UNIVERSITY OF SOUTHAMPTON

Faculty of Social Sciences
School of Mathematics

Polynomial Growth of Coarse Intervals in Coarse Median Spaces

by

Amina Assouda Ladjali

ORCiD: 0000-0001-5302-1591

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Abstract

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In this thesis, we explore the structure and geometry of coarse intervals in coarse median spaces, obtaining polynomial growth of coarse intervals as a consequence. We study finite rank bounded geometry coarse intervals in a quasi-geodesic coarse median space. We equip our coarse intervals with an ordering and then split our approach: we first consider rank 2 intervals and then turn our attention to higher rank intervals. In the rank 2 case, we introduce the concept of a coarse hyperplane in a coarse median space, a coarse analogue of hyperplanes in CAT(0) cube complexes, and use this as an important tool in proving three key properties: coarse hyperplanes intersected with a rank 2 coarse interval have co-dimension 1 in the coarse interval, coarse hyperplanes coarsely cover the whole coarse interval, and the intersection of a coarse hyperplane and coarse interval is 'almost' a coarse interval. We then use these three results and an inductive argument to show that rank 2 coarse intervals have quadratic growth. For the higher rank case, we introduce the important notion of a directly edge maximal point in a coarsely convex subset of a coarse interval. We then show that the length of a finite, incomparable sequence of directly edge maximal points, an antichain, associated to a coarsely convex subset is bounded above by the rank of the subset. Equipped with this result, we prove that an R-separated subset of directly edge maximal points equipped with a partial ordering can be decomposed into a union of chains via the aforementioned result and the application of Dilworth's Lemma. We then obtain two maps, $f = (f_i)$ and g, where f maps any point u in a coarse interval to a product of chains, which is isometrically embedded in $\mathbb{Z}^r$. Each chain gives the coordinate of u in that direction, i.e. f_i provides the ith coordinate of u. The map g maps the coordinates of u back into the interval by computing the minimum of these coordinates. Hence, we have shown that higher rank coarse intervals also have polynomial growth.

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I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. None of this work has been published before submission.

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Chapter 1

Introduction

Coarse median spaces (and groups) were introduced by Bowditch in Bowditch (2013a) and are generalisations of median algebras, i.e. they satisfy the axioms of a median algebra up to bounded distance. They are metric spaces equipped with a certain type of structure, the coarse median, that captures the concept of 'medianness' in the space. Specifically, given any three points in the space, there is a fourth point that is 'closer' to all three of them than any other point in the space, known as the coarse median of the three points.

The 'coarse' in coarse median spaces refers to the fact that the structure capturing medianness need not be exact, but rather can be understood only up to a certain level of approximation. This allows for the study of metric spaces that may not have a well-defined notion of exact medianness, but still exhibit analogous properties to those present in median algebras.

Finite median algebras are equivalent to (vertex sets of) finite CAT(0) cube complexes, Roller (2016), and therefore, we can informally think of coarse median spaces as coarsened versions of CAT(0) cube complexes. More intuitively, one can view a coarse median space $(X, d, \langle\rangle)$ as a metric space (X, d) equipped with a ternary operator $\langle\rangle$ (the coarse median), where finite subsets can be approximated by finite CAT(0) cube complexes in which the error is controlled by the metric. This comparison holds in the 'one-dimensional' case, where CAT(0) cube complexes are trees and hyperbolic space is 'coarsely tree-like.' An alternative characterisation of Bowditch's original definition of a coarse median space is provided in Niblo et al. (2019), where the need to approximate arbitrary finite subsets is replaced with only requiring subsets of cardinality at most 4. This characterisation can be viewed as a 'coarsening' of the axioms defining a median algebra, where the new coarse 4-point condition is a coarse analogue of the 4-point condition for median algebras.

The 'rank' of a coarse median space can be intuitively thought of as its dimension. A very helpful definition of rank, which we frequently refer to throughout the thesis, can be found in Niblo et al. (2019). Broadly, this says that a coarse median space has rank at most n if and only if it does not contain arbitrarily large $(n+1)$-dimensional coarse cubes, i.e. one of the edges of the cube must be trivial.

The notion of a coarse median space provides a unified approach of looking at different spaces, such as geodesic hyperbolic spaces and mapping class groups, which Bowditch showed are coarse median spaces of rank 1 (an alternative proof of this result was also given in Niblo et al. (2019)) and finite rank respectively, in Bowditch (2013a), and hence we are able to view all these spaces and groups under one umbrella. Coarse median spaces have many nice properties; for instance, coarse medians are preserved under quasi-isometry and direct products. Bowditch also shows that coarse medians are preserved under relative hyperbolicity (Bowditch (2013b)) and that asymptotic cones of coarse median spaces are topological median algebras Bowditch (2013a). Coarse median spaces therefore encompass a variety of interesting spaces and have many applications in geometric group theory.

In this thesis, we are concerned with the structure and geometry of coarse intervals in coarse median spaces. Coarse intervals are coarse analogues of median intervals, and in the CAT(0) cube complex case, these constitute the set of points lying on any edge path geodesic connecting a pair of points in the cube complex. The original definition was introduced by Bowditch in Bowditch (2013a) and a similar definition was given in Niblo et al. (2019), both of which are 'coarsenings' of the definition(s) of an interval in a median algebra. In a CAT(0) cube complex, the median of three points is the unique point in the intersection of the three intervals they define — this also holds coarsely in a coarse median space (Bowditch (2013a), Niblo et al. (2019)). Coarse intervals have not been explored in as much detail compared to coarse median spaces as a whole. At first sight, they are difficult to understand and thus are akin to 'black boxes', in the sense that we do not know what a coarse interval 'looks like'. Given two endpoints of the interval, x and y, we project points z in our coarse median space X onto the interval $[x, y]$ in order to explore what happens within.

Initially, proving that coarse intervals have polynomial growth was our motivation for studying the structure and geometry of coarse intervals. Through exploring, we have a deeper understanding of what coarse intervals look like and their inner structure. The idea is to use the rank to bound the number of points in an interval in order to get a clearer idea of its inner workings. We use the rank to 'cut up' the interval and obtain polynomial growth of coarse intervals as a consequence.

More specifically, given a quasi-geodesic uniformly discrete bounded geometry coarse median space X with finite rank coarse interval $[x, y] \subseteq X$, we equip $[x, y]$ with a coarsening of the partial ordering for median intervals given in Bowditch (2014). Loosely speaking, this says that given two points $a, b \in [x, y]$, a is 'coarsely less than' b

if the minimum of a and b, $\langle x, a, b\rangle$, is close to a (similarly for the maximum). We then divide our approach according to the rank of $[x, y]$: rank ≤ 2 and rank > 2.

Proving quadratic growth of rank 2 coarse intervals requires an important notion of a coarse hyperplane associated to two points in a coarse median space — they are a coarse analogue of hyperplanes in CAT(0) cube complexes. We present the definition below:

Definition 1.1. Let L be a constant. An *L-coarse hyperplane corresponding to a, b* in a coarse median space X, with $d(a, b)$ much greater than $2L$, divides the space into two half-spaces H_a and H_b, where $H_a = \{z \in X : a {\sim_L} \langle a, z, b\rangle\}$ and $H_b = \{z \in X : b {\sim_L} \langle a, z, b\rangle\}$, respectively. The coarse hyperplane itself is then defined to be

$$h_{ab} = X \setminus (H_a \cup H_b),$$

so h_{ab} partitions the complement into two disjoint pieces.

We then use this concept to prove three important results: coarse hyperplanes intersected with a rank 2 coarse interval have co-dimension 1 in the coarse interval, coarse hyperplanes coarsely cover the whole coarse interval, and the intersection of a coarse hyperplane and coarse interval is 'almost' a coarse interval. We then use these results and an inductive argument to show that rank 2 coarse intervals have quadratic growth:

Theorem 1.2. *Let X be a rank 2, uniformly discrete, uniformly locally finite, quasi-geodesic coarse median space. Then there exists a constant W such that for any coarse interval $[x, y] \subseteq X$, we have $\#[x, y] \leq W d(x, y)^2$, where $\#[x, y]$ denotes the cardinality of $[x, y]$.*

Extending the proof given in the rank 2 case proved more difficult than expected, and so we took inspiration from (Brodzki et al., 2009, Theorem 1.14) and its proof, which showed that intervals in CAT(0) cube complexes have polynomial growth. For the general rank n case, we introduce the key concept of a directly edge maximal point in a coarsely convex subset of a coarse interval; roughly speaking, these are points that can be thought of as 'maximal' points of the subset. More precisely, they are defined as follows.

Definition 1.3. Let X be a coarse median space and S be a δ-coarsely convex subset of $[x, y]$. Then a point $a \in S$ is said to be *directly edge maximal* with parameters C_1, C_2 if the following condition holds: for all n with $1 \leq n \leq r$ and $u_1, \ldots, u_n \in S$, if $a \sim_{C_1} \min(u_1, \ldots, u_n)$, then there exists an index $i \in \{1, \ldots, n\}$ such that $a \sim_{C_2} u_i$.

We then show that the length of a finite, incomparable sequence of directly edge maximal points, an antichain, associated to a coarsely convex subset S of a coarse interval $[x, y]$ is bounded above by the rank of S:

Theorem 1.4. *Given a coarse median space X, constant δ, rank constant* $C(\lambda)$*, and iterated* $(n+3)$*-point and symmetry constants F and G, respectively, there exists* $\lambda, E = C(\lambda), C_1$ *such that for all* C_2 *there exists M such that the following holds. Let* $[x, y] \subseteq X$ *be a coarse interval and* $S \subseteq [x, y]$ *be a δ-coarsely convex subset with rank at most* $r > 0$ *with respect to* $C(\lambda)$*. Let n be an integer such that* $1 \leq n \leq r+1$*. Given points* $u_1, \ldots, u_n \in S$ *which are n M-incomparable* (C_1, C_2)*-directly edge maximal points (an M-coarse antichain), where* $M = K(C_2 + \kappa_4) + 2H(0) + \kappa_4$*, then we obtain the following result. Define* $v_i = \min(u_1, \ldots, \hat{u}_i, \ldots, u_n)$ *for* $i \in \{1, \ldots, n\}$ *and* $v_0 = \min(u_1, \ldots, u_n)$*. Then the set* $\{v_0, v_1, \ldots, v_n\}$ *forms a* $(\lambda, E) - n$*-pod, where E is the non-triviality constant. Consequently, we have:*

$$\#\{u_1, \ldots, u_n\} \leq r,$$

which means that the size of any M-antichain n in S is bounded above by r, i.e. $n \leq r$*.*

Note that an n-pod can be defined as a configuration consisting of n line segments, each termed a 'leg', that extend from a common point, called the 'centre', to distinct end points in space. We derive what λ, E are in the proof of the theorem stated above (see proof of Theorem 6.9). A non-triviality constant in the context of an n-pod is a constant that sets a minimum distance between the points in the n-pod, ensuring that they are sufficiently spaced apart and are a non-trivial distance apart. This constant prevents any 'legs' of the n-pod from collapsing.

This then leads to proving that any R-separated subset of directly edge maximal points equipped with a partial ordering, where R-separated refers to the minimum distance between distinct points in a set being at least R, can be decomposed into a union of chains via Dilworth's Lemma Dilworth (1950) and the theorem above. We then obtain two maps, f and g; $f = (f_i)$ maps any point u in a coarse interval to a product of chains, which is isometrically embedded in $\mathbb{Z}^r$. Each chain gives the coordinate of u in that direction, i.e. the f_i provides the ith coordinate of u and can be thought of as 'slicing' the interval up. The map g simply maps the coordinates of u back into the interval by computing the minimum of these coordinates. Thus, we have shown that rank n coarse intervals also have polynomial growth and in addition, that the maps f and g have some nice properties, for instance, f is a quasi-morphism (in the sense of coarse medians).

Our investigation of coarse intervals was motivated by a result from Bowditch (Bowditch, 2014, Lemma 9.7), which proved that uniformly discrete coarse median space of bounded geometry and finite rank have polynomial growth. However, this result appeals to the asymptotic cone and we wanted to get more precise information about what happens on finite scales. Our result is a tightening of Bowditch's, but is a constructive argument bypassing the use of asymptotic cones, thus giving us a better insight as to what coarse intervals look like.

Coarse intervals can be thought of as a special case of coarse convex hulls, and this is explored in another paper of Bowditch's, Bowditch (a). An alternative proof of polynomial growth of coarse intervals is also given in (Bowditch, a, Theorem 1.4). The proof of this result has more assumptions on the coarse median space — in particular, the conditions on the space arise when considering projection maps to hyperbolic spaces, and the polynomial growth condition depends on the parameters associated to these hypotheses. In contrast, our argument makes much fewer assumptions on the space: we only consider finite rank bounded geometry coarse intervals in a quasi-geodesic coarse median space.

In addition, coarse median spaces have been studied from the perspective of intervals in Niblo et al. (2021). This paper shows that intervals play a fundamental role in determining the structure and geometry of coarse median spaces; for instance, the cardinality of intervals can be used as a substitute for measuring distance. It can also be shown that in a bounded geometry quasi-geodesic coarse median space, the metric is determined by the interval structure, which motivates the definition of a coarse median algebra; these can be viewed as a generalisation of discrete median algebras. A polynomial growth result for coarse intervals is also given here, but as a converse to Bowdtich's result, (Bowditch, 2014, Lemma 9.7). In particular, it shows that a polynomial bound on growth in coarse intervals does in fact characterise the rank.

Overall, we have gained a better understanding and picture of coarse intervals in this thesis, and as a consequence, have shown that coarse intervals have polynomial growth; however, our method of proving polynomial growth takes a very different approach compared to those taken in Bowditch (2014), Bowditch (a), Niblo et al. (2021). We focus on defining and using coarse hyperplanes, which we show are powerful structures and have parallel properties (in a coarse sense) to hyperplanes in CAT(0) cube complexes. Another very significant concept we introduce and use is that of a directly edge maximal point, which helps give us explicit maps in mapping coarse intervals to a product of chains, giving us coordinates for each point in a coarse interval. These two important ideas have not been introduced or used previously, at least not in this context, and there is certainly scope for further work when it comes to using these concepts in going forward with investigating coarse intervals. One example could be proving that results for directly edge maximal points also hold for indirectly edge maximal points, which are a generalisation of directly edge maximal points.

The structure of the thesis is as follows. In Chapters 2 and 3, we outline the necessary background information that will be required for the thesis, where we detail important definitions and properties associated to CAT(0) cube complexes, median algebras and coarse median spaces. In Chapter 4, we describe the necessary conditions needed on our coarse median space and intervals; in particular, we work with finite rank bounded geometry coarse intervals in a quasi-geodesic coarse median space. In Chapter 5, we introduce and define the notion of a coarse hyperplane associated to two points in a coarse median space and prove three main results associated to coarse hyperplanes, enabling us to prove that rank 2 coarse intervals have quadratic growth. In Chapter 6, we establish the notion of directly edge maximal points and detail their role in defining maps that provide coordinates for each point in a coarse interval. Finally, in the Appendix, we provide proofs of most of the main results of Chapter 5 from a median point of view.

Chapter 2

CAT(0) Cube Complexes and Median Algebras

2.1 Metrics and Geodesics

We start by defining certain key properties of metric spaces, such as bounded geometry and quasi-geodesicity, which are necessary conditions that we impose on our space.

Definition 2.1 (E.g. Niblo et al. (2021))**.** Let (X, d) be a metric space.

1. A subset $A \subseteq X$ is *bounded* if its diameter diam(A) = sup$\{d(x, y) \colon x, y \in A\}$ is finite, and A is a *net* in X if there exists some constant $C > 0$ such that for any $x \in X$, there exists some $a \in A$ such that $d(a, x) \leq C$.
2. The metric space (X, d) is said to be *uniformly discrete* if there exists a constant $C > 0$ such that for any $x \neq y \in X, d(x, y) > C$.
3. The metric space (X, d) is said to have *bounded geometry* if, for any $r > 0$, there exists some constant $n \in N$ such that the closed ball centred at x with radius r, $\#B(x, r) \leq n$ for any $x \in X$.

Definition 2.2 (Niblo et al. (2019), Niblo et al. (2021))**.** Let $(X, d), (Y, d')$ be metric spaces and $L, C > 0$ be constants.

1. An (L, C)-*large scale Lipschitz map* from (X, d) to (Y, d') is a map $f \colon X \to Y$ such that for any $x, x' \in X, d'(f(x), f(x')) \leq Ld(x, x') + C$.
2. A map $f \colon (X, d) \to (Y, d')$ is *bornologous* if there exists an increasing map $\rho_+ \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in X, d'(f(x), f(y)) \leq \rho_+(d(x, y))$.

3. An (L, C)-*quasi-isometric embedding* from (X, d) to (Y, d') is a map $f\colon X \to Y$ such that for any $x, x' \in X, L^{-1}d(x, x') - C \leq d'(f(x), f(x')) \leq Ld(x, x') + C$.

4. An (L, C)-*quasi-isometry* from (X, d) to (Y, d') is an (L, C)-large scale Lipschitz map $f\colon X \to Y$ such that there exists another (L, C)-large scale Lipschitz map $g\colon Y \to X$ with $f \circ g \sim CId_Y$ and $g \circ f \sim CId_X$.

Definition 2.3. (X, d) is said to be *quasi-geodesic* if there exist constants A_1 and A_2 such that for any two points $x, y \in X$, there exists a map $\gamma\colon [0, N] \to X$ with $\gamma(0) = x, \gamma(N) = y$ such that for any $s, t \in [0, N]$,

$$\frac{1}{A_1}|s - t| - A_2 \leq d(\gamma(s), \gamma(t)) \leq A_1|s - t| + A_2.$$

If (X, d) is (1,0)-quasi-geodesic, then we say that X is *geodesic.*

2.2 CAT(0) Cube Complexes

CAT(0) cube complexes are a class of metric spaces where the curvature is non-negative. There are two ways to define them; they are geodesic metric spaces equipped with a metric that satisfies the CAT(0) condition, or, they are simply connected cube complexes where every link is flag (this is a result of Gromov). They can also be thought of as higher dimensional analogues of simplicial trees. For further background information, one should refer to Bridson and Haefliger (1999), Chepoi (2000), Gromov (1987), Niblo and Reeves (1998) and Sageev (1995).

A *cube complex* is a polyhedral complex in which cells are isometric to unit Euclidean cubes and gluing maps are isometries between faces and cubes. One-dimensional cubes are called *edges*, two-dimensional cubes are called *squares* and a cube complex is *finite-dimensional* if there is a bound on the dimension of its cubes.

A cube complex can be equipped with a *geodesic metric*, in which the distance between two points is realised by the shortest path between them Bridson and Haefliger (1999). A geodesic metric space is said to be *CAT(0)* if it satisfies the *CAT(0) inequality*; this means that all its geodesic triangles are slimmer than the corresponding comparison triangles in Euclidean space. If there is a bound on the dimension of the cubes then the associated intrinsic geodesic metric is complete.

Let v be a vertex in a cube complex. The *link* of v, $lk(v)$, is the simplicial complex with a k-simplex for each corner of a $(k + 1)$-cube containing v. A *flag complex* is a simplicial complex with no 'empty' or 'missing' simplices; more formally, every clique in the 1-skeleton spans a simplex. A theorem of Gromov's provides a combinatorial characterisation of the CAT(0) condition Gromov (1987): a cube complex X is a *CAT(0)* *cube complex* if and only if it is simply connected and the link of each vertex is a flag complex (i.e. X is locally CAT(0), also known as *non-positively curved*).

The vertex set of a CAT(0) cube complex can also be endowed with the *edge path metric*, where the distance between a pair of points x and y is given by the minimum number of edges required to connect them. The *interval* between x and y is then defined to be the set of points lying on any edge path geodesic connecting x and y. To be precise, given $x, y \in V$, where V is the vertex set of a CAT(0) cube complex, the interval is given by

$$[x, y] = \{z \in V : d(x, y) = d(x, z) + d(z, y)\}.$$

Furthermore, a CAT(0) cube complex can be equipped with a set of *hyperplanes* where each edge intersects only one hyperplane. It is shown in Niblo and Reeves (1998) that hyperplanes can be treated as totally geodesic, codimension-1 subspaces; hence, they themselves have their own natural structure as CAT(0) cube complexes (however, they are not a sub-complex of the original CAT(0) cube complex). Each hyperplane divides the space into two *half-spaces* and the metric counts the number of hyperplanes separating two points. A pair of hyperplanes provides four possible half-space intersections; the hyperplanes *cross/intersect* if and only if each of these four half-space intersections is non-empty. An edge path connecting a point in one half-space to a point in the other must cross H; we say that H *separates* the two points. If a CAT(0) cube complex has finite dimension, then its dimension is the maximal number of pairwise intersecting hyperplanes. A set of vertices is *convex* if whenever it contains both x and y, it contains the interval $[x, y]$. Equivalently, a subset is convex if it is an intersection of half spaces, and we can redefine $[x, y]$ to be the intersection of all the half-spaces containing both x and y.

Note 2.4. Each n-dimensional cube in a CAT(0) cube complex defines n pairwise intersecting hyperplanes — which it crosses — and, conversely, a collection of n pairwise intersecting hyperplanes gives rise to a unique n-cube (which crosses these hyperplanes).

There is also the notion of midplanes Niblo and Reeves (1998) and carriers. A *midplane* of a cube $[-\frac{1}{2}, \frac{1}{2}]^n$ is its intersection with a codimension-1 coordinate hyperplane, so every n-cube contains n midplanes which pairwise intersect. The *carrier* of a hyperplane H is the union of all closed cubes C such that $H \cap C \neq \emptyset$. Unlike hyperplanes, the carrier is genuinely a sub-complex of the original CAT(0) cube complex. We will see later on that in the coarse world, coarse hyperplanes are coarsened versions of carriers.

More generally, hyperplanes are geometric interpretations of *walls*; where hyperplanes are a special feature of CAT(0) cube complexes, walls arise more generally in sets. To be specific, a *wall* Nica (2004) in a set X is a partition of X into two subsets called *half-spaces*. The set X is said to be a *space with walls* if X is equipped with a collection of walls, containing the trivial wall $\{\emptyset, X\}$, where any two distinct points are separated by a finite, non-zero number of walls. Note that a wall separates two distinct points $x, y \in X$ if x belongs to one of the half-spaces determined by the wall, while y belongs to the other half-space. Furthermore, observe that half-spaces are convex sets whose complements are also convex, just as half-spaces in hyperplanes are. We will discuss walls further in the next section on median algebras when we define the notion of dimension in these spaces.

Our main result is motivated by a theorem (Brodzki et al., 2009, Theorem 1.14) which states that an interval in a d-dimensional CAT(0) cube complex can be isometrically embedded into $\mathbb{R}^d$. We will show later that this also holds for coarse median intervals using the notion of directly edge maximal points and applying Dilworth's Lemma Dilworth (1950); given a point in a CAT(0) cube complex, we can equip it with a set of coordinates via hyperplanes depending on whether the point is before or after a certain hyperplane in each chain.

2.3 Median Algebras

Median algebras are sets equipped with a ternary operation — the median operation — satisfying a certain set of axioms. They can be thought of as algebraic abstractions of CAT(0) cube complexes; every finite median algebra is the vertex set of a finite CAT(0) cube complex and vice versa. In general, however, median algebras can be larger.

Medians were first introduced by Birkhoff and Kiss in the context of lattices, and Sholander and Isbell delved further into the relationship between median algebras and lattices, covering semilattices and modular and distributive lattices. However, we are interested in median algebras due to their natural connection with CAT(0) cube complexes; Röller pioneered the link between these two objects (Roller, 2016, Theorem 10.3) and we will explore their relationship in this section. There are a number of equivalent ways of defining median algebras Bandelt and Hedlíková (1983) but we will use the following definition as seen in Kolibiar and Marcisová (1974).

Definition 2.5. Let X be a set and $\langle\rangle\colon X^3 \to X$ be a ternary operation on X. Then $\langle\rangle$ is a *median operator* and the pair $(X, \langle\rangle)$ is a *median algebra* if for all $a, b, c, d \in X$, we have:

(M1) Localisation: $\langle a, a, b\rangle = a$;

(M2) Symmetry: $\langle a, b, c\rangle = \langle b, c, a\rangle = \langle b, a, c\rangle$;

(M3) The 4-point condition: $\langle a, b, c\rangle, b, d\rangle = \langle a, b, \langle c, b, d\rangle\rangle$.

Note 2.6. Alternatively, for (M3), we can use the 5-point condition instead:

$$\langle a, b, \langle c, d, e\rangle\rangle = \langle\langle a, b, c\rangle, \langle a, b, d\rangle, e\rangle,$$

for $e \in X$. The 5-point condition was introduced by Birkhoff and Kiss and the 4-point condition followed later, defined by Kolibiar and Marcisova'. However, these two conditions are actually equivalent and the proof of this result can be found in Kolibiar and Marcisová (1974), (Bowditch, b, Theorem 3.2.2, 4.2.1). Note that it is much easier to show that the 5-point condition implies the 4-point (set e to b), but the converse is more difficult.

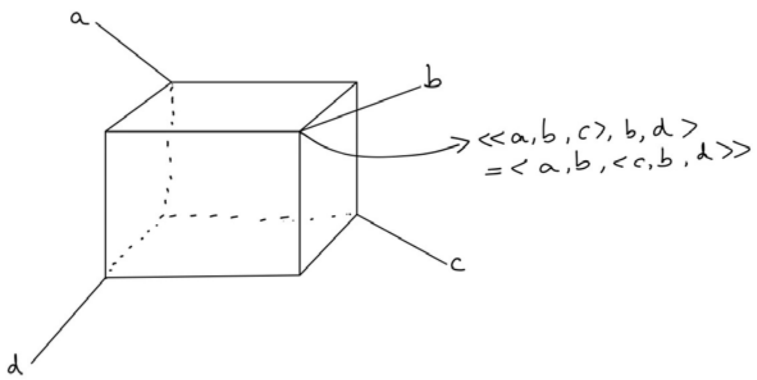


FIGURE 2.1: The CAT(0) cube complex for the free median algebra on $\{a, b, c, d\}$.

The figure above Niblo et al. (2019) illustrates the (M3) axiom, depicting the free median algebra generated by $\{a, b, c, d\}$. In particular, the *free median algebra* is defined as the following: let F be a median algebra, and let $A \subseteq F$. We say that F is *free* on A if any map, $\phi\colon A \to M$, into any median algebra, M, has a unique extension to a homomorphism, $\hat{\phi}\colon F \to M$ (that is, with $\hat{\phi}|A = \phi$). Observe that F is generated by A. Furthermore, if F is free on A and F' is free on A', then any bijection from A to A' induces a unique isomorphism from F to F'. Therefore, we can talk about 'the' free median algebra on a set X (assuming that it exists). Note that the free median algebra with finitely many generators is finite (Bowditch, b, Proposition 3.3.2).

We give some examples of median algebras Roller (2016) before moving onto intervals formulated in terms of medians.

Example 2.1. *(i) Let $\mathcal{S} = \{0, 1\}$; then $\mathcal{S}$ has a canonical median determined by taking a majority vote for each coordinate, i.e. the median is set to either 0 or 1, depending on which of these two values appears more frequently. Given $a, b, c \in \mathcal{S}$, at least two of these elements must agree, and by (M1) and (M2), this is the median, $\langle a, b, c\rangle$. To show that (M3) is satisfied, we must have either $d = e$, which implies that both sides of (M3) equate to d: $\langle d, e, \langle a, b, c\rangle\rangle = \langle d, d, \langle a, b, c\rangle\rangle = d$. On the other hand, we can have*

$d \neq e$, which implies that both sides of (M3) agrees with $\langle a,b,c\rangle$:
$\langle a, \langle b,d,e\rangle, \langle c,d,e\rangle\rangle = \langle a,b,c\rangle$.

(ii) Let $(X_i \colon i \in I)$ be a family of median algebras. Then the Cartesian product $\Pi_{i\in I} X_i$ with the median taken component-wise, i.e., $\langle (x_i),(y_i),(z_i)\rangle = (\langle x_i,y_i,z_i\rangle)$ is a median algebra.

An interval can now be restated in terms of the median operator; for any two points $x, y \in X$, the *interval between x and y* is given by

$$[x,y] := \{\langle x,z,y\rangle : z \in X\} = \{z \in X : \langle x,z,y\rangle = z\}.$$

Remark 2.7. Equality of the above two sets follows from axioms (M1)–(M3): if $c = \langle x,z,y\rangle$, then

$$\begin{aligned}\langle x,c,y\rangle &= \langle x, \langle x,z,y\rangle, y\rangle \\ &= \langle\langle x,y,x\rangle, \langle x,y,y\rangle, z\rangle \\ &= \langle x,y,z\rangle \\ &= c.\end{aligned}$$

One can think of $\langle x,z,y\rangle$ as the projection of z onto the interval $[x,y]$. Additionally, observe that $[x,y] \cap [y,z] \cap [z,x] = \{\langle x,y,z\rangle\}$; the proof of this and further properties of intervals (with their proofs) can be found in (Roller, 2016, (Int 1)-(Int 9)).

A subset $Y \subseteq X$ is *convex* if $[a,b] \subseteq Y$ for all $a, b \in Y$, i.e.
$Y = \{\langle x,y,z\rangle \colon x,y \in Y, z \in X\}$. The *convex hull* of Y is the smallest convex subset of X containing Y. A *subalgebra* is a subset of X closed under the median operation; note that any convex subset is a subalgebra. In addition, given a subset M of a median algebra X, there exists the smallest subalgebra containing M, the median closure of M.

Remark 2.8 (E.g. Špakula and Wright (2017)). The set of vertices of a CAT(0) cube complex gives rise to a median algebra in the following sense: the median of three points x, y, z is the unique vertex in the intersection of $[x,y] \cap [y,z] \cap [z,x]$. Equivalently, the median of x, y, z is the unique point lying on a geodesic between x and y, on a geodesic between y and z and a geodesic between z and x.

Note 2.9 ((Roller, 2016, Theorem 10.3)). An important thing to note — regarding the remark above and in general — is that an infinite median algebra with finite intervals can be naturally identified as the vertex set of an infinite dimensional CAT(0) cube complex and vice versa. This equivalence also holds true in the finite case, but we do not need to specify the condition of finite intervals as this already holds by definition.

There are two ways to define the rank of a median algebra. The first way is as follows: define the median n-cube to be $I^n = \{0,1\}^n$, where the median operator $\langle\rangle_n$ is given by the majority vote on each coordinate. Then the *rank* of a median algebra $(X, \langle\rangle)$ is the supremum of all n for which there is a subalgebra of $(X, \langle\rangle)$ isomorphic to the median algebra $(I^n, \langle\rangle_n)$.

The second way of defining the rank of a median algebra relies on walls, which we discussed earlier, and intuitively one can think of them as 'generalised hyperplanes'. Recall that a *wall* W is a partition $\{H^-(W), H^+(W)\}$ of X into two non-empty convex subsets. Two walls W, W' *cross* if each of the sets $\{H^-(W) \cap H^-(W')\}, \{H^-(W) \cap H^+(W')\}, \{H^+(W) \cap H^-(W')\}, \{H^+(W) \cap H^+(W')\}$ is non-empty. We say that (X, μ) has *rank at most n* if there is no collection of $n+1$ pairwise crossing walls of X.

Remark 2.10 (Niblo et al. (2019)). For the median algebra defined by the vertex set of a CAT(0) cube complex, the rank coincides with the dimension of the cube complex.

We can also view the (M3) axiom as an associativity axiom Niblo et al. (2019): given $b \in X$, the binary operator $(a, c) \mapsto a *_b c = \langle a, b, c\rangle$ is associative. It follows from the (M2) axiom that * is commutative and iterated projections leads to the iterated median, which was originally introduced in (Špakula and Wright, 2017, Definition 5.1) and which we define below.

Let $(X, \langle\rangle)$ be a median algebra with $y \in X$. The *iterated median operator* is defined as follows: for $x_1 \in X$, set

$$\langle x_1; y\rangle := x_1,$$

and for $x_1, \ldots, x_{k+1} \in X$ with $k \geq 1$, define

$$\langle x_1, \ldots, x_{k+1}; y\rangle := \langle\langle x_1, \ldots, x_k; y\rangle, x_{k+1}, y\rangle.$$

Note that this definition coincides with the original median operator, as $\langle x_1, x_2; y\rangle = \langle x_1, x_2, y\rangle$. The set $\{\langle x_1, \ldots, x_{k+1}; y\rangle : y \in X\}$ is the convex hull of the points x_i and intuitively, one should think of the iterated median $\langle x_1, \ldots, x_{k+1}; y\rangle$ as the projection of y onto the convex hull of the x_i. The iterated median operator is also symmetric in $x_1, \ldots, x_{k+1}$ and there are several other properties the iterated median operator possesses, as seen in (Špakula and Wright, 2017, Lemmas 5.2-5.3).

Note 2.11. In terms of the * operator defined above, the iterated median operator can be reformulated as follows:

$$\langle x_1, \ldots, x_k; y\rangle = x_1 *_y x_2 *_y \ldots *_y x_k.$$

Remark 2.12. We now provide two alternative definitions of median algebras:

1. A ternary operator defines a median if and only if it satisfies (M1), (M2) and *Isbell's condition* Isbell (1980): $\langle a, \langle a, b, c\rangle, \langle b, c, d\rangle\rangle = \langle a, b, c\rangle$. This says that $\langle a, b, c\rangle$ is in the interval $[a, \langle b, c, d\rangle]$.

2. Alternatively, $(X, \langle\rangle)$ is a median algebra if it satisfies (M1), (M2) and the *five-point condition* Birkhoff and Kiss (1947) (this is also axiom (M3)).

A *topological median algebra* is a topological space X equipped with the structure of a median algebra $\langle\rangle\colon X^3 \to X$, such that $\langle\rangle$ is continuous in the induced topology. When the topology on X comes from a metric d, then X is called a *metric median algebra*.

Lastly, an important definition is that of a median cube. This relies on the notion of a *median homomorphism*, which is a map between median algebras that is 'betweenness preserving' and respects medians. More precisely, given two median algebras $(X, \langle\rangle_X), (Y, \langle\rangle_Y), f\colon X \to Y$ is a *median homomorphism* if for all $x, y, z \in X$:

$$f(\langle x, y, z\rangle_X) = \langle f(x), f(y), f(z)\rangle_Y.$$

A *median cube* is a median homomorphism $\phi\colon \{0,1\}^n \to X$, such that

$$\phi(\langle x, y, z\rangle) = \langle \phi(x), \phi(y), \phi(z)\rangle.$$

Chapter 3

Coarse Median Spaces

Coarse median spaces were introduced by Bowditch in 2013 and can be thought of as ’coarsened’ versions of median algebras. The ’coarse’ in coarse median spaces refers to the fact that the structure capturing ‘medianness’ need not be exact. We will introduce Bowditch’s definition and an alternative reformulation given by Niblo-Wright-Zhang, along with diving into the notion of rank in a coarse median space, iterated coarse medians and a coarse analogue of median intervals, coarse intervals.

3.1 Definitions

We start off by providing Bowditch’s original definition of a coarse median space as seen in (Bowditch, 2013a, page 4), and then follow up with a reformulation of the definition, which is introduced in (Niblo et al., 2019, Theorem 4.12).

Definition 3.1. Let (X, d) be a metric space and $\langle\rangle\colon X^3 \to X$ be a ternary operation. Then $\langle\rangle$ is said to be a *coarse median* and $(X, d, \langle\rangle)$ is called a *coarse median space* if the following conditions hold:

(C1) There are constants $K \geq 1, H(0) \geq 0$, such that for all $a, b, c, a', b', c' \in X$ we have

$$d(\langle a, b, c\rangle, \langle a', b', c'\rangle) \leq K(d(a, a') + d(b, b') + d(c, c')) + H(0).$$

(C2) There is a function $H\colon \mathbb{N} \to [0, \infty)$ with the following property: suppose that $A \subseteq X$ with $1 \leq |A| \leq p < \infty$. Then there is a finite median algebra $(\Pi, \langle\rangle_\pi)$ and maps $\pi\colon A \to \Pi$ and $\sigma\colon \Pi \to X$ such that for all $x, y, z \in \Pi$ we have

$$d(\sigma(\langle x, y, z\rangle_\pi), \langle\sigma(x), \sigma(y), \sigma(z)\rangle) \leq H(p)$$

and

$$d(a, \sigma\pi a) \leq H(p)$$

for all $a \in A$.

Informally, (C1) says that $\langle \rangle$ is coarsely Lipschitz, while (C2) says that on finite sets, the coarse median looks like the median on a finite CAT(0) cube complex up to bounded distance.

Two coarse median operators, $\langle \rangle_1, \langle \rangle_2$, on (X, d) are said to be *uniformly close* if there is a uniform bound on the set of distances

$$\{d(\langle x, y, z\rangle_1, \langle x, y, z\rangle_2) \colon x, y, z \in X\}.$$

We refer to K, H as the *parameters* of $(X, d, \langle \rangle)$, which are not unique. Observe that there is a constant $\kappa_0 > 0$ (noted in Bowditch (2013a)) such that $\langle a, a, b\rangle \sim_{\kappa_0} a$ and $\langle a_1, a_2, a_3\rangle \sim_{\kappa_0} \langle a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\rangle$ for any permutation $\sigma \in S_3$. However, we can replace any coarse median operator by another to which it is uniformly close and which satisfies (M1) and (M2), i.e. $\kappa_0 = 0$. Therefore, without loss of generality, we may assume that $\langle \rangle$ satisfies the axioms (M1) and (M2).

We now provide an alternative characterisation of coarse median spaces as introduced in (Niblo et al., 2019, Theorem 3.1/Theorem 4.12). Here, only subsets of cardinality up to 4 need be considered, in contrast with Bowditch, where one needs to establish approximations for subsets of arbitrary cardinality.

Definition 3.2. Let (X, d) be a metric space and $\langle \rangle \colon X^3 \to X$ a ternary operation. Then $(X, d, \langle \rangle)$ is a coarse median space if and only if the following conditions hold:

(M1) $\langle a, a, b\rangle = a$ for any $a, b \in X$;

(M2) $\langle a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}\rangle = \langle a_1, a_2, a_3\rangle$, for any $a_1, a_2, a_3 \in X$ and σ a permutation;

(C1)′ There exists an affine control function $\rho \colon [0, \infty) \to [0, \infty)$ such that for all $a, a', b, c \in X$,

$$d(\langle a, b, c\rangle, \langle a', b, c\rangle) \leq \rho(d(a, a')),$$

where $\rho(t) = Kt + H(0)$.

(C2)′ There exists a constant $\kappa_4 > 0$ such that for any $a, b, c, d \in X$, we have

$$\langle\langle a, b, c\rangle, b, d\rangle \sim_{\kappa_4} \langle a, b, \langle c, b, d\rangle\rangle.$$

Similarly to the median case, there is also the notion of a ‘coarse 5-point condition’: there exists a constant $\kappa_5 > 0$ such that for any $a, b, c, d, e \in X$, we have

$$\langle a, b, \langle c, d, e \rangle \rangle \sim_{\kappa_5} \langle \langle a, b, c \rangle, \langle a, b, d \rangle, e \rangle. \tag{3.1}$$

Note 3.3. The parameters for a coarse median space as defined in Definition 3.2 is given by the 4-tuple of constants $(K, H(0), \kappa_4, \kappa_5)$ satisfying the axioms in Definition 3.2 together with estimate 3.1.

Definition 3.4.

1. Two points $x, y \in X$ are said to be *C-close* (with respect to the metric d) if $d(x, y) \leq C$. If x is C-close to y we write $x \sim_C y$.
2. Two points $x, y \in X$ are said to be *D-far* (with respect to the metric d) if $d(x, y) \geq D$. If x is D-far from y we write $x \leftrightarrow_D y$.

Similarly to that of a median cube, we also have the concept of a coarse cube. A map f between coarse median spaces $(X, d_X, \langle\rangle_X), (Y, d_Y, \langle\rangle_Y)$ is called an *L-quasi-morphism* if for any $a, b, c \in X$,

$$\langle f(a), f(b), f(c) \rangle_Y \sim_L f(\langle a, b, c \rangle_X),$$

where $L > 0$ is some constant. A *coarse cube* is then defined to be an L-quasi-morphism $\phi \colon \{0, 1\}^n \to X$.

3.2 The Rank of a Coarse Median Space

The rank can be intuitively thought of as the dimension of a coarse median space. More formally, in the context of Bowditch's original definition, a coarse median space $(X, d, \langle\rangle)$ is said to have *rank at most n* if we can choose the approximating median algebra in condition (C2), $(\Pi, \langle\rangle_\pi)$, to have rank at most n.

A general way to characterise the rank of a coarse median space is presented in (Niblo et al., 2019, Theorem 4.11). Informally, this says that a coarse median space has rank at most n if and only if it does not contain arbitrarily large $(n+1)$-dimensional coarse cubes. We can now present the complete definition of a coarse median space of rank $\leq n$ as seen in (Niblo et al., 2019, Theorem 4.12); this is the definition we will use for the remainder of the thesis.

Theorem 3.5. *We say that $(X, d, \langle\rangle)$ is a coarse median space of rank at most n if and only if conditions (M1), (M2), (C1)' and (C2)' hold, along with*

(C3)' $\forall \lambda > 0, \exists C = C(\lambda)$ *such that for any* $a, b \in X$, *any* $e_1, \ldots e_{n+1} \in [a, b]$ *with* $\langle e_i, a, e_j \rangle \sim_\lambda a$ *for all* $i \neq j$, *there exists* i *such that* $e_i \sim_C a$.

We think of condition (C3)' as the 'rank condition' — one should imagine a as a corner of an $(n+1)$-coarse cube and $e_1, \ldots, e_{n+1}$ as endpoints of edges adjacent to a.

While coarse intervals are generally introduced and defined in Section 3.4, we define below what it means for a coarse interval to have a certain rank for a given parameter. This definition is analogous to the Theorem above, applied specifically to coarse intervals, and is used extensively throughout the thesis.

Definition 3.6. Let X be a coarse median space and $[x,y] \subseteq X$ a coarse interval. Then $[x,y]$ is said to have rank at most n for a given parameter λ if there exists a constant $C = C(\lambda)$ depending on λ, such that for any $a,b \in [x,y], e_1, \ldots, e_{n+1} \in [a,b]$ with $\langle e_i, a, e_j \rangle \sim_\lambda a$ for all $i \neq j$, there exists i such that $e_i \sim_C a$.

Note 3.7. Note that Theorem 3.5 also holds for a median algebra, where all the 'close to' conditions are replaced by equality. More precisely, a median algebra $(X, \langle\rangle)$ has rank at most n if for any $a,b \in X$, any $e_1, \ldots, e_{n+1} \in [a,b]$ with $\langle e_i, a, e_j \rangle$ for all $i \neq j$, there exists i such that $e_i = a$.

Remark 3.8 (Niblo et al. (2019)). Note that for a median algebra endowed with an appropriate metric making it a coarse median space, the rank as a median algebra gives an upper bound for the rank as a coarse median space, however, these need not necessarily agree. For instance, a finite median algebra has rank 0 as a coarse median space.

3.3 Iterated Coarse Medians

Analogous to iterated medians in median algebras, we define the *iterated coarse median operator* (Niblo et al., 2019, Definition 2.15) for coarse median spaces.

Definition 3.9. Let $(X, d, \langle\rangle)$ be a coarse median space and $y \in X$. For $x_1 \in X$, define

$$\langle x_1; y \rangle : = x_1,$$

and for $k \geq 1$ and $x_1, \ldots, x_{k+1} \in X$, define

$$\langle x_1, \ldots, x_{k+1}; y \rangle : = \langle \langle x_1, \ldots, x_k; y \rangle, x_{k+1}, y \rangle.$$

Similarly to the median case, note that we still have $\langle x,y,z \rangle = \langle x,y;z \rangle$ for $x,y,z \in X$.

For further results/properties of the coarse median operator, one should refer to (Niblo et al., 2021, Lemma 2.14, Lemma 2.15, Lemma 2.16). The results we refer to the most in the thesis for coarse iterated medians are stated below:

Lemma 3.10. *(1) Let $(X, d, \langle\rangle)$ be a coarse median space with parameters $(K, H(0), \kappa_4, \kappa_5)$. Then there exists a constant F_n depending only on the parameters of X, such that for any $a, b, a_1, \ldots, a_n \in X$ we have:*

$$\langle a, b, \langle a_1, \ldots, a_{n-1}; a_n \rangle \rangle \sim_{F_n} \langle \langle a, b, a_1 \rangle, \ldots \langle a, b, a_{n-1} \rangle; a_n \rangle.$$

(2) *Let* $(X, d, \langle\rangle)$ *be a coarse median space with parameters* $(K, H(0), \kappa_4, \kappa_5)$. *Then for any* $n \in \mathbb{N}$, *there exists a constant* G_n *depending only on the parameters of* X, *such that for any* $a_1, \ldots, a_n, b \in X$ *and any permutation* $\sigma \in S_n$, *we have*

$$\langle a_{\sigma(1)}, \ldots, a_{\sigma(n)}; b\rangle \sim_{G_n} \langle a_1, \ldots, a_n; b\rangle.$$

(3) *Let* (X, d) *be a metric space with ternary operator* $\langle\rangle$ *satisfying (C1)' with parameter* ρ. *Then for any* n *there exists an increasing (affine) function function* ρ_n *depending on* ρ, *such that for any* $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \in X$:

$$d(\langle a_1, \ldots, a_n; a_0\rangle, \langle b_1, \ldots, b_n; b_0\rangle \leq \rho_n(\Sigma_{k=0}^{n} d(a_k, b_k)).$$

3.4 Coarse Intervals

In coarse median spaces, there exist coarse analogues of intervals in median algebras; one way to define them is as follows.

Definition 3.11 ((Niblo et al., 2019, Definition 2.20))**.** Given $(X, \langle\rangle)$, a set equipped with a ternary operator $\langle\rangle$, the *interval* between points x and y is defined to be:

$$[x, y] = \{\langle x, y, z\rangle : z \in X\}.$$

Compare this with Bowditch's definition, which states that a λ*-coarse interval* between points x and y in a coarse median space $(X, d, \langle\rangle)$ is defined to be

$$[x, y]_\lambda := \{z \in X : \langle x, y, z\rangle \sim_\lambda z\},$$

where λ is a constant.

Observe that $[x, y]_0 \subseteq [x, y]$, and for median algebras these two definitions of intervals actually coincide; see Remark 2.7. However, these two notions of interval do not always agree in a coarse median space. Throughout the thesis, we define coarse intervals as seen in Definition 3.11. Results concerning coarse intervals can be found in (Niblo et al., 2019, Lemma 2.21, Lemma 2.22).

Note 3.12. Given $a \in [x, y]$, $a \sim_{\kappa_4} \langle a, x, y\rangle$: since $a \in [x, y]$, we can write a as $a = \langle x, y, a'\rangle$ for some $a' \in [x, y]$. This in turn implies that

$$\begin{aligned} \langle x, y, a\rangle &= \langle x, y, \langle x, y, a'\rangle\rangle \\ &\sim_{\kappa_4} \langle\langle x, y, x\rangle, y, a'\rangle \\ &= \langle x, y, a'\rangle \\ &= a. \end{aligned}$$

Chapter 4

Prerequisites

4.1 Important Definitions

In this thesis, we assume that the coarse median spaces we study satisfy three key properties: they are quasi-geodesic, uniformly discrete, and have bounded geometry. These assumptions will be explicitly used in our proofs. Most of our results have been proven for both the median and coarse median cases — we will present our results for the median case in this thesis, but will avoid proving them here, as there are many similarities with the proofs of our coarse versions. We will instead place the proofs of our median statements in the Appendix.

Additionally, we will equip our coarse intervals with an adaptation of the partial ordering described in (Bowditch, 2014, Page 7) – although this is formulated for median algebras, we have adapted this partial ordering to coarse intervals, which is described in Definition 4.3. Before we introduce the ordering, we define what a distributive lattice is, as this is the context for which Bowditch's ordering is detailed in.

Definition 4.1. A *lattice* $(L, \leq)$ is a partially ordered set in which every pair of elements has a unique supremum (least upper bound or join) and infimum (greatest upper bound or meet). Thus, given $a, b \in L$, there exist unique elements $a \vee b$ (the join) and $a \wedge b$ (the meet), such that:

- $a \leq a \vee b$ and $b \leq a \vee b$,
- $a \wedge b \leq a$ and $a \wedge b \leq b$,
- if $x \leq a$ and $x \leq b$, then $x \leq a \wedge b$, and if $a \leq x$ and $b \leq x$, then $a \vee b \leq x$.

A lattice is *distributive* if the distributive law holds for all elements in the lattice. The distributive law states that, given $a, b, c \in L$

- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, and
- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

We now formally define the ordering in both the median and coarse median cases below.

Definition 4.2. Let X be a median algebra and let $x, y \in X$. We obtain a partial order as follows: given $a, b \in X$, then $a \leq b$ iff $\langle x, a, b \rangle = a$ — this gives us the concept of a minimum (with x as the basepoint). If we are in an interval, i.e. $a, b \in [x, y]$, then $a \leq b$ iff $\langle x, a, b \rangle = a$ or equivalently $\langle a, b, y \rangle = b$. The points $\langle x, a, b \rangle$ and $\langle a, b, y \rangle$ define the minimum and maximum of a, b respectively with respect to this order.

We extend the ordering above to the coarse median case:

Definition 4.3. Let X be a coarse median space and let $x, y \in X$. Fix a constant C and consider any two points $a, b \in X$. Then we can define a coarsening of a partial order as follows: let $x \in X$ (x is the basepoint); then $a \lesssim_C b$ iff $\langle x, a, b \rangle \sim_C a$, so we obtain the concept of a minimum. If we are in a coarse interval, i.e. $a, b \in [x, y]$, then $a \lesssim_C b$ iff $\langle x, a, b \rangle \sim_C a$ or alternatively, $\langle a, b, y \rangle \sim_B b$, where $B = K(C + \kappa_4) + 2H(0) + \kappa_4$ (by application of Lemma 4.7). The points $\langle x, a, b \rangle$ and $\langle a, b, y \rangle$ define the minimum and maximum of a, b respectively with respect to this order.

The ordering in Definition 4.2 is a partial order on median algebras, but we need to verify that it is a partial order on coarse median spaces too. We will show in Chapter 5 that the ordering in Definition 4.3 is actually a total order on M-separated subsets of rank 1 coarse intervals — see Lemma 5.2. We will show in a later chapter, Chapter 6, that this ordering is also a partial order on special subsets of coarse intervals.

We now introduce a very important definition from Bowditch (2014) that appears frequently in the thesis.

Definition 4.4. Let X be a coarse median space and $A \subseteq X$ be finite. Define

$$\text{sep}(A) = \min\{d(x, y) \colon x, y \in A, x \neq y\},$$

where sep(A) is the *separation constant*. Then we call A a *sep(A)-separated subset*.

4.2 Important Lemmas

These Lemmas are used regularly within the thesis, so we compile them all in one place here and reference them throughout. Let $[x, y]$ be a coarse interval in a coarse median space X and let $x \in X$ be the basepoint for the relation $\lesssim$ defined in Definition 4.3.

Lemma 4.5. *For constants A and B, if we define $C = f(A, B)$, then set $C = K(A + B) + 2H(0) + \kappa_4 + A$, such that if $a \lesssim_A b \lesssim_B c$, then $a \lesssim_C c$, where $a, b, c \in X$.*

Proof. Since $a \lesssim_A b$, this is equivalent to $\langle x, a, b\rangle \sim_A a$. This then implies that

$$\begin{aligned}
\langle x, a, c\rangle &\sim_{KA+H(0)} \langle c, x, \langle b, x, a\rangle\rangle \qquad \text{(using the (C1}'\text{) axiom)} \\
&\sim_{\kappa_4} \langle\langle c, x, b\rangle, x, a\rangle \qquad \text{(using the (C2}'\text{) axiom)} \\
&\sim_{KB+H(0)} \langle b, x, a\rangle \qquad \text{(using the (C1}'\text{) axiom)} \\
&\sim_A a \qquad \text{(by assumption).}
\end{aligned}$$

□

Lemma 4.6. *Given $a \in [x, y]$ and $b \in X$, $a \lesssim_{(K+1)\kappa_4+H(0)} \langle a, b, y\rangle$.*

Proof.

$$\begin{aligned}
\langle x, a, \langle y, a, b\rangle\rangle &\sim_{\kappa_4} \langle\langle x, a, y\rangle, a, b\rangle \qquad \text{(using the (C2}'\text{) axiom)} \\
&\sim_{K\kappa_4+H(0)} \langle a, a, b\rangle \qquad \text{(using the (C1}'\text{) axiom)} \\
&= a.
\end{aligned}$$

The last equality is due to the following: since $a \in [x, y]$, we can write a as $a = \langle x, y, a'\rangle$ for some $a' \in [x, y]$ — see Note 3.12. This in turn implies that

$$\begin{aligned}
\langle x, y, a\rangle &= \langle x, y, \langle x, y, a'\rangle\rangle \qquad \text{(by Note 3.12)} \\
&\sim_{\kappa_4} \langle\langle x, y, x\rangle, y, a'\rangle \qquad \text{(using the (C2}'\text{) axiom)} \\
&= \langle x, y, a'\rangle \\
&= a.
\end{aligned}$$

□

Lemma 4.7. *For some constant A, if we define $B = f(A)$, then set $B = K(A + \kappa_4) + 2H(0) + \kappa_4$, such that for any $a, b \in [x, y]$ with $\langle a, b, y\rangle \sim_A b$, we have $a \lesssim_B b$.*

Proof.

$$\begin{aligned}
\langle x, a, b\rangle &\sim_{KA+H(0)} \langle x, a, \langle a, b, y\rangle\rangle \qquad \text{(using the (C1}'\text{) axiom)} \\
&\sim_{(K+1)\kappa_4+H(0)} a,
\end{aligned}$$

where the last line is a consequence of Lemma 4.6. □

Chapter 5

Quadratic Growth of Rank 2 Coarse Intervals

In this chapter, we introduce the concept of a coarse hyperplane associated to two points in a coarse median space and use this definition to prove three results: coarse hyperplanes have co-dimension 1 with the interval, coarse hyperplanes coarsely cover the whole coarse interval, and the intersection of a coarse hyperplane and coarse interval is 'almost' a coarse interval. We then use these findings and an inductive argument to show that rank 2 coarse intervals have quadratic growth.

5.1 Coarse Ordering

The ordering in Definition 4.2 is a partial order on median algebras, but we need to verify that it is a partial order on coarse median spaces too. For the purposes of this chapter, we have shown that the ordering is actually a total order on M-separated subsets of rank 1 coarse intervals — see Lemma 5.2 below. We will show in a later chapter, Chapter 6, that this ordering is also a partial order on special subsets of coarse intervals.

Note 5.1. As we are purely working with rank 2 coarse intervals throughout this chapter, we explicitly define what it means for a coarse interval to achieve rank ≤ 2 under suitable parameters, using Theorem 3.5: given any $\lambda > 0$, there exists a constant $C = C(\lambda)$ such that for any $a, b \in [x, y]$, any $e_1, e_2, e_3 \in [a, b]$ with $\langle e_i, a, e_j \rangle \sim_\lambda a$ for all $i \neq j$, there exists i such that $e_i \sim_C a$.

Lemma 5.2. *Given a coarse median space X, rank parameter $C'(\lambda)$ and ordering parameter C, there exists M such that for any rank 1 coarse interval $[x, y] \subseteq X$ with respect to $C'(\lambda)$, any M-separated subset of $[x, y]$ is totally ordered.*

Proof. Let $a, b \in [x, y]$. Then $a \lesssim_C b$ iff $\langle x, a, b\rangle \sim_C a$. Comparability is satisfied due to the following: keeping Theorem 3.5 in mind, since $[x, y]$ is rank 1, we know that any bipod in this interval has a trivial side length. More formally, suppose that $a, b, \langle x, a, b\rangle$ form a bipod centred at $\langle x, a, b\rangle$; the point $\langle x, a, b\rangle \in [x, y]$ and the betweenness condition is satisfied:

$$\langle\langle x, a, b\rangle, a, b\rangle \sim_{\kappa_4} \langle x, a, b\rangle.$$

By Theorem 3.5, we know that either $\langle x, a, b\rangle \sim_{C'(\kappa_4)} a$ or $\langle x, a, b\rangle \sim_{C'(\kappa_4)} b$. Therefore, setting $C \geq C'(\kappa_4)$, we see that comparability holds: for any $a, b \in [x, y]$, either $a \lesssim_C b$ or $b \lesssim_C a$, that is, $a \sim_C \langle x, a, b\rangle$ or $b \sim_C \langle x, a, b\rangle$. In terms of anti-symmetry, if both $a \lesssim_C b$ and $b \lesssim_C a$, then $a \sim_C \langle x, a, b\rangle \sim_C b$ and we obtain $a \sim_{2C} b$.
For transitivity, let us assume that $a \lesssim_C b$, $b \lesssim_C c$ and $c \lesssim_C a$; then $a \sim_C \langle x, a, b\rangle$, $b \sim_C \langle x, b, c\rangle$ and $c \sim_C \langle x, a, c\rangle$. Using the coarse four-point condition — see the (C2)′ axiom in Definition 3.2 — we obtain

$$\begin{aligned} a &\sim_C \langle x, a, b\rangle \\ &\sim_{KC+H(0)} \langle x, a, \langle x, b, c\rangle\rangle \\ &\sim_{\kappa_4} \langle\langle x, a, b\rangle, x, c\rangle \\ &\sim_{KC+H(0)} \langle a, x, c\rangle \\ &\sim_C c. \end{aligned}$$

Now set $M = 2(KC + H(0) + C) + \kappa_4 + 1$ and take an M-separated subset in $[x, y]$ — then for all a, b, c in this M-separated subset, if $a \lesssim_C b$ and $b \lesssim_C c$, either $a \lesssim_C c$ or $c \lesssim_C a$. However, if $a \lesssim_C b$, $b \lesssim_C a$ and $c \lesssim_C a$, then by the calculation above we obtain $c \sim_M a \sim_M b \sim_M c$, which in turn implies that $a = b = c$, as the distances between a, b and c are less than M, and since we are in an M-separated subset the three points must be equal. We have ruled out $c \lesssim_C a$ and so we must have $a \lesssim_C c$. This choice of M means that both anti-symmetry and transitivity hold, therefore giving us a total ordering on M-separated subsets of $[x, y]$. □

5.2 Coarse Hyperplanes

We know how hyperplanes look and are defined in CAT(0) cube complexes, but this has not been formally extended to the coarse median world as of yet. Since hyperplanes are a very important structure of CAT(0) cube complexes and are a tool used in proving quadratic growth of intervals in these spaces, we formalise the concept of a coarse hyperplane below. The following picture gives an idea as to how a coarse hyperplane intuitively looks in a coarse interval (in the context of CAT(0) cube complexes):

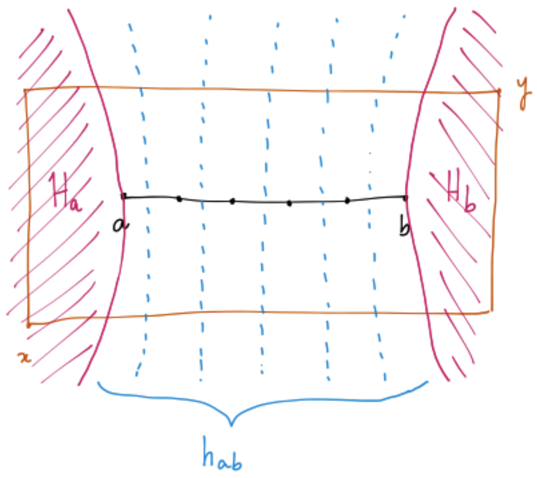


FIGURE 5.1: This is the two-dimensional case — $[x, y]$ is a two-dimensional interval and $[a, b]$ is a one-dimensional interval within it. The coarse hyperplane is represented by h_{ab} and H_a and H_b are the half-spaces corresponding to a and b respectively.

The definition is given more formally below.

Definition 5.3. Let L be a constant. An *L-coarse hyperplane corresponding to a, b* in a coarse median space X, with $d(a, b)$ much greater than $2L$, divides the space into two half-spaces H_a and H_b, where $H_a = \{z \in X : a \sim_L \langle a, z, b \rangle\}$ and $H_b = \{z \in X : b \sim_L \langle a, z, b \rangle\}$, respectively. The coarse hyperplane itself is then defined to be

$$h_{ab} = X \setminus (H_a \cup H_b),$$

so h_{ab} partitions the complement into two disjoint pieces.

Note that we require $d(a, b)$ much greater than $2L$ so that the coarse hyperplane is non-trivial and the half-spaces do not intersect.

Now that coarse hyperplanes have been formally defined, we want to use them to aid us in answering the following three important questions that will enable us to prove quadratic growth of coarse intervals:

1. In a CAT(0) cube complex, we know that hyperplanes have co-dimension 1 in the interval. Do these coarse hyperplanes intersected with a rank $\leq n$ coarse interval also have rank $\leq n - 1$ in the coarse interval? (Using the characterisation of rank as seen in Theorem 3.5.)

2. Do coarse hyperplanes coarsely cover the whole coarse interval $[x, y]$?

3. Is the intersection of a coarse hyperplane and coarse interval 'almost' a coarse interval, i.e. is $h_{ab} \cap [x, y]$ 'almost' a coarse interval?

We have answered the first question for rank n, although we primarily apply it in reference to rank 2 intervals. The second question has been solved for rank 2, but our argument naturally extends to higher rank. Question 3 has only been proven for rank 2, but this will suffice here as we use a different argument to prove polynomial growth of rank n coarse intervals. We then combine the solutions to these three questions and an inductive argument to prove that rank 2 coarse intervals have quadratic growth — this is shown in Section 5.8. In Chapter 6, we detail our alternative approach in proving that rank n coarse intervals have polynomial growth; this heavily relies on the notion of edge maximal points and the application of Dilworth's Lemma.

5.3 Notation and Strategy

We set-up some notation and discuss our overall strategy in proving quadratic growth of rank 2 coarse intervals.

Let x, y be points in our quasi-geodesic coarse median space X. We know that there is a quasi-geodesic ϕ connecting x and y, but this may not lie in the interval $[x, y]$. To rectify this problem, project $\phi(i)$ into the interval $[x, y]$, i.e. replace $\phi(i)$ by $\langle x, \phi(i), y \rangle$.

The endpoints are also preserved (i.e. x and y are still the initial and end points of this new path) and there is a uniform bound between consecutive points of the path:

$$\begin{aligned} d(\langle x, \phi(i), y \rangle, \langle x, \phi(i+1), y \rangle) &\le Kd(\phi(i), \phi(i+1)) + H(0) \\ &\le K(A_1 + A_2) + H(0), \end{aligned}$$

where the first inequality follows from applying the (C1)′ axiom (see Definition 3.2). The second inequality follows by using the right-hand inequality in the definition of a quasi-geodesic: $d(\phi(i), \phi(i+1)) \le A_1|i - (i+1)| + A_2 = A_1 + A_2$, where A_1 and A_2 are the quasi-geodesic constants.

Note 5.4. The new path $\langle x, \phi(i), y \rangle$ may not necessarily be quasi-geodesic. The upper bound follows from the (C1)′ axiom in Definition 3.2, but the problem lies with the lower bound: the quasi-geodesic ϕ could be made up of loops or long pieces that would project down onto a point in $[x, y]$ — we have no control of the parameters. Hence, the situation is more complex and one would need to demonstrate extra care in verifying whether $\langle x, \phi(i), y \rangle$ is a quasi-geodesic or not. However, this does not present a complication for us, as we have shown above that the new path $\langle x, \phi(i), y \rangle$ gives us a uniform bound between consecutive points and this is all we need.

We summarise the method described earlier into the following lemma:

Lemma 5.5. *Given a coarse median space X with $x, y \in X$ and a quasi-geodesic path ϕ connecting x and y, there exists a sequence of points a_i, where $i \in [0, N]$, defined as $a_i = \langle x, \phi(i), y \rangle$. For all $[x, y] \subseteq X$, the sequence of points $a_0, \ldots, a_N$ is a path contained in $[x, y]$ with $a_0 = x$ and $a_N = y$, where $a_i \sim_{K(A_1+A_2)+H(0)} a_{i+1}$ for all $i \in [0, N]$.*

We also want to make sure that we do not forget the big picture of this chapter, so that we can keep in mind what we are aiming to prove. The following image gives us an overall picture to keep in mind (from a CAT(0) cube complex viewpoint):

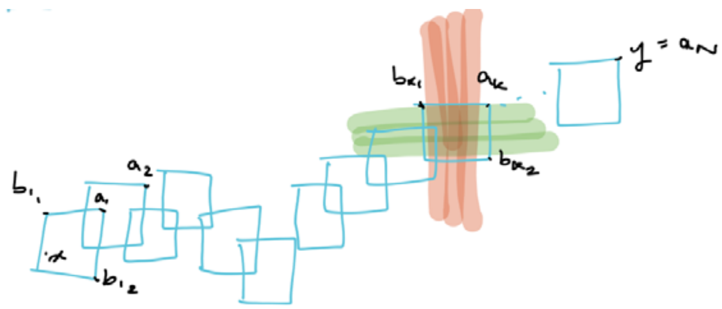


FIGURE 5.2: The rank 2 overall picture.

The idea in the above figure is to take maximal rank 'backwards' coarse cubes, where each top-right corner has a_k as a vertex (apart from $a_0 = x$). We assume the points $b_{k,1}, a_k, b_{k,2}$ form an r-pod of maximal r centred at a_k — note that $r = 1$ or 2 here — where a_k is a point of the path described in Lemma 5.5; observe that this path does not have to be monotone. More precisely, each r-pod is centred at a_k, with $b_{k,1}, b_{k,2} \in [x, a_k]$. The shaded-in orange and green represents the coarse hyperplanes associated to $[a_k, b_{k,i}]$; note that each $[a_k, b_{k,i}]$ is rank 1 (with $i \in \{1, 2\}$). We would then proceed to shade-in the coarse hyperplanes in the remaining intervals, hence indicating that these coarse hyperplanes do indeed coarsely cover $[x, y]$ and finally that the intersection of a coarse hyperplane and coarse interval is 'almost' a coarse interval. Combining these two results along with coarse hyperplanes having co-dimension 1 and an inductive argument would enable us to show that rank 2 coarse intervals have quadratic growth, which is one of the key results of the thesis.

However, there is a problem that can arise in our set-up which we wish to avoid:

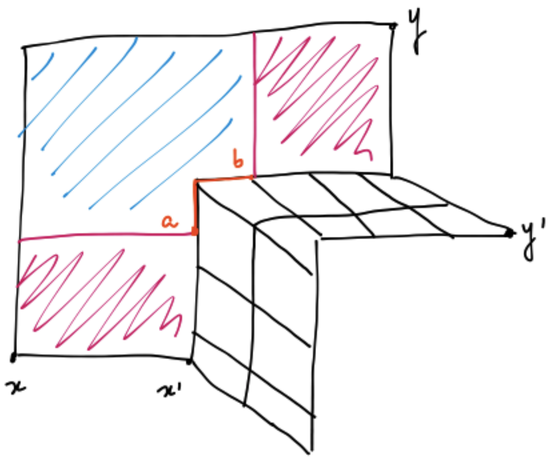


FIGURE 5.3: Although $[a, b]$ is one-dimensional, it contains a corner which presents an issue (cube complex viewpoint).

The problem with the $[a, b]$ interval containing a corner is that we do not have control over how 'fat' the hyperplanes are/can be. As seen above, the shaded blue part represents the coarse hyperplane corresponding to a and b and we can see that it is 'fat'. However, if we look at the corresponding hyperplane on the unshaded part of the image then it is 'thin', and notice that $[a, b]$ contains no corners here either. The presence of corners means that branching can occur within the interval; this is already a problem for the coarse interval h_{ab}, which is why we need to consider $h_{ab} \cap [x, y]$. Therefore, to allow us some degree of control as to how 'fat' the coarse hyperplanes can be, that is, to avoid branching from occurring in the interval, we will rule out this case when proving that coarse hyperplanes have co-dimension 1.

More formally, the median definition of a corner in rank 2 is given below.

Definition 5.6. Let a, b, x, y be elements of a median algebra X. Let $[a, b] \subseteq [x, y]$, where $[a, b]$ is one-dimensional. Then $c = \langle a, d, b \rangle$ (where $d \in [x, y]$) is a *corner* if the following all hold

- $a \neq \langle a, d, x \rangle$;
- $b \neq \langle b, d, y \rangle$;
- $c \neq a$;
- $c \neq b$.

Below is a geometric interpretation of our definition of a corner:

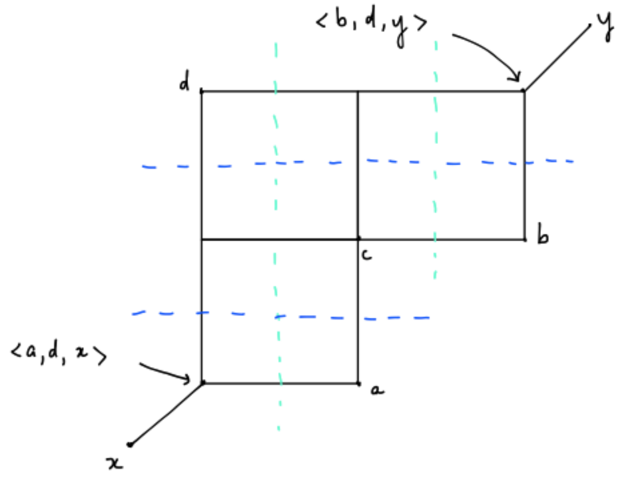


FIGURE 5.4: Two-dimensional configuration of a corner in a CAT(0) cube complex.

We now provide the coarse median definition of a corner in rank 2.

Definition 5.7. Let a, b, x, y be elements of a coarse median space X. Let $[a, b] \subseteq [x, y]$, where $[a, b]$ has rank 1 and $[x, y]$ has rank 2. Then $c = \langle a, d, b \rangle$ (where $d \in [x, y]$) is an *R-corner* for R sufficiently large if the following all hold:

- $a \leftrightarrow_R \langle a, d, x \rangle$;
- $b \leftrightarrow_R \langle b, d, y \rangle$;
- $c \leftrightarrow_R a$;
- $c \leftrightarrow_R b$.

In Chapter 5.5, we will show how to overcome the corner problem for rank 2.

5.4 Coarse Hyperplanes have Co-dimension 1

The objective of this section is to answer one of the main questions of this chapter, which is presented below. It states that coarse hyperplanes have co-dimension at least 1 with the coarse median space.

Theorem 5.8. *Given a coarse median space X, rank parameter $C(\lambda)$, ordering parameter G and quasi-morphism parameter E, there exist constants R, S such that the following holds. For all $[x, y] \subseteq X$, where rank $[x, y] \leq n$ with respect to $C(\lambda)$, if $[a, b] \subseteq [x, y]$, where $a \lesssim_G b$, has no R-corners, then an E-coarse cube in $[a, b]$ of rank equal to the rank of $[x, y]$ has a side of length at most S.*

The proof is divided into two cases and we describe these in detail in the following subsections.

5.4.1 Method

We outline our approach for showing that coarse hyperplanes have co-dimension 1. As mentioned in Section 5.2, although we prove this result for higher rank, we focus on applying it to rank 2 intervals.

We claim that the coarse hyperplanes intersected with the interval, $h_{ab} \cap [x, y]$, have rank $\leq n - 1$, where rank $[x, y] \leq n$. We now dive into the details of the proof; we have two cases as shown below along with a more general case, which is a combination of the two cases.

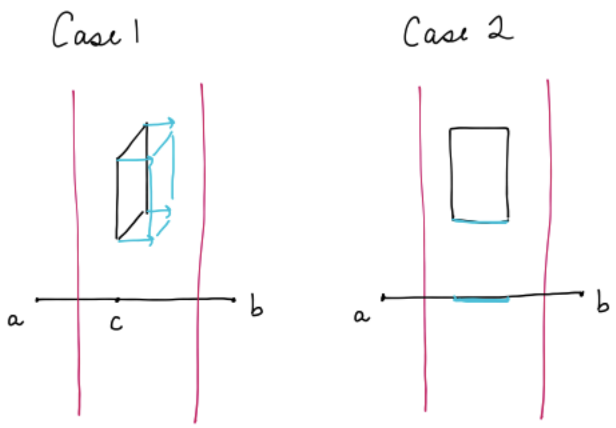


FIGURE 5.5: The two cases we need to prove to show that coarse hyperplanes have co-dimension 1 (in two-dimensions).

To be more precise, for case 1 we assume that none of the edges of the coarse cube line up in the direction of (a subinterval of) $[a, b]$, that is, all vertices project onto a neighbourhood of a single point c in $[a, b]$. The idea is to then project the vertices of the coarse cube in the direction of (a subinterval of) $[a, b]$, hence lifting it one rank higher, implying that the rank of the original coarse cube must have been one lower.

For Case 2, we assume (at least) one of the edges of the coarse cube lines up in the direction of a subinterval of $[a, b]$. Since $d(a, b)$ is bounded, this direction of the cube is trivial, so we are done for this case. The general case is then a combination of cases 1 and 2, which we focus on in the proof of Theorem 5.8.

For all cases, we are using the characterisation of rank as outlined in Theorem 3.5.

Note 5.9. Although we do not explicitly state the rank of $[a, b]$, we assume that there are no R-corners in Theorem 5.8 and thus this implicitly implies that the rank of $[a, b]$

must be 1. The lack of R-corners plays the role of the rank 1 condition here. Later in Section 5.5, when we attempt to bound the number of corners in an interval, we will explicitly assume that $[a, b]$ has rank 1.

5.4.2 Proof of Case 1

Lemma 5.10. *Given any coarse median space X and coarse interval $[x, y] \subseteq X$ with $a, b \in [x, y]$, for every R, ordering parameter G, quasi-morphism parameter E and F, there exists E'' such that for every non-triviality parameter D'', there exist D and L such that the following hold. Assume that $[a, b]$ does not have R-corners and that $a \lesssim_G b$, where G is the ordering parameter with respect to x as the basepoint. Let $C = \{u_1, \ldots, u_{2^d}\}$ be a rank d coarse cube in the hyperplane, that is, we have $\langle u_i, u_j, u_k \rangle \sim_E u_l$ for $i, j, k, l \in \{1, \ldots, 2^d\}$. Suppose that there exists a point $c \in [a, b]$ such that for all $i \in \{1, \ldots, 2^d\}$, the points u_i project onto an F-neighbourhood of c. Then, we can construct a coarse cube $\tilde{C} \subseteq [x, y]$ with rank $d + 1$ and quasi-morphism parameter E''. For all D'', assume that the u_i are all at least D apart, that is, for all $i \in \{1, \ldots, 2^d\}$, if $i \neq j$, then $d(u_i, u_j) \geq D$, where D is the non-triviality parameter for C. In addition, assume that $d(c, b)$ is non-trivial, in particular, $c \nleftrightarrow_L b$, where L is the parameter associated with the coarse hyperplane corresponding to a and b intersected with the interval, $h_{ab} \cap [x, y]$.*

Observe that when we project onto $c \in [a, b]$, we do not want c to be close to the endpoints of the interval; this is the same as wanting c to lie in the coarse hyperplane associated to a and b. Before we embark on the proof, we introduce the following lemma which we will make use of from here on.

Lemma 5.11. *Given a constant $P (= 2^d$ in this case) and a C-ordering, there exists a constant M, the separation parameter, such that P fixed points in an M-separated set with the ordering $\lesssim_C$ is a coarsening of a partial order.*

Proof of Lemma 5.10. Since c is not a corner, this means that c fails to satisfy at least one of the conditions in Definition 5.7. We cannot have $c \sim_R a$ or b as then the u_i will not be in the coarse hyperplane, they will be in H_a and H_b respectively, which are not a part of the coarse hyperplane. This implies that either $\langle a, u_i, x \rangle \sim_R a$ or $\langle b, u_i, y \rangle \sim_R b$, but these are dual cases so we will focus on $\langle a, u_i, x \rangle \sim_R a$ here since the $\langle b, u_i, y \rangle \sim_R b$ case can be proven very similarly.

The way we prove the lemma is the following: consider the G'-coarse ordering with c as the basepoint; then we can find an i_0 (that is not necessarily unique) in $\{1, \ldots, 2^d\}$ such that u_{i_0} is a maximal element with respect to the G'-ordering, where G' is the ordering parameter associated to the basepoint c. We also know such a u_{i_0} exists due to Lemma 5.11, so we have $\langle u_i, u_{i_0}, c \rangle \nleftrightarrow_{G'} u_{i_0}$ for all $i \neq i_0$. We then project the remaining vertices of C onto the interval $[c, u_{i_0}]$, that is, we set $v_i = \langle c, u_i, u_{i_0} \rangle$. We then obtain a cube that lies in $[c, u_{i_0}]$, which we denote $\hat{C} = \{v_1, \ldots, v_{2^d}\}$, with non-triviality parameter D' and quasi-morphism parameter E'. This cube may be smaller than C but will still be a coarse cube of the same rank so our original claim still holds. To be more specific, we show that the rank of $\hat{C} = \{v_1, \ldots, v_{2^d}\}$ remains the same by proving that $v_i \not\leftrightarrow_{D'} v_j$; this says that we have not collapsed any edges of C when projecting its vertices onto $[c, u_{i_0}]$, hence leaving the rank unchanged.

Note 5.12. 'Coarse medianness' is preserved (using the coarse five-point condition), and so $\hat{C} = \{v_1, \ldots, v_{2^d}\}$ will still be a coarse cube; we prove this below. (The calculation below also determines E'.)

$$\begin{aligned}
\langle v_i, v_j, v_k \rangle &\sim_{K\kappa_4 + H(0)} \langle \langle c, u_{i_0}, u_i \rangle, \langle c, u_{i_0}, u_j \rangle, \langle c, v_k, u_{i_0} \rangle \rangle \\
&\sim_{\kappa_5} \langle c, u_{i_0}, \langle u_i, u_j, \langle c, v_k, u_{i_0} \rangle \rangle \rangle \\
&\sim_{\kappa_5} \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, \langle c, u_{i_0}, v_k \rangle \rangle \rangle \\
&\sim_{K\kappa_4 + H(0)} \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, v_k \rangle \rangle \\
&= \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, \langle c, u_{i_0}, u_k \rangle \rangle \rangle \\
&\sim_{K\kappa_4 + H(0)} \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, u_k \rangle \rangle \\
&\sim_{\kappa_5} \langle c, u_{i_0}, \langle u_i, u_j, u_k \rangle \rangle \\
&\sim_{KE + H(0)} \langle c, u_{i_0}, u_l \rangle \\
&= v_l.
\end{aligned}$$

Now consider an arbitrary edge $\{v_i, v_j\} \in \hat{C}$. Then we can find a parallel edge containing u_{i_0} as a vertex since we are in a cube — if edges in the original cube are parallel then these correspond to parallel edges in the coarse cube. Call this edge $\{v_k, u_{i_0}\}$, such that (without loss of generality)

$$\begin{aligned}
\langle u_{i_0}, v_k, v_i \rangle &\sim_{E'} u_{i_0}, \\
\langle u_{i_0}, v_k, v_j \rangle &\sim_{E'} v_k.
\end{aligned}$$

This is the same as saying that $\{v_i, u_{i_0}, v_k, v_j\}$ forms a coarse 2-cube. Assume towards a contradiction that $v_i \sim_{D'} v_j$; then

$$\begin{aligned}
u_{i_0} &\sim_{E'} \langle u_{i_0}, v_k, v_i \rangle \\
&\sim_{KD' + H(0)} \langle u_{i_0}, v_k, v_j \rangle \\
&\sim_{E'} v_k.
\end{aligned}$$

Since we have $u_{i_0} = v_{i_0} \sim_{2E' + KD' + H(0)} v_k$, this implies that $v_k = \langle c, u_k, u_{i_0} \rangle \sim_{2E' + KD' + H(0)} u_{i_0}$, which in turn signifies that $u_{i_0} \lesssim_{G'} v_k$ (applying the ordering with c as the basepoint), where we set $G = 2E' + KD + H(0)$. However, this is a contradiction with respect to the G'-ordering, as we chose u_{i_0} to be maximal and so any edge containing u_{i_0} as a vertex cannot collapse (also because we stated earlier that $\langle u_i, u_{i_0}, c\rangle \leftrightarrow_{G'} u_{i_0}$ which prevents collapses). Therefore, we have shown that $v_i \leftrightarrow_{D'} v_j$ and so the coarse cube $\hat{C} = \{v_1, \ldots, v_{2^d}\}$ has the same rank as C.

When we pick u_{i_0}, it either falls into the first or second case, i.e. $\langle a, u_{i_0}, x\rangle \sim_R a$ or $\langle b, u_{i_0}, y\rangle \sim_R b$. Without loss of generality, let us assume that u_{i_0} satisfies the first case as the second case is dual to this.

The reason for projecting all the vertices of C onto $[c, u_{i_0}]$ is to ensure that all points are of the 'same type', i.e. we either have $\langle a, u_i, x\rangle \sim_R a$ for all i or $\langle b, u_i, y\rangle \sim_R b$ for all i. We want all our vertices to be of the same type because we can have $\langle a, u_i, x\rangle \sim_R a$ for some i and for other i we could have $\langle b, u_i, y\rangle \sim_R b$ instead, but we require our u_is to all satisfy the same condition. As a result of the projection we have

$$\begin{aligned}\langle a, v_i, x\rangle &= \langle a, \langle c, u_i, u_{i_0}\rangle, x\rangle \\ &\sim_{\kappa_5} \langle \langle a, c, x\rangle, \langle a, u_{i_0}, x\rangle, u_i\rangle \\ &\sim_{K(G+R)+H(0)} \langle a, a, u_i\rangle \\ &= a,\end{aligned}$$

where the penultimate approximation follows from $a \lesssim_G c$ (with respect to x as the basepoint in the ordering) and by assuming that u_{i_0} satisfies $\langle a, u_{i_0}, x\rangle \sim_R a$ (since we are focusing on this case). So without loss of generality, assume that the v_i satisfy $\langle a, v_i, x\rangle \sim_{R'} a$ for all i, where $R' = \kappa_5 + K(G+R) + H(0)$.

To summarise, we know

A. $\langle a, v_i, b\rangle \sim_F c$ for all i, where c is not a corner;

B. $\langle a, v_i, y\rangle \sim_{R'} v_i$ for all i.

Let $w_i = \langle b, v_i, y\rangle$. Note that $d(w_i, v_i)$ is non-trivial; this follows from $d(c, b)$ being non-trivial (which was stated in Lemma 5.10), hence we set $w_i \leftrightarrow_{D''} v_i$ for all i, where D'' is the non-triviality parameter for $\tilde{C}$. We will show that we can use the w_i to build a cube $\tilde{C}$ that is one rank higher than C. We have four cases to prove:

(i) $\langle v_i, v_j, v_k\rangle \sim_E v_l$;

(ii) $\langle v_i, v_j, w_k\rangle \sim_{E''} v_l$;

(iii) $\langle w_i, w_j, v_k\rangle \sim_{E''} w_l$;

(iv) $\langle w_i, w_j, w_k\rangle \sim_{E''} w_l$, where $i, j, k, l \in \{1, \ldots, 2^d\}$.

Proof of the four cases:

(i) Follows as we have already assumed that C is a coarse cube.

(ii)

$$\begin{aligned}
\langle v_i, v_j, w_k\rangle &= \langle v_i, v_j, \langle b, v_k, y\rangle\rangle \\
&\sim_{K(KR'+H(0))+H(0)} \langle v_i, v_j, \langle b, \langle a, v_k, y\rangle, y\rangle\rangle \quad \text{(using B.)} \\
&= \langle v_i, v_j, \langle\langle a, y, v_k\rangle, y, b\rangle\rangle \\
&\sim_{K\kappa_4+H(0)} \langle v_i, v_j, \langle a, y, \langle v_k, y, b\rangle\rangle \quad \text{(using 4-point)} \\
&\sim_{\kappa_5} \langle\langle v_i, v_j, y\rangle, \langle v_i, v_j, \langle v_k, y, b\rangle\rangle, a\rangle \quad \text{(using 5-point)} \\
&\sim_{K\kappa_5+H(0)} \langle\langle v_i, v_j, y\rangle, \langle\langle v_i, v_j, v_k\rangle, \langle v_i, v_j, y\rangle, b\rangle, a\rangle \quad \text{(using 5-point)} \\
&\sim_{K(KE'+H(0))+H(0)} \langle\langle v_i, v_j, y\rangle, \langle v_l, \langle v_i, v_j, y\rangle, b\rangle, a\rangle \\
&= \langle a, \langle v_i, v_j, y\rangle, \langle b, \langle v_i, v_j, y\rangle, v_l\rangle\rangle \\
&\sim_{\kappa_4} \langle\langle a, \langle v_i, v_j, y\rangle, b\rangle, \langle v_i, v_j, y\rangle, v_l\rangle \quad \text{(using 4-point)} \\
&\sim_{K\kappa_5+H(0)} \langle\langle\langle a, b, v_i\rangle, \langle a, b, v_j\rangle, y\rangle, \langle v_i, v_j, y\rangle, v_l\rangle \quad \text{(using 5-point)} \\
&\sim_{K(2KF+H(0))+H(0)} \langle\langle c, c, y\rangle, \langle v_i, v_j, y\rangle, v_l\rangle \quad \text{(using A.)} \\
&= \langle c, \langle v_i, v_j, y\rangle, v_l\rangle \\
&\sim_{KE'+H(0)} \langle c, \langle v_i, v_j, y\rangle, \langle v_i, v_j, v_k\rangle\rangle \\
&\sim_{\kappa_5} \langle v_i, v_j, \langle c, y, v_k\rangle\rangle \quad \text{(using 5-point)} \\
&\sim_{K(KF+H(0))+H(0)} \langle v_i, v_j, \langle y, v_k, \langle a, v_k, b\rangle\rangle\rangle \quad \text{(using A. to replace } c\text{)} \\
&\sim_{K\kappa_4+H(0)} \langle v_i, v_j, \langle\langle y, v_k, a\rangle, v_k, b\rangle\rangle \quad \text{(using 4-point)} \\
&\sim_{K(KR'+H(0))+H(0)} \langle v_i, v_j, \langle v_k, v_k, b\rangle\rangle \quad \text{(using B.)} \\
&= \langle v_i, v_j, v_k\rangle \\
&\sim_{E'} v_l.
\end{aligned}$$

(iii)

$$\begin{aligned}
\langle w_i, w_j, v_k\rangle &= \langle\langle b, v_i, y\rangle, \langle b, v_j, y\rangle, v_k\rangle \\
&\sim_{\kappa_5} \langle b, y, \langle v_i, v_j, v_k\rangle\rangle \quad \text{(using 5-point)} \\
&\sim_{KE'+H(0)} \langle b, y, v_l\rangle \\
&= w_l.
\end{aligned}$$

(iv)

$$\begin{aligned}
\langle w_i, w_j, w_k \rangle &= \langle \langle b, v_i, y \rangle, \langle b, v_j, y \rangle, \langle b, v_k, y \rangle \rangle \\
&\sim_{\kappa_5} \langle b, y, \langle v_i, v_j, \langle b, v_k, y \rangle \rangle \rangle \quad \text{(using 5-point)} \\
&= \langle b, y, \langle v_i, v_j, w_k \rangle \rangle \\
&\sim_{Kd(\langle v_i, v_j, w_k \rangle, v_l) + H(0)} \langle b, y, v_l \rangle \quad \text{(using (ii))} \\
&= w_l.
\end{aligned}$$

If B. is replaced with $\langle b, v_i, x \rangle \sim_{R'} v_i$ and $w_i = \langle a, v_i, x \rangle$, the proof follows similarly. Take E'' to be the error obtained by summing up the errors in part (iv) of the proof. Then $\tilde{C}$ is a coarse cube of rank $d+1$ with quasi-morphism parameter E'' and D'' the non-triviality parameter, as required. Observe that D'' feeds into the size of L, as we have built $\tilde{C}$ by extending C in the direction of the interval $[b, y]$. □

5.4.3 Proof of the General Case

Let C be a coarse cube contained in the coarse hyperplane corresponding to a, b and let V be the set of vertices of C.

When we project C onto $[a, b]$, we may see a cube of lower dimension as some vertices of C may project onto points that are close together. Since this (lower dimension) cube is contained in $[a, b]$, it will be 'small', as $d(a, b)$ is bounded and so this implies that we can bound the size of anything projected onto $[a, b]$.

Pick an edge of C, say $\{v_1, v_2\}$, and project it onto $[a, b]$; we then define $w_1 = \langle a, v_1, b \rangle$ and $w_2 = \langle a, v_2, b \rangle$. Given $v \in V$, projecting v directly onto $\{w_1, w_2\}$ is coarsely the same as first projecting v onto $\{v_1, v_2\}$ and then onto $\{w_1, w_2\}$: since C is a coarse cube, when we project v onto $\{v_1, v_2\}$, we obtain either v_1 or v_2. By definition of w_1, w_2, we then see that $\langle v, w_1, w_2 \rangle \sim w_1$ or w_2. More formally,

$$\begin{aligned}
\langle \langle v, v_1, v_2 \rangle, w_1, w_2 \rangle &= \langle \langle v, v_1, v_2 \rangle, \langle a, b, v_1 \rangle, \langle a, b, v_2 \rangle \rangle \\
&\sim_{\kappa_5} \langle a, b, \langle v_1, v_2, \langle v, v_1, v_2 \rangle \rangle \rangle \\
&\sim_{K\kappa_4 + H(0)} \langle a, b, \langle \langle v_2, v_1, v_2 \rangle, v_1, v \rangle \\
&= \langle a, b, \langle v_2, v_1, v \rangle \rangle \\
&\sim_{\kappa_5} \langle \langle a, b, v_2 \rangle, \langle a, b, v_1 \rangle, v \rangle \\
&= \langle w_2, w_1, v \rangle.
\end{aligned}$$

Since C is a coarse cube, we have $\langle v, v_1, v_2\rangle \sim_E v_1$ or v_2, where E is the quasi-morphism constant. Let us assume that $\langle v, v_1, v_2\rangle \sim_E v_1$; then

$$\begin{aligned}\langle\langle v, v_1, v_2\rangle, w_1, w_2\rangle &\sim_{KE+H(0)} \langle v_1, w_1, w_2\rangle\\ &= \langle v_1, \langle a, b, v_1\rangle, \langle a, b, v_2\rangle\rangle\\ &\sim_{\kappa_5} \langle a, b, \langle v_1, v_2, v_1\rangle\rangle\\ &= \langle a, b, v_1\rangle\\ &= w_1.\end{aligned}$$

The above calculation holds similarly when we instead assume that $\langle v, v_1, v_2\rangle \sim_E v_2$.

We now introduce some necessary notation and motivation for the main argument of the proof of Theorem 5.8.

Define $u_1 = \langle v_1, v_2, w_1\rangle, u_2 = \langle v_1, v_2, w_2\rangle$; then we can split $\{v_1, v_2\}$ into three sub-edges: it is made up of two 'vertical' edges $\{v_1, u_1\}, \{u_2, v_2\}$ and one 'horizontal' edge $\{u_1, u_2\}$. Here, a 'vertical' edge is an edge that projects onto a point in $[a, b]$ (see case 1) and a 'horizontal' edge is an edge that projects onto a subinterval of $[a, b]$ (see case 2), i.e. $\{u_1, u_2, w_1, w_2\}$ forms a coarse 2-cube. Additionally, $\{v_1, v_2\}$ could be a diagonal edge (in the sense that we can move vertically and horizontally), however, in the corresponding CAT(0) cube complex, it may look more intricate and actually be made up of three sub-edges.

If we take a parallel edge $\{x_1, x_2\}$ to $\{v_1, v_2\}$, then we obtain a coarse 2-cube comprised of these four points, but we can split this up into three coarse sub-2-cubes in the same manner we split $\{v_1, v_2\}$ into three sub-edges.

The reason we cut $\{v_1, v_2\}$ into three pieces is because we are only looking at an edge $\{w_1, w_2\}$ in the projection and not something of higher dimension. In particular, $\{v_1, u_1\}$ represents staying at w_1 (as it is a vertical edge), $\{u_1, u_2\}$ represents moving from w_1 to w_2 (as it is a horizontal edge), and $\{u_2, v_2\}$ represents staying at w_2 (as it is a vertical edge). Note that the movement from w_1 to w_2 is monotone.

In order to prove Theorem 5.8, we first need the following result.

Proposition 5.13. *Given C, v_1, v_2, u_1, u_2, the quasi-morphism constant E and the ordering parameter G referenced earlier, there exists M such that the following holds. Assume that there exists a co-dimension 1 face F of C (an $(n-1)$-coarse cube) such that $v_1 \in F$ but $v_2 \notin F$. Define q as the vertex of C opposite v_1 and p as the vertex of F opposite v_1. Given $f \in F$, f' the corresponding point in the face of C parallel to F, and $t \in \{v_1, u_1, u_2, v_2\}$, define the maps ϕ, ψ*

respectively:

$$\begin{aligned}\phi : F \times \{v_1, v_2\} &\to C \\ (f, v_1) &\mapsto \langle f, v_1, q\rangle \sim f \\ (f, v_2) &\mapsto \langle f, v_2, q\rangle \sim f',\end{aligned}$$

$$\begin{aligned}\psi : F \times \{v_1, u_1, u_2, v_2\} &\to X \\ (f, t) &\mapsto \langle f, t, q\rangle.\end{aligned}$$

Then the restriction of ψ to C equals ϕ (i.e. $\psi|_C = \phi$) and the map ψ is an M-quasi-morphism.

Having stated Proposition 5.13, we now provide the argument for the proof of Theorem 5.8:

Proof of Theorem 5.8. We can cut C up into three coarse sub-cubes with the same rank as C as outlined in Proposition 5.13; more specifically, we can split C into two 'vertical' cubes and one 'horizontal' cube. This approach of splitting into 'vertical' and 'horizontal' cubes works together to give us our required result (this is a combination of cases 1 and 2): suppose the vertical pieces of either vertical cube are non-trivial; we know that these cubes have the same rank as C, and thus applying case 1, we see that the vertical cubes have at least one rank lower than X, hence implying that C must also have at least one rank lower than X. If the vertical pieces have trivial lengths, then keeping case 2 in mind, we know by assumption that the edge $\{u_1, u_2\}$ lines up in the direction of $\{w_1, w_2\}$, and so will have bounded size (as $\{w_1, w_2\}$ has a bound on its size as it lies in $[a, b]$), that is, $\{u_1, u_2, w_1, w_2\}$ would form a coarse 2-cube, hence giving us a bound on the distance from u_1 to u_2 in terms of $d(w_1, w_2)$. This in turn implies that C has rank at least one lower than X. □

Note 5.14. Observe that $\{v_1, u_1, u_2, v_2\}$ all lie in a 'straight' line, in the sense that $u_1 \sim [v_1, u_2], u_2 \sim [u_1, v_2]$ (we also know that $u_1, u_2 \in [v_1, v_2]$). We prove this below:

1.

$$\begin{aligned}\langle v_1, u_1, u_2\rangle &= \langle v_1, \langle v_1, v_2, w_1\rangle, \langle v_1, v_2, w_2\rangle\rangle \\ &\sim_{\kappa_5} \langle v_1, v_2, \langle v_1, w_1, w_2\rangle\rangle \\ &\sim_{K\kappa_5 + H(0)} \langle v_1, v_2, w_1\rangle \\ &= u_1.\end{aligned}$$

2.

$$\begin{aligned}\langle u_1, u_2, v_2\rangle &= \langle\langle v_1, v_2, w_1\rangle, \langle v_1, v_2, w_2\rangle, v_2\rangle\\ &\sim_{\kappa_5} \langle v_1, v_2, \langle w_1, w_2, v_2\rangle\rangle\\ &= \langle v_1, v_2, \langle\langle v_1, a, b\rangle, \langle v_2, a, b\rangle, v_2\rangle\rangle\\ &\sim_{K\kappa_5+H(0)} \langle v_1, v_2, \langle a, b, \langle v_1, v_2, v_2\rangle\rangle\rangle\\ &= \langle v_1, v_2, \langle a, b, v_2\rangle\rangle\\ &= \langle v_1, v_2, w_2\rangle\\ &= u_2.\end{aligned}$$

We now present the proof of Proposition 5.13:

Proof of Proposition 5.13. We will show that

$$\psi(\langle f_i, f_j, f_k\rangle, \langle t_1, t_2, t_3\rangle) \sim \langle \psi(f_i, t_1), \psi(f_j, t_2), \psi(f_k, t_3)\rangle,$$

where $f_i, f_j, f_k \in F$, $\langle f_i, f_j, f_k\rangle \sim_E f_l$, $t_1 \lesssim_G t_2 \lesssim_G t_3$ with $t_i \in \{v_1, u_1, u_2, v_2\}$ and $1 \le i, j, k, l \le 2^n$.

The following important facts are needed before we can commence with the proof:

(i) $\langle f_i, f_j, q\rangle \sim [f_l, q]$ (and $f_l \sim [v_1, \langle f_i, f_j, q\rangle]$).

$$\begin{aligned}\langle f_l, \langle f_i, f_j, q\rangle, q\rangle &\sim_{KE+H(0)} \langle\langle f_i, f_j, f_k\rangle, \langle f_i, f_j, q\rangle, q\rangle\\ &\sim_{\kappa_5} \langle f_i, f_j, \langle f_k, q, q\rangle\rangle \qquad \text{(by the coarse 5-point condition)}\\ &= \langle f_i, f_j, q\rangle,\end{aligned}$$

as required.

(ii) $\langle v_1, p, u_1\rangle \sim v_1$.

$$\begin{aligned}\langle v_1, p, u_1\rangle &= \langle v_1, p, \langle v_1, v_2, w_1\rangle\rangle\\ &= \langle p, v_1, \langle v_2, v_1, w_1\rangle\rangle\\ &\sim_{\kappa_4} \langle\langle p, v_1, v_2\rangle, v_1, w_1\rangle\\ &\sim_{KE+H(0)} \langle v_1, v_1, w_1\rangle\\ &= v_1.\end{aligned}$$

Actually, $\langle v_1, p, t\rangle \sim v_1$ for $t \in \{v_1, u_1, u_2, v_2\}$.

(iii) $\langle f_l, \langle f_i, f_j, q\rangle, v_1\rangle \sim \langle f_l, \langle f_i, f_j, q\rangle, u_1\rangle$.

$$\begin{aligned}
\langle f_l, \langle f_i, f_j, q\rangle, v_1\rangle &\sim_{K(E+KE+H(0)+\kappa_4)+H(0)} \langle\langle f_i, f_j, f_k\rangle, \langle f_i, f_j, q\rangle, \langle v_1, p, u_1\rangle\rangle \\
&= \langle\langle f_i, f_j, f_k\rangle, f_m, \langle v_1, p, u_1\rangle\rangle \\
&\sim_{K\kappa_4+H(0)} \langle\langle f_i, f_j, f_k\rangle, \langle v_1, f_m, p\rangle, \langle v_1, p, u_1\rangle\rangle \\
&= \langle\langle f_i, f_j, f_k\rangle, \langle v_1, \langle f_i, f_j, q\rangle, p\rangle, \langle v_1, p, u_1\rangle\rangle \\
&\sim_{\kappa_5} \langle v_1, p, \langle\langle f_i, f_j, f_k\rangle, \langle f_i, f_j, q\rangle, u_1\rangle\rangle \\
&\sim_{\kappa_5} \langle\langle v_1, p, \langle f_i, f_j, f_k\rangle\rangle, \langle v_1, p, \langle f_i, f_j, q\rangle\rangle, u_1\rangle \\
&\sim_{2K\kappa_4+H(0)} \langle\langle f_i, f_j, f_k\rangle, \langle f_i, f_j, q\rangle, u_1\rangle \\
&\sim_{KE+H(0)} \langle f_l, \langle f_i, f_j, q\rangle, u_1\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\langle\langle f_i, f_j, f_k\rangle, \langle f_i, f_j, q\rangle, u_1\rangle &\sim_{KE+H(0)} \langle f_l, \langle f_i, f_j, q\rangle, u_1\rangle \\
&\sim_{(iii)} \langle f_l, \langle f_i, f_j, q\rangle, v_1\rangle \\
&\sim_{(i)} f_l.
\end{aligned}$$

Similarly, this result can be generalised as in the previous case: $\langle f_l, \langle f_i, f_j, q\rangle, v_1\rangle \sim \langle f_l, \langle f_i, f_j, q\rangle, t\rangle$ for $t \in \{v_1, u_1, u_2, v_2\}$.

We now proceed with the proof below. Let

$$\begin{aligned}
a' &= \langle f_i, t_1, q\rangle, \\
b' &= \langle f_j, t_2, q\rangle, \\
c' &= \langle f_k, t_3, q\rangle.
\end{aligned}$$

<u>We know:</u>

$$\langle\langle f_i, f_j, f_k\rangle, \langle t_1, t_2, t_3\rangle, q\rangle \sim_{K(E+G)+H(0)} \langle f_l, t_2, q\rangle.$$

<u>We want:</u>

$$\langle a', b', c'\rangle \sim_M \langle f_l, t_2, q\rangle.$$

Let $m \in [v_1, q]$. Set

$$\begin{aligned}
t &= \langle v_1, m, v_2\rangle \in [v_1, v_2] \\
f &= \langle v_1, m, p\rangle \in [v_1, p].
\end{aligned}$$

Then

$$\begin{aligned}\langle f,t,q\rangle &= \langle\langle v_1,m,p\rangle,\langle v_1,m,v_2\rangle,q\rangle\\ &\sim_{\kappa_5} \langle v_1,m,\langle p,v_2,q\rangle\rangle\\ &\sim_{KE+H(0)} \langle v_1,m,q\rangle\\ &\sim_{\kappa_4} m.\end{aligned}$$

Using the following approximation (based on the above calculation)

$$\langle\langle\langle a',b',c'\rangle,v_1,v_2\rangle,\langle\langle a',b',c'\rangle,v_1,p\rangle,q\rangle \sim_{KE+H(0)+\kappa_4+\kappa_5} \langle a',b',c'\rangle,$$

we will prove the claim by showing that

$$\langle\langle a',b',c'\rangle,v_1,p\rangle \sim f_l, \tag{5.1}$$
$$\langle\langle a',b',c'\rangle,v_1,v_2\rangle \sim t_2. \tag{5.2}$$

(5.1)

$$\begin{aligned}\langle\langle a',b',c'\rangle,v_1,p\rangle &\sim_{\kappa_5} \langle\langle v_1,p,a'\rangle,\langle v_1,p,b'\rangle,c'\rangle\\ &= \langle\langle v_1,p,\langle f_i,t_1,q\rangle\rangle,\langle v_1,p,\langle f_j,t_2,q\rangle\rangle,\langle f_k,t_3,q\rangle\rangle\\ &\sim_{2K\kappa_5+H(0)} \langle\langle\langle v_1,p,f_i\rangle,\langle v_1,p,t_1\rangle,q\rangle,\langle\langle v_1,p,f_j\rangle,\langle v_1,p,t_2\rangle,q\rangle,\langle f_k,t_3,q\rangle\rangle\\ &\sim_{K[(K(\kappa_4+(ii))+H(0))+(K(\kappa_4+(ii))+H(0))]+H(0)} \langle\langle f_i,v_1,q\rangle,\langle f_j,v_1,q\rangle,\langle f_k,t_3,q\rangle\rangle\\ &\sim_{2K\kappa_4+H(0)} \langle f_i,f_j,\langle f_k,t_3,q\rangle\rangle\\ &\sim_{\kappa_5} \langle\langle f_i,f_j,f_k\rangle,\langle f_i,f_j,q\rangle,t_3\rangle\\ &\sim_{KE+H(0)} \langle f_l,\langle f_i,f_j,q\rangle,t_3\rangle\\ &\sim_{(iii)} \langle f_l,\langle f_i,f_j,q\rangle,v_1\rangle\\ &\sim_{(i)} f_l.\end{aligned}$$

(5.2)

$$\begin{aligned}
\langle\langle a',b',c'\rangle,v_1,v_2\rangle &\sim_{\kappa_5} \langle\langle v_1,v_2,a'\rangle,\langle v_1,v_2,b'\rangle,c'\rangle\\
&= \langle\langle v_1,v_2,\langle f_i,t_1,q\rangle\rangle,\langle v_1,v_2,\langle f_j,t_2,q\rangle\rangle,\langle f_k,t_3,q\rangle\rangle\\
&\sim_{2K\kappa_5+H(0)} \langle\langle\langle v_1,v_2,f_i\rangle,\langle v_1,v_2,q\rangle,t_1\rangle,\langle\langle v_1,v_2,f_j\rangle,\langle v_1,v_2,q\rangle,t_2\rangle,\langle f_k,t_3,q\rangle\rangle\\
&\sim_{K[(2KE+H(0))+(2KE+H(0))]+H(0)} \langle\langle v_1,v_2,t_1\rangle,\langle v_1,v_2,t_2\rangle,\langle f_k,t_3,q\rangle\rangle\\
&\sim_{2K\kappa_4+H(0)} \langle t_1,t_2,\langle f_k,t_3,q\rangle\rangle\\
&\sim_{\kappa_5} \langle\langle t_1,t_2,t_3\rangle,\langle t_1,t_2,q\rangle,f_k\rangle\\
&\sim_{KG+H(0)} \langle t_2,\langle t_1,t_2,q\rangle,f_k\rangle\\
&\sim_{KG+H(0)} \langle t_2,t_2,f_k\rangle\\
&= t_2.
\end{aligned}$$

Note 5.15. We can deduce straight away that $\langle t_1,t_2,q\rangle \sim_G t_2$, as in the interval $[v_1,q]$, $t_1 \lesssim_G t_2$ means that $\langle v_1,t_1,t_2\rangle \sim_G t_1$ or equivalently, $\langle t_1,t_2,q\rangle \sim_{G'} t_2$, where G' is the constant depending on G derived from Lemma 4.7. We also know that $[v_1,v_2] \subseteq [v_1,q]$, so if $t_1,t_2 \sim [v_1,v_2]$, then $t_1 \lesssim_G t_2$ (in $[v_1,v_2]$) $\Leftrightarrow \langle v_1,t_1,t_2\rangle \sim_G t_1 \Leftrightarrow t_1 \lesssim_G t_2$ (in $[v_1,q]$) $\Leftrightarrow \langle t_1,t_2,q\rangle \sim_{G'} t_2$.

Thus we have shown that coarse hyperplanes have co-dimension at least 1 with X. □

5.5 Coarsened Corners

We mentioned briefly why containing corners in our intervals presents a problem for us, i.e. we do not have control over how 'fat' the hyperplanes can be.

In the proof of case 1 of Theorem 5.8, we assumed that there were no R-corners. Our previous strategy to overcome this issue was to only take the 'upper'/'forward' part of the hyperplane rather than the whole — we were sidestepping the corner problem by considering only the forward part, but we would miss pieces of the hyperplane in the covering process. The corner issue is an obstacle in proving that coarse hyperplanes coarsely cover coarse intervals, thus we now turn our attention to resolving this matter. We describe the corner problem in detail in the rank 2 case and proceed to show how this can be rectified. We answer this problem in both the context of CAT(0) cube complexes and coarse median spaces and present both versions of the statements of these results in this section; the proof of the median case can be found in the Appendix.

Below is the median formulation of our result.

Lemma 5.16. *Given a CAT(0) cube complex X, let $[x,y] \subseteq X$ be a 2-dimensional interval and $[a,b] \subseteq [x,y]$ be a 1-dimensional interval. Then $[a,b]$ has at most 1 corner.*

We now state the coarsened corner problem for coarse median spaces; the remainder of this section is devoted to proving the lemma below.

Lemma 5.17. *Given a coarse median space X, rank 2 parameter $C(\lambda)$, rank 1 parameter $C(\lambda')$ and ordering parameter G, there exists R such that for any rank 2 coarse interval $[x,y] \subseteq X$ with respect to $C(\lambda)$ and for any rank 1 coarse interval $[a,b] \subseteq [x,y]$ with respect to $C(\lambda')$ where $a \lesssim_G b$, $[a,b]$ has at most 1 R-corner.*

Recall the definition of a coarse corner (Definition 5.7) and the (C2) axiom in Bowditch's definition of a coarse median space (Definition 3.1). It states that there is a function $H\colon \mathbb{N} \to [0,\infty)$ with the following property: suppose that $A \subseteq X$ with $1 \leq |A| \leq p < \infty$. Then there is a finite median algebra $(\Pi, \langle\rangle_\pi)$ and maps $\pi\colon A \to \Pi$ and $\sigma\colon \Pi \to X$ such that for all $x,y,z \in \Pi$, we have

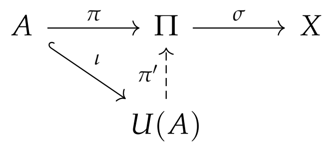
$$d(\sigma(\langle x,y,z\rangle_\pi), \langle \sigma(x), \sigma(y), \sigma(z)\rangle) \leq H(p)$$

and

$$d(a, \sigma\pi(a)) \leq H(p)$$

for all $a \in A$, i.e. that finite subsets of coarse median spaces can be approximated by finite CAT(0) cube complexes.

Here, σ is a $H(p)$-quasi-morphism and without loss of generality, we may assume that $\sigma\pi(a) = a$ (we suppose that the image of σ contains A exactly). Now, again without loss of generality, we can replace Π with $U(A)$, where $U(A)$ is the universal median algebra associated to A; this follows from the universal property of $U(A)$, which is pictured in the diagram below. An advantage of replacing Π with $U(A)$ is that we can explicitly draw and construct it.



Thus, let us take six points $\{x, a, b, y, d_1, d_2\} \in [x,y]$, where $[x,y]$ is a rank 2 interval in our coarse median space X, and define $c_i = \langle a, d_i, b\rangle, i = 1,2$. Assume that the following holds in $[a,b]$ (which we recall is rank 1): $x \lesssim_G a \lesssim_G c_1 \lesssim_G c_2 \lesssim_G b \lesssim_G y$, where G is the ordering parameter.

Associated to these six points, take $\{x', a', b', y', d_1', d_2'\} \in \Pi = U(A)$, where $U(A)$ is the universal median algebra on the six points $\{x', a', b', y', d_1', d_2'\}$ with $c_i' = \langle a', d_i', b'\rangle, i = 1,2$. Since $x \lesssim_G a \lesssim_G c_1 \lesssim_G c_2 \lesssim_G b \lesssim_G y$ coarsely holds in $[a,b]$, this implies the relations genuinely hold in $U(A)$: $a' \leq b'$ and $c_1' \leq c_2'$, so that overall we obtain $x' \leq a' \leq c_1' \leq c_2' \leq b' \leq y'$.

Note the following:

- We have a total ordering on M-separated subsets of $[a, b]$ by Lemma 5.2.
- We focus on the universal median algebra as this is the universal case.

The proof is split into three cases and these are as follows (note that we assume $c_1 \not\leftrightarrow_R c_2$ to avoid a trivial case, where R is the coarse corner constant as seen in Definition 5.7):

1. $c_2 \sim_N \langle c_2, d_1, y \rangle$ and c_1 is not a corner.
2. $c_1 \sim_P \langle c_1, d_2, x \rangle$ and c_2 is not a corner.
3. $c_1 \sim_S \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle$ and $c_2 \sim_T \langle x, c_2, \langle d_1, c_2, d_2 \rangle \rangle$ and neither c_1 nor c_2 are corners.

Proof of Lemma 5.17. We first need to check that the conditions of Theorem 3.5 for the interval $[c_1, d_2] \subseteq [x, y]$ holds (we could have also chosen $[c_2, d_1]$ instead). We need to show that $c_2, \langle c_1, d_2, x \rangle, \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle$ are close to $[c_1, d_2]$. By definition, $\langle c_1, d_2, x \rangle \in [c_1, d_2]$ so we just need to check the remaining points.

- $\langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle \sim_{\kappa_4} \langle \langle y, c_1, d_1 \rangle, c_1, d_2 \rangle \in [c_1, d_2]$.
-
$$\begin{aligned} c_2 &= \langle a, d_2, b \rangle \\ &\sim_{KG+H(0)} \langle \langle x, a, c_1 \rangle, d_2, b \rangle \\ &\sim_{\kappa_5} \langle \langle d_2, b, x \rangle, \langle d_2, b, a \rangle, c_1 \rangle \\ &= \langle \langle x, d_2, b \rangle, c_2, c_1 \rangle \\ &\sim_{\kappa_5} \langle \langle x, c_1, c_2 \rangle, \langle b, c_1, c_2 \rangle, d_2 \rangle \\ &\sim_{2KG+H(0)} \langle c_1, c_2, d_2 \rangle \in [c_1, d_2]. \end{aligned}$$

We now prove that $\{c_1, c_2, \langle c_1, d_2, x \rangle, \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle\}$ genuinely forms a tripod centred at c_1:

- $\langle \langle c_1, d_2, x \rangle, c_1, \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle \rangle \sim c_1$:

$$\begin{aligned} \langle \langle c_1, d_2, x \rangle, c_1, \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle \rangle &\sim_{\kappa_5} \langle \langle c_1, \langle c_1, d_2, x \rangle, y \rangle, \langle c_1, \langle c_1, d_2, x \rangle, c_1 \rangle, \langle d_1, c_1, d_2 \rangle \rangle \\ &= \langle \langle c_1, \langle c_1, d_2, x \rangle, y \rangle, c_1, \langle d_1, c_1, d_2 \rangle \rangle \\ &\sim_{K\kappa_4 + H(0)} \langle c_1, \langle d_1, c_1, d_2 \rangle, \langle \langle y, c_1, x \rangle, c_1, d_2 \rangle \rangle \\ &\sim_{K(K\kappa_4 + H(0)) + H(0)} \langle c_1, \langle d_1, c_1, d_2 \rangle, \langle c_1, c_1, d_2 \rangle \rangle \\ &= \langle c_1, c_1, \langle d_1, c_1, d_2 \rangle \rangle \\ &= c_1. \end{aligned}$$

- $\langle c_2, c_1, \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle \rangle \sim c_1$:

$$\begin{aligned}\langle c_2, c_1, \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle \rangle &\sim_{\kappa_5} \langle \langle c_1, c_2, y \rangle, \langle c_1, c_2, c_1 \rangle, \langle d_1, c_1, d_2 \rangle \rangle \\ &\sim_{KG+H(0)} \langle c_1, c_2, \langle d_1, c_1, d_2 \rangle \rangle \\ &\sim_{\kappa_4} \langle \langle c_2, c_1, d_1 \rangle, c_1, d_2 \rangle \\ &= \langle c_1, d_2, \langle \langle a, b, d_1 \rangle, \langle a, b, d_2 \rangle, d_1 \rangle \rangle \\ &\sim_{K\kappa_5+H(0)} \langle c_1, d_2, \langle a, b, \langle d_1, d_2, d_1 \rangle \rangle \rangle \\ &= \langle c_1, d_2, \langle a, b, d_1 \rangle \rangle \\ &= \langle c_1, d_2, c_1 \rangle \\ &= c_1.\end{aligned}$$

- $\langle c_2, c_1, \langle c_1, d_2, x \rangle \rangle \sim c_1$:

$$\begin{aligned}\langle c_2, c_1, \langle c_1, d_2, x \rangle \rangle &\sim_{\kappa_4} \langle \langle c_2, c_1, x \rangle, c_1, d_2 \rangle \\ &\sim_{KG+H(0)} \langle c_1, c_1, d_2 \rangle \\ &= c_1.\end{aligned}$$

Since we have shown that $\{c_1, c_2, \langle c_1, d_2, x \rangle, \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle\}$ genuinely forms a tripod, we can proceed with proving cases 1 and 2.

Case 1: If $c_2 \sim_N \langle c_2, d_1, y \rangle$, then c_1 is not a corner:

$$\begin{aligned}\langle b, d_1, y \rangle &\sim_{KG+H(0)} \langle \langle c_2, b, y \rangle, d_1, y \rangle \\ &\sim_{\kappa_4} \langle \langle d_1, y, c_2 \rangle, y, b \rangle \\ &\sim_{KN+H(0)} \langle c_2, y, b \rangle \\ &\sim_G b.\end{aligned}$$

Since $b \sim_{KG+KN+G+2H(0)+\kappa_4} \langle b, d_1, y \rangle, c_1$ is not a corner.

Case 2: If $c_1 \sim_P \langle c_1, d_2, x \rangle$ then c_2 is not a corner:

$$\begin{aligned}\langle a, d_2, x \rangle &\sim_{KG+H(0)} \langle \langle x, a, c_1 \rangle, d_2, x \rangle \\ &\sim_{\kappa_4} \langle \langle d_2, x, c_1 \rangle, x, a \rangle \\ &\sim_{KP+H(0)} \langle c_1, x, a \rangle \\ &\sim_G a.\end{aligned}$$

Since $a \sim_{KG+KP+G+2H(0)+\kappa_4} \langle a, d_2, x \rangle, c_2$ is not a corner.

Now that we have proven cases 1 and 2, we turn our attention to case 3. This case is slightly more complex and we now assume the following

- $c_1 \sim_S \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle$.

- $c_2 \sim_T \langle x, c_2, \langle d_1, c_2, d_2 \rangle \rangle$.

Case 3: Again, we begin by checking the conditions for the interval $[x, c_1]$ as seen in Theorem 3.5. We need to show that $a, \langle c_1, d_1, x \rangle, \langle c_1, d_2, x \rangle$ are in $[x, c_1]$. By definition, $\langle c_1, d_1, x \rangle, \langle c_1, d_2, x \rangle \in [x, c_1]$. Since $a \sim_G \langle x, a, c_1 \rangle$, this tells us that $a \in [x, c_1]$.
Next, we show that $\{c_1, a, \langle c_1, d_1, x \rangle, \langle c_1, d_2, x \rangle\}$ forms a tripod at c_1:

- $\langle \langle c_1, d_1, x \rangle, c_1, \langle c_1, d_2, x \rangle \rangle \sim c_1$: let

$$\begin{aligned} m_1 &= \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle \sim_S c_1 \\ m_2 &= \langle x, c_2, \langle d_1, c_2, d_2 \rangle \rangle \sim_T c_2. \end{aligned}$$

  By the coarse 5-point condition, $\langle \langle c_1, d_1, x \rangle, c_1, \langle c_1, d_2, x \rangle \rangle \sim_{\kappa_5} \langle c_1, x, \langle c_1, d_1, d_2 \rangle \rangle$.
  We know that $\langle x, m_2, c_1 \rangle \sim_{KT+H(0)} \langle x, c_2, c_1 \rangle$; we obtain
  $c_1 \sim_G \langle x, c_2, c_1 \rangle \sim_{KT+H(0)} \langle x, m_2, c_1 \rangle \sim_{\kappa_4} \langle x, c_1, \langle d_1, c_2, d_2 \rangle \rangle$. To sum up, we have

$$\begin{aligned} c_1 &\sim_{G+\kappa_4} \langle x, c_1, \langle d_1, c_2, d_2 \rangle \rangle \\ &\sim_{\kappa_5} \langle d_1, \langle x, c_1, c_2 \rangle, \langle x, c_1, d_2 \rangle \rangle \\ &\sim_{KG+H(0)} \langle d_1, c_1, \langle d_2, c_1, x \rangle \rangle \\ &\sim_{\kappa_4} \langle \langle d_1, c_1, d_2 \rangle, c_1, x \rangle. \end{aligned}$$

- $\langle a, c_1, \langle c_1, d_1, x \rangle \rangle \sim c_1$:

$$\begin{aligned} \langle a, c_1, \langle c_1, d_1, x \rangle \rangle &\sim_{\kappa_4} \langle \langle a, c_1, x \rangle, c_1, d_1 \rangle \\ &\sim_{KG+H(0)} \langle c_1, a, d_1 \rangle \\ &= \langle \langle a, d_1, b \rangle, a, d_1 \rangle \\ &\sim_{\kappa_4} \langle \langle a, d_1, a \rangle, d_1, b \rangle \\ &= \langle a, d_1, b \rangle \\ &= c_1. \end{aligned}$$

- $\langle a, c_1, \langle c_1, d_2, x\rangle\rangle \sim c_1$:

$$\begin{aligned}\langle a, c_1, \langle c_1, d_2, x\rangle\rangle &\sim_{\kappa_4} \langle\langle a, c_1, x\rangle, c_1, d_2\rangle \\ &\sim_{KG+H(0)} \langle a, c_1, d_2\rangle \\ &= \langle a, \langle a, d_1, b\rangle, d_2\rangle \\ &\sim_{\kappa_4} \langle\langle a, d_2, b\rangle, a, d_1\rangle \\ &= \langle c_2, a, d_1\rangle \\ &\sim_{KG+H(0)} \langle\langle c_1, c_2, b\rangle, a, d_1\rangle \\ &\sim_{\kappa_5} \langle\langle a, d_1, c_1\rangle, \langle a, d_1, b\rangle, c_2\rangle \\ &= \langle c_1, c_2, \langle a, d_1, c_1\rangle\rangle \\ &\sim_{\kappa_4} \langle\langle c_1, c_2, a\rangle, c_1, d_1\rangle \\ &\sim_{KG+H(0)} \langle c_1, c_1, d_1\rangle \\ &= c_1.\end{aligned}$$

All the conditions of the theorem hold, so we are in a position to show that we can only have at most one corner. Now,

(i) we must have $c_1 \not\leftrightarrow_R a$ because then c_1 would immediately fail to be a corner;

(ii) if $c_1 \sim_P \langle c_1, d_2, x\rangle$, then this is a case that has already been proven (see case 2);

(iii) this leaves us with $c_1 \sim_U \langle c_1, d_1, x\rangle$. Then c_1 is not a corner as

$$\begin{aligned}\langle a, d_1, x\rangle &\sim_{KG+H(0)} \langle\langle a, x, c_1\rangle, x, d_1\rangle \\ &\sim_{\kappa_4} \langle a, x, \langle c_1, x, d_1\rangle\rangle \\ &\sim_{KU+H(0)} \langle a, x, c_1\rangle \\ &\sim_G a.\end{aligned}$$

Set $R \geq \max(KG + Km + G + 2H(0) + \kappa_4)$, where $m \in \{N, P, U\}$. Therefore we have shown that in a rank 2 coarse interval, we cannot have more than one R-corner present. □

5.6 The Covering Problem

Now that the corner problem has been solved for rank 2, we can alter the proof of coarse hyperplanes having co-dimension 1 in the rank 2 case; since we know that there is at most one corner in each rank 2 interval, we can bound the number of corners. We then subdivide at the corners, allowing us to get rid of them and continue with our proof as given in Section 5.4.2.

We proceed by proving that coarse hyperplanes coarsely cover coarse intervals; we only prove this in the coarse median case rather than for CAT(0) cube complexes, too, as the argument requires a lot of care in dealing with the numerous parameters that arise. Here, we assume that $[a_k, b_{k,i}]$ is a rank 1 coarse interval, where a_k is defined in Lemma 5.5 and $b_{k,i} \in [x, a_k]$ in further detail below.

Suppose $x = a_0, \ldots, a_N = y$ is a path from x to y, where $a_j \sim_R a_{j+1}$ and $R = K(A_1 + A_2) + H(0)$ (R is 'small', refer to Lemma 5.5 for more details).

Assume that the points $a_k, b_{k,1}, \ldots, b_{k,r}$ form an r-pod of maximal r centred at a_k; note that $r = 1$ or 2 here. Suppose that $b_{k,i} \lesssim_{\kappa_4} a_k$ and that $d(b_{k,i}, a_k) \geq P$ (but not much larger) for $i = 1, \ldots, r$.

Consider a point $z \in [x, y]$ that is $\gtrsim_S a_k$ but not $\gtrsim_S a_{k+1}$, where $S \geq P + \kappa_4 + KL + H(0)$ (S is our largest parameter) and L is the hyperplane constant. Then the minimum of z and a_k, $\langle x, a_k, z \rangle$, must be approximately distance S from a_k, that is, $\langle x, a_k, z \rangle \sim_S a_k$, but this distance must be close to S. To be more precise, by the reverse triangle inequality, we have the following:

$$\begin{aligned} d(a_{k+1}, \langle x, a_k, z \rangle) &\geq d(a_{k+1}, \langle x, a_{k+1}, z \rangle) - d(\langle x, a_{k+1}, z \rangle, \langle x, a_k, z \rangle) \\ &> S - (KR + H(0)), \end{aligned}$$

where the last inequality follows from the fact that $z \not\gtrsim_S a_{k+1}$ and that $a_j \sim_R a_{j+1}$ used in combination with the (C1′) axiom from Definition 3.2. Now, again by the reverse triangle inequality, we have the following:

$$\begin{aligned} d(\langle x, a_k, z \rangle, a_k) &\geq d(\langle x, a_k, z \rangle, a_{k+1}) - d(a_{k+1}, a_k) \\ &> S - [(K+1)R + H(0)], \end{aligned}$$

where the last inequality follows from the fact that $z \not\gtrsim_S a_{k+1}$ and $a_j \sim_R a_{j+1}$.

As we are now using the 'backwards' cubes and the minimum of z and a_k is distance approximately S less than a_k, we want to show that z is in or before one of the coarse hyperplanes given by the edge $[a_k, b_{k,i}]$ for $i = 1, \ldots, r$.

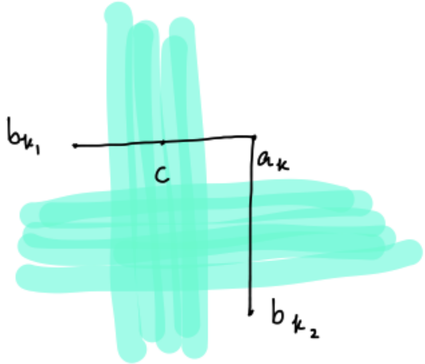


FIGURE 5.6: This shows the set-up for our two cases; for case 1, z lies before at least one of the coarse hyperplanes. In case 2, z is after the coarse hyperplanes, i.e. in the top right corner.

We summarise the above into a proposition below.

Proposition 5.18. *Given a coarse median space X, constants R, L and P, there exists S such that the following holds. For any coarse interval $[x, y] \subseteq X$, let $x = a_0, \ldots, a_N = y$ be a path from x to y, where $a_j \sim_R a_{j+1}$. For each $k \in \{0, \ldots, N\}$, assume that there exist points $b_{k,i} \in [x, a_k]$ such that $a_k, b_{k,1}, \ldots, b_{k,r}$ form an r-pod of maximal r centered at a_k, where $r = 1$ or 2. Suppose that $b_{k,i} \lesssim_{\kappa_4} a_k$ and that $d(a_k, b_{k,i}) \geq P$ (but not much larger than P) for $i = 1, \ldots, r$. Consider a point $z \in [x, y]$ such that $z \gtrsim_S a_k$ but $z \not\gtrsim_S a_{k+1}$; then z is in or before at least one of the coarse hyperplanes (with associated constant L) corresponding to $[a_k, b_{k,i}]$ for $i = 1, \ldots, r$.*

Proof. Since $z \gtrsim_S a_k$ but $z \not\gtrsim a_{k+1}$, this implies that $\langle x, a_k, z \rangle$ is at most S away from a_k but this distance is close to S.

We will show that it is not possible for $\langle z, a_k, b_{k,i} \rangle \sim_L a_k$ for $i = 1, \ldots, r$ while $\langle x, a_k, z \rangle \sim_S a_k$.

Assume towards a contradiction that $\langle z, a_k, b_{k,i} \rangle \sim_L a_k$ for $i = 1, \ldots, r$ (this says that z is after both hyperplanes, i.e. in the top right corner of Figure 5.7). Then we will prove that $\{a_k, b_{k,1}, \ldots, b_{k,r}, \langle x, a_k, z \rangle\}$ forms an $(r+1)$-pod centred at a_k.

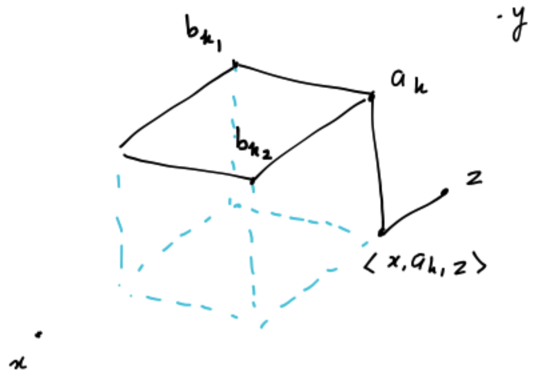


FIGURE 5.7: If $a_k, b_{k,1}, b_{k,2}, \langle x, a_k, z\rangle$ forms a tripod then we obtain a non-trivial coarse 3-cube, contradicting maximal dimensionality.

We need to verify that the conditions of Theorem 3.5 hold. The basepoint here is a_k and the interval we are focusing on is $[x, a_k]$. We already know that $a_k, b_{k,1}, \ldots, b_{k,r}, \langle x, a_k, z\rangle \in [x, a_k]$ by definition. Since $\langle x, a_k, z\rangle \in [x, a_k]$, this implies that $\langle x, a_k, z\rangle \sim_{\kappa_4} \langle x, \langle x, a_k, z\rangle, a_k\rangle$ (this follows from Note 3.12). However, by the ordering defined in Definition 4.3, this is equivalent to saying that $\langle x, a_k, z\rangle \lesssim_{\kappa_4} a_k$.

We are just left with proving that $\{a_k, b_{k,1}, \ldots, b_{k,r}, \langle x, a_k, z\rangle\}$ forms an $(r+1)$-pod centred at a_k.

- $$\begin{aligned}\langle b_{k,i}, a_k, \langle x, a_k, z\rangle\rangle &\sim_{\kappa_4} \langle\langle b_{k,i}, a_k, z\rangle, a_k, x\rangle \\ &\sim_{KL+H(0)} \langle a_k, a_k, x\rangle \\ &= a_k.\end{aligned}$$

- $$\langle b_{k,1}, a_k, b_{k,r}\rangle \sim_{\kappa_4+KL+H(0)} a_k,$$

Regarding the second point, we already know that these points form an r-pod by assumption, thus we also set $\langle b_{k,1}, a_k, b_{k,r}\rangle \sim_{\kappa_4+KL+H(0)} a_k$ for uniformity. Now, since $d(a_k, b_{k,i}) \geq P$, we will also assume that $d(\langle x, a_k, z\rangle, a_k) \geq P$ for uniformity. Putting this altogether, suppose that $P \leq d(\langle x, a_k, z\rangle, a_k) \leq P + \kappa_4 + KL + H(0)$, where we define $S \geq P + \kappa_4 + KL + H(0)$ (recall that $z \gtrsim a_k$). We additionally impose $d(b_{k,i}, a_k) \leq S$ for uniformity.

This tells us that the points $a_k, b_{k,1}, \ldots, b_{k,r}, \langle x, a_k, z\rangle$ form a non-trivial $(r+1)$-pod, implying that we missed a dimension and so can build our original coarse cube a dimension higher. This contradicts maximal dimensionality, which we originally assumed. □

Theorem 5.19. *Given a coarse median space X and rank parameter $C(\lambda)$, for all $R > 0$ and $L > 0$, let $x = a_0, \dots, a_N = y$ be a path from x to y with $a_j \sim_R a_{j+1}$ and let $[x, y] \subseteq X$ be a rank 2 interval with respect to $C(\lambda)$. Then there exist constants P, S, and Q and points $b_{k,i} \in [x, a_k]$ such that $S \geq d(a_k, b_{k,i}) \geq P$ and the union of the $L - [a_k, b_{k,i}]$ hyperplanes Q-cover $[x, y]$.*

Proof. Applying Proposition 5.18, we know that there exist points $b_{k,i} \in [x, a_k]$ such that $b_{k,i} \lesssim_{\kappa_4} a_k$ and that $P \leq d(a_k, b_{k,i}) \leq S$ for $i = 1, \dots, r$, where S was defined to be $S \geq P + \kappa_4 + KL + H(0)$. We conclude from the Proposition that for any $z \in [x, y]$ satisfying $z \gtrsim_S a_k$ but $z \not\gtrsim_S a_{k+1}$, then z is in or before at least one of the coarse hyperplanes (with associated constant L) corresponding to $[a_k, b_{k,i}]$ for $i = 1, \dots, r$.

Assume that

$$\langle z, a_k, b_{k,i} \rangle \sim_L b_{k,i},$$

where $i = 1, \dots, r$ (this says that z is before at least one of the hyperplanes). We will show that $z \gtrsim_S a_k$ and z before at least one of these hyperplanes implies that z is close to one of these hyperplanes.

We also know that $z \gtrsim_S a_k$, i.e.

$$\langle x, a_k, z \rangle \sim_S a_k,$$
$$\langle a_k, z, y \rangle \sim_S z.$$

Combining these two assumptions, we will show that z is close to either of the hyperplanes.

Take $c \in [a_k, b_{k,i}]$ with c not close to $a_k, b_{k,i}$ (so $c \in h_{a_k b_{k,i}}$). Note that $c \sim_{\kappa_4} \langle c, a_k, b_{k,i} \rangle$ by Note 3.12. We now prove the two claims below:

(i) $\langle z, c, y \rangle \sim z$;

(ii) $\langle z, c, y \rangle$ is in at least one of the hyperplanes.

(i)

$$\begin{aligned} \langle z, c, y \rangle &\sim_{KS+H(0)} \langle a_k, z, y \rangle \\ &\sim_S z \qquad \text{(since } z \gtrsim_S a_k\text{).} \end{aligned}$$

The first approximation follows since $d(b_{k,i}, a_k) \leq S \implies d(c, a_k) \leq S$. We can now define $Q = (K+1)S + H(0)$

(ii) Consider

$$\begin{aligned}\langle a_k, \langle z, c, y\rangle, b_{k,i}\rangle &\sim_{\kappa_5} \langle\langle a_k, b_{k,i}, z\rangle, \langle a_k, b_{k,i}, c\rangle, y\rangle \\ &\sim_{K(L+\kappa_4)+H(0)} \langle b_{k,i}, c, y\rangle \\ &\sim_{K(K\kappa_4+H(0)+\kappa_4)+K\kappa_4+\kappa_4+2H(0)} c \qquad (\text{as } b_{k,i} \lesssim c).\end{aligned}$$

We obtain the last approximation as follows: since $c \in [b_{k,i}, a_k]$, this implies that $c = \langle a_k, c', b_{k,i}\rangle$ for some $c' \in X$. Then

$$\begin{aligned}\langle x, b_{k,i}, c\rangle &= \langle x, b_{k,i}, \langle a_k, c', b_{k,i}\rangle\rangle \\ &\sim_{\kappa_4} \langle\langle x, b_{k,i}, a_k\rangle, c', b_{k,i}\rangle \\ &\sim_{K\kappa_4+H(0)} \langle b_{k,i}, c', b_{k,i}\rangle \qquad (\text{as } b_{k,i} \lesssim_{\kappa_4} a_k) \\ &= b_{k,i}.\end{aligned}$$

Now,

$$\begin{aligned}\langle b_{k,i}, c, y\rangle &\sim_{K(K\kappa_4+H(0)+\kappa_4)+H(0)} \langle\langle x, b_{k,i}, c\rangle, c, y\rangle \\ &\sim_{\kappa_4} \langle b_{k,i}, c, \langle x, c, y\rangle \\ &\sim_{K\kappa_4+H(0)} \langle b_{k,i}, c, c\rangle \\ &= c.\end{aligned}$$

This tells us that we should ensure that c is at least $L + \kappa_5 + K(L + \kappa_4) + K(K\kappa_4 + H(0) + \kappa_4) + K\kappa_4 + \kappa_4 + 3H(0)$ away from a_k and $b_{k,i}$ (which helps us to see why such a c exists) and that we should also let $P = 2[\kappa_5 + K(L + \kappa_4) + K(K\kappa_4 + H(0) + \kappa_4) + K\kappa_4 + \kappa_4 + 3H(0)] + R$, i.e. more than double the distance between c and $a_k, b_{k,i}$. □

Therefore, we have shown that coarse hyperplanes coarsely cover coarse intervals in the rank 2 case.

5.7 Coarse Hyperplanes are Coarsely Coarse Intervals

We now focus on answering our last question of the chapter: is the intersection of a coarse hyperplane and coarse interval 'almost' a coarse interval in itself? We divide our proof into three parts; we begin by showing that coarse hyperplanes satisfy the 'coarse convexity property' which is described below. We then prove an alternative version of Lemma 5.2 and apply it to coarse hyperplanes to show that M-separated subsets are totally ordered with respect to suitable parameters. We end with our key result of the section, which states that coarse hyperplanes are 'almost' coarse intervals themselves — see Theorem 5.25.

5.7.1 Coarse Convexity of Coarse Hyperplanes

Below, we show that the coarse hyperplanes satisfy the 'coarse convexity property', in other words, they are relatively coarsely convex.

Lemma 5.20. *For any coarse median space X, ordering parameter C and hyperplane constant L, there exists $L' > L$ such that the following holds. Given a rank 2 coarse interval $[x, y] \subseteq X$ and a rank 1 coarse interval $[a, b] \subseteq [x, y]$, let $h_{ab,L} \cap [x, y]$ and $h_{ab,L'} \cap [x, y]$ be the coarse hyperplanes intersected with $[x, y]$ corresponding to a, b with associated constants L and L', respectively. Then for all $p, q \in h_{ab,L'} \cap [x, y]$ and $w \in X$, the coarse hyperplanes $h_{ab,L} \cap [x, y]$ and $h_{ab,L'} \cap [x, y]$ satisfy the following:*

$$\forall p, q \in h_{ab,L'} \cap [x, y], \forall w \in X, \langle p, q, w\rangle_{[x,y]} := \langle\langle p, q, w\rangle, x, y\rangle \in h_{ab,L} \cap [x, y].$$

Proof. Let $p, q \in h_{ab,L'} \cap [x, y]$ and $w \in X$ and consider $\langle p, q, w\rangle$. We project p, q, w and $\langle p, q, w\rangle$ onto $[a, b]$ as this is the criteria for checking whether these points lie in the hyperplanes or not.

Denote

$$\begin{aligned} p' &= \langle p, a, b\rangle \\ q' &= \langle q, a, b\rangle \\ w' &= \langle\langle\langle p, q, w\rangle, x, y\rangle, a, b\rangle. \end{aligned}$$

We have

$$\begin{aligned} w' &= \langle\langle\langle p, q, w\rangle, x, y\rangle, a, b\rangle \\ &\sim_{\kappa_5} \langle\langle\langle p, x, y\rangle, \langle q, x, y\rangle, w\rangle, a, b\rangle \\ &\sim_{2K\kappa_4 + H(0)} \langle\langle p, q, w\rangle, a, b\rangle \\ &\sim_{\kappa_5} \langle p', q', w\rangle \\ &= w'', \end{aligned}$$

where the second approximation follows from the fact that $p, q \sim_{\kappa_4} [x, y]$. Without loss of generality, $p' \lesssim_C q'$ ($p', q' \in [a, b]$ which has rank 1), where C is the parameter associated to the coarse ordering.

Now, since $w'' \in [p', q']$ (by definition), we have

$$w'' = \langle p', q', w \rangle \sim_{2(\kappa_5 + K\kappa_4) + H(0)} w',$$

and so we obtain $w' \sim_{2(\kappa_5 + K\kappa_4) + H(0)} [p', q']$.
Combining the fact that $w' \sim [p', q']$, $[a, b]$ has rank 1 and that $p' \lesssim_C q'$, we obtain the following approximations (note that a is the basepoint with respect to the coarse ordering here):

$$\begin{aligned}
\langle a, w'', p' \rangle &= \langle a, \langle p', q', w \rangle, p' \rangle \\
&\sim_{\kappa_5} \langle \langle a, p', p' \rangle, \langle a, p', q' \rangle, w \rangle \\
&= \langle p', \langle a, p', q' \rangle, w \rangle \\
&\sim_{KC+H(0)} \langle p', p', w \rangle \\
&= p'.
\end{aligned}$$

Also,

$$\langle a, w', p', \rangle \sim_{Kd(w', w'') + H(0)} \langle a, w'', p' \rangle.$$

Altogether, we obtain

$$\langle a, w', p' \rangle \sim_{2K\kappa_5 + 2K^2\kappa_4 + KH(0) + \kappa_5 + KC + 2H(0)} p'.$$

Similarly, we get

$$\begin{aligned}
\langle w'', q', b \rangle &= \langle \langle p', q', w \rangle, q', b \rangle \\
&\sim_{\kappa_5} \langle \langle p', q', b \rangle, \langle q', q', b \rangle, w \rangle \\
&= \langle \langle p', q', b \rangle, q', w \rangle \\
&\sim_{KC+H(0)} \langle q', q', w \rangle \\
&= q'.
\end{aligned}$$

Again,

$$\langle w'', q', b \rangle \sim_{Kd(w', w'') + H(0)} \langle w', q', b \rangle.$$

Combining these two approximations,

$$\langle w'', q', b \rangle \sim_{2K\kappa_5 + 2K^2\kappa_4 + KH(0) + \kappa_5 + KC + 2H(0)} q'.$$

In other words, we have shown that $p' \sim_{2K\kappa_5+2K^2\kappa_4+KH(0)+\kappa_5+KC+2H(0)} [a, w']$ and similarly $q' \sim_{2K\kappa_5+2K^2\kappa_4+KH(0)+\kappa_5+KC+2H(0)} [w', b]$.

The last part of this argument requires us to bound the distance between a and p' in terms of the distance between a and w'. To be more specific, we want to show that

$$d(a, p') > L' \implies d(a, w') > L.$$

We prove this by looking at the contrapositive

$$d(a, w') \leq L \implies d(a, p') \leq L'.$$

We have the following (by the triangle inequality):

$$\begin{aligned} d(a, p') &\leq d(a, \langle a, w', p' \rangle) + d(\langle a, w', p' \rangle, p') \\ &= d(\langle a, a, p' \rangle, \langle a, w', p' \rangle) + d(\langle a, w', p' \rangle, p') \\ &\leq Kd(a, w') + H(0) + 2K\kappa_5 + 2K^2\kappa_4 + KH(0) + \kappa_5 + KC + 2H(0) \\ &\leq KL + H(0) + 2K\kappa_5 + 2K^2\kappa_4 + KH(0) + \kappa_5 + KC + 2H(0) \\ &= KL + 2K\kappa_5 + 2K^2\kappa_4 + KH(0) + \kappa_5 + KC + 3H(0). \end{aligned}$$

Similarly, we want to show that

$$d(q', b) > L' \implies d(w', b) > L.$$

Again, we prove this by looking at the contrapositive

$$d(w', b) \leq L \implies d(q', b) \leq L'.$$

The argument then follows in the same manner as the one for a and p' and so we obtain the same approximations as above.

To summarise, our aim is to show that the coarse hyperplanes $h_{ab,L'} \cap [x, y] \cap [x, y]$ and $h_{ab,L} \cap [x, y] \cap [x, y]$ have the 'coarse convexity property'. More specifically, we require $\forall L\ \exists L'$ such that $\forall a, b \in [x, y], \forall p, q \in h_{ab,L'} \cap [x, y], h_{ab,L} \cap [x, y]$ and $h_{ab,L'} \cap [x, y]$ satisfy

$$\forall p, q \in h_{ab,L'} \cap [x, y], \forall w \in X, \langle p, q, w \rangle_{[x,y]} := \langle \langle p, q, w \rangle, x, y \rangle \in h_{ab,L} \cap [x, y]. \tag{5.3}$$

Now, set $L' = KL + 2K\kappa_5 + 2K^2\kappa_4 + KH(0) + \kappa_5 + KC + 3H(0)$. Then with this value of L', we see that if $d(a, p') > L'$ then we deduce that $d(a, w') > L$, and this is the 'coarse convexity property' as required. □

5.7.2 An Alternative Version of Lemma 5.2

We state and prove an alternative version of Lemma 5.2 and apply it to the pair of coarse hyperplanes $h_{ab,L'} \cap [x,y]$ and $h_{ab,L} \cap [x,y]$.

Note 5.21.

- A subset B of a coarse median space X has rank 1 with respect to a parameter C' if the following holds: given $\lambda > 0$, there exists a constant $C' = C'(\lambda, \mu)$ such that for any $a, b \in B$, any $e_1, e_2 \sim_\mu [a,b]$ with $\langle e_1, a, e_2 \rangle \sim_\lambda a$, there exists $i \in \{1,2\}$ such that $e_i \sim_{C'} a$.
- As seen in Theorem 3.5, the endpoints of the interval also need to be contained in the subset we are focusing on, so one needs to exercise care when considering the rank of a subset compared to the rank of the whole space.

Lemma 5.22. *Given a coarse median space X and rank parameter $C'(\lambda, \mu)$, there exist λ, μ, M such that for all rank 1 coarse intervals $[x,y] \subseteq X$ and subsets $A \subseteq B \subseteq [x,y]$, if:*

1. *A and B possess the 'coarse convexity property' (5.3),*
2. *B is a rank 1 subset of X with respect to $C'(\lambda, \mu)$,*

then M-separated subsets of A are $C'(\lambda, \mu)$-totally ordered (for particular λ, μ).

Proof. We first show that comparability follows: let $a, b \in A$ and assume that they are incomparable. Then we obtain the bipod with points $a, \langle x,a,b \rangle, b$ centred at $\langle x,a,b \rangle$. The interval we are focusing on here is $[\langle x,a,b \rangle, \langle a,b,y \rangle]$. Since A, B have the 'coarse convexity property', we know that $\langle x,a,b \rangle \in B$, and so part of the bipod does not necessarily lie only in A, it can lie in B. We now check the conditions for Theorem 3.5 — the values of λ, μ will drop out as a result of this.

Note 5.23. We write $C'(\lambda, \mu) = C'$ for ease of notation.

1. We check whether $a, b \in [\langle x,a,b \rangle, \langle a,b,y \rangle]$:
   - $a \in [\langle x,a,b \rangle, \langle a,b,y \rangle]$:

$$\begin{aligned}\langle \langle x,a,b \rangle, a, \langle a,b,y \rangle \rangle &\sim_{\kappa_5} \langle a,b,\langle x,a,y \rangle \rangle \\ &\sim_{K\kappa_4 + H(0)} \langle a,b,a \rangle \\ &= a.\end{aligned}$$

   - $b \in [\langle x,a,b \rangle, \langle a,b,y \rangle]$:

$$\begin{aligned}\langle \langle x,a,b \rangle, b, \langle a,b,y \rangle \rangle &\sim_{\kappa_5} \langle a,b,\langle x,b,y \rangle \rangle \\ &\sim_{K\kappa_4 + H(0)} \langle a,b,b \rangle \\ &= b.\end{aligned}$$

2. We show that $\langle a, \langle x,a,b\rangle, b\rangle \sim \langle x,a,b\rangle$:

$$\begin{aligned}\langle a, \langle x,a,b\rangle, b\rangle &= \langle b,a,\langle b,a,x\rangle\rangle \\ &\sim_{\kappa_4} \langle\langle b,a,b\rangle, a, x\rangle \\ &= \langle b,a,x\rangle \\ &= \langle x,a,b\rangle.\end{aligned}$$

So we obtain $\lambda = \kappa_4$ and $\mu = K\kappa_4 + \kappa_5 + H(0)$.

However, B has rank 1, so Theorem 3.5 tells us that one direction of the bipod must be 'small.' Thus either

$$a \sim_{C'} \langle x,a,b\rangle \iff a \lesssim_{C'} b$$

or

$$b \sim_{C'} \langle x,a,b\rangle \iff b \lesssim_{C'} a.$$

Hence we see that either $a \lesssim_{C'} b$ or $b \lesssim_{C'} a$ and so comparability of points in A follows.

Now we can proceed with the full proof.

Let $a,b \in A \subseteq B \subseteq [x,y]$. Then $a \lesssim_{C'} b$ if and only if $\langle x,a,b\rangle \sim_{C'} a$. Since A, B have the 'coarse convexity property' and B has rank 1, comparability follows (see the argument above), so for any $a,b \in A$, either $a \lesssim_{C'} b$ or $b \lesssim_{C'} a$, i.e. $a \sim_{C'} \langle x,a,b\rangle$ or $b \sim_{C'} \langle x,a,b\rangle$. If both $a \lesssim_{C'} b$ and $b \lesssim_{C'} a$, then this implies that $a \sim_{C'} \langle x,a,b\rangle \sim_{C'} b$ and we obtain $a \sim_{2C'} b$.

For transitivity, let us assume that $a \lesssim_{C'} b$, $b \lesssim_{C'} c$ and $c \lesssim_{C'} a$; then $a \sim_{C'} \langle x,a,b\rangle$, $b \sim_{C'} \langle x,b,c\rangle$ and $c \sim_{C'} \langle x,a,c\rangle$. Using the coarse four-point condition, we obtain

$$\begin{aligned}a &\sim_{C'} \langle x,a,b\rangle \\ &\sim_{KC'+H(0)} \langle x,a,\langle x,b,c\rangle\rangle \\ &\sim_{\kappa_4} \langle\langle x,a,b\rangle, x, c\rangle \\ &\sim_{KC'+H(0)} \langle a,x,c\rangle \\ &\sim_{C'} c.\end{aligned}$$

Now set $M = 2(KC + H(0) + C') + \kappa_4 + 1$ and take an M-separated subset in A — then for all a,b,c in this M-separated subset, if $a \lesssim_{C'} b$ and $b \lesssim_{C'} c$, either $a \lesssim_{C'} c$ or $c \lesssim_{C'} a$ (by comparability). However, if $a \lesssim_{C'} b$, $b \lesssim_{C'} a$ and $c \lesssim_{C'} a$, then by the calculation above we obtain $c \sim_M a \sim_M b \sim_M c$, which implies that $a = b = c$, as the distances between a, b and c are less than M and since we are in an M-separated subset the three points must then be equal. We have ruled out $c \lesssim_{C'} a$ and so we must have $a \lesssim_{C'} c$. This choice of M means that both antisymmetry and transitivity hold, therefore giving us a $C'(\lambda, \mu)$-total ordering on M-separated subsets of A. □

5.7.3 Application of Lemma 5.22 to $h_{ab,L'} \cap [x,y]$ and $h_{ab,L} \cap [x,y]$

We can now apply Lemma 5.22 to the coarse hyperplanes $A = h_{ab,L'} \cap [x,y]$ and $B = h_{ab,L} \cap [x,y]$ — we know that $h_{ab,L'} \cap [x,y] \subseteq h_{ab,L} \cap [x,y] \subseteq [x,y]$ and we have also shown that they satisfy the 'coarse convexity property'. Furthermore, since we are working in rank 2, we have previously shown that coarse hyperplanes have co-dimension 1 and so in particular, $h_{ab,L} \cap [x,y]$ has rank 1. Consequently, we see that M-separated subsets in $h_{ab,L'} \cap [x,y]$ are C'-totally ordered.

Note 5.24. Observe that for fixed M, when taking a maximal M-separated subset, all points must be within distance M from each other.

Since M-separated subsets in $h_{ab,L'} \cap [x,y]$ are totally ordered and finite, this implies the existence of minimum and maximum elements. In particular, take a maximal M-separated subset (which is non-empty) — then this is coarsely the whole of $h_{ab,L'} \cap [x,y]$. More formally, we obtain the following 'sandwiching' result:

Theorem 5.25. *Given a coarse median space X and hyperplane constant L, there exists $L' > L$ (from Lemma 5.20) and M (derived from Lemma 5.22) such that the following holds. For any rank 1 coarse interval $[x,y] \subseteq X$ with $a, b \in [x,y]$, let $h_{ab,L} \cap [x,y]$ and $h_{ab,L'} \cap [x,y]$ be the coarse hyperplanes intersected with $[x,y]$ corresponding to a, b with associated constants L and L', respectively. Then there exist minimum and maximum points, $m_0, m_1 \in h_{ab,L'} \cap [x,y]$, such that $h_{ab,L'} \cap [x,y]$ sits inside the M-neighbourhood of $[m_0, m_1]$ (this is an M-coarsening of $h_{ab,L'} \cap [x,y]$), which in turn sits in $h_{ab,L} \cap [x,y]$.*

Note that the error for this 'sandwiching' result is M, the separation parameter.

5.8 Quadratic Growth of Rank 2 Coarse Intervals

We now combine our earlier results, Theorem 5.8, Lemma 5.17, Proposition 5.19 and Theorem 5.25, along with an inductive argument to show that rank 2 coarse intervals do indeed have quadratic growth. We can embed these coarse intervals into $\mathbb{R}^2$ similarly as in the case of CAT(0) cube complexes (see (Brodzki et al., 2009, Theorem 1.14)). We state our result below.

Theorem 5.26. *Let X be a uniformly discrete, uniformly locally finite quasi-geodesic coarse median space, and let C be a (rank) constant. Then there exists a constant W depending only on C and the local finiteness, quasi-geodesicity, and coarse median parameters of X, such that for any coarse interval $[x,y] \subseteq X$ of rank 2 with respect to C, we have*

$$\#[x,y] \leq Wd(x,y)^2,$$

where $\#[x,y]$ denotes the cardinality of $[x,y]$.

Proof. We induct on the rank of the coarse interval. Let $[x, y]$ be a rank 2 coarse interval; then we have shown that we can coarsely cover $[x, y]$ with coarse hyperplanes that are coarsely themselves intervals and have rank 1.

We have a sequence of $N + 1$ points $x = a_0, a_1, \ldots, a_N = y$ that are consecutive points in our path connecting x and y and are at most $K(A_1 + A_2) + H(0)$-separated from each other — see Lemma 5.5. By Proposition 5.19, there are at most two hyperplanes $h_{a_k,b_{k,i}} \cap [x, y]$, with $1 \leq k \leq N$, $i \in \{1, 2\}$, associated to each point a_k (except for $a_0 = x$), so we have at most $2N$ hyperplanes altogether that coarsely cover $[x, y]$, where $N \leq Zd(x, y)$ with $Z = A_1 + A_2$. By Theorem 5.8, we know that each $h_{a_k,b_{k,i}} \cap [x, y]$ has rank 1. We cover the interval $[x, y]$ by each $h_{a_k,b_{k,i}} \cap [x, y]$ with associated coarse hyperplane constant L' — Theorem 5.25 says that these L' hyperplanes sit inside an interval, call this interval $[s_k, S_k]$, which in turn sits inside each $h_{a_k,b_{k,i}} \cap [x, y]$ with associated coarse hyperplane constant L. We deduce from this that the intervals $[s_k, S_k]$ also cover the interval $[x, y]$. Now, the intervals $[s_k, S_k]$ are rank 1 as they sit inside the L hyperplanes, which we know are rank 1 by Theorem 5.8.

Since the results mentioned in the previous paragraph also apply to rank 1 intervals, we can use the same reasoning to conclude that the rank 1 intervals $[s_k, S_k]$ can be covered by rank 0 intervals. Since there are a linear number of these rank 0 intervals, each containing a bounded number of points due to bounded geometry, it follows that each rank 1 interval $[s_k, S_k]$ has linear growth.

Applying the inductive hypothesis, each $h_{a_k,b_{k,i}} \cap [x, y]$ has linear growth, i.e. these are linear in $d(x, y)$: $\#h_{a_k,b_{k,i}} \cap [x, y] \leq \rho d(s_k, S_k)$, for some ρ. This gives us a bound on the number of terms within each rank 1 $h_{a_k,b_{k,i}} \cap [x, y]$. Observe that $d(s_k, S_k) \leq 2Kd(x, y) + H(0)$: since both $s_k, S_k \sim [x, y]$, we can rewrite them as follows

$$s_k = \langle x, s'_k, y \rangle$$
$$S_k = \langle x, S'_k, y \rangle,$$

for some $s'_k, S'_k \in [x, y]$. We then obtain

$$s_k = \langle x, y, s'_k \rangle \sim_{Kd(x,y)+H(0)} \langle x, x, s'_k \rangle = x,$$

where we apply the (C1′) axiom from Definition 3.2. Replacing s_k with S_k, we also similarly obtain $S_k \sim_{Kd(x,y)+H(0)} x$. Bringing this together, we obtain $s_k \sim_{Kd(x,y)+H(0)} x \sim_{Kd(x,y)+H(0)} S_k$, that is, $d(s_k, S_k) \leq 2(Kd(x, y) + H(0))$. Now, putting all this together, we obtain $\#h_{a_k,b_{k,i}} \cap [x, y] \leq 2\rho(Kd(x, y) + H(0))$. Since X is uniformly discrete, $d(x, y) \geq 1$ for $x \neq y$. This means

$$\#h_{a_k,b_{k,i}} \cap [x, y] \leq 2\rho(Kd(x, y) + H(0)) \leq 2\rho(K + H(0))d(x, y).$$

We then multiply the number of coarse hyperplanes by the cardinality of each $h_{a_k,b_{k,i}} \cap [x,y]$, which gives us our quadratic bound:

$$\begin{aligned} 2N * \#h_{a_k,b_{k,i}} \cap [x,y] &\leq 2Zd(x,y) \cdot (2\rho K d(x,y) + 2\rho H(0)d(x,y)) \\ &= 4\rho Z K d(x,y)^2 + 4\rho Z H(0) d(x,y)^2 \\ &= 4\rho Z(K + H(0))d(x,y)^2. \end{aligned}$$

Let $W = 4\rho Z(K + H(0))$. This shows that $[x,y]$ has quadratic growth, as required. $\square$

Chapter 6

Structure of Rank n Coarse Intervals

We now extend our result in the previous chapter to the general rank n case. The method we use here differs from the rank 2 case, mainly due to the difficulty of extending some of the concepts we introduced for rank 2 to higher rank. Instead, we define the notion of a maximal edge point and apply Dilworth's Lemma using this concept to aid us in proving polynomial growth of rank n coarse intervals.

6.1 Maximal Edge Subsets and Dilworth's Lemma

In this section, we introduce the concept of a maximal edge point associated to a coarsely convex subset of a coarse interval; intuitively, these are points that can be thought of as 'maximal' points of the subset. We link these maximal edge points to antichains and then prove one of our key results: the length of any antichain in a maximal edge subset is bounded above by the rank of the subset.

We define maximal edge points below in both the median and coarse median cases. Suppose X is a coarse median space with parameters $K, H(0), \kappa_4, \kappa_5$ and let $[x, y]$ be a coarse interval in X. Let S be a coarsely convex subset of $[x, y]$ with rank $\leq r$ with respect to the rank constant $C(\lambda)$ — we clarify what it means for a subset of a coarse median space to be coarsely convex, as defined in (Bowditch, a, page 16). Finally, let $x \in X$ be the basepoint for the relation $\lesssim$.

Definition 6.1.
- Given a constant δ and a subset $S \subseteq X$, where X is a coarse median space, S is *δ-convex* if for all $a, b \in S$, $[a, b] \subseteq N(S, \delta)$, where $N(S, \delta)$ denotes the δ-neighbourhood of S.
- A subset $S \subseteq X$ is *coarsely convex* if there exists a δ such that S is δ-convex. In other words, if $a, b \in S$ and $x \in X$, then $d(\langle a, b, x\rangle, S) \leq \delta$.

Before we dive into defining maximal edge points, we state Dilworth's Lemma below.

Theorem 6.2 (Dilworth's Lemma)**.** *Let P be a finite partially ordered set. Then there exists a partition of P into a minimum number of chains $C_1, C_2, \ldots, C_k$ such that the size of the largest antichain in P is k. More generally, the size of the largest antichain in P is equal to the minimum number of chains required to cover all elements.*

6.1.1 Median Maximal Edge Points

Definition 6.3. Let X be a median algebra and $S \subseteq [x, y]$ a convex subset. Then a point $a \in S$ is said to be a *directly edge maximal point associated to* S if for any $u, v \in S$, where $u, v \geq a$ and $a = \langle a, u, v \rangle$, either $a = u$ or $a = v$. Equivalently, given $u, v \in S$ such that $a = \min(u, v) = \langle x, u, v \rangle$, then either $a = u$ or $a = v$.

We now consider three points $u, v, w \in S$ and show what it means for $a \in S$ to be a maximal edge point here.

$$\begin{aligned} a &= \min(u, v, w) \\ &= \langle u, v, w; x \rangle \\ &= \langle u, \langle v, w, x \rangle, x \rangle. \end{aligned}$$

If a is directly edge maximal, then either $a = u$ or $a = \langle v, w, x \rangle$. Choosing the latter, we have $a = \langle v, w, x \rangle$; since a is directly edge maximal, this implies that either $a = v$ or $a = w$. More generally, if a is directly edge maximal and $a = \min(u_1, \ldots, u_n)$, where $u_i \in S$, then either $a = u_1$ or $a = u_2$ and so on until $a = u_n$. Therefore, a is a directly edge maximal point in the median case if it is not the minimum of two points, or, equivalently, if a is not the minimum of at least two points.

We show that the length of a finite, incomparable sequence of directly edge maximal points, an antichain, associated to a convex subset S of an interval $[x, y]$ is bounded above by the rank of S. We focus on the cases when the rank is at most 1 and 2 as motivation for the coarse median context, then prove this theorem more generally by extending it to the coarse median world in the next section.

Prior to the statement of the theorem below, note the following:

- The notation $\hat{c_i}$ in the expression $c = \langle c_1, \ldots, \hat{c_i}, \ldots, c_n \rangle$ means the ith term is not present, i.e. $c = \langle c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n \rangle$.
- An n-pod can be defined as a configuration consisting of n line segments, each termed a 'leg', that extend from a common point, called the 'centre', to distinct end points in space.

Theorem 6.4. *Let X be a median algebra and $[x, y] \subseteq X$ be an interval. Suppose that S is a convex subset of $[x, y]$ with rank at most r for some integer r. Let n be an integer such that $1 \leq n \leq r+1$. Suppose there exist points $u_1, \ldots, u_n \in S$ which are n incomparable directly edge maximal points (an antichain). Define $v_0 = \min(u_1, \ldots, u_n)$ and for all $i \in \{1, \ldots, n\}, v_i = \min(u_1, \ldots, \hat{u}_i, \ldots, u_n)$. Then the set $\{v_0, v_1, \ldots, v_n\}$ forms an n-pod centred at v_0. This implies $n \leq r$, that is, the size of an antichain n in S is bounded above by r:*

$$\#\{u_1, \ldots, u_n\} \leq r.$$

Proof. rank of $S \leq 1$: we first show that $\{v_0, v_1, v_2\}$ forms a 2-pod/bipod; note that $v_0, v_1, v_2 \in S$ by convexity of S, as the v_i are constructed from the u_i which are elements of S. By definition, $v_0 = \min(u_1, u_2) = \langle u_1, u_2, x \rangle, v_1 = u_2, v_2 = u_1$. We now show the betweenness condition holds:

$$\begin{aligned}
\langle v_1, v_0, v_2 \rangle &= \langle v_1, \langle v_1, v_2, x \rangle, v_2 \rangle \\
&= \langle v_1, v_2, x \rangle \\
&= v_0.
\end{aligned}$$

Thus, the set $\{v_0, v_1, v_2\}$ forms a non-trivial bipod, where non-triviality is a consequence of the incomparability of u_1 and u_2. However, applying Note 3.7 gives us a contradiction to the rank of $S \leq 1$, as $\{v_0, v_1, v_2\}$ forming a non-trivial bipod implies that the rank of S would have needed to be at least 2 initially. Therefore, one side of the bipod $\{v_0, v_1, v_2\}$ must be trivial, i.e. $v_0 = v_1$ or $v_0 = v_2$ for S to have rank at most 1.

rank of $S \leq 2$: without loss of generality, fix i and let $i, j, k \in \{1, 2, 3\}$, where $i \neq j \neq k$. Define $w_i = \max(u_i, v_i) = \langle u_i, v_i, y \rangle$ and $z_j = \max(u_i, u_j) = \langle u_i, u_j, y \rangle$. Similarly to the rank 1 case, we begin by showing that $\{v_0, v_i, v_j, v_k\}$ forms a 3-pod/tripod; note that $v_0, v_i, v_j, v_k \in S$ by convexity of S, as the v_i are constructed from the u_i which are elements of S. By definition, $v_0 = \min(u_i, u_j, u_k) = \langle u_i, u_j, u_k; x \rangle$; we now show the betweenness condition holds:

$$\begin{aligned}
\langle v_i, v_0, v_j \rangle &= \langle v_i, \langle u_i, u_j, u_k; x \rangle, v_j \rangle \\
&= \langle \langle u_j, u_k, x \rangle, \langle \langle u_i, u_j, x \rangle, u_k, x \rangle, \langle u_i, u_k, x \rangle \rangle \\
&= \langle u_k, x, \langle u_j, \langle u_i, u_j, x \rangle, \langle u_i, u_k, x \rangle \rangle \rangle \\
&= \langle u_k, x, \langle u_i, x, \langle u_j, u_k, u_j \rangle \rangle \rangle \\
&= \langle u_k, x, \langle u_i, x, u_j \rangle \rangle \\
&= \langle u_i, u_j, u_k; x \rangle \\
&= v_0.
\end{aligned}$$

We also similarly have $\langle v_i, v_0, v_k \rangle = v_0$ and $\langle v_j, v_0, v_k \rangle = v_0$; the proof is analogous to the one above.

We next prove a statement that we will use for the remainder of the proof, $w_i = \min(\hat{z}_i, z_j, z_k)$:

$$\begin{aligned}
\min(\hat{z}_i, z_j, z_k) &= \min(z_j, z_k) \\
&= \langle z_j, z_k, x \rangle \\
&= \langle \langle u_i, u_j, y \rangle, \langle u_i, u_k, y \rangle, x \rangle \\
&= \langle u_i, y, \langle u_j, u_k, x \rangle \rangle \\
&= \langle u_i, y, v_i \rangle \\
&= w_i.
\end{aligned}$$

Note that $v_0 = \langle u_i, u_j, u_k; x \rangle = \langle \hat{u}_i, u_j, u_k, u_i; x \rangle = \langle v_i, u_i, x \rangle$. Suppose towards a contradiction that $v_0 = v_i$; then this implies that

$$\begin{aligned}
w_i &= \langle u_i, v_i, y \rangle \\
&= \langle u_i, v_0, y \rangle \\
&= \langle u_i, \langle v_i, u_i, x \rangle, y \rangle \\
&= \langle \langle y, u_i, x \rangle, u_i, v_i \rangle \\
&= \langle u_i, u_i, v_i \rangle \\
&= u_i.
\end{aligned}$$

By incomparability of the u_i, $u_i \neq z_j$, i.e. $u_i \neq \langle u_i, u_j, y \rangle$.

By assumption, u_i is directly edge maximal in S. Observe that $u_i = w_i = \min(\hat{z}_i, z_j, z_k) = \min(z_j, z_k)$, where $\min(\hat{z}_i, z_j, z_k) \in S$ by convexity of S, as the z_i are constructed from elements of S. Overall, we see that $u_i = \min(\hat{z}_i, z_j, z_k) = \min(z_j, z_k)$. Since u_i is directly edge maximal, $u_i = z_j$ or $u_i = z_k$, i.e. $u_i = \langle u_i, u_j, y \rangle$ or $u_i = \langle u_i, u_k, y \rangle$. However, $u_i = \langle u_i, u_j, y \rangle$ or $u_i = \langle u_i, u_k, y \rangle$ is a contradiction to incomparability, as we assumed that the u_i are incomparable, i.e. $u_i \neq \langle u_i, u_j, y \rangle$. Therefore, we must have $v_0 \neq v_i$.

Furthermore, applying Note 3.7 gives us a contradiction to the rank of $S \leq 2$, as $v_0 \neq v_i$ implies that $\{v_0, v_i, v_j, v_k\}$ forms a non-trivial tripod, meaning that the rank of S would have needed to be at least 3 initially. Therefore, one side of the tripod $\{v_0, v_i, v_j, v_k\}$ must be trivial for S to have rank at most 2. □

6.1.2 Coarse Maximal Edge Points

We now extend the median definition of maximal edge points to the coarse median world and divide them into direct and indirect; direct is analogous to the definition of maximal edge points we have seen earlier for medians.

Definition 6.5. Let X be a coarse median space and S be a δ-coarsely convex subset of $[x, y]$. Then a point $a \in S$ is said to be *directly edge maximal* with parameters C_1, C_2 if the following condition holds: for all n with $1 \leq n \leq r$ and $u_1, \ldots, u_n \in S$, if $a \sim_{C_1} \min(u_1, \ldots, u_n)$, then there exists an index $i \in \{1, \ldots, n\}$ such that $a \sim_{C_2} u_i$.

Observe that $n = 2$ would not work in Definition 6.5. While the minimum of n points $u_1, \ldots, u_n$ is the same as the minimum of the two points $\min(u_1, \ldots, u_{n-1})$ and u_n, the two-point version then only tells us that a would then be C_2-close to either u_n or $\min(u_1, \ldots, u_{n-1})$; however, in the latter case, a might not be C_2-close to any of $u_1, \ldots, u_{n-1}$.

Note that directly edge maximal points need slightly more care to deal with than their median counterparts; we demonstrate this in Theorem 6.9.

An interesting thing to consider is a more general notion of directly edge maximal, which we refer to as indirectly edge maximal points. They were introduced to guarantee the existence of a sufficient number of directly edge maximal points in specific situations, such as the one in the figure below.

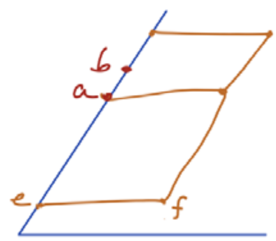


FIGURE 6.1: The motivation for indirectly edge maximal points.

Definition 6.6. Let X be a coarse median space and S be a δ-coarsely convex subset of $[x, y]$. Then a point $a \in S$ is said to be *indirectly edge maximal* with parameters C_1, C_2, D_1, D_2 if there exist $e, f \in S$ such that

- e is directly edge maximal in $[e, f]$ with parameters C_1, C_2;
- $\min(a, f) \sim_{D_1} e$;

- if there exists some $b \in S$ such that $\min(b, f) \sim_{D_1} e$, then $b \lesssim_{D_2} a$.

Although we introduce and define the concept of an indirectly edge maximal point, our focus in this chapter is to solely study the direct case, which we do from here on.

Remark 6.7. We denote the set of directly edge maximal points of a coarsely convex subset S of an interval $[x, y]$ by $\mathcal{M}_S$, which we call the *maximal edge subset associated to* S.

We prove our main result for coarse medians for directly edge maximal points; we show that the length of any antichain in the maximal edge subset associated to a coarsely convex subset S in a coarse interval $[x, y]$ is bounded above by the rank of S.

We will be using properties of the coarse iterated median operator as seen in Lemma 3.10. Call F_n the 'iterated $(n+3)$-point' constant and G_n the 'symmetry' constant associated to n points. In addition, we will also be making use of Theorem 3.5, which provides equivalent notions of rank in a coarse median space; note that this theorem also carries over to coarsely convex subsets, which we will apply below.

Note 6.8. Let X be a coarse median space. Then for any pair of points $a, b \in X$, a and b are said to be *M-incomparable*, for some constant M, if the following is satisfied:

- $\langle x, a, b \rangle \leftrightarrow_M a$;
- $\langle a, b, y \rangle \leftrightarrow_M b$;
- $\langle x, a, b \rangle \leftrightarrow_M b$;
- $\langle a, b, y \rangle \leftrightarrow_M a$.

Theorem 6.9. *Given a coarse median space X, coarse convexity constant δ, rank constant $C(\lambda)$, and iterated $(n+3)$-point and symmetry constants F and G, respectively, there exists $\lambda, E = C(\lambda), C_1$ such that for all C_2 there exists M such that the following holds. Let $[x, y] \subseteq X$ be a coarse interval and $S \subseteq [x, y]$ be a δ-coarsely convex subset with rank at most r with respect to $C(\lambda)$. Let n be an integer such that $1 \leq n \leq r+1$. Given points $u_1, \ldots, u_n \in S$ which are n M-incomparable (C_1, C_2)-directly edge maximal points (an M-coarse antichain), where $M = K(C_2 + \kappa_4) + 2H(0) + \kappa_4$, we obtain the following result. Define $v_i = \min(u_1, \ldots, \hat{u}_i, \ldots, u_n)$ for $i \in \{1, \ldots, n\}$ and $v_0 = \min(u_1, \ldots, u_n)$. Then the set $\{v_0, v_1, \ldots, v_n\}$ forms a $(\lambda, E) - n$-pod, where E is the non-triviality constant. Consequently, we have:*

$$\#\{u_1, \ldots, u_n\} \leq r,$$

which means that the size of any M-antichain, n, in S is bounded above by r, that is, $n \leq r$.

Proof. Having seen the proof when $[x, y]$ has rank at most 1 and 2 in the median algebra case, we concentrate on proving this theorem more generally for rank r. Fix i

and let $j \in \{1, \ldots, \hat{i}, \ldots, n\}$. Define $w_i = \max(u_i, v_i) = \langle u_i, v_i, y\rangle$ and $z_j = \max(u_i, u_j) = \langle u_i, u_j, y\rangle$.

We begin by showing that $v_0, v_1, \ldots, v_n$ genuinely forms a $\lambda - n$-pod for some λ to be determined; we use Theorem 3.5 to show this. Note that $v_0, v_1, \ldots, v_n \sim_\delta S$ by coarse convexity of S, as the v_i are constructed from the u_i which are elements of S. Therefore, we can now prove that betweenness holds:

Denote

$$t = \langle u_1, \ldots, \hat{u}_i, \hat{u}_j, \ldots, u_n; x\rangle.$$

Then we see that $v_i \sim_{G_r} \langle t, u_j, x\rangle$ and $v_j \sim_{G_r} \langle t, u_i, x\rangle$.

Next, we prove that $\langle v_i, v_j, x\rangle \sim \langle u_j, v_j, x\rangle$:

$$\begin{aligned}
\langle v_i, v_j, x\rangle &\sim_{2KG_r+H(0)} \langle\langle t, u_j, x\rangle, \langle t, u_i, x\rangle, x\rangle \\
&\sim_{F_3} \langle t, x, \langle u_j, u_i, x\rangle\rangle \\
&\sim_{\kappa_4} \langle\langle t, x, u_i\rangle, x, u_j\rangle \\
&\sim_{KG_r+H(0)} \langle v_j, x, u_j\rangle.
\end{aligned}$$

In addition, we also have

$$v_0 = \langle u_1, \ldots, u_n; x\rangle \sim_{G_{r+1}} \langle u_1, \ldots, \hat{u}_i, \ldots, u_n, u_i; x\rangle = \langle v_i, u_i, x\rangle.$$

Overall, we have shown that

$$\langle v_i, v_j, x\rangle \sim_{3KG_r+F_3+\kappa_4+2H(0)} \langle v_j, u_j, x\rangle \sim_{G_{r+1}} v_0.$$

Therefore, the betweenness condition holds as follows:

$$\begin{aligned}
\langle v_i, v_0, v_j\rangle &\sim_{KG_{r+1}+H(0)} \langle v_i, \langle v_j, u_j, x\rangle, v_j\rangle \\
&\sim_{\kappa_4} \langle\langle v_i, v_j, x\rangle, v_j, u_j\rangle \\
&\sim_{K(3KG_r+2H(0)+F_3+\kappa_4)+H(0)} \langle\langle v_j, u_j, x\rangle, v_j, u_j\rangle \\
&\sim_{\kappa_4} \langle v_j, u_j, x\rangle \\
&\sim_{G_{r+1}} v_0,
\end{aligned}$$

where $\lambda = (K+1)G_{r+1} + K(3KG_r + 2H(0) + F_3 + \kappa_4) + 2\kappa_4 + 2H(0)$.

We next prove a statement that we will use for the remainder of the proof, $w_i \sim \min(z_1, \ldots, \hat{z}_i, \ldots, z_n)$:

$$\begin{aligned}\min(z_1,\dots,\hat{z}_i,\dots,z_n) &= \langle\langle u_i,u_1,y\rangle,\dots,\langle u_i,u_n,y\rangle;x\rangle\\ &\sim_{F_{r+1}} \langle u_i,y,\langle u_1,\dots,\hat{u}_i,\dots,u_n;x\rangle\rangle\\ &= \langle u_i,y,v_i\rangle\\ &= w_i.\end{aligned}$$

Now, suppose towards a contradiction that $v_0 \sim_E v_i$; then this implies that

$$\begin{aligned}w_i &= \langle u_i,v_i,y\rangle\\ &\sim_{KE+H(0)} \langle u_i,v_0,y\rangle\\ &\sim_{KG_{r+1}+H(0)} \langle u_i,\langle v_i,u_i,x\rangle,y\rangle\\ &\sim_{\kappa_4} \langle\langle y,u_i,x\rangle,u_i,v_i\rangle\\ &\sim_{\kappa_4} u_i.\end{aligned}$$

Denote $E' = K(E+G_{r+1}) + 2H(0) + 2\kappa_4$. By M-incomparability (see Note 6.8) of the u_i, $u_i \not\leftrightarrow_M z_j$, that is, $u_i \not\leftrightarrow_M \langle u_i,u_j,y\rangle$.

By assumption, u_i is directly edge maximal in S with parameters (C_1,C_2). Observe that $u_i \sim_{E'} w_i \sim_{F_{r+1}} \min(z_1,\dots,\hat{z}_i,\dots,z_n)$, where $\min(z_1,\dots,\hat{z}_i,\dots,z_n) \in S$ by coarse convexity of S, as the z_i are constructed from elements of S. Choose $C_1 \geq E' + F_{r+1}$ — then we see that $u_i \sim_{C_1} \min(z_1,\dots,\hat{z}_i,\dots,z_n)$. Since u_i is directly edge maximal, $u_i \sim_{C_2} z_j$ for some $i \neq j$, i.e. $u_i \sim_{C_2} \langle u_i,u_j,y\rangle$. Note that we obtain $u_i \gtrsim_M u_j$, where $M = K(C_2+\kappa_4) + 2H(0) + \kappa_4$, by applying Lemma 4.7 with $A = C_2$. However, $u_i \sim_M \langle u_i,u_j,y\rangle$ is a contradiction to incomparability, as we assumed that the u_i are M-incomparable. Therefore, we must have $v_0 \not\leftrightarrow_E v_i$.

Furthermore, applying Theorem 3.5 gives us a contradiction to the rank of S, as $v_0 \not\leftrightarrow_E v_i$ implies that $\{v_0,v_1,\dots,v_n\}$ forms a non-trivial n-pod (or a non-trivial $r+1$-pod), meaning that the rank of S would have needed to be at least n (or $r+1$) initially. Therefore, one side of the $r+1$-pod $\{v_0,v_1,\dots,v_n\}$ must be trivial for S to have rank at most r, as required. □

6.2 Transitivity

6.2.1 Background

Again, suppose that X is a coarse median space with parameters $K, H(0), \kappa_4, \kappa_5$ and let $[x, y]$ be a finite rank coarse interval in X. Let S be a coarsely convex subset of $[x, y]$ with associated coarse convexity constant δ and rank $\leq r$ with respect to the rank constant $C(\lambda)$. Suppose that $\mathcal{M}_S \subseteq S$ is the maximal edge subset associated to S and consider any R-separated subset in $\mathcal{M}_S$ for some suitably large constant R, call it $\mathcal{Y}$.

Theorem 6.10. *Given a coarse median space X, coarse convexity constant δ and rank parameter $C(\lambda)$, there exists M (derived in the proof of Theorem 6.9) and R such that for any coarse interval $[x, y] \subseteq X$ and δ-coarsely convex subset $S \subseteq [x, y]$ with rank at most r with respect to $C(\lambda)$, we have the following. Let x be the basepoint for the relation $\lesssim_M$. Then, for all R-separated subsets $\mathcal{Y} \subseteq \mathcal{M}_S$, the relation $\lesssim_M$ has no loops of distinct points in $\mathcal{Y}$. That is, the transitive closure of $\lesssim_M$ is anti-symmetric on $\mathcal{Y}$.*

Proof. Observe that $M = K(C_2 + \kappa_4) + 2H(0) + \kappa_4$, as derived from Theorem 6.9. Now, suppose we have a loop of distinct points of size at most $2r + 1$ in $\mathcal{Y}$ — then by Lemma 4.5, $a_1 \lesssim_C a_{2r+1}$, where C is the constant constructed by repeated application of Lemma 4.5 and thus depends on M. We also have $a_{2r+1} \lesssim_M a_1$ as we assumed that we have a loop of size at most $2r + 1$. Putting these two inequalities together, we obtain $a_1 \sim_{C+M} a_{2r+1}$. Therefore, choosing $R \geq M + C$ implies that $a_i \sim_R a_j$; hence, we have chosen the separation constant R to be large enough such that there can be no loops of size at most $2r + 1$.

Suppose we have a loop

$$a_1 \lesssim_M a_2 \ldots \lesssim_M a_k \lesssim_M a_1,$$

where $a_i \in \mathcal{Y}$ and are distinct; we know that $k > 2r + 1$, as we have shown above that by our choice of R, there can be no loops of size at most $2r + 1$. Assume that a_i is M-incomparable to a_j, where $|i - j| > 1$; focusing on the 'worst case scenario', this then implies that a_{2i+1} is incomparable with a_{2j+1}, where $i, j \in \{0, \ldots, r\}$ and $i \neq j$. However, this gives us at least $r + 1$ incomparable points, an antichain, and by Theorem 6.9, we cannot have more than r M-incomparable points. This means that in the set $\{a_1, a_3, \ldots, a_{2r+1}\}$, not all of the elements are incomparable and so there must be comparable points within the set, say a_{2i+1} and a_{2j+1} are comparable. Zooming in on our original loop above, we have the sub-sequence

$$a_{2i+1} \lesssim_M \cdots \lesssim_M a_{2j+1}.$$

Comparability now implies two things: on one hand, we can have $a_{2j+1} \lesssim_M a_{2i+1}$, which means that we have a loop; however, this is a loop of length at most $2r + 1$ and

by our choice of R, this means that $a_{2i+1} \sim_R a_{2j+1}$ which implies that $a_{2i+1} = a_{2j+1}$. Therefore, we can have no loops in this scenario.
On the other hand, we can have $a_{2i+1} \lesssim_M a_{2j+1}$; this indicates that we can take a 'shortcut' and shorten the original sequence,

$$a_1 \lesssim_M \dots a_{2i+1} \lesssim_M a_{2j+1} \dots .$$

If this loop now has length at most $2r+1$, then we are done by our choice of R, as we know there are no loops at most this length. If our loop still has length greater than $2r+1$, then we can repeatedly apply shortcuts and shorten the loop until it has length at most $2r+1$ and then we are done.

Hence, there are no loops of distinct points in Y. □

Observe that, by Lemma 4.5, given

$$a_1 \lesssim_M a_2 \dots \lesssim_M a_{2r+1},$$

we obtain $a_1 \lesssim_C a_{2r+1}$, where C is the constant obtained from the proof of Theorem 6.10. Now, define the transitive closure of $\lesssim_M$ on $\mathcal{Y}$ to be $\lesssim_{M^+} = (\lesssim_M)^{2r}$. The transitive closure is, by definition, transitive, whereas for $(\lesssim_M)^{2r}$, this is a consequence of what is proved above, namely that we cannot have a chain of $2r+1$ points satisfying $\lesssim_M$ without there being some 'shortcuts'. Hence, $(\lesssim_M)^{2r}$ is the same as $(\lesssim_M)^{2r+k}$ for any $k \geq 0$. Moreover, if $a \lesssim_{M^+} b$, then this implies that $a \lesssim_C b$.

Reflexivity is satisfied by the original relation $\lesssim_M$, transitivity is now satisfied for $\lesssim_{M^+}$ (by definition of the transitive closure) and we have shown via Theorem 6.10 that anti-symmetry also holds — thus, we can conclude that $(\mathcal{Y}, \lesssim_{M^+})$ forms a partially ordered set. By Theorem 6.9, we know that the size of any antichain must be bounded above by r, the rank of S; note that if we have an antichain in the transitive closure then it is also an antichain for the original ordering. Applying Dilworth's Lemma (Theorem 6.2), we can now rewrite $\mathcal{Y}$ as a union of chains: $\mathcal{Y} = \bigcup_{i=1}^{r} \mathcal{C}_i$.

Remark 6.11. Let X be a coarse median space and let $\{\mathcal{C}_i\}_{i=1}^r$ be a collection of chains as above, where each $\mathcal{C}_i$ is totally ordered by the relation $\leq$. Note that each chain, $\mathcal{C}_i$, possesses a median structure as well as its inherited coarse median structure from X. On each $\mathcal{C}_i$, we have the following median structure: for each $i \in \{1, \dots, r\}$, given $a, b, c \in \mathcal{C}_i$, the total ordering on $\mathcal{C}_i$ implies, without loss of generality, that $a \leq b \leq c$. Therefore, define $\langle a, b, c \rangle_{\mathcal{C}_i} = b$.

We have the following maps:

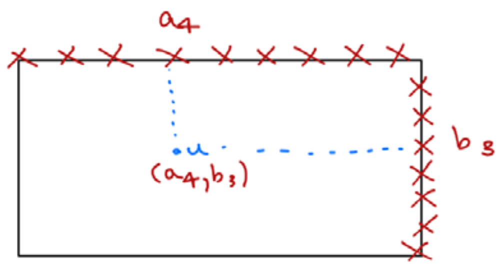


FIGURE 6.2: An intuitive view of the forwards map in the rank 2 case.

Definition 6.12 ((Forwards map)). Let X be a coarse median space and $\{\mathcal{C}_i\}_{i=1}^r$ be a collection of chains as defined above. Fix a constant C' (that will be determined soon). For each $i \in \{1, \ldots, r\}$, define a function $f_i \colon [x, y] \to \mathcal{C}_i$ as follows:

for each $u \in [x, y]$, there exists a point $f_i(u)$, given by the least point $a \in \mathcal{C}_i$ such that $u \lesssim_{C'} a$. Now, define a function

$$f \colon [x, y] \to \prod_{i=1}^{r} \mathcal{C}_i \qquad \text{by} \qquad f \colon u \mapsto (f_i(u))_{i=1,\ldots,r}.$$

In addition, $\prod_{i=1}^r \mathcal{C}_i$ is isometrically embedded into $\mathbb{Z}^r$ via the metric $d_{\mathcal{C}_i}$ with product median structure inherited from $\mathcal{C}_i$. Given $a, b \in \mathcal{C}_i$, the metric $d_{\mathcal{C}_i}$ is defined to be

$$d_{\mathcal{C}_i}(a, b) := \#[a, b]_{\mathcal{C}_i} - 1.$$

Informally, the $d_{\mathcal{C}_i}$ metric counts the number of steps from a to b.

Definition 6.13 ((Backwards map)). Let X be a coarse median space and $\{\mathcal{C}_i\}_{i=1}^r$ be a collection of chains as defined earlier. Define a function

$$g \colon \prod_{i=1}^{r} \mathcal{C}_i \to [x, y],$$

where, for all $(a_1, \ldots, a_r) \in \prod_{i=1}^r \mathcal{C}_i$, we define:

$$\begin{aligned} g((a_1, \ldots, a_r)) &= \min(a_1, \ldots, a_r) \\ &= \langle a_1, \ldots, a_r; x \rangle. \end{aligned}$$

Remark 6.14. For all constants R (defined in the proof of Theorem 6.10) and R-separated subsets $\mathcal{Y} \subseteq \mathcal{M}_S$, let $\mathcal{C}_i$ be a chain for each $i \in \{1, \ldots, r\}$, where $\mathcal{C}_i \in \mathcal{Y}$. Since each chain $\mathcal{C}_i$ sits in the original coarse median space X, it also carries a coarse median structure (as noted in Remark 6.11). We now show that each chain is a rank 1 piece in S.

Claim Let $\mathcal{C}_i \in \mathcal{Y}$ be a chain for each $i \in \{1, \ldots, r\}$ and $\lesssim_{M^+}$ be the partial ordering on $\mathcal{Y}$ (as defined earlier). Given $a, b, c \in \mathcal{C}_i$ with $a \lesssim_{M^+} b \lesssim_{M^+} c$, we obtain $b \sim \langle a, b, c \rangle$. Hence, for each i, C_i is a coarsely-median closed subset of X.

Proof. Since $a \lesssim_{M^+} b \lesssim_{M^+} c$, this also implies that $a \lesssim_C b \lesssim_C c$ (where C is defined in the proof of Theorem 6.10). We know that $a \sim_C \langle x, a, b \rangle$ and $b \sim_C \langle x, b, c \rangle$. Thus

$$\begin{aligned} b &\sim_C \langle x, b, c \rangle \\ &\sim_{KC+H(0)} \langle x, \langle a, b, y \rangle, c \rangle \\ &\sim_{\kappa_5} \langle \langle x, b, c \rangle, \langle x, y, c \rangle, a \rangle \\ &\sim_{K(C+\kappa_4)+H(0)} \langle b, c, a \rangle, \end{aligned}$$

as required. ☐

We now direct our attention to showing that f is actually a quasi-morphism — proving the following proposition will be a stepping stone towards proving this.

Proposition 6.15. *Given a coarse median space X, rank constant $C(\lambda)$ and constants δ, R, let $[x, y] \subseteq X$ be a coarse interval and let $S \subseteq [x, y]$ be a δ-coarsely convex subset with rank at most r with respect to $C(\lambda)$. For any R-separated subset $\mathcal{Y} \subseteq \mathcal{M}_S$, suppose $\mathcal{C}_i$ is a chain for each $i \in \{1, \ldots, r\}$, where $\mathcal{C}_i \in \mathcal{Y} \subseteq S$. Suppose $\lesssim_{M^+}$ is the partial ordering on $\mathcal{Y}$ (as defined above). Recall the function f_i and the metric $d_{\mathcal{C}_i}$ as defined in Definition 6.12. Let $a_1, a_2 \in \mathcal{C}_i$, where $a_1 \lesssim_{M+} a_2$, and let $h_1, h_2 \in S$, where $a_1 = f_i(h_1), a_2 = f_i(h_2)$. Then*

1. $f_i(\langle h_1, h_2, y \rangle) \sim a_2$;
2. $f_i(\langle x, h_1, h_2 \rangle) \sim a_1$,

where $\sim$ indicates close with respect to the metric $d_{\mathcal{C}_i}$.

To prove this proposition, we first need to show that transitivity 'almost' holds between elements of $\mathcal{Y}$ and S; we show this via the following lemma.

6.2.2 Proof of Transitivity

Lemma 6.16. *Given a coarse median space X, rank constant $C(\lambda)$ and constants δ and L, there exists P such that the following holds. For any coarse interval $[x, y] \subseteq X$, let $S \subseteq [x, y]$ be a δ-coarsely convex subset with rank at most r with respect to $C(\lambda)$. Suppose $\mathcal{C}_i$ is a chain for each $i \in \{1, \ldots, r\}$, with $\mathcal{C}_i \in \mathcal{Y} \subseteq S$, and that $\lesssim_{M^+}$ is the partial ordering on $\mathcal{Y}$ (as defined earlier). Recall the function f_i and the metric $d_{\mathcal{C}_i}$ as defined in Definition 6.12. Then for any $u \in S, a_1, a_2 \in \mathcal{C}_i$, if $u \lesssim_L a_1 \lesssim_{M^+} a_2 := f_i(u)$, this implies that $d_{\mathcal{C}_i}(a_1, a_2) \leq P$.*

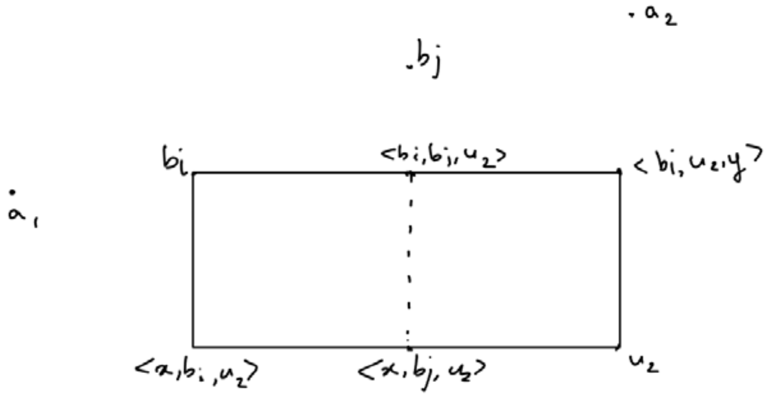


FIGURE 6.3: A visual representation of Lemma 6.16.

Proof. We have the following facts:

- $a_1 \lesssim_C b_1 \lesssim_C \ldots \lesssim_C b_k \lesssim_C a_2$, where $a_1, a_2, b_i \in \mathcal{Y}$ for $i \in \{1, \ldots, k\}$.
- C is the ordering error solely associated to elements of $\mathcal{Y}$ (derived in the proof of Theorem 6.10).
- C' is the ordering error associated to elements of S (introduced in Definition 6.12).
- The b_is are R-separated, i.e. $b_i \sim_R b_j$.
- $u \sim_{C'} \langle a_2, u, x\rangle$ (since $a_2 = f_i(u)$).
- $b_i \sim_C \langle b_i, a_2, x\rangle$.

The first bullet point follows as the transitive closure M^+ says that if $a_1 \lesssim_{M^+} b_1 \lesssim_{M^+} \ldots \lesssim_{M^+} b_k \lesssim_{M^+} a_2$, then this implies that $a_1 \lesssim_C b_1 \lesssim_C \ldots \lesssim_C b_k \lesssim_C a_2$. The last point is due to the fact that if $a_1 \lesssim_{M^+} b_1 \lesssim_{M^+} \ldots \lesssim_{M^+} b_k \lesssim_{M^+} a_2$, then this suggests that $b_i \lesssim_{M+} a_2$, which in turn tells us that $b_i \lesssim_C a_2$.

The key idea in this proof lies in showing that $\langle x, b_i, u\rangle \neq \langle x, b_j, u\rangle$. Thus, assume towards a contradiction that $\langle x, b_i, u\rangle = \langle x, b_j, u\rangle$. We will show that the three following statements hold, which will lead to a contradiction:

- $b_i \sim \langle b_i, b_j, \langle b_i, u, y\rangle\rangle$;
- b_i is 'far' from b_j;
- b_i is 'far' from $\langle b_i, u, y\rangle$.

The second point holds as we already know that the b_is are R-separated, i.e. $b_i \nleftrightarrow_R b_j$. We thus turn our attention to proving that the remaining two points hold.

Focusing on the first point, $b_i \sim \langle b_i, b_j, \langle b_i, u, y\rangle\rangle$, we begin by showing that $\langle\langle x, b_i, u\rangle, b_i, \langle b_i, u, y\rangle\rangle \sim b_i$:

$$\begin{aligned}\langle\langle x, b_i, u\rangle, b_i, \langle b_i, u, y\rangle\rangle &\sim_{\kappa_5} \langle b_i, u, \langle x, b_i, y\rangle\rangle \\ &\sim_{K\kappa_4+H(0)} \langle b_i, u, b_i\rangle \\ &= b_i.\end{aligned}$$

We now prove that $\langle\langle x, b_i, u\rangle, b_i, \langle b_i, u, y\rangle\rangle \sim \langle b_i, b_j, \langle b_i, u, y\rangle\rangle$:

$$\begin{aligned}\langle\langle x, b_i, u\rangle, b_i, \langle b_i, u, y\rangle\rangle &= \langle b_i, \langle x, b_j, u\rangle, \langle b_i, u, y\rangle\rangle \\ &\sim_{K\kappa_4+H(0)} \langle\langle b_i, x, y\rangle, \langle x, b_j, u\rangle, \langle b_i, u, y\rangle\rangle \\ &\sim_{\kappa_5} \langle b_i, y\langle x, u, \langle b_j, x, u\rangle\rangle\rangle \\ &\sim_{K\kappa_4+H(0)} \langle b_i, y\langle x, u, b_j\rangle\rangle \\ &\sim_{\kappa_5} \langle\langle x, b_i, y\rangle, \langle u, b_i, y\rangle, b_j\rangle \\ &\sim_{K\kappa_4+H(0)} \langle b_i, b_j, \langle b_i, u, y\rangle\rangle.\end{aligned}$$

Combining the two calculations above, we can now conclude that $b_i \sim_{K\kappa_4+H(0)+\kappa_5} \langle\langle x, b_i, u\rangle, b_i, \langle b_i, u, y\rangle\rangle = \langle\langle x, b_j, u\rangle, \langle b_i, u, y\rangle, b_i\rangle \sim_{3(K\kappa_4+H(0))+2\kappa_5} \langle b_i, b_j, \langle b_i, u, y\rangle\rangle$, i.e. $b_i \sim_{4(K\kappa_4+H(0))+3\kappa_5} \langle b_i, b_j, \langle b_i, u, y\rangle\rangle$, proving that the first points holds.

Now we focus on proving the last point holds, b_i is 'far' from $\langle b_i, u, y\rangle$. Observe that $d(\langle x, b_i, u\rangle, u)$ must be greater than C', as if $\langle x, b_i, u\rangle \sim_{C'} u$, this implies that $u \lesssim_{C'} b_i$; this contradicts the fact that $a_2 = f_i(u)$ and $a_2 \gtrsim_C b_i$. Thus we must have $d(\langle x, b_i, u\rangle, u) > C'$.

We now show that $\{b_i, \langle b_i, y, u\rangle, \langle b_i, x, u\rangle, u\}$ forms a coarse square:

(i) $\langle b_i, \langle x, b_i, u\rangle, u\rangle \sim_{\kappa_4} \langle x, b_i, u\rangle$.

(ii) $\langle b_i, \langle b_i, u, y\rangle, u\rangle \sim_{\kappa_4} \langle b_i, u, y\rangle$.

(iii)

$$\begin{aligned}\langle\langle x, b_i, u\rangle, b_i, \langle b_i, u, y\rangle\rangle &\sim_{\kappa_4} \langle b_i, u, \langle x, b_i, y\rangle\rangle \\ &\sim_{K\kappa_4+H(0)} \langle b_i, u, b_i\rangle \\ &= b_i.\end{aligned}$$

(iv)

$$\begin{aligned}\langle\langle x,b_i,u\rangle,u,\langle b_i,u,y\rangle\rangle &\sim_{\kappa_4} \langle b_i,u,\langle x,u,y\rangle\rangle\\ &\sim_{K\kappa_4+H(0)} \langle b_i,u,u\rangle\\ &= u.\end{aligned}$$

We have now shown that $\{b_i, \langle b_i,u,y\rangle, \langle x,b_i,u\rangle, u\}$ forms a $(K\kappa_4 + H(0) + \kappa_4)$-coarse square, so if

$$b_i \sim_{C_2} \langle b_i,u,y\rangle \implies u \sim_{C'} \langle x,b_i,u\rangle,$$

where $C' \geq K(C_2 + \kappa_4) + 2H(0) + \kappa_4$ (by Lemma 4.7).

The statement above is equivalent to the contrapositive:

$$\underbrace{d(u,\langle x,b_i,u\rangle) > C'}_{\text{we know this has to hold}} \implies d(b_i,\langle b_i,u,y\rangle) > C_2.$$

Therefore, we have shown that the third point holds true. Bringing all three points together, we have proven that $\{b_i, b_j, \langle b_i,u,y\rangle\}$ forms a non-trivial bipod centred at b_i. However, this is a contradiction to maximality of b_i, and therefore our initial assumption, $\langle x,b_i,u\rangle = \langle x,b_j,u\rangle$, was incorrect. Hence, we must actually have that $\langle x,b_i,u\rangle \neq \langle x,b_j,u\rangle$ for all $i \neq j$.

Lastly, having obtained a lower bound of C' for $d(u,\langle b_i,x,u\rangle)$, we now obtain an upper bound:

$$\begin{aligned}\langle x,b_i,u\rangle &\sim_{KC+H(0)} \langle x,\langle b_i,a_1,y\rangle,u\rangle\\ &\sim_{\kappa_5} \langle\langle x,u,a_1\rangle,\langle x,u,y\rangle,b_i\rangle\\ &\sim_{K(L+\kappa_4)+H(0)} \langle u,u,b_i\rangle\\ &= u.\end{aligned}$$

Therefore, $d(\langle x,b_i,u\rangle,u)$ is bounded, implying that there is a bounded (and thus finite) number P of $\langle x,b_i,u\rangle$s and thus b_is, and so $d(a_1,a_2)$ is bounded under d_{C_i}. More precisely, we use bounded geometry, as demonstrated below:

$$\begin{aligned} P &\geq \#B_{K(C+L+\kappa_4)+\kappa_5+2H(0)}(u) \\ &\geq \#\{\langle b_1, x, u\rangle, \ldots, \langle b_k, x, u\rangle\} \\ &= \#\{b_1, b_2, \ldots, b_k\} \\ &= k \\ &= d_{\mathcal{C}_i}(a_1, a_2) - 1. \end{aligned}$$

□

6.2.3 Application of Transitivity

We can now focus on proving Proposition 6.15.

Proof of Proposition 6.15. Before we launch into the proof, we set the following convention: $a \gtrsim_C b \iff b \lesssim_C a$. Note that a_1 is the least upper bound for h_1 and a_2 is the least upper bound for h_2.

1. Observe that $\langle h_1, h_2, y\rangle \gtrsim_A h_1$ and $\langle h_1, h_2, y\rangle \gtrsim_A h_2$ by Lemma 4.6, where $A = (K+1)\kappa_4 + H(0)$. We know that $a_2 \gtrsim_{C'} h_2$ and that for all $c \in \mathcal{C}_i$ that is an upper bound, i.e. for all $c \in \mathcal{C}_i, c \gtrsim_{C'} h_2 \implies a_2 \lesssim_{M^+} c \implies a_2 \lesssim_C c$.

   Thus,
   $$a_2 \gtrsim_{M^+} a_1 \gtrsim_{C'} h_1 \implies a_2 \gtrsim_B h_1$$
   by Lemma 4.5, where $B = K(C' + M^+) + 2H(0) + \kappa_4 + C'$.

   Need to show:

   (i) $a_2 \gtrsim_L \langle h_1, h_2, y\rangle$ for some constant L:

   $$\begin{aligned} \langle x, a_2, \langle h_1, h_2, y\rangle\rangle &\sim_{\kappa_5} \langle\langle x, a_2, h_1\rangle, \langle x, a_2, h_2\rangle, y\rangle \\ &\sim_{K(B+C')+H(0)} \langle h_1, h_2, y\rangle. \end{aligned}$$

   Hence, $L = K(B + C') + H(0) + \kappa_5$.

   (ii) a_2 is coarsely an upper bound for $\langle h_1, h_2, y\rangle$: suppose $f_i(h_2) = a_2 \lesssim_{M^+} f_i(\langle h_1, h_2, y\rangle)$. Then

   $$\langle h_1, h_2, y\rangle \lesssim_L a_2 \lesssim_{M^+} f_i(\langle h_1, h_2, y\rangle) \implies d_{\mathcal{C}_i}(a_2, f_i(\langle h_1, h_2, y\rangle)) \leq P,$$

   i.e. $f_i(\langle h_1, h_2, y\rangle) \sim_P a_2$, by applying Lemma 6.16.

(iii) a_2 is coarsely the least upper bound for $\langle h_1, h_2, y\rangle$: suppose $f_i(\langle h_1, h_2, y\rangle) \lesssim_{M^+} f_i(h_2) = a_2$. Then

$$h_2 \lesssim_A \langle h_1, h_2, y\rangle \lesssim_{C'} f_i(\langle h_1, h_2, y\rangle) \lesssim_{M^+} f_i(h_2) = a_2.$$

We have $h_2 \lesssim_A \langle h_1, h_2, y\rangle \lesssim_{C'} f_i(\langle h_1, h_2, y\rangle)$, and so by application of Lemma 4.5, we obtain $h_2 \lesssim_{L'} f_i(\langle h_1, h_2, y\rangle)$, where $L' = K(A + C') + 2H(0) + \kappa_4 + A$. Now,

$$h_2 \lesssim_{L'} f_i(\langle h_1, h_2, y\rangle) \lesssim_{M^+} f_i(h_2) = a_2 \implies d_{\mathcal{C}_i}(f_i(\langle h_1, h_2, y\rangle), f_i(h_2)) \leq P',$$

i.e. $f_i(\langle h_1, h_2, y\rangle) \sim_{P'} f_i(h_2) = a_2$, by applying Lemma 6.16.

Therefore, overall we obtain $d_{\mathcal{C}_i}(f_i(\langle h_1, h_2, y\rangle), f_i(h_2)) \leq \max(P, P')$, i.e. $f_i(\langle h_1, h_2, y\rangle) \sim_{\max(P,P')} f_i(h_2) = a_2$.

2. Similarly, observe that $\langle x, h_1, h_2\rangle \lesssim_A h_1$ and $\langle x, h_1, h_2\rangle \lesssim_A h_2$ by Lemma 4.6, where $A = (K+1)\kappa_4 + H(0)$. We again know that $a_1 \gtrsim_{C'} h_1$ and that for all $d \in \mathcal{C}_i, d \gtrsim_{C'} h_1 \implies a_1 \lesssim_{M^+} d \implies a_1 \lesssim_C d$.

<u>Need to show:</u>

(i) $a_1 \gtrsim_U \langle x, h_1, h_2\rangle$ for some constant U:

$$\begin{aligned}\langle x, a_1, \langle x, h_1, h_2\rangle\rangle &\sim_{\kappa_4} \langle\langle x, a_1, h_1\rangle, x, h_2\rangle \\ &\sim_{KC'+H(0)} \langle h_1, x, h_2\rangle.\end{aligned}$$

Thus, $U = KC' + H(0) + \kappa_4$.

(ii) a_1 is coarsely an upper bound for $\langle x, h_1, h_2\rangle$: suppose $f_i(h_1) = a_1 \lesssim_{M^+} f_i(\langle x, h_1, h_2\rangle)$. Then

$$\langle x, h_1, h_2\rangle \lesssim_U a_1 \lesssim_{M^+} f_i(\langle x, h_1, h_2\rangle) \implies d_{\mathcal{C}_i}(a_1, f_i(\langle x, h_1, h_2\rangle)) \leq Q,$$

i.e. $f_i(\langle x, h_1, h_2\rangle) \sim_Q a_1$, by applying Lemma 6.16.

(iii) a_1 is coarsely the least upper bound for $\langle x, h_1, h_2\rangle$: suppose $f_i(\langle x, h_1, h_2\rangle) \lesssim_{M^+} f_i(h_1) = a_1$. Let $b = f_i(\langle x, h_1, h_2\rangle)$. If we show that $b \gtrsim_{U'} h_1$ or $b \gtrsim_{U'} h_2$, then this will suffice to prove that $b \sim a_1$ in the $d_{\mathcal{C}_i}$ metric.

Since $\langle x, h_1, h_2\rangle \lesssim_{C'} b$, this is equivalent to $\langle x, \langle x, h_1, h_2\rangle, b\rangle \sim_{C'} \langle x, h_1, h_2\rangle$ and $\langle\langle x, h_1, h_2\rangle, b, y\rangle \sim_{C''} b$, where C'' is the constant derived from Lemma 4.7.

Let us consider $\{b, \langle b, h_1, y\rangle, \langle b, h_2, y\rangle\}$ — we show that this set forms a non-trivial bipod. Note that $b, h_1, h_2 \sim_\delta S$ by coarse convexity of S; we now check that $\{b, \langle b, h_1, y\rangle, \langle b, h_2, y\rangle\}$ genuinely forms a bipod centred at b.

$$
\begin{aligned}
b &\sim_{C''} \langle b, y, \langle x, h_1, h_2 \rangle \rangle \\
&\sim_{K\kappa_4+H(0)} \langle b, y, \langle h_1, h_2, \langle x, h_1, h_2 \rangle \rangle \rangle \\
&\sim_{K(K\kappa_4+H(0))+H(0)} \langle b, y, \langle b, y, \langle h_1, h_2, \langle x, h_1, h_2 \rangle \rangle \rangle \rangle \\
&\sim_{K\kappa_5+H(0)} \langle b, y, \langle \langle b, y, h_1 \rangle, \langle b, y, h_2 \rangle, \langle x, h_1, h_2 \rangle \rangle \rangle \\
&\sim_{\kappa_5} \langle \langle b, y, \langle b, y, h_1 \rangle \rangle, \langle b, y, \langle x, h_1, h_2 \rangle \rangle, \langle b, y, h_2 \rangle \rangle \\
&\sim_{K(\kappa_4+C'')+H(0)} \langle \langle b, y, h_1 \rangle, b, \langle b, y, h_2 \rangle \rangle.
\end{aligned}
$$

Thus, we see that $\{b, \langle b, h_1, y \rangle, \langle b, h_2, y \rangle\}$ forms a non-trivial bipod. However, $b \in \mathcal{C}_i$ implies that b is directly edge maximal, and so one side of the bipod must be trivial. This means that either $b \sim_{U'} \langle b, h_1, y \rangle$ or $b \sim_{U'} \langle b, h_2, y \rangle$.

If $b \sim_{U'} \langle b, h_1, y \rangle$, then this is equivalent to $b \gtrsim_{U'} h_1$. Hence,

$$h_1 \lesssim_{U'} b \lesssim_{M^+} a_1 = f_i(h_1) \implies d_{\mathcal{C}_i}(b, a_1) \leq Q',$$

i.e. $f_i(\langle x, h_1, h_2 \rangle) \sim_{Q'} a_1$, by applying Lemma 6.16.

If $b \sim_{U'} \langle b, h_2, y \rangle$, then this is equivalent to $b \gtrsim_{U'} h_2$; but $b \in \mathcal{C}_i$, and from the proof of the first statement, 1., above, this implies the following

$$h_2 \lesssim_{U'} b \lesssim_{M^+} a_1 \lesssim_{M^+} a_2 = f_i(h_2).$$

Since $\lesssim_{M^+}$ is genuinely transitive, we can simplify the above sequence:

$$h_2 \lesssim_{U'} b \lesssim_{M^+} a_2 \implies d_{\mathcal{C}_i}(b, a_2) \leq Q' \implies d_{\mathcal{C}_i}(b, a_1) \leq Q',$$

i.e. $f_i(\langle x, h_1, h_2 \rangle) \sim_{Q'} a_1$, by applying Lemma 6.16.

Therefore, overall we obtain $d_{\mathcal{C}_i}(f_i(\langle h_1, h_2, y \rangle), f_i(h_2)) \leq \max(Q, Q')$, i.e. $f_i(\langle h_1, h_2, y \rangle) \sim_{\max(Q,Q')} f_i(h_2) = a_2$.

□

We now turn our attention to showing that f_i is a quasi-morphism.

Theorem 6.17. *Given a coarse median space X, rank constant $C(\lambda)$ and coarse convexity constant δ, let $[x, y] \subseteq X$ be a coarse interval and let S be a δ-coarsely convex subset with rank at most r with respect to $C(\lambda)$. Suppose $\mathcal{C}_i$ is a chain for each $i \in \{1, \ldots, r\}$, with $\mathcal{C}_i \in \mathcal{Y} \subseteq S$. Recall the function f_i and the metric $d_{\mathcal{C}_i}$ as defined in Definition 6.12. Given $j \in \{1, 2, 3\}$, let $h_j \in S$ and $f_i(h_j) = a_j$, where $a_j \in \mathcal{C}_i$. Then f_i is a quasi-morphism,*

$$f_i(\langle h_1, h_2, h_3 \rangle) \sim \langle f_i(h_1), f_i(h_2), f_i(h_3) \rangle_{\mathcal{C}_i},$$

where $\sim$ *indicates close with respect to the metric* $d_{\mathcal{C}_i}$ *and* $\langle\rangle_{\mathcal{C}_i}$ *is defined in Remark 6.11.*

Proof. We show that $f_i(\langle h_1, h_2, h_3\rangle) \sim a_2$.

We again use the following convention: $a \gtrsim_C b \iff b \lesssim_C a$. Note that a_1 is the least upper bound for h_1, a_2 is the least upper bound for h_2 and a_3 is the least upper bound for h_3. From Proposition 6.15, we also know that $f_i(\langle h_1, h_2, y\rangle) \sim_{\max(P,P')} a_2$ and that $f_i(\langle x, h_1, h_2\rangle) \sim_{\max(Q,Q')} a_1$ in the $d_{\mathcal{C}_i}$ metric.

Need to show:

(i) $a_2 \gtrsim_J \langle h_1, h_2, h_3\rangle$ for some constant J:

$$\begin{aligned}\langle\langle h_1, h_2, h_3\rangle, \langle h_1, h_2, y\rangle, y\rangle &\sim_{\kappa_5} \langle h_1, h_2, \langle h_3, y, y\rangle\rangle \\ &= \langle h_1, h_2, y\rangle.\end{aligned}$$

Hence, $\langle h_1, h_2, h_3\rangle \lesssim_{\kappa_5} \langle h_1, h_2, y\rangle$. By the proof of Proposition 6.15(1i), we know that $\langle h_1, h_2, y\rangle \lesssim_L a_2$, where L is determined in the proof. Thus, we have

$$\langle h_1, h_2, h_3\rangle \lesssim_{\kappa_5} \langle h_1, h_2, y\rangle \lesssim_L a_2,$$

and so $\langle h_1, h_2, h_3\rangle \lesssim_J a_2$, where $J = K(\kappa_5 + L) + 2H(0) + \kappa_4 + \kappa_5$ by applying Lemma 4.5.

(ii) a_2 is coarsely an upper bound for $\langle h_1, h_2, h_3\rangle$: suppose $f_i(h_2) = a_2 \lesssim_{M^+} f_i(\langle h_1, h_2, h_3\rangle)$. Then

$$\langle h_1, h_2, h_3\rangle \lesssim_J a_2 \lesssim_{M^+} f_i(\langle h_1, h_2, h_3\rangle) \implies d_{\mathcal{C}_i}(a_2, f_i(\langle h_1, h_2, h_3\rangle)) \leq T,$$

i.e. $f_i(\langle h_1, h_2, h_3\rangle) \sim_T a_2$, by applying Lemma 6.16.

(iii) a_2 is coarsely the least upper bound for $\langle h_1, h_2, h_3\rangle$: suppose $f_i(\langle h_1, h_2, h_3\rangle) \lesssim_{M^+} f_i(h_2) = a_2$. Let $e = f_i(\langle h_1, h_2, h_3\rangle)$. If we show that $e \gtrsim_{J'} h_2$ or $e \gtrsim_{J'} h_3$, then this will suffice to prove that $e \sim a_2$ in the $d_{\mathcal{C}_i}$ metric.

Since $\langle h_1, h_2, h_3\rangle \lesssim_{C'} e$, this is equivalent to $\langle x, \langle h_1, h_2, h_3\rangle, e\rangle \sim_{C'} \langle h_1, h_2, h_3\rangle$ and $\langle\langle h_1, h_2, h_3\rangle, e, y\rangle \sim_{C''} e$, where C'' is the constant derived from Lemma 4.7.

Let us consider $\{e, \langle e, h_2, y\rangle, \langle e, h_3, y\rangle\}$ — we show that this set forms a non-trivial bipod. Note that $e, h_2, h_3 \sim_\delta S$ by coarse convexity of S; we now check that $\{e, \langle e, h_2, y\rangle, \langle e, h_3, y\rangle\}$ genuinely forms a non-trivial bipod centred at e.

$$\begin{aligned} e &\sim_{C''} \langle e, y, \langle h_1, h_2, h_3 \rangle\rangle \\ &\sim_{K\kappa_4 + H(0)} \langle e, y, \langle h_2, h_3, \langle h_1, h_2, h_3 \rangle\rangle\rangle \\ &\sim_{K(K\kappa_4 + H(0)) + H(0)} \langle e, y, \langle e, y, \langle h_2, h_3, \langle h_1, h_2, h_3 \rangle\rangle\rangle\rangle \\ &\sim_{K\kappa_5 + H(0)} \langle e, y, \langle \langle e, y, h_2 \rangle, \langle e, y, h_3 \rangle, \langle h_1, h_2, h_3 \rangle\rangle\rangle \\ &\sim_{\kappa_5} \langle \langle e, y, \langle e, y, h_2 \rangle\rangle, \langle e, y, \langle h_1, h_2, h_3 \rangle\rangle, \langle e, y, h_3 \rangle\rangle \\ &\sim_{K(\kappa_4 + C'') + H(0)} \langle \langle e, y, h_2 \rangle, e, \langle e, y, h_3 \rangle\rangle. \end{aligned}$$

Thus, we have shown that $\{e, \langle e, h_2, y\rangle, \langle e, h_3, y\rangle\}$ forms a non-trivial bipod. However, $e \in \mathcal{C}_i$ implies that e is directly edge maximal, and so one side of the bipod must be trivial. This means that either $e \sim_{J'} \langle e, h_2, y\rangle$ or $e \sim_{J'} \langle e, h_3, y\rangle$.

If $e \sim_{J'} \langle e, h_2, y\rangle$, then this is equivalent to $e \gtrsim_{J'} h_2$. Hence,

$$h_2 \lesssim_{J'} e \lesssim_{M^+} a_2 = f_i(h_2) \implies d_{\mathcal{C}_i}(e, a_2) \leq T',$$

i.e. $f_i(\langle h_1, h_2, h_3\rangle) \sim_{T'} a_2$, by applying Lemma 6.16.

If $e \sim_{J'} \langle e, h_3, y\rangle$, then this is equivalent to $e \gtrsim_{J'} h_3$; but $e \in \mathcal{C}_i$, so this implies the following

$$h_3 \lesssim_{J'} e \lesssim_{M^+} a_2 \lesssim_{M^+} a_3 = f_i(h_3).$$

Since $\lesssim_{M^+}$ is genuinely transitive, we can simplify the above sequence:

$$h_3 \lesssim_{J'} e \lesssim_{M^+} a_3 \implies d_{\mathcal{C}_i}(e, a_3) \leq T' \implies d_{\mathcal{C}_i}(e, a_2) \leq T',$$

i.e. $f_i(\langle h_1, h_2, h_3\rangle) \sim_{T'} a_2$, by applying Lemma 6.16.

Overall, we obtain

$$f_i(\langle h_1, h_2, h_3\rangle) \sim_{\max(T,T')} a_2 = \langle a_1, a_2, a_3\rangle_{\mathcal{C}_i} = \langle f_i(h_1), f_i(h_2), f_i(h_3)\rangle_{\mathcal{C}_i},$$

where $a_2 = \langle a_1, a_2, a_3\rangle_{\mathcal{C}_i}$, as we know that each $\mathcal{C}_i$ has a median structure (as defined in Remark 6.11). □

Now that we have shown that f_i is a $\max(T, T')$-quasi-morphism, we use this to prove that f is a quasi-morphism itself.

Corollary 6.18. *Given a coarse median space X, rank constant $C(\lambda)$ and coarse convexity constant δ, let $[x, y] \subseteq X$ be a coarse interval and let S be a δ-coarsely convex subset with rank at most r with respect to $C(\lambda)$. Suppose $\mathcal{C}_i$ is a chain for each $i \in \{1, \ldots, r\}$, with $\mathcal{C}_i \in \mathcal{Y} \subseteq S$. Recall the function f and the metric $d_{\mathcal{C}_i}$ as defined in Definition 6.12. Then for all*

$p, q, t \in S$, f is a quasi-morphism,

$$f(\langle p,q,t\rangle) \sim \langle f(p), f(q), f(t)\rangle_{\mathcal{C}_i},$$

where $\sim$ indicates close with respect to the metric $d_{\mathcal{C}_i}$ and $\langle\rangle_{\mathcal{C}_i}$ is defined in Remark 6.11.

Proof.

$$\begin{aligned}
f(\langle p,q,t\rangle) &= (f_i(\langle p,q,t\rangle))_{i=1,\ldots,r} \\
&= (f_1(\langle p,q,t\rangle), \ldots, f_r(\langle p,q,t\rangle)) \\
&\sim_{r\max(T,T')} (\langle f_1(p), f_1(q), f_1(t)\rangle, \ldots, \langle f_r(p), f_r(q), f_r(t)\rangle) \\
&= (\langle f_i(p), f_i(q), f_i(t)\rangle)_{i=1,\ldots,r} \\
&= \langle (f_i(p))_{i=1,\ldots,r}, (f_i(q))_{i=1,\ldots,r}, (f_i(t))_{i=1,\ldots,r}\rangle \\
&= \langle f(p), f(q), f(t)\rangle,
\end{aligned}$$

where the approximation is a consequence of f_i being a $\max(T, T')$-quasi-morphism and the penultimate equality follows by definition of the product median. □

We prove that f is bornologous - it suffices to show that the f_i are bornologous, as f being bornologous is an immediate consequence of this.

Theorem 6.19. *Given a coarse median space X, rank constant $C(\lambda)$ and coarse convexity constant δ, let $[x,y] \subseteq X$ be a coarse interval and let S be a δ-coarsely convex subset with rank at most r with respect to $C(\lambda)$. Suppose $\mathcal{C}_i$ is a chain for each $i \in \{1,\ldots,r\}$, with $\mathcal{C}_i \in \mathcal{Y} \subseteq S$. Recall the functions f_i defined in Definition 6.12 for each $i \in \{1,\ldots,r\}$. Let $h_1, h_2 \in S$ with $a_1 = f_i(h_1), a_2 = f_i(h_2)$, where $a_1, a_2 \in \mathcal{C}_i$. Then the maps f_i are bornologous for $i \in \{1,\ldots,r\}$.*

Proof. Bornologous is the condition that for all A there exists a B such that $d(h_1, h_2) \leq A$ implies that $d_{\mathcal{C}_i}(f_i(h_1), f_i(h_2)) \leq B$.

Hence, assume that h_1 and h_2 are close for some distance A, i.e. $h_1 \sim_A h_2$. Without loss of generality, let $a_1 \lesssim_{M^+} a_2$. We have $h_2 \sim_A h_1 \lesssim_{C'} a_1 \implies h_2 \lesssim_L a_1$ for some constant L. Putting this altogether, we obtain

$$h_2 \lesssim_L a_1 \lesssim_{M^+} a_2 \implies d_{\mathcal{C}_i}(a_1, a_2) \leq B,$$

i.e. $d_{\mathcal{C}_i}(f_i(h_1), f_i(h_2)) \leq B$ for some constant B by Lemma 6.16. □

We also prove that g is large scale Lipschitz.

Theorem 6.20. *Given a coarse median space X, rank constant $C(\lambda)$ and coarse convexity constant δ, let $[x,y] \subseteq X$ be a coarse interval and let S be a δ-coarsely convex subset with rank* *at most r with respect to $C(\lambda)$. Suppose $\mathcal{C}_i$ is a chain for each $i \in \{1, \dots, r\}$, with $\mathcal{C}_i \in \mathcal{Y} \subseteq S$. Recall the backwards map g defined in Definition 6.13:*

$$g\colon \prod_{i=1}^{r} \mathcal{C}_i \to [x,y], \qquad (a_1, \dots, a_r) \mapsto \langle a_1, \dots, a_r; x\rangle,$$

where $(a_1, \dots, a_r) \in \prod_{i=1}^{r} \mathcal{C}_i$. Then g is large scale Lipschitz.

Proof. Given $a = (a_1, \dots, a_r), a' = (a'_1, \dots, a'_r) \in \prod_{i=1}^{r} \mathcal{C}_i$, we have

$$\begin{aligned} d(g(a), g(a')) &= d(\langle a_1, \dots, a_r; x\rangle, \langle a'_1, \dots, a'_r; x\rangle) \\ &\leq K\Sigma_{j=1}^{r} d(a_j, a'_j) + H(0) \end{aligned}$$

for $j \in \{1, \dots, r\}$, where the inequality is a consequence of Lemma 3.10(3). □

Finally, we prove that $fg \sim id$ on $\mathrm{im}(f)$.

Theorem 6.21. *Given a coarse median space X, rank constant $C(\lambda)$ and coarse convexity constant δ, let $[x,y] \subseteq X$ be a coarse interval and let S be a δ-coarsely convex subset with rank at most r with respect to $C(\lambda)$. Suppose $\mathcal{C}_i$ is a chain for each $i \in \{1, \dots, r\}$, with $\mathcal{C}_i \in \mathcal{Y} \subseteq S$. Recall the forwards and backwards maps f and g introduced in Definitions 6.12 and 6.13, respectively. Additionally, recall the metric $d_{\mathcal{C}_i}$ defined in Definition 6.12. Given $(a_1, \dots, a_r)$, where each $a_i \in \mathcal{C}_i$, then $fg(a_1, \dots, a_r) \sim (a_1, \dots, a_r)$, that is, $fg \sim id$ on $\mathrm{im}(f)$, where $\sim$ indicates close with respect to the metric $d_{\mathcal{C}_i}$.*

Proof. By definition, $a_i = f_i(u)$, where $u \in S$ for $i \in \{1, \dots, r\}$, gives us a point in the image.

We are aiming to show that $fg(a_1, \dots, a_r)$ is close to $(a_1, \dots, a_r)$, which is the same as saying that for each coordinate i, the ith coordinate of $fg(a_1, \dots, a_r)$, which is $f_i(g(a_1, \dots, a_r))$, is close to the ith coordinate a_i of the point $(a_1, \dots, a_r)$.

Now, $g(a_1, \dots, a_r)$ is defined as the minimum of $(a_1, \dots, a_r)$, i.e. $g(a_1, \dots, a_r) = \langle a_1, \dots, a_r; x\rangle$. Since f_i is a $\max(T, T')$-quasi-morphism (Theorem 6.17), $f_i(g(a_1, \dots, a_r)) = f_i(\langle a_1, \dots, a_r; x\rangle) \sim_{H_r(\max(T,T'))} \langle f_i(a_1), \dots, f_i(a_r); f_i(x)\rangle_{\mathcal{C}_i}$, where $H_r(\max(T, T'))$ is derived by induction by application of the coarse iterated median, depending only on the parameters of X and $\max(T, T')$.

We show that $\langle f_i(a_1), \dots, f_i(a_r); f_i(x)\rangle_{\mathcal{C}_i}$ cannot be (much) smaller than a_i, and hence is close to a_i which is what we are aiming to prove; that is, $f_i(a_j) \sim a_i$ for $i \neq j$. Now, if we assume that $a_i \leq f_i(a_j)$, and by definition, we also know that $u \lesssim_{C'} a_i$, then putting these two together, we obtain $u \lesssim_{C'} a_i \leq f_i(a_j)$. Applying Lemma 6.16, we obtain $a_i \sim_P f_i(a_j)$ and we are done. Thus, we assume the reverse inequality, $f_i(a_j) \leq a_i$, and then apply Lemma 6.16 to deduce that these two points are close. Note that $f_i(a_i) = a_i$:

if $a_i \lesssim M^+ f_i(a_i)$, then by definition, $f_i(a_i)$ is the least point in $\mathcal{C}_i$ such that $a_i \lesssim_{C'} f_i(a_i)$. However, $a_i \lesssim_{C'} a_i$, and so $f_i(a_i) \lesssim_{M^+} a_i$, implying that $f_i(a_i) = a_i$.

Note that a_j and $f_i(a_j)$ are in different chains, but by definition of f_i, it always takes points — in this case a_j — to something satisfying $a_j \lesssim_{C'} f_i(a_j)$. Now notice that a_i itself was defined to be $f_i(u)$, while a_j is $f_j(u)$ (here is where we are using the fact that $(a_1, \ldots, a_r)$ is in the image). Hence, we are assuming that $f_i(a_j) \le a_i = f_i(u)$, and so Lemma 6.16 will apply as long as we can show that $u \lesssim_L f_i(a_j) = f_i(f_j(u))$ for some constant L. This holds, since f_i, f_j always take points to something greater (with constant C') by definition. More precisely, $f_j(u)$ is by definition the least point in $\mathcal{C}_j$ such that $u \lesssim_{C'} f_j(u)$. The point $f_i(f_j(u))$ is by definition the least point in $\mathcal{C}_i$ such that $f_j(u) \lesssim_{C'} f_i(f_j(u))$. Putting these two together and applying Lemma 4.5, we obtain $u \lesssim_L f_i(f_j(u))$, where $L = 2KC' + 2H(0) + \kappa_4 + C'$.

Therefore, by Lemma 6.16, we conclude that $f_i(a_j) \sim_P a_i$. This implies that $\langle f_i(a_1), \ldots, f_i(a_r); f_i(x)\rangle_{\mathcal{C}_i} \sim_{\rho_r(P)} a_i$, where we apply Lemma 3.10(3) and rewrite $a_i = \langle a_i, \ldots, a_i; a_i\rangle$. Hence, $fg(a_1, \ldots, a_r) \sim (a_1, \ldots, a_r)$, as required. To be exact:

$$\begin{aligned} f_i(g(a_1, \ldots, a_r)) &\sim_{H_r(\max(T,T'))} \\ &\sim_{\rho_r(P)} \langle f_i(a_1), \ldots, f_i(a_r); f_i(x)\rangle_{\mathcal{C}_i} \end{aligned}$$

Therefore,

$$\begin{aligned} f(g(a_1, \ldots, a_r) &= f(\langle a_1, \ldots, a_r; x\rangle) \\ &= (f_i(\langle a_1, \ldots, a_r; x\rangle))_{i=1}^r \\ &\sim_{rH_r(\max(T,T'))} (\langle f_i(a_1), \ldots, f_i(a_r); f_i(x)\rangle)_{i=1}^r \\ &\sim_{r\rho_r(P)} (a_i)_{i=1}^r. \end{aligned}$$

□

Appendix A

Appendix

We devote the Appendix to stating and proving median versions of most of our results for coarse median spaces; initially, the statements and proofs of our median results were used as motivation for the coarse median case and on most occasions, we could 'coarsen' the proof of the median case to provide us with a backbone (or more) for the proofs of the coarse median results. The proofs use similar techniques to ones in the coarse cases, but also provide good intuition when thinking about the coarse world, thus they are presented here for completeness and for clarity.

A.1 Co-dimension 1 of Coarse Hyperplanes

We begin by proving that coarse hyperplanes have co-dimension 1 in finite CAT(0) cube complexes, which we state below.

Theorem A.1. *Given a finite CAT(0) cube complex X and $[x, y] \subseteq X$, where rank $[x, y] \leq n$, consider the interval $[a, b] \subseteq [x, y]$, where $a \leq b$, has no corners (as defined in Definition 5.6). Then a cube in $[a, b]$ of rank equal to the rank of $[x, y]$ has a side that becomes trivial.*

The set-up for proving the theorem above is analogous to the set-up described in Section 5.4.1.

A.1.1 Proof of Case 1

Case 1 is much simpler to prove in the context of CAT(0) cube complexes compared to case 2. The proof is given below:

Lemma A.2. *Let X be a CAT(0) cube complex. Take the coarse hyperplane corresponding to a, b in the interval $[x, y]$, where $a, b, x, y \in X$. Let $C = \{u_1, \ldots, u_{2^d}\}$ be a rank d median cube* *in the hyperplane, that is, we have* $\langle u_i, u_j, u_k \rangle = u_l$ *for* $i, j, k, l \in \{1, \ldots, 2^d\}$. *Assume that* $[a, b]$ *has no corners and that* $a \leq b$. *Suppose also that the* u_i *project onto a point* $c \in [a, b]$ *for all* i *and assume that* $d(c, b)$ *is non-trivial, that is,* $c \neq a, b$. *Then we can construct a median cube* $\tilde{C} \subseteq [x, y]$ *with rank* $d + 1$.

Observe that when we project onto $c \in [a, b]$, we do not want c to be close to the endpoints of the interval; this is the same as wanting c to lie in the coarse hyperplane associated to a and b.

Proof of Lemma A.2. Since c is not a corner, this means that c fails to satisfy at least one of the conditions in Definition 5.6. We cannot have $c = a$ or b as then the u_i will not be in the coarse hyperplane, they will be in H_a and H_b respectively which are not a part of the coarse hyperplane. This implies that either $\langle a, v_i, x \rangle = a$ or $\langle b, v_i, y \rangle = b$, but these are dual cases so we will focus on $\langle a, v_i, x \rangle = a$ here since the case $\langle b, v_i, y \rangle = b$ can be proven very similarly.

The way we prove the lemma is the following: applying the ordering from Definition 4.2, set c as the new basepoint; then we can find an i_0 (that is not necessarily unique) in $\{1, \ldots, 2^d\}$ such that u_{i_0} is a maximal element with respect to this ordering, so we have $\langle u_i, u_{i_0}, c \rangle \neq u_{i_0}$ for all $i \neq i_0$. We then project the remaining vertices of C onto the interval $[c, u_{i_0}]$, i.e. we set $v_i = \langle c, u_i, u_{i_0} \rangle$ where $\{u_i, u_j\}$ forms an edge of C. We then obtain a cube that lies in $[c, u_{i_0}]$ which we denote $\hat{C} = \{v_1, \ldots, v_{2_d}\}$. This cube may be smaller than C but will still be a median cube of the same rank so our original claim still holds. To be more specific, we show that the rank of $\hat{C}$ remains the same by proving that $v_i \neq v_j$; this says that we have not collapsed any edges of C when projecting its vertices onto $[c, v_{i_0}]$, hence leaving the rank unchanged.

Note A.3. 'Medianness' is preserved (using the five-point condition), and so $\hat{C}$ will still be a median cube; we prove this below.

$$\begin{aligned}
\langle v_i, v_j, v_k \rangle &= \langle \langle c, u_{i_0}, u_i \rangle, \langle c, u_{i_0}, u_j \rangle, \langle c, v_k, u_{i_0} \rangle \rangle \\
&= \langle c, u_{i_0}, \langle u_i, u_j, \langle c, v_k, u_{i_0} \rangle \rangle \rangle \\
&= \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, \langle c, u_{i_0}, v_k \rangle \rangle \rangle \\
&= \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, v_k \rangle \rangle \\
&= \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, \langle c, u_{i_0}, u_k \rangle \rangle \rangle \\
&= \langle u_i, \langle c, u_{i_0}, u_j \rangle, \langle c, u_{i_0}, u_k \rangle \rangle \\
&= \langle c, u_{i_0}, \langle u_i, u_j, u_k \rangle \rangle \\
&= \langle c, u_{i_0}, u_l \rangle \\
&= v_l.
\end{aligned}$$

Now consider an arbitrary edge $\{v_i, v_j\} \in [c, u_{i_0}]$. Then we can find a parallel edge containing u_{i_0} as a vertex (since we are in a cube), call this edge $\{v_k, u_{i_0}\}$, such that (without loss of generality)

$$\langle u_{i_0}, v_k, v_i \rangle = u_{i_0}$$
$$\langle u_{i_0}, v_k, v_j \rangle = v_k.$$

This is the same as saying that $\{v_i, u_{i_0}, v_k, v_j\}$ forms a median square. Assume towards a contradiction that $v_i = v_j$; then

$$\begin{aligned} u_{i_0} &= \langle u_{i_0}, v_k, v_i \rangle \\ &= \langle u_{i_0}, v_k, v_j \rangle \\ &= v_k. \end{aligned}$$

Since we have $u_{i_0} = v_{i_0} = v_k$, this implies that $\langle c, u_k, u_{i_0} \rangle = u_{i_0}$ which in turn signifies that $u_{i_0} \leq v_k$ (applying the ordering with c as the basepoint). However, this is a contradiction as we chose u_{i_0} to be maximal and so any edge containing u_{i_0} as a vertex cannot collapse (also because we stated earlier that $\langle u_i, u_{i_0}, c \rangle \neq u_{i_0}$ which prevents collapses). Therefore, we have shown that $v_i \neq v_j$ and so the cube $\hat{C}$ has the same rank as C.

When we pick u_{i_0}, it either falls into the first or second case, i.e. $\langle a, u_{i_0}, x \rangle = a$ or $\langle b, u_{i_0}, y \rangle = b$. Without loss of generality, let us assume that u_{i_0} satisfies the first case as the second case is dual to this.

The reason for projecting all the vertices of C onto $[c, u_{i_0}]$ is to ensure that all points are of the 'same type', i.e. we either have $\langle a, u_i, x \rangle = a$ for all i or $\langle b, u_i, y \rangle = b$ for all i. We want all our vertices to be of the same type because we can have $\langle a, u_i, x \rangle = a$ for some i and for other i, $\langle b, u_i, y \rangle = b$ instead, but we require our u_is to all satisfy the same condition. As a result of the projection, we have

$$\begin{aligned} \langle a, v_i, x \rangle &= \langle a, \langle c, u_i, u_{i_0} \rangle, x \rangle \\ &= \langle \langle a, c, x \rangle, \langle a, u_{i_0}, x \rangle, u_i \rangle \\ &= \langle a, a, u_i \rangle \\ &= a, \end{aligned}$$

where the penultimate equality follows from $a \leq c$ (with respect to x as the basepoint in the ordering) and by assuming that u_{i_0} satisfies $\langle a, u_{i_0}, x \rangle = a$ (since we are focusing on this case). So without loss of generality, assume that the v_i satisfy $\langle a, v_i, x \rangle = a$ for all i.

To summarise, we know

A. $\langle a, v_i, b\rangle = c$ for all i, where c is not a corner;

B. $\langle a, v_i, y\rangle = v_i$ for all i.

Let $w_i = \langle b, v_i, y\rangle$. Observe that $d(w_i, v_i)$ is non-trivial; this follows from $d(c, b)$ being non-trivial (which was stated in Lemma A.2), hence we set $w_i \neq v_i$ for all i. Then we will show that we can use the w_i to build a cube $\tilde{C}$ that is one dimension higher than C. We have four cases to prove:

(i) $\langle v_i, v_j, v_k\rangle = v_l$;

(ii) $\langle v_i, v_j, w_k\rangle = v_l$;

(iii) $\langle w_i, w_j, v_k\rangle = w_l$;

(iv) $\langle w_i, w_j, w_k\rangle = w_l$, where $i, j, k, l \in \{1, \ldots, 2^d\}$.

(i) Follows as we have already assumed that C is a median cube.

(ii)

$$\begin{aligned}
\langle v_i, v_j, w_k\rangle &= \langle v_i, v_j, \langle b, v_k, y\rangle\rangle \\
&= \langle v_i, v_j, \langle b, \langle a, v_k, y\rangle, y\rangle\rangle \quad \text{(using B.)} \\
&= \langle v_i, v_j, \langle\langle a, y, v_k\rangle, y, b\rangle\rangle \\
&= \langle v_i, v_j, \langle a, y, \langle v_k, y, b\rangle\rangle \quad \text{(using 4-point)} \\
&= \langle\langle v_i, v_j, y\rangle, \langle v_i, v_j, \langle v_k, y, b\rangle\rangle, a\rangle \quad \text{(using 5-point)} \\
&= \langle\langle v_i, v_j, y\rangle, \langle\langle v_i, v_j, v_k\rangle, \langle v_i, v_j, y\rangle, b\rangle, a\rangle \quad \text{(using 5-point)} \\
&= \langle\langle v_i, v_j, y\rangle, \langle v_l, \langle v_i, v_j, y\rangle, b\rangle, a\rangle \\
&= \langle a, \langle v_i, v_j, y\rangle, \langle v_l, \langle v_i, v_j, y\rangle, b\rangle\rangle \\
&= \langle\langle a, \langle v_i, v_j, y\rangle, b\rangle, \langle a, \langle v_i, v_j, y\rangle, \langle v_i, v_j, y\rangle\rangle, v_l\rangle \quad \text{(using 5-point)} \\
&= \langle\langle\langle a, b, v_i\rangle, \langle a, b, v_j\rangle, y\rangle, \langle v_i, v_j, y\rangle, v_l\rangle \quad \text{(using 5-point)} \\
&= \langle\langle c, c, y\rangle, \langle v_i, v_j, y\rangle, v_l\rangle \quad \text{(using A.)} \\
&= \langle c, \langle v_i, v_j, y\rangle, v_l\rangle \\
&= \langle c, \langle v_i, v_j, y\rangle, \langle v_i, v_j, v_k\rangle\rangle \\
&= \langle v_i, v_j, \langle c, y, v_k\rangle\rangle \quad \text{(using 5-point)} \\
&= \langle v_i, v_j, \langle\langle a, v_k, b\rangle, y, v_k\rangle\rangle \quad \text{(using A. to replace } c\text{)} \\
&= \langle v_i, v_j, \langle\langle a, v_k, y\rangle, \langle v_k, y, v_k\rangle, b\rangle\rangle \quad \text{(using 5-point)} \\
&= \langle v_i, v_j, \langle v_k, v_k, b\rangle\rangle \quad \text{(using B.)} \\
&= \langle v_i, v_j, v_k\rangle \\
&= v_l.
\end{aligned}$$

(iii)

$$\begin{aligned}\langle w_i, w_j, v_k\rangle &= \langle\langle b, v_i, y\rangle, \langle b, v_j, y\rangle, v_k\rangle \\ &= \langle b, y, \langle v_i, v_j, v_k\rangle\rangle \quad \text{(using 5-point)} \\ &= \langle b, y, v_l\rangle \\ &= w_l.\end{aligned}$$

(iv)

$$\begin{aligned}\langle w_i, w_j, w_k\rangle &= \langle\langle b, v_i, y\rangle, \langle b, v_j, y\rangle, \langle b, v_k, y\rangle\rangle \\ &= \langle b, y, \langle v_i, v_j, \langle b, v_k, y\rangle\rangle\rangle \quad \text{(using 5-point)} \\ &= \langle b, y, \langle v_i, v_j, w_k\rangle\rangle \\ &= \langle b, y, v_l\rangle \quad \text{(using (ii))} \\ &= w_l.\end{aligned}$$

If B. is replaced with $\langle b, v_i, x\rangle = v_i$ and $w_i = \langle a, v_i, x\rangle$, the proof follows similarly. Thus $\tilde{C}$ is a median cube of rank $d+1$ as required. □

A.1.2 Proof of the General Case

Let C be a median cube contained in the coarse hyperplane corresponding to a, b and let V be the set of vertices of C.

When we project C onto $[a, b]$ we may see a cube of lower dimension as some vertices of C may project onto the same point. Since this (lower dimension) cube is contained in $[a, b]$, it will be 'small' as $d(a, b)$ is bounded and so this implies that we can bound the size of anything projected onto $[a, b]$.

Pick an edge of C, say $\{v_1, v_2\}$, and project down onto $[a, b]$; we then define $w_1 = \langle a, v_1, b\rangle$ and $w_2 = \langle a, v_2, b\rangle$. Given $v \in V$, projecting v directly onto $\{w_1, w_2\}$ is the same as first projecting v onto $\{v_1, v_2\}$ and then onto $\{w_1, w_2\}$: since C is a median cube, when we project v onto $\{v_1, v_2\}$, we obtain either v_1 or v_2. By definition of w_1, w_2, we then see that $\langle v, w_1, w_2\rangle = w_1$ or w_2. More formally,

$$\begin{aligned}\langle\langle v,v_1,v_2\rangle,w_1,w_2\rangle &= \langle\langle v,v_1,v_2\rangle,\langle a,b,v_1\rangle,\langle a,b,v_2\rangle\rangle\\ &= \langle a,b,\langle v_1,v_2,\langle v,v_1,v_2\rangle\rangle\rangle\\ &= \langle a,b,\langle\langle v_2,v_1,v_2\rangle,v_1,v\rangle\\ &= \langle a,b,\langle v_2,v_1,v\rangle\rangle\\ &= \langle\langle a,b,v_2\rangle,\langle a,b,v_1\rangle,v\rangle\\ &= \langle w_2,w_1,v\rangle.\end{aligned}$$

Since C is a median cube, we have $\langle v,v_1,v_2\rangle = v_1$ or v_2. Let us assume that $\langle v,v_1,v_2\rangle = v_1$; then

$$\begin{aligned}\langle\langle v,v_1,v_2\rangle,w_1,w_2\rangle &= \langle v_1,w_1,w_2\rangle\\ &= \langle v_1,\langle a,b,v_1\rangle,\langle a,b,v_2\rangle\rangle\\ &= \langle a,b,\langle v_1,v_2,v_1\rangle\rangle\\ &= \langle a,b,v_1\rangle\\ &= w_1.\end{aligned}$$

The above calculation holds similarly when we instead assume that $\langle v,v_1,v_2\rangle = v_2$.

We now introduce some necessary notation and motivation for the main argument of the proof of Theorem A.1.

Define $u_1 = \langle v_1,v_2,w_1\rangle, u_2 = \langle v_1,v_2,w_2\rangle$; then we can split $\{v_1,v_2\}$ into three sub-edges: it is made up of two 'vertical' edges $\{v_1,u_1\},\{u_2,v_2\}$ and one 'horizontal' edge $\{u_1,u_2\}$. Here, a 'vertical' edge is an edge that projects onto a point in $[a,b]$ (see case 1) and a 'horizontal' edge is an edge that projects onto a subinterval of $[a,b]$ (see case 2), i.e. $\{u_1,u_2,w_1,w_2\}$ forms a median square. Additionally, $\{v_1,v_2\}$ could be a diagonal edge, however in the corresponding CAT(0) cube complex it may look more intricate and be made up of three sub-edges.

If we take a parallel edge $\{x_1,x_2\}$ to $\{v_1,v_2\}$ then we obtain a median square comprised of these four points, but we can split this up into three median sub-squares in the same manner we split $\{v_1,v_2\}$ into three sub-edges.

The reason we cut $\{v_1,v_2\}$ into three pieces is because we are only looking at an edge $\{w_1,w_2\}$ in the projection and not something of higher dimension. In particular, $\{v_1,u_1\}$ represents staying at w_1 (as it is a vertical edge), $\{u_1,u_2\}$ represents moving from w_1 to w_2 (as it is a horizontal edge), and $\{u_2,v_2\}$ represents staying at w_2 (as it is a vertical edge). Note that the movement from w_1 to w_2 is monotone.

In order to prove Theorem A.1, we first need the following result.

Proposition A.4. *Let F be a co-dimension 1 face of C (so an $(n-1)$-cube) that contains v_1 but not v_2. Let q be the point in C opposite v_1 and p be the point in F opposite v_1. Define the maps ϕ, ψ:*

$$\begin{aligned} \phi : F \times \{v_1, v_2\} &\to C \\ (f, v_1) &\mapsto \langle f, v_1, q \rangle \sim f \\ (f, v_2) &\mapsto \langle f, v_2, q \rangle \sim f', \end{aligned}$$

where $f \in F$ and f' is the point in the parallel face to f;

$$\begin{aligned} \psi : F \times \{v_1, u_1, u_2, v_2\} &\to X \\ (f, t) &\mapsto \langle f, t, q \rangle, \end{aligned}$$

where $f \in F,\ t \in \{v_1, u_1, u_2, v_2\}$. Then $\psi|_C = \phi$ and ψ is a median morphism.

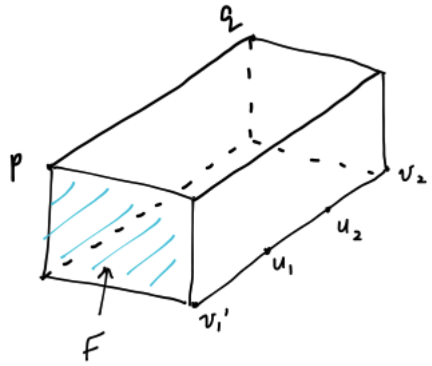


FIGURE A.1: This figure is a visual depiction of Chapter A.4; it shows that we can cut C up into three median 'vertical' and 'horizontal' sub-cubes. (Note that $v_1' = v_1$ here.)

Having stated Proposition A.4, we now provide the argument for the proof of Theorem A.1:

Proof. We can cut C up into three median sub-cubes with the same rank as C as seen in Proposition A.4; specifically, we can split C into two 'vertical' cubes and one 'horizontal' cube. This approach of splitting into 'vertical' and 'horizontal' cubes works together to give us our required result (this is a blend of cases 1 and 2): suppose the vertical pieces of either vertical cube are non-trivial; we know that these cubes have the same rank as C, and thus applying case 1 we see that the vertical cubes have one rank lower than X, hence implying that C must also have one rank lower than X. If the vertical pieces are trivial, then keeping case 2 in mind, we know by assumption that the edge $\{u_1, u_2\}$ lines up in the direction of $\{w_1, w_2\}$ and so will have bounded size (as $\{w_1, w_2\}$ has a bound on its size as it lies in $[a, b]$), i.e. $\{u_1, u_2, w_1, w_2\}$ would form a median square, hence giving us a bound on the distance from u_1 to u_2 in terms of $d(w_1, w_2)$. This in turn implies that C has rank (at most) one lower than X. □

Note A.5. Observe that $\{v_1, u_1, u_2, v_2\}$ all lie in a straight line, in the sense that $u_1 \in [v_1, u_2], u_2 \in [u_1, v_2]$ (we also know that $u_1, u_2 \in [v_1, v_2]$). We prove this below:

1.

$$\begin{aligned}\langle v_1, u_1, u_2\rangle &= \langle v_1, \langle v_1, v_2, w_1\rangle, \langle v_1, v_2, w_2\rangle\rangle \\ &= \langle v_1, v_2, \langle v_1, w_1, w_2\rangle\rangle \\ &= \langle v_1, v_2, w_1\rangle \\ &= u_1.\end{aligned}$$

2.

$$\begin{aligned}\langle u_1, u_2, v_2\rangle &= \langle\langle v_1, v_2, w_1\rangle, \langle v_1, v_2, w_2\rangle, v_2\rangle \\ &= \langle v_1, v_2, \langle w_1, w_2, v_2\rangle\rangle \\ &= \langle v_1, v_2, \langle\langle v_1, a, b\rangle, \langle v_2, a, b\rangle, v_2\rangle\rangle \\ &= \langle v_1, v_2, \langle a, b, \langle v_1, v_2, v_2\rangle\rangle\rangle \\ &= \langle v_1, v_2, \langle a, b, v_2\rangle\rangle \\ &= \langle v_1, v_2, w_2\rangle \\ &= u_2.\end{aligned}$$

We now present the proof of the above proposition.

One way of proving the proposition above would be to explicitly show that cutting C into three gives us three median sub-cubes of the same dimension. However, this would have yielded around 68 cases in total which is not efficient.

Proof of Proposition A.4. We will show that

$$\psi(\langle f_i, f_j, f_k\rangle, \langle t_1, t_2, t_3\rangle) = \langle \psi(f_i, t_1), \psi(f_j, t_2), \psi(f_k, t_3)\rangle,$$

where $f_i, f_j, f_k \in F, \langle f_i, f_j, f_k\rangle = f_l$, $t_1 \leq t_2 \leq t_3$ with $t_i \in \{v_1, u_1, u_2, v_2\}$ and $1 \leq i, j, k, l \leq 2^n$.

The following important facts are needed before we can commence with the proof:

(i) $\langle f_i, f_j, f_q \rangle \in [f_l, q] \qquad (\Leftrightarrow f_l \in [v_1, \langle f_i, f_j, q \rangle])$.

$$\begin{aligned}\langle f_l, \langle f_i, f_j, q \rangle, q \rangle &= \langle \langle f_i, f_j, f_k \rangle, \langle f_i, f_j, q \rangle, q \rangle \\ &= \langle f_i, f_j, \langle f_k, q, q \rangle \rangle \qquad \text{(by 5-point condition)} \\ &= \langle f_i, f_j, q \rangle,\end{aligned}$$

as required.

(ii) $\langle v_1, p, u_1 \rangle = v_1$.

$$\begin{aligned}\langle v_1, p, u_1 \rangle &= \langle v_1, p, \langle v_1, v_2, w_1 \rangle \rangle \qquad \text{(by definition of } u_1\text{)} \\ &= \langle p, v_1, \langle v_2, v_1, w_1 \rangle \rangle \\ &= \langle \langle p, v_1, v_2 \rangle, v_1, w_1 \rangle \qquad \text{(using 4-point condition)} \\ &= \langle v_1, v_1, w_1 \rangle \\ &= v_1.\end{aligned}$$

Actually, $\langle v_1, p, t \rangle = v_1$ for $t \in \{v_1, u_1, u_2, v_2\}$.

(iii) $\langle f_l, \langle f_i, f_j, q \rangle, v_1 \rangle = \langle f_l, \langle f_i, f_j, q \rangle, u_1 \rangle$.

$$\begin{aligned}\langle f_l, \langle f_i, f_j, q \rangle, v_1 \rangle &= \langle \langle f_i, f_j, f_k \rangle, \langle f_i, f_j, q \rangle, \langle v_1, p, u_1 \rangle \rangle \qquad \text{(by (ii))} \\ &= \langle \langle f_i, f_j, f_k \rangle, \langle v_1, \langle f_i, f_j, q \rangle, p \rangle, \langle v_1, p, u_1 \rangle \rangle\end{aligned}$$

(as $\langle f_i, f_j, q \rangle = f_m$ for some m and so will lie in $[v_1, p]$)

$$\begin{aligned}&= \langle v_1, p, \langle \langle f_i, f_j, f_k \rangle, \langle f_i, f_j, q \rangle, u_1 \rangle \rangle \qquad \text{(by 5-point)} \\ &= \langle \langle v_1, p, \langle f_i, f_j, f_k \rangle \rangle, \langle v_1, p, \langle f_i, f_j, q \rangle \rangle, u_1 \rangle \qquad \text{(by 5-point)} \\ &= \langle \langle f_i, f_j, f_k \rangle, \langle f_i, f_j, q \rangle, u_1 \rangle \\ &= \langle f_l, \langle f_i, f_j, q \rangle, u_1 \rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\langle \langle f_i, f_j, f_k \rangle, \langle f_i, f_j, q \rangle, u_1 \rangle &= \langle \langle f_i, f_j, f_k \rangle, \langle f_i, f_j, q \rangle, v_1 \rangle \\ &= \langle f_i, f_j, f_k \rangle \\ &= f_l \qquad \text{(by (i))}.\end{aligned}$$

This result can be generalised similarly to the previous case:
$\langle f_l, \langle f_i, f_j, q \rangle, v_1 \rangle = \langle f_l, \langle f_i, f_j, q \rangle, t \rangle$, for $t \in \{v_1, u_1, u_2, v_2\}$.

We proceed with the proof below. Let

$$a' = \langle f_i, t_1, q \rangle,$$
$$b' = \langle f_j, t_2, q \rangle,$$
$$c' = \langle f_k, t_3, q \rangle.$$

We know:

$$\langle \langle f_i, f_j, f_k \rangle, \langle t_1, t_2, t_3 \rangle, q \rangle = \langle f_l, t_2, q \rangle.$$

We want:

$$\langle a', b', c' \rangle = \langle f_l, t_2, q \rangle.$$

Let $m \in [v_1, q]$. Set

$$t = \langle v_1, m, v_2 \rangle \in [v_1, v_2]$$
$$f = \langle v_1, m, p \rangle \in [v_1, p].$$

Then

$$\begin{aligned} \langle f, t, q \rangle &= \langle \langle v_1, m, p \rangle, \langle v_1, m, v_2 \rangle, q \rangle \\ &= \langle v_1, m, \langle p, v_2, q \rangle \rangle \\ &= \langle v_1, m, q \rangle \\ &= m. \end{aligned}$$

Using the following identity (based on the above calculation)

$$\langle \langle \langle a', b', c' \rangle, v_1, v_2 \rangle, \langle \langle a', b', c' \rangle, v_1, p \rangle, q \rangle = \langle a', b', c' \rangle,$$

we will prove the claim by showing that

1. $$\langle \langle a', b', c' \rangle, v_1, p \rangle = f_l.$$

2. $$\langle \langle a', b', c' \rangle, v_1, v_2 \rangle = t_2.$$

1.

$$\begin{aligned}
\langle\langle a',b',c'\rangle,v_1,p\rangle &= \langle\langle v_1,p,a'\rangle,\langle v_1,p,b'\rangle,c'\rangle \qquad \text{(by 5-point)}\\
&= \langle\langle v_1,p,\langle f_i,t_1,q\rangle\rangle,\langle v_1,p,\langle f_j,t_2,q\rangle\rangle,\langle f_k,t_3,q\rangle\rangle\\
&= \langle\langle\langle v_1,p,f_i\rangle,\langle v_1,p,t_1\rangle,q\rangle,\langle\langle v_1,p,f_j\rangle,\langle v_1,p,t_2\rangle,q\rangle,\langle f_k,t_3,q\rangle\rangle \qquad \text{(by 5-point)}\\
&= \langle\langle f_i,v_1,q\rangle,\langle f_j,v_1,q\rangle,\langle f_k,t_3,q\rangle\rangle\\
&= \langle f_i,f_j,\langle f_k,t_3,q\rangle\rangle\\
&= \langle\langle f_i,f_j,f_k\rangle,\langle f_i,f_j,q\rangle,t_3\rangle \qquad \text{(by 5-point)}\\
&= \langle f_l,\langle f_i,f_j,q\rangle,t_3\rangle\\
&= \langle f_l,\langle f_i,f_j,q\rangle,v_1\rangle \qquad \text{(using fact (iii))}\\
&= f_l \qquad \text{(using fact (i)).}
\end{aligned}$$

2.

$$\begin{aligned}
\langle\langle a',b',c'\rangle,v_1,v_2\rangle &= \langle\langle v_1,v_2,a'\rangle,\langle v_1,v_2,b'\rangle,c'\rangle \qquad \text{(by 5-point)}\\
&= \langle\langle v_1,v_2,\langle f_i,t_1,q\rangle,\langle v_1,v_2,\langle f_j,t_2,q\rangle\rangle,\langle f_k,t_3,q\rangle\rangle \qquad \text{(by 5-point)}\\
&= \langle\langle\langle v_1,v_2,f_i\rangle,\langle v_1,v_2,q\rangle,t_1\rangle,\langle\langle v_1,v_2,f_j\rangle,\langle v_1,v_2,q\rangle,t_2\rangle,\langle f_k,t_3,q\rangle\rangle\\
&= \langle\langle v_1,v_2,t_1\rangle,\langle v_1,v_2,t_2\rangle,\langle f_k,t_3,q\rangle\rangle\\
&= \langle t_1,t_2,\langle f_k,t_3,q\rangle\rangle\\
&= \langle\langle t_1,t_2,t_3\rangle,\langle t_1,t_2,q\rangle,f_k\rangle \qquad \text{(by 5-point)}\\
&= \langle t_2,\langle t_1,t_2,q\rangle,f_k\rangle\\
&= \langle t_2,t_2,f_k\rangle\\
&= t_2.
\end{aligned}$$

Note A.6. We can deduce straight away that $\langle t_1,t_2,q\rangle = t_2$ as in the interval $[v_1',q]$, $t_1 \le t_2$ means that $\langle v_1',t_1,t_2\rangle = t_1$ or equivalently $\langle t_1,t_2,q\rangle = t_2$. We also know that $[v_1',v_2] \subseteq [v_1',q]$, so if $t_1,t_2 \in [v_1',v_2]$ then $t_1 \le t_2$ (in $[v_1',v_2]$) $\Leftrightarrow \langle v_1',t_1,t_2\rangle = t_1 \Leftrightarrow t_1 \le t_2$ (in $[v_1',q]$) $\Leftrightarrow \langle t_1,t_2,q\rangle = t_2$.

Thus we have shown that coarse hyperplanes have co-dimension at most 1 with X. □

A.2 The Corner Problem

We state and prove the median formulation of the corner problem below, which is again very similar to the coarse version.

Lemma A.7. *Let X be a CAT(0) cube complex. Let $[x,y]$ be a two-dimensional interval and let $[a,b]$ be a one-dimensional interval contained in $[x,y]$, where $a,b,x,y \in X$. Then $[a,b]$ has at most 1 corner.*

We begin by constructing the universal median algebra on the six points $\{x, a, b, y, d_1, d_2\}$ with $c_i = \langle a, d_i, b\rangle, i = 1, 2$. We have the following two relations: $a \leq b$ and $c_1 \leq c_2$, so that overall we obtain $x \leq a \leq c_1 \leq c_2 \leq b \leq y$.

Note A.8.
- We have a total ordering on $[a, b]$ as a result of it being one-dimensional and by applying the ordering outlined in Definition 4.2.
- We focus on the universal median algebra as this is the universal case (for any median algebra M with these points and relations there exists a unique map from the universal median algebra to M).

To construct a diagram of the universal median algebra described above, begin by considering the free median algebra case (so we use the same six points but without any relations) and note down all possible hyperplanes. Then impose the two relations given above which allows us to throw away any hyperplanes that don't satisfy these relations. Finally, use the equivalence between finite median algebras and finite CAT(0) cube complexes to build the CAT(0) cube complex consisting of the hyperplanes that satisfy the two relations. This yields the image below.

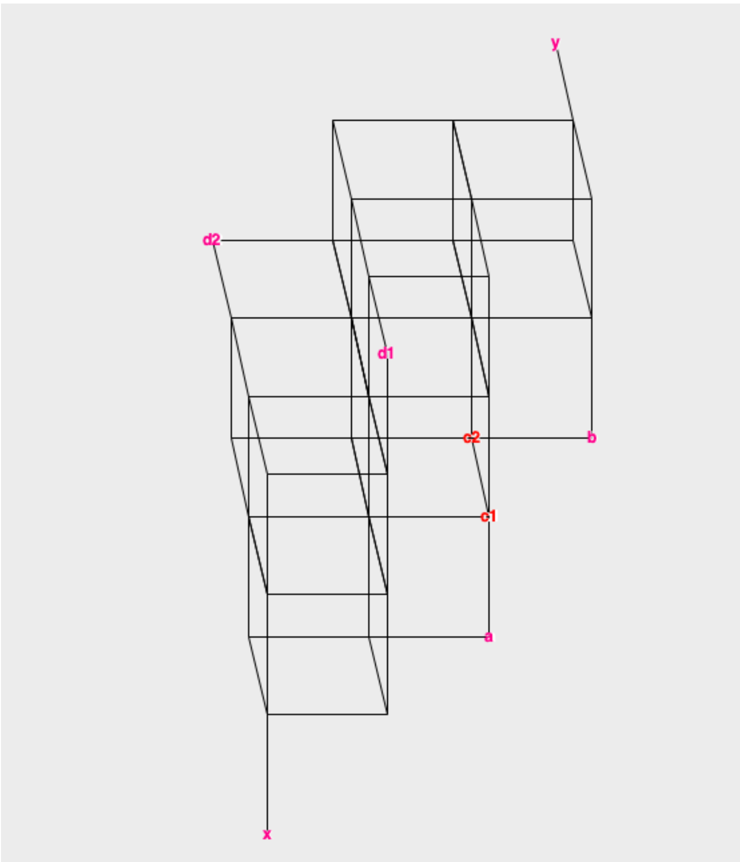


FIGURE A.2: The universal median algebra on the points $\{x, a, b, d_1, d_2, y\}$ with the relations $a \leq b$ and $c_1 \leq c_2$.

Collapsing any of the a, d_1, d_2 hyperplanes will immediately cause c_1 to not be a corner, so we avoid collapsing these for now as they are trivial cases.

The proof is split into three cases and are as follows (note that we assume $c_1 \neq c_2$ to avoid a trivial case):

1. Collapsing the ab and abd_2 hyperplanes (so $c_2 = \langle c_2, d_1, y \rangle$ and c_1 is not a corner).
2. Collapsing the d_2 and $d_1 d_2$ hyperplanes (so $c_1 = \langle c_1, d_2, x \rangle$ and c_2 is not a corner).
3. Collapsing the ab and $d_1 d_2$ hyperplanes (so $c_1 = \langle y, c_1, \langle d_1, c_1, d_2 \rangle$ and $c_2 = \langle x, c_2, \langle d_1, c_2, d_2 \rangle \rangle$ and neither c_1 nor c_2 are corners).

The central cuboids of the universal median algebra — which are duals of each other — are sufficient enough to focus on cases 1 and 2.

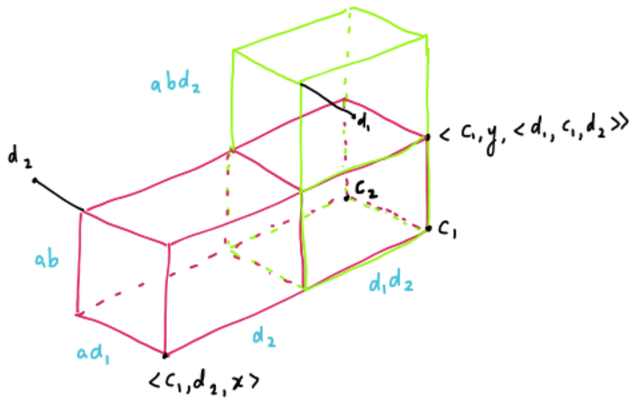


FIGURE A.3: The central cuboids in the universal median algebra. The text in blue represents hyperplanes.

Proof. We first need to check the conditions of Note 3.7 for the interval $[c_1, d_2]$ (we could have also chosen $[c_2, d_1]$ instead). We need to show that $c_2, \langle c_1, d_2, x \rangle, \langle y, c_1, \langle d_1, c_1, d_2 \rangle$ are in $[c_1, d_2]$. By definition, $\langle c_1, d_2, x \rangle \in [c_1, d_2]$ so we just need to check the remaining points.

- $\langle y, c_1, \langle d_1, c_1, d_2 \rangle = \langle \langle y, c_1, d_1 \rangle, c_1, d_2 \rangle \in [c_1, d_2]$.

-

$$\begin{aligned}
c_2 &= \langle a, d_2, b\rangle \\
&= \langle\langle x, a, c_1\rangle, d_2, b\rangle \\
&= \langle\langle d_2, b, x\rangle, \langle d_2, b, a\rangle, c_1\rangle \\
&= \langle\langle x, d_2, b\rangle, c_2, c_1\rangle \\
&= \langle\langle x, c_1, c_2\rangle, \langle b, c_1, c_2\rangle, d_2\rangle \\
&= \langle c_1, c_2, d_2\rangle \in [c_1, d_2].
\end{aligned}$$

We now prove that $\{c_1, c_2, \langle c_1, d_2, x\rangle, \langle y, c_1, \langle d_1, c_1, d_2\rangle\}$ forms a tripod centred at c_1:

- $\langle\langle c_1, d_2, x\rangle, c_1, \langle y, c_1, \langle d_1, c_1, d_2\rangle\rangle\rangle = c_1$;

$$\begin{aligned}
\langle\langle c_1, d_2, x\rangle, c_1, \langle y, c_1, \langle d_1, c_1, d_2\rangle\rangle\rangle &= \langle\langle c_1, \langle c_1, d_2, x\rangle, y\rangle, \langle c_1, \langle c_1, d_2, x\rangle, c_1\rangle, \langle d_1, c_1, d_2\rangle\rangle \\
&= \langle c_1, \langle d_1, c_1, d_2\rangle, \langle\langle c_1, y, c_1\rangle, \langle c_1, y, x\rangle, d_2\rangle\rangle \\
&= \langle c_1, \langle d_1, c_1, d_2\rangle, \langle c_1, c_1, d_2\rangle\rangle \\
&= \langle c_1, c_1, \langle d_1, c_1, d_2\rangle\rangle \\
&= c_1.
\end{aligned}$$

- $\langle c_2, c_1, \langle y, c_1, \langle d_1, c_1, d_2\rangle\rangle\rangle = c_1$;

$$\begin{aligned}
\langle c_2, c_1, \langle y, c_1, \langle d_1, c_1, d_2\rangle\rangle\rangle &= \langle\langle c_1, c_2, y\rangle, \langle c_1, c_2, c_1\rangle, \langle d_1, c_1, d_2\rangle\rangle \\
&= \langle c_1, c_2, \langle d_1, c_1, d_2\rangle\rangle \\
&= \langle\langle c_1, c_2, d_1\rangle, \langle c_1, c_2, c_1\rangle, d_2\rangle \\
&= \langle c_1, d_2, \langle c_1, c_2, d_1\rangle\rangle \\
&= \langle c_1, d_2, \langle\langle a, b, d_1\rangle, \langle a, b, d_2\rangle, d_1\rangle\rangle \\
&= \langle c_1, d_2, \langle a, b, \langle d_1, d_2, d_1\rangle\rangle\rangle \\
&= \langle c_1, d_2, \langle a, b, d_1\rangle\rangle \\
&= \langle c_1, d_2, c_1\rangle \\
&= c_1.
\end{aligned}$$

- $\langle c_2, c_1, \langle c_1, d_2, x\rangle\rangle = c_1$;

$$\begin{aligned}
\langle c_2, c_1, \langle c_1, d_2, x\rangle\rangle &= \langle c_1, c_2, c_1\rangle, \langle c_1, c_2, x\rangle, d_2\rangle \\
&= \langle c_1, c_1, d_2\rangle \\
&= c_1.
\end{aligned}$$

Since we have shown that $\{c_1, c_2, \langle c_1, d_2, x\rangle, \langle y, c_1, \langle d_1, c_1, d_2\rangle\}$ forms a tripod, we can proceed to proving cases 1 and 2.

Case 1: If $c_2 = \langle c_2, d_1, y \rangle$, then c_1 is not a corner:

$$\begin{aligned}
\langle b, d_1, y \rangle &= \langle c_2, b, y \rangle, d_1, y \rangle \\
&= \langle d_1, y, \langle c_2, y, b \rangle \rangle \\
&= \langle \langle d_1, y, c_2 \rangle, y, b \rangle \\
&= \langle c_2, y, b \rangle \\
&= b.
\end{aligned}$$

Since $b = \langle b, d_1, y \rangle, c_1$ is not a corner.

Case 2: If $c_1 = \langle c_1, d_2, x \rangle$, then c_2 is not a corner:

$$\begin{aligned}
\langle a, d_2, x \rangle &= \langle \langle x, a, c_1 \rangle, d_2, x \rangle \\
&= \langle d_2, x, \langle c_1, x, a \rangle \rangle \\
&= \langle \langle d_2, x, c_1 \rangle, x, a \rangle \\
&= \langle c_1, x, a \rangle \\
&= a.
\end{aligned}$$

Since $a = \langle a, d_2, x \rangle, c_2$ is not a corner.

Now that we have proven cases 1 and 2, we turn our attention to case 3. The two central cuboids are not enough to tell us about case 3, so we focus on the whole universal median algebra. After collapsing the ab and $d_1 d_2$ hyperplanes we are left with the below picture

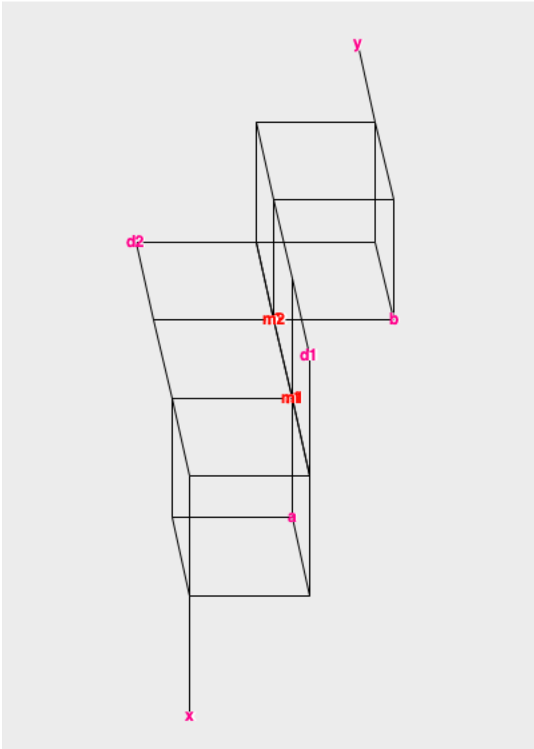


FIGURE A.4: This is the result of collapsing the ab and d_1d_2 hyperplanes in the universal median algebra, i.e. case 3.

We assume the following:

- $c_1 = \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle$.
- $c_2 = \langle x, c_2, \langle d_1, c_2, d_2 \rangle \rangle$.

Case 3: Again, we begin by checking the conditions for the interval $[x, c_1]$ as seen in Note 3.7. We need to show that a, $\langle c_1, d_1, x \rangle$, $\langle c_1, d_2, x \rangle$ are in $[x, c_1]$. By definition, $\langle c_1, d_1, x \rangle, \langle c_1, d_2, x \rangle \in [x, c_1]$. Since $a = \langle x, a, c_1 \rangle$ (due to the ordering seen early on), this tells us that $a \in [x, c_1]$. Next, we show the remaining part of the condition, namely

- $\langle \langle c_1, d_1, x \rangle, c_1, \langle c_1, d_2, x \rangle \rangle = c_1$; let

$$m_1 = \langle y, c_1, \langle d_1, c_1, d_2 \rangle \rangle = c_1$$
$$m_2 = \langle x, c_2, \langle d_1, c_2, d_2 \rangle \rangle = c_2.$$

By the 5-point condition, $\langle \langle c_1, d_1, x \rangle, c_1, \langle c_1, d_2, x \rangle \rangle = \langle c_1, x, \langle c_1, d_1, d_2 \rangle \rangle$.

We know that $\langle x, m_2, c_1\rangle = \langle x, c_2, c_1\rangle \iff \langle x, c_1, \langle d_1, c_2, d_2\rangle\rangle = c_1$. To sum up, we have

$$\begin{aligned} c_1 &= \langle x, c_1, \langle d_1, c_2, d_2\rangle\rangle \\ &= \langle d_1, \langle x, c_1, c_2\rangle, \langle x, c_1, d_2\rangle\rangle \\ &= \langle d_1, c_1, \langle d_2, c_1, x\rangle\rangle \\ &= \langle\langle d_1, c_1, d_2\rangle, c_1, x\rangle \\ &= \langle x, c_1, \langle d_1, c_1, d_2\rangle\rangle. \end{aligned}$$

- $\langle a, c_1, \langle c_1, d_1, x\rangle\rangle = c_1$;

$$\begin{aligned} \langle a, c_1, \langle c_1, d_1, x\rangle\rangle &= \langle\langle a, c_1, c_1\rangle, \langle a, c_1, x\rangle, d_1\rangle \\ &= \langle c_1, a, d_1\rangle \\ &= \langle\langle a, d_1, b\rangle, a, d_1\rangle \\ &= \langle b, \langle a, d_1, a\rangle, \langle a, d_1, d_1\rangle\rangle \\ &= \langle b, a, d_1\rangle \\ &= c_1. \end{aligned}$$

- $\langle a, c_1, \langle c_1, d_2, x\rangle\rangle = c_1$;

$$\begin{aligned} \langle a, c_1, \langle c_1, d_2, x\rangle\rangle &= \langle\langle a, c_1, c_1\rangle, \langle a, c_1, x\rangle, d_2\rangle \\ &= \langle c_1, a, d_2\rangle \\ &= \langle\langle a, d_1, b\rangle, a, d_2\rangle \\ &= \langle\langle a, d_2, a\rangle, \langle a, d_2, b\rangle, d_1\rangle \\ &= \langle a, c_2, d_1\rangle \\ &= \langle a, \langle c_1, c_2, b\rangle, d_1\rangle \\ &= \langle\langle a, d_1, c_1\rangle, \langle a, d_1, b\rangle, c_2\rangle \\ &= \langle c_1, c_2, \langle a, d_1, c_1\rangle\rangle \\ &= \langle\langle c_1, c_2, a\rangle, \langle c_1, c_2, c_1\rangle, d_1\rangle \\ &= \langle c_1, c_1, d_1\rangle \\ &= c_1. \end{aligned}$$

All the conditions of the theorem have been met, so we are in a position to show that we can only have at most one corner. Now,

(i) we must have $c_1 \neq a$ (i.e. the a hyperplane cannot collapse) because then c_1 would immediately fail to be a corner;

(ii) if $c_1 = \langle c_1, d_2, x \rangle$ (i.e. the d_2 hyperplane collapses), then this is a case that has already been proven (see case 2);

(iii) this leaves us with $c_1 = \langle c_1, d_1, x \rangle$ (i.e. the d_1 hyperplane collapses). Then c_1 is not a corner as

$$\begin{aligned}
\langle a, d_1, x \rangle &= \langle \langle a, x, c_1 \rangle, x, d_1 \rangle \\
&= \langle a, x, \langle c_1, x, d_1 \rangle \rangle \\
&= \langle a, x, c_1 \rangle \\
&= a.
\end{aligned}$$

Therefore we have shown that in the two-dimensional case, we cannot have more than one corner present. □

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