UNIVERSITY OF SOUTHAMPTON

Faculty of Social Sciences
School of Mathematical Sciences

Aspects of Holography

by

Zezhuang Hao
0000-0003-1037-3751

A thesis for the degree of
Doctor of Philosophy

April 2025

University of Southampton

Abstract

Faculty of Social Sciences
School of Mathematical Sciences

Doctor of Philosophy

Aspects of Holography

by Zezhuang Hao
0000-0003-1037-3751

In the thesis we studied various aspects of holography principle, especially in the context of real time holography and flat/CFT duality. In the first part, we discussed real-time holography within the embedding space formalism. Based on the previous work on the scalar fields, we presented matching conditions for on-shell integer spin fields when going from Euclidean to Lorentzian signature on AdS background. The main content of the second part is to develop the AdS/CFT correspondence into the flat/CFT correspondence. More precisely, for scalar fields, we constructed the dictionary between flat spacetime and the CFT on the boundary which works the same as AdS/CFT dictionary from the bottom-up point of view. After analysing the behaviour of scalar field modes on hyperbolic slices of Minkowski and performing the holographic renormalisation for the associated onshell action, we obtain a holography dictionary between the bulk theory and the corresponding dual theory on the celestial sphere. We propose that a single scalar field in the bulk is dual to two series of operators on the celestial sphere; the scaling dimension of these operators takes values on the principal series. Moreover, we will see that the two series of operators can be interpreted as ingoing and outgoing waves in the bulk. We illustrate our dictionary with the example of a single shock wave. The third part is basically the extension of construction of the flat/CFT dictionary from scalar fields to gravitational theories. Asymptotically flat spacetime is built up by asymptotically AdS hyperboloid slices in terms of Fefferman Graham coordinates together with soft modes propagating between different slices near the null boundary. Then we construct the flat holography dictionary based on studying Einstein equation at zero and first order and it turns out that these correspond to the description of hard and soft sector for the field theory from the boundary point of view. The explicit expression for energy-stress tensor is also determined by performing holographic renormalisation on the Einstein Hilbert action.

Contents

List of Figures

List of Tables

Acknowledgements

Thanks to my supervisor Prof. Marika Taylor for her invaluable guidance during the PhD study and also all the numbers of string and holography group for their kind help and insightful discussions. In any event, I would like to thank my friends and family. It is their love that makes me who I am.

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:
   (a) Zezhuang Hao. Holographic reconstruction of flat spacetime. *JHEP*, 09:060, 2024b.
   (b) Zezhuang Hao. Real time holography for higher spin theories. *JHEP*, 03: 168, 2024a.
   (c) Zezhuang Hao and Marika Taylor. Flat holography and celestial shockwaves. *JHEP*, 02:090, 2024.

Signed:.. Date:..................

To My Family.

子曰：“温故而知新，可以为师矣。”

——《论语·为政》

Chapter 1

Introduction

General relativity and quantum mechanics have been brought up for a century and they are believed to be the most fundamental rules which respectively govern the large scale structure of the universe and the microscopic interactions between elementary particles even though they are not compatible with each other. After their great success in predicting the observation from the lab, physicists spent long time looking for a unified theory of quantum and gravity, e.g. semiclassical field theory, supergravity, string theory. The quantum gravity theory that would please everyone has not been figured out yet while, during the extensive study of various proposed models, another fundamental principle which relates the dimension of spacetime, quantum and gravity effects has been found and caused a great attention in recent years, so called holography principle.

The idea of projecting the physical world to a lower dimension one living on the boundary exists for long while it has not been formally discussed in physics literature until the work ′t Hooft (1993); Susskind (1995), initiated by the study of the black hole entropy Christodoulou (1970); Penrose and Floyd (1971); Hawking (1971); Bekenstein (1973); Hawking (1975), which tells us that the entropy of a black hole is proportional to the area of its event horizon.

Based on such observation, one can further conclude that the degrees of freedom or information for a given system is bounded by its boundary area rather than its volume, which makes it possible to encode all the bulk information into the proposed boundary system. Such correspondence concerning the assignment of degrees of freedom is then developed to the duality between the subregion of the bulk and boundary, conjectured to be characterised by the Ryu-Takayanagi surface Ryu and Takayanagi (2006). Under proper assumptions, the conjecture was proven in Lewkowycz and Maldacena (2013); Faulkner et al. (2013) and then, taking the quantum effect into consideration, the concept of RT surface is generalised to the so-called quantum extremal surface Engelhardt and Wall (2015).

1.1 AdS/CFT Correspondence

It turns out that the structure of holography is more than the projection of degrees of freedom after the first concrete realization of the holography principle discovered by Maldacena (1999), called AdS/CFT correspondence. In that work, Maldacena pointed out that type IIB string theory on the $AdS_5 \times S^5$ background is dual to the $\mathcal{N} = 4$ super-Yang-Mills theory in 3+1 spacetime dimensions by studying the decoupling limit of the stack of D_3 branes in string theory and its corresponding low energy supergravity solution, which implies that, in additional to the reduction of dimension, the theory of quantum and gravity could also be relevant when comparing the theory in the AdS bulk with its boundary CFT correspondence. Such relation is similar to the relevance between space and time in gravity or the relevance between particles and waves in quantum mechanics.

However in practice, due to the lack of knowledge for the quantum gravity theory and the difficulty of studying strongly coupled gauge theory at low energy, one can first choose to investigate the AdS/CFT correspondence at the 't Hooft large N limit, under which the gauge theory will be simplified since the contribution from planar diagrams will become dominant if the number of colors N is large when keeping $\lambda = g_{YM}^2 N$ constant 't Hooft (1974). From the bulk side, we see that the string theory will become classical by comparing the map between parameters $g_s \sim g_{YM}^2$ and $\alpha'/L_{Ads}^2 \sim 1/\sqrt{g_s N}$. Moreover, by taking large value of λ, the string scale will become small compared with the AdS curvature and the classical gravity description is reliable, therefore the AdS/CFT correspondence becomes a weak/strong duality. In such case, the bulk theory is described by the semiclassical field theory and one can write down the effective action, decomposing the field at the boundary, and then map the data from asymptotic AdS infinity to the boundary CFT named AdS/CFT dictionary.

In the literature, there are mainly two ways to construct the AdS/CFT dictionary Witten (1998a); Gubser et al. (1998). One starts from the effective field theory on $AdS_5 \times S^5$ background while the other starts from AdS_5 thus they are called top-down and bottom-up approaches to AdS/CFT, respectively. At first sight, the bottom-up approach looks easier if one just considers the fields on the AdS_5 background but the supersymmetric information is lost due to the omission of the Kaluza-Klein fields on the S^5 sphere, e.g. we would obtain non-zero vacuum energy. Such issue is rescued in the work Skenderis and Taylor (2006a,b) by Skenderis and Taylor. They developed a KK reduction map which reduces all the fields in 10d to 5d in a gauge invariant way therefore concludes that the top-down and bottom-up approaches could be equivalent provided that proper reduction procedure is applied. For this thesis, we will adopt the bottom-up approach and ignore the KK fields on the internal space. In this case, the duality is clarified by the dictionary proposed by Witten

$$\exp\Big(-S_{\mathrm{AdS}_{d+1}}(\Phi)\Big)_{\Phi\sim\phi} = \Big\langle \exp -\int_{S^d} \phi\, \mathcal{O} \Big\rangle_{CFT}, \tag{1.1}$$

in which $S_{\mathrm{AdS_{d+1}}}(\Phi)$ is the action of the semi-classical theory in the bulk with scalar fields characterised by the boundary condition $\Phi \sim \rho^{-d+\Delta}\phi$ at large radius ρ of the Euclidean AdS spacetime. From the right-hand side, we can see that ϕ is dual to the source in the CFT theory and it is coupled to the operator $\mathcal{O}$. The scale dimension Δ of the operator and the mass M of the particle in the bulk preserve the relation $\Delta(d-\Delta) = M^2$, which is obtained by solving the equation of motion of Φ.

1.2 Extension beyond the large 't Hooft limit

AdS/CFT correspondence has been brought up over two decades and most of the checks are carried out in the large 't Hooft limit. At such limit the bulk theory tends to stay in the low energy region thus they are described by the well studied model. When the 't Hooft constant goes to the small limit the string scale becomes larger than the AdS scale thus the higher spin excitations and gravitational corrections should not be ignored while it was proposed that the spectra of string theory could be approximately described by the higher spin field theory and the boundary theory will be free, leading to the proposal that Vasiliev's higher spin Vasiliev (1990, 1999, 2003) theory in the AdS bulk is dual to the free $O(N)$ vector model on the boundary Sezgin and Sundell (2002); Klebanov and Polyakov (2002); Sezgin and Sundell (2005).

In the context of Yang-Mills theory, at the leading order in large N and for the purpose of all-light non-extremal operators, single trace operators are dual to single particle states on AdS. For the vector model, one needs to consider the current of spin s

$$J^s_{\mu_1\cdots\mu_s} = \phi^i \partial_{(\mu_1}\cdots\partial_{\mu_s)}\phi^i + \cdots \tag{1.2}$$

which are the bilinears in ϕ^i. These currents are proposed to be dual to the massless higher spin gauge fields in AdS. Such relation could be generalised to the case for fermionic fields in Leigh and Petkou (2003); Sezgin and Sundell (2005) and it has been verified by studying the three point correlation functions specifically Giombi and Yin (2010, 2013) with further extension to the Chern-Simons gauge theory Giombi et al. (2012); Aharony et al. (2012).

Recently a derivation of the HS/CFT duality is also given by constructing a map between the boundary fields in the bi-local form Das and Jevicki (2003) and higher spin fields on the bulk, so called AdS/CFT map de Mello Koch et al. (2019); Aharony et al. (2021a,b). Such derivation of the AdS/CFT correspondence in the context of higher spin and vector model duality starts from the study of the bilocal form of the $U(N)$

vector models. Given the N complex scalar fields $\phi_i(x)$ for the vector model, one first chooses to recast them into the bilocal function $G(x_1, x_2)$ written as

$$G(x_1, x_2) = \frac{1}{N} \sum_{I=1}^{N} \phi_i^*(x_1)\phi_i(x_2). \tag{1.3}$$

Further more, by changing the measure from $D\phi_i(x)$ to the bilocal measure $DG(x_1, x_2)$, the free partition function for the vector model of source J

$$Z_{\text{free}}[J] = \int \prod_{I=1}^{N} D\phi_I(x) \exp(-S[G, J]) \tag{1.4}$$

then can be rewritten into a partition function that depends on the bilocal function $G(x_1, x_2)$. Given the bulk higher spin field, one can construct the so called AdS/CFT map

$$\Phi_J(X, W) = \int dP_1 dP_2 \mathcal{M}_J(X, W|P_1, P_2)\eta(P_1, P_2) \tag{1.5}$$

where X, W, P are coordinates in the embedding space and the specific form of the propagator $\mathcal{M}_J(X, W|P_1, P_2)$ is determined in Aharony et al. (2021a,b). $\eta(X_1, X_1)$ is the perturbation of the function $G(x_1, x_2)$ of the order $1/\sqrt{N}$.

As we have already known, the action for the bulk higher spin theory is not found yet while the construction of AdS/CFT map provides us with a powerful tool to investigate the bulk theory from the study of vector model. Using the AdS/CFT map together with the bilocal form of the partition function, through the AdS/CFT dictionary, one then obtains the quadratic term of the bulk action as

$$S^{(2)}_{\text{local}}[\Phi_J] = \sum_{J=0}^{\infty} \frac{1}{\alpha_J} \int \frac{dX}{(\frac{d-1}{2})_J J!} \Phi_J(X, K_W)(\nabla_X^2 - M^2_{d-J-2.J})(\nabla_X^2 - M^2_{d+J,J})\Phi_J(X, W) \tag{1.6}$$

in which the value of the mass $M_{d-J-2,J}$ and $M_{d+J,J}$ depends on the pole structure of $\mathcal{M}_J$. As it is shown in the work Aharony et al. (2021a), the bulk two point functions $\langle \Phi_J \Phi_J \rangle$ are the difference of two propagators of on-shell spin J fields, one with positive propagator of mass $M_{d+j-2,J}$ while the other with negative propagator of mass $M_{d+j,J}$. The degrees of freedom described by the mass $M_{d-J-2,J}$ are associated to the physical modes while the modes associated to the mass $M_{d+j,J}$ are the ghost modes.

In their work, the AdS/CFT map is constructed between the Euclidean AdS and CFT while here in the chapter 2 of this thesis we will first obtain the Lorentzian version of AdS/CFT map based on the study of real time holography then further study its various implications on the understanding of holography principle.

1.3 R-T formula and Information Paradox

Based on the study of AdS/CFT correspondence, Ryu and Takayanagi proposed the relation concering the boundary entropy and bulk geometry as

$$\mathcal{S}_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}} \tag{1.7}$$

where $\mathcal{S}_A$ is the entanglement entropy for the given subregion A on the boundary. γ_A is the codimension two surface which minimizes the codimension one surface bounded by the subregion A on the boundary. In the context of quantum information theory, the entanglement entropy is given by the formula

$$\mathcal{S} = -n\partial_n(\log Z(n) - n\log Z(1))|_{n=1} \tag{1.8}$$

where $Z(n) = \text{Tr}\rho^n$. Furthermore, it was then realized by Lewkowycz and Maldacena that the calculation of the entanglement entropy at the boundary is equivalent to the calculation of gravitational action in the bulk S_{gr} with the help of the study of replica trick Holzhey et al. (1994); Calabrese and Cardy (2004, 2009) and AdS/CFT dictionary. The replica trick tells us that the treatment of $Z(n)$ is equivalent to the calculation of CFT partition function on the replica manifold. Therefore finally one can use the bulk gravitational data to reproduce the boundary entropy with the help of dictionary (1.1). Of course, to obtain the finite entropy, renormalisation procedure is required and we have Taylor and Woodhead (2016a,b)

$$\mathcal{S}_{\text{ren}} = n\partial_n(S_{\text{gr,ren}}(n) - nS_{\text{gr,ren}}(1))|_{n=1}. \tag{1.9}$$

Later, by taking the bulk gravitational loop effects into consideration, one can also establish the relation between the bulk data and the boundary quantum corrected entropy therefore one has

$$\mathcal{S}_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}} + \mathcal{S}_{\text{bulk-ent}} + \cdots \tag{1.10}$$

where $\mathcal{S}_{\text{bulk-ent}}$ is the entanglement entropy calculated in the bulk for the region bounded by the extremal surface γ_A and the boundary subregion A. $\cdots$ represents the counterterm contribution. Such calculation is performed on the static gravity background while it was then postulated by Engelhardt and Wall that one should take the gravitational back reaction into consideration thus they obtain

$$\mathcal{S}_A = \mathcal{S}_{\text{gen}}(\chi_A) \tag{1.11}$$

where χ_A is so called quantum extremal surface and it is determined by figuring out the extremal value of the generalized entropy $\mathcal{S}_{\text{gen}}$.

The concept of quantum extremal surface plays an important rule in the discussion of black hole information paradox. Before the Page time, the quantum extremal surface (QES) lies near the horizon while it turns out the QES could possibly lies inside the horizon at the late stage of the black hole evaporation Penington (2020); Almheiri et al. (2020b). Like the RT formula, such proposal could also be proofed using the AdS/CFT dictionary together the replica trick, provided the wormhole contribution is taken into consideration Penington et al. (2022); Almheiri et al. (2020a).

1.4 Flat Holography

Actually, there have been attempts to address the problems for developing a flat version of holographic principle dated even back to the birth of AdS/CFT in the talk given by Witten E.Witten (1998). During that talk, he discussed various obstacle to writing down the Flat/CFT dictionary. Conceptually, if one assumes that both of the quantum gravity theory and scattering amplitudes are dual to the CFTs on the boundary, then it will be hard to understand that why the quantum gravity theory should be equivalent to its own scattering amplitudes. From the technical point of view, the complexity of the geometric structure and the behaviour of fields at the two null boundaries in Minkowski space make it hard to write down the boundary correlation functions or to study the distribution of the degrees of freedom. Ultimately, he proposed that if the flat theory is dual to the boundary structure X, then X should be more complicated than a conventional CFT. The complicated nature of the structure X can also be seen from the study of symmetries of the asymptotic flat spacetime. Not like the AdS case, the isometry group for asymptotically flat space will reduce to the infinite BMS group Bondi et al. (1962); Sachs (1962b,a) rather than the Poincare group. Globally, the BMS group is generated by the supertranslations and superrotations in which supertranslations behaves like 1d translation while superrotations are characterised by $SL(2,\mathbb{C})$. After fifty years of study of BMS group, people realised that the superroations could be locally generalised to the Virasoro algebra, even with a central extension Barnich and Troessaert (2010a,b, 2011), which brings hope to construct the duality between the flat theory and the 2d CFT Belavin et al. (1984).

1.4.1 Celestial Holography

Based on another observation that the supertranslation Ward identity is equivalent to the Weinberg's soft graviton theorem on the celestial sphere Strominger (2014); He et al. (2015) when studying the symmetry of the graviton scattering amplitudes, Strominger with his collaborators then conjectured the duality between scattering amplitudes and celestial CFT so called celestial holography.

The story comes from the conjecture that the symmetry group for scattering process on the asymptotically flat background should be the subgroup coming from the total allowed symmetry group living at two null boundaries $\text{BMS}^0 \in \text{BMS}^+ \times \text{BMS}^-$. Then Strominger proposed that one should impose antipodal matching conditions at the null boundary $\mathcal{I}^{\pm}$ at the spatial infinity $\mathcal{I}^+_- = \mathcal{I}^-_+$. Such matching between physical quantities at two boundaries leads to the match between the conserved charges $Q^- = Q^+$ therefore it makes sense to consider the conservation law

$$\langle \text{out}|Q^+\mathcal{S} - \mathcal{S}Q^-|\text{in}\rangle = 0 \tag{1.12}$$

coming from the dynamical consequence of the commutation relation $[Q, H] = 0$ between the conserved charge and the Hamiltonian. Moreover, after decomposing the total charge into the soft and hard part we have

$$\langle \text{out}|Q^+_S\mathcal{S} - \mathcal{S}Q^-_S|\text{in}\rangle = -\langle \text{out}|Q^+_H\mathcal{S} - \mathcal{S}Q^-_H|\text{in}\rangle \tag{1.13}$$

where $Q^{\pm} = Q^{\pm}_S + Q^{\pm}_H$. It turns out that, after the quantization of the soft modes and rewriting the momentum of the particle in terms of the energy and a point on the celestial sphere, one can show that the above Ward identity is equivalent to the Weinberg's soft graviton theorem, i.e, the hard and soft charge will contribute to the hard and soft sector of the scattering process , respectively.

After that, the relation between 4d scattering amplitudes and 2d celestial CFT is extensively studied and the celestial dictionary is proposed to be

$$\langle \tilde{\text{out}}|\mathcal{S}|\tilde{\text{in}}\rangle = \langle \mathcal{O}^{\pm}_{\Delta_1,J_1}(z_1,\bar{z}_1)\dots\mathcal{O}^{\pm}_{\Delta_n,J_n}(z_n,\bar{z}_n)\rangle_{\text{CCFT}} \tag{1.14}$$

where $\mathcal{O}^{\pm}_{\Delta_1,J_1}$ are operators on the celestial sphere of scale dimension Δ_1 and spin J_1. $|\tilde{\text{in}}\rangle$ and $|\tilde{\text{out}}\rangle$ represent the transformed in and out going states. For massless particles, these are Mellin transforms thus one has

$$\langle \mathcal{O}^{\pm}_{\Delta_1,J_1}(z_1,\bar{z}_1)\dots\mathcal{O}^{\pm}_{\Delta_n,J_n}(z_n,\bar{z}_n)\rangle_{\text{CCFT}} = \prod_{i=1}^{n}\int_0^{\infty} dw_i^{\Delta_i-1}\langle \text{out}|\mathcal{S}|\text{in}\rangle. \tag{1.15}$$

More precisely, given the bulk field of spin s, one can project it onto the in or out going conformal basis $\Phi^{s,\pm}_{\Delta,J}$ by considering the inner product on the codimension one surface Σ, written as

$$\mathcal{O}^{s,\pm}_{\Delta,J}(z,\bar{z}) = i(\Phi^s(X),\ \Phi^{s,\pm}_{\Delta,J}(X;z,\bar{z}))_{\Sigma}, \tag{1.16}$$

in which the product is given by the Klein-Gordon norm on the codimension-one surface Σ. From the above celestial dictionary, one can deduce that the two point function take the form

$$\langle \mathcal{O}^{s,\pm}_{\Delta_1,J_1}(z_1,\bar{z}_1)\mathcal{O}^{s,\mp}_{\Delta_2,J_2}(z_2,\bar{z}_2)\rangle_{\text{CCFT}} \sim \delta(z_1 - z_2), \tag{1.17}$$

which is the main feature of the celestial CFT. This tells us the celestial CFT is not the same as the CFT studied in the standard field theory literature. One can also see this from the study of OPE coefficients or by checking the corresponding translation rules for the correlation functions.

Here, one should note the difference between celestial dictionary and the AdS/CFT dictionary. For the AdS/CFT, one first proposes the map between asymptotic bulk data and CFT context then the entire map between two theories is clarified by the bulk action and boundary partition function. For example, given the source and operator one can use the bulk action to reproduce all the CFT correlation functions. This is because all the dynamical information of the CFT is encoded in the partition function, which is equivalent to the study of bulk action through the AdS/CFT dictionary. For the celestial dictionary, the duality is classified by the map between all the bulk scattering amplitudes and celestial correlation functions. Following the idea of bootstrap that one can use three point function as the building block to recover all the higher point functions with the help of the OPE coefficients, the study of celestial OPE coefficients will be crucial in the context of celestial holography.

In addition to the matter field, one also has the background geometry. For asymptotically flat spacetimes, the boundary stress tensor is defined to be He et al. (2016)

$$T_{zz} = \frac{i}{2\pi G} \int d^2 w \frac{1}{z-w} D_w^2 D^{\bar{w}} N^{(1)}_{w\bar{w}} \tag{1.18}$$

where $N^{(1)}_{zz}$ is the zero mode of the News tensor. Although such definition will make the stress tensor non-local in terms of the bulk metric, one can check it will give us local boundary stress tensor Ward identities by considering the bulk soft graviton scattering theorem.

As we have mentioned, the Celestial CFT behaves in a different way from the standard CFT, which can also been seen from the corresponding celestial CFT OPE coefficients Pate et al. (2021); Guevara et al. (2021); Strominger (2021). The positive helicity gluons admit the holomorphic collineaer expansion

$$\mathcal{O}^{+,a}_{\Delta_1}(z_1,\bar{z}_1)\mathcal{O}^{+,b}_{\Delta_2}(z_2,\bar{z}_2) \sim \frac{if^{abc}}{z_{12}} C(\Delta_1,\Delta_2)\mathcal{O}^{+,c}_{\Delta_1+\Delta_2-1}(z_2,\bar{z}_2) + \cdots \tag{1.19}$$

and the expansion form is fixed by the $SL(2,\mathbb{C})$ symmetry and the leading soft theorem up to the coefficient $C(\Delta_1,\Delta_2)$ which could further be uniquely determined by considering the gluons transformation $\delta\mathcal{O}^{\pm,a}_{\Delta}$. Given the OPE, one can also study the algebra of soft operators and it turns out that the algebra obeys the relation

$$[\hat{R}^{k,a}_{n}, \hat{R}^{l,b}_{n'}] = if^{ab}_{c}\hat{R}^{k+l-1,c}_{n+n'}, \tag{1.20}$$

in which $R^{k,a}(z,\bar{z}) = \lim_{\epsilon\to 0}\epsilon\mathcal{O}^{+,a}_{k+\epsilon}(\bar{z})$ and $\hat{R}^{k,a}_n \sim R^{k,a}_n$. For gravity, one has the $w_{1+\infty}$ algebra written as

$$[w^p_m, w^q_n] = [m(q-1) - n(p-1)]w^{p+q-2}_{m+n} \tag{1.21}$$

with p, q running over positive, half-integral values $p, q = 1, \frac{3}{2}, \cdots$.

From the above discussion, one can see that the OPE coefficients are determined by imposing constraints from the symmetry and one should also expect there should exist a relation playing the role of crossing symmetry in the standard CFT that characterises the dynamical structure of the celestial CFT and make the construction of celestial dictionary complete.

The main theme of the development of the celestial holography could be summarized as a diagram called infrared triangle Strominger (2017); Raclariu (2021); Pasterski (2021), which reveals the equivalence between soft theorems, asymptotic symmetries and the memory effect. Till now, we have discussed the relation between the soft theorem and the asymptotic symmetries, which contribute to two of the three vertices on the infrared triangle and both of them can be written into the form of Ward identities on the celestial sphere. Another topic during the development of the celestial holography is the gravitational memory effect Strominger and Zhiboedov (2016). On the one hand, the memory effect is related to the soft theorem by the Fourier transform and, on the other hand, the change of the deviation between two nearby detectors induced by the passing through gravitational waves also measures transitions between BMS-inequivalent vacuum. Putting these three concepts together, the obtained structure is so called infrared triangle.

As we have seen, the celestial CFT exhibits a rich structure for people to study the scattering amplitudes, but there are some other issues which cause a great confusion. For example, people could not understand the reason why a real time flat theory should be dual to an Euclidean theory on the sphere and it is still hard to say if the celestial CFT is unitary or not since the scale dimension living on the principle series are complex.

1.4.2 Carrollian CFT

There are later developments which claim that the $4d$ scattering amplitudes should be dual to the $3d$ Carrollian CFT Donnay et al. (2022); Bagchi et al. (2022) thus the BMS symmetry is manifested and signatures from both sides will fit. Similar to the celestial holography, such duality is brought up based on the equivalence between the algebra Duval et al. (2014); Bagchi (2010)

$$\mathfrak{CCarr}_3 = \mathfrak{bms}_4 \tag{1.22}$$

where the Carrollian algebra $\mathfrak{CCarr}_3$ is obtained by taking the $c \to 0$ limit of the Poincare group. $\mathfrak{bms}_4$ is the algebra for the BMS group.

Moreover, given the bulk field of spin σ denoted as $\Phi(u, z, \bar{z})$, the flat/Carrollian dictionary is then proposed to be

$$\tilde{\mathcal{M}}(\{u_i, z_i, \bar{z}_i, h_i, \bar{h}_i, \epsilon_i\}) = \prod_i \langle \Phi^{\epsilon_i}_{h_i, \bar{h}_i}(u, z_i, \bar{z}_i) \rangle \tag{1.23}$$

where $h = \frac{\Delta+\sigma}{2}$ and $\bar{h} = \frac{\Delta-\sigma}{2}$. $\epsilon = \pm 1$ represent the in and out going modes. The transformed scattering amplitude $\tilde{M}$ is defined as Bagchi et al. (2022)

$$\tilde{\mathcal{M}}(\{u_i, z_i, \bar{z}_i, h_i, \bar{h}_i, \epsilon_i\}) = \prod_{i=n}^{n} \int_0^\infty dw_i w_i^{\Delta_i - 1} e^{-i\epsilon_i w_i u_i} S(\{\epsilon_i w_i, z_i, \bar{z}_i, \sigma_i\}). \tag{1.24}$$

One can see that the main difference between the celestial dictionary and Carrollian CFT dictionary is the factor $e^{i\epsilon w u}$ in the integral. The celestial CFT operators can be reproduced by performing the integral over the along the null boundaries Donnay et al. (2022)

$$\mathcal{O}^+_{\Delta,J}(z, \bar{z}) = i^\Delta \Gamma[\Delta] \int_{-\infty}^{+\infty} duu\, \Phi^+_{h,\bar{h}}(u, z, \bar{z}), \tag{1.25}$$

$$\mathcal{O}^-_{\Delta,J}(z, \bar{z}) = i^\Delta \Gamma[\Delta] \int_{-\infty}^{+\infty} dvv\, \Phi^-_{h,\bar{h}}(v, z, \bar{z}). \tag{1.26}$$

From the above discussion, one can see that the definition for the flat/Carroll field dictionary is basically parallel to the celestial dictionary while the boundary Carrollian field theory is better studied this time since now we have the action. Here, for 3d case, we start from the $c \to 0$ limit of the free theory for the scalar ϕ therefore the boundary action takes the form de Boer et al. (2023)

$$S = \frac{1}{2} \int dt d^2x (\dot{\phi}^2 - m^2 \phi^2) \tag{1.27}$$

where $\dot{\phi}$ is the derivative with respect to the time t. Give such action, one can write down the mode expansion for the scalar field

$$\phi = e^{imt} \int d^2k\, a_k\, e^{ik\cdot x} + c.c \tag{1.28}$$

and perform the canonical quantization following the standard treatment of the quantum field theory so that one can obtain the spectra of the particles.

Unlike standard QFT, where the momentum k is constrained by the on-shell condition while here the momentum could take arbitrary value, which leads to the infinite degeneracy for each energy level. For zero temperature, one can impose the cut-off for the degrees of freedom while, in the case for the nonzero temperature, all the degenerate states labeled by k at each energy level could be excited due to the thermal fluctuation. In the onshell case, k lies on a 1d curve while now k could take arbitrary values on the 2d plane. Therefore, it seems likely the standard regularization procedure in QFT will not work and the partition function for thermal Carrollian field will be ill defined [1]. Such issue will not effect the current definition of the flat/Carrollian dictionary while it will make the establishment for the flat/CFT dictionary like the AdS/CFT correspondence challenging.

In fact, there are lots of attempts de Haro et al. (2001a); de Boer and Solodukhin (2003); Mann and Marolf (2006); Costa (2012); Nguyen and Salzer (2021) aiming to construct the flat/CFT dictionary which will work the same way as the AdS/CFT dictionary. In the context of celestial holography, there are also recent discussions about the connection between Witten diagrams and scattering amplitudes Ball et al. (2019); Casali et al. (2022); Iacobacci et al. (2023); Bagchi et al. (2023a); Iacobacci et al. (2024); Melton et al. (2023) based on the geometry connection that the AdS spacetime could be treated as the hyperboloid embedded in the flat spacetime. Here the main goal of the chapter 3 and 4 is to develop a flat/CFT dictionary working the same way as the AdS/CFT dictionary based on such observation.

[1] The examiner believes that there should be a proper regularization procedure like the case for QFT to make the Carroll thermal partition function well-defined. As far as I know, the authors in de Boer et al. (2023) have attempted to find a regularization procedure which preserves the Carroll symmetry. But the result is negative.

Chapter 2

Real Time Holography

2.1 Real-Time Holographic in Embedding Space

In this section, we will discuss real-time holographic within the embedding space formalism. Beginning with a brief summary of the approach to real-time holography developed in Skenderis and van Rees (2009, 2008), which illustrates the discussions with the case of a free scalar field, we then lift the results to the embedding space and present the matching condition for higher spin fields.

2.1.1 Real-Time Holography: Review

In holography, when studying the field configuration or constructing the dictionary between two theories, people often choose to first specify the behaviour of the field at the boundary of AdS and then use them to write down the bulk-boundary propagator or the source of the operator which belongs to the boundary CFT Witten (1998b); Freedman et al. (1999); Costa et al. (2014), for further development one can see the application of Fefferman-Graham expansion Fefferman and Graham (1985); Graham (1999); Skenderis (2002).

This is enough for us to deal with Euclidean AdS/CFT since in Euclidean signature, spatial and time directions are indistinguishable and the data on the boundary will in principle encode the whole information of the field. As for the Lorentzian signature, specifying the behaviour of the field at the spatial boundary will no longer enable us to uniquely determine the bulk field due to the lack of information about the field at the far past or the far future, i.e., the boundary of the time direction. Moreover it is interesting to note that, if there is a black hole in the bulk, one can show that it is possible to determine such information on the boundary by specifying the behavior of the modes at the horizon Son and Starinets (2002); Herzog and Son (2003) .

In fact such issue has already been addressed by Hartle and Hawking Hartle and Hawking (1983) when studying the quantum gravity wavefunction of the universe. In order to specify the initial condition of the Lorentzian evolution they choose to glue part of Euclidean path integral to the initial codimension one surface of the Lorentzian spacetime. If we denote the action of the Euclidean and Lorentzian spacetime as S_E and S_L, the weight of the quantum gravity path integral can be represented as

$$e^{-S_E}\, e^{iS_L}, \tag{2.1}$$

in which the term e^{-S_E} can be treated as a norm factor resulting from the preparation of initial state. At quantum level, especially for the quantum field on the curved space time, this enables us to pick out a preferred vacuum and Hilbert space Wald (1994); Witten (2021a).

At the same time, on the field theory side, one can also construct various contours in complex time plane to calculate the corresponding vacuum-vacuum, thermal or out of time order correlation function Landsman and Van Weert (1987); Maldacena and Susskind (2013). And the field theory along these contours are continuous, for example, one can see the study of analyticity of the Wightman functions Osterwalder and Schrader (1973, 1975). This implies that on the bulk side the Euclidean action and Lorentzian action should also match smoothly. Rather than regarding the Euclidean action as a norm factor, one should instead also treat the Euclidean action as the dynamical part, i.e., consider the total action $-S_E + iS_L$, filling the contour with the bulk geometry, and impose the matching condition at the joint surface.

Before presenting the matching conditions, here we first clarify the concept that what we mean by matching two theories together. Consider two physical systems that live in two regions labelled by I and II, the dynamics of two systems are governed by the action S_I, S_{II}, respectively. Moreover, we denote the intersection of these two regions as Σ. It could be a purely mathematical surface or a physically measurable junction, like the domain wall and we propose the condition that two theories joint smoothly at the surface Σ to be

$$\delta S_I|_\Sigma = \delta S_{II}|_\Sigma, \tag{2.2}$$

in which $\delta S|_\Sigma$ represents the boundary term of the variation. It is obvious if $S_{II} = S_I$ then Σ will not exist physically. In our case, the region I and II now becomes spacetime with different signatures, the dynamics are governed by the action S_L, S_E. As for the full quantum gravity theory, the matching condition following from (2.2) is still unknown but if we just consider the scalar field, and in the context of saddle point approximation, we have

$$\phi_E|_\Sigma = \phi_L|_\Sigma, \qquad (i\partial_t\phi_L + \partial_\tau\phi_E)|_\Sigma = 0, \tag{2.3}$$

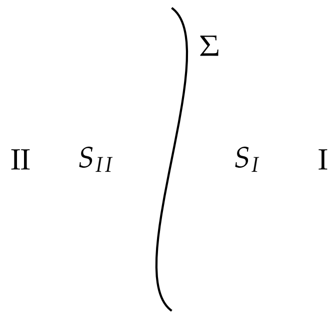


FIGURE 2.1: Two systems distribute in region I and II, joint at the surface Σ.

evaluated at the joint surface Σ in which subscripts E, L are used to represent that the field lie along the imaginary and real part of the contour and we will stick with such convention throughout the thesis.

Thus this motivates us, taking the vacuum-vacuum contour for example, to write down the duality relation Skenderis and van Rees (2009, 2008)

$$\langle 0|T \exp\left(-i\int d^d x\sqrt{-g}\phi_0\mathcal{O}\right)|0\rangle = \int_{\phi\sim\phi_0} DgD\phi \exp\left(-S_E + iS_L\right), \tag{2.4}$$

in which on the left hand side g represents the bulk metric while on the right hand it represents the induced conformal structure on the boundary. As for the scalar field lives in the bulk with boundary value ϕ_0, on the boundary CFT theory, it can be regarded as the source of the operator $\mathcal{O}$ while such correspondence are called real time gauge/-gravity duality. In practice, we evaluate the right hand side at the AdS saddle point and the formula has been used to calculate the Wightman functions and produced right results. Finite temperature correlation functions are also studied provided the matching condition for thermal contour is applied on the bulk side. Moreover, it is interesting to note that, by identifying pair of sources along the thermal contour van Rees (2009), one can recover the ingoing waves and retarded correlations functions when there is a black hole in the bulk.

As for the higher spin field, the matching condition will be quite complicated. Classically, the equation of motion for higher spin fields will be non-linear Vasiliev (1990, 1999, 2003) while one needs to impose more physical restrictions when dealing with the matching in the context of quantum theory. One can see the discussion of massless spin two field in Louko and Sorkin (1997); Kontsevich and Segal (2021); Witten (2021b). Here, we will only discuss the matching condition at the classic and linear level, i.e., the equation of motion are linear equations. In this case, we can write down the matching condition for higher spin fields by the investigation of the higher spin bulk-boundary propagator in embedding space.

2.1.2 Holographic Scalar Fields in Embedding Space Formalism

We start from the study of scalar fields in embedding space and one can find a brief introduction of embedding space together with solutions of the KG equation in Appendix A. For the embedding scalar bulk-boundary propagators, i.e., the Green function for the particle moving on the *AdS* background with given asymptotic limit, they are deduced to be

$$G^E_\Delta(X_1, X_2) = \frac{1}{(-2X_1 \cdot X_2 + i\epsilon)^\Delta_E} + Y^E_\Delta(X_1, X_2), \tag{2.5}$$

$$G^L_\Delta(X_1, X_2) = \frac{1}{(-2X_1 \cdot X_2 + i\epsilon)^\Delta_L} + Y^L_\Delta(X_1, X_2), \tag{2.6}$$

in which we use E and L to represent different values in the two signatures while X_1, X_2 are two points in embedding space. In our expression, propagators are separated into two terms, the first is the regularised sources with proper $i\epsilon$ prescription and the second is the contribution from normalizable modes denoted as Y_Δ. By solving the Klein-Gordon equation on the surface $X^2 = -1$, we obtain [1]

$$Y^L_\Delta(X) = \frac{1}{(2\pi)^d} \int dK\, e^{iK^\mu X_\mu / X^+}\, \theta(-K^2)\, B_L(K)\, (X^+)^{-d/2}\, J_\Delta(|K|X^+) \tag{2.7}$$

in which $K = (w, k^i)$ for $1 \leq i \leq d$ is the momentum space coordinate and J_Δ is the Bessel function written in terms of the scale dimension Δ. J_Δ s can be regarded as orthogonal basis of the normalizable modes and the coefficients $B(K)$ are determined by the boundary condition of the propagator. We can check that these two terms behave like z^Δ and $z^{d-\Delta}$ respectively when $z \to 0$ and they are two independent solutions of the asymptotic Klein-Gordon equation.

To obtain the Euclidean version of the normalizable modes Y^E_Δ, we first do the Wick rotation on Y^L_Δ, i.e., taking $X^0 \to -iX^0$, which will result in blow up modes when $X^0 \to \pm\infty$. To get rid of this, we use the absolute value of $|X^0 K_0|$ in the exponential term, which leads to

$$Y^E_\Delta(X) = \frac{1}{(2\pi)^d} \int dK\, e^{(-|X^0 K_0| + iX^i K_i)/X^+}\, \theta(-K^2)\, B_E(K)\, (X^+)^{-d/2}\, J_\Delta(|K|X^+). \tag{2.8}$$

Given the bulk-boundary propagator, we treat them as a set of complete basis so that in general we can expand an arbitrary scalar field $\Phi_\Delta(X)$ as

$$\Phi_\Delta(X) = \int dP\, C_\Delta(P)\, G_\Delta(X, P), \tag{2.9}$$

[1] In fact, it is subtle to lift a function from the AdS spacetime to the embedding space and there are various ways to do this. In the section 2.1.4, we will use BTZ solution as an example to discuss this in detail.

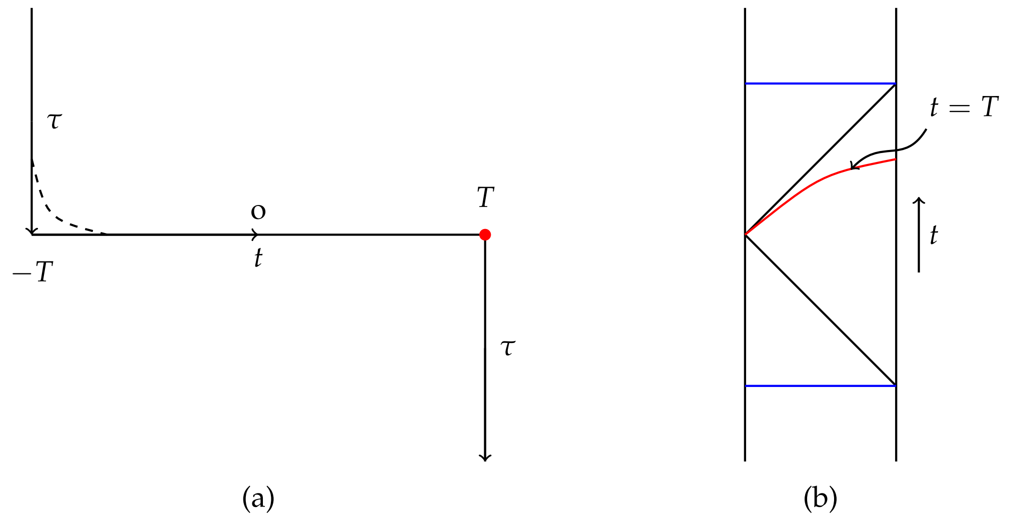


FIGURE 2.2: (a) A toy Wick rotation contour, vertical lines represent the imaginary time τ while horizontal lines represent the real time t. The red point at $(T, 0)$ is the matching surface Σ. Operators are inserted at the two corners and therefore $2T$ represents the time difference between two operators. The dashed curve means that we can deform the contour apart from the axes in principle. (b) The global AdS_2 is illustrated by the strip while our embedding coordinate covers the region between two blue lines, which is a hyperboloid. Moreover, the Poincaré coordinates covers half of the hyperboloid showed in the triangle. The red line represents the matching surface Σ at $t = T$.

in which the integral means that we are summing over the null rays and $C_\Delta(P)$ can be treated as the coefficients of the basis while Δ is the scale dimension of the dual boundary fields, related to the mass of $\Phi_\Delta(X)$. Noting that there are two sets of basis thus we have two possible ways of expansion

$$\Phi^E_\Delta(X) = \int dP\, C^E_\Delta(P)\, G^E_\Delta(X, P), \tag{2.10}$$

$$\Phi^L_\Delta(X) = \int dP\, C^L_\Delta(P)\, G^L_\Delta(X, P), \tag{2.11}$$

where one for imaginary time while the other for real time. We should keep in mind that these two kinds of expansions only work in their own proper region. Taking the contour in Fig.2.2 for example, which starts from $(-T, +\infty)$ and ends at $(T, -\infty)$ with two corners at $(\pm T, 0)$, we can interpret that two vertical lines are used to prepare the initial and final quantum state and the horizontal line represents the evolution of time from $-T$ to T.

To study the behavior of the fields along this contour, we should apply the Euclidean expansion Φ_E on τ axes and the Lorentzian expansion Φ_L on t axes. For the contour going through the plane, like the dashed line around $-T$, these two kinds of basis will mix and it will go beyond our discussion. Since the contour with corners are not smooth, to make the fields Φ^E_Δ and Φ^L_Δ consistent along the whole contour, matching conditions should be imposed at the singular surface. Consider the surface at $t = T$ and we denote it as Σ , illustrated in Fig.2.2, the matching conditions Skenderis and van Rees (2008, 2009) are

$$\Phi^E_\Delta(X)|_\Sigma = \Phi^L_\Delta(X)|_\Sigma, \tag{2.12}$$

$$\partial_\tau \Phi^E_\Delta(X)|_\Sigma + i\partial_t \Phi^L_\Delta(X)|_\Sigma = 0, \tag{2.13}$$

and these two conditions enable us to solve C^L_Δ, C^E_Δ and B_L, B_E. Since the basis of the source and the normalizable modes are independent, we will deal with B and C separately. First for the source terms, with the help of (A.8) and (A.7), we choose to push forward the derivative of the time to the embedding space

$$\partial_\tau \frac{1}{(-2X\cdot P)^\Delta_E} = \frac{2\Delta}{(-2X\cdot P)^{\Delta+1}_E}(-X^0 P^+ + X^+ P^0), \tag{2.14}$$

$$\partial_t \frac{1}{(-2X\cdot P)^\Delta_L} = \frac{2\Delta}{(-2X\cdot P)^{\Delta+1}_L}(X^0 P^+ - X^+ P^0). \tag{2.15}$$

Moreover, in order to use the equation (2.12) and (2.13) to solve the coefficients, we need to figure out the form of a function when it is restricted on the surface Σ from the Euclidean and Lorentzian point of view. For Lorentizian signature, since the contour lies exactly along the real t axes, the restricted function on Σ means that we take $t = T$. For the Euclidean signature, since the contour shifts away from the pure imaginary axes τ and in order to reflect such shift in the theory, we need to make the variable in the function shift as $X^0 \to X^0 + iT$ and then set $\tau = 0$. Moreover, after taking the coordinate rotation $P^E_0 = iP^L_0$ into consideration and substituting (2.14), (2.15) into the matching condition equations, we find the solution should be

$$C^E_\Delta = C^L_\Delta. \tag{2.16}$$

As for the normalizable term, by directly comparing the integrands, from (2.12) we have

$$B_L(K) + B^-_L(K) = B_E(K) + B^-_E(K), \tag{2.17}$$

while from (2.13) we obtain[2]

$$B_L(K) - B^-_L(K) = B_E(K) + B^-_E(K), \tag{2.18}$$

in which we define $B^-(w,k^i) = B(-w,k^i)$ and these two equations will lead to

$$B_E = B_L = 0, \tag{2.19}$$

[2] In fact here we take $T = 0$ otherwise there will be a phase factor $e^{\pm iwT}$ in front of the coefficients. Since B should only depend on the boundary conditions rather than the choice of contour, it will not change the result.

which tells us normalizable modes do not contribute to the bulk-boundary propagator.

2.1.3 Matching Conditions for Higher Spin Fields

Now, we are going to discuss the matching condition of higher spin fields and lift them to the embedding space. The AdS_{d+1} spacetime is regarded as a hyperboloid $X^2 = -1$ in the embedding space, as we have introduced before. Here we use the embedding coordinates in the light cone gauge $X^A = (X^+, X^-, X^\mu)$ and the AdS Poincaré coordinates $y^a = (y^\mu, z)$. Moreover, the Jacobian matrix between these two coordinates is given by

$$\frac{\partial X}{\partial y^\mu} = \frac{1}{z}(\ 0,\ 2y_\mu,\ \delta^\nu_\mu\ ) \qquad \text{and} \qquad \frac{\partial X}{\partial z} = (-\frac{1}{z^2},\ 1,\ 0,\ldots,0), \tag{2.20}$$

which tell us the rule to push back a vector H_A in embedding space to a vector H_a in AdS space as a submanifold thus we have

$$H_a(X) = \frac{\partial X^A}{\partial y^a} H_A(X), \tag{2.21}$$

where the variable X in the expression $H_a(X)$ is used to remind us that the tensor is written in terms of embedding coordinates. Noting that the dimension of AdS is lower than the embedding space, there is redundancy when we transform form H_A to H_a. To construct a one-to-one correspondence between symmetric tensors in these two spaces, we should impose the transverse condition $X^A H_A(X) = 0$ to eliminate the extra degree of freedom, while we can understand it as restricting the vector H_A tangent to the AdS submanifold.

Before writing down the matching conditions in embedding space, based on the study of scalar field matching, we first propose the two conditions in the AdS space

$$H^E_a(X)|_\Sigma = H^L_a(X)|_\Sigma, \tag{2.22}$$

$$\partial_\tau H^E_a(X)|_\Sigma + i\partial_t H^L_a(X)|_\Sigma = 0. \tag{2.23}$$

These two equations specify conditions at the joint surface Σ up to the first order. Since perturbatively physical equations with spin, like the curved background Maxwell equation, are often second order differential equations, the above conditions at the boundary enable us to determine the tensor field in a unique way. As for the first matching condition, using the Jacobian matrix, we can write it in terms of the embedding coordinate as

$$\left(\frac{\partial X^A}{\partial y^a}\right)^E H^E_A(X) = \left(\frac{\partial X^A}{\partial y^a}\right)^L H^L_A(X), \tag{2.24}$$

in which we omit the joint surface restriction and again we use E, L to distinguish Jacobian matrices in Euclidean and Lorentzian signature, respectively. As for the second matching condition, since the derivative with respect to time $\partial_t = \partial_\tau = \partial_0$ is involved, we need to study the matching conditions in different spacetime directions separately. Taking the Euclidean signature for example, for $a \neq 0$, we have

$$\partial_\tau \left(\frac{\partial X^A}{\partial y^a} H_A(X) \right) = \frac{\partial X^A}{\partial y^a} \, \partial_\tau H_A(X), \tag{2.25}$$

while for $a = 0$

$$\partial_\tau \left(\frac{\partial X^A}{\partial y^0} H_A(X) \right) = \frac{\partial X^A}{\partial y^0} \, \partial_\tau H_A(X) + 2X^+ H_+, \tag{2.26}$$

which tells us that ∂_τ will commute with the Jacobian matrix except for the $y^0 = \tau$ direction in which we get the extra $2X^+ H_-$ contribution. This leads us to define an operator valued matrix $\mathcal{M}$ as

$$\mathcal{M} := \frac{\partial X}{\partial y} \partial_\tau + \begin{pmatrix} 2X^+ & 0 \\ & \\ 0 & 0 \end{pmatrix} = \frac{\partial X}{\partial y} \partial_\tau + \mathcal{T}, \tag{2.27}$$

in which the second term $\mathcal{T}$ is a $(d+1) \times (d+2)$ matrix which has $(0, \ldots, d-1, z)$ as rows and $(-, +, 0, \ldots, d-1)$ as columns while $(0, -)$ is the only nonzero element. With the help of matrix $\mathcal{M}$, we can write (2.25) and (2.26) in a compact form as

$$\partial_\tau \left(\frac{\partial X^A}{\partial y^a} H_A(X) \right) = \mathcal{M}_a^A \, H_A, \tag{2.28}$$

and the second matching condition can be written as

$$\mathcal{M}^E \cdot H^E(X) + i \mathcal{M}^L \cdot H^L(X) = 0, \tag{2.29}$$

in which M^E is the matrix we have already discussed in (2.27) acting on the column vector H_A and M^L is

$$\mathcal{M}^L = \left(\frac{\partial X}{\partial y} \right)^L \partial_t - \begin{pmatrix} 2X^+ & 0 \\ & \\ 0 & 0 \end{pmatrix}. \tag{2.30}$$

Equation (2.24) together with (2.29) give us matching conditions for the vector fields while it is straight forward to generalise them to the high spin fields. Before doing that, we first hide the spin index into the polynomial via the operator K_A Costa et al. (2014)

$$H_{A_1 \ldots A_J}(X) = \frac{1}{J! (\frac{d-1}{2})_J} K_{A_1} \ldots K_{A_J} H(X, W), \tag{2.31}$$

in which $H(X,W) = W^{A_1} \dots W^{A_J} H_{A_1 \dots A_J}(X)$ is a polynomial in terms of W. The operator K_A is defined as

$$K_A = \left(\frac{d-1}{2} + W \cdot \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W^A}. \tag{2.32}$$

Moreover, we should note that K_A only involves the variable W provided $H_{A_1 \dots A_j}$ is a symmetric traceless tensor. Thus K_A acts on the polynomial $H(X,W)$ independently and we can write higher spin field matching conditions in terms of the embedding polynomials as

$$\begin{aligned} &\left(\frac{\partial X^{A_1}}{\partial y^{a_1}} K_{A_1} \right)^E \dots \left(\frac{\partial X^{A_J}}{\partial y^{a_J}} K_{A_J} \right)^E H^E(X,W) \\ = &\left(\frac{\partial X^{A_1}}{\partial y^{a_1}} K_{A_1} \right)^L \dots \left(\frac{\partial X^{A_J}}{\partial y^{a_J}} K_{A_J} \right)^L H^L(X,W) \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} &\sum_{i=1}^{J} \left(\frac{\partial X^{A_1}}{\partial y^{a_1}} K_{A_1} \right)^E \dots \left(T^{A_i}_{a_i} K_{A_i} \right)^E \dots \left(\frac{\partial X^{A_J}}{\partial y^{a_J}} K_{A_J} \right)^E H^E(X,W) \\ +i &\sum_{i=1}^{J} \left(\frac{\partial X^{A_1}}{\partial y^{a_1}} K_{A_1} \right)^L \dots \left(T^{A_i}_{a_i} K_{A_i} \right)^L \dots \left(\frac{\partial X^{A_J}}{\partial y^{a_J}} K_{A_J} \right)^L H^L(X,W) = 0, \end{aligned} \tag{2.34}$$

in which we take $T^{A_J}_{a_J} = M^{A_J}_{a_J}$ for short. At first sight, the matching conditions in embedding space are more complicated than those in AdS space since more operators and transform matrices are involved but we should note that the tensor field $H_{a_1 \dots a_J}$ will be simplified a lot once we write it into the form of polynomial $H(X,W)$.

Examples

Here we will see how the matching conditions work based on the investigation of the fields with spin $J=1$ and $J=2$. Given the equation of motion

$$\nabla^2 H_{\mu_1 \mu_2 \dots \mu_J} = M^2 H_{\mu_1 \mu_2 \dots \mu_J}, \tag{2.35}$$

together with the constraints $\nabla^\mu H_{\mu \dots \mu_J} = 0$ and $H^\mu_{\mu \dots \mu_J} = 0$, where ∇ is the covariant derivative on the AdS spacetime and $H_{\mu_1 \mu_2 \dots \mu_J}$ are the associated higher spin fields with mass M. We can obtain spin J bulk-to-boundary propagator of dimension Δ written as Costa et al. (2011, 2014)

$$G_{\Delta,J}(X,P;W,Z) = \frac{((-2P \cdot X)(W \cdot Z) + 2(W \cdot P)(Z \cdot X))^J}{(-2P \cdot X)^{\Delta+J}}, \tag{2.36}$$

in which X, P are points live on the bulk and boundary associated with the polynomial variables W, Z and the dictionary $M^2 = \Delta(\Delta - d) - J$ for the higher spin fields is also preserved. The above expression is determined by imposing the boundary condition on bulk Green function

$$\lim_{z_2\to 0} z_2^{-\Delta} G_{\Delta,J}(X_1, X_2; Z_1, Z_2) = G_{\Delta,J}(X_1, X_2^{\infty}; Z_1, Z_2). \tag{2.37}$$

Here we are abusing the notion $G_{\Delta,J}$. On the left it represents the bulk Green function, while the right hand side of (2.37) represents a bulk-boundary operator. The limit $z \to 0$ means that we are sending a bulk point $X = \frac{1}{z}(1, z^2 + y^2, y^\mu)$ to the boundary $X^\infty = (1, y^2, y^\mu)$ thus the explicit physical meaning of $G_{\Delta,J}$ depends on the position of its second point. Moreover, after taking the limit in (2.37), we can see the formula in (2.36) only encodes the information about the sources while the information about the normalizable modes proportional to $z^{d-\Delta}$ is absent.

In the following two examples, we will try to deal with the field of spin $J = 1$ and $J = 2$ separately and the above operators will be used extensively together with the higher spin field expansion

$$H(X, W) = \int \frac{dP}{J!(\frac{d}{2}-1)_J} C_{\Delta,J}(P, D_Z) G_{\Delta,J}(X, P; W, Z), \tag{2.38}$$

in which the higher spin fields and the bulk-to-boundary propagator are both written in terms of polynomials. The operator D_Z is defined as

$$D_Z^A = \left(\frac{d}{2} - 1 + Z \cdot \frac{\partial}{\partial Z}\right) \frac{\partial}{\partial Z_A} - \frac{1}{2} Z^A \frac{\partial^2}{\partial Z \cdot \partial Z}. \tag{2.39}$$

As for the $C_{\Delta,J}(P, D_Z)$, we can treat it as an operator polynomial $C_{\Delta,J}^{A_1\cdots A_J} D^Z_{A_1\cdots A_J}$ thus determining the matching condition for $H^{A_1\cdots A_J}$ is equivalent to determining the condition for $C_{\Delta,J}^{A_1\cdots A_J}$.

i) Spin $J = 1$ Match

For the spin 1 case, we first write down the expansion of fields in terms of the polynomial

$$H(X, W) = \frac{1}{\frac{d}{2} - 1} \int dP\, C^A(P)\, D_A^Z\, G_{\Delta,J=1}(X, P; W, Z), \tag{2.40}$$

in which $C^A = C^A_{\Delta,J=1}$ are coefficients carrying the tensor indexes. After substituting $G_{\Delta,J}$, D_A^Z into the integrand and applying K_A^W on the polynomial, we obtain the tensor in embedding space

$$H_A(X) = \int dP\, \frac{-2(P\cdot X)C_A + 2(C\cdot X)P_A}{(-2P\cdot X)^{\Delta+1}} \tag{2.41}$$

in which we can check that the transverse condition $X \cdot H = 0$ is automatically satisfied for arbitrary $C^A(P)$. Moreover, one can also directly see from the above expression that there is a redundancy of the coefficients

$$C^A(P) \longrightarrow C^A(P) + \lambda P^A, \tag{2.42}$$

which tells us a shift of the coefficient by λP^A will give us the same higher spin field. We call this the pure gauge of coefficients since it works the same way as the pure gauge for CFT spin fields and later we will see that the consistence of matching conditions requires us to fix the pure gauge.

Now, if we just consider the integrand, the matching condition (2.24) tells us that

$$-(P \cdot X)_E \, C^E_a + (C^E \cdot X)_E \, P_a = -(P \cdot X)_L \, C^L_a + (C^L \cdot X)_L \, P_a, \tag{2.43}$$

in which C^E_A, C^L_A are the tensor coefficients in Euclidean and Lorentzian signature and they have already been pulled back to the AdS space thus labelled by C^E_a, C^L_a. Here we should note that the above equation is restricted on the surface Σ which we did not write down explicitly for short. Since P_a serve as variables in the integrand, to solve the above matching condition for arbitrary P, we should make each term on both sides fit. Furthermore, noting that $(P \cdot X)_E = (P \cdot X)_L$ is guaranteed by the rotation of embedding coordinates, therefore the non trivial conditions are determined as

$$C^E_a|_\Sigma = C^L_a|_\Sigma, \qquad (C^E \cdot X)_E|_\Sigma = (C^L \cdot X)_L|_\Sigma. \tag{2.44}$$

The first equation is just the statement that the coefficients are continuous at the joint surface Σ in AdS spacetime and the second one can be simplified to $iC^E_0 = C^L_0$, which is the feature of Wick rotation on coefficient tensor fields associated with the rotation of embedding space coordinates.[3]

Here, we should stop to check that if the two conditions in (2.44) could be compatible. Actually, if we consider the coordinate transformation of time direction under the condition $\tau = iT, t = T$ and $C^E_0 = iC^L_0$, the relation we get in AdS spacetime should be $C^E_{a=0} = iC^L_{a=0}$ at the joint surface rather than the first one in (2.44) ($C^E_{a=0} = C^L_{a=0}$), which implies in fact the matching condition we should impose on the tensor field is that

$$H^E_a(X)|_\Sigma = H^L_a(X)|_\Sigma, \qquad \text{for} \qquad a \neq 0 \tag{2.45}$$

$$H^E_0(X)|_\Sigma = iH^L_0(X)|_\Sigma. \tag{2.46}$$

[3]Here $C_{A=0}$ represents the tensor in embedding space while later we will meet the zero component in AdS space and we denote it as $C_{a=0}$.

And the modification of the matching condition in time direction results from the convention when we are doing the calculation while we can treat it as the feature of the rotation of vector fields, which we do not need to consider in scalar field matching.

Physically, suppose that we have the higher spin action in the hand and try to match actions of different signatures together, the special matching condition (2.46) for time component will make the invariant term like

$$\eta^{ab} H_a^L H_b^L = \delta^{ab} H_a^E H_b^E \tag{2.47}$$

joint smoothly at the surface Σ. Thus now we can see that the condition (2.46) becomes obvious from physics point of view even though it was derived by checking the consistence of the propagator matching. Next, we come to study the second matching equation. Inspired by the above discussion, we propose the modified matching conditions as

$$\partial_\tau H_a^E(X)|_\Sigma + i\partial_t H_a^L(X)|_\Sigma = 0, \qquad \text{for} \qquad a \neq 0 \tag{2.48}$$

$$\partial_\tau H_0^E(X)|_\Sigma - \partial_t H_0^L(X)|_\Sigma = 0, \tag{2.49}$$

in which we have taken the rotation of H_0 into consideration. As for the derivative with respect to time, we need to substitute (A.8) and (A.7) into (2.41) and then obtain

$$\begin{aligned} \partial_\tau H_A(X) &= \int dP\, \frac{-2(-2X^0P_- + X^+P_0)C_A + 2(-2X^0C_- + X^+C_0)\, P_A}{(-2P\cdot X)_E^{\Delta+1}} \\ &+ \frac{2(\Delta+1)(-2X^0P_- + X^+P_0)(-2(X\cdot P)_E C_A + 2(C\cdot X)_E P_A)}{(-2X\cdot P)_E^{\Delta+2}} \end{aligned} \tag{2.50}$$

for Euclidean signature and

$$\begin{aligned} \partial_t H_A(X) &= \int dP\, \frac{-2(2X^0P_- + X^+P_0)\, C_A + 2(-2X^0C_- + X^+C_0)\, P_A}{(-2P\cdot X)_L^{\Delta+1}} \\ &+ \frac{2(\Delta+1)(2X^0P_- + X^+P_0)(-2(X\cdot P)_L\, C_A + 2(C\cdot X)_L\, P_A)}{(-2X\cdot P)_L^{\Delta+2}} \end{aligned} \tag{2.51}$$

for Lorentzian signature. It will be convenient to note the differences between them mainly come from the inner product $(\cdot)_{L/E}$ and the sign in front of $\pm X^0 P_-$. Then, using the matching condition (2.29), we have

$$\begin{aligned} &\Delta(-X\cdot P)_E\, (-2X^0P_- + X^+P_0)C_a + ((-X\cdot P)_E\, (-2X^0C_- + X^+C_0) \\ &+(\Delta+1)(C\cdot X)_E\, (-2X^0P_- + X^+P_0))P_a + i(\cdots)_L = 0, \end{aligned} \tag{2.52}$$

in which $a \neq 0$ and $(\cdots)_L$ represents the Lorentzian version of the formula we wrote down explicitly and one can check that the above equation is trivial provided the conditions in (3.120) are satisfied. The new restriction comes from the study of the zero component for $a = 0$, taking the extra contribution $2X^+H_-$ into consideration, we have

$$\Delta(-X\cdot P)_E\ (-2X^0P_- + X^+P_0)C_{a=0} + (P\cdot X)_E^2\ C_-X^+ - (P\cdot X)_E(C\cdot X)_E P_-X^+$$
$$+((-X\cdot P)_E\ (-2X^0C_- + X^+C_0) + (\Delta+1)(C\cdot X)_E\ (-2X^0P_- + X^+P_0))P_0$$
$$-(\cdots)_L = 0. \tag{2.53}$$

To solve it, one should impose the condition

$$C_-^E|_\Sigma = C_-^L|_\Sigma, \tag{2.54}$$

which gives us the proper gauge of the embedding coordinates on the joint surface Σ. To understand such an extra condition, first we consider the degrees of freedom of the matching conditions in terms of the coefficients C^A. Since we are dealing with $d+1$ dimensional AdS spacetime, there are $d+1$ equations for us to solve in (2.44), together with the gauge condition (2.54), we have $d+2$ matching conditions, which uniquely fix the $d+2$ coefficients C^A in embedding space. We can also understand this in a way that is similar to the direct match of H_A. As we have mentioned before, after imposing the transverse condition, there will be a one to one correspondence between the fields in the embedding space and the fields on the AdS surface. But the transverse conditions will not introduce any restriction on the coefficients therefore we have the pure gauge redundancy. Now we see that the pure gauge should be fixed when doing the matching.

ii) Spin $J = 2$ Match

For the spin 2 field, we just consider the symmetric and traceless tensor field H_{AB} for simplicity. In this case, the polynomial can be written as

$$H(X,W) = \frac{2}{(d+1)(d-1)} \int dP\ C^{AB}(P)\ D_A^Z D_B^Z\ G_{\Delta,J=2}(X,P;W,Z), \tag{2.55}$$

where $C^{AB} = C^{AB}_{\Delta J=2}$ and it corresponds to the tensor

$$H_{AB}(X,W) = \int dP\ \frac{4}{(-2X\cdot P)^{\Delta+2}}\ ((X\cdot P)^2 C_{AB} \tag{2.56}$$
$$+P_AP_BX^CX^DC_{CD} - (X\cdot P)P_AX^CC_{CB} - (X\cdot P)P_BX^CC_{CA}).$$

After imposing the matching condition, we have

$$(X\cdot P)_E^2\ C^E_{ab} + P_aP_bX^cX^dC^E_{cd} - (X\cdot P)_E\ P_a\ X^cC^E_{cb} - (X\cdot P)_E\ P_b\ X^cC^E_{ca}$$

$$= (X\cdot P)_L^2\ C^L_{ab} + P_aP_bX^cX^dC^L_{cd} - (X\cdot P)_L\ P_a\ X^cC^L_{cb} - (X\cdot P)_L\ P_b\ X^cC^L_{ca}, \tag{2.57}$$

in which the tensor C_{AB}, P_A, X_A together with the inner product have been pulled back to the AdS Poincaré coordinates. Similar to the study of spin $J=1$ case, we deduce the solution to be

$$C^E_{ab}|_\Sigma = C^L_{ab}|_\Sigma \quad \text{for} \quad a\neq 0, \qquad C^E_{0b}|_\Sigma = iC^L_{0b}|_\Sigma \quad \text{for} \quad a=0, \tag{2.58}$$

$$C^E_{ab}\ X^a|_\Sigma = C^L_{ab}\ X^a|_\Sigma, \tag{2.59}$$

in which the first implies the continuation of the tensor field, the second corresponds to the Wick rotation on the time direction and the third implies the match of inner product, and we note that they are compatible by imposing $C^E_{0B} = iC^L_{0B}$.

Now, we come to study the second matching condition and check if the restrictions proposed in (2.59) are enough. Substituting (A.8) into (2.56), we see that the derivative with respect to the Euclidean time is given by

$$\partial_\tau H_{AB}(X,W) = 4\int dP\ \frac{2}{(-X\cdot P)_E^{\Delta+2}}((-2X^0P_- + X^+P_0)((X\cdot P)_E\ C_{AB} \tag{2.60}$$

$$-P_{(A}X^CC_{CB)}) + P_AP_BX^D(-2X^0C_{-D} + X^+C_{0D}) - (X\cdot P)_EP_{(A}(-2X^0C_{-B)} + X^+C_{0B)}))$$

$$+\frac{2(\Delta+2)(-2X^0P_- + X^+P_0)}{(-2X\cdot P)_E^{\Delta+3}}\ ((X\cdot P)_E\ C_{AB} + P_AP_BX^CX^DC_{CD} - 2(X\cdot P)_E\ P_{(A}X^CC_{CB)}).$$

in which we use the convention for symmetrising the free tensor index $C_{(AB)} = \frac{1}{2}(C_{AB} + C_{BA})$. And, as for the Lorentzian time derivative, all we need to do is to change the sign in front of X^0P_- and the notion of inner product while we will not show that explicitly here. Consider the integrand, we will obtain

$$P_aP_b((\Delta+2)(-2X^0P_- + X^+P_0)X^cX^dC_{cd} + (-2P\cdot X)_EX^d(-2X^0C_{-d} + X^+C_{0d}))$$

$$-P_{(a}(X\cdot P)_E(\Delta(-2X^0P_- + X^+P_0)X^cC_{cb)} + (-2P\cdot X)_E\ (-2X^0C_{-b)} + X^+C_{0b)}))$$

$$+\Delta(-2X^0P_- + X^+P_0)(X\cdot P)_E^2\ C_{ab} + i(\cdots)_L = 0 \tag{2.61}$$

for $a, b \neq 0$ from the second the matching condition, which will be trivial provided the restrictions we imposed in (2.59) are satisfied. After all, to make the matching condition fits at the $a = 0$ and $b \neq 0$ direction, one should also impose the gauge constraint

$$C^{E}_{-b}|_{\Sigma} = C^{L}_{-b}|_{\Sigma}, \tag{2.62}$$

which results from the matching condition

$$\partial_\tau H^{E}_{0b} - \partial_t H^{L}_{0b} = 0. \tag{2.63}$$

2.1.4 Embedding the BTZ

As we have mentioned, there is not a unique way to lift a AdS solution to the embedding space. In this section, we will use BTZ solutions as an example to illustrate more detail on this. First, we will introduce BTZ black hole Banados et al. (1993, 1992) in embedding space via identifying points along the trajectory generated by the chosen Killing vector in AdS_3. We start from the embedding of Lorentzian AdS_3 given by

$$-(X^{-1})^2 - (X^0)^2 + (X^1)^2 + (X^2)^2 = -R^2, \tag{2.64}$$

in which the $SO(2,2)$ symmetry is manifested and R is related to the cosmological constant by $-\Lambda = R^{-2}$. The AdS_3 in embedding space has the metric

$$ds^2 = -(dX^{-1})^2 - (dX^0)^2 + (dX^1)^2 + (dX^2)^2, \tag{2.65}$$

and in order to get the black hole geometry, we introduce the Killing vector

$$\xi = \frac{r_+}{R}\left(X^{-1}\frac{\partial}{\partial X^2} + X^2\frac{\partial}{\partial X^{-1}}\right), \tag{2.66}$$

where r_+ is a constant characterising the size of horizon. Given the initial point P, the Killing vector will generate a curve $c(t) = e^{t\xi}P$ in which ξ serves as the tangent vector along $c(t)$. Here, we are going to identify the points on the curve such that $t \in 2\pi\mathbb{Z}$ and those points which are invariant under the transformation $e^{2\pi\mathbb{Z}\xi}$ will become singularities of the quotient space $AdS_3\backslash \sim$. Although the quotient space satisfies the Einstein equation at the regular points, we still need to get rid of the closed timelike curves, resulting from the identification procedure, to obtain a reasonable causal structure. According to the property of the Killing field, $\xi \cdot \xi$ is preserved along the curve $c(t)$, thus the necessary condition for the absent of closed timelike curve is $\xi^2 > 0$ everywhere in the manifold and in terms of the embedding coordinates, we have

$$(X^{-1})^2 - (X^2)^2 = X^+X^- > 0, \tag{2.67}$$

which gives us the black hole geometry with a proper causal structure. To see the effect of identification, we will introduce the (t, r, ϕ) coordinate defined as

$$X^{-1} = \frac{Rr}{r_+}\cosh(\frac{r_+}{R}\phi), \quad (2.68)$$

$$X^{2} = \frac{Rr}{r_+}\sinh(\frac{r_+}{R}\phi), \quad (2.69)$$

$$X^{0} = \frac{R}{r_+}\sqrt{r^2 - r_+^2}\,\sinh(\frac{r_+ t}{R^2}), \quad (2.70)$$

$$X^{1} = \frac{R}{r_+}\sqrt{r^2 - r_+^2}\,\cosh(\frac{r_+ t}{R^2}), \quad (2.71)$$

in which $-\infty < t, \phi < \infty$, $r \geq r_+$, and for $0 \leq r \leq r_+$ the expressions for X^{-1}, X^2 are the same while we have

$$X^{0} = \frac{-R}{r_+}\sqrt{r_+^2 - r^2}\,\cosh(\frac{r_+ t}{R^2}), \quad (2.72)$$

$$X^{1} = \frac{-R}{r_+}\sqrt{r_+^2 - r^2}\,\sinh(\frac{r_+ t}{R^2}), \quad (2.73)$$

for X^0, X^1. These two patches together will cover the AdS_3 and the Killing vector becomes

$$\xi = \frac{\partial}{\partial\phi} \quad (2.74)$$

once we have pushed it back to the hyperboloid described by (t, r, ϕ). Therefore, in the new coordinates, the identification of points under $e^{2\pi\mathbb{Z}\xi}$ is equivalent to imposing $\phi \cong \phi + 2\pi$ and then we obtain the black hole metric

$$ds^2 = -\frac{r^2 - r_+^2}{R^2}dt^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + \frac{r^2}{R^2}d\phi^2, \quad (2.75)$$

from which we can see that the horizon lies at $r = r_+$, thus in embedding coordinates, the horizon is

$$X^+X^- = R^2. \quad (2.76)$$

For simplicity, we usually choose to perform the coordinates transformation

$$t \longrightarrow \frac{r_+}{R^2}t, \qquad r \longrightarrow \frac{r}{r_+}, \qquad \phi \longrightarrow \frac{r_+}{R}\phi \quad (2.77)$$

and get

$$ds^2 = -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1} + r^2 d\phi^2, \quad (2.78)$$

with the periodic condition $\phi = \phi + \frac{2\pi r_{\pm}}{R}$. Moreover, to make the coordinates smooth at the horizon, we then introduce Kruskal coordinates (U, V) defined as

$$\begin{cases} U = \sqrt{\frac{r-1}{r+1}}\, e^{t} \qquad V = \sqrt{\frac{r-1}{r+1}}\, e^{-t} \qquad 1 \leq r \\ U = \sqrt{\frac{1-r}{r+1}}\, e^{t} \qquad V = \sqrt{\frac{1-r}{r+1}}\, e^{-t} \qquad 0 < r < 1, \end{cases}$$

Moreover, the metric becomes

$$ds^2 = \Omega^2(r)\; dUdV + r^2 d\phi^2, \tag{2.79}$$

in which $\Omega(r) = r + 1$ is the conformal factor and the horizon lies at $U = 0$ or $V = 0$.

Now, we come to introduce the solution of the Kelin-Gordon equation for the scalar field Φ on the BTZ black hole background (2.75), written as

$$\Box_G \Phi_\Delta - m^2 \Phi_\Delta = 0, \tag{2.80}$$

in which $\Box_G$ represents the Laplacian operator on the curved spacetime G. The detail of the solution is shown in the Appendix B and here we just discuss the results. As for the solution near the horizon $r = r_+$, there are two independent modes

$$\psi_\pm = e^{\frac{ir_\pm}{R^2}(\pm\omega t - kR\phi)} f_\Delta(\pm\omega, k, \frac{r}{r_+}), \tag{2.81}$$

from which we can see that the behaviour of the modes near the horizon depends on the frequency ω which characterizes the propagation of the modes along the circle, called left or right moving modes. If we consider the solution at infinity $r \to \infty$, the modes now become

$$\psi_\pm = e^{\frac{ir_\pm}{R^2}(\omega t - kR\phi}) f_{\Delta^\pm}(\omega, k, \frac{r}{r_+}), \tag{2.82}$$

which will be scale dimension dependent. We can see that these two modes will behave like $r^{-\Delta}$ and $r^{\Delta-2}$ asymptotically, corresponding to the source and normalizable modes, which have been studied in the vacuum AdS_3 case.

It is worthwhile to note that, although we obtain four different kinds of modes in total, this does not mean that the scalar field Φ should be the linear combination of these four modes. The reason that we obtain four modes here is that we are expanding the same function around different singular points thus the basis changes. Near horizon, the basis carries the information of the direction of the propagation while, at the infinity, the basis carries the information of asymptotic behavior of the field according to the radius r.

Embedding the Solution

In the above section, we have studied the embedding structure of the BTZ black hole and obtained the solution of Kelin-Gordon equation on AdS background while in this section we are going to lift the solution from the AdS hyperboloid to the embedding space.

There are various ways of extending a function on a hyperboloid to the embedding space. Since we can foliate the embedding space with AdS surfaces of different radii R, a natural embedding way is just treating the radii R as a variable and extending the solution obtained from some standard surface $R = R_0$ to the surfaces with different radii R. This is easy to do mathematically while we should note that the extended function will not be the solution of Kelin-Gordon equation on other AdS surfaces except for the surface R_0. Vice versa, to fully retain the physical meaning of the extended function, one can solve the KG equation (2.80) on each surface while it is hard to smoothly glue them together.

Here, instead of considering the extension of the solution directly. We consider the extension of the KG equation. Using the generator of the $SO(2,2)$ group

$$J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}, \tag{2.83}$$

we can construct the quadratic Casimir directly in embedding space Penedones (2016)

$$\frac{1}{2} J_{AB} J^{BA} \Phi_\Delta(X) = R^2 \nabla_{AdS} \Phi_\Delta(X), \tag{2.84}$$

in which $\nabla_{AdS} = \Box_G$ since BTZ black geometry is locally isometric to AdS_3. The above expression can be treated as the decomposition of an embedding operator along the AdS surfaces. Given the quadratic Casimir, the equation of motion in embedding now is proposed to be

$$\frac{1}{2} J_{AB} J^{BA} \Phi_\Delta(X) = m^2 R^2 \Phi_\Delta(X), \tag{2.85}$$

from which we see that the mass term m^2R^2 now depends on the radii of the surfaces, and it will be reduced to the KG equation at radii $R = 1$. Since $[J_{AB},\ X^2 + R^2] = 0$, we can solve the Equation (2.85) on each surface, following the same method of solving KG equation, provided we make

$$m^2 R^2 = \Delta_R(\Delta_R - 2). \tag{2.86}$$

Now, if we write solution in terms of the (r, t, ϕ, R) coordinates i.e, $\Phi(X) = \Phi(t, r, \phi, R)$, the modes will become

$$\psi_{\pm} = e^{\frac{ir_+}{R^2}(\pm\omega t - kR\phi)} f_{\Delta_R}(\pm\omega, k, \frac{r}{r_+}) \tag{2.87}$$

at the horizon while, at the infinity, the two modes are

$$\psi_{\pm} = e^{\frac{ir_+}{R^2}(\omega t - kR\phi)} f_{\Delta_R^{\pm}}(\omega, k, \frac{r}{r_+}), \tag{2.88}$$

in which we not only make the phase part R dependent but also the scale dimension R dependent. These solutions are smooth in embedding space and they are also solutions of the scalar KG equation on each AdS surface with mass mR, i.e, now the equation becomes

$$\Box_G \Phi_{\Delta_R} - m^2 R^2 \Phi_{\Delta_R} = 0, \tag{2.89}$$

which can be understood that we are considering excitation spectra or fluctuation of particle with mass of order R on each AdS surfaces.

BTZ Propagator

In this part, based on the study of the geometry of BTZ black holes, we will try to generalise the vacuum AdS propagators to the black hole case and discuss the thermal feature of the boundary theories. As we have already known, BTZ geometry is obtained from AdS_3 by the identification of the points $\phi \cong \phi + 2\pi$. In the embedding coordinates, using (2.68) and (2.69), one can deduce that this is equivalent to

$$X^{\pm} \cong e^{\pm 2\pi r_+} X^{\pm}. \tag{2.90}$$

Here, to study the identified points more carefully, we introduce the notion

$$\widetilde{X}^n := (e^{+2n\pi r_+} X^+,\ e^{-2n\pi r_+} X^-\ X^0,\ X^1), \tag{2.91}$$

in which we use the superscript n to represent the winding number of the coordinates. Moreover, we should note that the points $\widetilde{X}^n$ are distinguishable in the AdS_3 geometry while they form a cover of a single point X of BTZ black hole

$$X = \widetilde{X}^0 \cong \widetilde{X}^1 \cdots \widetilde{X}^n \cong \widetilde{X}^{n+1} \cdots . \tag{2.92}$$

Following such convention, one can directly write down the BTZ boundary-bulk propagator

$$G^{BTZ}_{\Delta,J}(X, P; W, Z) = \sum_{n=-\infty}^{\infty} \frac{((-2P \cdot \widetilde{X}^n)(W \cdot Z) + 2(W \cdot P)(Z \cdot \widetilde{X}^n))^J}{(-2P \cdot \widetilde{X}^n)^{\Delta+J}} \tag{2.93}$$

with the help of the method of images introduced in Keski-Vakkuri (1999); Kraus et al. (2003). Basically, the infinite sum over the winding number on the right hand side is used to construct a function that is invariant at the points $\widetilde{X}^n$ so that the relation (2.92) is manifested in the context of the BTZ geometry.

To see how this works in a more specific way, we choose to go back to the (t, r, ϕ) coordinates. The bulk points X are well defined in the previous section and here for the boundary points P, we write them as

$$P = (P^+,\ P^-,\ P^0,\ P^1) = (e^{r_+\phi'},\ e^{-r_+\phi'},\ \sinh(r_+t'),\ \cosh(r_+t')), \tag{2.94}$$

in which the light ray condition $P^2 = 0$ is satisfied and the (t', ϕ') are in fact coordinates of a cylinder. With these coordinates, for the scalar case, one can write the bulk boundary operator as

$$G^+_{\Delta,1}(X,P) = \sum_{n=-\infty}^{\infty} \frac{1}{\left(-\frac{\sqrt{r^2-r_+^2}}{r_+}\cosh(r_+\delta t) + \frac{r}{r_+}\cosh(r_+(\delta\phi + 2\pi n))\right)^{\Delta}} \tag{2.95}$$

for $r > r_+$ and $\delta\phi = \phi - \phi'$, $\delta t = t - t'$. This is the bulk-boundary propagator when the bulk point is outside the horizon. For the inside horizon propagator, we should use the coordinate 2.72, 2.73 and then obtain

$$G^-_{\Delta,1}(X,P) = \sum_{n=-\infty}^{\infty} \frac{1}{\left(-\frac{\sqrt{r_+^2-r^2}}{r_+}\sinh(r_+\delta t) + \frac{r}{r_+}\cosh(r_+(\delta\phi + 2\pi n))\right)^{\Delta}}. \tag{2.96}$$

For the boundary correlation functions, the polynomial of higher spin two point function

$$\langle \mathcal{O}_J(P_1)\mathcal{O}_J(P_2)\rangle(Z_1, Z_2) = \sum_{n=-\infty}^{\infty} \frac{((-2\widetilde{P}_1^n \cdot P_2)(Z_1 \cdot Z_2) + 2(\widetilde{P}_1^n \cdot Z_2)(P_2 \cdot Z_1))^J}{(-2\widetilde{P}_1^n \cdot P_2)^{\Delta+J}} \tag{2.97}$$

is obtained by the projection of the bulk-boundary propagator to the r^{Δ} term, in which $\widetilde{P}^n$ are defined as

$$\widetilde{P}^n := (e^{+2n\pi r_+}P^+,\ e^{-2n\pi r_+}P^-,\ P^0,\ P^1) \tag{2.98}$$

thus the temperature $T = r_+/2\pi$ can be deduced after going to the Euclidean signature $t = -i\tau$.

2.2 A Lorentzian AdS/CFT Map

In this section we will do the Wick rotation of CFT completeness relation by matching the scale dimension and spin in different signature properly then present a Lorentzian AdS/CFT map based the derivation of Euclidean AdS/CFT map in Aharony et al. (2021a,b). Given the Lorentzian AdS/CFT map, we will present the matching conditions for the quadratic action then propose the matching conditions for higher order action.

2.2.1 Scale Dimension and Spin

First we will discuss the behavior of scale dimension Δ and spin J in different signatures from both the conformal field theory and gravity theory point of view, which plays the central role in the context of Wick rotation and AdS/CFT matching.

	Symmetry Group	Principal Series	Parameter
Euclidean	$SO(d+1,1) \sim SO(1,1) + SO(d)$	$\mathcal{E}_{\Delta,\,J}$	$\Delta = \frac{d}{2} + i\mathbb{R}, \quad J \in \mathbb{Z}$
Lorentzian	$SO(d,2) \sim SO(1,1)+$ $SO(1,1) + SO(d-2)$	$\mathcal{P}_{\Delta,\,J,\,\lambda}$	$\Delta = \frac{d}{2} + i\mathbb{R}$ $J = \frac{d-2}{2} + i\mathbb{R},\ \lambda \in \mathbb{Z}$

TABLE 2.1: Harmonic analysis of the conformal symmetry for d spacetime dimensional Euclidean and Lorentzian conformal field theory. The principal series for Euclidean and Lorentzian signature are labelled by $\mathcal{E}$ and $\mathcal{P}$, respectively.

We start from the study of representation theory of the conformal symmetry group. As for the scale dimension induced by the dilaton operator D, it generates a noncompact direction, i.e noncompact subgroup $SO(1,1)$, in both signatures and takes the value $\Delta = \frac{d}{2} + i\mathbb{R}$ on the same principal series provided the representation is unitary while the story for the spin is different.

For the spin in Euclidean signature, it represents the compact group $SO(d)$ thus takes the integer value $J \in \mathbb{Z}$ and in Lorentzian signature it represents the noncompact time direction generated by the operator M_{01} therefore becomes continuous on the principal series $J = \frac{d-2}{2} + i\mathbb{R}$. The decomposition of the symmetry group and the range of parameters are summarized in Table.2.1. We should note that in Lorentzian signature we usually take the representation of compact group $SO(d-2)$ to be trivial therefore

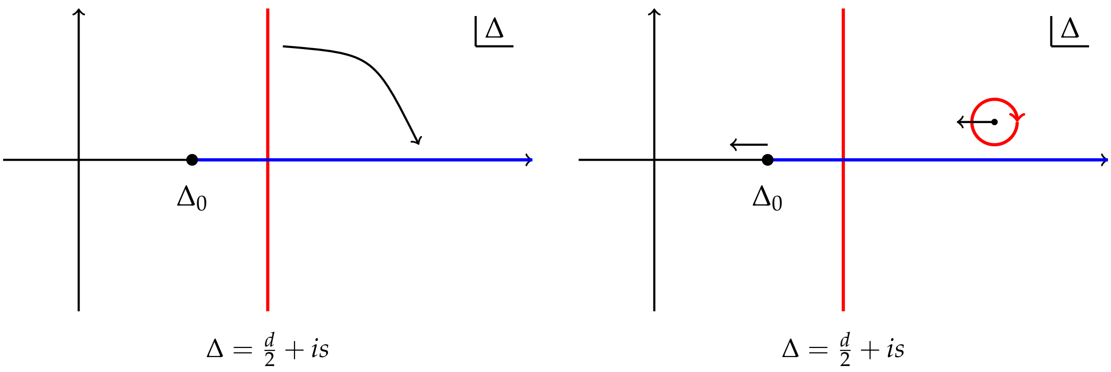


FIGURE 2.3: The analytic continuation of the scale dimension is shown on the left hand in which principal series lies on the red line and physical scale dimension distributes along the blue line on the real axes, with the lower bound Δ_0. In the right figure, the motion of the poles according to the changing of the lower bound Δ_0 is illustrated. In order to eliminate such effects caused by the motion of poles, an extra part of the contour winding around the singularity should be added into the original contour γ. The new one is denoted as γ_J and illustrated by the red curves.

make $\lambda = 1$. In the end, to get the physical operators, we need to analytically continue the Δ, J from principal series to real axes while the rotation of the scale dimension is shown in LHS of Fig.2.3.

Practically, in order to get an analytic function, the contour along the principal series is not enough and we need to deform the contour especially when we meet poles in the complex plane. For example, in the completeness relation (2.102) or (2.112), $\Delta_0 = \frac{d}{2}$ is defined on the principal series. But physically, as the lower bound of the scale dimension, Δ_0 itself depends on the theory thus it varies along the real axes according to the interaction picture we have. Note that Δ_0 serves as a parameter in the constant factor thus the position of the pole of $N_{\Delta,J}$ will move when we change the value of Δ_0. Taking these effects into consideration, if one wants to extend the equation apart from $\Delta = \frac{d}{2}$ analytically, extra contours going around the poles should be added properly, illustrated in RHS of Fig.2.3, and the Euclidean case is discussed in Aharony et al. (2021a). To distinguish the deformed contours from the original one γ, we denote them as γ_E and γ_L corresponding to the Euclidean and Lorentzian CFT [4].

The above discussion comes from the study of the symmetry group on field theory side, which must have implications in the bulk theory with gravity due to the correspondence. The connections come in when we try to construct a map between the bulk fields and the boundary fields via the decomposition (2.10) and (2.11). In (2.10) and (2.11), we just state without explanation that Δ should lie on the principal series even though harmonic analysis does not apply to the gravity theory. Here we point out that if one wants to complete the map between the bulk and boundary field, one should

[4]In fact we should impose that $\gamma_E = \gamma_L$ to make fields satisfy the match condition and we label them together as γ_J, which means that the deformation of the principal series depends on the spin J.

choose the contour γ_E in (2.10) and the contour γ_L in (2.11) and such choice of contour in the bulk theory can be interpreted as selecting proper modes of the bulk-boundary propagator, i.e, we have the correspondence

$$\text{Poles of } N_J(\Delta)\,,\; N_{\Delta,J} \qquad \longleftrightarrow \qquad \text{Bulk Modes,} \tag{2.99}$$

This is similar to the case we have met in the study of quasinormal modes Birmingham (2001); Birmingham et al. (2002, 2003) of BTZ black holes in which the pole of frequency in the BTZ solutions is related to the pole of boundary CFT correlation functions.

Moreover, in the study of four-point functions of the SYK model, Maldacena and Stanford (2016), the same pole structures arise from the calculation of eigenfunctions of the casimir operator. When expanding the four-point function as casimir eigenfunctions, the coefficients are written in terms of the eigenvalues $N_{\Delta,J} \sim \frac{k_c(h)}{1-k_c(h)}$ of the kernel. It has a pole at $k_c(h) = 1$ when $h = 2$ in which h plays the role of scale dimension. The behavior of the infinite term around the pole $h = 2$ is given by $\delta k_c(h = 2) \sim \frac{1}{\beta \mathcal{J}}$, which results from the broken of conformal structure. From the bulk point of view, instead of considering the exact AdS_2 geometry, one should study the near AdS_2 ($NAdS_2$) geometry Maldacena et al. (2016); Maldacena and Stanford (2016) and such broken of symmetry is described by the Goldstone boson mode so called dilaton.

Having discussed the role of scale dimension, we now come to the study of spin J while it will become more complicated even though we just stay on the CFT side. The subtlety firstly comes in when we try to analytically extend the spin J in Lorentzian signature apart from the principal series since the physical spin are discrete integer numbers Mack (1977) on real axes rather than a continuous interval and there is no mechanism telling us how the basis collapse or whether the physical basis is still complete.

We will also meet similar difficulty when doing rotation between two signatures. For example, in the study of partial wave decomposition of conformal four point functions Dobrev et al. (1977); Simmons-Duffin et al. (2018); Kravchuk and Simmons-Duffin (2018), we need to sum over all possible spin J for every representations. This is represented as a sum over non-negative integers in Euclidean signature. In Lorentzian signature, in order to make such completeness still valid, we need to sum over the principal series, which now becomes a continuous integral. We see that such a gap between integer and continuous number arises again while this time it can be resolved by applying the complex analysis techniques.

The key idea comes from the study of Sommerfeld-Waston transform Eden et al. (2002); Gribov (2003); Cornalba (2008), which tells us that it is possible to rewrite the sum of discrete numbers into an integral along the proper contour in the complex J plane.

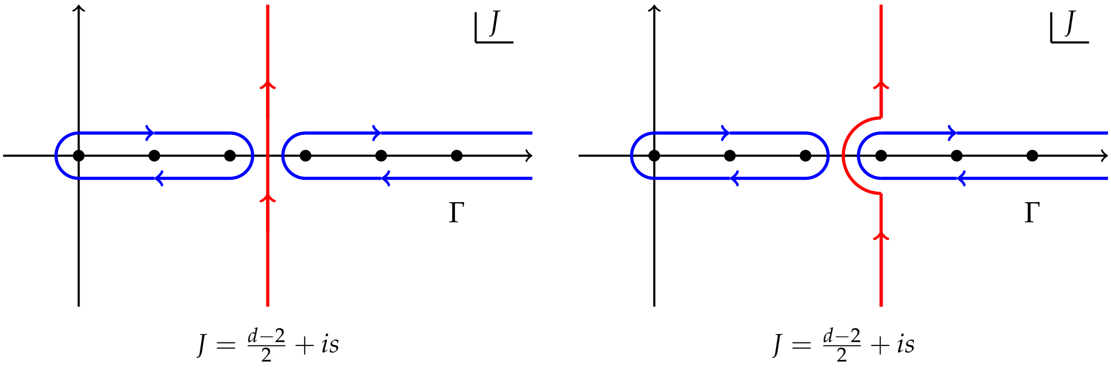


FIGURE 2.4: The contour in the complex J plane we meet during the Wick rotation is illustrated. The blue contour denoted as Γ is the one we used to take the place of the sum over non-negative spin while the red line is the spin principal series. Wick rotation can be understood as the deformation of Γ to the principal series. The figure on the left shows the configuration when the spacetime dimension d is odd so that the principal series will not pass through any integer point on the real axes. The figure on the right shows the even dimension case in which we need to deform the principal series around $(\frac{d-2}{2}, 0)$ in order to make contours not meet with each other so that the Wick rotation is continuous.

More precisely, we have

$$\sum_{J=0}^{\infty} \longrightarrow \frac{i}{2} \int_{\Gamma} \frac{dJ}{\sin(\pi J)}, \tag{2.100}$$

in which the contour Γ is shown in Fig.2.4. At this stage, we are still in Euclidean signature and the next thing we need to do is to deform the contour from Γ to the Lorentzian principal series. During the deformation, various physical phenomenon will show up and it usually depends on the pole structure of the integrand. Specifically, we can write the integrand into the product of the coefficients $N_J(\Delta)$ and the basis $I^E_{\Delta,J}$ while the pole structure is hidden in the analytic extension of $N_J(\Delta)$, labeled by $N(\Delta, J)$. After figuring out the Lorentzian version of the basis $I^L_{\Delta,J}$, one may write down the formula in Lorentzian signature. We summarise the Wick rotation procedure into the formula

$$\sum_{J=0}^{\infty} \int_{\gamma} \frac{d\Delta}{2\pi i} N_J(\Delta)\, I^E_{\Delta,J} \longrightarrow \frac{i}{2} \int_{\gamma_J} \frac{dJ}{\sin(\pi J)} \int_{\gamma} \frac{d\Delta}{2\pi i} N(\Delta, J)\, I^L_{\Delta,J}, \tag{2.101}$$

which is the combination of following four steps

i) Write the discrete sum $\sum$ in terms of the integral along Γ.

ii) Do the space time rotation on the basis $I^E_{\Delta,J}$ in order to get the Lorentzian expression $I^L_{\Delta,J}$.

iii) Analytically extend the coefficients $N_J(\Delta)$ to the complex plane to get $N(\Delta, J)$.

iv) Rotate the contour Γ to Lorentzian principal series γ_J while the pole structure should be taken into consideration.

The above method have already been used to study the CFT partial wave decomposition Hartman et al. (2016); Simmons-Duffin et al. (2018); Kravchuk and Simmons-Duffin (2018) and the poles are called Regge poles Gribov (2003); Cornalba (2008) in the context of scattering amplitude, which are connected to the CFT correlation functions in the Regge region, and we will use them to deal with the conformal completeness relation in the next section.

2.2.2 CFT Completeness Relation

In conformal field theory, three-point correlation functions are elementary building blocks of the higher point functions and it is believed that, after imposing proper constraints, they will encode the whole information of a CFT, which is the idea of conformal bootstrap. Moreover, according to the harmonic analysis of the conformal symmetry group $SO(d+1,1)$ for a d dimensional Euclidean CFT Dobrev et al. (1977); Karateev et al. (2019); Aharony et al. (2021a), we can regard three-point functions as the basis of the fields and the orthogonality of the basis is given by the completeness relation

$$\begin{aligned}\delta(x_1,x_3)\delta(x_2,x_4) &= \frac{1}{2}\sum_{J=0}^{\infty}\int_{\gamma}\frac{d\Delta}{2\pi i}\int d^d x_5\ N_J(\Delta) \\ &\times\ \langle O_{\Delta_0}(x_1)O_{\Delta_0}(x_2)O_{\Delta,J}^{\mu_1\cdots\mu_J}(x_5)\rangle\langle O_{\widetilde{\Delta}_0}(x_3)O_{\widetilde{\Delta}_0}(x_4)O^{\widetilde{\Delta},J}_{\mu_1\cdots\mu_J}(x_5)\rangle,\end{aligned} \tag{2.102}$$

in which x_i^μ for $0\le\mu\le d-1$ are spacetime coordinates and Δ is the scale dimension. Due to the harmonic analysis, the representation of $SO(d+1,1)$ will be unitary provided Δ lies on the principal series $\gamma=\frac{d}{2}+is$ and we denote the shadow transform of the operator $O_\Delta(x_i)$ as $O_{\widetilde{\Delta}}(x_i)$, in which $\widetilde{\Delta}=d-\Delta$. For simplicity, we will label them as O_i and $\widetilde{O}_i$ in the later discussion. Moreover, given a CFT, we should note that the scale dimension usually has a lower bound named Δ_0 and for free theory it takes the value $\frac{d-2}{2}$.

The completeness relation (2.102) can be used to expand local or bi-local fields in Euclidean signature while we need to do the Wick rotation on it to study the field theory in Lorentzian signature. In order to apply the Wick rotation skill, we first choose to integrate over x_1, x_4 on the two delta functions, making them a constant, and then focus on the integrand term on the RHS which is labelled as [5]

$$I_{\Delta,J} := \int d^d x_1 d^d x_4 d^d x_5\ \langle O_1O_2O_5\rangle\langle\widetilde{O}_3\widetilde{O}_4\widetilde{O}_5\rangle. \tag{2.103}$$

[5]More precisely, we should use $I^E_{\Delta,J}$.

The main task of the rest of this section is to find the expression of $I_{\Delta,J}$ in Lorentzian signature.

There are five spacetime points in the product of two three-points functions and we use the conformal symmetry to fix three of them, i.e, we set $x_2 = (0, \dots, 0)$, $x_3 = (1, 0, \dots, 0)$ and $x_5 = (\Lambda, 0, \dots, 0)$[6] and then obtain

$$I_{\Delta,J} = \int d^d x_1 d^d x_4 \; \langle O_1 O_2 O_5 \rangle \langle \widetilde{O}_3 \widetilde{O}_4 \widetilde{O}_5 \rangle \tag{2.104}$$

in which there are only two variables x_1 and x_4 left and they are explicitly shown in the integral. We should note that there will be an overall constant term arising from the volume of integral over x_5 when we choose the gauge but we do not need to consider that since it will go away once we ungauge the fixed points back to the integral in the end. To perform the Wick rotation, we introduce the normal Feynman continuation in which we take $x^0 = (i + \epsilon)t$, $u = x^1 - t$ and $v = x^1 + t$. Therefore we obtain

$$I_{\Delta,J} = -\frac{1}{4} \int dv_1 du_1 dv_4 du_4 d^{d-2} x_1 d^{d-2} x_4 \; \langle O_1 O_2 O_5 \rangle \langle \widetilde{O}_3 \widetilde{O}_4 \widetilde{O}_5 \rangle, \tag{2.105}$$

and the singularities at the coincident points $x_1 \sim x_2, x_5$, $x_4 \sim x_3, x_5$ are given by

$$u_{12} v_{12} + i\epsilon = 0 \qquad u_{15} v_{15} + i\epsilon = 0, \tag{2.106}$$

$$u_{43} v_{43} + i\epsilon = 0 \qquad u_{45} v_{45} + i\epsilon = 0. \tag{2.107}$$

As for the integral, we can think about fixing u_1 and u_4 and then investigating the behavior of $I_{\Delta,J}$ on the v_1 and v_4 complex plane. Due to the introduction of the $i\epsilon$ expression, singularities will shift apart from the real axes and their positions are determined by the equation (F.109) and (2.107). More explicitly, singularities of v will shift to the upper half plane if u is negative and vice versa.

Since we are interested in the nontrivial $I_{\Delta,J}$, the singularities $x_1 \sim x_2$ and $x_1 \sim x_5$ can not lie in the same half plane otherwise we can deform the integral contour along the real axes to the infinity and make $I_{\Delta,J}$ vanish. The same argument holds for the $x_4 \sim x_3$ and $x_4 \sim x_5$ singularities, which means that the sign of u_{12}, u_{15} and u_{43}, u_{45} should be different, i.e, it requires that

$$u_2 < u_1 < u_5, \qquad u_3 < u_4 < u_5. \tag{2.108}$$

The next step is to determine the deformation of the integral contours of v_1 and v_4 and we illustrate one way of the deformation in Fig.2.5, in which we let the v_1 integral go around $1 \sim 2$ and the v_4 integral go around $4 \sim 5$. This means that we restrict the

[6] Usually we take $\Lambda \longrightarrow \infty$ but here we make Λ a finite number.

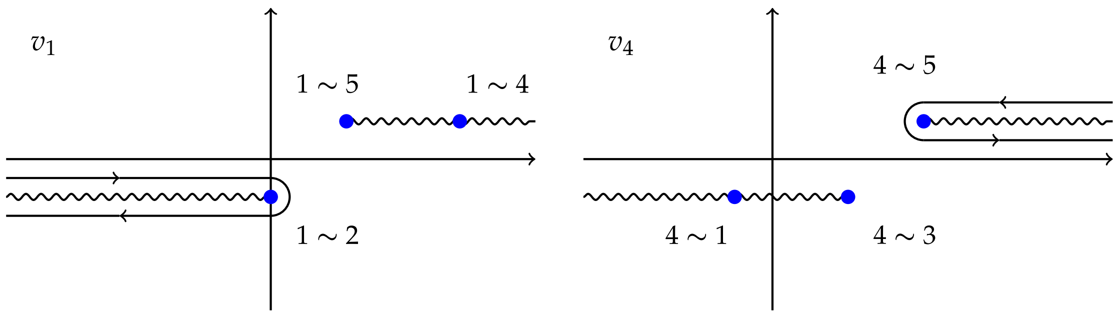


FIGURE 2.5: One possible choice of integral contour and branch cuts are shown in the v_1 and v_4 plane, in which blue points are coincident points in the correlation functions and wavy lines are branch cuts in the complex plane. In the v_1 plane the contour winding around $1 \sim 2$ will generate the commutator $[O_1, O_2]$ while the contour around $4 \sim 5$ in the v_4 plane will generate the commutator $[\widetilde{O}_5, \widetilde{O}_4]$.

integral in the region $v_1 \leq 0, v_4 \geq \Lambda$ and, at the same time, it induces the term

$$\langle [O_1, O_2]\; O_5 \rangle \langle \widetilde{O}_3\; [\widetilde{O}_5, \widetilde{O}_4] \rangle, \tag{2.109}$$

in which the winding of the contour around the branch cuts will produce a commutator $[,\ ]$ and the order of the operator in the commutator is determined by the direction of the contour. After taking all kinds of deformation into consideration, we obtain

$$(\langle [O_1, O_2]\; O_5 \rangle + \langle O_2\; [O_5, O_1] \rangle) \times (\langle \widetilde{O}_3\; [\widetilde{O}_5, \widetilde{O}_4] \rangle + \langle [\widetilde{O}_4, \widetilde{O}_3]\; \widetilde{O}_5 \rangle), \tag{2.110}$$

in which each term corresponds to a region in the (v_1, v_4) plane and they together contribute to $I_{\Delta,J}$. After all, we should sum over all possible Δ and J in $I_{\Delta,J}$ to get the completeness relation. Now we should note that such problem has already been extensively discussed and resolved in (2.101) and here we just write down the result in embedding space

$$\delta(P_1, P_3)\delta(P_2, P_4) = \int_{\gamma_J} \frac{idJ}{2\sin(\pi J)} \int \frac{dP}{J!(\frac{d}{2}-1)_J} \int_{\gamma} \frac{d\Delta}{2\pi i} N(\Delta, J) \tag{2.111}$$

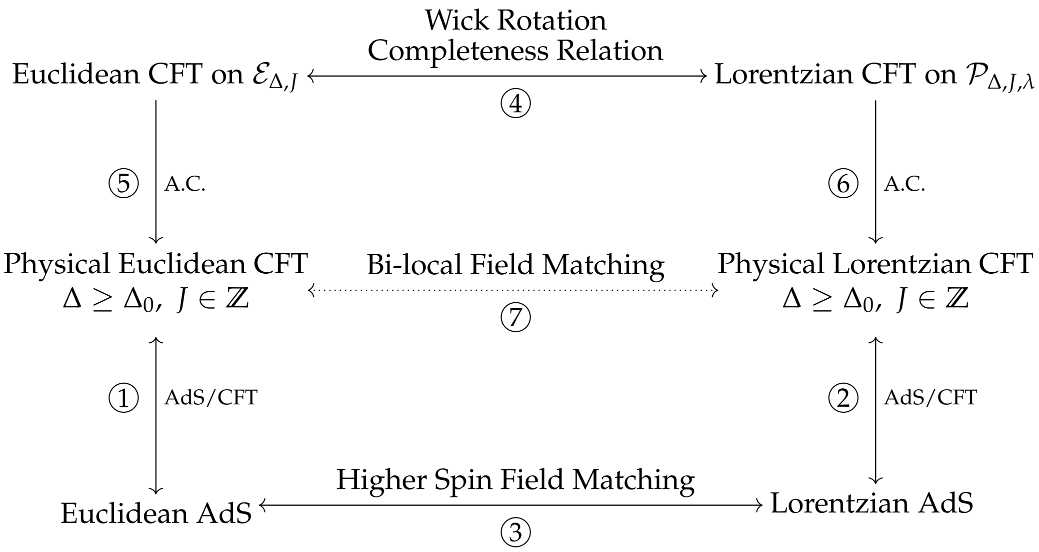
$$\times \quad (\langle [O_{\Delta_0}(P_1), O_{\Delta_0}(P_2)]\; O_{\Delta,J}(P, D_Z) \rangle + \langle O_{\Delta_0}(P_2)\; [O_{\Delta,J}(P, D_Z), O_{\Delta_0}(P_1)] \rangle)$$

$$\times \quad \langle O_{\widetilde{\Delta}_0}(P_3)\; [O_{\widetilde{\Delta},J}(P, Z), O_{\widetilde{\Delta}_0}(X_4)] \rangle + \langle [O_{\widetilde{\Delta}_0}(P_4), O_{\widetilde{\Delta}_0}(P_3)]\; O_{\widetilde{\Delta},J}(P, Z) \rangle) + \cdots .$$

in which P, Z are points in the embedding space corresponding to the boundary CFT, the overall constant $N(\Delta, J)$ will now become analytic function that depends on Δ, J and $\cdots$ represents the contribution from its poles. D_Z is the operator used to contract the spin indexes associated with the factor $\frac{1}{J!(\frac{d}{2}-1)_J}$.

Physical Basis

We have discussed that it is still unclear how to get physical spin from the principal series in Lorentzian signature while we found the connection between the Euclidean physical spin and the Lorentzian spin principal series via the Wick Rotation of the CFT completeness relation. Here we are going to directly rotate the physical Euclidean CFT to the physical Lorentzian CFT since spins are integers on both sides and we summarise all of these ideas in the following figure.



The work about the analytic continuation of Euclidean CFT labelled by ① and Euclidean AdS/CFT map labelled by ⑤ has already been discussed in Aharony et al. (2021a) while we have studied Wick Rotation of the completeness relation ④ and high spin field matching ③ in previous sections. In order to verify the AdS/CFT map in Lorentzian signature, denoted as ②, we need to find results in the physical Lorentzian CFT. Although the detail of analytic continuation of Lorentzian CFT is unclear, we find that it is possible to consider the direct correspondence between physical CFTs, labeled by ⑦.

⑦ can be regarded as the combination of ④, ⑤ and ⑥ and its net effect is just that we do the Wick rotation of space time while keep the spin physical. Now, we consider the behavior of spin in ④ and ⑥, in ④ we extend physical Euclidean spin to Lorentzian principal series while in ⑥ we need to extend spin on Lorentzian series to physical Lorentzian spin and in both cases the contour will move across poles on complex plane determined by $N(\Delta, J)$ but from opposite directions. Therefore we can see that the net contribution from the poles will be zero and then we obtain the physical

CFT completeness relation written as

$$\delta(P_1,P_3)\delta(P_2,P_4) = \sum_{J=0}^{\infty} \int \frac{dP}{J!(\frac{d}{2}-1)_J} \int_{\gamma} \frac{d\Delta}{2\pi i} N(\Delta,J) \tag{2.112}$$

$$\times \quad (\langle [O_{\Delta_0}(P_1), O_{\Delta_0}(P_2)]\ O_{\Delta,J}(P,D_Z)\rangle + \langle O_{\Delta_0}(P_2)\ [O_{\Delta,J}(P,D_Z), O_{\Delta_0}(P_1)]\rangle)$$

$$\times \quad (\langle O_{\tilde{\Delta}_0}(P_3)\ [O_{\tilde{\Delta},J}(P,Z), O_{\tilde{\Delta}_0}(P_4)]\rangle + \langle [O_{\tilde{\Delta}_0}(P_4), O_{\tilde{\Delta}_0}(P_3)]\ O_{\tilde{\Delta},J}(P,Z)\rangle).$$

2.2.3 AdS/CFT Map

In this part, we will use the CFT completeness relation to expand bi-local fields $\Phi(P_1,P_2)$ on the boundary, together with expansion of higher spin fields on the bulk, then construct a map between them in Lorentzian signature so called Lorentzian AdS/CFT map .

Bi-local Fields Matching

Before going into the derivation of Lorentzian AdS/CFT map, we first discuss the match of bi-local fields during the Wick rotation, as the building block of our construction. First, following the convention from the previous section, we denote the bi-local fields in Euclidean and Lorentzian signature as $\Psi_E(P_1,P_2)$ and $\Psi_L(P_1,P_2)$, respectively. From the holography point of view, the match of bi-local fields is conceptually different from the match of bulk fields. But if we just consider the match of two theories with actions, the matching rule (2.2) is universal. Therefore, consider the continuity of the field at the matching surface Σ, we immediately obtain the first matching condition, written as

$$\Psi_E(P_1,P_2)|_{\Sigma} = \Psi_L(P_1,P_2)|_{\Sigma}. \tag{2.113}$$

We should note that such matching condition is universal and it does not depend on how we construct the bi-local fields while, for the second matching condition, one needs to consider the detail structure of the theory. First, we consider the $O(N)$ vector model in which the bi-local field is defined as

$$\Psi(P_1,P_2) := \frac{1}{N}\sum_{I=1}^{N} \phi_I(P_1)\phi_I(P_2), \tag{2.114}$$

where ϕ_I are real scalar fields. Since for each scalar ϕ_I one should impose the condition (2.13), we then obtain the second matching condition, written as

$$\partial_{\tau_a}\Psi(P_1,P_2) + i\partial_{t_a}\Psi_L(P_1,P_2) = 0 \qquad \text{for} \qquad a = 1,2. \tag{2.115}$$

Then we consider the $U(N)$ vector model in which the bi-local field is given by

$$\Psi(P_1, P_2) := \frac{1}{N}\sum_{I=1}^{N} \phi_I^*(P_1)\phi_I(P_2), \tag{2.116}$$

where ϕ_I become complex with the conjugate ϕ_I^* while the matching condition of the complex conjugate field ϕ_I^* is the complex conjugate of (2.13). Therefore, for the $U(N)$ vector model, the second matching condition now becomes [7]

$$\partial_{\tau_1}\Psi(P_1, P_2) - i\partial_{t_1}\Psi_L(P_1, P_2) = 0 \tag{2.117}$$

$$\partial_{\tau_2}\Psi(P_1, P_2) + i\partial_{t_2}\Psi_L(P_1, P_2) = 0, \tag{2.118}$$

in which one can check that the matching conditions are hermitian in the sense that we treat 1,2 as the matrix indices. From the above discussion we can see that the explicit form of the second matching condition depends on the theory itself and there also exist physical models which do not require the second matching condition. For example, in SYK model the bi-local field is defined as

$$\Psi(P_1, P_2) := \frac{1}{N}\sum_{I=1}^{N} \langle \chi_I(P_1)\chi_I(P_2)\rangle, \tag{2.119}$$

in which χ_I are Majorana fermions. In this case, the requirement of the first matching condition is still necessary while there is no second matching condition since the on shell equation for χ_I is the first order differential equation. Moreover, one should impose proper reality and hermiticity condition on the spinor and action, respectively Nicolai (1978); Van Nieuwenhuizen and Waldron (1996).

Derivation

To make the Lorentzian results compatible with the Euclidean case, also for simplicity, we first define the Lorentzian three-point function basis as

$$\langle O_1 O_2 O_3\rangle_L := \langle [O_1, O_2]\ O_3\rangle + \langle O_2\ [O_3, O_1]\rangle, \tag{2.120}$$

$$\langle \widetilde{O}_1 \widetilde{O}_2 \widetilde{O}_3\rangle_L := \langle \widetilde{O}_1\ [\widetilde{O}_3, \widetilde{O}_2]\rangle + \langle [[\widetilde{O}_2, \widetilde{O}_1], \widetilde{O}_3]\rangle, \tag{2.121}$$

in which we can see that Lorentzian three-point function basis are in fact combinations of three-point functions and its shadow counterpart is not just the shadow transform of each operator while we should also take the effect of commutator into consideration. But $\langle O_1 O_2 O_3\rangle_L$ and $\langle \widetilde{O}_1 \widetilde{O}_2 \widetilde{O}_3\rangle_L$ are still orthogonal in the sense of (2.112). Therefore,

[7] We should note that in the bi-local form ϕ and ϕ^* are not independent if one requires the bi-local field is hermitian, i.e., $\Phi^*(P_2, P_1) = \Phi(P_1, P_2)$. If we treat ϕ and ϕ^* independently, there will be no sign difference in the matching condition as derived in the Appendix C.

from the bi-local field expansion

$$\Psi(P_1,P_2)_{E/L} = \sum_{J=0}^{\infty}\int_{\gamma_L}\frac{d\Delta}{2\pi i}\int\frac{dP}{J!(\frac{d}{2}-1)_J}\,\bar{C}^{E/L}_{\Delta,J}(P,D_Z)\langle O_{\Delta_0}(P_1),O_{\Delta_0}(P_2)O_{\Delta,J}(P,Z)\rangle_{E/L},$$

we can determine the coefficients $\bar{C}$ to be

$$\bar{C}^{E/L}_{\Delta,J}(P,Z) = \frac{N_{\Delta,J}}{2}\int dP_1 dP_2\,\Psi(P_1,P_2)\langle O_{\tilde{\Delta}_0}(P_1),O_{\tilde{\Delta}_0}(P_2)O_{\tilde{\Delta},J}(P,Z)\rangle_{E/L}. \tag{2.122}$$

As for the higher spin fields, we will focus on the study of transverse tensor fields and the transverse bulk completeness relation is given by

$$\delta^{TT}(X_1,X_2)(W_{12})^J = \int_{\gamma}\frac{d\Delta}{2\pi i}\int\frac{dP}{J!(\frac{d}{2}-1)_J}\frac{N_{\Delta,J}}{\alpha_J}G_{\Delta,J}(X_1,P;W_1,D_Z)G_{\tilde{\Delta},J}(X_2,P;W_2,Z),$$

which comes from the zero spin term of the full completeness relation and α_J are constants that depend on the spin. Therefore, given the off-shell tensor field expansion

$$H(X,W) = \int_{\gamma}\frac{d\Delta}{2\pi i}\int\frac{dP}{J!(\frac{d}{2}-1)_J}C_{\Delta,J}(P,D_Z)G_{\Delta,J}(X,P;W,Z), \tag{2.123}$$

we can use the bulk completeness relation to deduce the coefficient polynomial, which is given by

$$C_{\Delta,J}(P,Z) = \frac{N_{\Delta,J}}{\alpha_J}\frac{1}{(\frac{d-1}{2})_J J!}\int dX\,H(X,K_W)\,G_{\tilde{\Delta},J}(X,P;W,Z). \tag{2.124}$$

Now, we can construct a map between the bulk coefficients $C_{\Delta,J}$ and the CFT coefficients $\bar{C}_{\Delta,J}$, given by

$$C^E_{\Delta,J}(P,Z) = f^E_{\Delta,J}\,\bar{C}^E_{\Delta,J}(P,Z), \qquad C^L_{\Delta,J}(P,Z) = f^L_{\Delta,J}\,\bar{C}^L_{\Delta,J}(P,Z), \tag{2.125}$$

in which $f^E_{\Delta,J}$ and $f^L_{\Delta,J}$ are functions on the scale dimension and spin. Although one can propose the AdS/CFT map in Euclidean and Lorentzian signature separately, we will see that in fact $f^E_{\Delta,J}$ and $f^L_{\Delta,J}$ are not independent. According to the matching of bi-local field and the Wick rotation of the completeness relation, we know $\bar{C}^E_{\Delta,J} = \bar{C}^L_{\Delta,J}$ while we have $C^E_{-} = C^L_{-}$ from the study of higher spin field matching on the bulk. Taking these into consideration, we can conclude

$$f_{\Delta,J} := f^E_{\Delta,J} = f^L_{\Delta,J}, \tag{2.126}$$

which tells us that the AdS/CFT map should be invariant during the Wick rotation. Moreover, we can transfer the map between coefficients into the map between fields directly with the help of (2.122) and (2.124). We can deduce the CFT to AdS map to be

$$\begin{aligned} H(X,W) &= \frac{1}{2}\int_\gamma \frac{d\Delta}{2\pi i} f_{\Delta,J} N_{\Delta,J} \int \frac{dP}{J!(\frac{d}{2}-1)_J} \int dP_1 dP_2\, G^L_{\Delta,J}(X,P;W,D_Z) \\ &\times \langle O_{\tilde{\Delta}_0}(P_1), O_{\tilde{\Delta}_0}(P_2) O_{\tilde{\Delta},J}(P,Z)\rangle_L\, \Psi(P_1,P_2), \end{aligned} \tag{2.127}$$

and the AdS to CFT map to be

$$\begin{aligned} \Psi(P_1,P_2) &= \sum_{J=0}^{\infty} \int_\gamma \frac{d\Delta}{2\pi i} \frac{N_{\Delta,J}}{\alpha_J\, f_{\Delta,J}} \int \frac{dP}{(\frac{d-1}{2})_J (\frac{d}{2}-1)_J J!^2} \\ &\times \int dX \langle O_{\Delta_0}(P_1), O_{\Delta_0}(P_2) O_{\Delta,J}(P,Z)\rangle_L G^L_{\tilde{\Delta},J}(X,P;K_W,Z) H(X,W) \end{aligned} \tag{2.128}$$

therefore completes our derivation.

Here, we should note that the AdS/CFT map provides us with a machinery to build connections between bi-local fields and higher spin fields and the fields could be off-shell and both take values in embedding space. In order to imply that the higher spin fields live on the bulk AdS and the bi-local fields live on the boundary CFT, we need to specify their physical regions. That is to say, the physical region for H is the AdS surface $X^2 = -1$ and the physical region for Ψ is often taken to be light rays on the cone $X^2 = 0$. Moreover, for the on-shell AdS/CFT map, i.e, after solving equation of motion on the physical region, we will see that the on-shell AdS/CFT map is invertible in the large N limit.

2.3 Match Conditions for the Quadratic Action

As it was discussed in Aharony et al. (2021a), the AdS/CFT map mainly works for the off-shell field while the matching condition introduced in section 2.1 is applied to the on-shell field. In fact, on-shell fields belong to the subset of the off-shell fields. The expression (2.123) of the off-shell fields is basically the expansion of an arbitrary function by the given basis $G_{\Delta,J}$ thus all the information of a physical systems is encoded in the coefficients $C_{\Delta,J}$. Suppose that the mass of the spin J field is given by the scale dimension Δ_J, although it is hard to solve $C_{\Delta,J}$ directly, we can deduce the coefficient will take the form

$$C_{\Delta,J}(P) \to \delta(\Delta - \Delta_J) C_{\Delta_J}(P), \tag{2.129}$$

in which $C_{\Delta_J}(P)$ is the source of the spin J field. Therefore, we can see that the mass spectra of the theory is determined by the pole structure of these coefficients $C_{\Delta,J}$, which will give us the AdS/CFT dictionary, together with the CFT coefficients $\bar{C}_{\Delta,J}$. So in order to complete the construction of Lorentzian AdS/CFT map, one needs to specify the matching conditions for the off-shell field.

2.3.1 The Quadratic Action

The match condition for a generic off-shell field is hard to study without specifying the equation of motion or the action for a given theory. In this thesis we will focus on the study of a special class of off-shell fields which govern the higher spin field theory dual to the vector model on the boundary and they are described by the quadratic action shown in de Mello Koch et al. (2019); Aharony et al. (2021a). For simplicity, we just consider the scalar action S_E

$$S_E = \frac{1}{2}\int_{M_E} \sqrt{G}(\partial^\mu\partial^\nu\Phi\,\partial_\mu\partial_\nu\Phi + M_1^2\,\partial_\mu\Phi\,\partial^\mu\Phi + M_2^2\,\partial_\nu\Phi\,\partial^\nu\Phi + M_1^2M_2^2\,\Phi^2) \quad (2.130)$$

in which Φ is the off-shell scalar field propagating in the manifold M_E with AdS background described by the metric G while M_1 and M_2 are mass of the particles. Here we should note that they are spin dependent parameters coming from the pole structure of $C_{\Delta,J}$ and fixed in our context since we are dealing with scalar fields setting $J = 0$. Those off-shell scalar fields are not totally free while they should obey the equation of motion given by the action S_E, the variation is given by

$$\begin{aligned}\delta S_E &= \int_{M_E} \sqrt{G}(\partial^\mu\partial^\nu\Phi\,\partial_\mu\partial_\nu\delta\Phi + M_1^2\,\partial_\mu\Phi\,\partial^\mu\delta\Phi + M_2^2\,\partial_\nu\Phi\,\partial^\nu\delta\Phi + M_1^2M_2^2\,\Phi\,\delta\Phi) \\ &= \int_{\partial M_E} \sqrt{G}\left(\left(2\partial_t\,\partial^i\partial_i\Phi + M_1^2\,\partial_t\Phi + M_2^2\,\partial_t\Phi\right)\delta\Phi + \partial^t\partial_t\Phi\,\delta\partial_t\Phi\right) \qquad (2.131) \\ &\quad + \int_{M_E}\left(\partial^\nu\partial^\mu\sqrt{G}\,\partial_\nu\partial_\mu\Phi - M_1^2\partial_\mu\sqrt{G}\,\partial^\mu\Phi - M_2^2\partial^\nu\sqrt{G}\,\partial_\nu\Phi + \sqrt{G}\,M_1^2M_2^2\Phi\right)\delta\Phi\end{aligned}$$

in which the first term results from the boundary ∂M_E and the second term gives rise to the equation of motion written as

$$\frac{1}{\sqrt{G}}(\partial^\mu\partial^\nu\sqrt{G}\,\partial_\nu\partial_\mu\Phi - M_1^2\partial_\mu\sqrt{G}\,\partial^\mu\Phi - M_2^2\partial^\nu\sqrt{G}\,\partial_\nu\Phi) + M_1^2M_2^2\Phi = 0. \quad (2.132)$$

Given that the AdS/CFT map is also valid in Lorentzian signature, the Lorentzian version of the action S_L should exist and can be deduced to take the form of

$$S_L = -\frac{1}{2}\int_{M_L} \sqrt{-G}(\partial^\mu\partial^\nu\Phi\,\partial_\mu\partial_\nu\Phi + M_1^2\,\partial_\mu\Phi\,\partial^\mu\Phi + M_2^2\,\partial_\nu\Phi\,\partial^\nu\Phi + M_1^2M_2^2\,\Phi^2) \quad (2.133)$$

The matching condition of the total action $iS_L - S_E$ at the joint surface $\Sigma = \partial M_E = -\partial M_L$ is then deduced to be

$$\begin{aligned} i\left(2\partial_t\, \partial^i\partial_i\Phi_L + M_1^2\, \partial_t\Phi_L + M_2^2\, \partial_t\Phi_L\right)\delta\Phi_L + \left(2\partial_\tau\, \partial^i\partial_i\Phi_E + M_1^2\, \partial_\tau\Phi_E + M_2^2\, \partial_\tau\Phi_E\right)\delta\Phi_E &= 0, \\ i\partial_t^2\Phi_L\, \delta\partial_t\Phi_L - \partial_\tau^2\Phi_E\, \delta\partial_\tau\Phi_E &= 0, \end{aligned}$$

in which we have used the contraction relation $\partial_t^2 = -\partial^t\partial_t$ and $\partial_\tau^2 = \partial^\tau\partial_\tau$. Moreover, if we impose the condition that the charge $\partial^i\partial_i\Phi$ is conserved at the joint surface

$$i\partial_t\partial^i\partial_i\Phi_L + \partial_\tau\partial^i\partial_i\Phi_E = 0. \tag{2.134}$$

Then the matching conditions can be simplified to

$$\Phi_E - \Phi_L = 0, \tag{2.135}$$

$$i\partial_t\Phi_L + \partial_\tau\Phi_E = 0, \tag{2.136}$$

$$\partial_t^2\Phi_L + \partial_\tau^2\Phi_E = 0, \tag{2.137}$$

so called offshell matching conditions even though Φ_E, Φ_L now satisfy the quadratic equation (2.132). Moreover, after rewriting the equation into the form of

$$(\nabla^2_{AdS} - M_1^2)(\nabla^2_{AdS} - M_2^2)\Phi(X) = 0 \tag{2.138}$$

with the ∇_{AdS} on AdS background. Then one can obtain the solution in terms of the bulk boundary propagator written as

$$\Phi(X) = \int dP\, \frac{1}{(X\cdot P)^{\Delta_1}}\, C_{\Delta_1}(P) + \int dP \frac{1}{(X\cdot P)^{\Delta_2}}\, C_{\Delta_2}(P), \tag{2.139}$$

in which Δ_1 and Δ_2 are given by the relation

$$M_1^2 = \Delta_1(\Delta_1 - d), \qquad M_2^2 = \Delta_2(\Delta_2 - d). \tag{2.140}$$

$C_{\Delta_1}(P)$ and $C_{\Delta_2}(P)$ can be treated as coefficients or sources associated to the mode of scale dimension Δ_1, Δ_2. Therefore one can check the offshell match conditions (2.135) and (2.136) are solved by the matching of coefficients

$$C^E_{\Delta_1} = C^L_{\Delta_1}, \qquad C^E_{\Delta_2} = C^L_{\Delta_2} \tag{2.141}$$

and the condition (2.137) is then automatically preserved, which is shown in the appendix D. In fact, since we have seen that the offshell solution is the linear combination of two onshell fields, the offshell matching condition (2.134) can be rewritten in term of

$\partial_t^3, \partial_t^3$ after using the onshell equation of motion for each propagator with mass M_1, M_2.

Now we come back to the duality between the higher spin theory and the vector model. As it is known that the higher spin theory in the bulk is described by the Vasiliev equations and the corresponding action is still absent. Thus, with the help of AdS/CFT map, it is possible for us to reconstruct the bulk action from the action of boundary free vector model de Mello Koch et al. (2019); Aharony et al. (2021a). This is done by first decomposing the boundary action into the coefficients $\bar{C}_{\Delta,J}$ then mapping it to bulk coefficients $C_{\Delta,J}$ via the AdS/CFT map (2.125) therefore rewriting the action in terms of the bulk fields. The spectra is determined by the pole structure of $C_{\Delta,J}$ as shown in (2.129). It turns out that the bulk action is quadratic consisting of higher spin fields and the scalar part is described by (2.130) while the mass M_1 and M_2 are given by

$$\Delta_1 = d, \qquad \Delta_2 = d - 2. \tag{2.142}$$

From the above procedure, we can see that one of the mode described by C_{Δ_1} is physical and the mode described by C_{Δ_2} with negative mass $M_2^2 < 0$ is unphysical based on the observation that it will contribute to the offshell field Φ in a negative way $C_{\Delta_2} < 0$. Such unphysical modes are identified as ghost modes coming from the gauge fixing of the higher spin fields. Moreover, from the match condition (2.141), one can see that the physical and unphysical modes will behave independently during the Wick rotation.

Furthermore, as pointed out in the work Skenderis et al. (2009b,a) during the study of three-dimensional Einstein gravity, the higher-derivative terms in the action will introduce extra propagating degrees of freedom and they are identified as ghost modes. Those ghost modes make the theory unstable and violate the unitary condition. Such problem is rescued by considering the topologically massive gravity at the chiral point so that the left-moving sector will be gauge fixed and the extra degrees of freedom are then eliminated. Here for the higher spin fields, to make the theory physical, one can also choose to set the negative modes to zero in Euclidean signature by fixing the gauge of the higher spin fields , i.e $C^E_{\Delta_2} = 0$. This naturally tells us that the ghost modes will not contribute to the external legs of the Feynman diagrams since the source is turned off in real time by checking the match condition (2.141). However, the ghost modes will contribute to the Feynman diagram at the loop level while the detail goes beyond the study of classical match condition in this thesis [8].

[8] The matching condition developed in this thesis is the match for the classical field configuration and it tells us the vacuum is the Hartle-Hawking state. For the full description of the Hilbert space, one needs to study the match condition at quantum level.

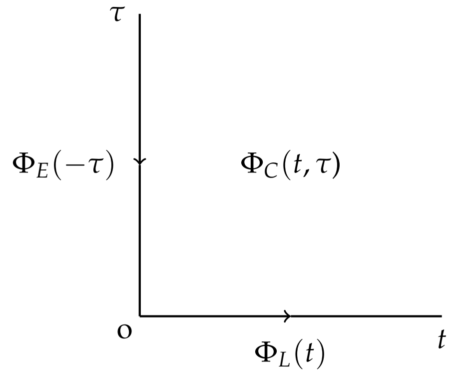


FIGURE 2.6: The complexification of the field Φ_E and Φ_L into a function Φ_C with complex variables is depicted.

2.3.2 Analyticity

At first sight, it looks confusing since, starting from different actions, we eventually arrive at the same matching condition (2.141) . As we will see, this becomes nature if one considers the behavior of the field and propagator over the whole complex time plane. The analytic property of the propagator $G_\Delta(X,P)$ implies that the condition (2.141) could be universal for arbitrary higher order matching.

The complex time plane is described by the variable $z = t + i\tau$, $\bar{z} = t - i\tau$ and the function on the complex plane is denoted as

$$\Phi_C(z,\bar{z}) = \Phi_C(t,\tau). \tag{2.143}$$

Although the complex time $t + i\tau$ will not make sense physically and we are just interested in the Euclidean field $\Phi_E(\tau)$ and the Lorentzian field $\Phi_L(t)$, we can still treat these two fields as living on the boundary of the complex field $\Phi_C(t,\tau)$. More precisely, as illustrated in Fig.2.6 we can impose the condition

$$\Phi_C(t,0) = \Phi_L(t), \qquad \Phi_C(0,\tau) = \Phi_E(-\tau), \tag{2.144}$$

which specify the boundary value of Φ_C while the value in the interior is still unknown. But after taking the first order matching condition (2.136) into consideration

$$\partial_t\Phi_L(t) - i\partial_\tau\Phi_E(\tau) = \partial_{\bar{z}}\Phi_C(t,\tau), \tag{2.145}$$

we find that a sufficient condition for the complex field is the analyticity ,i.e, an analytic complex function will naturally induced a pair of field Φ_L and Φ_E which satisfy the first order matching. We can also see that the second order matching condition (2.137) can be written as $\partial_z \partial_{\bar{z}} \Phi_C = 0$ thus now becomes trivial. Furthermore, using the relation

$$\partial_t = \frac{\partial_z + \partial_{\bar{z}}}{2}, \qquad \partial_t = i\frac{\partial_z - \partial_{\bar{z}}}{2}, \tag{2.146}$$

one can write the ∂_t, ∂_τ into the derivative ∂_z and $\partial_{\bar{z}}$ and check that the third order matching condition (2.134) becomes trivial provided $\partial_{\bar{z}} \Phi_C = 0$.

We obtain the necessary match conditions (2.135), (2.136) and (2.137) for the field by considering the action up to the quadratic order while higher order terms will show up if we sit down to study the complete bulk action. Therefore, we can propose that the higher order matching condition are generated by the derivatives ∂_z, $\partial_{\bar{z}}$ acting on the complex function and the matching condition for arbitrary higher order action is

$$\partial_{\bar{z}} \Phi_C(z, \bar{z}) = 0. \tag{2.147}$$

This assumption is reasonable since as for the boundary quantum field theory, we have the analytic Wightman functions so we should expect to get analytic fields on the bulk via the help of holography principle. Although the Wightman functions are characterised by a series of axioms Osterwalder and Schrader (1973, 1975) and here we start from the study of variation of the action δS. In this thesis, we are considering the match condition purely from the mathematical point of view i.e, match conditions in the bulk is dual to analytic properties of the Wightman function. A set of reconstruction axioms, for example the understanding of causality, is not established in the bulk. In some cases, higher derivative terms are related to the causality for a given theory while the detail relation is still not clear. It is interesting to explore the match conditions from the algebraic point of view, for example, establishing the Haag's theorem on the AdS background, and we leave this to further work.

Chapter 3

Flat Holography for Scalar Fields

The main goal of this chapter is to develop the AdS/CFT correspondence into the Flat/CFT correspondence thus bringing the holography principle and especially all the work on AdS/CFT to the measurable level. More precisely, we will finally construct the dictionary between flat spacetime and the CFT on the boundary, which works the same as (1.1).

Before going into the Flat/CFT dictionary, we first introduce another principle used many times in this thesis, which is the completeness relation of mode expansion for a generic physical field written as

$$\text{A Generic Solution} = \sum \text{Coefficient} \times \text{Modes}. \tag{3.1}$$

The mode expansion tells us that a generic field configuration can be decomposed into given modes for the linear physical system, and the information about the field is encoded in coefficients. These coefficients are determined by the boundary and initial data. The symbol $\sum$ means that we should sum over some proper set of modes, which is labeled by discrete or continuous numbers. Physically, for example, in quantum mechanics and quantum field theory, one always assumes that all the physical solutions are well behaved at the boundary; therefore, an inner product could be properly defined. Given the well-defined inner product structure and the existence of a self-adjoint operator, all modes form a complete set of basis for the physical solution space while the rigorous mathematical structure has been studied in the so-called Sturm-Liouville theory, but here we should note that the boundary conditions are often hard to specify or to check in the physical situation.

In physics, the mode expansion (3.1) is also called superposition principle and has been used widely, dating back to the birth of quantum mechanics. Here we will reconsider the mode expansion and find it is not as obvious as people thought it would be although it has been taken for granted for a long time. Taking the story of quantum field theory for example, the traditional mode is the plane wave mode $\Phi_K = e^{iK\cdot X}$ and all on-shell modes satisfying $-K^2 = M^2$ form a complete basis for the field describing particles of mass M. After the quantisation, coefficients for the plane wave are promoted to be creation and annihilation operators. Recently, except for plane waves, people have constructed a new kind of basis so called conformal basis Φ_Δ Pasterski et al. (2017); Pasterski and Shao (2017) to highlight the symmetry of Lorentz group $SO(1,3)$ and the unitarity of the representation of Lorentz group requires that Δ should lie on the principal series thus one assumes that all states on the principal series form a complete basis. In addition to the conformal basis, for this thesis, we are going to introduce another kind of mode based on the foliation of the Minkowski space as

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = -\tau^2, \tag{3.2}$$

in which one can treat it as the embedding of the AdS hyperboloid with radius $\tau \geq 0$. Given such foliation, one can further choose τ together with the coordinate on AdS surface as Minkowski space coordinates therefore recast the equation of motion for Minkowski into the AdS hyperboloid. Then, according to the superposition principle, one can claim that a generic field can be decomposed into modes Φ_k with effective mass k on the AdS surface. Since here the equation on AdS is not physical, k could take all the value in the complex plane and till now it is not clear how to determine which of them will form the necessary complete basis. We will use Klein-Gordon equation as an example to illustrate how the mode expansion (3.1) works in the context of AdS slicing (3.2) and discuss various possible choices of k in the section 3.1.

After a careful study of the mode analysis, one will be able to decompose the bulk action S for Minkowski space into k-mode components $S(k)$. To construct the Flat/CFT dictionary like (1.1), a technical issue ahead is that the one-shell action S^{onshell} is infinite due to the integral over the infinite spacetime volume and one needs to perform the renormalisation on S^{onshell} in order to make the action finite, denoted as S^{ren} or equivalently $S^{\text{ren}}(k)$. Such problem was addressed in the work Witten (1998a) then has been fully discussed by following work Henningson and Skenderis (1998); Balasubramanian and Kraus (1999); de Haro et al. (2001b). The developed systematic procedure is so called holography renormalisation. The basis idea of holography renormalisation is that one should treat the infinite part of the action in the bulk as IR divergences and introduce local counterterms S^{ct} to cancel the divergence, i.e. $S^{\text{ren}} = S^{\text{onshell}} + S^{\text{ct}}$. Such IR divergences in the bulk are dual to the UV divergences of the boundary QFT through the UV/IR connection Susskind and Witten (1998). The UV divergence in the bulk is dual to the IR divergence of the boundary QFT while it should be absent when working in the full context of the holography principle since the bulk quantum gravity theory is UV finite. As for the low energy effective description of the bulk theory, the UV divergence will appear and contribute to anomalous dimensions of CFT operators from boundary point of view. We will not discuss them in this thesis and one can see the straightforward treatment of UV divergences from the bulk side in the recent work Bañados et al. (2022). In the section 3.2, we will first decompose the field into AdS modes then apply the holography renormalisaton procedure on each single AdS surface thus complete the holography renormalisation for flat spacetime.

Given the flat holography renormalisation, one then obtains the dictionary between the effective theory on Minkowski and the CFT living on the boundary sphere. The CFT context can be read off from the renormalised action S^{ren}. It turns out a single bulk scalar field is dual to two series of CFT operators on the sphere with scale dimension living on the principal series. Later in section 3.3, we will see that we can decompose a massless field into in and out going shock waves and each of the shock wave is dual to one series of operators on the celestial sphere. One-point and two-point functions on the celestial sphere dual to the shock wave are also derived. Furthermore, we find that the full information of Minkowski could be stored in a pair of AdS hyperboloid, which forms a new kind of Cauchy surface.

3.1 Mode Analysis on Minkowski

In this section we consider solutions of the scalar field equation on Minkowski space and discuss how these can be used to construct a basis for scalar fields satisfying the given boundary conditions. We will begin our discussions with the familiar analysis within Minkowski coordinates before moving to Anti-de Sitter and de Sitter slicings.

We start from the study of solutions of KG equation written as

$$\left(\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial X_\mu} - M^2\right)\Phi_M(X) = 0, \tag{3.3}$$

in which M is the mass of the particle represented by a scalar field Φ_M and X^μ are coordinates of the Minkowski space $\mathbb{R}^{1,3}$ with signature $(-,+,+,+)$. Given the equation, one can directly write down a set of solutions

$$\phi_K(X) = e^{iK\cdot X} \tag{3.4}$$

in which $K^2 + M^2 = 0$ and K^μ are understood as the momentum of the particle. Thus the solution ϕ_K represents a single particle of mass M propagating freely in Minkowski space with momentum k. A generic solution Φ_M can be written as the superposition of all the single particle states

$$\Phi_M(X) = \sum_K a_k \phi_K(X) \tag{3.5}$$

in which a_k are coefficients that depend on k provided the on-shell condition is satisfied. Mathematically, ϕ_k are regarded as the basis that form the complete expansion of the solution space. Although the physical picture is maximally realized using the solution

ϕ_k, the symmetry of the spacetime given by the Lorentz group $SO(1,3) \cong SL(2,\mathbb{C})$ is not encoded here. To study the symmetry of the Lorentz group, one can use another set of basis so-called conformal basis ϕ_Δ defined by the translation relation

$$\phi_\Delta(\Lambda \cdot X; gw, g\bar{w}) \longrightarrow |cw+d|^{2\Delta}\phi_\Delta(X; w, \bar{w}), \tag{3.6}$$

in which $g \in SL(2,\mathbb{C})$ is defined by

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 0, \tag{3.7}$$

and $w, \bar{w}$ are coordinates of the spheres that lie along the null boundary of Minkowski space so-called Celestial sphere. In order to form the unitary representation, Δ is restricted on the principal series $1 + i\mathbb{R}$ and it can be further proved that all the functions ϕ_Δ lie on the principal basis are complete and orthonormal.

We have introduced two sets of basis one with maximally physical meaning while the other with significant geometry realization. Now we introduce another set of basis which will be convenient for us to bring the story into the context of AdS/CFT. The idea comes from the fact that one can treat the Euclidean AdS surface H_3 as the codimension one surface embedding in the whole space $\mathbb{R}^{1,3}$ and the equation of motion on the surface is reduced to

$$(\Delta_{H_3} - k^2)\ \phi_k(X) = 0, \tag{3.8}$$

in which k is the effective mass on the AdS surface with the induced Laplacian Δ_{H_3}. As we have discussed before, one can treat $\phi_k(X)$ as the basis and use them to construct the generic solution Φ_M. Those basis are summarized below.

$$\begin{array}{ccccc} & & \sum_k a_k \phi_k & & \\ & & \uparrow & & \\ & & \Phi_M & & \\ & \swarrow & & \searrow & \\ \sum_K a_K \phi_K & & & & \sum_\Delta a_\Delta \phi_\Delta \end{array} \tag{3.9}$$

3.1.1 Milne Slicing

Following this review, we now consider solution of the scalar field equation using Anti-de Sitter and de Sitter slicing of Minkowski space. We illustrate these slicings in Figure 3.1. Region $\mathcal{A}^\pm$ are foliated by Euclidean Anti-de Sitter (hyperbolic) surfaces while region $\mathcal{D}$ is foliated by de Sitter surfaces. To describe the region $\mathcal{A}$, which is sliced by

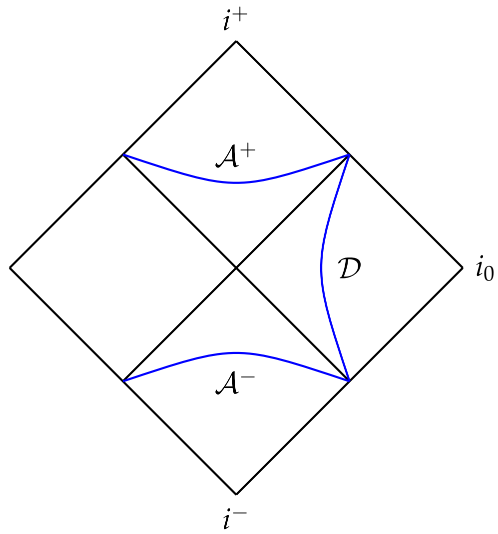


FIGURE 3.1: The Milne wedge $\mathcal{A}^\pm$ are sliced by AdS surfaces while the Rindler wedge $\mathcal{D}$ is foliated by dS surfaces.

hyperboloids, we use Milne coordinates written as

$$ds^2 = G_{\mu\nu} dX^\mu dX^\nu = -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{1+\rho^2} + 2\rho^2 \gamma_{z\bar{z}} dz d\bar{z} \right), \tag{3.10}$$

in which $\rho, \tau \in \mathbb{R}$. Here $z, \bar{z}, \gamma_{z\bar{z}}$ are complex coordinates and the metric is the standard metric on the sphere. τ is the radius of the AdS hyperboloid introduced in (3.2) and one only needs to take the positive part $\tau \geq 0$ to cover the single region $\mathcal{A}^+$. The near light cone region is given by $\tau \to 0$ while the null infinity is the region where $\tau \to \infty$. In such coordinates, the scalar equation can be separated into two equations

$$\left(\rho(\rho^2+1)\partial_\rho^2 + (3\rho^2+2)\partial_\rho - k^2\rho - \frac{l(l+1)}{\rho} \right) \phi_l(\rho, k) = 0, \tag{3.11}$$

$$\left(-3\frac{\partial_\tau}{\tau} - \partial_\tau^2 + \frac{\omega^2}{\tau^2} - M^2 \right) \psi(\tau, \omega) = 0, \tag{3.12}$$

where the first equation represents a particle of effective mass k on the hyperboloid and the second equation depends only on the time τ. Here l labels the usual discrete eigenvalue of scalar spherical harmonics $Y_m^l(z, \bar{z})$. Accordingly the scalar basis can be expressed as

$$f_{\omega,k,l,m}(\tau, \rho, z, \bar{z}) = \psi(\tau, \omega)\phi_l(\rho, k) Y_m^l(z, \bar{z}) \tag{3.13}$$

where the onshell condition requires $\omega = k$. As above, we will be interested in using this basis to represent fields with the same boundary conditions which are not necessarily onshell, hence we do not impose $\omega = k$ a priori. Any scalar satisfying the boundary conditions can be expressed as

$$\Phi(\tau, \rho, z, \bar{z}) = \sum_{l,m} \int d\omega dk \; f_{\omega,k,l,m}(\tau, \rho, z, \bar{z}) \tilde{\Phi}(\omega, k, l, m), \tag{3.14}$$

where $\tilde{\Phi}$ can be treated as coefficients and one can deduce them by applying the orthogonality relation of the basis

$$\int d\tau d\rho dz d\bar{z}\; w(\tau,\rho,z,\bar{z})\; f_{\omega,k,l,m}(\tau,\rho,z,\bar{z}) f_{\omega',k',l',m'}(\tau,\rho,z,\bar{z}) = \delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\delta(k-k') \tag{3.15}$$

with proper weight function w deduced form the equation of motion. In relation (3.14) we express the integrals abstractly; we will discuss how the domain of (ω, k) relates to boundary and regularity conditions below.

We can also define a basis on spatial slices

$$F_{k,l,m}(\rho,z,\bar{z}) = \phi_l(\rho,k) Y_m^l(z,\bar{z}) \tag{3.16}$$

Any scalar satisfying the equation of motion can be expressed as

$$\Phi(\tau,\rho,z,\bar{z}) = \sum_{l,m} \int dk\; F_{k,l,m}(\rho,z,\bar{z}) \bar{\Phi}(\tau,k,l,m), \tag{3.17}$$

where we have imposed the on-shell condition $\omega = k$ and reorganize the product of $\tilde{\Phi}(k,k,l,m)\psi(\tau,k)$ into $\bar{\Phi}(\tau,k,l,m)$.

Analogously we can transform only in the time direction i.e.

$$\Phi(\tau,\rho,z,\bar{z}) = \int d\omega\; \psi(\tau,\omega) \hat{\Phi}(\omega,\rho,z,\bar{z}), \tag{3.18}$$

where again we rewrite the data and make $\sum_{lm} \tilde{\Phi}(\omega,\omega,l,m) F_{k,l,m}(\rho,z,\bar{z})$ into $\hat{\Phi}(\omega,\rho,z,\bar{z})$. We will see later that those two are the most natural ways to read off the holographic data.

3.1.2 Explicit Modes

Massless Fields

Now let us turn to the explicit solution of the differential equations above. These have been discussed in the literature de Boer and Solodukhin (2003); Marolf (2007); Cheung et al. (2017b); Liu and Lowe (2021); Raclariu (2021), but here we will consider in further detail the role of regularity and boundary conditions. Let us consider the differential equation in time. It is useful to consider first the case of a massless field, so that the equation reduces to

$$\left(-3\frac{\partial_\tau}{\tau} - \partial_\tau^2 + \frac{\omega^2}{\tau^2}\right)\psi(\tau,\omega) = 0. \tag{3.19}$$

The generic solution takes the form

$$\psi(\tau,\omega) = \psi(\alpha_+)\tau^{-1+\alpha_+} + \psi(\alpha_-)\tau^{-1+\alpha_-} \tag{3.20}$$

where $\alpha_\pm$ are the two roots of

$$\alpha^2 = 1 + \omega^2 \tag{3.21}$$

Solutions are bounded $|\psi| < \infty$ at the null infinity $\tau \to \infty$ if either $|\text{Re}(\alpha_+)| \leq 1$ or $|\text{Re}(\alpha_-)| \leq 1$. More precisely, states that are localised in the center and vanish at the boundary take the value $|\text{Re}(\alpha)| < 1$, called bound states. States which could propagate to the infinity and have non-zero contribution at null boundary are called scattering states. We will study them in a separate way.

Scattering States

For scattering states, we should have either $\text{Re}(\alpha_+) = 1$ or $\text{Re}(\alpha_-) = 1$. Thus all solutions that are finite at the infinity have the form

$$\alpha = 1 + ip \tag{3.22}$$

with p real. We can write a general scattering state as

$$\psi(\tau, p) = \psi(p)e^{ip\ln\tau} \tag{3.23}$$

where p is real, $\alpha^2 = (1-p^2) + 2ip$ and $\omega^2 = 2ip - p^2$. Clearly each such mode is not real. If $\alpha_+ = 1 + ip$, then the corresponding second root of (3.21) is $\alpha_- = -(1+ip)$; the latter mode is bounded as $\tau \to \infty$ but is not bounded as $\tau \to 0$. Thus for a given real value of p the general solution takes the form

$$\psi(\tau, p) = \psi_+(p)\tau^{ip} + \psi_-(p)\tau^{-ip-2} \equiv \psi_+(p)f_+(\tau, p) + \psi_-(p)f_-(\tau, p) \tag{3.24}$$

To understand the orthogonality relation it is useful to first recall the standard relations for exponentials i.e.

$$\int_{-\infty}^{\infty} d(\ln\tau)e^{i(p-q)\ln\tau} = \int_0^{\infty} \frac{d\tau}{\tau}e^{i(p-q)\ln\tau} = 2\pi\delta(p-q) \tag{3.25}$$

The latter is equivalent to

$$\int_0^{\infty} d\tau w(\tau)f_+(\tau, p)f_-(\tau, q) = \delta(p-q) \tag{3.26}$$

where the weight function $w(\tau) = \tau$ is derived by expressing (3.19) in standard Sturm-Liouville form i.e.

$$\partial_\tau\left(P(\tau)\partial_\tau\psi\right) + Q(\tau)\psi = -\lambda w(\tau)\psi \tag{3.27}$$

where λ is the eigenvalue i.e. ω^2 and the coefficient functions $(P(\tau), Q(\tau))$ follow from (3.19).

Bound States on Principal Series

For bound states $|\psi| \to 0$ when $\tau \to \infty$, as we have mentioned, α should satisfy $|\mathrm{Re}(\alpha_\pm)| < 1$. Here we are just interested in the special case such that $\alpha_\pm$ are chosen to be

$$\alpha_\pm = \pm\, ip \tag{3.28}$$

for $p \in \mathbb{R}$, then τ modes $f_\pm$ will become

$$f_+(\tau, p) = \tau^{-1+\alpha^+} = \frac{e^{ip\ln\tau}}{\tau}, \qquad f_-(\tau, p) = \tau^{-1+\alpha^-} = \frac{e^{-ip\ln\tau}}{\tau}. \tag{3.29}$$

Now we impose further restriction on p so that make $p \geq 0$. This could always be done since $f_+(\tau, p) = f_-(\tau, -p)$ and one can treat such restriction as the reduction of the redundancy of the basis or the decomposition of the mode into positive and negative frequency components. For a generic function $\psi(\tau, p)$, we have the decomposition

$$\psi(\tau, p) = \psi(p) f_+(\tau, p) + \psi^*(p) f_-(\tau, p), \tag{3.30}$$

in which $\psi(p)$ are complex coefficients and $\psi(\tau, p)$ is now real. Given the weight function $w(\tau) = \tau$, one can check that

$$\int_0^\infty d\tau w(\tau) f_+(\tau, p) f_-(\tau, q) = 2\pi\, \delta(p - q) \tag{3.31}$$

and the relation

$$\int_0^\infty d\tau w(\tau) f_+(\tau, p) f_+(\tau, q) = \int_0^\infty d\tau w(\tau) f_-(\tau, p) f_-(\tau, q) = 2\pi\, \delta(p + q) = 0. \tag{3.32}$$

Later, we will see that those states are dual to operators on the celestial sphere with scale dimension Δ satisfying

$$\Delta = 1 + \alpha_+ = 1 + ip, \tag{3.33}$$

which is half of the principal series that forms the unitary representation of $SO(1,3)$ Dobrev et al. (1977). It is also worthwhile to note that the mode expansion (3.18) will become inverse Mellin transform if the τ modes take the form in (3.30).

Massive Fields

For non-zero mass the generic solution takes the form

$$\psi(\tau, \omega) = \psi(\alpha_+) \frac{J_{\alpha_+}(M\tau)}{\tau} + \psi(\alpha_-) \frac{J_{\alpha_-}(M\tau)}{\tau} \tag{3.34}$$

where $\alpha_\pm$ are again the roots of (3.21). Here we assume that $\alpha_\pm$ are generic complex numbers, in which case the two Bessel functions expressed in this form are manifestly linearly independent. For integer α the second solution will be expressed in the form of the second Bessel function Y_α. Solutions that are bounded as $\tau \to 0$ have

$$\mathrm{Re}(\alpha) \geq 1, \tag{3.35}$$

since $J_\alpha(M\tau) \sim \tau^\alpha$ as $\tau \to 0$. Using this limit of $J_\alpha(x)$ as $x \to 0$, the mode functions clearly reduce to those above as $M \to 0$. The behaviour of Bessel functions at large τ are complicated and we will leave this to further investigation.

As we have seen, the value of $\alpha(k)$ or equivalently Δ are often related to the behaviour of the solution near the light cone or null boundary. For example in section 3 of de Boer and Solodukhin (2003), they argued that the action should be regular around the light cone thus, for the modes behaves as $\tau^{-1+\alpha}$, one requires $\mathrm{Re}(\alpha) \geq 0$ which is a weaker condition than the boundedness condition (3.35).

In section 2 of Marolf (2007), the regularity of the solution is studied from the normalization point of view. The behaviour of the field at null boundary and light cone are both studied and it is argued that the solution should be oscillatory in order to make the mode normalizable. In our context, the oscillatory condition means that α should be complex, i.e. $\mathrm{Im}(\alpha) \neq 0$. Furthermore, Marolf also argued that the oscillatory fields should be separated into two parts. One is dynamical and it is normalizable according to the Klein-Gordon norm while the other part is not normalizable and is used to specify the boundary condition of the system. In additonal to the Klein-Gordon norm, the other kind of paring between the oscillatory modes is also introduced in order to study the inner product structure between all the modes.

More rigorous study of the asymptotic behaviour of the solution for Klein-Gordon equation in math literature are shown in Vasy (2013); Baskin et al. (2015, 2018). The boundedness of the solution for Schwarzschild case is shown in the gravity literature, so called Kay–Wald boundedness theorem Kay and Wald (1987) and one can see the review in Dafermos and Rodnianski (2013). For Minkowski case, stability for the Einstein equation is first shown in Christodoulou and Klainerman (1993). Then for scalar-Einstein case when the matter field propagating on the asymptotically Minkowski background, the stability is also proofed provided the decay of the fields is under well controlled at the boundary Lindblad and Rodnianski (2010), which leads to the constraints on the real part of the scale dimension i.e. $\mathrm{Re}(\alpha)$. This is similar to the study of Breitenlohner-Freedman bound for AdS spacetime Breitenlohner and Freedman (1982a,b). For the stability of Minkowski, a sharp bound for α is not found yet while, according to the above discussion, we summarise the possible range of as [1]

$$0 \leq \mathrm{Re}(\alpha) \leq 1, \qquad \mathrm{Im}(\alpha) \neq 0. \tag{3.36}$$

However, given the fact that α has two solutions satisfying $\alpha_+ + \alpha_- = 0$, we will have $\mathrm{Re}(\alpha_\pm) = 0$ if one requires both of the modes $f_\pm(\tau, p)$ live in the bound (3.36). For other choices of α, one of the two modes will be stable while the other will not.

3.1.3 Radial Equation

Now let us turn to the radial equation (3.11). It is important to distinguish between solutions to the equation for all radial values, and the asymptotic expansions from which the holographic dictionaries are constructed. The general solution to the radial equation can be written as

$$\phi_l(\rho, k) = \phi(k)\mathrm{csch}\eta \; P_l^\beta(\coth\eta) + \varphi(k)\mathrm{csch}\eta \; Q_l^\beta(\coth\eta), \tag{3.37}$$

in which $\rho = \sinh\eta$ and (P, Q) are associated Legendre functions. Note that the range of η is the same as that for ρ i.e. $0 \leq \eta < \infty$. The order of the function is given by

$$\beta^2 = 1 + k^2 \tag{3.38}$$

where here we do not assume that β is real. In fact, since $\coth(\eta) \geq 1$ over the domain of interest, it is more useful to write the general solution in terms of the hypergeometry functions as shown in the appendix F thus here it is convenient to choose the basis as

$$\phi_l(\rho, k) = \phi_l^+(k)\mathrm{csch}\eta \; P_l^{\beta_+}(\coth\eta) + \phi_l^-(k)\mathrm{csch}\eta \; P_l^{\beta_-}(\coth\eta), \tag{3.39}$$

where $\beta_\pm$ are the two (complex) roots of (3.38), with $(\beta_+ + \beta_-) = 0$.

To understand the regularity and boundedness conditions it is useful to consider the first the $l = 0$ solutions which can be written in terms of elementary functions as

$$\phi_0(\rho, k) = \phi^+(k)\frac{1}{\rho}(\rho + \sqrt{\rho^2+1})^{\beta_+} + \phi^-(k)\frac{1}{\rho}(\rho + \sqrt{\rho^2+1})^{\beta_-} \tag{3.40}$$

where $(\beta_+ + \beta_-) = 0$. A mode is bounded as $\rho \to \infty$ provided that $\mathrm{Re}(\beta) \leq 1$. However, no single mode is bounded as $\rho \to 0$. One can combine modes to obtain

[1] We should note that the upper bound comes from the boundedness of the modes $|\psi| \leq \infty$ at the null infinity while for the stability of Minkowski space the condition will usually be stronger than the boundedness. For example, in the work Lindblad and Rodnianski (2010), the decay behaviour of the field is required to be $|\psi| < \tau^{-1}$ thus the real part of α could only be zero after taking the lower bound into consideration. Here we choose to present the wider range for α although it not clear to us whether the value $0 < |\mathrm{Re}(\alpha)| \leq 1$ are physical and stable or not.

fields that are bounded as $\rho \to 0$:

$$\frac{1}{\rho} \sinh \left(\beta_+ \ln(\rho + \sqrt{\rho^2 + 1}) \right). \tag{3.41}$$

The orthogonality condition for $l = 0$ is obtained as above from writing the radial equation in Sturm-Liouville form (3.27), so that the coefficient and weight functions are given by

$$P(\rho) = \rho^2(\rho^2 + 1)^{\frac{1}{2}} \qquad w(\rho) = \frac{\rho^3}{(\rho^2 + 1)^{\frac{3}{2}}}. \tag{3.42}$$

Therefore we have

$$\int_0^\infty d\rho w(\rho) \mathcal{F}(\rho, \beta_+) \mathcal{F}(\rho, \beta_-) = 1 \qquad \beta_+ + \beta_- = 0 \tag{3.43}$$

where

$$\mathcal{F}(\rho, \beta_\pm) = \frac{1}{\rho}(\rho + \sqrt{\rho^2 + 1})^{\beta_\pm}. \tag{3.44}$$

By using the orthogonal relation (3.43), one can define the inner products between the regular physical spatial modes for $l = 0$. A full treatment for the construction of regular solutions in the cases for $l > 0$ is shown in the Appendix F and the study for the inner product is shown in the work Laddha et al. (2022). Till now, we have discussed the modes with various choices of the value of α or β and their corresponding physical interpretation while we should note that it is not clear which of them will form the necessary complete basis for the bulk fields and a generic principle to find out such a basis is still absent. Later we will see that different k-modes contribute to the correlation function living on the boundary celestial sphere in a different way according to the detail of the interaction. Here, we assume that given the detail of the theory a proper subset $\mathcal{P}$ of k always exists that enable us to perform the mode decomposition thus the modes form a complete basis and the superposition principle will work. In the rest of this thesis, we will focus on the study of onshell fields therefore the condition $\alpha = \beta$ is automatically imposed.

3.2 Holography

The purpose of this section is to develop a detailed holographic dictionary between the bulk theory in asymptotically Minkowski spacetimes and the putative dual theory, associated with null infinity. We will develop the dictionary using the example of a test scalar field in the fixed Minkowski background. Our approach will be based on the principles of AdS/CFT (1.1), i.e, writing a defining holographic relation of the form

$$\exp\left(iS^{\text{ren}}(\Phi)\right) = \left\langle \exp\, - \int_{\partial M} \mathcal{J}\, \mathcal{O} \right\rangle_{QFT}. \tag{3.45}$$

Here $S(\Phi)$ is the action of the bulk theory with scalar field Φ. Taking into account IR divergences, we will need to renormalise this action and $S^{\text{ren}}(\Phi)$ is the renormalised version of $S(\Phi)$; an important part of this section will be establishing the principles underlying the renomalisation procedure. On the right hand side of (3.45) we denote $\mathcal{J}$ and $\mathcal{O}$ as the source and operator in the quantum field theory at the boundary. Again this should be viewed as a renormalised expression. The detail of the mapping between the asymptotic bulk boundary condition and CFT sources together with operators are summarized in section 3.2.4

Following the story of the construction of the dictionary for AdS/CFT Witten (1998a); Freedman et al. (1999); Klebanov and Witten (1999), here we are also going to specify the source and operator by decomposing the data of the bulk field Φ into coefficients when doing the expansion at the boundary. It turns out that we need two series of operators $\{\mathcal{J}, \mathcal{O}\}$ and $\{\tilde{\mathcal{J}}, \tilde{\mathcal{O}}\}$ on the celestial sphere in order to reconstruct the bulk field by checking the renormalised action specifically and the way that they are coupled is determined by the causal and dynamical structure of the bulk theory. The new feature for the Flat/CFT dictionary is that we are reducing two spacetime dimensions at once and the dictionary is built between the bulk theory with the notion of time and the boundary Euclidean theory on the sphere thus the factor i plays an important role here when considering the emergence of time and the unitarity of the CFT.

3.2.1 Holographic Dictionary

We begin by reviewing the usual holographic dictionary for scalar fields on Euclidean AdS_3. Using the same coordinates for Euclidean AdS_3 as above i.e.

$$ds^2_{\text{AdS}_3} = g_{ij}dx^i dx^j = \left(\frac{d\rho^2}{1+\rho^2} + 2\rho^2\gamma_{z\bar{z}}dzd\bar{z}\right), \tag{3.46}$$

the boundary is at $\rho \to \infty$ and the boundary metric is manifestly spherical. Now consider a massive scalar field with action

$$S_{\text{AdS}_3} = \frac{1}{2}\int d^3x\sqrt{g}\left((\partial\varphi)^2 + m^2\varphi^2\right), \tag{3.47}$$

where g is the determinant of the Euclidean AdS_3 metric above. The onshell action is thus

$$S^{\text{onshell}}_{\text{AdS}_3} = \frac{1}{2}\int_{\partial AdS_3} d\Sigma^i \varphi\partial_i\varphi. \tag{3.48}$$

The asymptotic expansion of an onshell field takes the form

$$\varphi(\rho, z) = \rho^{\Delta-2}\left(\varphi(z) + \cdots\right) + \rho^{-\Delta}\left(\tilde{\varphi}(z) + \cdots\right) \tag{3.49}$$

where $\varphi(z)$ is the source for the dual operator $\mathcal{O}_\varphi(z)$ of dimension Δ, where $m^2 = \Delta(\Delta-2)$. When Δ is integral the asymptotic expansions contain logarithmic terms, which are related to the contact terms in two point functions discussed below.

One uses the asymptotic expansion of the onshell field to compute the explicit value of the regulated onshell action, from which one can construct covariant counterterms and the renormalised action

$$S^{\text{ren}}_{\text{AdS}_3} = \mathcal{L}_{\rho\to\infty}\left(S^{\text{onshell}}_{\text{AdS}_3} + S^{\text{ct}}_{\text{AdS}_3}\right) \tag{3.50}$$

The covariant counterterms are of the form

$$S^{\text{ct}}_{\text{AdS}_3} = -\frac{1}{2}(\Delta-2)\int_{\partial AdS_3} d^2x\sqrt{h}\varphi^2 + \cdots \tag{3.51}$$

where h is the determinant of the induced metric at the boundary.

In terms of the complex AdS coordinate (3.46), the $\text{AdS}_3/\text{CFT}_2$ dictionary the can be written as

$$\exp\Big(-S_{\text{AdS}_3}(\Phi)\Big) = \Big\langle \exp\ -\int_{S^2} dz^2\varphi(z)\ \mathcal{O}(z)\ \Big\rangle, \tag{3.52}$$

in which we have reorganised the factor $\frac{1}{2}$ resulting from the transformation between the complex and real (Poincaré) coordinates (E.69) into the rescaling of the source $\varphi(z)$. The expectation value of the dual operator is then defined as the variation of the renormalised action with respect to the source, expressed in terms of $\tilde{\varphi}$ as

$$\langle\mathcal{O}(z)\rangle_\varphi = 2(1-\Delta)\Omega_2(z)\tilde{\varphi}(z) + C(\varphi). \tag{3.53}$$

Here the function $C(\varphi)$ denotes contributions to the one point correlation function that are expressed in terms of the source; such contributions arise whenever Δ is integral and its exact form depends on the regularization scheme. Again, the weight $\Omega_2(z)$ comes from the transformation between (E.69) and (3.46). As usual the two point function can be obtained by functionally differentiating with respect to the source $\varphi(z)$ i.e.

$$\langle\mathcal{O}(z)\mathcal{O}(z')\rangle_\varphi = -2(1-\Delta)\Omega_2(z)\frac{\delta\tilde{\varphi}(z)}{\delta\varphi(z')} + \cdots \tag{3.54}$$

where the ellipses contribute only to contact terms in the correlation function and the renormalisation factor $2(\Delta-1)$ can be deduced by the study of bulk-boundary propagator which is briefly reviewed in the Appendix E.

Given the bulk-boundary propagator $K(\rho,z;z')$, a generic regular field in the bulk with boundary behaviour $\varphi(\rho,z)\sim\varphi(z)$ can be expressed as

$$\varphi(\rho,z) = \int_{S^2}\frac{1}{2}dz'd\bar{z}'\ K(\rho,z;z')\varphi(z'), \tag{3.55}$$

in which we have taken coordinate transformation factor $\frac{1}{2}$ into consideration and assume $\rho \to \infty$. With the help of the AdS/CFT propagator, one can deduce the CFT two-point function in a quick way. For example, the AdS_3 onshell action can be written as

$$S^{\text{onshell}}_{\text{AdS}_3} = \frac{1}{2}\int_{S^2} d\Omega_2\, R^2\sqrt{1+R^2}\,\left(\varphi(\rho,z)\partial_\rho\varphi(\rho,z)\right)_{\rho=R} \tag{3.56}$$

$$= -\frac{\Delta}{2\pi}\int_{S^2}\int_{S^2} d^2z d^2z'\,\frac{\varphi(z)\varphi(z')}{|z-z'|^{2\Delta}}, \tag{3.57}$$

in which in the second line we have used the expression (3.55) and the contraction relation (E.79) for the propagators. Following the similar procedure, one can also deduce $S^{\text{ct}}_{\text{AdS}_3}$ and we have

$$S^{\text{ct}}_{\text{AdS}_3} = -\frac{1}{2}(\Delta-2)\int_{S^2} d\Omega_2\, R^2\,\left(\varphi(\rho,z)\varphi(\rho,z)\right)_{\rho=R} \tag{3.58}$$

$$= -\frac{\Delta-2}{2\pi}\int_{S^2}\int_{S^2} d^2z d^2z'\,\frac{\varphi(z)\varphi(z')}{|z-z'|^{2\Delta}}, \tag{3.59}$$

therefore, according to the dictionary (3.52), the renormalised two-point function now becomes

$$\langle\mathcal{O}(z)\mathcal{O}(z')\rangle = -\frac{\delta^2 S^{\text{ren}}_{\text{AdS}_3}}{\delta\varphi(z)\delta\varphi(z')} = \frac{c_\Delta}{|z-z'|^{2\Delta}}, \tag{3.60}$$

where c_Δ takes the value

$$c_\Delta = \frac{2(\Delta-1)}{\pi}. \tag{3.61}$$

3.2.2 Holography Dictionary for Milne

In this section we turn to scalar fields in the Milne coordinates then proceed to perform the holography renormalisation for Minkowski spacetime. The action for the massive scalar field is

$$S = \frac{1}{2}\int_0^\infty d\tau\int_0^\infty d\rho\int dz d\bar{z}\sqrt{-G}\,\left((\partial\Phi)^2 + M^2\Phi^2\right), \tag{3.62}$$

in which G is given by (3.10) together with scalar fields Φ and we have restricted the integration region to $\mathcal{A}^+$. As usual we can express the onshell action as the exact term

$$S^{\text{onshell}} = \frac{1}{2}\int_0^\infty d\tau\int_0^\infty d\rho\int dz d\bar{z}\sqrt{-G}D^\mu(\Phi\partial_\mu\Phi), \tag{3.63}$$

which can be expressed as boundary terms thus we have $D^\mu = \frac{1}{\sqrt{-G}}\partial_\nu\sqrt{-G}G^{\nu\mu}$. The philosophy of the celestial holography approach is to foliate the spacetime with spacelike surfaces, and throughout this section we will work in this approach, analysing divergences at the spatial boundaries of each slice.

Accordingly, let us focus on the radial boundary as $\rho \to \infty$. Using the Milne form of the metric the onshell boundary terms are

$$S^{\text{onshell}} = \frac{1}{2}\int_0^\infty d\tau\tau \int_{\partial AdS_3} d\Sigma^i \Phi(\tau, x^i)\partial_i\Phi(\tau, x^i) \tag{3.64}$$

where the second integral is expressed in terms of the boundary of the Euclidean AdS_3 metric (3.46). Here we should note that, strictly speaking, the value of onshell action shown in (3.63) and (3.64) are not the same since we have ignored the integral over spatial direction at the fixed hyperboloids $\tau = 0$ and $\tau = +\infty$. After taking the other Milne wedge $\mathcal{A}^-$ into consideration, the difference is then determined by the integral over $\Phi(\tau = \pm\infty, \rho, z, \bar{z})\partial_\tau\Phi(\tau = \pm\infty, \rho, z, \bar{z})$ in which $\Phi(\tau = \pm\infty, \rho, z, \bar{z})$ are the initial and final data imposed for a given physical system since $\tau = \pm\infty$ are null boundaries of Minkowski space. If one proposes that the initial and final states for the physical system are vacuum, then we have $\Phi(\tau = \pm\infty, \rho, z, \bar{z}) = 0$ thus there will be no difference between (3.63) and (3.64). For scattering processes, the initial and final states are the in and out going states while one can assume the difference will contribute to the action in a small and finite way therefore leads to a proper $i\epsilon$ perscription of the quantum theory Weinberg (1995). Here we will only study the onshell action in the form of (3.64) and the explicit expression for this is

$$S^{\text{onshell}} = \frac{1}{2}\int_0^\infty d\tau\tau \int_{\partial AdS_3} d\Omega_2 R^2(1+R^2)^{\frac{1}{2}} \left(\Phi(\tau, \rho, z, \bar{z})\partial_\rho\Phi(\tau, \rho, z, \bar{z})\right)_{\rho=R} \tag{3.65}$$

where the boundary is regulated at $\rho = R$ and $d\Omega_2$ is the integration measure over the unit two sphere.

Given the onshell action, we can further decompose it into the k mode components by introducing the k mode function $f(\tau, \rho, z, \bar{z}; k)$ given by

$$\Phi(\tau, \rho, z, \bar{z}) = \int_{\mathcal{P}} dk f(\tau, \rho, z, \bar{z}; k) \tag{3.66}$$

and then we can use such decomposition of fields to transform the onshell action into k mode space after rewriting all the fields in the action in terms of f. More precisely, we can define the (k, k') mode of the action

$$S^{\text{onshell}}(k, k') := \frac{1}{2}\int_0^\infty \tau d\tau \int_{\partial AdS_3} d\Omega_2\ R^3(f(\tau, \rho, z, \bar{z}; k)\partial_\rho f(\tau, \rho, z, \bar{z}; k'))_{\rho=R} \tag{3.67}$$

and one can check at large R we have

$$S^{\text{onshell}} = \int_{\mathcal{P}} dk \int_{\mathcal{P}'} dk'\ S^{\text{onshell}}(k, k'), \tag{3.68}$$

where the double integral over the set $\mathcal{P}$ come from the fact that the onshell action for free particles are quadratic in terms of Φ. Moreover, we can treat the (k, k') mode of the action $S^{\text{onshell}}(k, k')$ as the onshell action which describes the interaction between a pair of modes (k,k'). Later we will see that $S^{\text{onshell}}(k,k')$ is proportional to the delta function if the domain of the integral over k takes the value such that $\beta_+(k) = i\mathbb{R}^+$ thus we have

$$S^{\text{onshell}}(k,k') = \delta(k-k')S^{\text{onshell}}(k,k) \tag{3.69}$$

and for simplicity we denote $S^{\text{onshell}}(k,k)$ as $S^{\text{onshell}}(k)$. In such convention, the onshell action then can be expressed as

$$S^{\text{onshell}} = \int_{\mathcal{P}} dk\, S^{\text{onshell}}(k), \tag{3.70}$$

which will be used as the standard form of the k mode decomposition of the action for free particles.

Before performing the renormalisation on $S^{\text{onshell}}(k)$, let us consider asymptotic solutions of the equation (3.11) as $\rho \to \infty$. The generic form for the asymptotic solution

$$\begin{aligned} \phi_l(\rho;k) &= \phi_l(\rho;\beta_+(k)) + \phi_l(\rho;\beta_-(k)) \\ &\equiv \rho^{\beta_+-1}\left(\phi_l^+(k) + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right) + \rho^{\beta_--1}\left(\phi_l^-(k) + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right) \end{aligned} \tag{3.71}$$

where $(\beta_+ + \beta_-) = 0$ and without loss of generality we will assume that $\text{Re}(\beta_+) \geq \text{Re}(\beta_-)$. Physical constraints and the inner product structure of these modes have already been studied in Section 3.1.3 and Appendix F. Here, instead of using l modes on the sphere we can express a general solution for the spatial part of the scalar for fixed k as

$$\phi(\rho,z,\bar{z};k) = \phi(\rho,z,\bar{z};\beta_+) + \phi(\rho,z,\bar{z};\beta_-) \tag{3.72}$$

by summing over the harmonic indices (l,m) thus the asymptotics of each solution are of the form

$$\phi(\rho,z,\bar{z};\beta_\pm) = \rho^{\beta_\pm-1}\left(\phi^\pm(z,\bar{z};k) + \mathcal{O}\left(\frac{1}{\rho^2}\right)\right). \tag{3.73}$$

Combining modes of a fixed value of k we obtain

$$f(\tau,\rho,z,\bar{z};k) = f_+(\tau,k)\phi(\rho,z,\bar{z};k) + f_-(\tau,k)\tilde{\phi}(\rho,z,\bar{z};k) \tag{3.74}$$

$$\begin{aligned} &= \tau^{\beta_+-1}\phi(\rho,z,\bar{z};\beta_+) + \tau^{\beta_--1}\tilde{\phi}(\rho,z,\bar{z};\beta_+) \\ &\quad + \tau^{\beta_+-1}\phi(\rho,z,\bar{z};\beta_-) + \tau^{\beta_--1}\tilde{\phi}(\rho,z,\bar{z};\beta_-) \end{aligned} \tag{3.75}$$

where the fields $\tilde{\phi}(\rho,z,\bar{z};\beta_\pm)$ have the properties (3.72) and (3.73) and well see the explicit expression for them in the next section.

Now let us return to the four-dimensional k mode action. The regulated action for modes of fixed k contains the terms

$$S^{\text{onshell}}(k) = \frac{1}{2}\int_0^\infty d\tau\tau \int_{\partial AdS_3} d\Omega_2 \Big((\beta_+ - 1)R^{2\beta_+}\Phi_s(\tau,z,\bar{z};k)^2 \\ + (\beta_- - 1)R^{2\beta_-}\Phi_v(\tau,z,\bar{z};k)^2 - 2\Phi_s(\tau,z,\bar{z};k)\Phi_v(\tau,z,\bar{z};k) + \cdots \Big), \quad (3.76)$$

where the boundary of the AdS slice is regulated as $\rho = R$ and the ellipses denote terms that are suppressed by at least $1/R^2$. We introduce a shorthand notation for the combinations of terms in the asymptotic radial expansions:

$$\begin{aligned} \Phi_s(\tau,z,\bar{z};k) &= \tau^{\beta_+-1}\phi^+(z,\bar{z};k) + \tau^{\beta_--1}\tilde{\phi}^+(z,\bar{z};k) \\ \Phi_v(\tau,z,\bar{z};k) &= \tau^{\beta_+-1}\phi^-(z,\bar{z};k) + \tau^{\beta_--1}\tilde{\phi}^-(z,\bar{z};k) \end{aligned} \quad (3.77)$$

Let us suppose that $\text{Re}(\beta_+) > 0$, in which case $\text{Re}(\beta_-) < 0$. In this case the first term in (3.76) will be divergent as $R \to \infty$, but the second term will vanish; all power law divergences will be of the form $R^{2\beta_+-2n}$ with n an integer.

As above, we can remove divergences with counterterms. These counterterms should be expressed in terms of quantities that are intrinsic to the regulated boundary, and they should be covariant with respect to the bulk diffeomorphism at $\rho = 0$ thus make $\Phi(\tau, R, z, \bar{z})$ transform as a scalar field. Here in fact, the background metric already uses a preferred slicing of the four-dimensional metric, i.e. a specific coordinate choice for time, and therefore we would not expect the counterterms to preserve full three-dimensional covariance of the boundary. In practice this means that the counterterms are expressed in the form

$$\begin{aligned} S^{\text{ct}} &= \int_0^\infty d\tau \int_{\partial AdS_3} d^2z\sqrt{-\bar{\gamma}}\left(a_1\Phi(\tau,R,z,\bar{z})^2 + a_2(\partial_z\partial_\tau\Phi(\tau,R,z,\bar{z}))^2 + \cdots\right) \\ &= \int_0^\infty d\tau\tau \int_{\partial AdS_3} d\Omega_2 \left(a_1\Phi(\tau,R,z,\bar{z})^2 + a_2(\partial_z\partial_\tau\Phi(\tau,R,z,\bar{z}))^2 + \cdots\right) \end{aligned} \quad (3.78)$$

where $\bar{\gamma}_{\tau\tau} = -1,\ \bar{\gamma}_{z\bar{z}} = \tau^2\gamma_{z\bar{z}}$ is the induced metric on the boundary of Milne wedge at $\rho = R$ (with the curvature radius being independent of τ) and the derivative ∂_z only acts on the celestial sphere. As we can see, the covariant of the bulk diffeomorphism at the surface $\rho = R$ shown in the first line is broken by fixing the gauge of coordinates in the second line. By construction these counterterms will remove the divergences because the analytic structure on the celestial sphere is precisely as described above for AdS_3/CFT_2. Indeed, matching with the dictionary above one obtains

$$\Delta_k = 1 + \beta_+ \quad (3.79)$$

and the finite terms in the renormalized action include

$$S^{\text{ren}}(k) = \beta_+ \int_0^\infty d\tau\tau \int_{\partial AdS_3} d\Omega_2 \Phi_s(\tau,z,\bar{z};k)\Phi_v(\tau,z,\bar{z};k) \tag{3.80}$$

$$= \beta_+ \int_0^\infty d\tau\tau \int_{\partial AdS_3} d\Omega_2 (\tau^{\beta_+-1}\phi^+(z,\bar{z};k) + \tau^{\beta_--1}\tilde{\phi}^+(z,\bar{z};k)) \times(\tau^{\beta_+-1}\phi^-(z,\bar{z};k) + \tau^{\beta_--1}\tilde{\phi}^-(z,\bar{z};k)). \tag{3.81}$$

As above, there would be additional finite terms in the action if β_+ were to be real and integer valued, but this is not the case of interest here. According to the dictionary given in (3.45), one can see that β_+ should be a pure imaginary number $\beta_+ \in i\mathbb{R}$ in order to make sure that CFT correlation functions given by the right hand side of (3.45) are real. Moreover, given the relation (3.79), we know that the scale dimension of the operator on the celestial sphere should take the value on the principal series $\Delta_k = 1 + i\mathbb{R}$[2].

Using the orthogonality relations for the τ eigenfunctions (3.31), (3.32) we can explicitly compute the τ integrals as

$$S^{\text{ren}}(k) = \beta_+ \int_{\partial AdS_3} d\Omega_2 (\phi^+(z,\bar{z};k)\tilde{\phi}^-(z,\bar{z};k) + \tilde{\phi}^+(z,\bar{z};k)\phi^-(z,\bar{z};k)) + \cdots \tag{3.82}$$

and one can also see that the $\delta(k-k')$ will come out if one choose to use $S^{\text{onshell}}(k,k')$ rather than $S^{\text{onshell}}(k)$. From this expression we can read off that there are two operators of dimension Δ_k with corresponding expectation values and sources:

$$\langle \mathcal{O}(z,\bar{z};k)\rangle = 2i\beta_+\Omega_2(z)\phi^-(z,\bar{z};k) \qquad \mathcal{J}(z,\bar{z};k) = \tilde{\phi}^+(z,\bar{z};k) \tag{3.83}$$

$$\langle \tilde{\mathcal{O}}(\ddagger,\bar{\ddagger};\|)\rangle = 2i\beta_+\Omega_2(z)\tilde{\phi}^-(z,\bar{z};k) \qquad \tilde{\mathcal{J}}(z,\bar{z};k) = \phi^+(z,\bar{z};k)$$

These two operators have the same two dimensional CFT scaling dimension, but are associated with different evolution in the τ direction.

A generic massless field Φ will be expressed as an integral over k, with the corresponding renormalized action being

$$S^{\text{ren}} = \int_{\mathcal{P}} dk S^{\text{ren}}(k) \tag{3.84}$$

$$= \int_{\mathcal{P}} dk\, \beta_+(k) \int_{\partial AdS_3} d\Omega_2 (\phi^+(z,\bar{z};k)\tilde{\phi}^-(z,\bar{z};k) + \tilde{\phi}^+(z,\bar{z};k)\phi^-(z,\bar{z};k))$$

The field Φ is thus dual to two continuous series of operators, labelled by k, whose sources and expectation values are given above in (3.83).

[2]In fact $\Delta_k = 1 + i\mathbb{R}^+$ if β_+ takes the value in $i\mathbb{R}^+$ and we assume that such k modes will form the necessary complete basis when one performs the mode decomposition following the discussion in section 3.1. There are also shadow operators given by the shadow transformation $\Delta_k \to 2 - \Delta_k$ so that the value of scale dimension will cover the whole principal series.

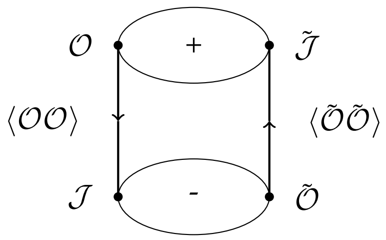


FIGURE 3.2: Propagators between two copies of operators are illustrated in the figure. Each disk represents one copy of AdS_3 hyperboloid with S^2 boundry drawn as a circle. + and - represent that the operators are obtained from the decomposition of the τ^{β_+-1} or τ^{β_--1} modes.

3.2.3 Correlation Functions

In this section we are going to study correlation functions in the context of Flat/CFT in a more precise way. Propagators for free fields $\langle \mathcal{O}\mathcal{O}\rangle$, $\langle \tilde{\mathcal{O}}\tilde{\mathcal{O}}\rangle$ are deduced and also represented in the language of diagrams. For higher point correlation functions in an interacting theory, the interactions are described by internal vertices of the diagrams. We use the $\Phi^3(X)$ interaction as an example to see how the operators of different scale dimensions are coupled with each other.

Following the previous approach in the context of AdS/CFT, we again choose to decompose the bulk fields into the bulk-boundary propagator

$$\begin{aligned}\Phi(\tau,\rho,z,\bar{z}) \;&=\; \frac{1}{2\sqrt{2}}\int_{\mathcal{P}} dk \int_{S^2} dz'd\bar{z}' \,\big(\tau^{\beta_+-1}K(\rho,z;\, z',\beta_+)\phi^+(z,\bar{z};k) \\ &\qquad +\tau^{\beta_--1}K(\rho,z;\, z',\beta_+)\tilde{\phi}^+(z,\bar{z};k)\big), \qquad (3.85)\end{aligned}$$

in which $\partial AdS_3 = S^2$ and $\phi^+(z,\bar{z};k)$, $\tilde{\phi}^+(z,\bar{z};k)$ can be treated as a pair of sources on the boundary as introduced in 3.83. Given such expression, from the onshell action

$$S^{\text{onshell}}(\Phi) = -\frac{1}{2}\int_0^\infty d\tau \int_{S^2} d\Omega_2\; R^2\sqrt{1+R^2}\;(\Phi(\tau,\rho,z,\bar{z})\partial_\rho\Phi(\tau,\rho,z,\bar{z}))_{\rho=R} \quad (3.86)$$

we have the k-mode component

$$S^{\text{onshell}}(k) = \frac{\Delta_k}{2\pi}\int_{S^2}\int_{S^2} d^2z d^2z' \frac{\phi^+(z,\bar{z};k)\tilde{\phi}^+(z,\bar{z};k)}{|z-z'|^{2\Delta_k}}, \quad (3.87)$$

in which we have integrated out the τ variable and the orthorgonal relations for the τ-modes are also applied. After performing the holographic renormalisation introduced in the previous section, the counterterm is then deduced to be

$$S^{\text{ct}}(\Phi) = -\frac{1}{2}(\Delta_k-2)\int_0^\infty d\tau \int_{S^2} d\Omega_2\; R^2\sqrt{1+R^2}\;(\Phi(\tau,\rho,z,\bar{z})\Phi(\tau,\rho,z,\bar{z}))_{\rho=R} + \cdots \quad (3.88)$$

with k-mode

$$S^{\rm ct}(k) = -\frac{1}{2\pi}(\Delta_k - 2)\int_{S^2}\int_{S^2} d^2z d^2z' \frac{\phi^+(z,\bar{z};k)\tilde{\phi}^+(z,\bar{z};k)}{|z-z'|^{2\Delta_k}}. \tag{3.89}$$

Given the Flat/CFT dictionary, in order to obtain the two-point function of the operator $\mathcal{O}$, we need to do the variation with respect to the corresponding source $\mathcal{J} = \tilde{\phi}^+$ twice therefore get

$$\langle \mathcal{O}(z,\bar{z};k)\mathcal{O}(z',\bar{z}';k)\rangle = \frac{i\delta^2 S^{\rm ren}}{\delta\tilde{\phi}^+(z)\tilde{\phi}^+(z')} = \int d^2z'' \frac{\delta\phi^+(z'',\bar{z}'';k)}{\delta\tilde{\phi}^+(z,\bar{z};k)} \frac{c_k}{|z''-z'|^{2\Delta_k}}, \tag{3.90}$$

in which $c_k = 2i(1-\Delta_k)/\pi$. The variation between two functions are not well defined while at least we should note that such value could not be zero since ϕ^+ and $\tilde{\phi}^+$ are not independent. Expanding them in terms of spherical harmonics, one will see that the variation of the two sources with respect to the basis can be written as

$$\delta\phi^+(z,\bar{z};k) = \sum_{l\neq 0,m} a^+_{lm}(k)\delta Y^l_m(z,\bar{z}) \qquad \delta\tilde{\phi}^+(z,\bar{z};k) = \sum_{l\neq 0,m} a^-_{lm}(k)\delta Y^l_m(z,\bar{z}) \tag{3.91}$$

in which the coefficients $a^{\pm}_{lm}(k)$ come from the decomposition of the bulk fields Φ and they are determined by assigning data on the Cauchy hypersurface chosen as the initial time [3]. More discussion of the coefficients can be found in Appendix G or Section 3.3. Physically, we can treat the deviation of the basis δY^l_m from the spherical harmonics as the deformation of the background geometry away from the flat case. Therefore one can define the variation between two sources as

$$\frac{\delta\phi^+(z,\bar{z};k)}{\delta\tilde{\phi}^+(z',\bar{z}';k)} := \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^+_{lm}(k)}{a^-_{lm}(k)}\delta(z-z'), \tag{3.92}$$

in which the factor $N_k = \sum_{l\neq 0,m} 1$ is introduced for normalization and one can interpret it as the measure of the discrete parameter space (l,m). Following such convention, then we obtain the two-point function

$$\langle \mathcal{O}(z,\bar{z};k)\mathcal{O}(z',\bar{z}';k)\rangle = \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^+_{lm}(k)}{a^-_{lm}(k)}\frac{c_k}{|z-z'|^{2\Delta}}, \tag{3.93}$$

and

$$\langle \tilde{\mathcal{O}}(z,\bar{z};k)\tilde{\mathcal{O}}(z',\bar{z}';k)\rangle = \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^-_{lm}(k)}{a^+_{lm}(k)}\frac{c_k}{|z-z'|^{2\Delta}}. \tag{3.94}$$

These two kinds of propagators carry the dynamical information of the physical system in the bulk. From the boundary point of view, they describe the coupling of the two series of operators and we represent such relation in Fig.3.2. For the higher point

[3] Actually, one should further impose Lorentz invariance, causality condition and the cluster decomposition principle on these coefficients when using quantum field theory to calculate scattering amplitudes of particles.

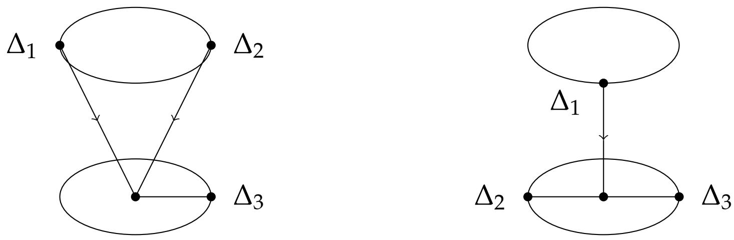


FIGURE 3.3: For $\lambda\Phi^3(X)$ interaction, two kinds of k-mode contribution for the three point function $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$are shown in the figure. The diagram on the left represents mode contribution like $\phi_1\phi_2\tilde{\phi}_3$ while the mode contribution like $\phi_1\tilde{\phi}_2\tilde{\phi}_3$ are shown on the right.

correlation functions, one needs to take the interaction of particles into consideration. Suppose that we have turned on the Φ^3 interaction of coupling constant λ, then we can write the action as

$$\lambda \int d^4X\,\Phi^3(X) = \lambda \int_{AdS_3} d^3x\sqrt{g} \int_{\mathcal{P}} dk_1dk_2dk_3 \int_0^\infty d\tau\frac{1}{\tau}(\tau^{\beta_+^1+\beta_+^2-\beta_+^3-1}\phi(z,\bar{z},\rho;k_1) \times\phi(z,\bar{z},\rho;k_2)\tilde{\phi}(z,\bar{z},\rho;k_3) + \tau^{\beta_+^1-\beta_+^2-\beta_+^3-1}\phi(z,\bar{z},\rho;k_1)\tilde{\phi}(z,\bar{z}\rho;k_2)\tilde{\phi}(z,\bar{z}\rho;k_3)+\cdots), \tag{3.95}$$

in which we have decomposed the fields into the integral over k-modes and collected all the τ-modes. To discuss the integral over τ modes in a more precise way, we write the value of β_+ as a complex number into the real and imaginary part

$$\sqrt{1+k_i^2} \equiv \beta_+^i = \gamma_i + ip_i, \tag{3.96}$$

therefore the integral, taking the first one $\phi_1\phi_2\tilde{\phi}_3$ for example, becomes

$$\int_0^\infty d\tau\frac{1}{\tau}\,\tau^{\gamma_1+\gamma_2-\gamma_3-1}e^{i(p_1+p_2-p_3)\ln\tau} \sim \delta(p_1+p_2-p_3) \tag{3.97}$$

after imposing the condition for the real part

$$\gamma_1+\gamma_2-\gamma_3 = 1. \tag{3.98}$$

In such the case, the interacting part in the action then can be reduced to

$$\delta(p_1+p_2-p_3)\,\lambda \int_{AdS_3} d^3x\sqrt{g}\,\phi(z,\bar{z},\rho;k_1)\phi(z,\bar{z},\rho;k_2)\tilde{\phi}(z,\bar{z},\rho;k_3) \tag{3.99}$$

and its contribution to the three-point function is shown in the left hand side Figure 3.3. For the $\phi_1\tilde{\phi}_2\tilde{\phi}_3$ contribution in (3.95), if one imposes the condition

$$\gamma_1-\gamma_2-\gamma_3 = 1 \tag{3.100}$$

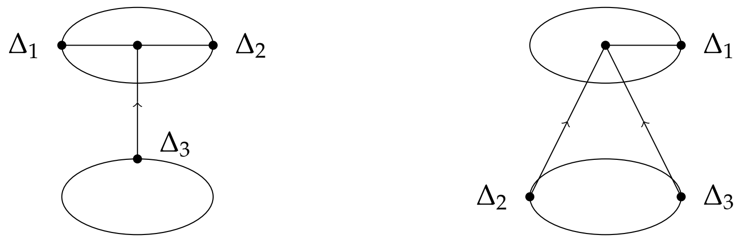


FIGURE 3.4: For $\lambda\Phi^3(X)$ interaction, two kinds of k-mode contribution for the three point function $\langle\tilde{\mathcal{O}}\tilde{\mathcal{O}}\tilde{\mathcal{O}}\rangle$ are shown in the figure. The diagram on the left represents mode contribution like $\phi\phi\tilde{\phi}$ while the mode contribution like $\phi\tilde{\phi}\tilde{\phi}$ are shown on the right.

therefore the interaction takes the form

$$\delta(p_1 - p_2 - p_3)\, \lambda \int_{AdS_3} d^3x\sqrt{g}\phi(z,\bar{z},\rho;k_1)\tilde{\phi}(z,\bar{z},\rho;k_2)\tilde{\phi}(z,\bar{z},\rho;k_3) \tag{3.101}$$

and the diagram is shown in the right hand side of Figure 3.3. In the figure, we have seen that the internal vertex is inserted in the $-$ disk since we are calculating the three point function $\langle\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ generated by the source $\tilde{\phi}^+$ living on the - disk. For the three point function $\langle\tilde{\mathcal{O}}\tilde{\mathcal{O}}\tilde{\mathcal{O}}\rangle$ we can also show the $\phi\phi\tilde{\phi}$ and $\phi\tilde{\phi}\tilde{\phi}$ interaction in terms of diagrams but now the internal vertex is inserted on the $+$ disk shown in Figure 3.4.

It is interesting to note that the diagrams introduced here can be treated as the intermediate between Feynman and Witten diagrams. If one collapses the two disks in the diagram, i.e ignoring the dynamical or the causal structure of the system, then we will obtain the Witten diagram which is often illustrated as a single disk. From the other hand, if one tries to sum over all the diagrams of different k modes, then one will recover the Feynman diagrams which enable us to study the scattering amplitudes for particles.

3.2.4 Holographic Dictionary for Onshell Scalar Fields

In this section we collate the results above and summarise the process for reading off the holographic data corresponding to an onshell scalar field $\Phi(\tau,\rho,z,\bar{z})$. In general the Flat/CFT dictionary is given by

$$\exp\left(iS^{\text{ren}}(\Phi)\right) = \left\langle \exp\ -\int_{S^2}\int_{\mathcal{P}} \left(\mathcal{J}_\Delta\ \mathcal{O}_\Delta + \tilde{\mathcal{J}}_\Delta\tilde{\mathcal{O}}_\Delta\right) \right\rangle_{CFT}. \tag{3.102}$$

To map the data between two sides, we first express the scalar field as a linear superposition of frequency modes, i.e.

$$\Phi(\tau,\rho,z,\bar{z}) = \int_{\mathcal{P}} dk\tau^{\beta_+-1}\phi(\rho,z,\bar{z};k) + \int_{\mathcal{P}} dk\tau^{\beta_--1}\tilde{\phi}(\rho,z,\bar{z};k) \tag{3.103}$$

where following (3.75) the two classes of modes can be expressed as

$$\begin{aligned}\phi(\rho,z,\bar{z};k) &= \phi(\rho,z,\bar{z};\beta_+)+\phi(\rho,z,\bar{z};\beta_-);\\ \tilde{\phi}(\rho,z,\bar{z};k) &= \tilde{\phi}(\rho,z,\bar{z};\beta_+)+\tilde{\phi}(\rho,z,\bar{z};\beta_-).\end{aligned} \tag{3.104}$$

These fields have asymptotic expansions

$$\begin{aligned}\phi(\rho,z,\bar{z};k) &= \rho^{\beta_+-1}\phi^+(z,\bar{z};k)+\rho^{\beta_--1}\phi^-(z,\bar{z};k)+\cdots\\ \tilde{\phi}(\rho,z,\bar{z};k) &= \rho^{\beta_+-1}\tilde{\phi}^+(z,\bar{z};k)+\rho^{\beta_--1}\tilde{\phi}^-(z,\bar{z};k)+\cdots\end{aligned} \tag{3.105}$$

from which one can read off expectation values and sources according to:

$$\begin{aligned}\langle\mathcal{O}(z,\bar{z};k)\rangle &= -2i\beta_+\Omega_2(z)\phi^-(z,\bar{z};k) \qquad \mathcal{J}(z,\bar{z};\beta)=\tilde{\phi}^+(z,\bar{z};k)\\ \langle\tilde{\mathcal{O}}(\ddagger,\bar{\ddagger};\|)\rangle &= -2i\beta_+\Omega_2(z)\tilde{\phi}^-(z,\bar{z};k) \qquad \tilde{\mathcal{J}}(z,\bar{z};k)=\phi^+(z,\bar{z};k)\end{aligned} \tag{3.106}$$

The decomposition of the field (3.103) follows from the orthogonality relations:

$$\begin{aligned}\phi(\rho,z,\bar{z};k) &= \frac{1}{2\pi}\int_0^\infty d\tau\;\tau^{\beta_-}\Phi(\tau,\rho,z,\bar{z})\\ \tilde{\phi}(\rho,z,\bar{z};k) &= \frac{1}{2\pi}\int_0^\infty d\tau\;\tau^{\beta_+}\Phi(\tau,\rho,z,\bar{z}).\end{aligned} \tag{3.107}$$

To calculate the two-point function and reduce the data to single AdS surface, we need to check the expression of $\phi(\rho,z,\bar{z};\beta_\pm)$ and $\tilde{\phi}(\tau,\rho,z,\bar{z};\beta_\pm)$ explicitly. Given the AdS modes as the basis, ϕ, $\tilde{\phi}$ are characterised by the coefficient $a^+_{lm}(k)$ $a^-_{lm}(k)$ written as

$$\phi^\pm(\rho,z,\bar{z};k) = \sum_{lm} a^+_{lm}(k)\;\phi_l(\rho;\beta_\pm)Y^l_m(z,\bar{z}) \tag{3.108}$$

$$\tilde{\phi}^\pm(\rho,z,\bar{z};k) = \sum_{lm} a^-_{lm}(k)\;\phi_l(\rho;\beta_\pm)Y^l_m(z,\bar{z}), \tag{3.109}$$

where we have chosen the normalisation for the spatial function as $\phi_l(\rho;\beta_+)=\rho^{\beta_+-1}+\cdots$. In such case, one can then write the sources in terms of spherical harmonic functions as

$$\phi^+(z,\bar{z};k) = \sum_{l,m} a^+_{lm}(k)Y^l_m(z,\bar{z}) \tag{3.110}$$

$$\tilde{\phi}^+(z,\bar{z};k) = \sum_{l,m} a^-_{lm}(k)Y^l_m(z,\bar{z}). \tag{3.111}$$

and the two copies of propagators are given by

$$\langle\mathcal{O}(z,\bar{z};k)\mathcal{O}(z',\bar{z}';k)\rangle = \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^+_{lm}(k)}{a^-_{lm}(k)}\frac{c_k}{|z-z'|^{2\Delta}} \tag{3.112}$$

$$\langle\tilde{\mathcal{O}}(z,\bar{z};k)\tilde{\mathcal{O}}(z',\bar{z}';k)\rangle = \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^-_{lm}(k)}{a^+_{lm}(k)}\frac{c_k}{|z-z'|^{2\Delta}}. \tag{3.113}$$

Such results work for Minkowski spacetime. For asymptotically Minkowski spacetime, one needs to do the harmonics analysis and the results are shown in Appendix G as (G.133) and (G.134).

Moreover, for higher point functions and theories with interaction. It is convenient to represent the correlation functions using two copies of disks labeled by + and -. For the correlators constructed out of the operator $\mathcal{O}$, the interactions are described by the internal vertices inserted on the - disk while for the $\tilde{\mathcal{O}}$ correlators, the points will be inserted on the + disk, i.e, all the internal vertices of each diagram can only exist in one of the disk. For the external legs, the one connects two points on a single disk is described by the standard AdS/CFT propagator. While for the ones that connect two disks, we need to take the extra factors constructed out of the coefficients $a^{\pm}_{lm}(k)$ into consideration. If we assume that the legs between two disks have directions and they always flow into the internal points, the leg that starts from the ith external vertex on the + disk then ends at the internal point the on - disk will contribute to a factor

$$i(+) \longrightarrow \bullet(-) \qquad\qquad \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^{+}_{lm}(k_i)}{a^{-}_{lm}(k_i)} \tag{3.114}$$

and the one starts from the ith external vertex on the - disk then ends at the internal point the on + disk will contribute to a factor

$$i(-) \longrightarrow \bullet(+) \qquad\qquad \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^{-}_{lm}(k_i)}{a^{+}_{lm}(k_i)}. \tag{3.115}$$

3.3 Shock Waves and Their Holographic Interpretation

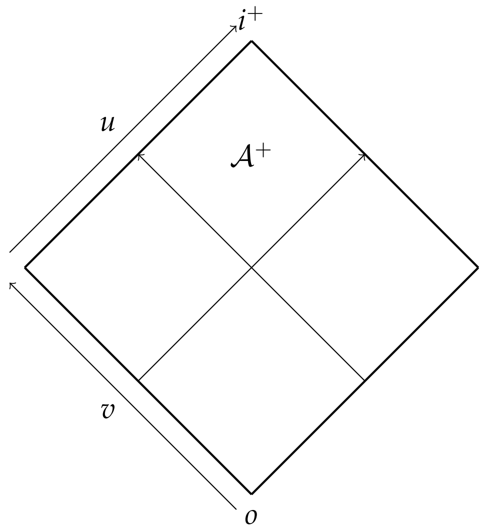


FIGURE 3.5: In and out going shock waves propagating in region $\mathcal{A}^+$ are shown in the figure.

In this section we consider the holographic interpretation of a shock wave. Here in our context, the shock waves are scalar shocks that could either distribute on a spherical shell or localise along the null geodesic of a massless particle. For the spherical shock wave, it describes the wave caused by a point like source then propagates in spacetime following a homogeneous way. For the second kind of shock wave, it could be treated as the approximation for the signals traveling at the speed of light like the laser beam [4]. In fact, to make photons trapped in the beam, one should take gravity effects into consideration and it turns out such shock wave will induce backreactions on the metric studied in Aichelburg and Sexl (1971); Dray and 't Hooft (1985). Then the massive case is also studied in a perturbative way. However, in our situation, the shape of shock waves will be less important while the ingoing and outgoing behaviour of the wave will be crucial. To construct the shock wave solutions, we start from the Minkowski metric written as

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2\, d\Omega_2^2 \\ &= -dudv + r^2\, d\Omega_2^2, \end{aligned} \tag{3.116}$$

where t, r are the time and radial directions and $d\Omega_2^2$ is the standard 2 sphere metric. In the second line, the retarded and advanced coordinates u, v are defined as

$$u = t - r, \qquad v = t + r. \tag{3.117}$$

A massless particle which is described by the field Φ satisfies

$$-4\partial_u \partial_v \Phi + \frac{1}{r^2} \Box_{S^2} \Phi = 0. \tag{3.118}$$

Let us consider a spherically symmetric solution so that the equation reduces to

$$\partial_u \partial_v \Phi = 0, \tag{3.119}$$

and general solutions are given by

$$\Phi(u, v) = \phi(u) + \tilde{\phi}(v) \tag{3.120}$$

where ϕ and $\tilde{\phi}$ are arbitrary functions of u, v.

An interesting physical solution is the spherical shock wave. A shock wave emitted from the boundary and propagating along the null ray as illustrated in Figure 3.5 is described by Φ_s^{in}

$$\Phi_s^{in}(v) = \phi_0\, \delta(v - v_0), \qquad v_0 > 0. \tag{3.121}$$

[4]We assume such method can be generalised to gauge fields or we are dealing with a high energy beam of bosonic particles with small mass. Maybe for massive particles, one should consider the ingoing and outgoing wavepackets.

We can view the shock wave solution as a specific linear combination of plane wave solutions. Furthermore, to make the wave really localise along the null ray, one could consider the gravitational shock wave $\Phi_g^{in} = \phi_0 \delta(v - v_0)\delta(z - z_0)$ while it is not a solution for KG equation in flat spacetime and it only exists when the gravitational effect is taken into consideration. Here, to study the flat/CFT dictionary in a simple way, we choose to use the spherical shock wave as an example to perform the calculation and the results for Φ_s^{in} is obtained by inserting the factor $\delta(z - z_0)$ behind [5].

To express the shock wave in terms of modes adapted to the hyperbolic slicing, we need to transform the coordinates (3.116) into Milne coordinates using

$$t^2 - r^2 = \tau^2, \qquad \rho\tau = r, \tag{3.122}$$

in which we should note that the Milne coordinates will only cover the region $\mathcal{A}^+$ if both (τ, ρ) are required to be positive $\rho, \tau \geq 0$. The near light cone region is described by $\tau \to 0$ and the asymptotic region is given by $\tau \to \infty$. In Milne coordinates, the shock wave can be expressed as

$$\Phi_s^{in} = \phi_0\, \delta(\rho\tau + \tau\sqrt{1+\rho^2} - v_0) = \phi_0 \delta(\tau e^{\eta} - v_0). \tag{3.123}$$

We can now decompose this solution into modes as described in the previous section, resulting in

$$\begin{aligned} \Phi^{in}(\rho, z, \bar{z}; \beta_+) &= \frac{\phi_0}{2\pi} v_0^{\beta_-} e^{-(1+\beta_-)\eta}, \\ \Phi^{in}(\rho, z, \bar{z}; \beta_-) &= \frac{\phi_0}{2\pi} v_0^{\beta_+} e^{-(1+\beta_+)\eta}. \end{aligned} \tag{3.124}$$

The fields are independent of the sphere coordinates. One can immediately read off the coefficients of the asymptotic expansion using the relation $\rho = \sinh\eta$ as

$$\begin{aligned} \phi^+(z, \bar{z}; k) &= \frac{\phi_0}{2\pi} 2^{\beta_+ - 1} v_0^{\beta_-} \qquad \phi^-(z, \bar{z}; k) = 0; \\ \tilde{\phi}^+(z, \bar{z}; k) &= 0 \qquad \tilde{\phi}^-(z, \bar{z}; k) = \frac{\phi_0}{2\pi} 2^{-\Delta} v_0^{\beta_+}. \end{aligned} \tag{3.125}$$

This means that the operators $\mathcal{O}(z, \bar{z}; k)$ has no source or expectation value, but the operators $\tilde{\mathcal{O}}(z, \bar{z}; k)$ have both: the sources are $\phi^+(z, \bar{z}; k)$ while

$$\langle \tilde{\mathcal{O}}(z, \bar{z}; k) \rangle = -i\beta_+ \frac{\phi_0}{\pi} 2^{-\Delta} v_0^{\beta_+}. \tag{3.126}$$

[5] Here we assume that the localised shock waves propagate on the background where the holography principle still works.

It is straightforward to repeat the same exercise for a shock wave propagating along an orthogonal null ray i.e.

$$\Phi_s^{out}(u) = \phi_0\, \delta(u - u_0), \qquad u_0 > 0. \tag{3.127}$$

This can be expressed in terms of the Milne coordinates as

$$\Phi_s^{out} = \phi_0\, \delta(\tau\sqrt{1+\rho^2} - \tau\rho - u_0) = \phi_0 \delta(\tau e^{-\eta} - u_0). \tag{3.128}$$

Decomposing into modes one finds

$$\begin{aligned} \Phi^{out}(\rho, z, \bar{z}; \beta_+) &= \frac{\phi_0}{2\pi} u_0^{\beta_-} e^{(1+\beta_-)\eta}, \\ \Phi^{out}(\rho, z, \bar{z}; \beta_-) &= \frac{\phi_0}{2\pi} u_0^{\beta_+} e^{(1+\beta_+)\eta}. \end{aligned} \tag{3.129}$$

One can then read off the coefficients of the asymptotic expansion using the relation $\rho = \sinh\eta$ as

$$\begin{aligned} \phi^+(z, \bar{z}; k) &= 0 \qquad \phi^-(z, \bar{z}; k) = \frac{\phi_0}{2\pi} 2^{\Delta} u_0^{\beta_+}; \\ \tilde{\phi}^+(z, \bar{z}; k) &= \frac{\phi_0}{2\pi} 2^{1-\beta_+} u_0^{\beta_-} \qquad \tilde{\phi}^-(z, \bar{z}; k) = 0. \end{aligned} \tag{3.130}$$

This means that the operators $\tilde{\mathcal{O}}(\ddagger, \bar{\ddagger}; \|)$ has no source or expectation value, but the operators $\mathcal{O}(z, \bar{z}; k)$ have both: the sources are $\tilde{\phi}^+(z, \bar{z}; k)$ while

$$\langle O(z, \bar{z}; k) \rangle = -i\beta_+ \frac{\phi_0}{\pi} 2^{\Delta} u_0^{\beta_+}. \tag{3.131}$$

Thus we can understand the two sets of dual operators as describing modes propagating in (u, v) directions respectively:

$$\begin{aligned} \Phi(u) &\to \{\mathcal{O}(z, \bar{z}; k), \tilde{\phi}^+(z, \bar{z}; k)\}; \\ \Phi(v) &\to \{\tilde{\mathcal{O}}(\ddagger, \bar{\ddagger}; \|), \prec^+(\ddagger, \bar{\ddagger}; \|)\}. \end{aligned} \tag{3.132}$$

As for the two point functions, the structure will become complicated and one needs to take the gravitational effect into consideration. For the spherical shock wave, it is the solution for KG equation in Minkowski but the two-point function will become trivial since the solution takes constant value on the sphere and the method we have introduced in the section 3.2 will not work. It does not mean that the dual theory on the boundary will become trivial while we need to take the gravity backreaction into consideration in order to investigate the correlation function at higher order if one treats the constant ϕ_0 as a small parameter. After backreaction from the matter, the metric then becomes

$$G'_{\mu\nu} = G_{\mu\nu} + \delta G_{\mu\nu}, \tag{3.133}$$

which G is the metric for Minkowski and the deformation caused by the matter is denoted as δG. They are governed by Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R\, G'_{\mu\nu} = T_{\mu\nu}, \tag{3.134}$$

in which $R_{\mu\nu}$ is the Ricci curvature for G' and $T_{\mu\nu}$ is the stress tensor determined by the scalar profile and in our case it is the shock wave Φ_s thus we can see that the stress tensor is of the order ϕ_0^2. One can treat it as the Newtonian constant $\phi_0^2 \sim G_N$ which is not explicitly shown in the equation. From the above equation, one can also see that the deformation δG also goes as the order of ϕ_0^2 [6]. Given the deformed background then the scalar fluctuation $\delta\Phi$ on the shock wave profile is determined by the KG equation

$$\Box_{G'} \Phi' = 0, \tag{3.135}$$

where $\Phi' = \Phi_s + \delta\Phi$. Here we should note that, although Φ_s is constant on the celestial sphere while the fluctuation $\delta\Phi$ is not necessary constant and it depends on the further specification of the data at the initial time. Therefore both of the vacuum expectation value and source will receive correction of order ϕ_0^2 coming from $\delta\Phi$ and the two-point function now becomes

$$\langle \mathcal{O}(z;k)\mathcal{O}(z';k)\rangle_s = \langle \mathcal{O}(z;k)\mathcal{O}(z';k)\rangle + \phi_0^2\, F(z,z';k), \tag{3.136}$$

where $\langle \cdots \rangle_s$ represents that the operators are now inserted on the shock wave background rather than the Minkowski vacuum $\langle \cdots \rangle$ and the higher order correction is of the order ϕ_0^2, i.e. G_N. Its specific form is given by the function $F(z,z';k)$ determined by the variation $\delta\Phi$. From above discussion, we know that all the spherical solutions without considering gravity effect in Minkowski are degenerate from boundary point of view and one needs to consider the variation of the scalar field in order to distinguish all the spherical solutions. The broken of the spherical symmetry caused by the gravity effect will enable us to calculate two-point functions at leading order and then introduce subleading terms characterised by the function $F(z,z';k)$. For the localised shock wave, one needs to figure out the background and then check if the holography principle still works on such background, which depends on the definition of asymptotic flat as well as the ability of holography principle and such work goes beyond the scope of this thesis.

[6] In fact we have $\langle T^{\mathrm{CFT}}_{\mu\nu}\rangle \sim \delta G$ in which $T^{\mathrm{CFT}}_{\mu\nu}$ is the stress tensor of the dual CFT theory on the celestial sphere. The specific expression relies on the holographic renormalisation of Einstein-Hilbert action which has been done in Graham-Fefferman coordinates for AdS case de Haro et al. (2001b).

3.3.1 Coefficients

After the study of the dual correlation functions on the boundary. Here we will use the shock wave model as an example to study the bulk field in a direct way following the mode analysis introduced in section 3.1 and try to determine the coefficients of those modes. It is easier for the spherical shock waves since they are constant on the sphere and only the zero mode will contribute when performing the mode expansion. The analysis for localised shock wave will be harder since the analysis for the $l \geq 1$ mode will be difficult and we will leave the mode analysis for Φ_g for further investigation.

Massless Fields

From the discussion in section 3.1.3 and appendix F, we have seen that the zero mode $l = 0$ on the AdS hyperboloid has two independent solutions at large radius $\rho = \sinh\eta \to \infty$

$$\phi_0(\eta;\beta_+) = \frac{e^{\beta_+\eta}}{\sinh\eta}, \qquad \phi_0(\eta;\beta_-) = \frac{e^{\beta_-\eta}}{\sinh\eta}. \tag{3.137}$$

The regular solution at $\rho = 0$, denoted as $\phi_r(\eta;k)$, is the linear combination of them with ratio $C_0^-(k)/C_0^+(k) = -1$ [7] thus it can be written as

$$\phi_r(\eta;k) = \frac{1}{\sqrt{\pi}}\frac{\sinh\beta_+\eta}{\sinh\eta}. \tag{3.138}$$

One can check that ϕ_r is regular for arbitrary β_- since $\phi_r \sim \beta_+$ at $\eta = 0$. Here we are interested in the principal series case $\beta_+ = ik$ for $k \geq 0$ and we assume that the result for other value can be obtained by the analytic continuation of β_+.

For the ingoing waves, one has the expansion

$$\Phi^{in}(v) = \int_{\mathcal{P}} dk\ (a^+_{in}(k)\ \tau^{-1+\beta_+} + a^-_{in}(k)\ \tau^{-1+\beta_-})\ \phi_r(\eta,k), \tag{3.139}$$

in which $a^{\pm}_{in}(k)$ is the pair of coefficients that we are going to determine. To calculate these coefficients, one should first note the orthogonal relation

$$\int_{-\infty}^{+\infty} \sinh^2\eta\ \phi_r^*(\eta;k)\phi_r(\eta;k) = \delta(k-k'), \tag{3.140}$$

in which ϕ_r^* is the complex conjugate of ϕ_r. Given the above relation, one can project out the η dependent part by performing the integral

$$\phi_0 \int_{-\infty}^{+\infty} d\eta\ \delta(\tau e^{\eta} - v_0)\sinh^2\eta\ \phi_r^*(\eta;k) \tag{3.141}$$

[7] One can obtain this by the direct observation of the liner combination of $\phi_0(\eta;\beta_\pm)$ or by checking the formula of $C_0^{\pm}(k)$ for odd β in the Appendix F.

therefore coefficients $a^{\pm}(k)$ are then deduced to be

$$a_{in}^{+}(k) = \frac{\phi_0 v_0^{-\beta_+}}{4\sqrt{\pi}}, \qquad a_{in}^{-}(k) = -\frac{\phi_0 v_0^{\beta_+}}{4\sqrt{\pi}}, \tag{3.142}$$

in which we have omitted the correction term of order τ^2. The fact that we get extra terms in additional to the modes $\tau^{-1+\beta_\pm}$ implies that the basis we have chosen is not complete. Here we assume that the mode expansion is done near the Milne horizon thus $\tau \to 0$ and the higher order term will be subleading. For the outgoing shock wave, following similar procedure, one has the expansion

$$\Phi^{out}(u) = \int_{\mathcal{P}} dk \; (a_{out}^{+}(k)\; \tau^{-1+\beta_+} + a_{out}^{-}(k)\; \tau^{-1+\beta_-})\; \phi_r(\eta, k), \tag{3.143}$$

in which the corresponding coefficients $a_{out}^{\pm}(k)$ are determined to be

$$a_{out}^{+}(k) = \frac{\phi_0 u_0^{\beta_+}}{4\sqrt{\pi}}, \qquad a_{out}^{-}(k) = -\frac{\phi_0 u_0^{-\beta_+}}{4\sqrt{\pi}}. \tag{3.144}$$

Massive Fields

Now we turn to the study of massive particles. First we try to make the particle slightly massive and then investigate the perturbative behaviour of the solution around the spherical shock wave. Similar to the study of massive KG equation, we choose to write the equation of motion for massive particle as

$$(\partial_u \partial_v + \lambda M^2)\Phi_M(X) = 0, \tag{3.145}$$

in which M is a constant and λ is a small parameter that represents the mass of the particles is small. Then we can write down a particular set of solutions for the particles with high momentum to the first order of λ as

$$\Phi_M(X) = \phi_0 \delta(v - v_0) + \lambda f(u, v), \tag{3.146}$$

in which $f(u,v)$ is a function of u, v. To determine $f(u,v)$, one should substitute the solution into the equation, solve it order by order in λ, and then obtain

$$\Phi_M(X) = \phi_0 \delta(v - v_0) - u\; \lambda\; \phi_0 M^2\; \theta(v - v_0), \tag{3.147}$$

in which $\theta(v - v_0)$ is the step function supported in the region $v > v_0$. The step function correction term tells us that, by adding a small amount of mass, the shock wave will be no longer localised along some spherical shell and propagate along the null direction while it will have a tail spreading over the whole region $v > v_0$ [8]. If we define the coefficients of massive field expanded by the modes on AdS surfaces as $a^{\pm}_M(k)$, i.e.

$$\Phi_M(u,v) = \int_{\mathcal{P}} dk \; (a^+_M(k)\; \tau^{-1+\beta_+} + a^-_M(k)\; \tau^{-1+\beta_-})\; \phi_r(\eta,k). \tag{3.148}$$

Then one can write $a^{\pm}_M(k)$ by the order λ as

$$a^{\pm}_M(k) = a^{\pm}_0(k) + \lambda\; a^{\pm}_1(k) + \lambda^2 a^{\pm}_2(k) + \cdots, \tag{3.149}$$

in which $a^{\pm}_0(k)$ are the coefficients for massless case we have discussed before

$$a^{\pm}_0(k) = a^{\pm}_{in}(k) \tag{3.150}$$

and $a^{\pm}_i(k)$ are higher order terms. Taking the solution in (3.147) for example, to calculate $a^{\pm}_1(k)$ one should evaluate the integral

$$\int_{-\infty}^{+\infty} d\eta \;\; \sinh^2\eta \; e^{-\eta}\tau \; \phi^*_r(\eta;k)\theta(\tau e^{\eta} - v_0), \tag{3.151}$$

in which we still use the massless solution as the basis when performing the perturbative expansion. The above integral will vanish when τ goes to zero thus one can conclude that

$$a^{\pm}_1(k) = 0, \tag{3.152}$$

which tells us coefficients are stable at the massless case. It shows that, for the modes we are interested in, the mass of particle will not play a crucial role and make significant contribution thus the shock wave model is still a good approximation for particles with small mass.

3.3.2 Cauchy Problem and Scattering

In section 3.2, we start from the holographic renormalisation for the onshell action in region $\mathcal{A}^+$ then conclude that the theory in flat spacetime $\mathcal{A}^+$ is dual to the CFT on the celestial sphere S^+_2 located at the future null boundary. To study the whole Minkowski space, in principle, one should consider the action in the region $\mathcal{A}^+ \cup \mathcal{D} \cup \mathcal{A}^-$ while it was conjectured in the work de Boer and Solodukhin (2003) that all the information of Minkowski could be classified by specifying the data on the two copies of AdS hyperboloid in $\mathcal{A}^+$ and $\mathcal{A}^-$ therefore it is enough to fully reconstruct the bulk theory using the holographic CFT data on the celestial sphere S^+_2 and S^-_2. In particular, the scattering amplitudes in Minkowski can also be constructed by studying the states on these

[8]However, comparing with the non-stable behaviour of the wavepacket, we expect solution (3.147) will only work as an approximation for the evolution of high momentum particles at the early stage, and there should also be spatial dependence part in the function $\delta(v - v_0)$ to reflect the fact that the massive particles with high momentum tend to move along a timelike trajectory near the light cone.

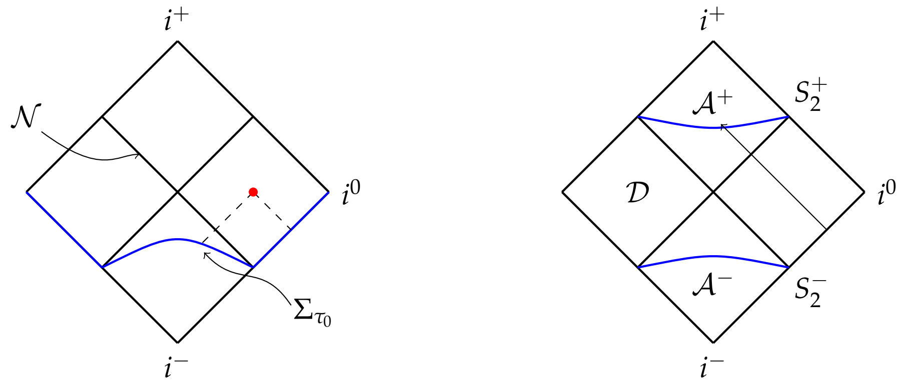


FIGURE 3.6: A single AdS surface together with part of null boundary form a Cauchy surface for the whole Minkowski spacetime shown the left hand side. For example, to determine the field configuration at the red point, one needs to specify the data on both of the AdS hyperboloid and null boundary. The shock wave will transform the data from the null boundary to the other AdS surface in region $\mathcal{A}^+$ so that two copies of AdS surfaces are equivalent to a Cauchy surface, which is illustrated in the right figure.

two AdS hyperboloid although we know the fact that they are not the standard Cauchy surface. Here based on the study of AdS and dS modes, we will reconsider the distribution of information in Minkowski and a physical proof of the above conjecture will also be illustrated by doing a thought experiment on the shock wave model.

Following the principle of the mode expansion, to study the local behaviour of the solution $\Phi_M(X)$ in region $\mathcal{A}$ denoted as $\Phi_M^A(X)$, one can expand the solution in terms of modes propagating on the AdS slicing. As we have studied in the section 3.1, the solution Φ_M^A can be represented by the linear combination of modes with effective mass k provided that there is a set $\mathcal{P}$ of k in which all the modes together form a complete basis of the solution space, written as

$$\Phi_M^A(X) = \sum_l \int_{\mathcal{P}_A} dk\, a_l(k)\, \psi^A(\tau;k) F_{kl}^A(\rho, z, \bar{z}), \tag{3.153}$$

in which $a_l(k)$ are coefficients and the label l is used to represent the other internal variables. $F_{kl}^A(\rho, z, \bar{z}) = \phi_l(\rho;k) Y_m^l(z, \bar{z})$ are the spatial modes introduced before while $(\rho, z, \bar{z})$ is the coordinate of AdS hyperboloid. For the same reason, we can choose to decompose the solution in region $\mathcal{D}$, denoted as $\Phi_M^D(X)$, into dS modes $\psi^D(\rho;k) F_{kl}^D(\tau, z, \bar{z})$ thus it can be written as

$$\Phi_M^D(X) = \sum_l \int_{\mathcal{P}_D} dk\, b_l(k)\, \psi^D(\rho;k) F_{kl}^D(\tau, z, \bar{z}), \tag{3.154}$$

in which we should note that the position of variable τ and ρ are switched since we are using them to label the timelike and spacelike direction.

Before imposing the initial condition of the solution $\Phi_M(X)$, we first consider the analytic continuation of the field $\Phi_M(X)$ from the region $\mathcal{A}^-$ into the region $\mathcal{D}$ via the null surface $\mathcal{N}$ shown in Figure 3.6. Given the field configuration $\Phi_M^A(X)$, one can perform the analytic continuation by making $k \to ik$ across the null surface $\mathcal{N}$ then obtain $\Phi_M^D(X)$. In terms of the coefficients, that is to say

$$\{a_l(k)\} = \{b_l(k)\} \tag{3.155}$$

in which we use the notion $\{\}$ to represent the information contained in the modes and the equal sign means that one can determine all the $b_l(k)$ s given the set of $a_l(k)$ or vice versa.

To study the initial condition, or to determine the coefficients, first we need to choose a proper codimension one surface to set up the initial data. For the field $\Phi_M^A(X)$, one can choose the AdS slicing $X^2 = -\tau_0^2$, denoted by Σ_{τ_0} as the Cauchy surface for region $\mathcal{A}^-$ therefore the field in region $\mathcal{A}^-$ is uniquely determined given the initial data f_i, g_i

$$\Phi_M(\tau_0, \rho, z, \bar{z}) = f_i(\rho, z, \bar{z}), \qquad n^i \partial_i \Phi_M = g_i(\rho, z, \bar{z}) \tag{3.156}$$

where n^i denotes the further unit normal of Σ_{τ_0}. For the field in region $\mathcal{D}$, the data on the surface Σ_{τ_0} is not enough for us to uniquely fix the field configuration $\Phi_M^D(X)$. One also needs to specify the data along the null boundary so that they form the Cauchy surface of the whole Minkowski together with the surface Σ_{τ_0}, which means one needs more data to determine Φ_M^D comparing to $\Phi_M^A(X)$. Since we have already known that fields in the region $\mathcal{A}$, $\mathcal{D}$ are fully determined by $\{a_l(k)\}$ and $\{b_l(k)\}$, we conclude that

$$\{a_l(k)\} \subset \{b_l(k)\}, \tag{3.157}$$

where the symbol $\subset$ means that one can determine all the coefficients $a_l(k)$ given the set of $b_l(k)$ while the other direction is not true anymore, which implies that there are modes not governed by the analytic continuation thus one has $\mathcal{P}_A \subset \mathcal{P}_D$.

Furthermore, based on the calculation in the previous section, we see that, for the massless particle, one can construct the shock wave as the solution of the Klein-Gordon equation. The shock waves propagate along the null direction and they are localised around the trajectory of the massless particles. Moreover, these shock waves that start from the null infinity then go through the AdS slicing surfaces in region $\mathcal{A}^+$ enable the exchange of information between the observer living in some particular AdS surface in region $\mathcal{A}^+$ and the observer on the null boundary. For example, the observer at the boundary can send the information of the initial position and momentum of the particle to the observer in region $\mathcal{A}^+$ via the shock wave and the observer in region $\mathcal{A}^+$ can read out these information by determining the coefficients $a^{\pm}(k)$. Thus two copies of AdS surface in region $\mathcal{A}^-$ and $\mathcal{A}^+$, respectively, form a structure that is equivalent to the Cauchy surface since we know that one AdS Surface in region $\mathcal{A}^-$ together with half of the null boundary $v > 0$ carry a complete set of data for one to determine the field configuration in the whole space time.

Chapter 4

Flat Holography for Gravity

In this chapter, we will develop the flat/CFT dictionary between the $d+2$ dimensional Einstein gravity and the d dimensional CFT on the celestial sphere. To study the gravitational theory on asymptotically flat background in a formal way, the first step is to specify the definition of asymptotic flatness. The general guideline for defining asymptotic flatness is that it must make the spacetime close enough to flat case while the deformations should at the same time contain enough non trivial physical contents for us to investigate. For example, the well studied Bondi gauge Bondi et al. (1962); Sachs (1962b,a) in gravity literature. To establish the flat/CFT dictionary, one also needs to construct a mapping of data between the bulk and boundary but there is no clue on how to decompose the bulk data in Bondi gauge. Noting that Minkowski spacetime is foliated by AdS hyperboloids and the specific map of data for AdS/CFT is clearly studied in terms of Fefferman Graham coordinates Fefferman and Graham (1985); Graham (2000); Fefferman and Graham (2011); Henningson and Skenderis (1998); de Haro et al. (2001b); Balasubramanian and Kraus (1999), therefore in this thesis we choose to define the asymptotically flat spacetime in terms of Fefferman Graham like coordinates written as

$$\begin{aligned} ds^2 &= -d\tau^2 + \tau^2 \left(\frac{d^2\rho}{4\rho^2} + \rho g_{ij}(\rho, x) dx^i dx^j \right) \\ &\quad + \frac{\tau}{\rho^2} m(\rho, x) d^2\rho + \tau\rho\, \sigma_{ij}(\rho, x) dx^i dx^j + \tau A_i(\rho, x) d\rho dx^i + \cdots , \end{aligned} \tag{4.1}$$

where the first line is the leading contribution to the spacetime coming from the asymptotic AdS slices and the second line is the subleading contribution for large τ. We will see that most of the non trivial physical results in addition to the AdS/CFT duality would come from the existence of such subleading sector. In section E, we will explore such gauge in a careful way by determining the asymptotic symmetries and solving Einstein equation at different order of τ and ρ. The strategy here is that we choose to expand the parameters $g_{ij}(\rho, x), m(\rho, x), \sigma_{ij}(\rho, x), A_i(\rho, x)$ in terms of $1/\rho$ and determine the constraints between the coefficients $g_{ij}^{(2k)}(x), m^{(2k)}(x), \sigma_{ij}^{(2k)}(x), A_i^{(2k)}(x)$ by solving the Einstein equations $R_{\mu\nu} = 0$ at the zero and first order of $1/\tau$.

After the study of asymptotically flat spacetime, for $d = 2$, we propose the flat/CFT dictionary as

$$\exp\left(iS_{\text{gr,ren}}[G]\right) = \left\langle \exp \frac{1}{2}\int_{S^2} d^2x\sqrt{\bar{G}}\,\bar{G}^{ij}\,T_{ij}\right\rangle \tag{4.2}$$

where G and $\bar{G}$ are the bulk metric and the background metric for the CFT, respectively. T_{ij} is the energy-stress tensor of the boundary CFT. To make such dictionary well defined and work the same way as the AdS/CFT dictionary, one further needs to perform proper renormalisation procedure on the bulk gravitational action $S_{\text{gr}}[G]$ making it finite and to specify the exact map of data between two sides. Here the exact map means that given the bulk metric we should be able determine boundary data $\bar{G}$ and T_{ij} or vice versa. These are two main obstacles during the development of flat holography dictionary and we will discuss them in section 4.2.

In the context of AdS/CFT, one needs to perform holographic renormalisation in order to obtain the finite renormalised action $S_{\text{gr,ren}}[G]$. The infinity coming from the integral over the whole spacetime is treated as the IR divergence and is regulated by choosing the AdS spatial radius ρ as the IR cut off. Here for flat spacetime, we have one more timelike non-compact direction labeled by τ and we simply choose to impose a bound L on it, i.e $\tau \in [0, L]$. After performing the holographic renormalisation in the given interval $[0, L]$, we obtain the renormalised gravitational action and then further propose the map of data between bulk and boundary as

$$\bar{G}_{ij} = g_{ij}^{(0)} + \frac{1}{L}\sigma_{ij}^{(0)} + \cdots, \tag{4.3}$$

where $\bar{G}_{ij} = g_{ij}^{(0)}$ is originally the AdS/CFT map and now we are taking the soft sector into consideration treating L as the energy cut-off of the boundary CFT. Using the flat/CFT dictionary, we obtain a specific expression for the energy stress tensor $\langle T_{ij}\rangle$, given by

$$\begin{aligned}\langle T_{ij}\rangle = \frac{iL^2}{16\pi G_N}(g_{ij}^{(2)} - g_{ij}^{(0)}\text{Tr}g_{(2)}) - \frac{iL}{8\pi G_N}\Big((\text{Tr}\,\sigma^{(2)} - \sigma_{kl}^{(0)}\,g_{(2)}^{kl})\,g_{ij}^{(0)} \\ +\text{Tr}g^{(2)}\sigma_{ij}^{(0)} - \sigma_{ij}^{(2)} + 2m^{(0)}(g_{ij}^{(2)} - g_{ij}^{(0)}\text{Tr}g^{(2)})\Big) + \cdots.\end{aligned} \tag{4.4}$$

Moreover, by considering the anomalies of the energy-stress tensor $\langle T_i^i\rangle$, the central charge is then determined to be

$$c = \frac{i3L^2}{4G_N} + \frac{iML3(\alpha-2)}{2G_N} \tag{4.5}$$

where $M = m^{(0)}$ and α are constants characterising the behaviour of asymptotically flat spacetime. The behaviour of the leading term is already argued in the work Cheung et al. (2017a); Pasterski and Verlinde (2022); Ogawa et al. (2023) and here we determine its value in a precise way. With the help of flat/CFT dictionary, we can in principle determine all the subleading contributions and in this thesis we just present the first order term.

The Einstein equations will impose constraints on the parameters in the metric and those constrains on the boundary will correspond to the Ward identity of the stress tensor. The story for AdS/CFT or for the hard sector is that the bulk equation of motion implies that the conservation of energy-stress tensor $\nabla^j \langle T_{ij} \rangle = 0$. By checking the Einstein equations at the first order, we find that it will not give us the conservation of the soft energy-stress tensor while we have

$$\bar{\nabla}^j \langle T_{ij} \rangle = \frac{iL}{16\pi G_N}(8\nabla_i m^{(2)} + 4M\nabla_i \mathrm{Tr} g^{(2)} - \sigma^{mk}_{(0)} \nabla_m g^{(2)}_{ki}), \tag{4.6}$$

where $\bar{\nabla}_i, \nabla_i$ are the covariant derivative with respect to $\bar{G}_{ij}, g^{(0)}_{ij}$, respectively. Thus one can interpret the soft modes as the radiation modes which generate flow of energy at the null boundary leading to a non conserved energy-stress tensor from the boundary point of view.

4.1 Fefferman Graham Gauge

In this section, we will start from the coordinate for asymptotically flat spacetime then recast it to the asymptotically AdS form. For the spacetime of dimension $d+2$, we first introduce the standard coordinate X^μ for $\mu = 0, 1, \cdots, d+1$ in which the flat metric takes the form $\eta_{00} = -1, \eta_{11} = \cdots \eta_{d+1,d+1} = 1$. The Euclidean AdS of dimension $d+1$ can be regarded as the hyperboloid embedded in the $d+2$ flat spacetime given by the relation [1]

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 = -\tau^2, \tag{4.7}$$

in which $\tau \geq 0$ is the radius of the AdS surface.

Motivated by such foliation, we now choose to write the asymptotically flat spacetime given by the metric $G(X)$ into the form

$$ds^2 = G_{\mu\nu}(X) dX^\mu dX^\nu = -d\tau^2 + \tau^2 \hat{G}_{ab}(\tau, y) dy^a dy^b,$$

where $a, b = 1, \cdots, d+1$ and we have fixed the gauge in order to make the $d\tau dy^a$ term vanish. One can always manage to find such gauge by performing local diffeomorphism transformation. For flat spacetime, the metric $\hat{G}_{ab}$ will be independent of τ and reduced to the metric for AdS_{d+1} described by y while it will become asymptotically

[1] Such patch is called Milne wedge for Minkowski while one can study Rindler wedge by the analytic continuation of the radius $\tau \to i\tau$. The Rindler wedge is sliced by dS hypersurfaces.

AdS when the spacetime is near flat. At the null boundary, we can expand the metric $\hat{G}_{ab}(\tau, y)$ around $1/\tau = 0$ written as

$$\hat{G}_{ab}(\tau, y) = \hat{G}^0_{ab}(y) + \hat{G}^1_{ab}(y)\frac{1}{\tau} + \cdots = \hat{G}^0_{ab} + h_{ab}(\tau, y), \tag{4.8}$$

in which $\hat{G}^0_{ab}(y)$ is the metric for $d+1$ dimensional asymptotically AdS while $\hat{G}^n_{ab}(y)$ are higher order corrections and they together form a complete description of asymptotically flat spacetime. We put all the higher order corrections into the term h_{ab}. In this thesis, to study asymptotically AdS spacetime G^0_{ab}, we choose to use Fefferman and Graham coordinates written as

$$\begin{aligned} \hat{G}^0_{ab} dy^a dy^b &= \frac{1}{r^2}(dr^2 + g_{ij}(r,x)dx^i dx^j) && (4.9) \\ &= \frac{d\rho^2}{4\rho^2} + \rho\, g_{ij}(x,\rho)dx^i dx^j, && (4.10) \end{aligned}$$

in which $y = (r, x^i)$ for $i = 1, \cdots d$ are coordinates on the AdS hyperboloid and $\rho = 1/r^2$. The d dimensional metric $g_{ij}(x,\rho)$ has been extensively studied in the AdS/CFT literature and the main method is that one can organise the data by doing expansion of the order ρ for $\rho \to \infty$. The leading term will contribute to the AdS spacetime while lower order terms are asymptotically AdS corrections.

4.1.1 Asymptotic Symmetries

To illustrate the spacetime structure introduced above in a more precise way, for simplicity, we take $d = 2$ as an example therefore the spacetime becomes asymptotically Minkowski. In this case, the metric is then given by

$$\begin{aligned} ds^2 &= -d\tau^2 + \tau^2\left(\frac{d^2\rho}{4\rho^2} + \rho\, dzd\bar{z} + \cdots\right) \\ &\quad + \frac{\tau}{\rho^2} m(\rho,z,\bar{z})d^2\rho + \tau\rho\, \sigma_{z\bar{z}}(\rho,z,\bar{z})dzd\bar{z} + 2\tau A_z(\rho,z,\bar{z})d\rho dz + \tau\rho\, \sigma_{zz}(\rho,z,\bar{z})dzdz \\ &\quad + \text{c.c} + \cdots, \end{aligned} \tag{4.11}$$

in which the Minkowski space is written in terms of Milne coordinates $(\tau, \rho, z, \bar{z})$ at the first line while the first order deviations with respect to τ are described by the functions $m(\rho,z,\bar{z})$, $A_z(\rho,z,\bar{z})$, $\sigma_{z\bar{z}}(\rho,z,\bar{z})$, $\sigma_{zz}(\rho,z,\bar{z})$ together with the corresponding complex conjugates. The dots in the first line represent the asymptotically AdS deformation while the dots in the third line represent higher order contributions according to $1/\tau$. Written in the form of $G_{\mu\nu}$, the metric is

$$G_{\rho\rho} = \frac{\tau^2}{4\rho^2} + \frac{\tau}{\rho^2}m + \mathcal{O}(\tau^0) \tag{4.12}$$

for $\rho\rho$ component and for $z\bar{z}$ component we have

$$G_{z\bar{z}} = \frac{\tau^2\rho}{2} + \tau\rho\ \sigma_{z\bar{z}} + \mathcal{O}(\tau^0). \tag{4.13}$$

At leading order, $G_{\rho z}$ G_{zz} will vanish while for asymptotic flat spacetime there could be subleading contribution like

$$G_{\rho z} = \tau A_z + \mathcal{O}(\tau^0) \qquad G_{\rho\bar{z}} = \tau A_{\bar{z}} + \mathcal{O}(\tau^0) \tag{4.14}$$

and

$$G_{zz} = \tau\rho\ \sigma_{zz} + \mathcal{O}(\tau^0) \qquad G_{\bar{z}\bar{z}} = \tau\rho\ \sigma_{\bar{z}\bar{z}} + \mathcal{O}(\tau^0). \tag{4.15}$$

Now, we will study asymptotic symmetry which is the diffeomorphism that preserves the metric of the form shown in (4.11). Given the Killing vector $\xi(\tau, \rho, z, \bar{z})$, the variation of the metric $\delta G_{\tau\rho}$ is then deduced to be

$$\begin{aligned}\delta G_{\tau\rho} &= -\partial_\rho\xi^\tau + \frac{\tau^2}{4\rho^2}\partial_\tau\xi^\rho \\ &\quad + \frac{\tau}{\rho^2} m\partial_\tau\xi^\rho + \tau A_z\partial_\tau\xi^z + \tau A_{\bar{z}}\partial_\tau\xi^{\bar{z}} + \mathcal{O}(\tau^0)\end{aligned} \tag{4.16}$$

and for $\delta G_{\tau z}$ we have

$$\begin{aligned}\delta G_{\tau z} &= -\partial_z\xi^\tau + \frac{\tau^2\rho}{2}\partial_\tau\xi^{\bar{z}} \\ &\quad + \tau A_z\partial_\tau\xi^\rho + \tau\rho\ \sigma_{zz}\partial_\tau\xi^z + \tau\rho\ \sigma_{z\bar{z}}\partial_\tau\xi^{\bar{z}} + \mathcal{O}(\tau^0),\end{aligned} \tag{4.17}$$

in which we have shown the variation up to $\mathcal{O}(\frac{1}{\tau})$ and higher order terms are omitted. For other terms we have

$$\delta G_{z\bar{z}} = \frac{\tau^2}{2}\xi^\rho + \frac{\tau^2\rho}{2}(\partial_z\xi^z + \partial_{\bar{z}}\xi^{\bar{z}}) + \mathcal{O}(\tau), \tag{4.18}$$

$$\delta G_{zz} = \tau^2\rho\ \ \partial_z\xi^{\bar{z}} + \mathcal{O}(\tau), \tag{4.19}$$

$$\delta G_{\rho z} = \frac{\tau^2}{4\rho^2}\partial_z\xi^\rho + \frac{\tau^2\rho}{2}\partial_\rho\xi^{\bar{z}} + \mathcal{O}(\tau), \tag{4.20}$$

$$\delta G_{\rho\rho} = \frac{\tau^2}{2\rho^3}(\rho\partial_\rho\xi^\rho - \xi^\rho) + \mathcal{O}(\tau). \tag{4.21}$$

To find out the asymptotic symmetries, one need to determine the Killing vector ξ which can be expanded as

$$\xi^\mu(\tau, \rho, z, \bar{z}) = \xi_0^\mu(\rho, z, \bar{z}) + \frac{1}{\tau}\xi_1^\mu(\rho, z, \bar{z}) + \cdots \tag{4.22}$$

where ξ_k^μ for $k \in N$ are coefficients associated to the term $1/\tau^k$. For $\delta G_{\tau\rho}$ and $\delta G_{\tau z}$ they have to vanish since we are working in the gauge $G_{\tau\rho} = G_{\tau z} = 0$. From the expression

(4.16) and (4.17), at the order of $\mathcal{O}(1)$ we obtain

$$\partial_z \xi_0^\tau + \frac{\rho}{2}\xi_1^{\bar{z}} = 0 \qquad \partial_\rho \xi_0^\tau + \frac{1}{4\rho^2}\xi_1^\rho = 0. \tag{4.23}$$

Moreover, we have

$$A_z \xi_1^\rho + \rho\, \sigma_{zz}\xi_1^z + \rho\, \sigma_{\bar{z}\bar{z}}\xi_1^{\bar{z}} = 0 \tag{4.24}$$

and

$$m\, \xi_1^\rho + \rho^2 A_z \xi_1^z + \rho^2 A_{\bar{z}}\xi_1^{\bar{z}} = 0 \tag{4.25}$$

when considering the $\delta G_{\tau\rho} = \delta G_{\tau z} = 0$ of the order τ^2. For δG_{zz} and $\delta G_{\bar{z}\bar{z}}$, the contribution at the order $1/\tau$ should vanish in order to preserve the condition of asymptotic flatness therefore we have

$$\partial_z \xi_0^{\bar{z}} = \partial_{\bar{z}} \xi_0^z = 0. \tag{4.26}$$

For the same reason we have

$$\rho\partial_\rho \xi_0^\rho - \xi_0^\rho = 0 \tag{4.27}$$

when considering the component $G_{\rho\rho}$ in the (4.21). Then the killing vector can be written as

$$\xi^\tau = \chi(z,\bar{z}), \tag{4.28}$$
$$\xi^\rho = 0, \tag{4.29}$$
$$\xi^z = Y^z(z,\bar{z}), \tag{4.30}$$
$$\xi^{\bar{z}} = Y^{\bar{z}}(z,\bar{z}), \tag{4.31}$$

where χ is an arbitrary function on $z, \bar{z}$ and we have $\partial_z Y^{\bar{z}} = \partial_{\bar{z}} Y^z = 0$ therefore the BMS group Bondi et al. (1962); Sachs (1962b,a) is recovered at leading order. The transformation of the spacetime metric under such symmetry group is then given by

$$\delta g_{z\bar{z}} = g_{z\bar{z}}(\partial_z Y^z + \partial_{\bar{z}} Y^{\bar{z}}) + Y^z \partial_z g_{z\bar{z}} + Y^{\bar{z}}\partial_{\bar{z}} g_{z\bar{z}} \tag{4.32}$$
$$\delta g_{zz} = 2g_{zz}\, \partial_z Y^z + Y^z \partial_z g_{zz} + Y^{\bar{z}}\partial_{\bar{z}} g_{zz} \tag{4.33}$$
$$\delta A_z = Y^z \partial_z A_z + Y^{\bar{z}}\partial_{\bar{z}} A_z + A_z \partial_z Y^z \tag{4.34}$$
$$\delta \sigma_{zz} = Y^z \partial_z \sigma_{zz} + Y^{\bar{z}}\partial_{\bar{z}}\sigma_{zz} + 2\sigma_{zz}\partial_z Y^z + 2\chi g_{zz} \tag{4.35}$$
$$\delta \sigma_{z\bar{z}} = Y^z \partial_z \sigma_{z\bar{z}} + Y^{\bar{z}}\partial_{\bar{z}}\sigma_{z\bar{z}} + (\partial_z Y^z + \partial_{\bar{z}} Y^{\bar{z}})\sigma_{z\bar{z}} + 2\chi g_{z\bar{z}} \tag{4.36}$$

where we have used the complex metric $g_{z\bar{z}}$ for the hard sector. From above translation rules, one can see that the superrotation part described by Y^z, $Y^{\bar{z}}$ will act on the leading and subleading part of the metric while the supertranslation part described by χ will only act on the subleading part.

4.1.2 Equation of Motion

Now we turn to study the dynamics of the gravitational system. The general $d+2$ dimensional asymptotic flat spacetime is described by the coordinate [2]

$$\begin{aligned} ds^2 &= -d\tau^2 + \tau^2 \left(\frac{d^2\rho}{4\rho^2} + \rho g_{ij}(\rho, x) dx^i dx^j \right) \\ &+ \frac{\tau}{\rho^2} m(\rho, x) d^2\rho + \tau\rho\ \sigma_{ij}(\rho, x) dx^i dx^j + \tau A_i(\rho, x) d\rho dx^i + \cdots \end{aligned} \tag{4.37}$$

in terms of the real coordinates (τ, ρ, x^i) for $i = 1, \ldots, d$. In the first line, the metric is mainly built up from asymptotically AdS hyperboloids. It describes the hard sector of the gravitational theory and manifests the superrotation symmetry. The second line is the subleading contribution to the asymptotically flat spacetime according to $1/\tau$. It describes the soft modes coming from the radiation and manifests the superrotation and supertranslation symmetry.

In this section, we are going to determine the constraints on the metric g_{ij} and the soft parameters by checking the Einstein equations $R_{\mu\nu} = 0$ at different orders. Starting with the connections, they are given by

$$\Gamma^{\tau}_{ab} = \tau \hat{G}_{ab} + \frac{1}{2}\tau^2 \partial_\tau \hat{G}_{ab} \tag{4.38}$$

and

$$\Gamma^{a}_{b\tau} = \frac{\delta^a_b}{\tau} + \frac{1}{2}\hat{G}^{ac}\partial_\tau \hat{G}_{cb} \tag{4.39}$$

while we have $\Gamma^{\tau}_{\tau\tau} = \Gamma^{\tau}_{\tau a} = \Gamma^{a}_{\tau\tau}$=0 and $\Gamma^{a}_{bc} = \hat{\Gamma}^{a}_{bc}[\hat{G}]$. In our definition the Ricci tensor is now given by

$$R_{ab}[G] = R^{d+1}_{ab}[\hat{G}] - \Gamma^{\tau}_{ab}\Gamma^{c}_{c\tau} + 2\Gamma^{c}_{a\tau}\Gamma^{\tau}_{bc} - \partial_\tau \Gamma^{\tau}_{ab}, \tag{4.40}$$

which can be further decomposed into

$$R_{ab}[G] = R^{d+1}_{ab}[\hat{G}] - d\hat{G}_{ab} \tag{4.41}$$

$$-\frac{d+1}{2}\tau\partial_\tau \hat{G}_{ab} - \frac{1}{2}\tau^2\partial^2_\tau \hat{G}_{ab} - \frac{\tau}{2}\hat{G}_{ab}\hat{G}^{cd}\partial_\tau \hat{G}_{cd} \tag{4.42}$$

$$+\frac{1}{2}\tau^2 \partial_\tau \hat{G}_{cb}\partial_\tau \hat{G}_{da}\hat{G}^{cd}. \tag{4.43}$$

[2] The goal of this chapter is to fully reconstruct the bulk gravitational theory together with the static background therefore the solution is not separated into in and out going modes like what we did for the scalar fields. In principle, we can introduce in and out going gravitational waves when dealing with the theory in a perturbative way therefore making the results compatible with the previous chapter.

For $\tau\tau$ component we have

$$R_{\tau\tau}[G] = \frac{1}{\tau}\hat{G}^{ab}\partial_\tau\hat{G}_{ab} + \frac{1}{2}\partial_\tau(\hat{G}^{ab}\partial_\tau\hat{G}_{ab}) + \frac{1}{4}\hat{G}^{ac}\hat{G}^{bd}\partial\tau\hat{G}_{ab}\partial_\tau\hat{G}_{cd} \tag{4.44}$$

while for τa one can deduce that

$$R_{a\tau}[G] = \frac{1}{2}\hat{\nabla}_a(\hat{G}^{bc}\partial_\tau\hat{G}_{bc}) - \frac{1}{2}\hat{\nabla}_b(\hat{G}^{bc}\partial_\tau\hat{G}_{ca}), \tag{4.45}$$

where the covariant derivative is with respect to the metric $\hat{G}_{ab}$. In practice, to study the equation of motion at different orders, we choose to write the Ricci curvature perturbatively according to the expansion (4.8), which means we have

$$R_{\mu\nu}[G^0 + h] = R^0_{\mu\nu}[G^0] + \frac{1}{\tau}R^1_{\mu\nu} + \cdots \tag{4.46}$$

where the zero order mainly comes from the hard sector of the metric described by G^0 and the soft modes will go into the first or higher order terms.

4.1.2.1 Zero Order

For convenience, we will denote $\hat{G}^0_{ab}(y)$ as $\hat{G}_{ab}(y)$ in this subsection and consider the equation of motion at leading order of $1/\tau$. In such case, connections involving τ component are given by $\Gamma^\tau_{ab} = \tau\hat{G}_{ab}, \Gamma^a_{b\tau} = \frac{1}{\tau}\delta^a_b$ and $\Gamma^\tau_{\tau\tau} = \Gamma^\tau_{\tau a} = \Gamma^a_{\tau\tau} = 0$. For the connections not involving τ component denoted as Γ^a_{bc}, they are given by the direct $d+1$ dimensional calculation using the AdS metric $\tau^2\hat{G}_{ab}(y)$. Now, to solve the vacuum Einstein equation with zero cosmology constant

$$R_{\mu\nu}[G] = 0, \tag{4.47}$$

we should deduce the Ricci curvature R written as $R[G] = R^a_a[G] + R^\tau_\tau[G]$. One can easily verify that $R_{\tau\mu} = 0$, while for R_{ab} one has

$$R_{ab}[G] = R^{(d+1)}_{ab}[\hat{G}] - d\hat{G}_{ab}, \tag{4.48}$$

in which we have introduced the notion $R^{(d+1)}[\hat{G}]$ to denote that the Ricci curvature induced on the $d+1$ dimensional AdS hyperboloid. Therefore, the Ricci curvature for near flat spacetime is then deduced to be

$$R[G] = R^{(d+1)}[\hat{G}] - d(d+1) \tag{4.49}$$

and the Einstein equation in (4.47) is equivalent to

$$R^{(d+1)}_{ab}[\hat{G}] - \frac{1}{2}R^{(d+1)}[\hat{G}]\ \hat{G}_{ab} = \Lambda\hat{G}_{ab} \tag{4.50}$$

for

$$\Lambda = -\frac{d(d-1)}{2}. \tag{4.51}$$

We can treat Λ as the effective cosmology constant and τ as the effective AdS radius since we are recasting the curvature $R[G]$ for $d+2$ dimensional near flat metric $\hat{G}_{\mu\nu}$ into the induced curvature $R^{(d+1)}[\hat{G}]$ for $d+1$ dimensional asymptotically AdS metric $\hat{G}_{ab}(y)$.

In terms of Fefferman Graham gauge, the study of equation of motion at zero order is equivalent to the study of the differential equation for $g_{ij}(x,\rho)$. The function $g_{ij}(x,\rho)$ are determined by Graham (2000); Fefferman and Graham (2011); Henningson and Skenderis (1998); de Haro et al. (2001b)

$$\begin{aligned} &\rho^2(2\rho g''_{ij} + 4g'_{ij} - 2\rho g^{lm} g'_{mj} g'_{li} + \rho g^{lm} g'_{lm}\, g'_{ij}) + R_{ij}[g] + (d-2)\rho^2 g'_{ij} + \rho^2 g^{lm} g'_{lm}\, g_{ij} = 0 \\ &\nabla_i(\, g^{lm} g'_{lm}) - \nabla^j g'_{ij} = 0 \\ &g^{ij}(\rho g''_{ij} + 2g'_{ij}) - \frac{1}{2}\rho g^{ik} g^{jm} g'_{ij} g'_{km} = 0 \end{aligned} \tag{4.52}$$

according to the equation (4.47) or (4.50) and the covariant derivative is with respect to the metric g_{ij}. A brief study of such equation is shown in appendix I and here we have $g'_{ij} = \partial_\rho g_{ij}$. Moreover, for even d, one has the expansion

$$g_{ij}(x,\rho) = g^{(0)}_{ij} + \rho^{-1} g^{(2)}_{ij} + \cdots + \rho^{-d/2} g^{(d)}_{ij} + c_{ij}\rho^{-d/2}\log\rho + \cdots \tag{4.53}$$

when ρ goes to infinity. Here, following the convention from previous literature, the superscript $2k$ in the coefficients $g^{(2k)}_{ij}$ are used to keep track of the order of r. Or equivalently, k is used to keep track of the order of ρ. Coefficients $g^{(2k)}_{ij}$ are uniquely determined by the lower order terms via checking the equation (4.52) at the order of ρ^k while such procedure will fail until $2k = d$. In such case, the equation of motion will only allow us to determine $\mathrm{Tr} g^{(d)}$ and it also leaves us freedom to introduce the traceless algorithm term parameterised by c_{ij}. Fox example, by checking the first two equations at leading order, one can obtain the relation

$$R_{ij}[g^{(0)}] = (d-2) g^{(2)}_{ij} + g^{(0)}_{ij}\mathrm{Tr} g^{(2)} \tag{4.54}$$

together with

$$\nabla_i(\mathrm{Tr} g^{(2)}) - \nabla^j g^{(2)}_{ji} = 0 \tag{4.55}$$

where the covariant derivative ∇_i and the trace Tr now are with respective to the metric $g^{(0)}_{ij}$ while we will keep such convention in the following part of this thesis. One can see that $g^{(2)}_{ij}$ is fully determined as the function of $g^{(0)}_{ij}$ for $d \neq 2$ while only the trace part is fixed when $d = 2$.

4.1.2.2 First Order

We have studied $R^0_{\mu\nu}$ in the previous section while here we are going to deal with $R^1_{\mu\nu}$ therefore determine equation of motion at first order. It is easier to calculate the $R_{\tau\tau}$ and $R_{\tau a}$ components by checking the formula (4.44) and (4.45). The $R_{\tau\tau} = 0$ will be trivial at first order while for $R^1_{\tau a}$, we have

$$R^1_{\tau a} = \frac{1}{2}\hat{\nabla}_a(\hat{G}^{bc}_0 \hat{G}^1_{bc}) - \frac{1}{2}\hat{\nabla}^b \hat{G}^1_{ab} \tag{4.56}$$

where the covariant derivative $\hat{\nabla}_a$ is with respect to the metric $\hat{G}^0_{ab}$. For the R_{ab} components, we have

$$R^1_{ab} = R^{d+1,1}_{ab} - \frac{d+1}{2}\hat{G}^1_{ab} + \frac{1}{2}\hat{G}^1_{ab} G^{cd}_0 G^1_{cd} \tag{4.57}$$

from which one can see that the first order contribution $R^{d+1,1}_{ab}$ of R^{d+1}_{ab} will also contribute to the first order R^1_{ab} therefore it will make the results more complicated. The strategy here is to determine the first order term in R^{d+1}_{ab} then add the other terms in (4.57) which also contribute at the first order.

Such soft sector is described by the first order term $h_{\mu\nu}$. More precisely, it is determined by the parameter m, σ_{ij} and A_i once the gauge is fixed. To simplify the calculation further, now we consider the expansion of parameter m, σ_{ij} and A_i by the order $1/\rho$. Taking the parameter $m(\rho, x)$ for example, we have

$$m(\rho, x) = m^{(0)}(x) + \frac{1}{\rho} m^{(2)}(x) + \cdots, \tag{4.58}$$

in which $m^{(0)}(x)$ is the leading term while $m^{(2)}(x)$ is the subleading contribution. $m^{(0)}$, $m^{(2)}$ describe the zero order and first order contribution to the soft sector according to the spatial radius of AdS hyperboloid ρ. For the parameter σ_{ij}, A_i we adopt similar convention and the corresponding coefficients are denoted as $A^{(2k)}_i$ and $\sigma^{(2k)}_{ij}$.

Therefore following the equation of motion given by the Einstein equation explicitly showed in the appendix J, we obtain the constraints

$$\text{Tr}\ \sigma^{(0)} = 4dm^{(0)} \tag{4.59}$$

by considering $R_{\rho\rho} = R_{\rho\tau} = 0$. Moreover we have

$$A^{(0)}_i = 0 \tag{4.60}$$

by checking the equation of motion $R_{i\tau} = 0$ at the zero order of $1/\rho$. For the equation $R_{ij} = 0$, again at leading order, we obtain

$$\frac{d-1}{2}\sigma^{(0)}_{ij} + \frac{1}{2} g^{(0)}_{ij}\ \text{Tr}\ \sigma^{(0)} - 2(2d-1)m^{(0)} g^{(0)}_{ij} = 0, \tag{4.61}$$

which is compatible with the constraint (4.59) after taking the trace on both sides. One can also obtain the relation

$$\nabla^j \sigma^{(0)}_{ji} = 4\nabla_i m^{(0)} \tag{4.62}$$

after acting the covariant derivative ∇^j on both sides.

Now we consider the Einstein equations at first order of $1/\rho$ in order to determine $m^{(2)}$, $A^{(2)}_i$ and $\sigma^{(2)}_{ij}$. By checking the equation of motion $R_{i\tau} = 0$ at the first order, we have

$$2\nabla_i m^{(0)} = A^{(2)}_i \tag{4.63}$$

while for $R_{\rho i} = 0$ we obtain the relation

$$\frac{3d-1}{2} A^{(2)}_i - d\nabla_i m^{(0)} + \frac{1}{4}\nabla^j \sigma^{(0)}_{ij} - \frac{1}{4}\nabla_i \mathrm{Tr}\sigma^{(0)} = 0. \tag{4.64}$$

Given the above equation, we can deduce that $A^{(2)}_i = 0$ after using the relation (4.59),(4.62) and (4.63). This tells us that the parameter $m^{(0)}$ should be a constant and the zero order coefficient $\sigma^{(0)}_{ij}$ is conserved with respect to the metric $g^{(0)}_{ij}$, written as

$$\mathrm{Tr}\sigma^{(0)} = 4dM \qquad \nabla^j \sigma^{(0)}_{ij} = 0 \tag{4.65}$$

where we have denoted the parameter $m^{(0)}$ as constant M. For the equation of motion $R_{\rho\tau} = 0$ and $R_{\rho\rho} = 0$, they will give us the relations

$$dm^{(2)} + \frac{1}{4}\mathrm{Tr}\, \sigma^{(2)} - \mathrm{Tr} g_{(2)} m^{(0)} = 0 \tag{4.66}$$

and

$$\frac{d}{2} m^{(2)} + \frac{1}{8}\mathrm{Tr}\, \sigma^{(2)} - \frac{1}{8} g^{ij}_{(2)} \sigma^{(0)}_{ij} = 0. \tag{4.67}$$

From these two equations one can see that, in order to make $m^{(2)}$ and $\sigma^{(2)}_{ij}$ solvable, one should further impose constraint on $g^{ij}_{(2)}\sigma^{(0)}_{ij}$ thus we have

$$g^{ij}_{(2)} \sigma^{(0)}_{ij} = 4\mathrm{Tr} g^{(2)} m^{(0)}. \tag{4.68}$$

To obtain $\sigma^{(2)}_{ij}$, one needs to check $R_{ij} = 0$ explicitly. The equation is more involved and here we just present the result

$$\begin{aligned}\frac{3-d}{2}\sigma^{(2)}_{ij} + (6-4d) m^{(2)} g^{(0)}_{ij} - 6 g^{(2)}_{ij} m^{(0)} - \frac{1}{2}(\mathrm{Tr}\sigma^{(2)} - g^{lk}_{(2)}\sigma^{(0)}_{lk}) g^{(0)}_{ij} \\ +4\mathrm{Tr} g^{(2)} m^{(0)} g^{(0)}_{ij} - \mathrm{Tr} g^{(2)} \sigma^{(0)}_{ij} + \frac{1}{2}\mathrm{Tr}\sigma^{(0)} g^{(2)}_{ij} + \delta R_{ij} = 0,\end{aligned} \tag{4.69}$$

where $R_{ij}[g+\frac{1}{\tau}\sigma] = R_{ij}[g] + \delta R_{ij}$ [3] and we can see that, given the value of $m^{(2)}$, $\sigma_{ij}^{(2)}$ is determined by solving the equation. After taking the trace on both sides, we have

$$\frac{3-2d}{2}\mathrm{Tr}\sigma^{(2)} + (6-4d)dm^{(2)} - 6\mathrm{Tr}g^{(2)}\, m^{(0)} + \frac{d}{2}g_{(2)}^{lm}\sigma_{lm}^{(0)} + 2dm^{(0)}\mathrm{Tr}g^{(2)} = 0, \quad (4.70)$$

from which we will obtain the same relation as (4.68) after the substitution of (4.66) or (4.67).

From above calculation, we see that the value of $m^{(2k)}$ are related to the trace of soft metric coefficients $\mathrm{Tr}\sigma^{(2k)}$. More precisely, by considering the constraint (4.59), (4.66) and (4.68), one can find that they obey the more compact relation

$$\mathrm{Tr}^{(2k)}(\sigma - 4mg) = 0 \quad (4.71)$$

where $\mathrm{Tr}^{(2k)}$ is defined as the trace over the metric $g_{ij}^{(2k)}$ and we denote $\mathrm{Tr}^{(0)} = \mathrm{Tr}$. The relation between (4.71) and the Einstein equations is not clear but we expect this is true when going to the higher order.

Further more, by checking the equation of motion $R_{i\tau} = 0$ at the order of $1/\rho^2$, we obtain the relations involving $\nabla^j\sigma_{ij}^{(2)}$ written as

$$\begin{aligned}(2-d)A_i^{(4)} + 2\nabla_i m^{(2)} + \frac{1}{2}\nabla_i(\mathrm{Tr}\sigma^{(2)} - g_{(2)}^{kl}\sigma_{kl}^{(0)})\\ -\frac{1}{2}\nabla^j\sigma_{ij}^{(2)} + \frac{1}{4}\nabla_k g_{(2)}^{mk}\sigma_{mi}^{(0)} + \frac{1}{4}\nabla_i g_{(2)}^{mn}\sigma_{mn}^{(0)} = 0\end{aligned} \quad (4.72)$$

where we have used the relation (4.55) and the last two terms come from the variation of the connection $\delta\Gamma^i_{jk}$. It turns out that, although a little bit tedious, equation (4.72) will be useful for us to study Ward identities of the boundary conformal field theory with the help of flat/CFT dictionary. Moreover, by studying the equation of motion $R_{i\rho} = 0$ at second order of $1/\rho^2$ one should be able to determine $A_i^{(4)}$ once $m^{(2)}$ or equivalently Tr $\sigma^{(2)}$ is fixed.

4.2 The flat/CFT Dictionary

The Einstein-Hilbert action for the gravitational theory on a four dimensional asymptotically flat manifold M with boundary ∂M is given by Gibbons and Hawking (1977)

$$S_{\mathrm{gr}}[G] = \frac{1}{16\pi G_N}\left(\int_M d^4X\sqrt{-G}\, R[G] + \int_{\partial M} d^3X\sqrt{-\gamma}\, 2K\right), \quad (4.73)$$

in which K is the trace of the second fundamental form and γ is the induced metric on the boundary. To evaluate the action, we first choose to use the equation of motion

[3] More precisely, we have $\delta R_{ij} = \frac{1}{2}(\nabla^m\nabla_m\sigma_{ij} + \nabla_i\nabla_j\sigma_m^m - \nabla_k\nabla_i\sigma_j^k - \nabla_k\nabla_j\sigma_i^k)$.

(4.47) and set the boundary as the surface with constant AdS spatial radius $\rho = 1/\epsilon$. Then the regulated action is given by

$$S_{\text{gr,reg}} = \frac{1}{16\pi G_N} \int_0^\infty d\tau \int_{S^2} d^2x \sqrt{-\gamma}\, 2K \Big|_{\rho=1/\epsilon} \tag{4.74}$$

where we have $\gamma_{\tau\tau} = -1, \gamma_{ij} = \tau^2 \hat{G}_{ij}$.[4]

Moreover, to calculate the integral over K, one should notice the relation $K_{ij} = \hat{\nabla}_i n_j$ thus we obtain [5]

$$\int_0^\infty d\tau \int d^2x \sqrt{-\gamma}\, 2K \Big|_{\rho=1/\epsilon} = \int_0^\infty \tau d\tau \int_{S^2} d^2x \sqrt{\hat{G}} \frac{\hat{G}^{\rho\mu}}{\sqrt{\hat{G}^{\rho\rho}}} (2\hat{G}^{ij}\partial_i \hat{G}_{j\mu} - \hat{G}^{ij}\partial_\mu \hat{G}_{ij}) \Big|_{\rho=1/\epsilon} \tag{4.75}$$

in which n^μ is the outward unit normal for the boundary ∂M. In our case, for the boundary $\rho = 1/\epsilon$, the only non-zero component is $n^\rho = \sqrt{G^{\rho\rho}}$. To see the divergent part of the regulated action in a more precise way, for even d, one can use the expansion for g shown in (4.53) and extract the infinite part written as

$$S_{\text{gr,reg}} = \frac{1}{16\pi G_N} \int_0^\infty d\tau \tau \int_{S^2} d^2x \sqrt{g_{(0)}} \left(\epsilon^{-1} a_1(\tau, x) + a_0(\tau, x) + \log\epsilon\, b(\tau, x) + \mathcal{O}(\epsilon^0) \right), \tag{4.76}$$

where a_i and b are the corresponding coefficients. To get the renormalised action $S_{\text{gr,ren}}$, one should introduce the local and covariant counterterm $S_{\text{gr,ct}}$ to eliminate the divergence, which takes the form

$$S_{\text{gr,ct}} = \int_0^\infty d\tau \int_{S^2} d^2x\, f(\tau, z)\sqrt{-\gamma} + \int_0^\infty d\tau \int_{S^2} d^2x\, g(\tau, z)\sqrt{-\gamma}\, R\,[\gamma] + \cdots \tag{4.77}$$

where f, g are scalar functions of τ, x and they are determined by the coefficients a_i, b in (4.76). Given the renormalised action $S_{\text{gr,ren}} = S_{\text{gr,reg}} + S_{\text{gr,ct}}$ together with the dictionary

$$\exp\, (iS_{\text{gr,ren}}[G]) = \left\langle \exp \frac{1}{2} \int_{S^2} d^2x \sqrt{\bar{G}}\, \bar{G}^{ij}\, T_{ij} \right\rangle \tag{4.78}$$

the CFT stress tensor T_{ij} is then deduced to be

$$\langle T_{ij} \rangle = \lim_{\epsilon \to 0} \frac{2i}{\sqrt{\bar{G}}} \frac{\delta S_{\text{gr,ren}}}{\delta \bar{G}^{ij}}, \tag{4.79}$$

where $\bar{G}_{ij}$ is the background metric of the boundary CFT. Before going into the detail of holographic renormalisation, here we briefly discuss the structure of renormalised

[4]In fact, there are three components that belong to the boundary ∂M. One is at $\rho = 1/\epsilon$ while the other two are at $\tau = 0$ and $\tau = \infty$. In thesis we focus on the renormalisation of the divergence at $\rho = 1/\epsilon$. For the integral along the surface of constant τ, we treat them as the assignment of initial and final data. The treatment of the integral at the constant time surface is equivalent to the procedure that we fix the initial modes by hand or inserting proper $i\varepsilon$ description in the path integral. More rigorously, like the treatment for real time holography Skenderis and van Rees (2009, 2008); Hao (2024a), one can choose to glue an Euclidean cap at $\tau = 0$ surface and make divergence cancelled.

[5]We are abusing the notion here and $\hat{\nabla}$ means the covariant derivative with respect to the metric $\hat{G}_{ij}$.

gravity effective action. Through the renormalisation procedure, the IR divergence is regulated by choosing the AdS spatial radius $\rho = 1/\epsilon$ as the low energy cutoff. However, for the flat spacetime, we also have timelike direction labelled by τ and here we treat it as the UV cutoff from boundary point of view by specifying the range of integral $0 \leq \tau \leq L$. Therefore, organized by the powers of L, the action takes the form

$$S_{\text{gr,reg}} = S^0_{\text{gr,reg}} + S^1_{\text{gr,reg}} + \cdots . \tag{4.80}$$

where $S^0_{\text{gr,ren}}$ is the leading contribution while $S^0_{\text{gt,ren}}$ is subleading. After performing the integral of τ during the holographic renormalisation, we have

$$S_{\text{gr,ren}} = L^2 S^0_{\text{gr,ren}} + L S^1_{\text{gr,ren}} + \mathcal{O}(\log L), \tag{4.81}$$

in which $S^0_{\text{gr,ren}}$ is the contribution to the renormalised action of the order L^2 and $S^1_{\text{gr,ren}}$ is the lower order term. We identify $S^0_{\text{gr,ren}}$ as the hard sector since it comes from the AdS hyperboloid while the soft sector is identified as $S^1_{\text{gr,ren}}$ coming from soft modes in the metric.

Given the dictionary (4.78), to perform the calculation and make it work the same as the AdS/CFT dictionary, we need the specific map between the boundary and bulk data and here we propose the relation to be

$$\bar{G}_{ij} = g^{(0)}_{ij} + \frac{1}{L}\sigma^{(0)}_{ij} + \cdots \tag{4.82}$$

where the boundary background metric is expanded by the order of energy cut off L given by the bulk data $g^{(0)}_{ij}$ and $\sigma^{(0)}_{ij}$.

4.2.1 Hard Sector

In this section, we choose to perform the holographic renormalisation for the hard sector ignoring the soft contribution from h_{ij}. It turns out the treatment of the hard sector is equivalent to the linear summation over all the AdS hyperboloid contribution and one can regard this part as the review of AdS/CFT holographic renormalisation. At zero order, the onshell action takes the form

$$S_{\text{gr,reg}} = \frac{-1}{16\pi G_N} \int_0^L d\tau \tau \int_{S^2} d^2x \sqrt{g}\, 2\rho^2 \left(\frac{2}{\rho} + g^{ij}\partial_\rho g_{ij} \right) \Big|_{\rho = 1/\epsilon} \tag{4.83}$$

and the counterterm is given by

$$S_{\text{gr,ct}} = -\frac{d-1}{8\pi G_N} \int_0^L d\tau \int_{S^2} d^2x \, \frac{1}{\tau}\sqrt{-\gamma}. \tag{4.84}$$

Now, following the above discussion, we will show the result for $d = 2$. In such case, $g_{(0)ij}$ is the metric on the sphere and the regulated stress tensor T_{ij}^{reg} on the celestial sphere is therefore given by

$$T_{ij}^{\text{reg}} = \frac{iL^2}{16\pi G_N}(K_{ij} - \gamma_{ij}K) = \frac{iL^2}{16\pi G_N}\rho\left((d-1)g_{ij} + \rho g_{ij}g^{lk}\partial_\rho g_{lk} - \rho\partial_\rho g_{ij}\right)\Big|_{\rho=1/\epsilon}, \tag{4.85}$$

in which L is the upper bound of the integral over τ and we have set $\rho = 1/\epsilon$. After the subtraction of the counterterm

$$T_{ij}^{\text{ct}} = -\frac{iL^2}{16\pi G_N}\frac{(d-1)g_{ij}}{\epsilon} + \cdots, \tag{4.86}$$

one then obtains the stress tensor

$$\langle T_{ij}\rangle = \frac{iL^2}{16\pi G_N}(g_{ij}^{(2)} - g_{ij}^{(0)}\text{Tr}g^{(2)}) \tag{4.87}$$

by taking the limit $\epsilon \to 0$. Moreover, with the help of the relation

$$g_{(2)ij} = \frac{1}{2}(Rg_{(0)ij} + t_{ij}), \qquad \text{Tr}\, t = -R, \tag{4.88}$$

where $R = R\left[g_{(0)}\right]$ and t_{ij} is a conserved symmetric tensor $\nabla^i t_{ij} = 0$, we have

$$\langle T_{ij}\rangle = \frac{iL^2}{32\pi G_N}\, t_{ij}. \tag{4.89}$$

Therefore, after taking the trace, the Weyl anomaly is then deduced to be

$$\langle T_i^i\rangle = -\frac{c}{24\pi}R, \tag{4.90}$$

in which c is the central charge on the celestial sphere

$$c = \frac{i3L^2}{4G_N}. \tag{4.91}$$

One can see the central charge will approach $i\infty$ as argued in Cheung et al. (2017a); Pasterski and Verlinde (2022); Ogawa et al. (2023) if one treats L as the scale of energy. The complex central charge and stress tensor make the boundary CFT different from the conventional CFT and the physical interpretation is yet not clear to us. However, taking the previous work for scalar case into consideration, the framework for the CFT with complex scale dimensions, stress tensor and central charge should be consistent. Moreover, these complex values will not ruin the local properties of the QFT, which can be seen by the study of the anomaly and Ward identities of the dual complex CFT theory.

The infinite behaviour of the central charge can be easily understood given the detail of the flat/CFT dictionary, which has been extensively studied in the case for scalar fields studied in chapter 3. The onshell scalar field is studied by the separation of variables which splits the τ direction and other coordinates on the AdS hypersurfaces and it turns out that one could decompose a single scalar field into infinite number of modes labelled by the complex number k making the scale dimension of the dual operator live on the principal series. Here we have the metric $G_{\mu\nu}$ while it is hard to apply the variable separation method to split the nonlinear Einstein equation into the τ dependent part and the other part which describes the equation of motion on the AdS hyperboloid therefore decompose the metric into the form of

$$G_{\mu\nu}(\tau,\rho,x) \longrightarrow G_{\mu\nu}(\rho,x;k) \tag{4.92}$$

labelled by the parameter k. But here it is still reasonable to assume that the bulk metric is dual to infinite number of operators on the boundary described by stress-tensor modes denoted as $T_{ij}(x;k)$ and the energy-stress tensor calculated here are in fact the summation of all these modes. Each mode will contribute to the central charge in a finite way while the total effect will become infinite after summing over all the modes labelled by k treated as the frequency space dual to the τ direction.

4.2.2 Soft Sector

Now, based on our study of hard sector, we move on to the study of soft sector. In order to obtain the next leading order correction $S^1_{\text{gr,ren}}$, one needs to consider the higher order terms in $1/\tau$ of the onshell action (4.75).

$$\begin{aligned} S^0_{\text{gr,reg}} + S^1_{\text{gr,reg}} = \frac{-1}{16\pi G_N}\int_0^L d\tau\tau \int_{S^2} d^2x\sqrt{\hat{G}}\frac{4\rho}{\sqrt{\hat{G}^{\rho\rho}}}\big((d+\rho g^{ij}\partial_\rho g_{ij}) \\ -\frac{1}{\tau}(2\nabla^i A_i - g^{ij}\rho\partial_\rho\sigma_{ij} + \sigma^{ij}\rho\partial_\rho g_{ij})\big), \end{aligned} \tag{4.93}$$

therefore regulated action at the first order now becomes

$$\begin{aligned} S^1_{\text{gr,reg}} = \frac{L}{8\pi G_N}\int_{S^2} d^2z\ \sqrt{g_{(0)}}\ \rho\ \Big(2\nabla^i A_i - \rho g^{ij}\partial_\rho\sigma_{ij} + \sigma^{ij}\rho\partial_\rho g_{ij} \\ -(2m+2dM)(d+\rho g^{ij}\partial_\rho g_{ij})\Big)\Big|_{\rho=1/\epsilon} \end{aligned} \tag{4.94}$$

where the integral over τ has already been performed and the contribution in the second line comes from the determinant $\sqrt{\hat{G}}$ and the norm vector factor $\sqrt{G_{\rho\rho}}$. Together with the expression of the extrinsic curvature

$$K_{ij} = -\rho(\rho\partial_\rho g_{ij} + g_{ij}) + \frac{\rho}{\tau}\Big(\nabla_i A_j + \nabla_j A_j - \rho\partial_\rho\sigma_{ij} - \sigma_{ij} + 2m(\rho\partial_\rho g_{ij} + g_{ij})\Big), \tag{4.95}$$

one can obtain the soft stress tensor $T_{ij}^{1,\text{reg}}$

$$T_{ij}^{1,\text{reg}} = \frac{iL}{8\pi G_N}\rho\Big((\nabla_i A_j + \nabla_j A_i - \rho\partial_\rho\sigma_{ij} - \sigma_{ij}) + (\rho g^{lk}\partial_\rho g_{lk} + d)\sigma_{ij} \\ -(2\nabla_i A^i - \rho g^{lk}\partial_\rho\sigma_{lk} + \rho\sigma^{lk}\partial_\rho g_{lk})g_{ij} + 2m(\rho\partial_\rho g_{ij} + g_{ij}) - 2m(\rho g^{lk}\partial_\rho g_{lk} + d)g_{ij}\Big) \tag{4.96}$$

by checking the first order of $K_{ij} - \gamma_{ij}K$ then performing the integral over τ like we have done for the hard sector. From the above expression one can see the stress tensor will go to infinity at large ρ. One of the divergent term comes from $\nabla_i A_j$ while the other term comes from the first order metric on the sphere σ_{ij}. However, taking the constraint $A_i^{(0)} = A_i^{(2)} = 0$ and the counterterm

$$S_{\text{gr,ct}} = -\frac{d-1}{8\pi G_N}\int_0^L d\tau \int_{S^2} d^2x\, \tau\sqrt{\hat{G}}\,\left(4\rho^2\hat{G}_{\rho\rho}\right)^{-\frac{1}{2}} \tag{4.97}$$

into consideration, we have

$$T_{ij}^{\text{ct}} = \frac{iL^2}{16\pi G_N}(1-d)\rho\big(g_{ij} + \frac{2}{L}(\sigma_{ij} - 2mg_{ij})\big) + \cdots \tag{4.98}$$

therefore the corresponding finite renormalised stress tensor at first order becomes

$$\langle T_{ij}^1\rangle = -\frac{iL}{8\pi G_N}\Big((\text{Tr}\,\sigma^{(2)} - \sigma_{kl}^{(0)}\, g_{(2)}^{kl})\, g_{ij}^{(0)} + \text{Tr}g^{(2)}\sigma_{ij}^{(0)} \\ -\sigma_{ij}^{(2)} + 2m^{(0)}(g_{ij}^{(2)} - g_{ij}^{(0)}\text{Tr}g^{(2)})\Big). \tag{4.99}$$

4.2.3 Ward Identities

Given the flat/CFT dictionary and the specific expression of the energy-stress tensor, now we turn to study Ward identities concerning $\langle T_{ij}\rangle$ with the help of constraints on the gravity metric studied before. For Weyl anomaly, we will perform the calculation from the boundary point of view which means that the indices now are raised and lowered by the metric $\bar{G}_{ij}$. After taking the trace, for the soft stress tensor, we have[6]

$$\langle T_i^i\rangle^1 = -\frac{iL}{16\pi G_N}\Big(2(d-1)\big(\text{Tr}\sigma^{(2)} - \sigma_{ij}^{(0)} g_{(2)}^{ij} - 2m^{(0)}\text{Tr}g^{(2)}\big) \\ +\text{Tr}g^{(2)}\text{Tr}\sigma^{(0)} - \sigma_{ij}^{(0)} g_{(2)}^{ij}\Big), \tag{4.100}$$

which is equivalent to

$$\langle T_i^i\rangle^1 = \frac{iL}{\pi G_N}m^{(2)} \tag{4.101}$$

[6]Here $\langle T_i^i\rangle^1$ represents the first order of $\bar{G}^{ij}\langle T_{ij}\rangle$, i.e $\langle T_i^i\rangle^1 = \langle T_{ij}^1\rangle g_{(2)}^{ij} - \langle T_{ij}^0\rangle\sigma_{(0)}^{ij}$.

after using the relation (4.66), (4.68) and setting $d = 2$. As we have studied, $m^{(2)}$ could be an arbitrary scalar function therefore it will contribute to the anomaly at the subleading order in an arbitrary way. At the same time, one should note that the form of the anomaly is highly constrained in two dimensional conformal field theory Deser and Schwimmer (1993); Bonora et al. (1986), the anomaly should be proportional to the Euler density, which means one should consider a special class of asymptotically flat spacetime in order to make the boundary field theory conformal. Here, we choose to consider a set of solutions of $m^{(2)}$ written as

$$\mathrm{Tr}\sigma^{(2)} = \alpha M \mathrm{Tr} g^{(2)} \qquad dm^{(2)} = (1 - \frac{\alpha}{4}) M \mathrm{Tr} g^{(2)} \tag{4.102}$$

satisfying the constraint (4.66) and (4.67) for a real parameter α. Such choice could be treated as the fix of gauge for soft sector like the gauge of asymptotic AdS hyperboloids are fixed in terms of Fefferman Graham coordinates where $G^0_{\rho\rho} = 1/4\rho^2$. Therefore, we will have

$$\langle T^i_i \rangle^1 = -\frac{iL}{16\pi G_N}(2\alpha - 8)\mathrm{Tr} g^{(2)} m^{(0)}. \tag{4.103}$$

In such case, we can treat the contribution from $m^{(2)}$ as part of the central charge at subleading order. To determine the central charge at the order of $1/L$, we can use the relation

$$\langle T^i_i \rangle = -\frac{c}{24\pi} R\ [\bar{G}]. \tag{4.104}$$

where $R[\bar{G}] = R[g_0] + \frac{1}{L}(g^{ij}_{(0)} \delta R_{ij} - \sigma^{ij}_{(0)} R_{ij}[g_{(0)}])$. By checking the formula (4.103) and (4.104) specifically, we have

$$c = \frac{i3L^2}{4G_N} + \frac{iML3(\alpha - 2)}{2G_N} \tag{4.105}$$

from which we can see the central charge will have first order correction that depends on the geometry of spacetime characterised by the parameter M and α while we leave higher order correction for further investigation.

Before going to study the conservation laws of energy-stress tensor, we first recall some lessons learnt from the AdS/CFT correspondence. For the asymptotically AdS case, the spacetime behaves like a box and no particle could finally reach the infinity while this fits the calculation that the dual CFT energy-stress tensor is conserved. Following our definition of asymptotic flatness, the hard sector is built up by the AdS hyperboloid therefore the dual energy-stress tensor on the celestial sphere is expected to be conserved at leading order, written as

$$\bar{\nabla}^j \langle T_{ij} \rangle^0 = 0, \tag{4.106}$$

which can be deduced using (4.68) and (4.87). Comparing with the AdS spacetime, one of the main feature for asymptotically flat spacetime is that there could be gravitational radiation at the boundary and thus the system is not strictly closed i.e, energy could be carried in or away by particles passing through the null boundary. Now, for the stress tensor at subleading order, we have

$$\bar{\nabla}^j \langle T_{ij} \rangle^1 = -\frac{iL}{16\pi G_N} \big(2\nabla_i (\mathrm{Tr}\sigma^{(2)} - \sigma^{(0)}_{kl} g^{kl}_{(2)}) - 2\nabla^j \sigma^{(2)}_{ji} + \sigma^{mk}_{(0)} \nabla_m g^{(2)}_{ki} + \nabla^j \mathrm{Tr} g_{(2)} \sigma^{(0)}_{ij} \big) \tag{4.107}$$

where we have used the the relation (4.55) and the covariant derivative $\bar{\nabla}_i$ is with respect to the background metric $\bar{G}_{ij}$ of the boundary CFT. Moreover, taking the constraint (4.72) into consideration, we have

$$\bar{\nabla}^j \langle T_{ij} \rangle = \frac{iL}{16\pi G_N} (8\nabla_i m^{(2)} + 4M\nabla_i \mathrm{Tr} g^{(2)} - \sigma^{mk}_{(0)} \nabla_m g^{(2)}_{ki}), \tag{4.108}$$

from which we can see that the stress tensor is not conserved at the subleading order due to the existence of soft modes therefore we can interpret such soft modes as the radiation modes which generate the flow of energy through the null boundary.

Chapter 5

Conclusions

In this thesis, we have used the AdS/CFT dictionary to develop a holographic dictionary between flat space and celestial CFT. The key steps in our approach are transforming bulk fields from time to frequency representation, and using the usual AdS/CFT dictionary on spatial hyperbolic slices of the fields in mixed representation of frequency/hyperbolic spatial coordinates. We have shown that a single scalar field propagating in Minkowski is dual to two series of operators on the celestial sphere with scale dimensions on the principal series. One can physically interpret the two sets of operators as ingoing and outgoing modes.

Asymptotically (locally) flat spacetimes have as asymptotic symmetries the (extended) BMS groups at the null boundaries. Therefore, the total symmetry for given observable quantities should be $\mathrm{BMS}^+ \times \mathrm{BMS}^-$ since we have the null boundaries at far past and far future and we proposed that such symmetry is manifested by these two series of operators. Moreover in the work Strominger (2014), it was proposed that the symmetry which a quantum gravity scattering matrix should preserve is the subgroup $\mathrm{BMS}^0 \subset \mathrm{BMS}^+ \times \mathrm{BMS}^-$ by matching two null boundaries at the spatial infinity i^0. This fits with our observation that the two series of operators are dual to the ingoing and outgoing shock waves in the bulk and they are related by physical processes that occur in the center. From the boundary point of view, we can see that these two series of operators are coupled with each other.

There has recently been considerable discussion of the role of Carollian symmetry in flat space holography Hartong (2016); Ravera (2019); Bagchi et al. (2019); Bergshoeff et al. (2020); Hansen et al. (2022); Donnay et al. (2022); Bagchi et al. (2022); de Boer et al. (2023); Bagchi et al. (2023b,a); Nguyen and West (2023); Saha (2023a,b). It would be interesting to explore how the structure of the holographic dictionary for the metric can be interpreted in term of Carollian structure. In the context of Carollian CFTs, one can introduce the notion of Carrollian time t_c as the dual of effective mass $t_c \sim k$ and thus the series of correlation functions on the celestial sphere can be viewed as dual to a 3d correlation function, i.e.

$$\langle \mathcal{O}(z,\bar{z};k)\mathcal{O}(z',\bar{z}';k)\rangle \longleftrightarrow \langle \mathcal{O}(z,\bar{z},t_c)\mathcal{O}(z',\bar{z}',t_c)\rangle, \tag{5.1}$$

and these are related by the integral transform

$$\mathcal{O}(t_c,z,\bar{z}) = \int_{\mathcal{P}} dk\, G(t_c,k)\ \mathcal{O}(z,\bar{z};k), \tag{5.2}$$

in which the Green function $G(t_c,k)$ would be determined by the definition of Carollian time t_c together with the dynamical structure of the system. As we can see, it is easier to study the distribution of the scale dimensions and construct the dictionary using the operators in k space while it may be more convenient to study the symmetries and the evolution of the system in the proposed $3d$ spacetime. We will not go into detail of the integral transform and leave the explicit form of $G(t_c,k)$ for further investigation.

The key feature for the proposed flat/celestial CFT dictionary is that it reduces two dimensions from the bulk to boundary celestial sphere. The duality relates the bulk theory to a Euclidean CFT on the sphere, with the time dependence captured by the map of a single bulk 4d field to an infinite tower of CFT operators. Many subtle questions remain about the recovery of unitarity from the dual perspective. The scale dimensions of the CFT operators are complex therefore the Euclidean CFT is not unitary, yet many of the standard results used extensively in two dimensional CFTs, such as Cardy's formula, rely on unitarity. Recovery of unitarity from the dual perspective would rely on understanding how the boundary data in k space can be reinterpreted in the t_c domain. In particular, this would be necessary to explore how black hole information is recovered at the quantum level.

In our construction of the flat/CFT dictionary, one can see that the boundary correlation functions are determined by the coefficients $a^{\pm}_{lm}(k)$ which carry the information about the bulk solution. These coefficients are determined by specifying the data on the Cauchy surface of initial time and they govern the dynamical evolution of the system. To construct a proper defined quantum field theory, one should understand how constraints such as causality, Lorentz invariance and the cluster decomposition principle are related to this data. We will leave deeper exploration of such relations to further work.

We noted that one may use the data on two copies of the Euclidean AdS hyperboloid together with the equation of motion to reconstruct the linearised field in the whole Minkowski spacetime. However, one should note that these two AdS surfaces are not Cauchy surfaces according to the standard definitions given by Hawking and Ellis (2023); Wald (1984). A deeper understanding of the underlying structure will be helpful to study scattering amplitudes and the causal properties of spacetime.

Regarding the gravity part, we have developed the holographic renormalisation procedure for the gravitational action on the asymptotically flat background then obtain the flat/CFT dictionary between the $d+2$ dimensional theory in the bulk and the d dimensional CFT on the celestial sphere. Based on the construction of flat/CFT dictionary, we then obtain a precise map between the asymptotic bulk data and the conformal energy stress tensor. By considering the conformal anomalies, we deduce the value of the central charge up to the first subleading order, which comes from the soft sector of the energy stress tensor. It turns out that the central charge takes complex value approaching infinity. Such behaviour has already been argued for long while here, given the flat/CFT duality relation, we may ask if the complex central charge implies that the boundary CFT is not unitary.

From the literature, the definition of unitarity has two interpretations. One is that the transformations generated by the Hermitian conserved charges are unitary and it is equivalent to the reflection positivity of the correlation functions once the theory is Wick rotated to the Euclidean signature. The other definition for unitary is that the conformal blocks form the unitary representation of the conformal group based on the study of harmonic analysis of the Lorentz group. In the context of Celestial holography, the standard answer for the unitary problem is that the 2d correlators with scale dimensions on the principal series form the unitary representation of the Lorentz group $SO(1,3)$.

This is fine if one wants to construct the boundary celestial CFT by matching the symmetries between the bulk scattering amplitudes and the boundary field theory. But following the construction in this thesis, by extending the AdS/CFT dictionary to the flat case, we claim that the boundary theory is the CFT with complex scale dimension and central charge. This forces us to investigate the unitary problem following the first kind of definition which is the conventional unitary problem for a field theory. At first sight, the 2d CFT with complex scale dimension will violate the reflection positivity condition therefore will not be unitary but we are not assuming that the 2d Euclidean CFT obtained by the dictionary comes from the Lorentzian 1+1 CFT. As we have explained, one needs to sum over all the 2d operators as showed in (5.2) in order to reconstruct a 1+2 dimensional real time theory. The problem one should ask is that if such 1+2 dimensional theory is unitary or not. The answer is not clear to us while we think the answer is probably yes since these operators in 3d will form a unitary representation of Lorentz group (After performing the summation, we should consider the symmetry group $\mathfrak{bms}_4$.) therefore the theory defined by these operators should be unitary.

In the gravity part, we have obtained a complex central charge but this time it should not be a surprise since we have already known we are dealing with a CFT consisting of complex scale dimension operators. Discussing the unitary problem for a complex charge in our familiar real scale dimension CFT will not make too much sense. Given the complex central charge and stress tensor, one immediate question is whether the results are compatible with the CFT of complex scale dimension. Then what might be the physical interpretation in the bulk. As we have mentioned, the unitary problem should be studied when the real time 3d theory is well established and we leave such problem to the further discussion. Moreover, we can see the value of central charge is expressed as power expansion according to the energy cut off L and the physical implication of such expansion is also not clear from either bulk or boundary point of view.

The introduction of the cut-off L can be treated as an additional input when performing the renormalisation for the gravitational action. For scalar case in chapter 3, we also have the integral over τ from zero to infinity, while there is no need to introduce the cut off to make the action finite. That is because the integral over τ together with the weight function and τ modes will produce orthogonal relations telling us the coupling between different modes labelled by k. We have seen such orthogonal relation by solving Klein-Gordon equation explicitly while here the treatment for Einstein equation will be much harder as we have briefly discussed in section 4.2. In this thesis, we have not studied each graviton mode in a microscopic way while we choose to study those infinite number of modes macroscopically by introducing the cut-off L.

As we have seen, for the flat/CFT duality, most of the nontrivial results come from the contribution of the soft sector. It will lead to a non-conserved stress-energy tensor from the boundary point of view. Such stress tensor makes the behaviour of boundary CFT more complicated while it enables us to investigate the gravitational bulk radiation using the boundary data. Therefore the interpretation of non-conserved part of the energy-stress tensor is more like the introduction of heat bath or matter fields studied in the AdS/CFT correspondence.

The definition of asymptotic flatness is clarified in the whole thesis as (4.37) while one may ask if we could consider the asymptotically flat spacetime in a more general sense. It is interesting to explore how the renormalisation works if Fefferman Graham gauge is broken. For example, one can consider the case that $G^0_{\rho\rho}$ takes arbitrary form or $G^0_{\rho i} \neq 0$. For the spacetime in (4.37), the choice of spatial radius ρ on the AdS hyperboloid as the IR regulator is straight forward since it will not break the asymptotic symmetry while the development of holographic renormalisation will become more complicated if one wants to deal with more general metric.

For the soft sector, we also meet the similar problem like the gauge fixing of the AdS hyperboloid and this comes from the freedom of the choice of $\mathrm{Tr}\sigma^{(2)}$ or equivalently $m^{(2)}$. As we have seen, the trace part $\mathrm{Tr}\sigma^{(2)}$ tends to contribute to the subleading part of the anomalies of the stress tensor $\langle T^i_i \rangle$ in a arbitrary way while the form of Weyl anomaly is highly constrained from the CFT point of view. After the holographic renormalisation, to make the field theory coming from the bulk gravitational theory conformal, we have to further fix the gauge of $\mathrm{Tr}\sigma^{(2)}$ as shown in (4.102). Here we have the freedom to do so but this leads to the problem that if all the asymptotically flat gravitational theory is dual to the CFT on the boundary. In fact, the definition of asymptotically flat spacetime is a vague concept in gravity as we have discussed in the introduction. In addition to the Ricci flat condition $R_{\mu\nu} = 0$, one should also make the spacetime approach to flat at infinity so that recover enough flat space results and properties. On the other hand, the CFT is well studied thus such mismatching makes the construction of flat/CFT duality challenging.

Although there are various unresolved technical and conceptual challenges in constructing the flat/CFT dictionary, one can see that the main structure of the flat/CFT dictionary is already established. On the one hand, we can see that the dictionary works the same as the AdS/CFT dictionary, on the other hand, the results strongly suggest that the boundary theory behaves like a series of CFTs except for some complex features therefore we adopt the name flat/CFT for the new duality relation during the whole thesis and we hope the flat/CFT will play an important role just like its parent AdS/CFT but this time it will lead us to the real physical world.

A Embedding space

Bulk embedding space

In this section we review the embedding of hyperbolic space and AdS space into flat space in one dimension higher, focussing in particular on the differences between Euclidean AdS (hyperbolic space) and Lorentzian AdS.

We view H^{d+1} as a spacelike surface in $R^{d+1,1}$. The coordinates of $R^{d+1,1}$ are denoted as $X = (X^+, X^-, X^\mu)$ with $\mu = 0, \cdots (d-1)$, with the metric being:

$$ds^2 = -dX^- dX^+ + \delta_{\mu\nu} dX^\mu dX^\nu \tag{A.3}$$

Then H^{d+1} can be embedded as the spacelike surface $X^2 = -1$, which can be parameterized in Poincaré coordinates as

$$X = \frac{1}{z}\left(1, z^2 + y^2, y^\mu\right). \tag{A.4}$$

Note that as $z \to 0$ the embedding approaches $\frac{1}{z}P$ where P is null.

Lorentzian AdS$_{d+1}$ is embedded into $R^{d,2}$ as follows. The coordinates of $R^{d,2}$ are denoted as $X = (X^+, X^-, X^\mu)$ with $\mu = 0, \cdots (d-1)$, with the metric being:

$$ds^2 = -dX^- dX^+ + \eta_{\mu\nu} dX^\mu dX^\nu \tag{A.5}$$

AdS$_{d+1}$ can be embedded as the surface $X^2 = -1$ with signature (d,1), which can again be parameterized in Poincaré coordinates as (E.69) but with the induced metric now being

$$ds^2 = \frac{1}{z^2}\left(dz^2 + \eta_{\mu\nu} dy^\mu dy^\nu\right). \tag{A.6}$$

To distinguish different time directions it is convenient to use t and τ to label the Lorentzian and Euclidean time directions; they share the same coordinate expression $t, \tau = y^0 = X^0/X^+$. We will find it convenient to treat X^0 as a complex number with t and τ the real and imaginary part, respectively. In the embedding space the derivatives with respect to t and τ can be expressed as

$$\frac{\partial}{\partial t} = \frac{\partial X^\mu}{\partial t}\frac{\partial}{\partial X^\mu} = -\frac{2t}{z}\frac{\partial}{\partial X^-} + \frac{1}{z}\frac{1}{\partial X^0} = -2X^0\frac{\partial}{\partial X^-} + X^+\frac{\partial}{\partial X^0}, \tag{A.7}$$

$$\frac{\partial}{\partial \tau} = \frac{\partial X^\mu}{\partial \tau}\frac{\partial}{\partial X^\mu} = \frac{2\tau}{z}\frac{\partial}{\partial X^-} + \frac{1}{z}\frac{1}{\partial X^0} = 2X^0\frac{\partial}{\partial X^-} + X^+\frac{\partial}{\partial X^0}. \tag{A.8}$$

Conformal light cone

In this section we summarise the embedding of flat space into the conformal light cone of a flat space in two dimensions higher, focussing on the differences between Euclidean and Lorentzian flat space.

We denote the coordinates of the embedding space as $P = (P^+, P^-, P^\mu)$ with $\mu = 0, \cdots (d-1)$. In the case of Euclidean d-dimensional flat space the metric on the embedding space $R^{d+1,1}$ is

$$ds^2 = -dP^+dP^- + \delta_{\mu\nu}dP^\mu dP^\nu \tag{A.9}$$

The relation between the embedding coordinates and the flat space coordinates x^μ is determined by the conformal light cone conditions:

$$P^2 = 0 \qquad P = \lambda P \tag{A.10}$$

which are solved by $P = (1, x^2, x^\mu)$ with $x^2 = x^\mu x_\mu$.

For Lorentzian d-dimensional flat space the metric on the embedding space $R^{d,2}$ is

$$ds^2 = -dP^+dP^- + \eta_{\mu\nu}dP^\mu dP^\nu \tag{A.11}$$

The relation between the embedding coordinates and the flat space coordinates x^μ is still determined by the conformal light cone conditions (A.10).

Solutions

Here we briefly present the solutions that are used in section 2.1.2 and one can see more detail in Skenderis and van Rees (2009). Given the real time action on AdS_{d+1} background

$$S = -\frac{1}{2}\int d^{d+1}x\sqrt{-G}(\partial_\mu\Phi\partial^\mu\Phi + m^2\Phi^2) \tag{A.12}$$

with the metric G in E.57, we have the equation of motion

$$z^{d+1}\partial_z(z^{-d+1}\partial_z\Phi) + z^2\Box_0\Phi - m^2\Phi = 0 \tag{A.13}$$

with the solution in momentum space $q = (\omega, \vec{k})$

$$e^{-i\omega t+\vec{k}\cdot\vec{x}}z^{d/2}K_l(qz), \qquad e^{-i\omega t+\vec{k}\cdot\vec{x}}z^{d/2}I_l(qz), \tag{A.14}$$

in which K_l, I_l are two types of Bessel functions. $\Delta = \frac{d}{2} + l$ is defined as $m^2 = \Delta(\Delta - d)$. For spacelike momenta $q^2 > 0$ these modes are well behaved, while for timelike momenta, one need to specify how the contour of ω winds around the branch cuts of

$q = \sqrt{q^2}$. Usually we use the $i\epsilon$ prescription

$$q_\epsilon = \sqrt{-\omega^2 + \vec{k}^2 - i\epsilon} \tag{A.15}$$

to specify that we are applying the Feynman contours C over the complex ω plane. Moreover, the solutions behave like

$$z^{\frac{d}{2}} K_l \sim \frac{z^{d/2-l}}{q^l}, \qquad z^{\frac{d}{2}} I_l \sim \frac{z^{\frac{d}{2}+l}}{q^{-l}} \tag{A.16}$$

for $z \longrightarrow 0$ and

$$z^{\frac{d}{2}} K_l \sim \sqrt{\frac{z^{d-1}}{q}} e^{-qz}, \qquad z^{\frac{d}{2}} I_l \sim \sqrt{\frac{z^{d-1}}{q}} e^{qz} \tag{A.17}$$

when $z \longrightarrow \infty$. According to their asymptotics behavior, K_l are called source modes and contributes to the bulk-boundary propagator X_L as

$$X_L(t, \vec{x}, z) = \frac{1}{(2\pi)^d} \int_C d\omega \int d\vec{k} e^{-i\omega t + i\vec{k}\cdot\vec{x}} \frac{2^{l+1} q_\epsilon^l}{\Gamma(l)} z^{\frac{d}{2}} K_l(q_\epsilon z), \tag{A.18}$$

which leads to the familiar space time expression

$$X_L = i\Gamma(l)\Gamma(l + \frac{d}{2})\pi^{-\frac{d}{2}} \frac{z^{l+\frac{d}{2}}}{(-t^2 + \vec{x}^2 + z^2 + i\epsilon)^{l+\frac{d}{2}}}. \tag{A.19}$$

As for I_l, they are called normalizable modes and we should note they will become infinite at $z = \infty$ when q is space like. Therefore, we only consider the time like contribution

$$Y_L(t, \vec{x}, z) = \frac{1}{(2\pi)^d} \int_C d\omega \int d\vec{k} e^{-i\omega t + i\vec{k}\cdot\vec{x}} \, \theta(-q^2) b(\omega, \vec{k}) \, z^{\frac{d}{2}} J_l(|q|z), \tag{A.20}$$

in which $b(\omega, \vec{k})$ are undetermined coefficients and J_l is defined as $I_l(-i|q|z) = e^{-i\pi l/2} \times J_l(|q|z)$. One can obtain the Euclidean version by doing the Wick rotation $t = -i\tau$. For source contribution we have

$$X_E = i\Gamma(l)\Gamma(l + \frac{d}{2})\pi^{-\frac{d}{2}} \frac{z^{l+\frac{d}{2}}}{(\tau^2 + \vec{x}^2 + z^2 + i\epsilon)^{l+\frac{d}{2}}} \tag{A.21}$$

and the normalizable term is

$$Y_E(t, \vec{x}, z) = \frac{1}{(2\pi)^d} \int_C d\omega \int d\vec{k} e^{-|\omega\tau| + i\vec{k}\cdot\vec{x}} \, \theta(-q^2) b(\omega, \vec{k}) \, z^{\frac{d}{2}} J_l(|q|z), \tag{A.22}$$

in which we have chosen the positive frequency when $\tau > 0$. After lifting these to the embedding space, one should be able to recover the expressions shown at the beginning of section 2.1.2 by making

$$G_\Delta = X + Y. \tag{A.23}$$

B Solving BTZ

We start from the Klein-Gordon equation for the scalar field Φ, written as

$$\Box_G \Phi - m^2 \Phi = 0, \tag{B.24}$$

in which the Laplacian operator $\Box_G$ has the form of

$$\Box_G \Phi = \frac{1}{\sqrt{-G}} \partial_\mu (\sqrt{-G} G^{\mu\nu} \partial_\nu \Phi). \tag{B.25}$$

After using the BTZ black hole metric (2.78), we obtain the equation of motion in terms of the (r, t, ϕ) coordinates

$$-\frac{1}{r^2-1}\partial_t^2 \Phi + \frac{1}{r^2}\partial_\phi^2 \Phi + (r^2-1)\partial_r^2 \Phi + \frac{3r^2-1}{r}\partial_r \Phi - \Delta(\Delta-2) = 0, \tag{B.26}$$

in which we make $m^2 = \Delta(\Delta - 2)$. The modes of the above equation are given by

$$\psi = e^{i\omega t - ik\phi} f_\Delta(\omega, k, r), \tag{B.27}$$

in which ω, k tell us how the modes propagate along the circle and $f_\Delta(\omega, k, r)$ is given by

$$f_\Delta(\omega, k, r) = C_{\omega k \Delta} \left(1 - \frac{1}{r^2}\right)^{-\frac{i\omega}{2}} r^{-\Delta} H(\frac{1}{r^2}) \tag{B.28}$$

where $C_{\omega k \Delta}$ are normalization constants while the function $H(\frac{1}{r^2})$ is determined by Euler's hypergeometric differential equation

$$z(1-z)H'' + (\Delta - (\Delta + 1 - i\omega)z)H' - \frac{1}{4}(\Delta - i(\omega - k))(\Delta - i(\omega + k))H = 0, \tag{B.29}$$

where we have $z = \frac{1}{r^2}$ and the solutions are built from hypergeometric functions $F(a, b; c; z)$. Therefore in this case we have

$$a = \frac{\Delta}{2} - \frac{i}{2}(\omega + k), \qquad b = \frac{\Delta}{2} - \frac{i}{2}(\omega - k), \qquad c = \Delta = l + 1. \tag{B.30}$$

Note that hypergeometric functions are locally expressed as power series and it converges when $|z| < 1$ while the function over the whole complex z plane can be obtained by the analytic continuation. Moreover, for physical systems, ω and k are usually non-integral thus the form of the solutions are mainly determined by the value of Δ. Now, we discuss the solutions in two cases.

i) *None of the numbers c, $c - a - b$; $a - b$ is equal to an integer.*

In this case, solutions can be expanded as combination of two independent power series at each singular point $z = 0, 1, \infty$. Here we will only consider the behavior of the solution around the horizon $z = 1$ and the infinity $z = 0$. At $z = 1$, solutions depend on the frequency ω and two independent solutions are given by

$$H_{-\omega}(z) = F(\frac{\Delta}{2} - \frac{i}{2}(\omega + k), \frac{\Delta}{2} - \frac{i}{2}(\omega - k); 1 - i\omega; 1 - z) \tag{B.31}$$

and

$$H_{\omega}(z) = (1-z)^{i\omega} F(\frac{\Delta}{2} + \frac{i}{2}(\omega - k), \frac{\Delta}{2} + \frac{i}{2}(\omega + k); 1 + i\omega; 1 - z). \tag{B.32}$$

Taking these two into consideration at the same time, we can write the modes as

$$\psi_{\pm} = e^{\pm i\omega t - ik\phi} f_{\Delta}(\pm\omega, k, r), \tag{B.33}$$

in which

$$f_{\Delta}(\pm\omega, k, r) = C_{\omega k \Delta} \left(1 - \frac{1}{r^2}\right)^{-\frac{i\omega}{2}} r^{-\Delta} H_{\pm\omega}(\frac{1}{r^2}) \tag{B.34}$$

At $z = 0$, we also have two independent solutions while they now depend on the scale dimension Δ and we write them as

$$H_{\Delta^+}(z) = F(\frac{\Delta}{2} - \frac{i}{2}(\omega + k), \frac{\Delta}{2} - \frac{i}{2}(\omega - k); \Delta; z) \tag{B.35}$$

and

$$H_{\Delta^-}(z) = z^{1-\Delta} F(1 - \frac{\Delta}{2} - \frac{i}{2}(\omega + k), 1 - \frac{\Delta}{2} - \frac{i}{2}(\omega - k); 2 - \Delta; z). \tag{B.36}$$

Similar to the $z = 1$ case, we can write the modes as

$$\psi_{\pm} = e^{i\omega t - ik\phi} f_{\Delta^{\pm}}(\omega, k, r), \tag{B.37}$$

in which

$$f_{\Delta^{\pm}}(\omega, k, r) = C_{\omega k \Delta} \left(1 - \frac{1}{r^2}\right)^{-\frac{i\omega}{2}} r^{-\Delta} H_{\Delta^{\pm}}(\frac{1}{r^2}). \tag{B.38}$$

ii) $c = \Delta = l + 1$ *is an integer for* $l = 1, 2, 3, \ldots$

In this case, we will try to get the solutions of the equation from two ways. From one hand, we can apply the formula in Abramowitz et al. (1988) for integer l directly and then obtain the fundamental system of the solution, given by

$$H_{1(0)}(z) = F(a, b; l + 1, z), \tag{B.39}$$

and

$$H_{2(0)}(z) = F(a,b;l+1;z)\ \mathrm{ln}z + \sum_{n=1}^{m} \frac{(a)_n(b)_n}{(1+l)_n n!} z^n (\psi(a+n) - \psi(a) + \psi(b+n) \tag{B.40}$$

$$-\psi(b) - \psi(l+1+n) + \psi(l+1) - \psi(n+1) + \psi(1)) - \sum_{n=1}^{\infty} \frac{(n-1)!(-m)_n}{(1-a)_n(1-b)_n} z^{-n}.$$

These are two independent solutions at $z = 0$ and one can obtain the solution at $z = 1, \infty$ through the analytic continuation.

From the other hand, we can treat the integer case $\Delta = l+1$ as the limit of the general case $\Delta = l+1+\delta$ after taking $\delta \to 0$. This allows us to use the solutions for non-integer Δ thus their physical meanings are retained. To take the $\delta \to 0$ limit, we need to take care of the coefficients of the power series since they may have poles at integer Δ. Taking the solution $H_{-\omega}$ for example. If we expand $H_{-\omega}$ around $z = 0$ as power series, the term $\Gamma(k-l)$ will show up in the denominator of the coefficients of z^k which leads to the poles when $k - l = 1$. In fact, we can resolve those poles by choosing the hypergeometric function transformation $z \to 1 - z$ for integer Δ then get

$$H_{-\omega}(z) = \frac{\Gamma(l)\Gamma(a+b-l)}{\Gamma(a)\Gamma(b)} z^{-l} \sum_{n=0}^{l-1} \frac{(a-l)_n(b-l)_n}{n!(1-m)_n} z^n - \frac{(-1)^l \Gamma(a+b-l)}{\Gamma(a-l)\Gamma(b-l)} \tag{B.41}$$

$$\times \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(m+n)!} z^{-n} \left(\mathrm{ln}z - \psi(n+1) - \psi(n+l+1) + \psi(a+n) + \psi(b+n)\right), \tag{B.42}$$

from which we can see that the infinite term in the coefficients are transformed to the function lnz. Given the above solution, we choose the normalization constants to be

$$C_{\pm\omega kl} = \frac{\Gamma(\frac{1}{2}(l+1) + \frac{i}{2}(\pm\omega - k))\Gamma(\frac{1}{2}(l+1) + \frac{i}{2}(\pm\omega + k))}{\Gamma(l)\Gamma(1 \pm i\omega)} \tag{B.43}$$

so that the coefficient of the leading term r^{l-1} in $\psi_\pm$ turns to be 1. Therefore, the solution can be written as

$$\psi = e^{\pm i\omega t - ik\phi}(r^{l-1} + \cdots + \alpha(\pm\omega, k, l) r^{-l-1}[\ln(r^2) + \beta(\pm\omega, k, l)] + \cdots), \tag{B.44}$$

in which we just show the r^{l-1} and r^{-l-1} term and the coefficients α, β are given by

$$\alpha(\pm\omega, k, l) = (-1)^l \frac{(\frac{i}{2}(\pm\omega + k) + \frac{1}{2}(1+l))_l (\frac{i}{2}(\pm\omega - k) + \frac{1}{2}(1+l))_l}{l!(l-1)!}, \tag{B.45}$$

$$\beta(\pm\omega, k, l) = -\psi(\frac{i}{2}(\pm\omega + k) + \frac{1}{2}(1+l)) - \psi(\frac{i}{2}(\pm\omega - k) + \frac{1}{2}(1+l)) + \psi(1) + \psi(l+1). \tag{B.46}$$

C Complex Scalar Matching

In this section, we will derive the matching conditions for the complex scalar fields. We start from writing down the action for the complex scalar ϕ. In Lorentzian signature we have

$$S_L = \frac{1}{2}\int_{M_L} \sqrt{-G}(-\partial_\mu \phi_L^* \, \partial^\mu \phi_L - m^2 \phi_L^* \phi_L) \tag{C.47}$$

while for Euclidean signature the action is

$$S_E = \frac{1}{2}\int_{M_E} \sqrt{G}(\partial_\mu \phi_E^* \, \partial^\mu \phi_E + m^2 \phi_E^* \phi_E), \tag{C.48}$$

in which we denote the Lorentzian spacetime and Euclidean spacetime as M_L and M_E joint at the codimension one surface Σ. As we have discussed, the continuation of the state implies the first matching condition $\phi_E = \phi_L$. Moreover, the stationarity of the total on-shell action with respect to ϕ and ϕ^* tells us

$$\frac{\delta}{\delta\phi}\,(\, iS_L - S_E) = \int_\Sigma \sqrt{K}(-i\partial_t \phi_L^* - \partial_t \phi_E^*) = 0, \tag{C.49}$$

$$\frac{\delta}{\delta\phi^*}\,(\, iS_L - S_E) = \int_\Sigma \sqrt{K}(-i\partial_t \phi_L - \partial_t \phi_E) = 0, \tag{C.50}$$

in which K is the intrinsic curvature induced on Σ. Therefore, we obtain the second matching condition written as

$$i\partial_t \phi_L + \partial_t \phi_E = 0 \qquad \text{and} \qquad i\partial_t \phi_L^* + \partial_t \phi_E^* = 0. \tag{C.51}$$

D Quadratic Matching

Here we present the detail of the verification of (2.137). The Euclidean and Lorentzian propagator are given by

$$G_\Delta^E(X,P) = \frac{1}{(-2X\cdot P)_E^\Delta} \qquad G_\Delta^L(X,P) = \frac{1}{(-2X\cdot P)_L^\Delta}. \tag{D.52}$$

Acting ∂_t^2 and ∂_τ^2 on them, we obtain

$$\partial_\tau^2 \frac{1}{(-2X\cdot P)_E^\Delta} = \frac{4\Delta(\Delta+1)}{(-2X\cdot P)_E^{\Delta+2}}(-X^0P^- + X^+P^0)^2 - \frac{2\Delta}{(-2X\cdot P)_E^{\Delta+1}} X^+P^- \tag{D.53}$$

and

$$\partial_t^2 \frac{1}{(-2X\cdot P)_L^\Delta} = \frac{4\Delta(\Delta+1)}{(-2X\cdot P)_L^{\Delta+2}}(X^0P^- - X^+P^0)^2 + \frac{2\Delta}{(-2X\cdot P)_L^{\Delta+1}} X^+P^-. \tag{D.54}$$

Acting ∂_t^3 and ∂_τ^3 on the propagator we have

$$\partial_\tau^3 \frac{1}{(-2X\cdot P)_E^\Delta} = \frac{8\Delta(\Delta+1)(\Delta+2)}{(-2X\cdot P)_E^{\Delta+2}}(-X^0P^- + X^+P^0)^3 \\ -X^+P^- \frac{8\Delta(\Delta+1)}{(-2X\cdot P)_E^{\Delta+1}}(-X^0P^- + X^+P^0) \qquad \text{(D.55)}$$

and

$$\partial_t^3 \frac{1}{(-2X\cdot P)_L^\Delta} = \frac{8\Delta(\Delta+1)(\Delta+2)}{(-2X\cdot P)_L^{\Delta+2}}(X^0P^- - X^-P^0)^3 \\ +X^+P^- \frac{8\Delta(\Delta+1)}{(-2X\cdot P)_L^{\Delta+1}}(X^0P^- - X^+P^0). \qquad \text{(D.56)}$$

One can directly check that (2.137) is true at the joint and surface $t = T$, $\tau = iT$, after taking the rotation of coordinates $P_0^E = iP_0^L$ into consideration.

E Coordinates

In this section, we will introduce various kinds of coordinates for Minkowski space that are convenient for us to reduce the data to the AdS hyperboloid, which are used many times in this thesis. The flat space time is described by the metric $\eta_{\mu\nu}$ for $\mu, \nu = 0, 1, 2, 3$, with diagonal elements $\eta_{00} = -1$ $\eta_{11} = \eta_{22} = \eta_{33} = 1$, written us

$$ds^2 = \eta_{\mu\nu}dX^\mu dX^\nu = -(dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2, \qquad \text{(E.57)}$$

in which (X^0, X^1, X^2, X^3) are the chosen coordinates. Here we just focus on the four dimensional spacetime and the codimension one AdS_3 hypersurface characterised by the radius τ can be treated as the embedding

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 = -\tau^2, \qquad \text{(E.58)}$$

where we should note that here the flat Minkowski space is the physical space and the AdS_3 surfaces are introduced for the decomposition of data. The timelike wedge in Minkowski which can be foliated by the AdS surfaces are so called Milne wedge.

Moreover, given such foliation, one can introduce global coordinates $(\tau, \eta, \theta, \phi)$ to cover the Milne wedge. The transformation is given by

$$X^0 = \tau\ \cosh\eta, \qquad \text{(E.59)}$$
$$X^1 = \tau\ \sin\theta\ \sin\phi\ \sinh\eta, \qquad \text{(E.60)}$$
$$X^2 = \tau\ \sin\theta\ \cos\phi\ \sinh\eta, \qquad \text{(E.61)}$$
$$X^3 = \tau\ \cos\theta \sinh\eta, \qquad \text{(E.62)}$$

in which one can see the relation (E.58) is automatically satisfied. (θ, ϕ) are coordinates on the sphere and τ is the radius of the AdS surface. The spatial distance from the origin on the hyperboloid is described by $\sinh\eta$. In the global coordinate, the metric now becomes

$$ds^2 = -d\tau^2 + \tau^2 \left(d\eta^2 + \sinh^2\eta \, d\Omega_2^2 \right), \tag{E.63}$$

in which the metric on the standard sphere S^2 is given by

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{E.64}$$

$$= \frac{4}{(1+z\bar{z})^2} \, dz d\bar{z} = 2\gamma_{z\bar{z}} \, dz d\bar{z}. \tag{E.65}$$

The complex coordinates $(z, \bar{z})$ on the plane are obtained by the stereographic projection from the sphere

$$z = e^{i\phi} \tan\frac{\theta}{2} \qquad \bar{z} = e^{-i\phi} \tan\frac{\theta}{2}. \tag{E.66}$$

As we have mentioned, the value $\sinh\eta$ makes more sense as a physical quantity thus one can define $\rho = \sinh\eta$ then the metric now becomes

$$ds^2 = -d\tau^2 + \tau^2 \left(\frac{d\rho^2}{1+\rho^2} + 2\rho^2 \gamma_{z\bar{z}} dz d\bar{z} \right), \tag{E.67}$$

which is the standard form of Milne coordinates in the literature.

To study a single AdS surface, sometimes it is more convenient to introduce Poincare coordinates (t, x, y) defined as

$$t = \frac{1}{X^0 + X^3}, \qquad x = \frac{X^1}{X^0 + X^3}, \qquad y = \frac{X^2}{X^0 + X^3}, \tag{E.68}$$

and after setting $\tau = 1$, one can pull back the metric to the AdS surface then obtain

$$ds^2_{\mathrm{AdS}_3} = \frac{dt^2 + dx^2 + dy^2}{t^2} = \frac{dt^2 + d\omega d\bar{\omega}}{t^2}, \tag{E.69}$$

in which $\omega = x + iy$. In terms of global coordinates, the Poincare coordinates can be written as

$$t = \frac{1}{\cosh\eta + \cos\theta \sinh\eta}, \tag{E.70}$$

$$x = \frac{\sin\phi \sin\theta \sinh\eta}{\cosh\eta + \cos\theta \sinh\eta}, \tag{E.71}$$

$$y = \frac{\cos\phi \sin\theta \sinh\eta}{\cosh\eta + \cos\theta \sinh\eta}, \tag{E.72}$$

and $(\omega, \bar{\omega})$ takes the form of

$$\omega = \frac{e^{i\phi} \sin\theta \sinh\eta}{\cosh\eta + \cos\theta \sinh\eta}, \qquad \bar{\omega} = \frac{e^{-i\phi} \sin\theta \sinh\eta}{\cosh\eta + \cos\theta \sinh\eta}. \tag{E.73}$$

One should note that, in the large η limit, $(\omega, \bar{\omega})$ will tend to $(z, \bar{z})$ thus it becomes complex coordinates of celestial sphere on the boundary.

In terms of Poincare coordinates, the boundary-bulk propagator $K_\Delta(t, x, y; x', y')$ for massless fields is given by

$$K(t, x, y; x', y') = \frac{1}{\pi}\left(\frac{t}{t^2 + (x - x')^2 + (y - y')^2}\right)^\Delta = \frac{1}{\pi}\left(\frac{t}{t^2 + (\omega - z')(\bar{\omega} - \bar{z}')}\right)^\Delta, \tag{E.74}$$

in which (x', y') are the points on the boundary and $z' = x' + iy'$. Following the dictionary for the massless particles of dimension two, we have $\Delta = 2$. The propagator could also be written in terms of global coordinates and at large ρ, one can check it takes the form

$$K^{\rho=\infty}(\rho, z; z') = \frac{(1 + z\bar{z})^\Delta}{\pi\rho^\Delta} \frac{1}{|z - z'|^{2\Delta}} = \frac{2}{\pi\Omega_2(z)\ \rho^\Delta} \frac{1}{|z - z'|^{2\Delta}}, \tag{E.75}$$

in which $\Omega_2(z) dz d\bar{z} = d\Omega_2$ is the volume form of the standard sphere in terms of complex coordinates. If we treat $K(\rho, z, z)$ as a distribution i.e., just consider the behaviour under the integral over $z, \bar{z}$, the boundary-bulk propagators are in fact equivalent to the delta function between boundary points Witten (1998a), i.e, we have

$$\frac{(1 + z\bar{z})^\Delta}{\pi\rho^\Delta} \frac{1}{|z - z'|^{2\Delta}} + \cdots = K(\rho, z, z') \xrightarrow{\rho=\infty} \delta(z - z'), \tag{E.76}$$

in which we have done the expansion of $K(\rho, z; z')$ at large radius ρ. Therefore, a generic field in the bulk with boundary behaviour $\varphi(\rho, z) \sim \varphi(z)$ can be expressed as

$$\varphi(\rho, z) = \int_{S^2} \frac{1}{2} dz' d\bar{z}'\ K(\rho, z; z') \varphi(z'), \tag{E.77}$$

in which we used the relation $dz d\bar{z} = 2 dx dy$ and for the value of the field at large ρ we just need to consider the first term in (E.76) thus make $K = K^{\rho=\infty}$. Here we should note that, by considering the property of the Green function

$$\int d^2 z' \delta(z - z') \delta(z' - z'') = \delta(z - z'') \tag{E.78}$$

we have the contracting relation for the propagator

$$\int_{S^2} \Omega_2(z') dz' d\bar{z}' \left(\frac{2}{\pi\Omega_2(z')\rho}\right)^\Delta \frac{1}{|z - z'|^{2\Delta}} \frac{1}{|z' - z''|^{2\Delta}} = \frac{4}{\pi} \frac{1}{|z - z''|^{2\Delta}}, \tag{E.79}$$

which turns out to be useful in simplifying the calculation.

F Flat Solutions

In this section, we will present the solution of equation on the AdS hyperboloid (3.11) written as

$$\left(-\partial_\eta^2 - 2(\coth\eta)\partial_\eta + l(l+1)\text{csch}^2\eta + k^2\right)\phi_l(\eta;k) = 0, \tag{F.80}$$

for $\rho = \sinh\eta$. Solutions can be found at the boundary and origin respectively and written as the expansion of proper basis. We should note that the basis at the origin and boundary are dependent and they are related via transformation, which we will see in the end of this section. Mode solutions for Lorentzian AdS have been studied in Balasubramanian et al. (1999) while solutions for dS modes haven been studied in Liu and Lowe (2021); Laddha et al. (2022).

Behaviour at the boundary

For the solution at the boundary, we first choose to write them in terms of hypergeometric functions and then transform them into the associated Legendre functions. In order to transform the equation into the standard form for hypergeometric functions, we write the solution into the form of

$$\phi_{lk}(\eta) = \frac{f_{\beta l}(\frac{1}{\sinh^2\eta})}{\sinh^{\beta+1}\eta}, \tag{F.81}$$

in which $\beta^2 = 1 + k^2$ and f depends on η for $\eta \geq 0$. Now the equation (F.80) becomes

$$4x(x+1)f''_{\beta l}(x) + 2(2(1+\beta) + (3+2\beta)x)f'_{\beta l}(x) - (l(l+1) - \beta(\beta+1))f_{\beta l}(x) = 0, \tag{F.82}$$

in which x is defined as

$$x = \frac{1}{\sinh^2\eta}. \tag{F.83}$$

Here we should note that the above equation is still not in the form of hypergeometric equation because of the $x(x+1)$ term in front of $f''_{\beta l}(\eta)$. Thus we further do the transformation $x \to x-1$ then obtain the equation

$$x(1-x)p''_{\beta l}(x) - \frac{1}{2}((3+2\beta)x - 1)p'_{\beta l}(x) + \frac{1}{4}(l(l+1) - \beta(1+\beta))p_{\beta l}(x) = 0, \tag{F.84}$$

in which $p_{\beta l}(x)$ is defined as

$$p_{\beta l}(x) = f_{\beta l}(x-1). \tag{F.85}$$

Given the equation (F.84), one can write down the solution at $x = 1$ as

$$p_{\beta l}(x) = {}_2F_1\left(\frac{1}{2} + \frac{l}{2} + \frac{\beta}{2}, -\frac{l}{2} + \frac{\beta}{2}\, ;\, 1+\beta\, ;\, 1-x\right) \tag{F.86}$$

therefore the $f_{\beta l}(\eta)$ is then deduced to be

$$f_{\beta l}\left(\frac{1}{\sinh^2\eta}\right) = {}_2F_1\left(\frac{1}{2}+\frac{l}{2}+\frac{\beta}{2}, -\frac{l}{2}+\frac{\beta}{2}\ ;\ 1+\beta\ ;\ -\frac{1}{\sinh^2(\eta)}\right). \tag{F.87}$$

Furthermore, after applying the transformation for hypergeomtric functions

$${}_2F_1(\frac{a+c-1}{2}, \frac{c-a}{2};\ c\ ;4z(1-z)) = (1-z)^{1-c}{}_2F_1(1-a, a\ ;\ c\ ;z) \tag{F.88}$$

for

$$c = 1+\beta, \qquad a = l+1, \qquad z = \frac{1}{2}(1-\coth\eta) \tag{F.89}$$

to the solution (F.87), we get

$$\phi_{l\beta}(\eta) = 2^{\beta}\frac{e^{-\beta\eta}}{\sinh(\eta)}\ {}_2F_1(-l, l+1\ ;\ 1+\beta\ ;\ \frac{1}{2}(1-\coth\eta)). \tag{F.90}$$

Noting that β could take both of the value $\beta_{\pm} = \pm\sqrt{1+k^2}$, one finally concludes the two independent solutions are

$$\phi_l(\eta;\beta_+) = \frac{\Gamma(1-\beta_+)}{(-2)^{\beta_+}}\frac{P_l^{\sqrt{1+k^2}}(\coth\eta)}{\sinh\eta}, \qquad \phi_l(\eta;\beta_-) = \frac{\Gamma(1-\beta_-)}{(-2)^{\beta_-}}\frac{P_l^{-\sqrt{1+k^2}}(\coth\eta)}{\sinh\eta}, \tag{F.91}$$

in which we have taken the factor $\Gamma(1\pm\beta)$ into consideration.

Behaviour at the origin

To study the behaviour of the solution at the origin, denoted as $\chi_l(\eta;k)$, we choose to write the function into the form of

$$\chi_l(\eta;k) = \sinh^a\eta\ f_{\beta l}(\sinh^2\eta) \tag{F.92}$$

in which a should satisfy the relation

$$a(a+1) = l(l+1) \tag{F.93}$$

so that the equation can be recast into the hypergeometric form

$$x(1-x)q''_{\beta l}(x) - \frac{1}{2}(-1+2(2+a)x)q'_{\beta l}(x) + \frac{1}{4}(\beta^2-a^2-2a)q_{\beta l}(x) = 0, \tag{F.94}$$

in which again β^2 takes the value $1+k^2$ and the function $q_{l\beta}(x)$ is defined as

$$q_{\beta l}(x) = f_{\beta l}(x-1). \tag{F.95}$$

Given the hypergeometic equation, solutions are then deduced to be

$$f_{\beta l}(\sinh^2\eta) = {}_2F_1(\frac{1}{2}+\frac{a}{2}+\frac{\beta}{2}, \frac{1}{2}+\frac{a}{2}-\frac{\beta}{2}\ ;\ \frac{1}{2}\ ;\ \cosh^2\eta), \tag{F.96}$$

in which we have set $\beta = \sqrt{1+k^2}$. More precisely, for $a = l$ we have

$$\chi_l^1(\eta;k) = \sinh^l(\eta)\ {}_2F_1(\frac{1}{2}+\frac{l}{2}+\frac{\beta}{2}, \frac{1}{2}+\frac{l}{2}-\frac{\beta}{2}\ ;\ \frac{1}{2}\ ;\ \cosh^2\eta) \tag{F.97}$$

while for $a = -1-l$ the solution becomes

$$\chi_l^2(\eta;k) = \sinh^{-l-1}(\eta)\ {}_2F_1(-\frac{l}{2}-\frac{\beta}{2}, -\frac{l}{2}+\frac{\beta}{2}\ ;\ \frac{1}{2}\ ;\ \cosh^2\eta). \tag{F.98}$$

Here, we are just interested in the solution $\chi_l^l(\eta;k)$ since it is the regular solution around the origin for $l \geq 0$ and one can verify that, by using the transformation rule for hypergeometric function

$$\begin{aligned} {}_2F^1(a,b;c;z) &= \frac{(1-z)^{-a}\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(b-a)} {}_2F^1\left(a, c-b; a-b+1; \frac{1}{1-z}\right) && \text{(F.99)} \\ &+ (1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} {}_2F^1\left(b, c-a; b-a+1; \frac{1}{1-z}\right), && \text{(F.100)} \end{aligned}$$

it can be written in terms of the solution at the boundary as

$$\chi_l^1(\eta;k) = C_l^+(k)\phi_l(\eta;\beta_+) + C_l^-(k)\phi_l(\eta;\beta_-), \tag{F.101}$$

in which $C_l^\pm(k)$ are the coefficients given by

$$C_l^+(k) = (-i)^{1+l+\beta}\frac{\Gamma(\frac{1}{2})\Gamma(-\beta)}{\Gamma(\frac{1}{2}+\frac{l}{2}-\frac{\beta}{2})\Gamma(-\frac{l}{2}-\frac{\beta}{2})} \tag{F.102}$$

and

$$C_l^-(k) = (-i)^{1+l-\beta}\frac{\Gamma(\frac{1}{2})\Gamma(\beta)}{\Gamma(\frac{1}{2}+\frac{l}{2}+\frac{\beta}{2})\Gamma(-\frac{l}{2}+\frac{\beta}{2})}. \tag{F.103}$$

Ratio

In order to obtain the CFT two-point function on the celestial sphere, one should calculate the functional derivative of the one-point function with respective to the source. Moreover, with the help of AdS/CFT dictionary, the functional derivative is given by the ratio of coefficients, written as

$$\frac{C_l^-(k)}{C_l^+(k)} = (-1)^\beta \frac{\Gamma(\beta)\Gamma(\frac{1}{2}+\frac{l}{2}-\frac{\beta}{2})\Gamma(-\frac{l}{2}-\frac{\beta}{2})}{\Gamma(-\beta)\Gamma(\frac{1}{2}+\frac{l}{2}+\frac{\beta}{2})\Gamma(-\frac{l}{2}+\frac{\beta}{2})}. \tag{F.104}$$

To simplify above expression, we first use the recurrence formula for Gamma function given by

$$\Gamma(-\frac{l}{2}-\frac{\beta}{2})(-\frac{l}{2}-\frac{\beta}{2})(-\frac{l}{2}+1-\frac{\beta}{2})\cdots(\frac{l}{2}-1-\frac{\beta}{2})=\Gamma(\frac{l}{2}-\frac{\beta}{2}) \tag{F.105}$$

and

$$\Gamma(-\frac{l}{2}+\frac{\beta}{2})(-\frac{l}{2}+\frac{\beta}{2})(-\frac{l}{2}+1+\frac{\beta}{2})\cdots(\frac{l}{2}-1+\frac{\beta}{2})=\Gamma(\frac{l}{2}+\frac{\beta}{2}) \tag{F.106}$$

to transform $\Gamma(-\frac{l}{2}\pm\frac{\beta}{2})$ into $\Gamma(\frac{l}{2}\pm\frac{\beta}{2})$. Therefore, one can write the ratio (F.104) into the form of

$$(-1)^{\beta}\,\frac{\Gamma(\beta)\Gamma(\frac{1}{2}+\frac{l}{2}-\frac{\beta}{2})\Gamma(\frac{l}{2}-\frac{\beta}{2})}{\Gamma(-\beta)\Gamma(\frac{1}{2}+\frac{l}{2}+\frac{\beta}{2})\Gamma(\frac{l}{2}+\frac{\beta}{2})}\times\frac{(-\frac{l}{2}+\frac{\beta}{2})(-\frac{l}{2}+1+\frac{\beta}{2})\cdots(\frac{l}{2}-1+\frac{\beta}{2})}{(-\frac{l}{2}-\frac{\beta}{2})(-\frac{l}{2}+1-\frac{\beta}{2})\cdots(\frac{l}{2}-1-\frac{\beta}{2})}. \tag{F.107}$$

After applying the Legendre duplication formula

$$\Gamma(z)\Gamma(z+\frac{1}{2})=2^{1-2z}\sqrt{\pi}\Gamma(2z) \tag{F.108}$$

for $z=\frac{l}{2}\pm\frac{\beta}{2}$, we have

$$\Gamma(\frac{1}{2}+\frac{l}{2}\pm\frac{\beta}{2})\Gamma(\frac{l}{2}\pm\frac{\beta}{2})=\sqrt{\pi}2^{1-(l\pm\beta)}\Gamma(l\pm\beta). \tag{F.109}$$

For the part on right of (F.107), one should notice that

$$\begin{aligned}&\frac{(-\frac{l}{2}+\frac{\beta}{2})(-\frac{l}{2}+1+\frac{\beta}{2})\cdots(\frac{l}{2}-1+\frac{\beta}{2})}{(-\frac{l}{2}-\frac{\beta}{2})(-\frac{l}{2}+1-\frac{\beta}{2})\cdots(\frac{l}{2}-1-\frac{\beta}{2})}\\&=(-1)^{l}\frac{(-\frac{l}{2}+1-\frac{\beta}{2})(-\frac{l}{2}+2-\frac{\beta}{2})\cdots(\frac{l}{2}-\frac{\beta}{2})}{(-\frac{l}{2}-\frac{\beta}{2})(-\frac{l}{2}+1-\frac{\beta}{2})\cdots(\frac{\beta}{2}-1-\frac{\beta}{2})}=(-1)^{l}\frac{\beta-l}{\beta+l}\end{aligned} \tag{F.110}$$

After substituting (F.109) and (F.110) into (F.107), one has

$$\frac{C_l^-(k)}{C_l^+(k)} = (-1)^{l+\beta}\,2^{2\beta}\,\frac{\Gamma(\beta)\Gamma(l-\beta)(\beta-l)}{\Gamma(-\beta)\Gamma(l+\beta)(\beta+l)} \tag{F.111}$$

$$= (-1)^{l+\beta+1}\,4^{\beta}\,\frac{\Gamma(\beta)\Gamma(l-\beta+1)}{\Gamma(-\beta)\Gamma(l+\beta+1)} \tag{F.112}$$

$$= (-1)^{l+\beta+1}\,4^{\beta}\,\frac{B(\beta,l-\beta+1)}{B(-\beta,l+\beta+1)}, \tag{F.113}$$

where we have written the result in terms of Beta function $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ in the third line.

G Harmonic Modes

Like the Fourier transform between the spacetime and momentum, the transformation between the discrete mode variables (l, m) and complex coordinates $(z, \bar{z})$ on the sphere

$$(l, m) \longleftrightarrow (z, \bar{z}) \tag{G.114}$$

are related by the spherical harmonics $Y^l_m(z, \bar{z})$. More precisely, as shown in (3.16), the transformation is realised via the expansion

$$F_k(\rho, z, \bar{z}) := \sum_{lm} F_{k,l,m}(\rho, z, \bar{z}) = \sum_{l,m} \phi_l(\rho; k) Y^l_m(z, \bar{z}) \tag{G.115}$$

in which $F_k(\rho, z, \bar{z})$ is the spatial k mode that depends on $(\rho, z, \bar{z})$ and $\phi_l(\rho; k)$ is the associated expression in the mode variables (ρ, l, m). The m dependence is suppressed since the equation of motion on the AdS hyperboloid does not depend on m [1]. Given the solution $\phi_l(\rho; \beta_\pm)$ and their asymptotic expansion at infinity

$$\phi_l(\rho; \beta_\pm) = \rho^{\beta_\pm - 1}(\phi^\pm_l(k) + \mathcal{O}(\frac{1}{\rho^2})), \tag{G.116}$$

one can immediately obtain the dictionary for AdS/CFT in the form of mode variables (m, l), written as

$$\hat{\mathcal{J}}_{lm}(k) = \phi^+_l(k) \qquad \langle \hat{\mathcal{O}}_{lm}(k) \rangle = -2i\beta_+ \, \phi^-_l(k) \qquad \text{for} \qquad -l \leq m \leq l, \tag{G.117}$$

in which $\hat{\mathcal{J}}_{lm}(k)$ and $\langle \hat{\mathcal{O}}_{lm}(k) \rangle$ are the corresponding source and one-point function that lives on the boundary celestial sphere. Here they are not required to be physical operators and sources thus we can treat them as virtual particles by construction. In terms of $(z, \bar{z})$ coordinates, they should have the form of

$$\hat{\mathcal{J}}(z, \bar{z}; k) = \sum_{l,m} \phi^+_l(k) Y^l_m(z, \bar{z}), \qquad \langle \hat{\mathcal{O}}(z, \bar{z}; k) \rangle = -\sum_{lm} 2i\beta_+ \, \phi^-_l(k) Y^l_m(z, \bar{z}). \tag{G.118}$$

Here we should note that $\phi_l(\rho; \beta_\pm)$ are two independent solutions at the boundary and they are singular at the origin. The regular solution can be obtained by directly solving the equation at the origin so called $\chi^1(\eta; k)$. They are solutions of the same equation at different singular points so $\phi_l(\rho; \beta_\pm)$, $\chi^1(\eta; k)$ are not independent. The transformation between them are given by

$$\chi^1_l(\eta; k) = C^+_l(k)\phi_l(\eta; \beta_+) + C^-_l(k)\phi_l(\eta; \beta_-), \tag{G.119}$$

[1] In fact, it is more appropriate to use the notion $\phi_{lm}(\rho; k)$ here even though the solution dose not depend on m explicitly.

in which we have chosen solutions at the boundary as the basis and $C_l^{\pm}(k)$ are coefficients determined in (F.101) and (F.102). Given the above relation, one can get the functional derivative between the source and one-point function thus higher point functions can be determined. For the two-point function, we have

$$\langle \hat{\mathcal{O}}_{lm}(k)\ \hat{\mathcal{O}}_{l',m'}(k)\rangle = -\frac{\delta\langle \hat{\mathcal{O}}_{lm}(k)\rangle_J}{\delta \hat{\mathcal{J}}_{l'm'}(k)}\bigg|_{J=0} = -\delta^l_{l'}\delta^m_{-m'}\ 2i\beta_+\frac{C_l^-(k)}{C_l^+(k)}, \tag{G.120}$$

in which the two-point function is written in terms of mode variables (l,m). Here, we should note the value of $\langle\mathcal{O}\rangle_J$ is scheme dependent and we assume that proper regularization procedure in the mode space (l,m) exists so that (G.120) is true for two-point function, like what has been done in momentum space Freedman et al. (1992); Skenderis (2002). To go back to the complex coordinates on the celestial sphere, one can do the sum over spherical harmonics $Y^l_m(z,\bar{z})$ then obtain

$$\langle \hat{\mathcal{O}}(z,\bar{z};k)\ \hat{\mathcal{O}}(z',\bar{z}';k)\rangle = -2i\beta_+\sum_{lm}\frac{C_l^-(k)}{C_l^+(k)}Y^l_m(z,\bar{z})Y^l_{-m}(z',\bar{z}'). \tag{G.121}$$

Here we should note that the two-point function on the sphere is obtained by summing over two discrete variables (l,m) while one can also just do the sum over variable m and obtain the l-mode source, one-point function

$$\hat{\mathcal{J}}_l(z,\bar{z};k) = \sum_m \phi_l^+(k)Y^l_m(z,\bar{z}) \qquad \langle\hat{\mathcal{O}}_l(z,\bar{z};k)\rangle = -2i\beta_+\sum_m \phi_l^-(k)Y^l_m(z,\bar{z}), \tag{G.122}$$

and the corresponding two-point function is given by

$$\langle \hat{\mathcal{O}}_l(z,\bar{z};k)\ \hat{\mathcal{O}}_l(z',\bar{z}';k)\rangle = -2i\beta_+\sum_{m}\frac{C_l^-(k)}{C_l^+(k)}Y^l_m(z,\bar{z})Y^l_{-m}(z',\bar{z}'). \tag{G.123}$$

Two-point Funciton

To study the dictionary for flat space in a more precies way, we consider a generic k mode $f(\tau,\rho,z,\bar{z};k)$ for on-shell field $\Phi(\tau,\rho,z,\bar{z})$ defined in (3.66), or equivalently

$$f(\tau,\rho,z,\bar{z};k) = \sum_{lm}\int d\omega f_{w,k,l,m}(\tau,\rho,z,\bar{z})\tilde{\Phi}(w,k,l,m). \tag{G.124}$$

Following the notion in (3.17), we choose to decompose the k mode into the spatial modes therefore $f(\tau,\rho,z,\bar{z};k)$ now takes the form

$$\begin{aligned} f(\tau,\rho,z,\bar{z};k) &= \sum_{lm} \bar{\Phi}(\tau,k,l,m)\phi_l(\rho;k)Y^l_m(z,\bar{z}) \quad \text{(G.125)} \\ &= \sum_{lm}(a^+_{lm}(k)f_+(\tau,k)+a^-_{lm}(k)f_-(\tau,k))\ (\phi_l(\rho;\beta_+)+\phi_l(\rho;\beta_-))Y^l_m(z,\bar{z}) \\ &= f_+(\tau,k)\phi(\rho,z,\bar{z};\beta_+)+f_-(\tau,k)\tilde{\phi}(\rho,z,\bar{z};\beta_+) \quad \text{(G.126)} \\ & \quad f_-(\tau,k)\phi(\rho,z,\bar{z};\beta_-)+f_-(\tau,k)\tilde{\phi}(\rho,z,\bar{z};\beta_-), \end{aligned}$$

in which $\bar{\Phi}(\tau,k,l,m)$ and $\phi_l(\rho,k)$ are modes that depend on τ and ρ. In the first line, we have summed over the two discrete variables l,m and also made the τ-mode l,m dependent by introducing the coefficients $a^{\pm}_{lm}(k)$ [2]. They are determined by the initial data. In the third line, we rearrange them into the τ mode functions and highlight their asymptotic behaviour according to $\beta_\pm$. The function $\phi(\rho,z,\bar{z},\beta_\pm)$, $\tilde{\phi}(\rho,z,\bar{z},\beta_\pm)$ are given by

$$\begin{aligned} \phi(\rho,z,\bar{z};\beta_\pm) &= \sum_{lm} a^+_{lm}(k)\ \phi_l(\rho;\beta_\pm)Y^l_m(z,\bar{z}) \quad \text{(G.127)} \\ \tilde{\phi}(\rho,z,\bar{z};\beta_\pm) &= \sum_{lm} a^-_{lm}(k)\ \phi_l(\rho;\beta_\pm)Y^l_m(z,\bar{z}). \quad \text{(G.128)} \end{aligned}$$

Moreover, using the asymptotic expansion (G.116) for $\phi_l(\rho;\beta_\pm)$ we obtain the leading contribution for $\phi(\rho,z,\bar{z};\beta_\pm)$ and $\tilde{\phi}(\rho,z,\bar{z};\beta_\pm)$ written as

$$\begin{aligned} \phi^\pm(z,\bar{z};k) &= \sum_{lm} a^+_{lm}(k)\ \phi^\pm_l(k)Y^l_m(z,\bar{z}) \quad \text{(G.129)} \\ \tilde{\phi}^\pm(z,\bar{z};k) &= \sum_{lm} a^-_{lm}(k)\ \phi^\pm_l(k)Y^l_m(z,\bar{z}). \quad \text{(G.130)} \end{aligned}$$

Now, given the above asymptotic expansion, we rewrite the Flat/CFT dictionary (3.83) into

$$\mathcal{J}(z,\bar{z};k) = \sum_{lm} a^-_{lm}(k)\mathcal{J}_{lm}(k)Y^m_l(z,\bar{z}) \qquad \langle\mathcal{O}(z,\bar{z};k)\rangle = \sum_{lm} a^+_{lm}(k)\langle\mathcal{O}_{lm}(k)\rangle Y^l_m(z,\bar{z}), \quad \text{(G.131)}$$

$$\tilde{\mathcal{J}}(z,\bar{z};k) = \sum_{lm} a^+_{lm}(k)\hat{\mathcal{J}}_{lm}(k)Y^m_l(z,\bar{z}) \qquad \langle\tilde{\mathcal{O}}(z,\bar{z};k)\rangle = \sum_{lm} a^-_{lm}(k)\langle\hat{\mathcal{O}}_{lm}(k)\rangle Y^l_m(z,\bar{z}), \quad \text{(G.132)}$$

from which we can see there is a pair of source and one-point function $\{\mathcal{J},\mathcal{O}\}$, $\{\tilde{\mathcal{J}},\tilde{\mathcal{O}}\}$ and they are combination of the source and one-point functions introduced in the AdS/CFT dictionary. Here we should note that the source and one-point functions $\{\mathcal{J},\mathcal{O}\}$, $\{\tilde{\mathcal{J}},\tilde{\mathcal{O}}\}$ now become physical and their existence does not rely on the AdS/CFT dictionary i.e., one could study them without writing them in terms of AdS

[2] Here, we should note that coefficients $a^{\pm}_{lm}(k)$ play the same role as $\psi_\pm(p)$ in (3.24) or $\psi(p)$ in (3.30) for fixed l,m.

modes $\{\hat{\mathcal{J}}_{lm}, \hat{\mathcal{O}}_{lm}\}$. Given the above dictionary, one can deduce the two-point function

$$\langle\tilde{\mathcal{O}}(z,\bar{z};k)\tilde{\mathcal{O}}(z',\bar{z}';k)\rangle = \frac{1}{N_k}\sum_{lm}\frac{a^-_{lm}(k)}{a^+_{lm}(k)}\langle\hat{\mathcal{O}}_{lm}(k)\hat{\mathcal{O}}_{l,-m}(k)\rangle Y^l_m(z,\bar{z})Y^l_{-m}(z',\bar{z}'), \quad \text{(G.133)}$$

$$\langle\mathcal{O}(z,\bar{z};k)\mathcal{O}(z',\bar{z}';k)\rangle = \frac{1}{N_k}\sum_{lm}\frac{a^+_{lm}(k)}{a^-_{lm}(k)}\langle\hat{\mathcal{O}}_{lm}(k)\hat{\mathcal{O}}_{l,-m}(k)\rangle Y^l_m(z,\bar{z})Y^l_{-m}(z',\bar{z}'), \quad \text{(G.134)}$$

in which the $(l.m)$ mode two-point functions are given in(G.120). Here we should note that $a^-_{lm}/a^+_{lm} = a^+_{lm}/a^-_{lm} = 0$ if $a^-_{lm} = 0$ or $a^+_{lm} = 0$ and the coefficients satisfy $a^\pm_{lm} = a^\pm_{l,-m}$ if one assumes that the fields are invariant under the parity transformation on the sphere $\phi = -\phi$.

During the calculation, we assume the coefficients $a_{lm}(k)$ determined by the initial data are (l,m) dependent. In fact, we can simplify the coefficients if there is a rotating symmetry for the solution on the sphere thus the coefficients will be m independent and we label them as $a_l(k)$. In this case, the k mode will be written as

$$f(\tau,\rho,z,\bar{z};k) = \sum_{lm}(a^+_l(k)f_+(\tau,k) + a^-_l(k)f_-(\tau,k))\,(\phi_l(\rho;\beta_+) + \phi_l(\rho;\beta_-))Y^l_m(z,\bar{z}). \quad \text{(G.135)}$$

The Flat/CFT dictionary remains the same while the source and one-point function will be written in terms of the shorter form

$$\mathcal{J}(z,\bar{z};k) = \sum_l a^-_l(k)\hat{\mathcal{J}}_l(z,\bar{z};k) \qquad \langle\mathcal{O}(z,\bar{z};k)\rangle = \sum_l a^+_l(k)\langle\hat{\mathcal{O}}_l(z,\bar{z};k)\rangle, \quad \text{(G.136)}$$

$$\tilde{\mathcal{J}}(z,\bar{z};k) = \sum_l a^+_l(k)\hat{\mathcal{J}}_l(z,\bar{z};k) \qquad \langle\tilde{\mathcal{O}}(z,\bar{z};k)\rangle = \sum_l a^-_l(k)\langle\hat{\mathcal{O}}_l(z,\bar{z};k)\rangle, \quad \text{(G.137)}$$

in which $\{\mathcal{J}_l, \mathcal{O}_l\}$ are the l mode source and one-point function defined in (G.122). As for the two-point function, following the standard functional derivative procedure, we have

$$\langle\tilde{\mathcal{O}}(z,\bar{z};k)\tilde{\mathcal{O}}(z',\bar{z}';k)\rangle = \frac{\delta\langle\tilde{\mathcal{O}}(z,\bar{z};k)\rangle_J}{\delta\tilde{\mathcal{J}}_k(z',\bar{z}';k)}\bigg|_{J=0} = \frac{1}{N_k}\sum_l\frac{a^-_l(k)}{a^+_l(k)}\langle\hat{\mathcal{O}}_l(z,\bar{z};k)\hat{\mathcal{O}}_l(z',\bar{z}';k)\rangle, \quad \text{(G.138)}$$

$$\langle\mathcal{O}(z,\bar{z};k)\mathcal{O}(z',\bar{z}';k)\rangle = \frac{\delta\langle\mathcal{O}(z,\bar{z};k)\rangle_J}{\delta\mathcal{J}(z',\bar{z}';k)}\bigg|_{J=0} = \frac{1}{N_k}\sum_l\frac{a^+_l(k)}{a^-_l(k)}\langle\hat{\mathcal{O}}_l(z,\bar{z};k)\hat{\mathcal{O}}_l(z',\bar{z}';k)\rangle, \quad \text{(G.139)}$$

in which the l-mode two-point function on the right hand side are given by (G.123). From the above discussion, one can see that it is not possible to simplify the coefficients $a_l(k)$ further and make them l independent otherwise the pair of source and one-point function will become linearly dependent and be reduced to one copy.

Boundary-Bulk Propagator

Following the study of the mode expansion of the fields, we know that the source of the fields can be expanded by the spherical harmonics on the sphere with coefficients $a^{\pm}_{lm}(k)$, written as

$$\mathcal{J}(z,\bar{z};k) = \tilde{\phi}^{+}(z,\bar{z};k) = \sum_{lm} a^{-}_{lm}(k) Y^{l}_{m}(z,\bar{z}), \tag{G.140}$$

$$\tilde{\mathcal{J}}(z,\bar{z};k) = \phi^{+}(z,\bar{z};k) = \sum_{lm} a^{+}_{lm}(k) Y^{l}_{m}(z,\bar{z}). \tag{G.141}$$

Therefore, with the help of the bulk-boundary propagator $K(\rho,z;z')$, one can then write the spatial mode $\phi(\rho,z,\bar{z};k)$ and $\phi(\rho,z,\bar{z};k)$ into the form of

$$\begin{aligned}
\phi(\rho,z,\bar{z};k) &= \frac{1}{2}\int dz'd\bar{z}' K(\rho,z;z')\phi^{+}(z',\bar{z}';k) = \frac{1}{2}\sum_{lm} a^{+}_{lm}(k)\int dz'd\bar{z}' K(\rho,z;z') Y^{l}_{m}(z',\bar{z}'),\\
\tilde{\phi}(\rho,z,\bar{z};k) &= \frac{1}{2}\int dz'd\bar{z}' K(\rho,z;z')\tilde{\phi}^{+}(z',\bar{z}';k) = \frac{1}{2}\sum_{lm} a^{-}_{lm}(k)\int dz'd\bar{z}' K(\rho,z;z') Y^{l}_{m}(z',\bar{z}').
\end{aligned}$$

Given such expression, together with the Flat/CFT dictionary, the one-point functions are now deduced to be

$$\mathcal{O}(z,\bar{z};k) = -2i\beta_{+}\Omega_2(z)\phi^{-}(z,\bar{z};k) = -\frac{2i\beta_{+}}{\pi}\sum_{lm} a^{+}_{lm}(k)\int dz'd\bar{z}'\frac{1}{|z-z'|^{2\Delta}} Y^{l}_{m}(z',\bar{z}'), \tag{G.142}$$

$$\tilde{\mathcal{O}}(z,\bar{z};k) = -2i\beta_{+}\Omega_2(z)\tilde{\phi}^{-}(z,\bar{z};k) = -\frac{2i\beta_{+}}{\pi}\sum_{lm} a^{-}_{lm}(k)\int dz'd\bar{z}'\frac{1}{|z-z'|^{2\Delta}} Y^{l}_{m}(z',\bar{z}'). \tag{G.143}$$

Moreover, by doing the functional variation with respect to the source $\mathcal{J}(z,\bar{z};k)$ and $\tilde{\mathcal{J}}(z,\bar{z};k)$, one should be able to obtain the two point functions. The functional variation between the one-point function and the source can be transformed into variation between spherical harmonics since both of the operator $\mathcal{O}$ and the source $\mathcal{J}$ are now written in terms of harmonic function Y^{l}_{m}. At first, as a kind of approximation, we assume that the boundary-bulk propagator is a function that do not depend on the spherical harmonics, then the two-point functions become

$$\begin{aligned}
\langle \mathcal{O}(z,\bar{z};k)\mathcal{O}(z',\bar{z}';k)\rangle &= 2i\beta_{+}\frac{a^{+}_{0}(k)}{a^{-}_{0}(k)} + \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^{+}_{lm}(k)}{a^{-}_{lm}(k)}\frac{c_k}{|z-z'|^{2\Delta}},\\
\langle \tilde{\mathcal{O}}(z,\bar{z};k)\tilde{\mathcal{O}}(z',\bar{z}';k)\rangle &= 2i\beta_{+}\frac{a^{-}_{0}(k)}{a^{+}_{0}(k)} + \frac{1}{N_k}\sum_{l\neq 0,m}\frac{a^{-}_{lm}(k)}{a^{+}_{lm}(k)}\frac{c_k}{|z-z'|^{2\Delta}},
\end{aligned} \tag{G.144}$$

in which $c_k = 2i\beta_{+}/\pi$ is the renormalised factor. Here we should note the $l = 0$ term is a constant since Y^{0}_{0} is a constant function and the functional variation is then reduced to the ratio of coefficients. One can always set such term to zero by shifting the fields with a constant and make $a^+_{00}(k)$ or $a^-_{00}(k)$ zero. We choose to keep them here since in the discussion of shock waves, the fields take constant value on the sphere and the only contribution will be the $l = 0$ term. Form the expression (G.144), one can see that the two-point function is of scale dimension Δ and the coefficients of the bulk fields $a^{\pm}_{lm}(k)$ are encoded in the boundary operator as the ratios of the linear combination.

Now, we will determine the functional variation in a more precise way by decomposing the bulk-boudary into the harmonics modes

$$\Omega_2(z)K(\rho, z; z') = \sum_{lm} K_{lm}(\rho) Y^l_m(z, \bar{z}) Y^l_{-m}(z', \bar{z}') \tag{G.145}$$

in which the function $K(\rho)$ can be treated as the coefficients and the weight $\Omega_2(z)$ is introduced here for later convenience. Given such decomposition, the one-point function can be written into the form

$$\mathcal{O}(z, \bar{z}; k) = -2i\beta_+ \sum_{lm} a^+_{lm}(k) \int dz' d\bar{z}' K_{lm} Y^l_m(z, \bar{z}) \left(Y^l_{-m}(z', \bar{z}')\right)^2, \tag{G.146}$$

$$\tilde{\mathcal{O}}(z, \bar{z}; k) = -2i\beta_+ \sum_{lm} a^-_{lm}(k) \int dz' d\bar{z}' K_{lm} Y^l_m(z, \bar{z}) \left(Y^l_{-m}(z', \bar{z}')\right)^2, \tag{G.147}$$

in which $K_{lm} = \rho^{\Delta} K_{lm}(\rho)$. Therefore, by calculating the functional variation with respect to the spherical harmonics Y^l_m, one then obtain the two point function

$$\langle \mathcal{O}(z, \bar{z}; k) \mathcal{O}(z', \bar{z}'; k) \rangle = \frac{4i\beta_+}{N_k} \sum_{lm} \frac{a^+_{lm}(k)}{a^-_{lm}(k)} K_{lm} Y^l_m(z, \bar{z}) Y^l_{-m}(z', \bar{z}') \tag{G.148}$$

$$\langle \tilde{\mathcal{O}}(z, \bar{z}; k) \tilde{\mathcal{O}}(z', \bar{z}'; k) \rangle = \frac{4i\beta_+}{N_k} \sum_{lm} \frac{a^-_{lm}(k)}{a^+_{lm}(k)} K_{lm} Y^l_m(z, \bar{z}) Y^l_{-m}(z', \bar{z}'). \tag{G.149}$$

One can check such expression is equivalent to the result obtained from the mode analysis calculation by making

$$4i\beta_+ K_{lm} \equiv \delta^l_{l'} \delta^m_{-m'} \langle \hat{\mathcal{O}}_{lm}(k) \hat{\mathcal{O}}_{l'm'}(k) \rangle, \tag{G.150}$$

or equivalently we have

$$2K_{lm} = -\frac{C^-_l(k)}{C^+_l(k)}. \tag{G.151}$$

H Asymptotic symmetries

Given the Killing vector ξ, we can then further write down the variation of the metric as

$$\mathcal{L}_\xi G_{\mu\nu} = \xi^\sigma \partial_\sigma G_{\mu\nu} + G_{\mu\sigma}\,\partial_\nu \xi^\sigma + G_{\nu\sigma}\,\partial_\mu \xi^\sigma, \tag{H.152}$$

in which the expression is true for generic metric. In this thesis we will work in the Feffermam Graham gauge which means we have $G_{\tau\tau} = -1$ and $G_{\tau a} = 0$. In such gauge, for $\tau\tau$ component, we have

$$\mathcal{L}_\xi G_{\tau\tau} = \xi^\sigma \partial_\sigma G_{\tau\tau} + 2G_{\tau\sigma}\,\partial_\tau \xi^\sigma = -2\partial_\tau \xi^\tau \tag{H.153}$$

while for the $\tau\rho$ and τz component we have

$$\begin{aligned} \mathcal{L}_\xi G_{\tau\rho} &= G_{\tau\sigma}\,\partial_\rho \xi^\sigma + G_{\rho\sigma}\,\partial_\tau \xi^\sigma && \text{(H.154)} \\ &= -\partial_\rho \xi^\tau + G_{\rho\rho}\partial_\tau \xi^\rho + G_{\rho z}\partial_\tau \xi^z + G_{\rho\bar{z}}\partial_\tau \xi^{\bar{z}} && \text{(H.155)} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_\xi G_{\tau z} &= G_{\tau\sigma}\,\partial_z \xi^\sigma + G_{z\sigma}\,\partial_\tau \xi^\sigma && \text{(H.156)} \\ &= -\partial_z \xi^\tau + G_{\rho z}\partial_\tau \xi^\rho + G_{zz}\partial_\tau \xi^z + G_{z\bar{z}}\partial_\tau \xi^{\bar{z}}. && \text{(H.157)} \end{aligned}$$

For the spatial $\rho z\bar{z}$ part, we have

$$\mathcal{L}_\xi G_{z\bar{z}} = \xi^\sigma \partial_\sigma G_{z\bar{z}} + G_{z\sigma}\partial_{\bar{z}} \xi^\sigma + G_{\bar{z}\sigma}\partial_z \xi^\sigma \tag{H.158}$$

$$\mathcal{L}_\xi G_{zz} = \xi^\sigma \partial_\sigma G_{zz} + 2G_{z\sigma}\partial_z \xi^\sigma \tag{H.159}$$

$$\mathcal{L}_\xi G_{\rho\rho} = \xi^\sigma \partial_\sigma G_{\rho\rho} + 2G_{\rho\sigma}\partial_\rho \xi^\sigma \tag{H.160}$$

$$\mathcal{L}_\xi G_{\rho z} = \xi^\sigma \partial_\sigma G_{\rho z} + G_{\rho\sigma}\partial_z \xi^\sigma + G_{z\sigma}\partial_\rho \xi^\sigma. \tag{H.161}$$

I Feffermam and Graham Coordinates

In terms of Fefferman and Graham coordinates (r, x_i). The metric for asymptotic AdS spacetime takes the form

$$\hat{G}_{rr} = \frac{1}{r^2}, \qquad \hat{G}_{ij} = \frac{g_{ij}}{r^2}, \qquad \hat{G}_{ri} = 0, \tag{I.162}$$

in which the asymptotic behaviour is described by the function $g_{ij}(r, x_i)$. In such coordinates, the connection is then given by

$$\Gamma^r_{rr} = -\frac{1}{r}, \qquad \Gamma^r_{ri} = \Gamma^i_{rr} = 0. \tag{I.163}$$

For the other components that involve r component, in terms of the function g_{ij}, the connections can be written as

$$\Gamma^r_{ij} = -\frac{r^2}{2}\,\partial_r \hat{G}_{ij} = \frac{g_{ij}}{r} - \frac{1}{2}\partial_r g_{ij} \tag{I.164}$$

and

$$\Gamma^i_{rj} = \frac{1}{2}\hat{G}^{ik}\partial_r \hat{G}_{kj} = -\frac{\delta^i_j}{r} + \frac{1}{2}g^{ik}\partial_r g_{kj} \tag{I.165}$$

For the connections that do not involve the r component, they are determined by the function g_{ij} and one can treat them as the connection of g_{ij}, i.e, $\Gamma^i_{jk} = \hat{\Gamma}^i_{jk}[g]$.

Given the connections, we can use them to calculate the Ricci tensor following the definition (J.171), the rr component is given by

$$\begin{aligned} R^{d+1}_{rr}[\hat{G}] &= \partial_r \Gamma^k_{rk} + \Gamma^l_{rk}\Gamma^k_{rl} - \Gamma^r_{rr}\Gamma^k_{kr} \\ &= \frac{1}{2}\,g^{ij}\,\partial^2_r\,g_{ij} - \frac{1}{2r}g^{ij}\partial_r g_{ij} - \frac{1}{4}g^{ij}\,g^{lm}\,\partial_r g_{il}\,\partial_r g_{jm} + \frac{d}{r^2} \end{aligned} \tag{I.166}$$

and the ir components is determined to be

$$\begin{aligned} R^{d+1}_{ir}[\hat{G}] &= \partial_i \Gamma^k_{rk} - \partial_k \Gamma^k_{ri} + \Gamma^k_{rl}\Gamma^l_{ik} - \Gamma^m_{ri}\Gamma^k_{km} \qquad \text{(I.167)} \\ &= \frac{1}{2}\partial_i(\,g^{lm}\,\partial_r g_{lm}) - \frac{1}{2}\partial_k(\,g^{kl}\,\partial_r g_{il}) + \frac{1}{2}g^{km}\,\Gamma^l_{ik}\,\partial_r g_{lm} - \frac{1}{2}g^{mk}\,\Gamma^l_{lm}\,\partial_r g_{ik}. \end{aligned}$$

Moreover, in terms of covariant derivative ∇_i with respective to the metric g_{ij}, R_{ir} can be simplified to

$$R^{d+1}_{ir}[\hat{G}] = \frac{1}{2}\nabla_i(g^{lm}\,\partial_r g_{lm}) - \frac{1}{2}\nabla^j\partial_r g_{ji}. \tag{I.168}$$

The ij component is given by

$$\begin{aligned} R_{ij}^{d+1}[\hat{G}] &= R_{ij}[g] - \partial_r \Gamma_{ij}^r + \Gamma_{ir}^k \Gamma_{jk}^r + \Gamma_{il}^r \Gamma_{jr}^l - \Gamma_{ij}^r \Gamma_{lr}^l - \Gamma_{ij}^r \Gamma_{rr}^r \\ &= R_{ij}[g] + \frac{1}{2}\partial_r^2 g_{ij} + \frac{d}{r^2} g_{ij} + \frac{1-d}{2r}\partial_r g_{ij} - \frac{1}{2} g^{km}\partial_r g_{ki}\partial_r g_{mj} \\ &\quad + \frac{1}{4}\partial_r g_{ij} g^{lm}\partial_r g_{lm} - \frac{1}{2r} g_{ij} g^{lm}\partial_r g_{lm}, \end{aligned} \tag{I.169}$$

in which the induced Ricci tensor of g_{ij} is denoted as $R_{ij}[g]$.

J Equation of Motion

To calculate the Ricci tensor, we will use the convention

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\rho} \tag{J.170}$$

so that the tensor is given by

$$R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} - \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu}. \tag{J.171}$$

For d dimensional spacetime, in terms of the Milne coordinates, the Einstein equation at linear level could be written as

$$R_{\tau\tau}[G] = \frac{4\rho^2}{\tau}\partial_\tau h_{\rho\rho} + 2\rho^2\partial_\tau^2 h_{\rho\rho} + \frac{1}{2}g^{ij}\partial_\tau^2 h_{ij} + \frac{1}{\tau}g^{ij}\partial_\tau h_{ij} \tag{J.172}$$

$$\begin{aligned} R_{\tau i}[G] = {} & 2\rho^2\nabla_i\partial_\tau h_{\rho\rho} + \frac{1}{2\rho}\nabla_i(\, g^{jm}\partial_\tau h_{jm}) - \frac{1}{2\rho}\partial_\tau\nabla^k h_{ki} - 2\rho^2\partial_\rho\partial_\tau h_{i\rho} \\ & -4\rho\partial_\tau h_{i\rho} - 2\rho^2\partial_\tau h_{\rho i}\Gamma^a_{a\rho} \end{aligned} \tag{J.173}$$

in which we have

$$\Gamma^a_{a\rho} = \frac{d-2}{2}\frac{1}{\rho} + \frac{1}{2}g^{lm}\partial_\rho g_{lm} \tag{J.174}$$

and $R_{\tau\rho}$ is deduced to be

$$\begin{aligned} R_{\tau\rho}[G] = {} & -\frac{1}{2\rho^2}g^{ij}\partial_\tau h_{ij} + \frac{1}{2\rho}\partial_\rho(g^{ij}\partial_\tau h_{ij}) - 2\rho\partial_\tau h_{\rho\rho} - \frac{1}{2\rho}\partial_\tau\nabla^i h_{i\rho} \\ & -2\rho^2\partial_\tau h_{\rho\rho}\Gamma^a_{a\rho} + \frac{1}{2\rho}g^{ij}\partial_\tau h_{jk}\Gamma^k_{i\rho}. \end{aligned} \tag{J.175}$$

For other components, with the help of the expansion of the Ricci curvature,

$$R^{d+1}_{ab}[\hat{G}^{(0)} + h] = R^{d+1}_{ab}[\hat{G}^{(0)}] + \frac{1}{2}(\hat{\nabla}^2 h_{ab} + \hat{\nabla}_a\hat{\nabla}_b h - \hat{\nabla}_c\hat{\nabla}_a h^c_b - \hat{\nabla}_c\hat{\nabla}_b h^c_a) + \mathcal{O}(h^2) \tag{J.176}$$

where $\hat{\nabla}_a$ is the covariant derivative with respect to the metric $\hat{G}^{(0)}_{ab}$. Then one can obtain

$$\begin{aligned} R_{\rho\rho}[G] = {} & -dh_{\rho\rho} - \frac{d+2}{2}\tau\partial_\tau h_{\rho\rho} - \frac{1}{2}\tau^2\partial_\tau^2 h_{\rho\rho} - \frac{\tau}{8\rho^3}g^{ij}\partial_\tau h_{ij} + (4\rho^2\Gamma^i_{\rho j}\Gamma^j_{i\rho} - \frac{g^{ij}}{\rho^2}\Gamma^\rho_{ij} \\ & +\frac{g^{ij}}{\rho}\Gamma^k_{i\rho}\Gamma^\rho_{jk} - 4(d+\rho g^{ij}\partial_\rho g_{ij}))h_{\rho\rho} - (2\rho(d+\rho g^{ij}\partial_\rho g_{ij}) + \frac{g^{ij}}{2\rho}\Gamma^\rho_{ij})\partial_\rho h_{\rho\rho} \\ & +\frac{1}{2\rho}\nabla^i\nabla_i h_{\rho\rho} - \frac{1}{\rho}\nabla_i\partial_\rho h^i_\rho - \frac{2}{\rho}\Gamma^k_{i\rho}\nabla^i h_{k\rho} - \frac{1}{\rho^2}\nabla_i h^i_\rho + \frac{1}{2\rho}\partial_\rho^2 h^i_i - \frac{1}{2\rho^2}\partial_\rho h^i_i \\ & +\frac{1}{\rho}\Gamma^i_{\rho j}\partial_\rho h^j_i + \frac{1}{2\rho^3}h^i_i + \frac{1}{\rho}g^{ij}\Gamma^k_{i\rho}\Gamma^l_{j\rho}h_{kl} - \frac{1}{\rho^2}\Gamma^i_{\rho j}h^j_i - \frac{1}{\rho}\Gamma^i_{\rho j}\Gamma^k_{\rho i}h^j_k \end{aligned} \tag{J.177}$$

where we have $h^i_j = g^{ik}h_{kj}$ and $h^i_\rho = g^{ik}h_{k\rho}$. The connections used here are given by $\Gamma^\rho_{\rho\rho} = -\frac{1}{\rho}$, $\Gamma^i_{j\rho} = \frac{\delta^i_j}{2\rho} + \frac{1}{2}g^{ik}\partial_\rho g_{kj}$ and $\Gamma^\rho_{ij} = -2\rho^2 g_{ij} - 2\rho^3\partial_\rho g_{ij}$. For the components of $R_{i\rho}$, R_{ij} we have

$$\begin{aligned}
R_{i\rho}[G] = &-\frac{d+1}{2}\tau\partial_\tau h_{i\rho} - \frac{1}{2}\tau^2\partial_\tau^2 h_{i\rho} - dh_{i\rho} - 2\rho\nabla_i h_{\rho\rho} - 2\rho^2\nabla_k(\Gamma^k_{i\rho}h_{\rho\rho}) - \frac{1}{2\rho}\nabla^m(\Gamma^\rho_{mi}h_{\rho\rho}) \\
&-2\rho^2\Gamma^a_{a\rho}\nabla_i h_{\rho\rho} + \frac{1}{2\rho}(\nabla^m\nabla_m h_{i\rho} - \nabla_k\nabla_i h^k_\rho) + 2\rho^2\partial_\rho(\Gamma^j_{\rho i}h_{\rho j}) - \frac{1}{2\rho}\partial_\rho(\Gamma^\rho_{ij}h^j_\rho) \\
&+\frac{3}{2}\Gamma^\rho_{mi}\Gamma^k_{n\rho}g^{mn}h_{k\rho} + \frac{g^{mn}}{2\rho}\Gamma^\rho_{mn}\Gamma^j_{\rho i}h_{j\rho} + \frac{1}{2\rho}g^{mn}\Gamma^k_{m\rho}\Gamma^\rho_{nk}h_{i\rho} + \frac{1}{2\rho}\Gamma^j_{k\rho}\Gamma^\rho_{ij}h^k_\rho \\
&-\frac{1}{2\rho^2}g^{mn}\Gamma^\rho_{mn}h_{i\rho} + 2\rho^2\Gamma^j_{k\rho}\Gamma^k_{j\rho}h_{\rho i} + \Gamma^a_{a\rho}(-2\rho h_{\rho i} + 2\rho^2\Gamma^j_{\rho i}h_{\rho j} + 2\rho^2\Gamma^k_{i\rho}h_{\rho k} - \frac{1}{2\rho}\Gamma^\rho_{ij}h^j_\rho) \\
&+8\rho\Gamma^j_{i\rho}h_{\rho j} - 2h_{i\rho} + \frac{1}{\rho}\Gamma^\rho_{ij}\partial_\rho h^j_\rho - 2\rho\partial_\rho h_{i\rho} - 2\rho^2\Gamma^a_{a\rho}\partial_\rho h_{\rho i} + \frac{1}{\rho}\nabla_k(\Gamma^l_{i\rho}h^k_l) \\
&-\frac{1}{2\rho}\Gamma^k_{m\rho}\nabla^m h_{ik} + \frac{1}{2\rho}\Gamma^k_{j\rho}\nabla_i h^j_k - \frac{1}{2\rho}\nabla^j(\Gamma^k_{j\rho}h_{ik}) - \frac{1}{2\rho}\nabla_j(\Gamma^j_{\rho k}h^k_i) + \frac{1}{2\rho}\Gamma^j_{k\rho}\nabla_j h^k_i \\
&+\frac{1}{2\rho^2}\nabla_k h^k_i - \frac{1}{2\rho^2}\nabla_i h - \frac{1}{2\rho}\Gamma^k_{i\rho}\nabla_k h + \frac{1}{2\rho}\nabla_i\partial_\rho h - \frac{1}{2\rho}\nabla_j\partial_\rho h^j_i
\end{aligned} \tag{J.178}$$

and

$$\begin{aligned}
R_{ij}[G] = &-dh_{ij} - \frac{d+1}{2}\tau\partial_\tau h_{ij} - \frac{1}{2}\tau^2\partial_\tau^2 h_{ij} - \frac{\tau}{2}g_{ij}(g^{mn}\partial_\tau h_{mn} + 4\rho^3\partial_\tau h_{\rho\rho}) \\
&+\partial_\rho(4\rho^2\Gamma^\rho_{ij}h_{\rho\rho}) - 2\rho^2(\Gamma^k_{\rho i}\Gamma^\rho_{kj}h_{\rho\rho} + \Gamma^k_{\rho j}\Gamma^\rho_{ki}h_{\rho\rho}) + \frac{1}{\rho}\Gamma^\rho_{ni}\Gamma^\rho_{nj}g^{mn}h_{\rho\rho} + 2\rho^2\nabla_i\nabla_j h_{\rho\rho} \\
&+4\rho^2\Gamma^a_{a\rho}\Gamma^\rho_{ij}h_{\rho\rho} - \Gamma^\rho_{ij}\partial_\rho(2\rho^2 h_{\rho\rho}) - 2\partial_\rho(\rho^2\nabla_i h_{\rho j}) - 2\partial_\rho(\rho^2\nabla_j h_{\rho i}) \\
&-2\rho^2\nabla_k(\Gamma^k_{i\rho}h_{j\rho}) - 2\rho^2\nabla_k(\Gamma^k_{j\rho}h_{i\rho}) + \frac{1}{2\rho}\Gamma^\rho_{kj}\nabla_i h^k_\rho + \frac{1}{2\rho}\Gamma^\rho_{ki}\nabla_j h^k_\rho + \frac{1}{\rho}\nabla_k(\Gamma^\rho_{ij}h^k_\rho) \\
&-2\rho^2\Gamma^a_{a\rho}(\nabla_i h_{j\rho} + \nabla_j h_{i\rho}) - 2\rho^2\Gamma^k_{\rho i}h_{\rho j} - 2\rho^2\Gamma^k_{\rho j}h_{\rho i} - 2\rho^2\Gamma^k_{\rho j}\nabla_i h_{k\rho} - 2\rho^2\Gamma^k_{\rho i}\nabla_j h_{k\rho} \\
&-\frac{1}{2\rho}\nabla^n(\Gamma^\rho_{ni}h_{\rho j}) - \frac{1}{2\rho}\nabla^n(\Gamma^\rho_{nj}h_{i\rho}) - \frac{1}{2\rho}\Gamma^\rho_{mi}\nabla^m h_{\rho j} - \frac{1}{2\rho}\Gamma^\rho_{mj}\nabla^m h_{\rho i} \\
&-\frac{1}{2\rho}(\nabla_k\nabla_i h^k_j + \nabla_k\nabla_j h^k_i) + \frac{1}{2\rho}\nabla_i\nabla_j h + \frac{1}{2\rho}\nabla_m\nabla^m h_{ij} + 2\rho^2\partial_\rho^2 h_{ij} + 2\rho\partial_\rho h_{ij} \\
&-2\rho^2\big(\partial_\rho(\Gamma^k_{\rho i}h_{kj}) + \partial_\rho(\Gamma^k_{\rho j}h_{ki}) + \Gamma^k_{\rho i}\partial_\rho h_{kj} + \Gamma^k_{\rho j}\partial_\rho h_{ki}\big) + \frac{1}{2\rho^2}\Gamma^\rho_{ij}h - \frac{1}{2\rho}\Gamma^\rho_{ij}\partial_\rho h \\
&+\frac{1}{2}\Gamma^\rho_{ki}\partial_\rho(\frac{1}{\rho}h^k_j) + \frac{1}{2}\Gamma^\rho_{kj}\partial_\rho(\frac{1}{\rho}h^k_i) - \frac{1}{2}\partial_\rho(\frac{1}{\rho}\Gamma^\rho_{ki}h^k_j) - \frac{1}{2}\partial_\rho(\frac{1}{\rho}\Gamma^\rho_{kj}h^k_i) \\
&+2\rho^2(\Gamma^k_{\rho i}\Gamma^m_{\rho k}h_{mj} + \Gamma^k_{\rho j}\Gamma^m_{\rho k}h_{mi} + 2\Gamma^k_{\rho i}\Gamma^m_{\rho j}h_{mk}) - \frac{1}{2\rho}(\Gamma^\rho_{ki}\Gamma^m_{\rho j} + \Gamma^\rho_{kj}\Gamma^m_{\rho i})h^k_m \\
&+\frac{1}{2\rho}(\Gamma^\rho_{ki}\Gamma^k_{\rho m}h^m_j + \Gamma^\rho_{kj}\Gamma^k_{\rho m}h^m_i + \Gamma^k_{\rho i}\Gamma^\rho_{km}h^m_j + \Gamma^k_{\rho j}\Gamma^\rho_{km}h^m_i) - 2\rho\Gamma^k_{\rho i}h_{kj} - 2\rho\Gamma^k_{\rho j}h_{ki} \\
&-\frac{1}{2\rho}\Gamma^a_{a\rho}(\Gamma^\rho_{im}h^m_j + \Gamma^\rho_{jm}h^m_i) + \frac{g^{mn}}{2\rho}\Big(\Gamma^\rho_{mn}(\Gamma^k_{i\rho}h_{kj} + \Gamma^k_{\rho j}h_{ki}) + \Gamma^k_{n\rho}(\Gamma^\rho_{mj}h_{ki} \\
&+\Gamma^\rho_{mi}h_{kj}) - \Gamma^\rho_{mn}\partial_\rho h_{ij}\Big).
\end{aligned} \tag{J.179}$$

Given the Einstein tensor, then one should be able to deduce h_{ab} exactly by solving the Einstein equation $R_{\mu\nu} = 0$. For simplicity, we consider the equation at the leading order which means for h_{ab} we have

$$h_{\rho\rho} = \frac{1}{\rho^2\tau}m(\rho,z,\bar{z}), \qquad h_{ij} = \frac{\rho}{\tau}\sigma_{ij}(\rho,z,\bar{z}), \qquad h_{\rho i} = \frac{1}{\tau}A_i(\rho,z,\bar{z}) \tag{J.180}$$

where higher order terms of $1/\tau$ are omitted. Substituting these into the Einstein equation, we have the equation for m, σ_{ij} and A_i written as

$R_{\rho\tau} = 0$

$$\frac{1}{2}\nabla^i A_i - \frac{1}{4}g^{ij}\sigma_{ij} - \frac{\rho}{2}\partial_\rho(g^{ij}\sigma_{ij}) + m(d + \rho g^{ij}\partial_\rho g_{ij}) - \frac{\rho}{4}\sigma^{mn}\partial_\rho g_{mn} = 0, \tag{J.181}$$

$R_{i\tau} = 0$

$$2\nabla_i m + \frac{1}{2}\nabla_i(\sigma^m_m) - \frac{1}{2}\nabla^k\sigma_{ki} - 2\rho^2\partial_\rho A_i - \rho A_i(d + 2 + \rho g^{lm}\partial_\rho g_{lm}) = 0, \tag{J.182}$$

$R_{\rho\rho} = 0$

$$\begin{aligned}
&\frac{1}{8}\sigma^m_m - \frac{d}{2}m + (4\rho^2\Gamma^i_{j\rho}\Gamma^j_{i\rho} - \frac{1}{\rho^2}g^{ij}\Gamma^\rho_{ij} + \frac{g^{ij}}{\rho}\Gamma^k_{i\rho}\Gamma^\rho_{jk} - 4(d + \rho g^{ij}\partial_\rho g_{ij}))m \\
&-(2(d + \rho g^{ij}\partial_\rho g_{ij}) + \frac{g^{ij}}{2\rho^2}\Gamma^\rho_{ij})(-2m + \rho\partial_\rho m) + \frac{1}{2\rho}\nabla^i\nabla_i m - \rho\nabla_i\partial_\rho A^i - 2\rho\Gamma^k_{i\rho}\nabla^i A_k \\
&-\nabla_i A^i + \frac{\rho}{2}(\partial_\rho\sigma^m_m + \rho\partial^2_\rho\sigma^m_m) + \rho^2\,\Gamma^i_{\rho j}\partial_\rho\sigma^j_i + \rho^2 g^{ij}\Gamma^k_{i\rho}\Gamma^l_{j\rho}\sigma_{kl} - \rho^2\Gamma^i_{\rho j}\Gamma^k_{\rho i}\sigma^j_k = 0,
\end{aligned} \tag{J.183}$$

$R_{\rho i} = 0$

$$\begin{aligned}
&-\frac{d+5}{2}A_i + 2\rho^2\partial_\rho(\Gamma^l_{i\rho}A_l) - \frac{1}{2\rho}\partial_\rho(\Gamma^\rho_{il}A^l) + \frac{1}{2\rho}\Gamma^\rho_{ki}\Gamma^k_{\rho j}A^j + 2\rho^2\Gamma^k_{\rho j}\Gamma^j_{k\rho}A_i \\
&+\Gamma^a_{a\rho}(2\rho^2\Gamma^k_{i\rho}A_k - \frac{1}{2\rho}\Gamma^\rho_{ij}A^j - 2\rho A_i + 2\rho^2\Gamma^k_{\rho i}A_k) + \frac{g^{mn}}{2\rho}\Gamma^\rho_{mn}(\Gamma^j_{i\rho}A_j - \frac{1}{\rho}A_i) \\
&+\frac{3g^{mn}}{2\rho}\Gamma^k_{m\rho}\Gamma^\rho_{ni}A_k + \frac{g^{mn}}{2\rho}\Gamma^\rho_{nk}\Gamma^k_{m\rho}A_i - 2\rho^2\Gamma^a_{a\rho}\partial_\rho A_i - 2\rho\partial_\rho A_i + 8\rho\Gamma^j_{i\rho}A_j + \frac{1}{\rho}\Gamma^\rho_{ij}\partial_\rho A^j \\
&+\frac{1}{2\rho}(\nabla^m\nabla_m A_i - \nabla_k\nabla_i A^k) - \frac{2}{\rho}\nabla_i m - \nabla_k(2\Gamma^k_{i\rho}m) - \frac{1}{2\rho^3}\nabla^k(\Gamma^\rho_{ki}m) - 2\Gamma^a_{a\rho}\nabla_i m \\
&-\frac{1}{2}\Gamma^k_{m\rho}\nabla^m\sigma_{ik} + \frac{1}{2}\Gamma^k_{j\rho}\nabla_i\sigma^j_k + \nabla_k(\Gamma^l_{i\rho}\sigma^k_l) - \frac{1}{2}\nabla^j(\Gamma^k_{j\rho}\sigma^k_i) - \frac{1}{2}\nabla_j(\Gamma^j_{\rho k}\sigma^k_i) + \frac{1}{2}\Gamma^j_{k\rho}\nabla_j\sigma^k_i \\
&-\frac{1}{2}\Gamma^k_{i\rho}\nabla_k\sigma^m_m + \frac{1}{2}\nabla_i\partial_\rho\sigma^m_m - \frac{1}{2}\nabla_j\partial_\rho\sigma^j_i = 0,
\end{aligned} \tag{J.184}$$

$R_{ij} = 0$

$$\begin{aligned}
&\frac{-d+3}{2}\sigma_{ij} + 6\rho\partial_\rho\sigma_{ij} + 2\rho^2\partial_\rho^2\sigma_{ij} + \frac{1}{2\rho}\nabla^m\nabla_m\sigma_{ij} + \frac{1}{2\rho}\nabla_i\nabla_j\sigma_m^m - \frac{1}{2\rho}(\nabla_k\nabla_i\sigma_j^k + \nabla_k\nabla_j\sigma_i^k)\\
&+\frac{1}{2}g_{ij}(\sigma_m^m + 4m) - 2\rho\nabla_k(\Gamma^k_{i\rho}A_j) - 2\rho\nabla_k(\Gamma^k_{\rho j}A_i) + \frac{1}{2\rho^2}\Gamma^\rho_{kj}\nabla_i A^k + \frac{1}{2\rho^2}\Gamma^\rho_{ki}\nabla_j A^k + \frac{1}{\rho^2}\nabla_k(\Gamma^\rho_{ij}A^k)\\
&-4\nabla_i A_j - 4\nabla_j A_i - 2\rho(\nabla_i\partial_\rho A_j + \nabla_j\partial_\rho A_i) - 2\rho\Gamma^a_{a\rho}(\nabla_i A_j + \nabla_j A_i) - 2\rho\Gamma^k_{\rho i}\nabla_k A_j - 2\rho\Gamma^k_{\rho j}\nabla_k A_i\\
&-2\rho\Gamma^k_{\rho j}\nabla_i A_k - 2\rho\Gamma^k_{\rho i}\nabla_j A_k - \frac{1}{2\rho^2}\nabla^n(\Gamma^\rho_{ni}A_j) - \frac{1}{2\rho^2}\nabla^n(\Gamma^\rho_{nj}A_i) - \frac{1}{2\rho^2}\Gamma^\rho_{mi}\nabla^m A_j - \frac{1}{2\rho^2}\Gamma^\rho_{mj}\nabla^m A_i\\
&-\frac{2\Gamma^\rho_{ij}}{\rho}\partial_\rho m + \frac{4}{\rho}\partial_\rho(\Gamma^\rho_{ij}m) + 4\Gamma^a_{a\rho}\Gamma^\rho_{ij}\frac{m}{\rho} - 2(\Gamma^k_{\rho j}\Gamma^\rho_{ki} + \Gamma^k_{\rho i}\Gamma^\rho_{kj})\frac{m}{\rho} + \frac{1}{\rho^4}g^{mn}\Gamma^\rho_{mj}\Gamma^\rho_{ni}m + 2\nabla_i\nabla_j\frac{m}{\rho}\\
&-2\rho(\partial_\rho(\rho\Gamma^k_{\rho i}\sigma_{kj}) + \partial_\rho(\rho\Gamma^k_{\rho j}\sigma_{ki}) + \Gamma^k_{\rho i}\partial_\rho(\rho\sigma_{kj}) + \Gamma^k_{\rho j}\partial_\rho(\rho\sigma_{ki})) + 2\rho^2(\Gamma^k_{\rho i}\Gamma^m_{\rho k}\sigma_{mj} + \Gamma^k_{\rho j}\Gamma^m_{\rho k}\sigma_{mi}\\
&+2\Gamma^k_{\rho i}\Gamma^m_{\rho j}\sigma_{mk}) - \frac{1}{2\rho}(\Gamma^\rho_{ki}\Gamma^m_{\rho j} + \Gamma^\rho_{kj}\Gamma^m_{\rho i})\sigma^k_m - \frac{1}{2\rho}\Gamma^a_{a\rho}(\Gamma^\rho_{im}\sigma^m_j + \Gamma^\rho_{jm}\sigma^m_i) + \frac{1}{2\rho}\Gamma^\rho_{ki}\partial_\rho\sigma^k_j + \frac{1}{2\rho}\Gamma^\rho_{kj}\partial_\rho\sigma^k_i\\
&+\frac{1}{2\rho}(\Gamma^\rho_{ki}\Gamma^k_{\rho m}\sigma^m_j + \Gamma^\rho_{kj}\Gamma^k_{\rho m}\sigma^m_i + \Gamma^k_{\rho i}\Gamma^\rho_{km}\sigma^m_j + \Gamma^k_{\rho j}\Gamma^\rho_{km}\sigma^m_i) - \frac{1}{2\rho}\partial_\rho(\Gamma^\rho_{ki}\sigma^k_j) - \frac{1}{2\rho}\partial_\rho(\Gamma^\rho_{ki}\sigma^k_i)\\
&+\frac{g^{mn}}{2\rho}(\Gamma^\rho_{mn}(\Gamma^k_{i\rho}\sigma_{kj} + \Gamma^k_{\rho j}\sigma_{ki}) + \Gamma^k_{n\rho}(\Gamma^\rho_{mj}\sigma_{ki} + \Gamma^\rho_{mi}\sigma_{kj})) - \frac{g^{mn}}{2\rho^2}\Gamma^\rho_{mn}\partial_\rho(\rho\sigma_{ij}) - \frac{1}{2\rho}\Gamma^\rho_{ij}\partial_\rho\sigma^m_m\\
&-2\rho\Gamma^k_{\rho i}\sigma_{kj} - 2\rho\Gamma^k_{\rho j}\sigma_{ki}.
\end{aligned} \tag{J.185}$$

References

Milton Abramowitz, Irene A Stegun, and Robert H Romer. Handbook of mathematical functions with formulas, graphs, and mathematical tables, 1988.

Ofer Aharony, Guy Gur-Ari, and Ran Yacoby. d=3 Bosonic Vector Models Coupled to Chern-Simons Gauge Theories. *JHEP*, 03:037, 2012. .

Ofer Aharony, Shai M. Chester, and Erez Y. Urbach. A Derivation of AdS/CFT for Vector Models. *JHEP*, 03:208, 2021a. .

Ofer Aharony, Shai M. Chester, and Erez Y. Urbach. AdS from CFT for scalar QED. 9 2021b.

P. C. Aichelburg and R. U. Sexl. On the Gravitational field of a massless particle. *Gen. Rel. Grav.*, 2:303–312, 1971. .

Ahmed Almheiri, Thomas Hartman, Juan Maldacena, Edgar Shaghoulian, and Amirhossein Tajdini. Replica Wormholes and the Entropy of Hawking Radiation. *JHEP*, 05:013, 2020a. .

Ahmed Almheiri, Raghu Mahajan, Juan Maldacena, and Ying Zhao. The Page curve of Hawking radiation from semiclassical geometry. *JHEP*, 03:149, 2020b. .

Máximo Bañados, Ernesto Bianchi, Iván Muñoz, and Kostas Skenderis. Bulk renormalization and the AdS/CFT correspondence. 8 2022.

Arjun Bagchi. Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories. *Phys. Rev. Lett.*, 105:171601, 2010. .

Arjun Bagchi, Aditya Mehra, and Poulami Nandi. Field Theories with Conformal Carrollian Symmetry. *JHEP*, 05:108, 2019. .

Arjun Bagchi, Shamik Banerjee, Rudranil Basu, and Sudipta Dutta. Scattering Amplitudes: Celestial and Carrollian. *Phys. Rev. Lett.*, 128(24):241601, 2022. .

Arjun Bagchi, Prateksh Dhivakar, and Sudipta Dutta. AdS Witten diagrams to Carrollian correlators. *JHEP*, 04:135, 2023a. .

Arjun Bagchi, Kedar S. Kolekar, and Ashish Shukla. Carrollian Origins of Bjorken Flow. *Phys. Rev. Lett.*, 130(24):241601, 2023b. .

Vijay Balasubramanian and Per Kraus. A Stress tensor for Anti-de Sitter gravity. *Commun. Math. Phys.*, 208:413–428, 1999. .

Vijay Balasubramanian, Per Kraus, and Albion E. Lawrence. Bulk versus boundary dynamics in anti-de Sitter space-time. *Phys. Rev. D*, 59:046003, 1999. .

Adam Ball, Elizabeth Himwich, Sruthi A. Narayanan, Sabrina Pasterski, and Andrew Strominger. Uplifting AdS_3/CFT_2 to flat space holography. *JHEP*, 08:168, 2019. .

Maximo Banados, Claudio Teitelboim, and Jorge Zanelli. The Black hole in three-dimensional space-time. *Phys. Rev. Lett.*, 69:1849–1851, 1992. .

Maximo Banados, Marc Henneaux, Claudio Teitelboim, and Jorge Zanelli. Geometry of the (2+1) black hole. *Phys. Rev. D*, 48:1506–1525, 1993. . [Erratum: Phys.Rev.D 88, 069902 (2013)].

Glenn Barnich and Cedric Troessaert. Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited. *Phys. Rev. Lett.*, 105:111103, 2010a. .

Glenn Barnich and Cedric Troessaert. Aspects of the BMS/CFT correspondence. *JHEP*, 05:062, 2010b. .

Glenn Barnich and Cedric Troessaert. BMS charge algebra. *JHEP*, 12:105, 2011. .

Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of radiation fields in asymptotically minkowski space. *American Journal of Mathematics*, 137(5):1293–1364, 2015.

Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of scalar waves on long-range asymptotically minkowski spaces. *Advances in Mathematics*, 328:160–216, 2018.

Jacob D. Bekenstein. Black holes and entropy. *Phys. Rev. D*, 7:2333–2346, 1973. .

A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys. B*, 241:333–380, 1984. .

Eric Bergshoeff, José Manuel Izquierdo, and Luca Romano. Carroll versus Galilei from a Brane Perspective. *JHEP*, 10:066, 2020. .

Danny Birmingham. Choptuik scaling and quasinormal modes in the anti–de sitter space/conformal-field theory correspondence. *Physical Review D*, 64(6):064024, 2001.

Danny Birmingham, Ivo Sachs, and Sergey N Solodukhin. Conformal field theory interpretation of black hole quasinormal modes. *Physical review letters*, 88(15):151301, 2002.

Danny Birmingham, Ivo Sachs, and Sergey N Solodukhin. Relaxation in conformal field theory, hawking-page transition, and quasinormal or normal modes. *Physical Review D*, 67(10):104026, 2003.

H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner. Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems. *Proc. Roy. Soc. Lond. A*, 269: 21–52, 1962. .

L. Bonora, P. Pasti, and M. Bregola. WEYL COCYCLES. *Class. Quant. Grav.*, 3:635, 1986. .

Peter Breitenlohner and Daniel Z. Freedman. Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity. *Phys. Lett. B*, 115:197–201, 1982a. .

Peter Breitenlohner and Daniel Z. Freedman. Stability in Gauged Extended Supergravity. *Annals Phys.*, 144:249, 1982b. .

Pasquale Calabrese and John Cardy. Entanglement entropy and conformal field theory. *J. Phys. A*, 42:504005, 2009. .

Pasquale Calabrese and John L. Cardy. Entanglement entropy and quantum field theory. *J. Stat. Mech.*, 0406:P06002, 2004. .

Eduardo Casali, Walker Melton, and Andrew Strominger. Celestial amplitudes as AdS-Witten diagrams. *JHEP*, 11:140, 2022. .

Clifford Cheung, Anton de la Fuente, and Raman Sundrum. 4D scattering amplitudes and asymptotic symmetries from 2D CFT. *JHEP*, 01:112, 2017a. .

Clifford Cheung, Anton de la Fuente, and Raman Sundrum. 4d scattering amplitudes and asymptotic symmetries from 2d cft. *Journal of High Energy Physics*, 2017(1):1–55, 2017b.

D. Christodoulou. Reversible and irreversible transforations in black hole physics. *Phys. Rev. Lett.*, 25:1596–1597, 1970. .

Demetrios Christodoulou and Sergiu Klainerman. The global nonlinear stability of the minkowski space. *Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi" Séminaire Goulaouic-Schwartz"*, pages 1–29, 1993.

Lorenzo Cornalba. Eikonal methods in ads/cft: Regge theory and multi-reggeon exchange, 2008.

Miguel S. Costa, Joao Penedones, David Poland, and Slava Rychkov. Spinning Conformal Correlators. *JHEP*, 11:071, 2011. .

Miguel S. Costa, Vasco Goncalves, and João Penedones. Spinning AdS Propagators. *JHEP*, 09:064, 2014. .

R. N. Caldeira Costa. Holographic Reconstruction and Renormalization in Asymptotically Ricci-flat Spacetimes. *JHEP*, 11:046, 2012. .

Mihalis Dafermos and Igor Rodnianski. Lectures on black holes and linear waves. *Clay Math. Proc.*, 17:97–205, 2013.

Sumit R Das and Antal Jevicki. Large-n collective fields and holography. *Physical Review D*, 68(4):044011, 2003.

Jan de Boer and Sergey N. Solodukhin. A Holographic reduction of Minkowski space-time. *Nucl. Phys. B*, 665:545–593, 2003. .

Jan de Boer, Jelle Hartong, Niels A. Obers, Watse Sybesma, and Stefan Vandoren. Carroll stories. 7 2023.

Sebastian de Haro, Kostas Skenderis, and Sergey N. Solodukhin. Gravity in warped compactifications and the holographic stress tensor. *Class. Quant. Grav.*, 18:3171–3180, 2001a. .

Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis. Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence. *Commun. Math. Phys.*, 217:595–622, 2001b. .

Robert de Mello Koch, Antal Jevicki, Kenta Suzuki, and Junggi Yoon. AdS Maps and Diagrams of Bi-local Holography. *JHEP*, 03:133, 2019. .

Stanley Deser and A. Schwimmer. Geometric classification of conformal anomalies in arbitrary dimensions. *Phys. Lett. B*, 309:279–284, 1993. .

V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov. *Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory*, volume 63. 1977. .

Laura Donnay, Adrien Fiorucci, Yannick Herfray, and Romain Ruzziconi. A carrollian perspective on celestial holography. *arXiv preprint arXiv:2202.04702*, 2022.

Tevian Dray and Gerard 't Hooft. The Gravitational Shock Wave of a Massless Particle. *Nucl. Phys. B*, 253:173–188, 1985. .

C. Duval, G. W. Gibbons, and P. A. Horvathy. Conformal Carroll groups and BMS symmetry. *Class. Quant. Grav.*, 31:092001, 2014. .

Richard John Eden, Peter Vincent Landshoff, David I Olive, and John Charlton Polkinghorne. *The analytic S-matrix*. Cambridge University Press, 2002.

Netta Engelhardt and Aron C. Wall. Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime. *JHEP*, 01:073, 2015. .

E.Witten. Baryons and branes in anti de sitter space, 1998. https://www.on.kitp.ucsb.edu/online/strings98/witten/.

Thomas Faulkner, Aitor Lewkowycz, and Juan Maldacena. Quantum corrections to holographic entanglement entropy. *JHEP*, 11:074, 2013. .

Charles Fefferman and C. Robin Graham. Conformal invariants. In *Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984*, number S131 in Astérisque. Société mathématique de France, 1985. URL http://www.numdam.org/item/AST_1985__S131__95_0/.

Charles Fefferman and C. Robin Graham. The ambient metric. *Ann. Math. Stud.*, 178: 1–128, 2011.

Daniel Z. Freedman, Kenneth Johnson, and Jose I. Latorre. Differential regularization and renormalization: A New method of calculation in quantum field theory. *Nucl. Phys. B*, 371:353–414, 1992. .

Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Correlation functions in the CFT(d) / AdS(d+1) correspondence. *Nucl. Phys. B*, 546:96–118, 1999. .

G. W. Gibbons and S. W. Hawking. Action Integrals and Partition Functions in Quantum Gravity. *Phys. Rev. D*, 15:2752–2756, 1977. .

Simone Giombi and Xi Yin. Higher spin gauge theory and holography: the three-point functions. *Journal of High Energy Physics*, 2010(9):1–80, 2010.

Simone Giombi and Xi Yin. The Higher Spin/Vector Model Duality. *J. Phys. A*, 46: 214003, 2013. .

Simone Giombi, Shiraz Minwalla, Shiroman Prakash, Sandip P. Trivedi, Spenta R. Wadia, and Xi Yin. Chern-Simons Theory with Vector Fermion Matter. *Eur. Phys. J. C*, 72:2112, 2012. .

C Robin Graham. Volume and area renormalizations for conformally compact einstein metrics. *arXiv preprint math/9909042*, 1999.

C. Robin Graham. Volume and area renormalizations for conformally compact Einstein metrics. *Rend. Circ. Mat. Palermo S*, 63:31–42, 2000.

Vladimir N Gribov. *The theory of complex angular momenta: Gribov lectures on theoretical physics*. Cambridge University Press, 2003.

S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett. B*, 428:105–114, 1998. .

Alfredo Guevara, Elizabeth Himwich, Monica Pate, and Andrew Strominger. Holographic symmetry algebras for gauge theory and gravity. *JHEP*, 11:152, 2021. .

Dennis Hansen, Niels A. Obers, Gerben Oling, and Benjamin T. Søgaard. Carroll Expansion of General Relativity. *SciPost Phys.*, 13(3):055, 2022. .

Zezhuang Hao. Real time holography for higher spin theories. *JHEP*, 03:168, 2024a. .

Zezhuang Hao. Holographic reconstruction of flat spacetime. *JHEP*, 09:060, 2024b. .

Zezhuang Hao and Marika Taylor. Flat holography and celestial shockwaves. *JHEP*, 02:090, 2024. .

James B Hartle and Stephen W Hawking. Wave function of the universe. In *EUCLIDEAN QUANTUM GRAVITY*, pages 310–325. World Scientific, 1983.

Thomas Hartman, Sachin Jain, and Sandipan Kundu. Causality Constraints in Conformal Field Theory. *JHEP*, 05:099, 2016. .

Jelle Hartong. Holographic Reconstruction of 3D Flat Space-Time. *JHEP*, 10:104, 2016. .

S. W. Hawking. Gravitational radiation from colliding black holes. *Phys. Rev. Lett.*, 26: 1344–1346, 1971. .

S. W. Hawking. Particle Creation by Black Holes. *Commun. Math. Phys.*, 43:199–220, 1975. . [Erratum: Commun.Math.Phys. 46, 206 (1976)].

Stephen W. Hawking and George F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2 2023. ISBN 978-1-00-925316-1, 978-1-00-925315-4, 978-0-521-20016-5, 978-0-521-09906-6, 978-0-511-82630-6, 978-0-521-09906-6. .

Temple He, Vyacheslav Lysov, Prahar Mitra, and Andrew Strominger. BMS supertranslations and Weinberg's soft graviton theorem. *JHEP*, 05:151, 2015. .

Temple He, Prahar Mitra, and Andrew Strominger. 2D Kac-Moody Symmetry of 4D Yang-Mills Theory. *JHEP*, 10:137, 2016. .

M. Henningson and K. Skenderis. The Holographic Weyl anomaly. *JHEP*, 07:023, 1998. .

Christopher P Herzog and Dam Thanh Son. Schwinger-keldysh propagators from ads/cft correspondence. *Journal of High Energy Physics*, 2003(03):046, 2003.

Christoph Holzhey, Finn Larsen, and Frank Wilczek. Geometric and renormalized entropy in conformal field theory. *Nucl. Phys. B*, 424:443–467, 1994. .

Lorenzo Iacobacci, Charlotte Sleight, and Massimo Taronna. From celestial correlators to AdS, and back. *JHEP*, 06:053, 2023. .

Lorenzo Iacobacci, Charlotte Sleight, and Massimo Taronna. Celestial Holography Revisited II: Correlators and Källén-Lehmann. 1 2024.

Denis Karateev, Petr Kravchuk, and David Simmons-Duffin. Harmonic Analysis and Mean Field Theory. *JHEP*, 10:217, 2019. .

Bernard S Kay and Robert M Wald. Linear stability of schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere. *Classical and Quantum Gravity*, 4(4):893, 1987.

Esko Keski-Vakkuri. Bulk and boundary dynamics in btz black holes. *Physical Review D*, 59(10), Mar 1999. ISSN 1089-4918. . URL http://dx.doi.org/10.1103/PhysRevD.59.104001.

Igor R. Klebanov and Edward Witten. AdS / CFT correspondence and symmetry breaking. *Nucl. Phys. B*, 556:89–114, 1999. .

I.R Klebanov and A.M Polyakov. Ads dual of the critical o(n) vector model. *Physics Letters B*, 550(3-4):213–219, Dec 2002. ISSN 0370-2693. . URL http://dx.doi.org/10.1016/S0370-2693(02)02980-5.

Maxim Kontsevich and Graeme Segal. Wick rotation and the positivity of energy in quantum field theory. *The Quarterly Journal of Mathematics*, 72(1-2):673–699, 2021.

Per Kraus, Hirosi Ooguri, and Stephen Shenker. Inside the horizon with AdS / CFT. *Phys. Rev. D*, 67:124022, 2003. .

Petr Kravchuk and David Simmons-Duffin. Light-ray operators in conformal field theory. *Journal of High Energy Physics*, 2018(11):1–111, 2018.

Alok Laddha, Siddharth G Prabhu, Suvrat Raju, and Pushkal Shrivastava. Squinting at massive fields from infinity. *arXiv preprint arXiv:2207.06406*, 2022.

Nicolaas P Landsman and Ch G Van Weert. Real-and imaginary-time field theory at finite temperature and density. *Physics Reports*, 145(3-4):141–249, 1987.

Robert G. Leigh and Anastasios C. Petkou. Holography of the N=1 higher spin theory on AdS(4). *JHEP*, 06:011, 2003. .

Aitor Lewkowycz and Juan Maldacena. Generalized gravitational entropy. *JHEP*, 08:090, 2013. .

Hans Lindblad and Igor Rodnianski. The global stability of minkowski space-time in harmonic gauge. *Annals of mathematics*, pages 1401–1477, 2010.

Chang Liu and David A. Lowe. Conformal wave expansions for flat space amplitudes. *JHEP*, 07:102, 2021. .

Jorma Louko and Rafael D Sorkin. Complex actions in two-dimensional topology change. *Classical and Quantum Gravity*, 14(1):179, 1997.

Gerhard Mack. All unitary ray representations of the conformal group su (2, 2) with positive energy. *Communications in Mathematical Physics*, 55(1):1–28, 1977.

Juan Maldacena. The large-n limit of superconformal field theories and supergravity. *International journal of theoretical physics*, 38(4):1113–1133, 1999.

Juan Maldacena and Douglas Stanford. Remarks on the sachdev-ye-kitaev model. *Physical Review D*, 94(10):106002, 2016.

Juan Maldacena and Leonard Susskind. Cool horizons for entangled black holes. *Fortsch. Phys.*, 61:781–811, 2013. .

Juan Maldacena, Douglas Stanford, and Zhenbin Yang. Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space. *PTEP*, 2016(12):12C104, 2016. .

Robert B. Mann and Donald Marolf. Holographic renormalization of asymptotically flat spacetimes. *Class. Quant. Grav.*, 23:2927–2950, 2006. .

Donald Marolf. Asymptotic flatness, little string theory, and holography. *JHEP*, 03:122, 2007. .

Walker Melton, Atul Sharma, and Andrew Strominger. Celestial Leaf Amplitudes. 12 2023.

Kevin Nguyen and Jakob Salzer. Celestial IR divergences and the effective action of supertranslation modes. *JHEP*, 09:144, 2021. .

Kevin Nguyen and Peter West. Carrollian conformal fields and flat holography. 5 2023.

Hermann Nicolai. A possible constructive approach to (super-φ3) 4 (i). euclidean formulation of the model. *Nuclear Physics B*, 140(2):294–300, 1978.

Naoki Ogawa, Tadashi Takayanagi, Takashi Tsuda, and Takahiro Waki. Wedge holography in flat space and celestial holography. *Phys. Rev. D*, 107(2):026001, 2023. .

Konrad Osterwalder and Robert Schrader. Axioms for euclidean green's functions. *Communications in mathematical physics*, 31(2):83–112, 1973.

Konrad Osterwalder and Robert Schrader. Axioms for euclidean green's functions ii. *Communications in Mathematical Physics*, 42(3):281–305, 1975.

Sabrina Pasterski. Lectures on celestial amplitudes. *Eur. Phys. J. C*, 81(12):1062, 2021. .

Sabrina Pasterski and Shu-Heng Shao. Conformal basis for flat space amplitudes. *Phys. Rev. D*, 96(6):065022, 2017. .

Sabrina Pasterski and Herman Verlinde. Chaos in celestial CFT. *JHEP*, 08:106, 2022. .

Sabrina Pasterski, Shu-Heng Shao, and Andrew Strominger. Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere. *Phys. Rev. D*, 96(6):065026, 2017. .

Monica Pate, Ana-Maria Raclariu, Andrew Strominger, and Ellis Ye Yuan. Celestial operator products of gluons and gravitons. *Rev. Math. Phys.*, 33(09):2140003, 2021. .

Joào Penedones. Tasi lectures on ads/cft. *New Frontiers in Fields and Strings*, Nov 2016. . URL http://dx.doi.org/10.1142/9789813149441_0002.

Geoff Penington, Stephen H. Shenker, Douglas Stanford, and Zhenbin Yang. Replica wormholes and the black hole interior. *JHEP*, 03:205, 2022. .

Geoffrey Penington. Entanglement Wedge Reconstruction and the Information Paradox. *JHEP*, 09:002, 2020. .

R. Penrose and R. M. Floyd. Extraction of rotational energy from a black hole. *Nature*, 229:177–179, 1971. .

Ana-Maria Raclariu. Lectures on celestial holography. *arXiv preprint arXiv:2107.02075*, 2021.

Lucrezia Ravera. AdS Carroll Chern-Simons supergravity in 2 + 1 dimensions and its flat limit. *Phys. Lett. B*, 795:331–338, 2019. .

Shinsei Ryu and Tadashi Takayanagi. Holographic derivation of entanglement entropy from AdS/CFT. *Phys. Rev. Lett.*, 96:181602, 2006. .

R. Sachs. Asymptotic symmetries in gravitational theory. *Phys. Rev.*, 128:2851–2864, 1962a. .

R. K. Sachs. Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times. *Proc. Roy. Soc. Lond. A*, 270:103–126, 1962b. .

Amartya Saha. $w_{1+\infty}$ and Carrollian Holography. 8 2023a.

Amartya Saha. Carrollian approach to 1 + 3D flat holography. *JHEP*, 06:051, 2023b. .

E. Sezgin and P. Sundell. Massless higher spins and holography. *Nucl. Phys. B*, 644: 303–370, 2002. . [Erratum: Nucl.Phys.B 660, 403–403 (2003)].

E. Sezgin and P. Sundell. Holography in 4D (super) higher spin theories and a test via cubic scalar couplings. *JHEP*, 07:044, 2005. .

David Simmons-Duffin, Douglas Stanford, and Edward Witten. A spacetime derivation of the lorentzian ope inversion formula. *Journal of High Energy Physics*, 2018(7):1–40, 2018.

Kostas Skenderis. Lecture notes on holographic renormalization. *Classical and Quantum Gravity*, 19(22):5849, 2002.

Kostas Skenderis and Marika Taylor. Holographic Coulomb branch vevs. *JHEP*, 08:001, 2006a. .

Kostas Skenderis and Marika Taylor. Kaluza-Klein holography. *JHEP*, 05:057, 2006b. .

Kostas Skenderis and Balt C. van Rees. Real-time gauge/gravity duality. *Phys. Rev. Lett.*, 101:081601, 2008. .

Kostas Skenderis and Balt C. van Rees. Real-time gauge/gravity duality: Prescription, Renormalization and Examples. *JHEP*, 05:085, 2009. .

Kostas Skenderis, Marika Taylor, and Balt C. van Rees. AdS boundary conditions and the Topologically Massive Gravity/CFT correspondence. *AIP Conf. Proc.*, 1196(1): 266, 2009a. .

Kostas Skenderis, Marika Taylor, and Balt C. van Rees. Topologically Massive Gravity and the AdS/CFT Correspondence. *JHEP*, 09:045, 2009b. .

Dam T Son and Andrei O Starinets. Minkowski-space correlators in ads/cft correspondence: Recipe and applications. *Journal of High Energy Physics*, 2002(09):042, 2002.

Andrew Strominger. On BMS Invariance of Gravitational Scattering. *JHEP*, 07:152, 2014. .

Andrew Strominger. Lectures on the Infrared Structure of Gravity and Gauge Theory. 3 2017.

Andrew Strominger. w(1+infinity) and the Celestial Sphere. 5 2021.

Andrew Strominger and Alexander Zhiboedov. Gravitational Memory, BMS Supertranslations and Soft Theorems. *JHEP*, 01:086, 2016. .

Leonard Susskind. The World as a hologram. *J. Math. Phys.*, 36:6377–6396, 1995. .

Leonard Susskind and Edward Witten. The Holographic bound in anti-de Sitter space. 5 1998.

Gerard 't Hooft. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys. B*, 72: 461, 1974. .

Gerard 't Hooft. Dimensional reduction in quantum gravity. *Conf. Proc. C*, 930308:284–296, 1993.

Marika Taylor and William Woodhead. Renormalized entanglement entropy. *JHEP*, 08: 165, 2016a. .

Marika Taylor and William Woodhead. The holographic F theorem. 4 2016b.

Peter Van Nieuwenhuizen and Andrew Waldron. On euclidean spinors and wick rotations. *Physics Letters B*, 389(1):29–36, 1996.

Balt C. van Rees. Real-time gauge/gravity duality and ingoing boundary conditions. *Nucl. Phys. B Proc. Suppl.*, 192-193:193–196, 2009. .

M. A. Vasiliev. Nonlinear equations for symmetric massless higher spin fields in (A)dS(d). *Phys. Lett. B*, 567:139–151, 2003. .

Mikhail A. Vasiliev. Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions. *Phys. Lett. B*, 243:378–382, 1990. .

Mikhail A. Vasiliev. Higher spin gauge theories: Star product and AdS space. pages 533–610, 10 1999. .

András Vasy. Microlocal analysis of asymptotically hyperbolic and kerr-de sitter spaces (with an appendix by semyon dyatlov). *Inventiones mathematicae*, 194:381–513, 2013.

Robert M. Wald. *General Relativity*. Chicago Univ. Pr., Chicago, USA, 1984. .

Robert M Wald. *Quantum field theory in curved spacetime and black hole thermodynamics*. University of Chicago press, 1994.

Steven Weinberg. *The quantum theory of fields*, volume 1. Cambridge university press, 1995.

Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998a. .

Edward Witten. Anti de sitter space and holography. *arXiv preprint hep-th/9802150*, 1998b.

Edward Witten. Why Does Quantum Field Theory In Curved Spacetime Make Sense? And What Happens To The Algebra of Observables In The Thermodynamic Limit? 12 2021a.

Edward Witten. A note on complex spacetime metrics. *arXiv preprint arXiv:2111.06514*, 2021b.

To use bibliography as well as the references section use the `multibbl` package.