An input-output "fundamental lemma" for quarter-plane causal 2D models

Paolo Rapisarda, *Member, IEEE*, Yueqing Zhang, *Member, IEEE*

***Abstract*—If an input-output data trajectory generated by a 2D quarter-plane causal system is "sufficiently informative", then any system trajectory restriction (an "unfolding") is a finite linear combination of data unfoldings. We also design experiments to generically obtain sufficiently informative data.**

***Index Terms*—Data-driven control; Linear systems; Computational methods**

I. Introduction

In [2] we showed that special restrictions ("*unfoldings*", see Section IV therein) of input-state (i-s) trajectories of a controllable quarter-plane causal 2D system are linear combinations of unfoldings of one "sufficiently informative" (i-s) trajectory. Assuming that the state is measurable postulates an insight about the system structure that is at odds with a truly data-driven approach, where problems should be formulated at the level of *external* (input and output) variables. In this paper we address such weakness and we show that given sufficiently informative *input-output* (i-o) data, *any* i-o unfolding is a linear combination of *data* unfoldings, thus providing a *data-driven parametrization* of restrictions of *i-o* trajectories. In the 1D case the relevance of such parametrizations for simulation, control, and signal processing is well known, see [9]. Our results for quarter-plane causal 2D systems have the potential of delivering a comparable impact, given the wide use of such models in image-processing, sensor networks, and iterative learning control (see [13], [5], [7], [17], [18]; data-driven approaches to other classes of nD systems are in [1], [10]). Our main results are stated in terms of input-output variables and their properties *only*: state-space representations (specifically, Fornasini-Marchesini ones) are only used in the proofs. In this paper we also state a sufficient *persistency of excitation* condition for sufficient informativity.

The paper is structured as follows: in Section II we gather some background material; we also introduce data matrices, unfoldings, and informativity for identification. Section III contains a $2D$-version of the "fundamental lemma" of [20]. In Section IV we design input sequences corresponding to sufficiently informative input-output data. In Section V we summarize our results and illustrate our current research.

Notation

$\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are respectively the set of natural, integer and real numbers, and $\mathbb{Z}^2 := \mathbb{Z} \times \mathbb{Z}$. $\mathbb{R}^n$ is the space of n-dimensional vectors with real entries. $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices with real entries; and $\mathbb{R}^{n\times\infty}$ the set of real matrices with n rows and an infinite number of columns. The transpose of $M \in \mathbb{R}^{n\times m}$ is denoted by $M^\top$ and its pseudoinverse by $M^\dagger$; the image of M is denoted by $\operatorname{im}(M)$.

If A_i, $i = 1,\ldots,n$, are matrices with the same number of columns, we define $\operatorname{col}(A_i)_{i=1,\ldots,n} := \begin{bmatrix} A_1^\top & \ldots & A_n^\top \end{bmatrix}^\top$. We denote the set $\{w : \mathbb{Z}^2 \to \mathbb{R}^q\}$ of q-dimensional doubly-indexed sequences by $(\mathbb{R}^q)^{\mathbb{Z}^2}$, and the set $\{w : \mathbb{Z} \to \mathbb{R}^q\}$ by $(\mathbb{R}^q)^{\mathbb{Z}}$. If $w_i \in (\mathbb{R}^{q_i})^{\mathbb{Z}^2}$, $i = 1,\ldots,n$, we define $\operatorname{col}(w_i)_{i=1,\ldots n}(k,\ell) := \begin{bmatrix} w_1(k,\ell)^\top & \ldots & w_n(k,\ell)^\top \end{bmatrix}^\top \in \left(\mathbb{R}^{\sum_{i=1}^n q_i}\right)^{\mathbb{Z}^2}$. Analogous notation is used for sequences $w_i \in (\mathbb{R}^{q_i})^{\mathbb{Z}}$, $i = 1,\ldots,n$. $\mathbb{R}[z]$ is the ring of polynomials with real coefficients in z; $\mathbb{R}[z_1, z]$ the ring of polynomials with real coefficients in z_1, z, and $\mathbb{R}^{n\times m}[z_1, z]$ the ring of $n \times m$ matrices with entries in $\mathbb{R}[z_1, z]$. Given $S \subset \mathbb{R}[z_1, z_2]$, we denote by $\langle S \rangle$ the module generated by the elements of S. The same notation is used for modules of the ring $\mathbb{R}^{1\times m}[z_1, z]$ of polynomial row vectors with m entries.

We denote by σ_i, $i = 1, 2$, the shifts on $(\mathbb{R}^q)^{\mathbb{Z}^2}$: $(\sigma_1 w)(i,j) := w(i+1,j)$ and $(\sigma_2 w)(i,j) := w(i,j+1)$. We define σ_i^{-1}, $i = 1, 2$ by $\left(\sigma_1^{-1} w\right)(i,j) := w(i-1,j)$ and $\left(\sigma_2^{-1} w\right)(i,j) := w(i,j-1)$. We denote the composition of σ_1 and σ_2^{-1} by $\sigma := \sigma_1 \circ \sigma_2^{-1}$.

II. Background material

A. Fornasini-Marchesini second models

The Fornasini-Marchesini second model (referred to as FM in the rest of the paper) is described by the equations:

$$\begin{aligned} \sigma_1 x &= A_1 x + A_2 \sigma x + B_1 u + B_2 \sigma u \\ y &= Cx + Du \,, \end{aligned} \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i = 1, 2$ and $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$; the *state* $x(i,j) \in \mathbb{R}^n$, the *input* $u(i,j) \in \mathbb{R}^m$, and the *output* $y(i,j) \in \mathbb{R}^p$. The other standard representations of quarter-plane causal $2D$ systems are Roesser models (see [15]). These are equivalent to the FM ones (see [6]); thus we use (1) without loss of generality.

We associate to (1) three sets of trajectories:

- the *input-output behavior* defined by

$$\mathfrak{B} := \left\{\operatorname{col}(u,y) : \mathbb{Z}^2 \to \mathbb{R}^{m+p} \mid \exists\, x : \mathbb{Z}^2 \to \mathbb{R}^n \text{ s.t. } \operatorname{col}(u,x,y) \text{ satisfies (1)}\right\} \;; \tag{2}$$

- the *input-state behavior* defined by

$$\mathfrak{B}_{x,u} := \left\{\operatorname{col}(u,x) : \mathbb{Z}^2 \to \mathbb{R}^{m+n} \mid \operatorname{col}(u,x) \text{ satisfies the first equation (1)}\right\} \;; \tag{3}$$

P. Rapisarda and Y. Zhang are with the Electrical Power Engineering group, ECS, University of Southampton, Great Britain (e-mail: pr3@ecs.soton.ac.uk, Yueqing.Zhang@soton.ac.uk).

- the *input-state-output behavior* defined by

$$\mathfrak{B}_{x,u,y} := \{\operatorname{col}(u,x,y) : \mathbb{Z}^2 \to \mathbb{R}^{m+n+p} \mid \operatorname{col}(u,x,y) \text{ satisfies (1)}\} \ . \quad (4)$$

The first equation in (1) can be equivalently written as

$$(\sigma_1 I_n - A_1 - A_2\sigma)\, x + (-B_1 - B_2\sigma)\, u = 0 \, .$$

Define $A(z) := A_1 + A_2 z$ and $B(z) := B_1 + B_2 z$; then $\mathfrak{B}_{x,u} = \ker R(\sigma_1, \sigma)$, where

$$R(z_1, z) := \begin{bmatrix} z_1 - A(z) & -B(z) \end{bmatrix} \in \mathbb{R}^{n\times(n+m)}[z_1, z] \, .$$

We denote by $\mathcal{L}_k$ the k-th *diagonal line* in $\mathbb{Z}\times\mathbb{Z}$:

$$\mathcal{L}_k := \{(i,j) \in \mathbb{Z}^2 \mid i + j = k\} \ , \ k = 0, \ldots, N \ ,$$

and define $\mathcal{L}_{0:N} := \bigcup_{i=0,\ldots,N} \mathcal{L}_i$. Given $f : \mathbb{Z}^2 \to \mathbb{R}^q$, we denote by $f_{|\mathcal{L}_k}$ the restriction of f to $\mathcal{L}_k$. We associate with $f_{|\mathcal{L}_k}$ the $1D$ sequence with i-th term $f_{k+i,-i}$, $i = 0, \ldots$.

We define global reachability (see [3]).

Definition 1. *The model* (1) *is* globally reachable *if* $\forall\ x^* : \mathbb{Z} \to \mathbb{R}^n$ *there exist* $N \in \mathbb{N}$, $u : \mathcal{L}_{0:N} \to \mathbb{R}^m$ *and* $x : \mathcal{L}_{0:N} \to \mathbb{R}^n$ *such that* $x_{|\mathcal{L}_0} = 0$, $\operatorname{col}(x,u) \in \mathfrak{B}_{x,u}$ *and* $x_{|\mathcal{L}_{N+1}} = x^*$.

The following characterization of global reachability is used in Section IV to establish experiment design results.

Theorem 1. *The following statements are equivalent:*

1) *The FM model* (1) *is globally reachable;*
2) $\operatorname{rank}\begin{bmatrix} B(z) & A(z)B(z) & \ldots & A(z)^{n-1}B(z)\end{bmatrix} = n$*;*
3) *If* $v(z)\begin{bmatrix} B(z) & A(z)B(z) & \ldots & A(z)^{n-1}B(z)\end{bmatrix} = 0$ *for* $v \in \mathbb{R}^{1\times n}[z]$*, then* $v = 0$.

B. Data matrices and unfoldings

Let $\operatorname{col}(\hat{u}, \hat{y}) \in \mathfrak{B}$; we define the *data set* as $\operatorname{col}(\hat{u}, \hat{y})_{\mathcal{L}_{0:N}}$.

Given $j \in \mathbb{N}$, we denote by $\mathcal{H}_j(\hat{y}_{|\mathcal{L}_k})$ the block-Hankel matrix with $(j+1)p$ rows and an infinite number of columns:

$$\mathcal{H}_j(\hat{y}_{|\mathcal{L}_k}) := \begin{bmatrix} \cdots & \hat{y}_{k-1,1} & \hat{y}_{k,0} & \cdots \\ \cdots & \hat{y}_{k,0} & \hat{y}_{k+1,-1} & \cdots \\ \cdots & \hat{y}_{k+1,-1} & \hat{y}_{k+2,-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \hat{y}_{k+j-1,-j+1} & \hat{y}_{k+j,-j} & \cdots \end{bmatrix} ; \quad (5)$$

we define $\mathcal{H}_j(\hat{u}_{|\mathcal{L}_k}) \in \mathbb{R}^{(j+1)m\times\infty}$ analogously. Note that each column of (5) consists of $(j+1)$ consecutive values of $\hat{y}_{|\mathcal{L}_k}$. Analogous considerations hold for $\mathcal{H}_j(\hat{u}_{|\mathcal{L}_k}) \in \mathbb{R}^{(j+1)m\times\infty}$.

We define the *data matrix* $\mathbb{D}_N(\operatorname{col}(\hat{u}, \hat{y}))$ by

$$\mathbb{D}_N(\operatorname{col}(\hat{u},\hat{y})) := \left[\begin{array}{c} \mathcal{H}_N(\hat{u}_{|\mathcal{L}_0}) \\ \mathcal{H}_{N-1}(\hat{u}_{|\mathcal{L}_1}) \\ \vdots \\ \mathcal{H}_0(\hat{u}_{|\mathcal{L}_N}) \\ \hline \mathcal{H}_N(\hat{y}_{|\mathcal{L}_0}) \\ \mathcal{H}_{N-1}(\hat{y}_{|\mathcal{L}_1}) \\ \vdots \\ \mathcal{H}_0(\hat{y}_{|\mathcal{L}_N}) \end{array}\right] \in \mathbb{R}^{(m+p)\frac{(N+1)(N+2)}{2}\times\infty} \, . \quad (6)$$

The columns of (6) are constructed by "unfolding" the values of $\hat{u}$ and $\hat{y}$ on an equilateral triangle of $\mathbb{Z}^2$ with vertex at $(k+N, -k)$ and side length $N+1$. We call the restriction of $\hat{u}$ and $\hat{y}$ on any such equilateral triangle an N*-unfolding of* $\operatorname{col}(\hat{u}, \hat{y})$ *at* $(k, -k)$.

Example 1. Consider the lattice depicted in Figure 1 and set $N = 2$. In Figure 1 we use different colors to distinguish the points on $\mathcal{L}_0$ (green), $\mathcal{L}_1$ (blue) and $\mathcal{L}_2$ (red). We compute

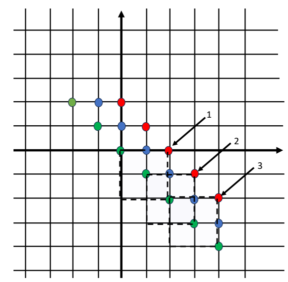

Fig. 1. Lattice for the construction of $\mathbb{D}_2$ for Example 1.

$\mathcal{H}_2(\hat{u}_{|\mathcal{L}_0})$, $\mathcal{H}_1(\hat{u}_{|\mathcal{L}_1})$ and $\mathcal{H}_0(\hat{u}_{|\mathcal{L}_2})$ and stack these matrices obtaining (7).

$$\begin{bmatrix} \mathcal{H}_2(\hat{u}_{|\mathcal{L}_0}) \\ \mathcal{H}_1(\hat{u}_{|\mathcal{L}_1}) \\ \mathcal{H}_0(\hat{u}_{|\mathcal{L}_2}) \end{bmatrix} = \left[\begin{array}{cc:c:cc} \cdots & \hat{u}_{0,0} & \hat{u}_{1,-1} & \hat{u}_{2,-2} & \cdots \\ \cdots & \hat{u}_{1,-1} & \hat{u}_{2,-2} & \hat{u}_{3,-3} & \cdots \\ \cdots & \hat{u}_{2,-2} & \hat{u}_{3,-3} & \hat{u}_{4,-4} & \cdots \\ \cdots & \hat{u}_{1,0} & \hat{u}_{2,-1} & \hat{u}_{3,-2} & \cdots \\ \cdots & \hat{u}_{2,-1} & \hat{u}_{3,-2} & \hat{u}_{4,-3} & \cdots \\ \cdots & \hat{u}_{2,0} & \hat{u}_{3,-1} & \hat{u}_{4,-2} & \cdots \end{array}\right] \quad (7)$$

The first column of such matrix corresponds to the "equilateral triangle" of $\mathbb{Z}^2$ labelled "1" in Figure 1, consisting of

$$\{(0,0), (1,-1), (2,-2), (1,0), (2,-1), (2,0)\} \ ;$$

the second column, to the triangle labelled "2", consisting of

$$\{(1,-1), (2,-2), (3,-3), (2,-1), (3,-2), (3,-1)\} \ ;$$

the third one, to that labelled "'3" in Figure 1.

The definition of the matrices $\mathcal{H}_2(\hat{y}_{|\mathcal{L}_0})$, $\mathcal{H}_1(\hat{y}_{|\mathcal{L}_1})$ and $\mathcal{H}_0(\hat{y}_{|\mathcal{L}_2})$, and consequently of $\begin{bmatrix} \mathcal{H}_2(\hat{y}_{|\mathcal{L}_0}) \\ \mathcal{H}_1(\hat{y}_{|\mathcal{L}_1}) \\ \mathcal{H}_0(\hat{y}_{|\mathcal{L}_2}) \end{bmatrix}$, is analogous. ■

C. Informativity for identification

The set of *left-annihilators of the data* is defined by

$$\mathcal{N}(\operatorname{col}(\hat{u},\hat{y})_{\mathcal{L}_{0:N}}) := \left\{\eta \in \mathbb{R}^{1\times(m+p)}[z_1, z] \mid \eta(\sigma_1, \sigma)\operatorname{col}(\hat{u},\hat{y})_{|\mathcal{L}_{0:N}} = 0\right\} \ ,$$

and we denote by $\langle \mathcal{N}(\operatorname{col}(\hat{u},\hat{y})_{\mathcal{L}_{0:N}}\rangle$ the module of $\mathbb{R}^{1\times(m+p)}[z_1, z]$ generated by its elements.

We define $\mathcal{N}(\mathfrak{B})$, the *module of annihilators* of $\mathfrak{B}$, by

$$\mathcal{N}(\mathfrak{B}) := \left\{\eta \in \mathbb{R}^{1\times(m+p)}[z_1, z] \mid \eta(\sigma_1, \sigma)\operatorname{col}(u, y) = 0 \ \forall \ \operatorname{col}(u,y) \in \mathfrak{B}\right\} \ .$$

It is a standard result in $2D$ behavioral system theory that given a kernel representation $\ker R(\sigma_1, \sigma) = \mathfrak{B}$ with $R \in$

$\mathbb{R}^{g\times(m+p)}[z_1,z]$, $\mathcal{N}(\mathfrak{B})$ consists of the module generated by the rows of $R(z_1,z)$.

"Sufficient richness" of the data is defined as follows.

Definition 2. *The data* $\mathrm{col}(\widehat{u},\widehat{y})_{\mathcal{L}_{0:N}}$ *are* informative for identification *if* $\langle\mathcal{N}(\mathrm{col}(\widehat{u},\widehat{y})_{\mathcal{L}_{0:N}})\rangle = \mathcal{N}(\mathfrak{B})$.

$\langle\mathcal{N}(\mathrm{col}(\widehat{u},\widehat{y})_{\mathcal{L}_{0:N}})\rangle \supseteq \mathcal{N}(\mathfrak{B})$, since each element in $\mathcal{N}(\mathfrak{B})$ annihilates all trajectories in $\mathfrak{B}$, in particular $\mathrm{col}(\widehat{u},\widehat{y})$. Informativity for identification implies the opposite inclusion: *all* annihilators of *all* trajectories of $\mathfrak{B}$ belong to $\langle\mathcal{N}(\mathrm{col}(\widehat{u},\widehat{y})_{\mathcal{L}_{0:N}})\rangle$. This property was characterized in Theorem 2 in [14] for *autonomous* quarter-plane causal systems. The result was generalized to the case when $y=x$ (directly measurable state variable) in Theorem 1 of [2]. In the next section we extend such characterization to *input-output* data.

III. An i-o 'fundamental lemma'

Let $M(z)=M_0+M_1z+\ldots+M_rz^r\in\mathbb{R}^{g\times q}[z]$ and let $L\geqslant r$. The *coefficient matrix* of $M(z)$, denoted by $\mathrm{coeff}(M(z))$, is the $g\times q(L+1)$ matrix defined by

$$\mathrm{coeff}(M(z)) := \begin{bmatrix} M_0 & M_1 & \ldots & M_r & 0_{g\times q} & \ldots & 0_{g\times q}\end{bmatrix} .$$

Note that $M(z)=\mathrm{coeff}(M(z))\,\mathrm{col}\left(z^iI_q\right)_{i=0,\ldots,L}$. Note also that if $f:\mathbb{Z}^2\to\mathbb{R}^q$ then for every $k\in\mathbb{Z}$ it holds that

$$M(\sigma)f_{|\mathcal{L}_k} = \mathrm{coeff}(M(z))\,\mathrm{col}\left(\sigma^i f_{|\mathcal{L}_k}\right)_{i=0,\ldots,L} . \tag{8}$$

Given (1), we define for $k=0,\ldots,N-1$ the matrices

$$\begin{aligned}\mathcal{D}_k &:= \begin{bmatrix} D & 0 & \ldots & 0\\ 0 & D & \ldots & 0\\ \vdots & \ldots & \ddots & \vdots\\ 0 & 0 & \ldots & D\end{bmatrix}\\ &= \mathrm{col}(\mathrm{coeff}\left(z^iD\right))_{i=0,\ldots,N-1-k}\\ \mathcal{O}_k &:= \mathrm{col}(\mathrm{coeff}\left(Cz^iA(z)^k\right))_{i=0,\ldots,N-1-k} .\end{aligned}$$

Note that $\mathcal{D}_k\in\mathbb{R}^{(N-k)p\times(N-k)m}$ and $\mathcal{O}_k\in\mathbb{R}^{(N-k)p\times(N-k)n}$. We also define the matrices $\mathcal{M}_{k,j}$ by $\mathcal{M}_{0,j}:=\mathcal{D}_0$, $j=0,\ldots,k-1$; and for $k=1,\ldots,N-1$

$$\mathcal{M}_{k,j} := \mathrm{col}(\mathrm{coeff}\left(Cz^iA(z)^{k-1-j}B(z)\right))_{i=0,\ldots,N-1-k} .$$

Finally, we define

$$\mathcal{S} := \left[\begin{array}{ccccc} I & 0 & \ldots & 0 & 0\\ 0 & I & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \ldots & I & 0\\ \hline \mathcal{D}_0 & 0 & \ldots & 0 & \mathcal{O}_0\\ \mathcal{M}_{1,0} & \mathcal{D}_1 & \ldots & 0 & \mathcal{O}_1\\ \mathcal{M}_{2,0} & \mathcal{M}_{2,1} & \ldots & 0 & \mathcal{O}_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathcal{M}_{N-1,0} & \mathcal{M}_{N-1,1} & \ldots & \mathcal{D}_{N-1} & \mathcal{O}_{N-1}\end{array}\right] . \tag{9}$$

The following result is instrumental to establish the main result of this paper.

Proposition 1. *Define* $\mathcal{S}$ *by* (9). *The following statements are equivalent:*

1) $\mathrm{col}(u,x,y)$ *is a trajectory of* (1)*;*
2) *The following equation holds:*

$$\left[\begin{array}{c}\mathrm{col}(\sigma^iu_{|\mathcal{L}_0})_{i=0,\ldots,N-1}\\ \mathrm{col}(\sigma^iu_{|\mathcal{L}_1})_{i=0,\ldots,N-2}\\ \vdots\\ u_{|\mathcal{L}_{N-1}}\\ \hline \mathrm{col}(\sigma^iy_{|\mathcal{L}_0})_{i=0,\ldots,N-1}\\ \mathrm{col}(\sigma^iy_{|\mathcal{L}_1})_{i=0,\ldots,N-2}\\ \mathrm{col}(\sigma^iy_{|\mathcal{L}_2})_{i=0,\ldots,N-3}\\ \vdots\\ \widehat{y}_{|\mathcal{L}_{N-1}}\end{array}\right] = \mathcal{S}\begin{bmatrix}\mathrm{col}(\sigma^iu_{|\mathcal{L}_0})_{i=0,\ldots,N-1}\\ \mathrm{col}(\sigma^iu_{|\mathcal{L}_1})_{i=0,\ldots,N-2}\\ \vdots\\ u_{|\mathcal{L}_{N-1}}\\ \mathrm{col}(\sigma^ix_{|\mathcal{L}_0})_{i=0,\ldots,N}\end{bmatrix} \tag{10}$$

Moreover, let $\mathrm{col}(u,x)\in\mathfrak{B}_{x,u}$*; then* $\mathrm{col}(u,y)$ *defined by* (10) *is an input-output trajectory of* $\mathfrak{B}$.

Proof. x and y are solutions of (1) if and only if their restrictions to consecutive diagonal lines $\mathcal{L}_i$ satisfy the equations

$$\begin{aligned} x_{|\mathcal{L}_i} &= A(\sigma)^ix_{|\mathcal{L}_0} + \sum_{j=0}^{i-1}A(\sigma)^{i-j-1}B(\sigma)u_{|\mathcal{L}_j} \qquad (11)\\ y_{|\mathcal{L}_i} &= CA(\sigma)^ix_{|\mathcal{L}_0} + \sum_{j=0}^{i-1}CA(\sigma)^{i-j-1}B(\sigma)u_{|\mathcal{L}_j} + Du_{|\mathcal{L}_i}\end{aligned}$$

The equivalence of statements 1) and 2) follows. The second claim is straightforward. □

The main result of this section is the following.

Theorem 2. *Let* $\mathrm{col}(\widehat{u},\widehat{x},\widehat{y})$ *be a trajectory of* (1).

Every linear combination of a finite number of columns of

$$\mathbb{D}_{N-1}(\mathrm{col}(\widehat{u},\widehat{y})) = \left[\begin{array}{c}\mathrm{col}(\sigma^i\widehat{u}_{|\mathcal{L}_0})_{i=0,\ldots,N-1}\\ \mathrm{col}(\sigma^i\widehat{u}_{|\mathcal{L}_1})_{i=0,\ldots,N-2}\\ \vdots\\ \widehat{u}_{|\mathcal{L}_{N-1}}\\ \hline \mathrm{col}(\sigma^i\widehat{y}_{|\mathcal{L}_0})_{i=0,\ldots,N-1}\\ \mathrm{col}(\sigma^i\widehat{y}_{|\mathcal{L}_1})_{i=0,\ldots,N-2}\\ \vdots\\ \widehat{y}_{|\mathcal{L}_{N-1}}\end{array}\right] , \tag{12}$$

is an N*-unfolding of a trajectory of* $\mathfrak{B}$*. Moreover, if*

$$\mathrm{rank}\left(\begin{bmatrix}\mathrm{col}(\sigma^i\widehat{u}_{|\mathcal{L}_0})_{i=0,\ldots,N-1}\\ \mathrm{col}(\sigma^i\widehat{u}_{|\mathcal{L}_1})_{i=0,\ldots,N-2}\\ \vdots\\ \widehat{u}_{|\mathcal{L}_{N-1}}\\ \mathrm{col}(\sigma^i\widehat{x}_{|\mathcal{L}_0})_{i=0,\ldots,N-1}\end{bmatrix}\right) = \frac{N(N+1)}{2}m + Nn , \tag{13}$$

then every N*-unfolding of every trajectory* $\mathrm{col}(u,y)\in\mathfrak{B}$ *is a linear combination of a finite number of columns of* (12).

Proof. Any linear combination of the columns of (12) is the unfolding of a finite linear combination, with the same coefficients, of σ-shifts of $(\widehat{u},\widehat{y})\in\mathfrak{B}$. Since $\mathfrak{B}$ is linear and $\alpha_\ell\in\mathbb{R}$, $\ell=0,\ldots,N$, then

$$\mathrm{col}(\widehat{u},\widehat{y})\in\mathfrak{B} \Longrightarrow \sum_{\ell=0}^{N}\alpha_\ell\sigma^\ell\,\mathrm{col}(\widehat{u},\widehat{y})\in\mathfrak{B} .$$

This proves the first part of the claim. We prove the second part of the claim. Using the shift-invariance of $\mathfrak{B}$ we assume without loss of generality that the given unfolding of $\operatorname{col}(u,y) \in \mathfrak{B}$ involves the first N diagonal lines. Since $\operatorname{col}(u,y) \in \mathfrak{B}$, there exists a trajectory x such that $\operatorname{col}(u,x,y) \in \mathfrak{B}_{u,x,y}$; moreover, $\operatorname{col}(u,y)_{\mathcal{L}_{0:N-1}}$ and $\operatorname{col}(u,x)_{\mathcal{L}_{0:N-1}}$ are related by (10). Such relation defines an analogous one between every unfolding of $\operatorname{col}(u,y)_{\mathcal{L}_{0:N-1}}$ and a corresponding unfolding of $\operatorname{col}(u,x)_{\mathcal{L}_{0:N-1}}$, i.e. between each column of the matrix on the left-hand side of (10) and the corresponding column of the matrix appearing on the right-hand side of (10). Such unfolding of $\operatorname{col}(u,x)_{\mathcal{L}_{0:N-1}}$ is a real vector with $\sum_{j=1}^{N} jm + Nn$ components; since by assumption (13) the matrix on the right-hand side of (10) is surjective, such vector can be written as a linear combination of its columns. Combining the columns of $\mathbb{D}_{N-1}(\operatorname{col}(\widehat{u},\widehat{y}))$ in (12) with the same coefficients yields the given $\operatorname{col}(u,y)$ unfolding. This concludes the proof. $\square$

Example 2. Consider the SISO system described by

$$\begin{aligned} \sigma_1 x &= \begin{bmatrix} 0 & \frac{1}{10} \\ \frac{9}{10} & 0 \end{bmatrix} x + \begin{bmatrix} 0 & \frac{1}{10} \\ \frac{2}{10} & 0 \end{bmatrix} \sigma x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma u \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x \,. \end{aligned} \tag{14}$$

It is straightforward to verify that

$$\begin{aligned} &(9 + 11\sigma + 2\sigma^2)y - 100\sigma_1^2 y \\ &+(100 + 40\sigma u + 10\sigma^2)u + (200 + 100\sigma)\sigma_1 u = 0 \,, \end{aligned} \tag{15}$$

equivalently for $i \in \mathbb{Z}$ it holds that

$$\begin{aligned} &(9 + 11\sigma + 2\sigma^2)y_{|\mathcal{L}_i} - 100 y_{|\mathcal{L}_{i+2}} \\ &+(100 + 40\sigma u + 10\sigma^2)u_{|\mathcal{L}_i} + (200 + 100\sigma)u_{|\mathcal{L}_{i+1}} = 0 \,. \end{aligned}$$

We generate 102-samples long random sequences $\widehat{x}_{|\mathcal{L}_0}$ and $\widehat{u}_{|\mathcal{L}_i}$, $i = 0, \ldots, 3$. We generate the corresponding sequences $\widehat{x}_{|\mathcal{L}_i}$, $i = 1,2,3$ via (14). The 18×97 matrix

$$H(\widehat{u},\widehat{x}) := \begin{bmatrix} \operatorname{col}\left(\sigma^i \widehat{u}_{|\mathcal{L}_0}\right)_{i=0,\ldots,3} \\ \operatorname{col}\left(\sigma^i \widehat{u}_{|\mathcal{L}_1}\right)_{i=0,\ldots,2} \\ \operatorname{col}\left(\sigma^i \widehat{u}_{|\mathcal{L}_2}\right)_{i=0,1} \\ \widehat{u}_{|\mathcal{L}_3} \\ \operatorname{col}\left(\sigma^i \widehat{x}_{|\mathcal{L}_0}\right)_{i=0,\ldots,3} \end{bmatrix}$$

has rank 18: the condition (13) is satisfied for $N = 4$. The 20×97 matrix

$$H(\widehat{u},\widehat{y}) := \begin{bmatrix} \operatorname{col}\left(\sigma^i \widehat{u}_{|\mathcal{L}_0}\right)_{i=0,\ldots,3} \\ \operatorname{col}\left(\sigma^i \widehat{u}_{|\mathcal{L}_1}\right)_{i=0,\ldots,2} \\ \operatorname{col}\left(\sigma^i \widehat{u}_{|\mathcal{L}_2}\right)_{i=0,1} \\ \widehat{u}_{|\mathcal{L}_3} \\ \operatorname{col}\left(\sigma^i \widehat{y}_{|\mathcal{L}_0}\right)_{i=0,\ldots,3} \\ \operatorname{col}\left(\sigma^i \widehat{y}_{|\mathcal{L}_1}\right)_{i=0,\ldots,2} \\ \operatorname{col}\left(\sigma^i \widehat{y}_{|\mathcal{L}_2}\right)_{i=0,1} \\ \widehat{y}_{|\mathcal{L}_3} \end{bmatrix} ,$$

has rank 17; its left-annihilators are associated with the coefficients of the difference equation (15).

From Theorem 2 it follows that every 4-unfolding of the input-output system trajectories is generated by the columns of $H(\widehat{u},\widehat{y})$. To verify this, we generate another finite set of samples following the same procedure but with a different random initial condition $x'_{|\mathcal{L}_0}$ and different random inputs $u'_{|\mathcal{L}_i}$, $i = 0, \ldots, 3$, corresponding to $H(u',x')$ and $H(u',y')$. It can be verified that each column of $H(u',y')$ is a linear combination of the columns of $H(\widehat{u},\widehat{y})$, as stated in the second part of Theorem 2. $\square$

We restate the result of Theorem 2 more explicitly; to do this, given N and $\operatorname{col}(u,y) \in \mathfrak{B}$, we denote by $f(u,y)(j,k)$ the N-unfolding of $\operatorname{col}(u,y)$ whose left-most vertex is (j,k):

$$f(u,y)(j,k) := \begin{bmatrix} u_{j,k} \\ u_{j+1,k-1} \\ \vdots \\ u_{j+N-1,k-N+1} \\ \vdots \\ u_{j+N-1,k} \\ y_{j,k} \\ y_{j+1,k-1} \\ \vdots \\ y_{j+N-1,k-N+1} \\ \vdots \\ y_{j+N-1,k} \end{bmatrix} . \tag{16}$$

Corollary 1. *Assume that the rank condition* (13) *is satisfied. Define* $M := \operatorname{rank}\left(\mathbb{D}_{N-1}(\operatorname{col}(\widehat{u},\widehat{y}))\right)$ *and let the vectors* V_i, $i = 1, \ldots, M$ *form a basis for the image of* $\mathbb{D}_{N-1}(\operatorname{col}(\widehat{u},\widehat{y}))$*:*

$$\operatorname{im} \mathbb{D}_{N-1}(\operatorname{col}(\widehat{u},\widehat{y})) = \operatorname{im} \underbrace{\begin{bmatrix} V_1 & \ldots & V_M \end{bmatrix}}_{=:V} .$$

Let $\operatorname{col}(u,y) \in \mathfrak{B}$ *and let* $N' \in \mathbb{N}$. *For every* $(j,k) \in \mathbb{Z}^2$ *there exist* $\alpha_i(j',k') \in \mathbb{R}$, $i = 1, \ldots, M$, $j' = j, \ldots, j+N-1$, $k' = k, \ldots, k+N-1$, *such that*

$$\begin{aligned} &\begin{bmatrix} f(u,y)(j,k) & \ldots & \ldots & f(u,y)(j+N',k-N') \end{bmatrix} \\ &= V \begin{bmatrix} \alpha_1(j,k) & \ldots & \alpha_1(j+N',k+N') \\ \vdots & \ldots & \vdots \\ \alpha_M(j,k) & \ldots & \alpha_M(j+N',k+N') \end{bmatrix} . \end{aligned} \tag{17}$$

Remark 1. Further research is needed to investigate under which conditions the converse of the statement in Corollary 1 holds. The problem consists in determining conditions on the data $\operatorname{col}(\widehat{u},\widehat{y})$, N, N', and on $\alpha_i(j',k') \in \mathbb{R}$, $i = 1, \ldots, M$, $j' = j, \ldots, j+N-1$, $k' = k, \ldots, k+N-1$ such that the left-hand side of (17) is a "frame" consisting of the values of a system trajectories on $N-1$ consecutive diagonal lines and N' horizontal ones.

A necessary condition is that the sequence $\alpha_i(j',k')$ yields a matrix on the left-hand side of (17) consists of adjacent unfoldings of some $2D$-sequence. Such condition is satisfied in the following case. Let D be a matrix of rank $M = \operatorname{rank}\left(\mathbb{D}_{N-1}(\operatorname{col}(\widehat{u},\widehat{y}))\right)$ consisting of a finite number M' of *adjacent* columns of $\mathbb{D}_{N-1}(\operatorname{col}(\widehat{u},\widehat{y}))$. It is straightforward to verify (see the first part of Theorem 2) that for every $r\in\mathbb{N}$, $r<M'$, and every choice of α_i, $i=0,\dots,r$, the matrix

$$\mathbb{D}_{N-1}(\operatorname{col}(\widehat{u},\widehat{y}))\begin{bmatrix}\alpha_0 & 0 & \dots & 0\\ \alpha_1 & \alpha_0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_r & \alpha_{r-1} & \dots & \alpha_0\\ 0 & \alpha_r & \ddots & \vdots\\ 0 & 0 & \ddots & \alpha_{r-1}\\ 0 & 0 & \dots & \alpha_r\end{bmatrix},$$

consists of successive unfoldings of a trajectory $(u,y)\in\mathfrak{B}$, namely $\operatorname{col}(u,y)=\sum_{i=0}^{r}\alpha_i\sigma^i\operatorname{col}(\widehat{u},\widehat{y})$. Note that unless $\mathfrak{B}$ is finite-dimensional (and consequently autonomous i.e. without input variables, see [4], [19]), there exist system trajectories that are not finite linear combinations of shifts of $\operatorname{col}(\widehat{u},\widehat{y})$. □

Remark 2. Equation (17) is an *image representation*, with

$$\alpha(j,k):=\begin{bmatrix}\alpha_1(j,k) & \dots & \alpha_M(j,k)\end{bmatrix}^\top,$$

being the sequence of *latent variable* values. Equation (17) provides a *parametrization* of all unfoldings of system trajectories in terms of a *finite* matrix V computed *directly* from sufficiently informative input-output data. Such parametrization is currently being used to solve the *finite-extent LQ-optimal control problem* for $2D$ quarter-plane causal models, defined on $[0,N]\times[0,M]\subset\mathbb{Z}^2$, where $N,M\in\mathbb{N}$ (a Roesser model-based formulation of this problem, a characterization of optimality and computational methods for its solution are illustrated in [11], [12]). It can be shown that a substantial class of finite-extent LQ-optimal control problems can be reduced using (17) to the solution of a quadratic optimization problem with equality constraints arising from the boundary conditions. Such results will be presented elsewhere. □

IV. Persistency of excitation and experiment design

Theorem 2 is the counterpart of Theorem 1 p. 327 of [20]: every finite "window" of values of an i-o trajectory (i.e. an unfolding) produced by a quarter-plane causal system is the linear combination of analogous windows computed from "sufficiently informative" data. The notion of "sufficient informativity" is characterized in the rank condition (13) that involves the *state* trajectory corresponding to $\operatorname{col}(\widehat{u},\widehat{y})$. If the state is not directly measurable such condition cannot be directly checked, and alternative sufficient conditions are needed. To this purpose we introduce *persistency of excitation*.

Definition 3. *Let* $L\in\mathbb{N}$. *A signal* $u:\mathcal{L}_{0:L-1}\to\mathbb{R}^m$ *is* persistently exciting of order k *if*

$$\begin{bmatrix}\operatorname{col}\left(\sigma^i u_{|\mathcal{L}_0}\right)_{i=0,\dots,L+k}\\ \operatorname{col}\left(\sigma^i u_{|\mathcal{L}_1}\right)_{i=0,\dots,L+k-1}\\ \vdots\\ \operatorname{col}\left(\sigma^i u_{|\mathcal{L}_{L-1}}\right)_{i=0,\dots,L-1}\end{bmatrix}, \tag{18}$$

has full row rank.

Proposition 2. *Assume that* (1) *is globally controllable, that* $x_{|\mathcal{L}_0}=0$, *and let* $L\geqslant n$. *If* $u_{|\mathcal{L}_{0:L-1}}$ *is persistently exciting of order* k, *then*

$$\operatorname{col}\left(\sigma^i x_{|\mathcal{L}_L}\right)_{i=0,\dots,k} \tag{19}$$

has full row rank.

Proof. We define

$$\mathcal{C}(z):=\begin{bmatrix}A(z)^{L-1}B(z) & \dots & A(z)B(z) & B(z)\end{bmatrix}.$$

With this position, it follows from the first equation in (11) and the assumption $x_{|\mathcal{L}_0}=0$ that

$$\sigma^i x_{|\mathcal{L}_L}=\sigma^i\mathcal{C}(\sigma)\begin{bmatrix}u_{|\mathcal{L}_0}\\ u_{|\mathcal{L}_1}\\ \vdots\\ u_{|\mathcal{L}_{L-1}}\end{bmatrix},\ i=0,\dots,k.$$

We rewrite such equations using $\operatorname{coeff}\left(z^i\mathcal{C}(z)\right)$ (see (8)):

$$\sigma^i x_{|\mathcal{L}_L}=\operatorname{coeff}\left(z^i\mathcal{C}(z)\right)\begin{bmatrix}\operatorname{col}\left(\sigma^j u_{|\mathcal{L}_0}\right)_{j=i,\dots,L+i}\\ \operatorname{col}\left(\sigma^j u_{|\mathcal{L}_1}\right)_{j=i,\dots,L+i-1}\\ \vdots\\ \operatorname{col}\left(\sigma^j u_{|\mathcal{L}_{L-1}}\right)_{j=i,\dots,i+1}\end{bmatrix},$$

$i=0,\dots,k$. From this equation it follows that $\operatorname{col}\left(\sigma^i x_{|\mathcal{L}_L}\right)_{i=0,\dots,k}$ equals

$$\operatorname{col}\left(\operatorname{coeff}\left(z^i\mathcal{C}(z)\right)_{i=0,\dots,k}\right)\begin{bmatrix}\operatorname{col}\left(\sigma^j u_{|\mathcal{L}_0}\right)_{j=0,\dots,L+k}\\ \operatorname{col}\left(\sigma^j u_{|\mathcal{L}_1}\right)_{j=0,\dots,L+k-1}\\ \vdots\\ \operatorname{col}\left(\sigma^j u_{|\mathcal{L}_{L-1}}\right)_{j=0,\dots,k+1}\end{bmatrix}.$$

We show that global controllability of (1) implies that

$$\operatorname{col}\left(\operatorname{coeff}\left(z^i\mathcal{C}(z)\right)_{i=0,\dots,k}\right) \tag{20}$$

has full row rank. Assume that row vectors v_j exist, $j=0,\dots,k$, such that

$$\begin{bmatrix}v_0 & \dots & v_k\end{bmatrix}\operatorname{col}\left(\operatorname{coeff}\left(z^i\mathcal{C}(z)\right)_{i=0,\dots,k}\right)=0;$$

then $\left(v_0+v_1z+\ldots+v_kz^k\right)\mathcal{C}(z)=0$. Since $L\geqslant n$ and since (1) is globally controllable it follows (Theorem 1) that rank $\mathcal{C}(z)=n$, which implies $v_i=0$, $i=1,\dots,k$. Consequently, (20) has full row rank.

To conclude the proof of the claim, assume that

$$\begin{bmatrix}v_0 & \dots & v_k\end{bmatrix}\operatorname{col}\left(\sigma^i x_{|\mathcal{L}_L}\right)_{i=0,\dots,k}=0;$$

then

$$\begin{bmatrix}v_0 & \dots & v_k\end{bmatrix}\operatorname{col}\left(\operatorname{coeff}\left(z^i\mathcal{C}(z)\right)_{i=0,\dots,k}\right) \tag{21}$$

left-annihilates

$$\begin{bmatrix} \operatorname{col}\left(\sigma^j u_{|\mathcal{L}_0}\right)_{j=0,\dots,L+k} \\ \operatorname{col}\left(\sigma^j u_{|\mathcal{L}_1}\right)_{j=0,\dots,L+k-1} \\ \vdots \\ \operatorname{col}\left(\sigma^j u_{|\mathcal{L}_{L-1}}\right)_{j=0,\dots,k+1} \end{bmatrix}.$$

By persistency of excitation we conclude that (21) is the zero-vector; because of the assumption of global controllability we conclude that $v_j = 0$, $j = 0, \dots, k$, and consequently that the matrix (19) has full row rank. This concludes the proof. $\square$

To state our next result, we need to introduce the concept of *algebraic genericity*. Let $\mathcal{L}$ be a linear finite-dimensional space; then given a basis $\{\ell_i\}_{i=1,\dots,d}$ for $\mathcal{L}$, every $\ell \in \mathcal{L}$ can be written as $\ell = \sum_{i=1}^{d} x_i \ell_i$ for some coefficients x_i in the field on which $\mathcal{L}$ is defined. A map $p : \mathcal{L} \to \mathbb{R}$ is a *polynomial* if $p(\ell)$ is a polynomial in the variables x_i, $i = 1, \dots, d$. An *algebraic variety* is a subset $\mathcal{V}$ of $\mathcal{L}$ consisting of all zeroes of some polynomial p. A subset $\mathcal{S} \subset \mathcal{L}$ is called *generic* if there is a proper algebraic variety $\mathcal{V} \subsetneq \mathcal{L}$ such that $\mathcal{S} \supset (\mathcal{L} \backslash \mathcal{V})$.

Proposition 3. *Assume that* (1) *is globally controllable, that* $x_{|\mathcal{L}_0} = 0$, *and let* $L \geqslant n$. *If* $u_{|\mathcal{L}_{0:L-1}}$ *is persistently exciting of order* k, *then generically for every* $j = 0, 1, \dots$ *the matrix*

$$\begin{bmatrix} \operatorname{col}(\sigma^i u_{|\mathcal{L}_L})_{i=0,\dots,k} \\ \operatorname{col}(\sigma^i u_{|\mathcal{L}_{L+1}})_{i=0,\dots,k-1} \\ \vdots \\ u_{|\mathcal{L}_{L+k}} \\ \operatorname{col}(\sigma^i x_{|\mathcal{L}_L})_{i=0,\dots,k} \end{bmatrix} \tag{22}$$

has full row rank.

Proof. The row rank of (22) is not full if and only if all its maximal order minors are zero. For a fixed $x_{\mathcal{L}_L}$, each of these minors is a polynomial in the variables $u_{\mathcal{L}_{L+\ell}}(i_1, i_2)$, $\ell = 0, \dots, k$, $(i_1, i_2) \in \mathbb{Z}^2$. It follows that (22) has not full rank if and only if the intersection of the varieties of such polynomials is non-empty. The intersection of such varieties is a proper algebraic variety itself; the claim follows. $\square$

Remark 3. To achieve sufficiently informative data, one can proceed as follows. Starting with the system at rest, i.e. $\operatorname{col}(u_{|\mathcal{L}_j}, y_{|\mathcal{L}_j}) = 0$, $j \leqslant 0$, one applies a persistently exciting signal of order N at $\mathcal{L}_j$, $j = 0, \dots, N$ obtaining a full rank matrix (19) (see Proposition 2). Subsequently, one applies a persistently sequence $u_{\mathcal{L}_{L+\ell}}$, $\ell = 0, \dots, N$ (for example, consisting of random values). It follows from Proposition 3 that generically the persistency of excitation condition

$$\operatorname{rank}\left(\begin{bmatrix} \operatorname{col}(\sigma^i u_{|\mathcal{L}_L})_{i=0,\dots,N-1} \\ \operatorname{col}(\sigma^i u_{|\mathcal{L}_{L+1}})_{i=0,\dots,N-2} \\ \vdots \\ u_{|\mathcal{L}_{L+N-1}} \\ \operatorname{col}(\sigma^i x_{|\mathcal{L}_L})_{i=0,\dots,N-1} \end{bmatrix}\right) = \frac{N(N+1)}{2} m + Nn$$

is satisfied. From Theorem 2 it follows that any unfolding of any system trajectory is expressible as a linear combination of the columns of $\mathbb{D}_{N-1}(\operatorname{col}(u, y))$. $\square$

V. Conclusions and further work

In Theorem 2 we generalized the results of [2] to *input-output* measurements. We also illustrated how sufficiently informative data can be generated (see Remark 3).

Current research aims to apply these results to the data-driven simulation problem (see [8]) and to the data-driven "finite-horizon" LQ-optimal control problem (see Remark 2).

References

[1] Bharti, A. and D. Pal, "Data informativity for identification and data-driven simulation for scalar time-relevant discrete 2D systems", *Proc. 26th MTNS*, 2024.

[2] Chu, B., Rapisarda, P., Rocha, P., "Data-driven analysis and control of 2D Fornasini-Marchesini models", *IEEE Control Systems Letters*, vol. 8, pp. 664-669, 2024.

[3] Fornasini, E. and Marchesini, G., "Doubly-indexed dynamical systems: state-space models and structural properties", *Math. Systems Theory*, vol. 12, pp. 59–72, 1978.

[4] Fornasini, E., Rocha, P., and Zampieri, S., "State Space Realization of 2-D Finite-Dimensional Behaviours", *SIAM Journal on Control and Optimization*, vol. 31, no. 6, pp. 1502-1517, 1993.

[5] Dewasurendra, D.A. and Bauer, P.H. "A novel approach to grid sensor networks", *15th IEEE Int. Conf. Electr., Circ. Syst.*, 2008.

[6] Gałkowski, K., "The Fornasini-Marchesini and the Roesser models: algebraic methods for recasting", *IEEE Trans. Aut. Contr.*, vol. 41, no. 1, pp. 107-112, 1996.

[7] Kurek, J.E. and Zaremba, M.B., "Iterative learning control synthesis based on 2-D system theory", *IEEE Trans. Aut. Contr.*, vol. 38, no. 1, pp. 121-125, 1993.

[8] Markovsky, I. and Rapisarda, P. "Data-driven simulation and control", *Int. J. Contr.*, vol. 81, no. 12, pp. 1946-1959, 2008.

[9] Markovsky, I., Dörfler, F., "Behavioral systems theory in data-driven analysis, signal processing, and control", *Annual Reviews in Control*, vol. 52, pp. 42-64, 2021.

[10] M. Mukherjee and D. Pal, "Data-based stability analysis of strongly autonomous discrete nD systems", *Proc. 26th MTNS*, 2024.

[11] Ntogramatzidis, L. and Cantoni, M., "On the sufficiency of finite-frame LQR optimality conditions for 2-D Roesser models", *Proc. ECC*, pp. 3897-3899, 2007.

[12] Ntogramatzidis, L. and Cantoni, M., "LQ optimal control for 2D Roesser models of finite extent", *Systems & Control Letters*, vol. 58, no. 7, pp. 482-490, 2009.

[13] Piou, J.E., Dumanian, A.J., "Application of the Fornasini-Marchesini first model to data collected on a complex target model", *Proc. 2014 ACC*, pp. 2279-2284, 2014.

[14] Rapisarda, P. and Pal, D., "Informativity for Identification for 2D State-Representable Autonomous Systems, with Applications to Data-Driven Simulation", *Proc. 62nd IEEE CDC*, 2023.

[15] Roesser, R., "A discrete state-space model for linear image processing", *IEEE Trans. Aut. Contr.*, vol. 20, no. 1, pp. 1-10, 1975.

[16] Rogers, E. "The Fornasini-Marchesini model- its role in the analysis and control of physical systems", *Proc. 2015 IEEE 9th International Workshop on Multidimensional (nD) Systems (nDS)*, Vila Real, Portugal, pp. 1-6, 2015.

[17] Wang, Y., Zhou, D. and Gao, F., "Iterative learning reliable control of batch processes with sensor faults", *Chemical Engineering Science*, vol. 63, no. 4, pp. 1039-1051, 2008.

[18] Zhang, H. and Wang, Y., "Performance assessment for batch processes under ILC based on 2-D Fornasini-Marchesini model", *Proc. 2016 35th Chinese Control Conference (CCC)*, Chengdu, China, pp. 3206-3211, 2016.

[19] Valcher, M., "Characteristic Cones and Stability Properties of Two-dimensional Autonomous Behaviors", *IEEE Transactions on Circuits and Systems, Part I*, vol. CAS-47, pp. 290–302, 2000.

[20] Willems, J.C., Rapisarda, P., Markovsky, I and De Moor, B.L.M., "A note on persistency of excitation", *Syst. Contr. Lett.*, vol. 54, no. 4, pp. 325-329, 2005.