$ and for each $f \in F$, $q_f \in \mathbb{N}$ is the capacity of f. W can be interpreted as a set of workers and F as a set of firms. For each agent $i \in F$ ($i \in W$), there exists a weak preference $\succsim_i$ over $W \cup \{\emptyset\}$ (over $F \cup \{\emptyset\}$), with $\succ_i$ and $\sim_i$ as its strict and indifference part respectively. Agent i's set of possible preferences are denoted by $\mathcal{L}_i^F$, if $i \in F$, and $\mathcal{L}_i^W$, if $i \in W$. We make the following assumptions on $\mathcal{L}_f^F$ and $\mathcal{L}_w^W$:

Assumption 1 **(no indifference to unemployment/vacancy)**: For no $(w, f) \in W \times F$ and $(\succsim_w, \succsim_f) \in \mathcal{L}_w^W \times \mathcal{L}_f^F$, we have $w \sim_f \emptyset$ or $f \sim_w \emptyset$.

Assumption 2 **(responsive preferences)**: For all $f \in F$, $S, T, K \subseteq W$ with $S \cap K = T \cap K = \emptyset$ and $|S|, |T| \leq 1$, we have that $S \cup K \succsim_f T \cup K \iff S \succsim_f T$.

Given $\succsim_f \in \mathcal{L}_f^F$, we say that $w \in W$ is **acceptable** for $f \in F$, if $w \succ_f \emptyset$ (similarly we define acceptability for any $f \in F$, given $\succsim_w \in \mathcal{L}_w^W$). A *matching* is a function $\mu : W \rightarrow F \cup \{\emptyset\}$ such that, for all $f \in F$, $|\mu^{-1}(f)| \leq q_f$. Let the set of all possible matches be $\mathcal{M}$. We now extend preferences over partners to preferences over matchings in the usual way: For all $f \in F$, $w \in W$, $\succsim_f \in \mathcal{L}_f^F$ and $\succsim_w \in \mathcal{L}_w^W$:

$$\mu R_f \bar{\mu} \iff \mu^{-1}(f) \succsim_f \bar{\mu}^{-1}(f) \text{ and } \mu R_w \bar{\mu} \iff \mu(w) \succsim_w \bar{\mu}(w)$$

Then, let $R = ((R_f)_{f \in F}, (R_w)_{w \in W})$ be a preference profile and $\mathcal{R}$ be the set of all possible preference profiles.

A matching $\mu \in \mathcal{M}$ is *individually rational* if, for all $w \in W$ we have $\mu(w) \succsim_w \emptyset$ and, for all $f \in F$ and $k \in \mu^{-1}(f)$, $k \succsim_f \emptyset$. The next definition is the standard concept of blocking adapted for the particular environment:

Definition 4.1 Given $\succsim_f \in \mathcal{L}_f^F$ and $\succsim_w \in \mathcal{L}_w^W$, for any matching $\mu \in \mathcal{M}$, $(w, f) \in W \times F$ is called a *blocking pair* if:

(i) $f \succ_w \mu(w)$, and

[15] As in the seminal studies of Gale and Shapley (1962) or Roth and Sotomayor (1992).

[16] See Irving (1994), Abdulkadiroğlu et al. (2009), Domaniç et al. (2017), Chen and Li (2019), or Bonifacio et al. (2023) for example.

(ii) there exists $w' \in \mu^{-1}(f)$ such that $w \succ_f w'$, or $|\mu^{-1}(f)| < q_f$ and $w \succ_f \emptyset$.

Given a preference profile $R \in \mathcal{R}$, a matching $\mu \in \mathcal{M}$ is called *stable* if it is individually rational and there exists no blocking pair. We now define two well-known social choice rules in this environment.

Definition 4.2 The stable rule is defined as $S : \mathcal{R} \rightrightarrows \mathcal{M}$ such that, for all $R \in \mathcal{R}$, $S(R) = \{\mu \in \mathcal{M} | \mu \text{ is stable at } R\}$.

Definition 4.3 The (strongly) Pareto optimal rule is defined as $PO : \mathcal{R} \rightrightarrows \mathcal{M}$ such that, for all $R \in \mathcal{R}$,

$$PO(R) = \{\mu \in \mathcal{M} | \text{ there exists no } \mu' \in \mathcal{M} \text{ such that for all } i \in N, \mu' R'_i \mu \text{ and for some } j \in N, \mu' P_j \mu\}.$$

If only strict preferences are allowed, stability guarantees Pareto optimality. However, in our environment where indifferences are allowed this is not necessarily the case. Therefore, one is interested in implementing the intersection of S and PO as defined[17] below:

Definition 4.4 The Pareto stable rule $PS : \mathcal{R} \rightrightarrows \mathcal{M}$ is defined as $PS(R) = \{\mu \in \mathcal{M} | \mu \in S(R) \cap PO(R)\}$.

Erdil and Ergin (2017) showed that PS is non-empty and constructed an algorithm to find the set of Pareto stable matchings. However, as is well-known in matching environments, stability is not compatible with strategy-proofness. We show that PS is implementable in strong core, which recovers some optimism on its implementation. Moreover, we show that it is not implementable in weak core. These two results clearly outline the expansion of the set of implementable goals, when coalitions form under the strong core behavioral protocol. Our next proposition and example clarify this point:

Proposition 2 *PS is implementable in strong core.*

Proof We need to show that PS satisfies weak SC-monotonicity and unanimity with respect to some $Y \equiv \mathcal{M} \supseteq PS(\mathcal{R})$. We start with the former one. Consider $R, R' \in \mathcal{R}$ such that $\mu \in PS(R) \backslash PS(R')$. Now, suppose that the premises of weak SC-monotonicity are satisfied that is, for all $K \in \mathcal{N}_0$, we have $[I_K(\mu, R) \cup SL_K(\mu, R)] \subseteq I_K(\mu, R') \cup SL_K(\mu, R')$.

Now, since $\mu \notin PS(R')$, there are two possibilities: either (i) μ is not Pareto optimal, or (ii) μ is not stable. We consider each case separately:

(i) μ is not Pareto optimal that is, $\mu \notin PO(R')$: Then, there exists $\mu' \in \mathcal{M}$ such that, for all $i \in N$, $\mu' R'_i \mu$ and for some $j \in N$, $\mu' P'_j \mu$. Thus, we have that $\mu' \notin I_N(\mu, R') \cup SL_N(\mu, R')$ and, by our assumption $\mu' \notin [I_N(\mu, R) \cup SL_N(\mu, R)]$. In such case though, we have that for all $i \in N$, $\mu' R_i \mu$ and for some $j \in N$, $\mu' P_j \mu$, which contradicts that $\mu \in PS(R)$.

[17] See also Sotomayor (2011).

(ii) μ is not stable that is, $\mu \notin S(R')$: Here, we distinguish two further subcases:

(a) μ is not individually rational: **Case 1**: Suppose there exists $w \in W$ such that $\emptyset \succ_w \mu(w)$. Now, our assumption that for all $K \in \mathcal{N}_0$, $[I_K(\mu, R) \cup SL_K(\mu, R)] \subseteq [I_K(\mu, R') \cup SL_K(\mu, R')]$ implies that for all $i \in N$, $L_i(\mu, R) \subseteq L_i(\mu, R')$. Take any μ' such that $\mu'(w) = \emptyset$. Then, $\mu' \notin L_w(\mu, R')$ which, by our assumption, implies $\mu' \notin L_w(\mu, R)$, a contradiction. **Case 2**: Suppose there exists $f \in F$ and $k \in \mu^{-1}(f)$, where $\emptyset \succ_f k$. Now, take $\bar{\mu} \in \mathcal{M}$ where:

- For all $w \in W\backslash\{k\}$, $\mu(w) = \bar{\mu}(w)$,
- for all $\tilde{f} \in F\backslash\{f\}$, $\mu^{-1}(\tilde{f}) = \mu^{-1}(\tilde{f})$
- $\bar{\mu}^{-1}(f) = \mu^{-1}(f) \setminus \{k\}$, and
- $\bar{\mu}(k) = \emptyset$.

Now, from Assumption 2, $\bar{\mu} P'_f \mu$ and, since for all $i \in N$, $L_i(\mu, R) \subseteq L_i(\mu, R')$, we have $\bar{\mu} P_f \mu$, a contradiction.

(b) There exists a blocking pair: Then, there exists $(w, f) \in W \times F$ and a matching $\bar{\mu} \in \mathcal{M}$, where $\bar{\mu}(w) = f \neq \mu(w)$, such that $\bar{\mu} P'_f \mu$ and $\bar{\mu} P'_w \mu$. Then, since for all $i \in N$, $L_i(\mu, R) \subseteq L_i(\mu, R')$, we have $\bar{\mu} P_f \mu$ and $\bar{\mu} P_w \mu$, again a contradiction.

We now show that PS satisfies unanimity. Consider a preference profile $R \in \mathcal{R}$ and $\mu \in \mathcal{M}$ such that, for all $i \in N$, $\mathcal{M} \subseteq L_i(\mu, R)$. Then, clearly, μ is stable and strongly Pareto Optimal, so $\mu \in PS(R)$. This completes the proof.

□

To show that PS is not implementable in weak core, we present the following example:

Example 4.1 Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2\}$ $\mathcal{R} = \{R, R'\}$ and $q_{f_1} = q_{f_2} = 1$. The preferences are as below:

R				R'			
R_{f_1}	R_{f_2}	R_{w_1}	R_{w_2}	R_{f_1}	R_{f_2}	R_{w_1}	R_{w_2}
w_1	w_2	f_2	f_1	w_1	w_2	f_1 f_2	f_1 f_2
w_2	w_1	f_1	f_2	w_2	w_1	$\emptyset$	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$		

Let $\mu = ((f_1, w_2), (f_2, w_1))$ and $\mu' = ((f_1, w_1), (f_2, w_2))$ (which means that in μ f_1 is matched with w_2 and so on) and notice that $PS(R) = \{\mu, \mu'\}$, while $PS(R') = \{\mu'\}$. With the help of the following table where preferences on matchings are shown, one can verify that for all $i \in F \cup W$, $L_i(\mu, R) \subseteq L_i(\mu, R')$.[18]

[18] Notice all other matchings where an agent is unmatched are never socially optimal and their relative ranking does not change. Thus, we have omitted them and replaced them with dots.

R				R'			
R_{f_1}	R_{f_2}	R_{w_1}	R_{w_2}	R_{f_1}	R_{f_2}	R_{w_1}	R_{w_2}
$\mu' \ldots$	$\mu' \ldots$	$\mu \ldots$	$\mu \ldots$	$\mu' \ldots$	$\mu' \ldots$	$\mu\mu' \ldots$	$\mu\mu' \ldots$
$\mu \ldots$	$\mu \ldots$	$\mu' \ldots$	$\mu' \ldots$	$\mu \ldots$	$\mu \ldots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		

Hence, PS does not satisfy Maskin-monotonicity and is thus not implementable in weak core.

5 Double implementation in weak and strong core

So far we have made specific assumptions regarding the coalition formation process. Either a coalition forms when all its members strictly gain, in which case the relevant solution concept is the weak core or, a coalition forms when at least one member gains with the others not being worse off, in which case we apply the strong core. Apart from being quite extreme (it might not be the case that the same coalition formation protocol applies to all coalitions), these assumptions can also be criticized as unrealistic, as it is hardly plausible for the planner to be aware of the protocol in advance.

Therefore, in this section we study implementation of SCRs that is robust to whether agents behave according to the strong core or the weak core behavioral assumption. The relevant concept that we will use is *double implementation in weak and strong core*. Formally, a rights structure $\Gamma = (S, h, \gamma)$ *doubly implements* an SCR ϕ if, for all $R \in \mathcal{R}$, $h(C(\Gamma, R)) \subseteq \phi(R) \subseteq h(SC(\Gamma, R))$. Notice now that we demand each socially optimal outcome x to be a strong core outcome and, each weak core outcome to be socially optimal. As every strong core outcome is also a weak core one, the nestedness of the above sets implies equality.

Surprisingly, we show that double implementation in equivalent to weak core implementation, as our next theorem shows:

Theorem 2 *The following are equivalent:*

(i) An SCR ϕ is doubly implementable.
(ii) An SCR ϕ satisfies Maskin-monotonicity and unanimity with respect to some $Y \supseteq \phi(\mathcal{R})$.

Proof See Appendix I. □

An immediate observation follows:

Corollary 2 *An SCR ϕ is implementable in weak core if and only if it is doubly implementable.*

Proof Immediate implication of Theorems 1 and 2. □

At this point, a natural question arises: Does the equivalence between double and weak core implementation hold for the special case of codes of rights? And, how does double implementation relate to strong core implementation in such case? Our following results show that the answer to the first question is negative, as we show that double implementation by codes of rights restricts the set of implementable SCRs compared to both weak and strong core implementation (by codes of rights).

Before proceeding it is fruitful to review the result of Korpela et al. (2020) who characterize weak core implementation by codes of rights. Their key condition is called *strong-monotonicity*. An extra definition that we will use is that for all $K \in \mathcal{N}_0$, $x \in X$ and $R \in \mathcal{R}$, $L_K(x, R) \equiv \cup_{i \in K} L_i(x, R)$. Their main conditions are the following:

Definition 5.1 An SCR ϕ satisfies *strong-monotonicity* if for all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $K \in \mathcal{N}_0$

$$\bigcap\nolimits_{R \in \phi^{-1}(x)} L_K(x, R) \subseteq L_K(x, R'),$$

we have $x \in \phi(R')$.

Definition 5.2 An SCR ϕ satisfies *unanimity* if for all $R \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $i \in N$ we have $X \subseteq L_i(x, R)$, then $x \in \phi(R)$.

A few comments are in order: First, it is straightforward to see that unanimity implies unanimity with respect to Y (but the converse is not true in general). This is because, in a rights structures environment, unanimity with respect to the whole set of outcomes X is not a necessary condition. Second, similarly to our previous discussion on SC-monotonicity, strong-monotonicity takes into account all possible preference profiles in which $x \in \phi(R)$. The following theorem characterizes implementation in weak core by codes of rights:

Theorem 3 *[Korpela et al. (2020)] The following are equivalent:*

(i) An SCR ϕ is implementable in weak core by codes of rights.
(ii) An SCR ϕ satisfies strong-monotonicity and unanimity.

For our results on this issue, first, we characterize double implementation in codes of rights by identifying a condition that we call *D-monotonicity*. Then, we show that while D-monotonicity implies strong-monotonicity and SC-monotonicity (Proposition 3), the converses are not true (Examples 5.1 and 5.2). The definition of D-monotonicity can be found below:

Definition 5.3 An SCR ϕ satisfies *D-monotonicity* if for all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $K \in \mathcal{N}_0$

$$\bigcap\nolimits_{x \in \phi^{-1}(R)} [I_K(x, R) \cup SL_K(x, R)] \subseteq L_K(x, R'),$$

we have $x \in \phi(R')$.

We are now ready to characterize double implementation by codes of rights:

Theorem 4 *The following are equivalent:*

(i) An SCR ϕ is doubly implementable by codes of rights.
(ii) An SCR ϕ satisfies D-monotonicity and unanimity.

Proof See Appendix I. □

The next proposition logically relates double implementation with weak and strong implementation by codes of rights:

Proposition 3 *The following implications are true:*

(i) D-monotonicity implies strong-monotonicity.
(ii) D-monotonicity implies SC-monotonicity.

Proof The proof is obvious, as it follows directly from the respective implementation concepts. □

While we have shown that D-monotonicity implies strong and SC-monotonicity, the converses of these implications are not true. This is established by the following examples.

Example 5.1 Let $N = \{1, 2\}$, $X = \{x, y\}$, $\mathcal{R} = \{R, R', R''\}$, while $\phi(R) = \phi(R') = \{x, y\}$ and $\phi(R'') = \{y\}$. The preference profiles are shown below:

R		R'		R''	
R_1	R_2	R'_1	R'_2	R''_1	R''_2
xy	y	y	xy	y	y
	x	x		x	x

First, note that strong monotonicity is satisfied in this example. To see this, notice that $\bigcap_{R\in\phi^{-1}(x)} L_{\{1,2\}}(x, R) = \{x, y\} \nsubseteq L_{\{1,2\}}(x, R'') = \{y\}$, thus its premise is not fulfilled and it holds vacuously. Next, we claim that D-monotonicity fails. For this, note that $x \in [\phi(R) \cap \phi(R')] \setminus \phi(R'')$ and that:

- $\bigcap_{R\in\phi^{-1}(x)} L_1(x, R) = \{x\} = L_1(x, R'')$.
- $\bigcap_{R\in\phi^{-1}(x)} L_2(x, R) = \{x\} = L_2(x, R'')$.
- $\bigcap_{R\in\phi^{-1}(x)}[I_{\{1,2\}}(x, R) \cup SL_{\{1,2\}}(x, R)] = \{x\} = L_{\{1,2\}}(x, R')$.

Hence, ϕ is implementable in weak core by codes of rights but not doubly implementable (by codes of rights).

Example 5.2 Let $N = \{1, 2\}$, $X = \{x, y\}$, $\mathcal{R} = \{R, R', R''\}$, while $\phi(R) = x$, $\phi(R') = \{x, y\}$ and $\phi(R'') = \{y\}$. The preference profiles are shown below:

R		R'		R''	
R_1	R_2	R'_1	R'_2	R''_1	R''_2
y	x	x	y	xy	y
x	y	y	x		x

Now note that ϕ satisfies SC-monotonicity since we have $\cap_{R\in\phi^{-1}(x)}[I_{\{1,2\}}(x, R) \cup SL_{\{1,2\}}(x, R)] = \{x, y\} \nsubseteq \{x\} = I_{\{1,2\}}(x, R'') \cup SL_{\{1,2\}}(x, R'')$. However, ϕ does not satisfy D-monotonicity. This is because $x \in [\phi(R) \cap \phi(R')] \setminus \phi(R'')$ and:

- $\cap_{R\in\phi^{-1}(x)} L_1(x, R) = \{x\} \subseteq L_1(x, R'') = \{x, y\}$.
- $\cap_{R\in\phi^{-1}(x)} L_2(x, R) = \{x\} = L_1(x, R'')$.
- $\cap_{R\in\phi^{-1}(x)}[I_{\{1,2\}}(x, R) \cup SL_{\{1,2\}}(x, R)] = \{x, y\} = L_{\{1,2\}}(x, R'')$.

Thus, ϕ is implementable in strong core by codes of rights but not doubly implementable (by codes of rights).

We find the results above interesting in their own right: while in rights structures, double implementation is equivalent to weak core implementation, this is not the case for a codes of rights environment, where robustness to the identification of the coalition formation protocol comes at an additional cost to the social planner. This result derives again from the fact that in strong core coalitions matter. Indeed, one can notice that, if we restrict attention to individual-based codes of rights, then D-monotonicity collapses to singleton strong-monotonicity[19] and the equivalence between double and weak core implementation by (individual-based) codes of rights is restored. Since the same applies to SC-monotonicity which, under individual-based codes of rights, collapses to D-monotonicity as well, double implementation by individual-based codes of rights is also equivalent to strong core implementation (by individual-based codes of rights).

To further enhance our result, we show that, in general, the logical distance between double and weak core implementation by codes of rights is inherent to the notion and, apparently, persistent. To outline this, we add another layer of robustness, that of *external stability*:

Definition 5.4 Given a code of rights γ and a preference profile $R \in \mathcal{R}$, a set $Z \subseteq X$ is externally stable if, for all $x \in X \setminus Z$, there exists $y \in Z$ and $K \in \gamma(x, y)$ such that $yP_K x$.

Then, naturally, given a codes of rights γ and a preference profile $R \in \mathcal{R}$ we define the externally stable weak (strong) core $EC(\gamma, R)$ $(ESC(\gamma, R))$ as the set of weak (strong) core states, such that $EC(\gamma, R)$ $(ESC(\gamma, R))$ is an externally stable set. The relevant implementation notion is straightforward: a codes of rights doubly-implements an SCR ϕ with external stability if it doubly implements ϕ and for all $R \in \mathcal{R}$, $C(\gamma, R) = SC(\gamma, R)$ is an externally stable set.

[19] That would be the following condition: For all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $i \in N$, $\bigcap_{R\in\phi^{-1}(x)} L_i(x, R) \subseteq L_i(x, R')$, then $x \in \phi(R')$. As shown in Korpela et al. (2020), this is a necessary and (almost) sufficient condition for implementation in weak core with individual-based codes of rights.

Table 1 Logical connections between implementation conditions in rights structures and codes of rights environments

	Weak core implementation		Double implementation		Strong core implementation
Rights structures and external stability	Winner Maskin-monotonicity	$\Longleftrightarrow$	**Winner Maskin-monotonicity**	$\Rightarrow$	**Winner weak SC-monotonicity**
	$\Downarrow$		$\Downarrow$		$\Downarrow$
Rights structures	Maskin-monotonicity	$\Longleftrightarrow$	**Maskin-monotonicity**	$\Rightarrow$	**Weak SC-monotonicity**
	$\Uparrow$		$\Uparrow$		$\Uparrow$
Codes of rights	Strong-monotonicity	$\Leftarrow$	**D-monotonicity**	$\Rightarrow$	SC-monotonicity
	$\Uparrow$		$\Uparrow$		$\Uparrow$
Codes of rights and external stability	Winner strong-monotonicity	$\Leftarrow$	**Winner D-monotonicity**		**Winner SC-monotonicity**

External stability captures the idea that for any outcome that is not in the core (whether weak or strong), there should be an outcome in the core and a coalition effective for such change, such that each agent in the coalition prefers that change to take place. In other words, if for some reason we end up in a non-equilibrium state, there is always a (direct) path that will take us to an equilibrium one. Korpela et al. (2020) characterize externally stable implementation by codes of rights in weak core using a condition they call *winner strong-monotonicity*. To show that the extra requirement of external stability does not close the gap between double and weak core implementation by codes of rights, we provide a full characterization of externally stable double implementation by codes of rights. The first condition is a variant of winner strong-monotonicity which we call *winner D-monotonicity* and we define below:

Definition 5.5 An SCR ϕ satisfies winner D-monotonicity if for all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $K \in \mathcal{N}_0$

$$\bigcap_{R\in\phi^{-1}(x)}[I_K(x,R)\cup SL_K(x,R)]\cap\phi(R')\subseteq L_K(x,R'),$$

then $x \in \phi(R')$.

The second condition is found in Korpela et al. (2020) and is called *no simultaneous domination*:

Definition 5.6 An SCR ϕ satisfies no simultaneous domination if for all $x \in X$, $R \in \mathcal{R}$ and $y \in X \setminus \phi(R)$, there exists $i \in N$ and $y \in \phi(R)$ such that $yP_i x$.

No simultaneous domination guarantees that, for any non-socially optimal outcome, there exists a socially optimal one preferred by some agent. It is a slightly stronger condition that unanimity, as shown in the proposition below:

Proposition 4 *No simultaneous domination implies unanimity.*

Proof Suppose that ϕ satisfies no simultaneous domination and assume that for some $R \in \mathcal{R}$ and $x \in X$, we have for all $i \in N$, $X \subseteq L_i(x, R)$. Now, for the sake of contradiction, assume $x \notin \phi(R)$. Then, by no simultaneous domination, there must exist $j \in N$ and $y \in \phi(R)$, such that $yP_j x$. However, this contradicts our previous assumption. This completes the proof. □

Our characterization theorem for externally stable double implementation by codes of rights follows:

Theorem 5 *The following are equivalent:*

(i) An SCR ϕ satisfies winner D-monotonicity and no simultaneous domination.
(ii) An SCR ϕ is doubly implementable with external stability by codes of rights.

Proof See Appendix I. □

Again, it is straightforward to show that winner D-monotonicity implies D-monotonicity, yet the converse is not true. Therefore, we can conclude that the extra requirement of external stability does not close the gap between double implementation and weak core implementation by codes of rights.

To close our results section, and to better place them along the rest of the literature on implementation by rights structures, we present the logical connections between the several implementability conditions in a rights structures/codes of rights environment.[20] They are summarized in the following table (our contributions are in bold):

6 Concluding remarks

In this paper, we have fully characterized implementation in strong core by rights structures. After Koray and Yildiz (2018) and Korpela et al. (2020) have shown that coalitional rights do not matter when we consider the weak core implementation by rights structures, we have shown that coalitions are essential when we consider the strong core instead. Moreover, we have shown that there are social choice rules that are implementable in strong core, which are not implementable in weak core. An especially interesting case of such rule was shown in our application in a many-to-one matching environment with indifferences. Thus, we simultaneously outlined the expansion of the implementable social choice rules that occurs with the strong core as an equilibrium concept, as well as the fundamental importance of coalitional rights for institution design.

For robustness, we also considered double implementation in weak and strong core, which proved to be equivalent to implementation in weak core. This implies that the social planner cannot exploit the benefits of the strong core when considering double implementation. This equivalence breaks down in the special case of double implementation by codes of rights. This highlights that in a realistic mechanism design situation, the robustness of double implementation comes at a greater cost to the social planner. This result is robust to the addition of external stability as another desideratum.

Our results have completed the picture of implementation by rights structures and codes of rights using the weak and the strong core as solution concepts. However, we feel that this field is far from fully explored as there are many fruitful avenues for further research. One such direction is the issue of farsightedness which, apart from Korpela et al. (2021), has been little explored yet.

[20] For completeness, we also characterize externally stable strong core and double implementation by rights structures, as well as externally stable strong core implementation by codes of rights. When there is no logical operator between conditions, it means they are logically independent. Our formal results, including some results from Korpela et al. (2018) and Korpela et al. (2020), as well as the proofs, can be found in Appendix II.

Appendix I

Proof of Theorem 1(A)

The original proof can be found in Korpela et al. (2018). We restate it below for completeness.

Proof (i) $\Rightarrow$ (ii): Suppose that ϕ is implementable in weak core by $\Gamma = (S, h, \gamma)$ and let $Y = h(S)$. First, we show that ϕ satisfies unanimity. Let for all $i \in N$, $Y \subseteq L_i(x, R)$, for some $R \in \mathcal{R}$ and $x \in Y$. Suppose, for the sake of contradiction, that $x \notin \phi(R)$. Then, since Γ implements ϕ in weak core, $h(C(\Gamma, R)) = \phi(R)$, so there exists $K \in \mathcal{N}_0$ such that, for all $i \in K$, $h(t)P_i h(s) = x$, for some $t \in S$. But, this contradicts our assumption above.

Now, we show that ϕ satisfies Maskin-monotonicity. Let for all $i \in N$, $L_i(x, R) \cap Y \subseteq L_i(x, R')$ and suppose $x \notin \phi(R')$. First, since $x \in \phi(R)$, there exists $s \in S$, $h(s) = x$. Then, since $x \notin \phi(R')$, there exists $t \in S$ and $K \in \mathcal{N}_0$ such that, for all $i \in N$, $y \equiv h(t)P_i' h(s) = x$. Then, for all $i \in K$, $y \notin L_i(x, R')$ which, by our assumption, implies that $y \notin Y$, or that for all $i \in K$, $y \notin L_i(x, R)$. The first is rejected by the definition of Y. Assume the second. Then, we have that $s \notin C(\Gamma, R)$ and hence $x \notin \phi(R)$, a contradiction.

(ii) $\Rightarrow$ (iii): Suppose ϕ satisfies Masking-monotonicity and unanimity with respect to Y. Now, consider the following rights structure:

- $S = \{(x, R) \in Y \times \mathcal{R} | x \in \phi(R)\} \cup Y$.
- For all $s \in S$, $h((x, R)) = h(x) = x$.
- For all $i \in N$ and $s, t \in S$:

  (I) If $s = (x, R)$, then $\{i\} \in \gamma(s, t) \iff xR_i h(t)$.
  (I) If $s = x$, then $\{i\} \in \gamma(s, t)$.

We show that Γ implements ϕ in weak core:

Part 1: for all $R \in \mathcal{R}$, $\phi(R) \subseteq h(C(\Gamma, R))$.

Let $s = (x, R)$. Then, $x \in \phi(R)$. So, for all t and $i \in N$, we have $\{i\} \in \gamma(s, t)$ if and only if $xR_i h(t)$, so it must be that $s \in C(\Gamma, R)$.

Part 2: for all $R \in \mathcal{R}$, $h(C(\Gamma, R)) \subseteq \phi(R)$.

Consider $s \in C(\Gamma, R)$. We distinguish the following cases:

(a) $s = (y, R')$: Then, by the construction of Γ, we have that for all i, $L_i(y, R') \cap Y \subseteq L_i(x, R)$. Since ϕ satisfies monotonicity with respect to Y, we must have that $y \in \phi(R)$.
(b) $s = y$: Now, from the design of Γ, we have that for all i, $Y \subseteq L_i(y, R)$. Then, unanimity applies, so $y \in \phi(R)$. Finally, the implication (iii) $\Rightarrow$ (i) is trivial. This completes the proof.

□

Proof of Theorem 1(B)

Proof (i) $\Rightarrow$ (ii): Assume that ϕ is implementable in strong core by $\Gamma = (S, h, \gamma)$ and suppose that for some $R, R' \in \mathcal{R}, x \in \phi(R) \backslash \phi(R')$, while $[I_K(x, R) \bigcup SL_K(x, R)] \cap Y \subseteq I_K(x, R') \bigcup SL_K(x, R')$ holds for all $K \in \mathcal{N}_0$.

Let $Y \equiv h(S) \supseteq \phi(\mathcal{R})$. Since $x \in \phi(R)$, there exists s, such that $h(s) = x$. Furthermore, since $x \notin \phi(R')$, there exists t and $K \in \gamma(s, t)$, such that for all $i \in K$, $h(t) \equiv y R_i' x$ and for some $j \in K$, $y P_j' x$. This implies that $y \notin \cup_{i \in K} SL_i(x, R')$ and $y \notin \cap_{i \in K} L_i(x, R') \supseteq I_K(x, R')$, so $y \notin I_K(x, R')$. By our assumption, we have either that $y \notin Y$, which is rejected by the definition of Y, or that $y \notin I_K(x, R) \bigcup SL_K(x, R)$. But then, we have that for all $i \in K$, $y R_i x$ and there exists $j \in K$ such that $y P_j x$, which contradicts that $x \in \phi(R)$. Thus ϕ satisfies weak SC-monotonicity with respect to Y.

The proof of the necessity of unanimity is very similar to the Proof of Theorem 1(A) and is thus omitted.

(ii) $\Rightarrow$ (i): Assume that ϕ satisfies weak SC-monotonicity and unanimity with respect to Y and consider $\Gamma = (S, h, \gamma)$ as in the proof of Theorem 1(A), with the exception that rule (I) is substituted with:

(Ia) If $s = (x, R)$, then $K \in \gamma(s, t) \iff$ for all $i \in K$, $x I_i h(t)$ or there exists $j \in K$, such that $x P_j h(t)$.

We now show that Γ implements ϕ in strong core:

Part 1: for all $R \in \mathcal{R}$, $\phi(R) \subseteq h(SC(\Gamma, R))$.

Let $s = (x, R)$. Then, $x \in \phi(R)$. Then, for all t and $K \in \mathcal{N}_0$, we have $K \in \gamma(s, t) \iff$ for all $i \in K$, $x I_i h(t)$, or there exists $j \in K$, $x P_j h(t)$. We clearly have that $s \in SC(\Gamma, R)$.

Part 2: for all $R \in \mathcal{R}$, $h(SC(\Gamma, R)) \subseteq \phi(R)$.

Consider $s \in SC(\Gamma, R)$. We now distinguish the following cases:

(a) $s = (y, R')$: Then, we have that for all K, $[I_K(y, R') \bigcup SL_K(y, R')] \cap Y \subseteq I_K(y, R) \bigcup SL_K(y, R)$. Thus, by weak SC-monotonicity, we have $y \in \phi(R)$.

(b) $s = y$: By the design of γ, we have that for all i, $Y \subseteq L_i(y, R)$. Thus, unanimity applies, so $y \in \phi(R)$. This completes the proof.

□

Proof of Theorem 2

Proof (i)$\Rightarrow$(ii): Let ϕ be doubly implementable by $\Gamma = (S, h, \gamma)$ and let $Y = h(S)$. Now consider $R, R' \in \mathcal{R}$ such that, for all $i \in N$, we have $L_i(x, R) \cap Y \subseteq L_i(x, R')$, for some $x \in \phi(R)$. Suppose $x \notin \phi(R')$.

Since $x \in \phi(R)$, there exists $s \in SC(\Gamma, R)$, such that $h(s) = x$. Now, since $x \notin \phi(R')$, we have $s \notin C(\Gamma, R')$, so there exists $K \in \mathcal{N}_0$ and $t \in S$ with $K \in \gamma(s, t)$, where $y \equiv h(t) \notin L_K(x, R')$, so for all $i \in K$, we have $y \notin L_i(x, R')$. By our assumption, this implies either that $y \notin Y$ (which is rejected by the definition of Y), or that for all $i \in K$, $y \notin L_i(x, R)$. However, in such case, we have $K \in \gamma(s, t)$ and for all $i \in K$, $y \notin L_i(x, R)$ which implies $s \notin SC(\Gamma, R) \subseteq C(\Gamma, R)$, a contradiction.

Showing the necessity of unanimity is similar to the previous proofs and is thus omitted.

(ii)⇒(i): Assume that an SCR ϕ satisfies Maskin-monotonicity and unanimity. We now show that $\Gamma = (S, h, \gamma)$ as defined in the proof of Theorem 2 doubly implements ϕ.

Part 1: for all $R \in \mathcal{R}$, $\phi(R) \subseteq h(SC(\Gamma, R))$.

Let $s = (x, R)$. Then, $x \in \phi(R)$ and, for all $t \in S$ and $K \in \mathcal{N}_0$, we have $K \in \gamma(s, t)$ if and only if $h(t) \in I_K(x, R) \cup SL_K(x, R)$ and thus, $x \in SC(\Gamma, R) \subseteq C(\Gamma, R)$.

Part 2: for all $R \in \mathcal{R}$, $h(C(\Gamma, R)) \subseteq \phi(R)$.

We distinguish the following cases:

(a) $s = (y, R') \in C(\Gamma, R)$: Then, we have that for all $K \in \mathcal{N}_0$ and $t \in S$ with $K \in \gamma(s, t)$, $[I_K(y, R') \cup SL_K(y, R')] \cap Y \subseteq I_K(x, R) \cup SL_K(x, R)$. Since this holds for all $K \in \mathcal{N}_0$, let $K = \{i\}$. Then, we have that for all $i \in N$, $L_i(x, R') \cap Y \subseteq L_i(x, R)$. By Maskin-monotonicity, we have $x \in \phi(R)$ as required.

(b) $s = y \in C(\Gamma, R)$: As before, we have for all $i \in N$, $Y \subseteq L_i(y, R)$ and by unanimity we have $y \in \phi(R)$ as required.

□

Proof of Theorem 4

(i)⇒(ii): Let ϕ be doubly implementable by γ. Consider $R, R' \in \mathcal{R}$ with $x \in \phi(R)$ such that, for all $K \in \mathcal{N}_0$, we have $\bigcap_{R \in \phi^{-1}(x)}[I_K(x, R) \cup SL_K(x, R)] \subseteq L_K(x, R')$. For the sake of contradiction, assume $x \notin \phi(R')$

Since $x \in \phi(R)$, we have $x \in SC(\gamma, R)$. Now, since $x \notin \phi(R')$, it is not possible that $x \in C(\Gamma, R')$ and there exists $y \in X$, $K \in \mathcal{N}_0$ with $K \in \gamma(x, y)$ such that for all $i \in K$ we have $y P'_i x$. Then, by our assumption, there exists $R'' \in \mathcal{R}$ with $x \in \phi(R'')$ such that $y \notin I_K(x, R'') \cup SL_K(x, R'')$. This implies that for all $i \in K$, $y R''_i x$ and for some $j \in K$, $y P''_j x$. However, since $K \in \gamma(x, y)$, this contradicts that $x \in SC(\gamma, R'') = \phi(R'')$, a contradiction.

Again, the proof of necessity of unanimity is similar to the previous proofs and is thus omitted.

(ii)⇒(i):

Assume that an SCR ϕ satisfies D-monotonicity and unanimity. We now show that γ as defined below doubly implements ϕ. For all $x, y \in X$:

- If $x \in \phi(\mathcal{R})$, then for all $K \in \mathcal{N}_0$, $K \in \gamma(x, y) \iff y \in \bigcap_{R \in \phi^{-1}(x)}[I_K(x, R) \cup SL_K(x, R)]$.
- If $x \notin \phi(\mathcal{R})$, then for all $K \in \mathcal{N}_0$, $K \in \gamma(x, y)$.

We distinguish the following cases:

Part 1: For all $R \in \mathcal{R}$, $\phi(R) \subseteq SC(\gamma, R)$.

Let $x \in \phi(R)$. Then, clearly, since by construction we have $K \in \gamma(x, y) \iff y \in \bigcap_{R \in \phi^{-1}(x)}[I_K(x, R) \cup SL_K(x, R)]$, it must be that $y \in SC(\gamma, R)$.

Part 2: For all $R \in \mathcal{R}$, $C(\gamma, R) \subseteq \phi(R)$.

We distinguish the following cases:

(a) $x \in \phi(\mathcal{R})$. Then, $x \in \phi(R'')$ for some $R'' \in \mathcal{R}$. Since $x \in C(\gamma, R)$, from the construction of the codes of rights γ we have $\bigcap_{R \in \phi^{-1}(x)}[I_K(x, R'') \cup SL_K(x, R'')] \subseteq L_K(x, R)$. By D-monotonicity, we have $x \in \phi(R)$ as required.

(b) $x \notin \phi(\mathcal{R})$. In such case, by construction, for all $K \in \mathcal{N}_0$ and $y \in X$, $K \in \gamma(x, y)$, so for x to be in $C(\gamma, R)$, we have for all $K \in \mathcal{R}$, $X \subseteq L_K(x, R)$. Since this holds for all K, it holds for all singleton coalitions too. Now, the premises of unanimity are fulfilled, thus $x \in \phi(R)$ as well. This completes the proof.

Proof of Theorem 5

(i)⇒(ii):

Suppose ϕ is doubly implementable with external stability by codes of rights. First, we show that ϕ satisfies no simultaneous domination. This is straightforward to show: Suppose that this is not the case. So, consider $R \in \mathcal{R}$ and suppose there exists $y \in X \setminus \phi(R)$ such that, for all $i \in N$ and $x \in \phi(R)$, yR_ix. Fix $x \in \phi(R)$. Then, by implementability, $x \in EC(\gamma, R)$ and thus external stability of $EC(\gamma, R)$ is violated.

Second, we show that ϕ satisfies winner D-monotonicity. Suppose that this is not the case and assume there exist $R, R' \in \mathcal{R}$ and $x \in \phi(R) \setminus \phi(R')$ such that, for all $K \in \mathcal{N}_0$, $\bigcap_{R \in \phi^{-1}(x)}[I_K(x, R) \cup SL_K(x, R)] \cap \phi(R') \subseteq L_K(x, R')$. Now, $x \notin \phi(R') = EC(\gamma, R')$ so, by external stability, there exists $y \in \phi(R')$ and $K \in \gamma(x, y)$ such that for all $i \in K$, yP'_ix. Now, this implies that $y \notin L_K(x, R')$ which, by our assumption above (and given that $y \in \phi(R')$), implies $y \notin \bigcap_{R \in \phi^{-1}(x)}[I_K(x, R) \cup SL_K(x, R)]$. Then, there exists $\bar{R} \in \mathcal{R}$ with $x \in \phi(\bar{R})$ such that for all $i \in K$, $y\bar{R}_ix$ and there exists $j \in K$ with $y\bar{P}_jx$. However, since $K \in \gamma(x, y)$, this contradicts that $x \in SC(\gamma, \bar{R})$.

(ii)⇒(i):

Assume that an SCR ϕ satisfies winner D-monotonicity and no simultaneous domination. It is straightforward to show that γ as defined in the proof of Theorem 5 doubly implements ϕ with external stability and the proof is omitted. We will now show that, for any $R \in \mathcal{R}$, $C(\gamma, R)$ is an externally stable set. Suppose that this is not the case. Then, there exists $R \in \mathcal{R}$ and $y \in X \setminus C(\gamma, R)$ such that for all $x \in \phi(R)$ and $K \in \gamma(y, x)$ we have $x \in L_K(y, R)$. We distinguish the following cases:

(a) $y \in \phi(\mathcal{R})$: Then, there exists $\bar{R} \in \mathcal{R}$ such that $y \in \phi(\bar{R})$. Now, by double implementability, $y \notin \phi(R)$, so $y \in \phi(\bar{R}) \setminus \phi(R')$. By winner D-monotonicity then, there exists $K \in \mathcal{N}_0$ and $x \in \bigcap_{\bar{R} \in \phi^{-1}(y)}[I_K(y, \bar{R}) \cup SL_K(y, \bar{R})]$ such that $x \notin L_K(y, R)$. However, by the construction of γ and the violation of external stability that we assumed above, we have that for all $x \in \phi(R)$ and all $K \in \mathcal{N}_0$, $x \in \bigcap_{\bar{R} \in \phi^{-1}(y)}[I_K(y, \bar{R}) \cup SL_K(y, \bar{R})]$ implies $x \in L_K(x, R)$, which is a contradiction.

(b) $y \notin \phi(\mathcal{R})$: In such case, from the construction of γ, we have for all $i \in N$ and $x \in X$, $\{i\} \in \gamma(y, x)$. Now, since $y \in C(\gamma, R)$, by the construction of γ, we have for all $i \in N$, yR_ix, which contradicts that ϕ satisfies no simultaneous domination. This completes the proof.

Appendix II: Supplemental results

Externally stable weak core implementation

In this section, we characterize weak core implementation (by rights structures) with external stability. First, we re-define the notion for the general case of rights structures:

Definition 6.1 Given a rights structure $\Gamma = (S, h, \gamma)$ and a preference profile $R \in \mathcal{R}$, a set of states $T \subseteq S$ is externally stable if, for all $s \in S \setminus T$, there exists $t \in T$ and $K \in \gamma(s, t)$ such that, for all $i \in K$, $h(t) P_i h(s)$.

Our relevant characterization conditions are the following:

Definition 6.2 An SCR ϕ satisfies winner Maskin-monotonicity if, for all $R, R' \in \mathcal{R}$, and $x \in \phi(R)$, whenever for all $i \in N$

$$L_i(x, R) \cap \phi(R') \subseteq L_i(x, R'),$$

then we have $x \in \phi(R')$.

Definition 6.3 An SCR ϕ satisfies no simultaneous domination with respect to $Y \supseteq \phi(\mathcal{R})$ if for all $R \in \mathcal{R}$ and $x \in Y \setminus \phi(R)$, there exists $y \in \phi(R)$ and $i \in N$ such that $y P_i x$.

Winner Maskin-monotonicity is a variant of Maskin-monotonicity adapted to account for external stability. For the reader who has become familiar with our results already, it will be straightforward to see that winner Maskin-monotonicity implies Maskin-monotonicity, but not the other way around. No simultaneous domination with respect to Y is slightly different than the no simultaneous domination condition that we have used for the codes of rights environment. It is straightforward to show that it implies unanimity with respect to Y. Our characterization theorem goes as follows:

Theorem 6 *The following are equivalent:*

(i) An SCR ϕ satisfies winner Maskin-monotonicity and no simultaneous domination with respect to some $Y \supseteq \phi(\mathcal{R})$.
(ii) An SCR ϕ is implementable in weak core with external stability.

Proof The proof is very similar to the proof of Theorem 1 (with the same rights structure for the sufficiency part) by taking external stability into account. Therefore, we will only show the necessity of no simultaneous domination with respect to Y and winner Maskin-monotonicity. We start with the former. Let ϕ be implementable in weak core with external stability by $\Gamma = (S, h, \gamma)$ and suppose that it does not satisfy no simultaneous domination with respect to any set Y. We will show that this cannot be the case. Let $Y = h(S) \supseteq \phi(\mathcal{R})$. Then, take $R \in \mathcal{R}$ and suppose there exists $y \in Y \setminus \phi(R)$ such that, for all $i \in N$ and $x \in \phi(R)$, $y R_i x$. Fix $x \in \phi(R)$. By the definition of Y, there exist $s \in S$, $h(s) = y$ and $t \in S$, $h(t) = x$. Then, by implementability, $x \in \phi(R)$ implies $s \in EC(\Gamma, R)$ so, for all $i \in N$ we have $y = h(s) R_i h(t) = x$. Clearly, external stability of $EC(\Gamma, R)$ is violated.

To show the necessity of winner Maskin-monotonicity, suppose that $x \in \phi(R) \setminus \phi(R')$ and for all $i \in N$, $L_i(x, R) \cap \phi(R') \subseteq L_i(x, R')$. First, note that, since $x \in \phi(R)$, there exists $s \in EC(\Gamma, R)$ such that $h(s) = x$. Since $x \notin \phi(R')$, we have $s \notin EC(\Gamma, R')$ so, by external stability, there exists $t \in EC(\Gamma, R')$ and $K \in \gamma(s, t)$ such that for all $i \in K$, $y \equiv h(t) P_i h(s)$. This implies that $y \notin L_i(x, R')$ and, by our assumption, we have for all $i \in N$, $y \notin \phi(R')$, or $y \notin L_i(x, R)$. The first case is rejected since $t \in EC(\Gamma, R') = \phi(R')$. The second one is also rejected since for all $i \in K$, $y \notin L_i(x, R)$ and $K \in \gamma(s, t)$ implies that $x \notin EC(\Gamma, R)$. This completes the proof. □

Externally stable strong core implementation

The relevant condition for externally stable strong core implementation is the following:

Definition 6.4 An SCR ϕ satisfies winner SC-monotonicity if, for all $R, R' \in \mathcal{R}$, and $x \in \phi(R)$, whenever for all $K \in \mathcal{N}_0$

$$I_K(x, R) \cup SL_K(x, R) \cap \phi(R') \subseteq I_K(x, R') \cup SL_K(x, R'),$$

then we have $x \in \phi(R')$.

Theorem 7 *The following are equivalent:*

(i) An SCR ϕ satisfies winner SC-monotonicity and no simultaneous domination with respect to $Y \supseteq \phi(\mathcal{R})$.
(ii) An SCR ϕ is implementable in strong core with external stability.

Proof The proof is very similar to previous proofs and thus omitted. □

While external stable strong core implementation is more demanding than strong core implementation, some interesting SCRs are implementable. In our finite model, it can be shown that the rule that assigns the set of strongly Pareto optimal outcomes to each preference profile is implementable in strong core with external stability.[21]

Externally stable double implementation

Not very surprisingly, double implementation by external stability is equivalent to weak core implementation by external stability (in the case of rights structures). Our theorem below clarifies:

Theorem 8 *The following are equivalent:*

(i) An SCR ϕ satisfies winner Maskin-monotonicity and no simultaneous domination with respect to some $Y \supseteq \phi(\mathcal{R})$.
(ii) An SCR ϕ is doubly implementable with external stability.

Proof Again, the proof is very similar to previous proofs and is thus omitted. □

21 A similar result can be shown for externally stable weak core implementation and the rule that assigns the set of weakly Pareto optimal outcomes.

Externally stable strong core implementation by codes of rights

In this section we characterize implementation in strong core by codes of rights and external stability. Our relevant condition is the following:

Definition 6.5 An SCR ϕ satisfies winner SC-monotonicity if for all $R, R' \in \mathcal{R}$ and $x \in \phi(R)$, whenever for all $K \in \mathcal{N}_0$ we have

$$\bigcap_{R \in \phi^{-1}(x)}[I_K(x, R) \cup SL_K(x, R)] \cap \phi(R') \subseteq [I_K(x, R') \cup SL_K(x, R')], \text{ then we have}$$

$x \in \phi(R')$.

Our theorem follows:

Theorem 9 *The following are equivalent:*

(i) An SCR ϕ satisfies winner SC-monotonicity and no simultaneous domination.
(ii) An SCR ϕ is implementable in weak core with external stability.

Proof The proof works exactly like previous proofs and is thus omitted. □

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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