SPECTRA OF SUBRINGS OF COHOMOLOGY GENERATED BY CHARACTERISTIC CLASSES FOR FUSION SYSTEMS

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Abstract. If $\mathcal{F}$ is a saturated fusion system on a finite p-group S, we define the Chern subring $\mathrm{Ch}(\mathcal{F})$ of $\mathcal{F}$ to be the subring of $H^*(S;\mathbb{F}_p)$ generated by Chern classes of $\mathcal{F}$-stable representations of S. We show that $\mathrm{Ch}(\mathcal{F})$ is contained in $H^*(\mathcal{F};\mathbb{F}_p)$ and apply a result of Green and the first author to describe its maximal ideal spectrum in terms of a certain category of elementary abelian subgroups. We obtain similar results for various related subrings, including those generated by characteristic classes of $\mathcal{F}$-stable S-sets.

1. Introduction

Let G be a finite group and k be a field of characteristic p. Quillen's description [15] of the spectrum of the mod-p cohomology ring of G has been extremely useful in representation theory. The support variety of a kG-module M is a subvariety of the spectrum, subvarieties correspond to ideals in the cohomology ring, and the ideal defining the support variety is the kernel of the ring homomorphism $H^*(G) = \mathrm{Ext}_{\mathbb{F}_pG}(\mathbb{F}_p, \mathbb{F}_p) \to \mathrm{Ext}_{kG}(M, M)$. Just as ideals of $H^*(G)$ correspond to subvarieties of the spectrum, subrings of $H^*(G)$ correspond to quotients of the spectrum, with subrings over which $H^*(G)$ is integral corresponding to quotients with finite fibres. In [12] Green and the first named author gave a description of the spectra of subrings of $H^*(G)$ that are both 'large' and 'natural', paying particular attention to the subring generated by Chern classes of representations of G. A later article applied these results to the subring of $H^*(G)$ generated by characteristic classes of homomorphisms from G to the symmetric group Σ_n for all n [13].

Our aim is to extend many of these results concerning subrings of $H^*(G)$ to analogous results for subrings of the cohomology of a saturated fusion system. Recall that a *fusion system* $\mathcal{F}$ on a finite p-group S is a category with $\mathrm{Ob}(\mathcal{F}) = \{P \leq S\}$ and $\mathrm{Mor}(\mathcal{F})$ a set of injective group homomorphisms between subgroups satisfying some weak axioms. $\mathcal{F}$ is *saturated* if it satisfies two additional 'Sylow' axioms which hold whenever $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of a group G with Sylow p-subgroup S, where morphisms are given by G-conjugation maps. The cohomology $H^*(\mathcal{F})$ of a saturated fusion system $\mathcal{F}$ is defined to be the subring of $\mathcal{F}$-stable elements in $H^*(S)$ (see Section 3.1).

We now fix a finite p-group S and let $\mathcal{F}$ be a saturated fusion system on S. We define categories of elementary abelian subgroups of S by stipulating that an injective homomorphism $f \in \mathrm{Hom}(E_1, E_2)$ is in

$\mathcal{E}(\mathcal{F})$ iff there exists $\varphi \in \mathcal{F}$ such that $f(e) = \varphi(e)$ for all $e \in E_1$;

$\mathcal{E}'(\mathcal{F})$ iff for each $e \in E_1$ there exists $\varphi \in \mathcal{F}$ such that $f(e) = \varphi(e)$;

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$\mathcal{E}'_{\mathbb{R}}(\mathcal{F})$ iff for each $e \in E_1$ there exists $\varphi \in \mathcal{F}$ such that $f(e) \in \{\varphi(e), \varphi(e^{-1})\}$;
$\mathcal{E}'_P(\mathcal{F})$ iff $\langle e \rangle$ and $\langle f(e) \rangle$ are $\mathcal{F}$-conjugate for all $e \in E_1$;
$\mathcal{A}(\mathcal{F})$ iff $f(U)$ is $\mathcal{F}$-conjugate to U for all $U \leq E_1$.

Here by $\langle e \rangle$ we mean the subgroup generated by e. Note that $f \in \mathcal{E}(\mathcal{F})$ is equivalent to $f \in \mathrm{Hom}_{\mathcal{F}}(E_1, E_2)$ and that φ depends on the choice of e for the categories $\mathcal{E}'(\mathcal{F})$ and $\mathcal{E}'_{\mathbb{R}}(\mathcal{F})$.

Our results describe various spectra of subrings of $H^*(\mathcal{F})$ in terms of the above categories. Assume that k is algebraically closed, and for a finitely generated commutative $\mathbb{F}_p$-algebra R write $V_R(k) := \mathrm{Hom}(R, k)$ for the variety of ring homomorphisms from R to k with the Zariski topology generated by closed sets of form

$$\{\phi \in \mathrm{Hom}(R, k) \mid \ker(\phi) \supseteq I\},$$

for an ideal $I \trianglelefteq R$. Note that any ring homomorphism $f : R \to R'$ determines a mapping of varieties

$$f^* : V_{R'}(k) \to V_R(k), \text{ given by } \phi \mapsto \phi \circ f.$$

Moreover, if R' is a finitely generated $f(R)$-module, then f^* has finite fibres. Note also that there is a continuous map

$$V_R(k) \to \mathrm{Spec}(R), \text{ given by } \phi \mapsto \ker(\phi).$$

Depending on the choice of k this can be made surjective. Since $H^*(S)$ is a graded finitely generated $\mathbb{F}_p$-algebra, $h^*(S) := H^*(S)/\sqrt{0}$ is a finitely generated *commutative* $\mathbb{F}_p$-algebra (here $\sqrt{0}$ denotes the ideal generated by elements which square to 0) and we write

$$X_S(k) := V_{h^*(S)}(k) = \mathrm{Hom}(h^*(S), k)$$

for the associated variety. Note that a group homomorphism $f : P \to Q$ induces a continuous map $f_* : X_P(k) \to X_Q(k)$ between the associated varieties. With the above terminology, Linckelmann has shown [14] that there exists a homeomorphism

$$\varinjlim_{\mathcal{E}(\mathcal{F})} X_E(k) \to V_R(k)$$

where $R = H^*(\mathcal{F}) \subseteq H^*(S)$. This is an analogue of Quillen's description of the spectrum of $H^*(G)$ mentioned above. In Section 3.2 we consider subrings of $H^*(\mathcal{F})$ generated by Chern classes of $\mathcal{F}$-stable ordinary representations of S: those for which the associated character is constant on $\mathcal{F}$-conjugacy classes (see Section 2). Our first main result may be viewed as an analogue of [12, Proposition 7.1] for fusion systems:

Theorem 1.1. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let R be the subring of $H^*(\mathcal{F})$ generated by Chern classes of:*

(1) *representations of S;*
(2) *real representations of S;*
(3) *permutation representations of S,*

which are $\mathcal{F}$-stable. Then in each case, there is a homeomorphism

$$\varinjlim_{\mathcal{C}(R)} X_E(k) \to V_R(k)$$

where the category $\mathcal{C}(R)$ is:

$$(1)\ \mathcal{E}'(\mathcal{F});\ (2)\ \mathcal{E}'_{\mathbb{R}}(\mathcal{F});\ (3)\ \mathcal{E}'_P(\mathcal{F}).$$

To prove Theorem 1.1, we first observe that in each case the subring R is both large and natural (see Definition 4.2). We then apply a result of Green and the first named author, Theorem 4.3, to deduce the existence of a category $\mathcal{C}(R)$ as in the conclusion of Theorem 1.1. To describe the morphisms in $\mathcal{C}(R)$ we exploit the fact that they are uniquely determined by how they interact with the characters of the representations we consider (see Lemma 4.4). In case (3), we rely on the existence of an explicit basis for the ring of $\mathcal{F}$-stable permutation characters of S determined by Reeh [16].

In [13] the authors study, for a finite group G, the variety for the subring $\mathcal{S}(G)$ of $H^*(G)$ generated by the images of the maps $\rho^* : H^*(\Sigma_n) \to H^*(G)$ (there the ring $\mathcal{S}(G)$ was denoted by $S_h(G)$ which clashes with our use of S as a Sylow p-subgroup). Our second main result may be regarded as an analogue of [13, Theorem 2.6] for fusion systems:

Theorem 1.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let R be the subring of $H^*(S)$ generated by characteristic classes of $\mathcal{F}$-stable permutations of S. Then there is a homeomorphism:*

$$\underset{\mathcal{A}(\mathcal{F})}{\underrightarrow{\operatorname{colim}}}\, X_E(k) \to V_R(k).$$

Our argument to prove Theorem 1.2 is an adaptation of that found in [13], for finite groups, and relies on particular properties of Reeh's basis of $\mathcal{F}$-stable permutation characters of S. As for the subrings considered in Theorem 1.1, we show in Corollary 6.5 that the ring R in Theorem 1.2 can be described in terms of images in cohomology of maps between classifying spaces.

We close the introduction with some remarks pertaining to a possible extension of Theorems 1.1 and 1.2 to the case of fusion systems on infinite groups. Indeed, the main result of [12] is concerned with varieties for the cohomology of *any* compact Lie group. The fusion system of a such a group is a particular example of a *p-local compact group* which is a saturated fusion system on a discrete p-toral group (a p-group with a finite index infinite torus $(\mathbb{Z}/p^\infty)^n$, for some $n \geq 1$). There is a version of Quillen stratification for such groups (see [4, Theorem 5.1]), and it has been shown that certain classes of p-local compact groups, for example those coming from finite loop spaces and p-compact groups, admit unitary embeddings, at least in one of two possible senses (see [7]). Observe that the existence of unitary embeddings for compact Lie groups is a key ingredient in the proof of [12, Proposition 2.2].

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2. $\mathcal{F}$-STABLE REPRESENTATIONS AND GROUP ACTIONS

We adopt standard notation for fusion systems as found, for example, in [2]. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S.

2.1. **The ring of $\mathcal{F}$-stable representations.** Recall that an object $P \leq S$ is *$\mathcal{F}$-centric* if for all morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, we have $C_S(\varphi(P)) \leq \varphi(P)$. Let $\mathcal{F}^c$ denote the set of $\mathcal{F}$-centric subgroups.

Definition 2.1. The *orbit category* $\mathcal{O} = \mathcal{O}(\mathcal{F})$ of $\mathcal{F}$ is the category defined via:

(a) $\mathrm{ob}(\mathcal{O}) = \{P \mid P \leq S\}$;
(b) for each $P, Q \leq S$, $\mathrm{Hom}_{\mathcal{O}}(P, Q) = \mathrm{Rep}_{\mathcal{F}}(P, Q) := \mathrm{Hom}_{\mathcal{F}}(P, Q)/\mathrm{Inn}(Q)$ is the set of $\mathrm{Inn}(Q)$-orbits of $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ (with action given by right composition of morphisms).

The centric orbit category $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}$ with object set $\mathcal{F}^c$.

Definition 2.2. An ordinary character χ of S is *$\mathcal{F}$-stable* if for all $g \in S$ and morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(\langle g \rangle, S)$, $\chi(\varphi(g)) = \chi(g)$. That is, χ takes the same value on all members of each $\mathcal{F}$-conjugacy class of S. Denote by $\mathrm{C}(\mathcal{F})$ the subring of $\mathrm{C}(S)$ (the character ring of S) consisting of $\mathcal{F}$-stable characters. Also, for a natural number n, denote by $\mathrm{C}_n(S)$ and $\mathrm{C}_n(\mathcal{F})$ the subsets of characters of degree n.

Following [8], for any group G, let $\mathrm{Rep}_n(G) = \mathrm{Rep}(G, U(n))$ denote the set of isomorphism classes of n-dimensional ordinary representations of G. Let $R(G)$ denote the representation ring of G.

Definition 2.3. A complex representation ρ of S is *$\mathcal{F}$-fusion preserving* if $\rho|_P = \rho|_{\varphi(P)} \circ \varphi \in \mathrm{Rep}_n(P)$ for any $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$; let $\mathrm{Rep}_n(\mathcal{F})$ denote the set of isomorphism classes of n-dimensional complex $\mathcal{F}$-fusion preserving representations of S

Note that $\rho \in \mathrm{Rep}_n(\mathcal{F})$ if and only if $\chi_\rho \in \mathrm{C}_n(\mathcal{F})$ where χ_ρ is the character associated to ρ. Using the Alperin-Goldschmidt fusion theorem for fusion systems, one can show:

Proposition 2.4. *Let $\mathcal{F}$ be a saturated fusion system over S. Then*
$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)} \mathrm{Rep}_n(P) \cong \mathrm{Rep}_n(\mathcal{F}).$$

Proof. This is a straightforward modification of the argument used to prove [8, Proposition 3.6] with $\mathcal{F}^{cr}$ replaced by $\mathcal{F}^c$. □

Now let $R(\mathcal{F})$ be the subring of $\mathcal{F}$-stable representations in $R(S)$ and $\mathrm{C}(\mathcal{F})$ be the Grothendieck group of
$$\bigcup_{n=1}^{\infty} \mathrm{C}_n(\mathcal{F}).$$

Write $S^{\mathcal{F}}$ for a set of $\mathcal{F}$-conjugacy class representatives of S.

Theorem 2.5. $\mathrm{C}(\mathcal{F}) \otimes \mathbb{C}$ *is equal to the space of* $\mathbb{C}$*-class functions on* $S^{\mathcal{F}}$.

Proof. See [3, Lemma 2.1]. □

From this result, we easily deduce that two elements are $\mathcal{F}$-conjugate if their character values coincide:

Corollary 2.6. *If $s, t \in S$ are such that $\chi(s) = \chi(t)$ for all $\chi \in \mathrm{C}(\mathcal{F})$, then s and t are $\mathcal{F}$-conjugate.*

2.2. **The ring of $\mathcal{F}$-stable S-sets.** If S acts on a finite set X and $\phi : P \to S$ is a homomorphism, denote by ${}^{\phi}X$ the P-set X with action given by $p \cdot x = \phi(p)x$, where the right side is the original action of S on X.

Definition 2.7. Let X be a finite S-set.

(1) X is said to be *$\mathcal{F}$-stable* if for every $P \leq S$ and every morphism $\phi : P \to S$ in $\mathcal{F}$ the P-sets X and ${}^{\phi}X$ are isomorphic.
(2) X is said to be *linearly $\mathcal{F}$-stable* if the associated permutation character is $\mathcal{F}$-stable.

Plainly any $\mathcal{F}$-stable S-set is linearly $\mathcal{F}$-stable, but the converse is not true. For example [12, Section 7] discusses an example of this phenomenon for $G = \mathrm{GL}(3, \mathbb{F}_p)$. Note that we may, equivalently define a homomorphism $\rho : S \to \Sigma_n$ associated to an S-set X of cardinality n, to be $\mathcal{F}$-stable if for all $P \leq S$ and all morphisms $\phi : P \to S$ in $\mathcal{F}$, the morphisms $\rho|_P$ and $\rho \circ \phi$ differ by an inner automorphism of Σ_n.

If X is an S-set and $Q \leq S$, let $\Phi_Q(X) = |X^Q|$ denote the number of Q-fixed points of X. To prove Theorem 1.2 we shall need to know that there are sufficiently many $\mathcal{F}$-stable permutation representations. Reeh [16] shows the following:

Proposition 2.8. *For each $P \leq S$ there exists an $\mathcal{F}$-stable S-set α_P with the properties that:*

(1) $\Phi_Q(\alpha_P) = 0$ *unless Q is $\mathcal{F}$-subconjugate to P;*
(2) $\Phi_{P'}(\alpha_P) = |N_S(P')/P'|$ *when P' is a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate of P; and*
(3) $\alpha_P \cong \alpha_Q$ *as S-sets if P and Q are $\mathcal{F}$-conjugate.*

Proof. See [16, Proposition 4.8]. □

In fact, Reeh shows that the S-sets α_Q as Q ranges over a set of $\mathcal{F}$-conjugacy class representatives of subgroups of S can be chosen to form an additive basis for the Burnside ring of $\mathcal{F}$-stable S-sets, but we will not need this.

3. Cohomology of fusion systems and the Chern subring

As in the previous section, we let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. As shown by Chermak [10], to $\mathcal{F}$ we may associate a unique (up to isomorphism) centric linking system $\mathcal{L}$ whose p-completed nerve plays the role of the classifying space of $\mathcal{F}$. In particular, when $\mathcal{F} = \mathcal{F}_S(G)$ is the fusion system of a finite group with Sylow p-subgroup S, we have $|\mathcal{L}|_p^\wedge \simeq BG_p^\wedge$. The triple $(S, \mathcal{F}, \mathcal{L})$ is sometimes referred to as a *p-local finite group.*

3.1. **The cohomology ring of a fusion system.** Following [6, Section 5], we define the cohomology of $\mathcal{F}$ as follows:

Definition 3.1. The subring $H^*(\mathcal{F};\mathbb{F}_p)$ of *$\mathcal{F}$-stable elements* of $H^*(S,\mathbb{F}_p)$ is the preimage in $H^*(S,\mathbb{F}_p)$ of the natural map

$$H^*(S,\mathbb{F}_p) \to \varprojlim_{\mathcal{O}(\mathcal{F}^c)} H^*(-;\mathbb{F}_p).$$

From [8, Theorem 4.2], we obtain:

Theorem 3.2. *There is an isomorphism*

$$H^*(|\mathcal{L}|_p^\wedge;\mathbb{F}_p) \cong \varprojlim_{\mathcal{O}(\mathcal{F}^c)} H^*(BP;\mathbb{F}_p).$$

In particular, the rings $H^*(\mathcal{F};\mathbb{F}_p)$ and $H^*(|\mathcal{L}|_p^\wedge;\mathbb{F}_p)$ are isomorphic.

In [8, Theorem 5.3], for $m > 0$, the authors prove the following:

Theorem 3.3. *There is a natural map*

$$\psi_m : [|\mathcal{L}|_p^\wedge, BU(m)_p^\wedge] \to \mathrm{Rep}_m(\mathcal{F})$$

satisfies:

(1) *for each $\rho \in \mathrm{Rep}_m(\mathcal{F})$ and sufficiently large $M > 0$, $\rho \oplus M\mathrm{reg} \in \mathrm{im}(\psi_{m+M|S|})$;*
(2) *if $f_1, f_2 \in [|\mathcal{L}|_p^\wedge, BU(m)_p^\wedge]$ are such that $\psi_m(f_1) = \psi_m(f_2)$ then $f_1 \oplus h \simeq f_2 \oplus h$ for some $h \in [|\mathcal{L}|_p^\wedge, BU(n)_p^\wedge]$ with $\psi_n(h) = N\mathrm{reg}$ (some $N \geq 0$).*

Here reg denotes the regular representation of S and $\oplus$ is the Whitney sum. Note that we have strengthened the *statement* of (1) above compared to that given in [8, Theorem 5.3]; there it is claimed only that there exists some $M > 0$, but the argument given proves the stronger claim which we will require.

3.2. **Chern classes of $\mathcal{F}$-stable representations.** Write

$$\mathbb{F}_p[c_1, c_2, \ldots, c_n] = H^*(BU(n);\mathbb{F}_p) \cong \mathbb{F}_p[x_1, \ldots, x_n]^{\Sigma_n},$$

where c_i has degree $2i$ and the isomorphism is given by sending c_i to the ith symmetric polynomial. For any finite group P, a unitary representation $\rho : P \to U(n)$ induces a map $\hat{\rho} : BP \to BU(n)$ whose homotopy class depends on the conjugacy class of ρ. We thus obtain a map

$$\rho^* : H^*(BU(n);\mathbb{F}_p) \to H^*(BP;\mathbb{F}_p) = H^*(P;\mathbb{F}_p)$$

and so define the ith *Chern class* of ρ to be $c_i(\rho) := \rho^*(c_i) \in H^{2i}(P;\mathbb{F}_p)$.

In particular, if $\rho \in \mathrm{Rep}_n(\mathcal{F}) \subseteq \mathrm{Rep}_n(S)$ then for each $1 \leq i \leq n$, we have $c_i(\rho) \in H^{2i}(S;\mathbb{F}_p)$. In fact, we have:

Proposition 3.4. *For $\mathcal{F}$ and ρ as above, $c_i(\rho) \in H^{2i}(\mathcal{F};\mathbb{F}_p)$ for each $1 \leq i \leq n$.*

Proof. By Proposition 2.4 we can regard ρ as a tuple

$$(\rho_P)_P \in \varprojlim_{\mathcal{O}(\mathcal{F}^c)} \mathrm{Rep}_n(P)$$

given by an $\mathcal{O}(\mathcal{F}^c)$-compatible family of representations. For each $\mathcal{F}$-centric subgroup P, restriction induces commutative diagrams

$$\begin{array}{ccc} H^*(BU(n);\mathbb{F}_p) & \xrightarrow{\rho|_P} & H^*(P;\mathbb{F}_p) \\ & \searrow^{\rho^*} & \uparrow \\ & & H^*(S;\mathbb{F}_p) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^*(BU(n);\mathbb{F}_p) & \xrightarrow{(\rho|_P)} & \varprojlim_{\mathcal{O}(\mathcal{F}^c)} H^*(BP;\mathbb{F}_p) \\ & \searrow^{\rho^*} & \uparrow\cong \\ & & H^*(\mathcal{F};\mathbb{F}_p) \end{array}$$

and thus using Definition 3.1, we have $c_i(\rho) = (c_i(\rho|_P))_P \in H^{2i}(\mathcal{F};\mathbb{F}_p)$. □

Thus for $\rho \in \mathrm{Rep}_n(\mathcal{F})$, it makes sense to define the *ith Chern class* of ρ to be $c_i(\rho) \in H^*(\mathcal{F};\mathbb{F}_p)$. We may further define $c_\bullet(\rho) = 1 + c_1(\rho) + \cdots + c_n(\rho)$ to be the *total Chern class* of ρ. This definition is extended to virtual representations by setting $c_i(-\rho) = \rho^*(c_i')$ where $c_\bullet' = 1 + c_1' + c_2' + \cdots$ is the unique power series in $\mathbb{F}_p[[c_1, \ldots, c_n]]$ satisfying $c_\bullet' c_\bullet = 1$. In particular, for each i it follows that $c_i(-\rho)$ is expressible as a polynomial in the Chern classes $c_j(\rho)$ for $j \leq i$.

Definition 3.5. The *Chern subring* $\mathrm{Ch}(\mathcal{F})$ of $H^*(\mathcal{F};\mathbb{F}_p)$ is the subring generated by the $c_i(\rho)$ for all i and virtual representations ρ, or equivalently for all representations ρ.

There is an alternative definition of the Chern subring using the classifying space for the linking system, which we will temporarily denote by $\mathrm{Ch}'(\mathcal{F})$. The mod-$p$ cohomology of $BU(n)^\wedge_p$ is of course a polynomial ring $\mathbb{F}_p[c_1, \ldots, c_n]$, and $\mathrm{Ch}'(\mathcal{F})$ is defined to be the subring of $H^*(|\mathcal{L}|^\wedge_p;\mathbb{F}_p)$ generated by the images in cohomology of all maps $f : |\mathcal{L}|^\wedge_p \to BU(n)^\wedge_p$ for all $n \geq 1$. Similarly, for a finite group G define $\mathrm{Ch}'(G)$ to be the subring of $H^*(G;\mathbb{F}_p)$ generated by the images in cohomology of all maps $f : BG \to BU(n)$.

Proposition 3.6. *We have* $\mathrm{Ch}'(G) = \mathrm{Ch}(G)$ *and* $\mathrm{Ch}'(\mathcal{F}) = \mathrm{Ch}(\mathcal{F})$.

Proof. Each n-dimensional representation ρ gives rise to a map $B\rho : BG \to BU(n)$, and so $\mathrm{Ch}(G) \subseteq \mathrm{Ch}'(G)$. In general, not every map $f : BG \to BU(n)$ arises in this way (see [1, Proposition 1.18]), but this is true stably, in the sense that there is a virtual representation ρ' whose Chern classes coincide with those of f, as shown in [1, Theorem 1.10]. It follows that $\mathrm{Ch}(G) = \mathrm{Ch}'(G)$.

By the statement for groups we see that $\mathrm{Ch}(S) = \mathrm{Ch}'(S)$, from which it follows that $\mathrm{Ch}'(\mathcal{F}) \subseteq \mathrm{Ch}(\mathcal{F})$. It remains to establish the opposite inclusion. The direct analogue of [1, Theorem 1.10] for fusion systems is not known. Instead, we show that given any $\theta \in \mathrm{Rep}_n(\mathcal{F})$, there exists $N \geq n$ and $f \in [|\mathcal{L}|^\wedge_p, BU(N)^\wedge_p]$ so that for each $i \leq n$, $c_i(\theta) = c_i(f^*)$.

For any n, note that $c_\bullet(p\theta) = (c_\bullet(\theta))^p$, and so inductively one sees that $c_i(p^k\theta)$ can only be non-zero when p^k divides i. Now, let ρ denote the regular representation of S. By Theorem 3.3 one can pick $p^k > n$ sufficiently large so that $\theta \oplus p^k\rho$ is realized by a map $|\mathcal{L}|^\wedge_p \to BU(N)^\wedge_p$, where $N = n + p^k|S|$. Each Chern class of $\theta \oplus p^k\rho$ is contained in $\mathrm{Ch}'(\mathcal{F})$, and for $i \leq n$, $c_i(\theta \oplus p^k\rho) = c_i(\theta)$. □

Note that $\mathrm{Ch}(\mathcal{F})$ is finitely generated by [12, Proposition 2.1].

4. Varieties and Quillen stratification

4.1. **The Green-Leary category of elementary abelian subgroups.** Following [12, Section 6], to a subring R of the cohomology ring of a finite group can be associated a certain diagram $\mathcal{C}(R)$ of elementary abelian subgroups, and this is used to recover the maximal ideal spectrum of R under mild conditions.

Definition 4.1. Let S be a finite group and R be a subring of $H^*(S;\mathbb{F}_p)$. Let $\mathcal{C} = \mathcal{C}(R)$ be the category whose objects are the elementary abelian subgroups of S, and where $f \in \mathrm{Hom}_{\mathcal{C}}(E_1, E_2)$ if and only if the corresponding diagram

$$\begin{array}{ccc} R & \xleftarrow{\quad \mathrm{Id} \quad} & R \\ \big\downarrow{\scriptstyle \mathrm{res}} & & \big\downarrow{\scriptstyle \mathrm{res}} \\ h^*(E_1;\mathbb{F}_p) & \xleftarrow[f^*]{} & h^*(E_2;\mathbb{F}_p) \end{array} \tag{4.1}$$

commutes.

As in [12, Section 6], we also define:

Definition 4.2. Let S be a finite group and R be a subring of $H^*(S;\mathbb{F}_p)$.

(1) R is *large* if it contains the Chern classes of a positive multiple of the regular representation of S;
(2) R is *natural* if it is generated by homogeneous elements and closed under the action of the Steenrod algebra.

We remark that the definition of *large* given above is a simplification of the one used in [12]. In the case when S is not finite (e.g., S a compact Lie group or a p-toral group), a large subring is one that contains the Chern classes of a virtual representation of non-zero degree whose restriction to every elementary abelian p-subgroup of S is regular.

In [12, Theorem 6.1] the authors prove:

Theorem 4.3. *Let S be a finite group and R be a subring of $H^*(S;\mathbb{F}_p)$. If R is large and natural then the map*

$$\underset{\mathcal{C}(R)}{\underrightarrow{\mathrm{colim}}}\, X_E(k) \to V_R(k)$$

is a homeomorphism.

Note that if $R = H^*(S;\mathbb{F}_p)$ then Theorem 4.3 is due to Quillen [15]. One tool for describing $\mathcal{C}(R)$ is [12, Lemma 7.2], which we restate here for convenience:

Lemma 4.4. *Let S be a finite group, let A be an additive subgroup of $R(S)$ containing the regular representation and let $R = R_A$ be the subring of $H^*(S)$ generated by Chern classes of elements of A. Then R is large and natural. Furthermore $f : E_1 \to E_2$ is a morphism in $\mathcal{C}(R)$ if and only if $\chi(e) = \chi(f(e))$ for all $e \in E_1$ and all characters χ of elements of A.*

For example, if R is the subring generated by the Chern class of the regular representation then R is large and natural, and $\mathcal{C}(R)$ is the category of all injective maps between elementary abelian subgroups by [12, Lemma 6.2].

4.2. **Quillen stratification for fusion systems.** Now let S be a finite p-group and $\mathcal{F}$ be a saturated fusion system on S. We first apply Theorem 4.3 to reinterpret Linckelmann's description of the spectrum of the cohomology ring of a fusion system. Recall that a subgroup $P \leq S$ is said to be $\mathcal{F}$-*subconjugate* to $Q \leq S$ if some $\mathcal{F}$-conjugate of P is contained in Q.

Proposition 4.5. *Let $\mathcal{F}$ be a saturated fusion system on S and E be an elementary abelian subgroup of S. Let σ_E be a homogeneous element in $h^*(E;\mathbb{F}_p)$ satisfying*

$$\mathrm{res}^E_F(\sigma_E) = 0 \text{ for each } F < E.$$

Then,

1. *for any $\eta \in h^*(E;\mathbb{F}_p)^{\mathrm{Aut}_{\mathcal{F}}(E)}$, there is $\eta' \in H^*(\mathcal{F};\mathbb{F}_p)$ such that $\mathrm{res}^S_E(\eta') = (\sigma_E \cdot \eta)^{p^a}$; and*
2. *there exists $\rho_E \in h^*(\mathcal{F};\mathbb{F}_p)$ such that $\mathrm{res}^S_E(\rho_E) = (\sigma_E)^{p^a}$ and $\mathrm{res}^S_F(\rho_E) = 0$ for all subgroups F to which E is not $\mathcal{F}$-subconjugate.*

Proof. See [14, Proposition 6]. □

Following [14], let $X_{\mathcal{F}}(k) = V_{h^*(\mathcal{F};\mathbb{F}_p)}(k)$ denote the maximal ideal spectrum of $H^*(\mathcal{F};\mathbb{F}_p)$ and, for a subgroup $Q \leq S$, set $X_{\mathcal{F},Q}(k) := (\mathrm{res}^S_Q)^*(X_Q(k))$ where res^S_Q is the restriction map $H^*(\mathcal{F};\mathbb{F}_p) \to H^*(Q;\mathbb{F}_p)$. Finally set

$$X^+_Q(k) := X_Q(k) \setminus \bigcup_{R<Q} (\mathrm{res}^Q_R)^*(X_R(k)), \text{ and } X_{\mathcal{F},Q}(k)^+ = r^*_Q(X^+_Q(k)).$$

The existence of an element σ_E satisfying the conditions in Proposition 4.5 is shown in [5, Section 5.6] in the discussion which precedes [5, Lemma 5.6.2] and from this Linckelmann deduces in [14, Theorem 1(i)] that

$$X_{\mathcal{F}}(k) = \bigcup_E X_{\mathcal{F},E}(k) = \coprod_E X^+_{\mathcal{F},E}(k),$$

is a union of locally closed subvarieties, where E runs through a set of $\mathcal{F}$-isomorphism class representatives of elementary abelian subgroups of S. Equivalently, (c.f. [5, Corollary 5.6.4]) we have the following result:

Theorem 4.6. *The natural map*

$$\underrightarrow{\mathrm{colim}}_{\mathcal{E}(\mathcal{F})} X_E(k) \to X_{\mathcal{F}}(k)$$

is an inseparable isogeny.

In particular, we have:

Proposition 4.7. *Suppose $R = H^*(\mathcal{F};\mathbb{F}_p) \subseteq H^*(S;\mathbb{F}_p)$. Then,*

1. *R is large and natural;*
2. *$\mathcal{C}(R)$ is exactly $\mathcal{E}(\mathcal{F})$.*

Proof. R is large since the regular representation of S is $\mathcal{F}$-stable, and naturality follows because the action of the Steenrod algebra on the cohomology of each pair of subgroups $P, Q \leq S$ commutes with the maps induced by any homomorphism $\phi : P \to Q$ in $\mathcal{F}$.

A consequence of the argument in [14] which proves Theorem 4.6 is that the morphisms $\varphi \in \mathrm{Hom}_{\mathcal{F}}(E_1, E_2)$ are exactly those for which the diagram (4.1) commutes with $R = H^*(\mathcal{F}; \mathbb{F}_p)$, proving (2). □

5. Chern classes of $\mathcal{F}$-stable representations

We now apply Theorem 4.3 to describe the spectra of the subrings of $H^*(\mathcal{F})$ determined by various classes of $\mathcal{F}$-stable representations of S. The reader is referred to Section 1 for definitions of the categories $\mathcal{E}(\mathcal{F})$, $\mathcal{E}'(\mathcal{F})$, $\mathcal{E}'_{\mathbb{R}}(\mathcal{F})$ and $\mathcal{E}'_P(\mathcal{F})$. We start with the collection of all $\mathcal{F}$-stable representations.

Proposition 5.1. *If* $R = \mathrm{Ch}(\mathcal{F}) \subseteq H^*(\mathcal{F}; \mathbb{F}_p)$ *then* $\mathcal{C}(R) = \mathcal{E}'(\mathcal{F})$.

Proof. Since $A = R(\mathcal{F}) \subseteq R(S)$ is an additive subgroup of the representation ring of S generated by genuine representations, and containing the regular representation, and since $\mathrm{Ch}(\mathcal{F})$ is exactly the subring of $H^*(\mathcal{F}; \mathbb{F}_p)$ generated by Chern classes of elements of A, we have by Lemma 4.4 that:

(1) R is large and natural;
(2) $f \in \mathrm{Mor}_{\mathcal{C}(R)}(E_1, E_2)$ if and only if $\chi(e) = \chi(f(e))$ for all $e \in E_1$ and all characters χ of elements of A.

Hence $\mathcal{C}(R) = \mathcal{E}'(\mathcal{F})$ by Corollary 2.6 and so

$$\underset{\mathcal{E}'(\mathcal{F})}{\underrightarrow{\mathrm{colim}}}\, X_E(k) \to V_{\mathrm{Ch}(\mathcal{F})}(k)$$

is a homeomorphism by Theorem 4.3. □

5.1. **Real representations.** The *real Chern subring* $\mathrm{Ch}_{\mathbb{R}}(\mathcal{F})$ is defined to be the subring of $H^*(S)$ generated by Chern classes of real $\mathcal{F}$-stable representations of S. There are other possible definitions for $\mathrm{Ch}_{\mathbb{R}}(\mathcal{F})$, such as the (possibly larger) ring containing the Chern classes of all complex representations whose characters are real, which we shall temporarily denote by $\mathrm{Ch}''_{\mathbb{R}}(\mathcal{F})$, and the (possibly smaller) ring generated by the images of those maps $f : |\mathcal{L}|^\wedge_p \to BU(n)^\wedge_p$ that factor through the map $BO(n)^\wedge_p \to BU(n)^\wedge_p$ for some n, which we temporarily denote by $\mathrm{Ch}'_{\mathbb{R}}(\mathcal{F})$. We first show that all these choices lead to homeomorphic varieties:

Proposition 5.2. *The inclusions* $\mathrm{Ch}'_{\mathbb{R}}(\mathcal{F}) \to \mathrm{Ch}_{\mathbb{R}}(\mathcal{F}) \to \mathrm{Ch}''_{\mathbb{R}}(\mathcal{F})$ *induce homeomorphisms*

$$V_{\mathrm{Ch}''_{\mathbb{R}}(\mathcal{F})}(k) \to V_{\mathrm{Ch}_{\mathbb{R}}(\mathcal{F})}(k) \to V_{\mathrm{Ch}'_{\mathbb{R}}(\mathcal{F})}(k)$$

of the associated varieties.

Proof. There are inclusions $U(n) \to SO(2n) \to U(2n)$. If M has trace λ, then the image of M under this composite has trace $\lambda + \overline{\lambda}$. Thus if ρ is any complex representation with real character, then 2ρ is a real representation. The total Chern characters are related by $c_\bullet(2\rho) = c_\bullet(\rho)^2$. In the case when $p \neq 2$, it follows that each $c_i(\rho)$ is in the subring generated by the Chern classes of 2ρ, and so for $p \neq 2$, $\mathrm{Ch}''_{\mathbb{R}}(\mathcal{F}) = \mathrm{Ch}_{\mathbb{R}}(\mathcal{F})$. In the case when $p = 2$, $c_{2i}(2\rho) = c_i(\rho)^2$, and since elements of the algebraically closed field k of characteristic two have unique square roots, any ring homomorphism from $\mathrm{Ch}_{\mathbb{R}}(\mathcal{F})$ to k extends uniquely to one from $\mathrm{Ch}''_{\mathbb{R}}(\mathcal{F})$ to k.

Let $\rho : S \to O(n)$ be a real representation of S that is $\mathcal{F}$-stable. The regular representation of S is of course a real representation, and so by the argument used in Proposition 3.6 there exists some large N and $f : |\mathcal{L}|_p^\wedge \to BU(N)_p^\wedge$ so that $c_i(f) = c_i(\rho)$ for $i \leq n$. The composite of f with the inclusion map $i : BU(N)_p^\wedge \to BSO(2N)_p^\wedge$ has $c_\bullet(i \circ f) = (c_\bullet(f))^2$, and so by the argument used in the previous paragraph a ring homomorphism from $\mathrm{Ch}'_{\mathbb{R}}(\mathcal{F})$ to the algebraically closed field k of characteristic $p > 0$ extends uniquely to a homomorphism from $\mathrm{Ch}_{\mathbb{R}}(\mathcal{F})$ to k. □

We now obtain, just as in [12], a description of the variety of $\mathrm{Ch}_{\mathbb{R}}(\mathcal{F})$.

Proposition 5.3. *If $R = \mathrm{Ch}_{\mathbb{R}}(\mathcal{F}) \subseteq H^*(\mathcal{F}; \mathbb{F}_p)$, then $\mathcal{C}(R) = \mathcal{E}'_{\mathbb{R}}(\mathcal{F})$.*

Proof. The regular real representation of S is $\mathcal{F}$-stable, and so $\mathrm{Ch}_{\mathbb{R}}(\mathcal{F})$ is both large and natural by Lemma 4.4. Moreover, it is easily seen to be natural and so Theorem 4.3 may be applied. If χ is a real character, then $\chi(g) = \chi(g^{-1})$ for any $g \in S$. Hence $\mathcal{E}'_{\mathbb{R}}(\mathcal{F})$ is contained in $\mathcal{C}(R)$. By the argument given in [12, Proposition 7.1 and Proposition 7.2], to establish the reverse inclusion it suffices to show that characters of real $\mathcal{F}$-stable representations separate the conjugacy classes of pairs $\{g, g^{-1}\}$. Let g, h be elements of S and suppose that h is not $\mathcal{F}$-conjugate to either g or to g^{-1}. We need to construct a real $\mathcal{F}$-stable character which takes different values on g and h. We first claim that there is an $\mathcal{F}$-stable character χ with

$$\chi(h) \neq \chi(g) \text{ and } \chi(h) \neq \chi(g^{-1}) = \overline{\chi(g)}. \tag{5.1}$$

To see this, note that Corollary 2.6 implies there are χ_1 and χ_2 with $\chi_1(h) \neq \chi_1(g)$ and $\chi_2(h) \neq \chi_2(g^{-1})$. Now at least one of the three characters χ_1, χ_2, and $\chi_1 + \chi_2$ satisfies (5.1). Indeed, the only way that χ_1 can fail is if $\chi_1(h) = \chi_1(g^{-1})$ and the only way that χ_2 can fail is if $\chi_2(h) = \chi_2(g)$. In this case we have that $\chi_1(h) + \chi_2(h) = \chi_1(h) + \chi_2(g) \neq \chi_1(g) + \chi_2(g)$ and similarly $\chi_1(h) + \chi_2(h) = \chi_1(g^{-1}) + \chi_2(h) \neq \chi_1(g^{-1}) + \chi_2(g^{-1})$.

Now both the sum $\chi + \overline{\chi}$ and the product $\chi\overline{\chi}$ are $\mathcal{F}$-stable real characters and we claim that at least one of these two characters will take different values on g and h. If this is not the case then writing $\alpha = \chi(g)$ and $\beta = \chi(h)$, we have

$$\beta\overline{\beta} = \alpha\overline{\alpha} \text{ and } \beta + \overline{\beta} = \alpha + \overline{\alpha}.$$

Hence $\{\chi(h), \overline{\chi}(h)\} = \{\chi(g), \overline{\chi}(g)\}$ is the solution set for the equation

$$x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha} = x^2 - (\beta + \overline{\beta})x + \beta\overline{\beta},$$

which contradicts $\chi(h) \notin \{\chi(g), \overline{\chi(g)}\}$. □

5.2. **Permutation representations.** For a saturated fusion system $\mathcal{F}$ on a p-group S, define the *linear permutation Chern subring* $\mathrm{Ch}_P(\mathcal{F})$ to be the subring of $H^*(S)$ generated by the Chern classes of all $\mathcal{F}$-stable linear permutation representations. Define also the *permutation Chern subring* $\mathrm{Ch}'_P(\mathcal{F})$ to be the subring of $H^*(S)$ generated by the linearizations of all $\mathcal{F}$-stable permutation representations. By Lemma 4.4, both these rings are large and natural and from the definitions, $\mathrm{Ch}'_P(\mathcal{F}) \subseteq \mathrm{Ch}_P(\mathcal{F})$. One consequence of the next result is that this inclusion induces a homeomorphism of varieties.

Proposition 5.4. *If $R = \mathrm{Ch}_P(\mathcal{F})$ and $R' = \mathrm{Ch}'_P(\mathcal{F})$, then $\mathcal{C}(R) = \mathcal{C}(R') = \mathcal{E}'_P(\mathcal{F})$.*

Proof. Since $R' \subseteq R$, $\mathcal{C}(R) \subseteq \mathcal{C}(R')$. Suppose $f : E_1 \to E_2$ is a morphism in $\mathcal{C}(R')$ and let $e \in E_1$. By Lemma 4.4, $\chi(e) = \chi(f(e))$ for all characters χ of $\mathcal{F}$-stable permutation representations.

If $\langle e \rangle$ and $\langle f(e) \rangle$ are $\mathcal{F}$-conjugate, then $f(e)$ and e^i are $\mathcal{F}$-conjugate for some $i \geq 1$ and $\chi(f(e)) = \chi(e^i) = \chi(e)$ for all permutation characters χ of $\mathcal{F}$-stable S-sets. Thus $\mathcal{E}'_P(\mathcal{F}) \subseteq \mathcal{C}(R)$.

Now suppose that $\langle e \rangle$ and $\langle f(e) \rangle$ are not $\mathcal{F}$-conjugate, and let $\langle e' \rangle$ be a fully $\mathcal{F}$-normalised $\mathcal{F}$-conjugate of $\langle e \rangle$. We show that $f \notin \mathcal{C}(R')$. Suppose, to the contrary, that $f \in \mathcal{C}(R')$ and let χ be the permutation character associated to the $\mathcal{F}$-stable S-set $\alpha_{\langle e \rangle}$ from Proposition 2.8. Since $\langle f(e) \rangle$ and $\langle e \rangle$ are not $\mathcal{F}$-conjugate, we have that $\Phi_{\langle f(e) \rangle}(\alpha_{\langle e \rangle}) = 0$ by Proposition 2.8(1). On the other hand,

$$\Phi_{\langle e' \rangle}(\alpha_{\langle e \rangle}) = |N_S(\langle e' \rangle)/\langle e' \rangle| \neq 0$$

by Proposition 2.8(2). If $i \geq 1$ is such that e'^i is $\mathcal{F}$-conjugate to e, then since χ is $\mathcal{F}$-stable,

$$0 \neq |\mathrm{Fix}_{\alpha_{\langle e \rangle}}(e'^i)| = \chi(e'^i) = \chi(e) = \chi(f(e)) = |\mathrm{Fix}_{\alpha_{\langle e \rangle}}(f(e))|.$$

Since $\langle f(e) \rangle$ is cyclic of order p this implies that $\Phi_{\langle f(e) \rangle}(\alpha_{\langle e \rangle}) \neq 0$, a contradiction.

We have thus shown that $\mathcal{E}'_P(\mathcal{F}) \subseteq \mathcal{C}(R) \subseteq \mathcal{C}(R') \subseteq \mathcal{E}'_P(\mathcal{F})$, as required. □

Proof of Theorem 1.1. This follows on combining Propositions 5.1, 5.3 and 5.4. □

6. Characteristic classes of $\mathcal{F}$-stable permutations

In this section, $\mathcal{F}$ is a saturated fusion system on a finite p-group S. We adapt the discussion in [13] to the setting of fusion systems.

Lemma 6.1. *Let X be an $\mathcal{F}$-stable S-set and $\rho_X : S \to \Sigma_n$ be an associated homomorphism. Then the induced map*

$$\rho_X^* : H^*(\Sigma_n) \to H^*(S)$$

depends only on X and has image contained in $H^*(\mathcal{F})$.

Proof. Since the images of two choices of ρ_X differ only by an inner automorphism of Σ_n, ρ_X^* depends only on X. Denote by $A_n(P)$ the set of isomorphism classes of P-sets of order n for a finite group P, and by $A_n(\mathcal{F})$ the set of isomorphism classes of $\mathcal{F}$-stable S-sets of order n. Then arguing exactly as in Proposition 2.4, we have a bijection

$$\varprojlim_{\mathcal{O}(\mathcal{F}^c)} A_n(P) \cong A_n(\mathcal{F}).$$

Now the argument in Proposition 3.4 with $U(n)$ replaced by Σ_n yields the result. □

Definition 6.2. Let $\mathcal{S}(\mathcal{F})$ be the *permutation subring* of $H^*(S)$ generated by $\mathrm{im}(\rho_X^*)$ for all $\mathcal{F}$-stable S-sets.

Just as for the Chern subring, there is an analogous topological definition: $\mathcal{S}'(\mathcal{F})$ is defined to be the subring of $H^*(|\mathcal{L}|_p^\wedge; \mathbb{F}_p) \leq H^*(S; \mathbb{F}_p)$ generated by the images in cohomology of all maps $f : |\mathcal{L}|_p^\wedge \to B(\Sigma_n)_p^\wedge$ for all $n \geq 1$. A map f as above defines an $\mathcal{F}$-stable S-set of cardinality n and so there is an inclusion map $\mathcal{S}'(\mathcal{F}) \to \mathcal{S}(\mathcal{F})$. We have been unable to establish a direct analogue of Proposition 3.6, but we shall show below that this inclusion map induces an isomorphism of varieties, a result that appeared as a conjecture in an earlier version of this article.

The connection between $\mathcal{F}$-stable S-sets and maps of p-completed classifying spaces is rather more subtle than for $\mathcal{F}$-stable unitary representations; see the PhD thesis of Matthew Gelvin [11] for a discussion. Nevertheless, in [9, Proposition 7.8] it is shown that for any $\mathcal{F}$-stable S-set X, there is $k \geq 0$ and a map $f : |\mathcal{L}|_p^\wedge \to B(\Sigma_n)_p^\wedge$, where $n = p^k|X|$, so that the $\mathcal{F}$-stable S-set induced by f is p^kX, the disjoint union of p^k copies of X.

Recall the definition of the category $\mathcal{A} = \mathcal{A}(\mathcal{F})$ in Section 1 which is analogous to the category $\mathcal{A}_h$ in [13, Section 3] by [13, Lemma 3.2]. We apply Proposition 2.8 to prove the following analogue of [13, Lemma 2.7] for fusion systems:

Lemma 6.3. *Let $E_1, E_2 \leq S$ and $f : E_1 \to E_2$ be an injective group homomorphism. Then $f \in \mathcal{A}$ if and only if for every $x \in \mathcal{S}(\mathcal{F})$, the class $\mathrm{res}^S_{E_1}(x) - f^*\mathrm{res}^S_{E_2}(x)$ lies in the nilradical of $H^*(E_1)$. Moreover the same statement holds with $\mathcal{S}'(\mathcal{F})$ instead of $\mathcal{S}(\mathcal{F})$.*

Proof. Suppose $f \in \mathcal{A}$. Then E_1 and $f(E_1)$ are $\mathcal{F}$-conjugate, so that by Proposition 2.8(3), the $\mathcal{F}$-stable S-sets α_{E_1} and $\alpha_{f(E_1)}$ are isomorphic. Denoting (respectively) by ρ_1 and ρ_2 the corresponding S-representations, we have $\rho_1|_{E_1} \cong \rho_2|_{f(E_1)} \circ f$ and so there exists some $\sigma \in \Sigma_{|S|}$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & f(E_1) \\ \downarrow{\scriptstyle \rho_1|_{E_1}} & & \downarrow{\scriptstyle \rho_2|_{f(E_1)}} \\ \Sigma_{|S|} & \xrightarrow{c_\sigma} & \Sigma_{|S|} \end{array} \tag{6.1}$$

commutes. Hence $\mathrm{res}^S_{E_1} - f^*\mathrm{res}^S_{E_2}$ kills $\mathrm{im}(\rho_1^*)$. Since $\mathcal{S}'$ is a subring of $\mathcal{S}$, this argument proves the 'only if' part of the claim for $\mathcal{S}'$ too.

Conversely, suppose that $f \notin \mathcal{A}$. Then there exists $U \leq E_1$ such that U is not $\mathcal{F}$-conjugate to $f(U)$. If U is not fully $\mathcal{F}$-normalised, then let $\varphi \in \mathrm{Hom}_\mathcal{F}(U, S)$ be such that $\varphi(U)$ is fully $\mathcal{F}$-normalised. Then, since $\mathcal{F}$ is saturated, there exists $\widetilde{\varphi} \in \mathrm{Hom}_\mathcal{F}(N_\varphi, S)$ which extends φ. Since $E_1 \leq C_S(U) \leq N_\varphi$ we obtain a map

$$f' := f \circ (\widetilde{\varphi}|_{E_1})^{-1} \in \mathrm{Hom}(\varphi(E_1), E_2).$$

Plainly $f' \notin \mathcal{A}$ since $f \notin \mathcal{A}$, so replacing f by f' and U by $\varphi(U)$ if necessary, we may assume that U is fully $\mathcal{F}$-normalised. Then by Proposition 2.8, $\Phi_{f(U)}(\alpha_U) = 0 \neq \Phi_U(\alpha_U)$. Let X be the direct sum of α_U and the regular representation. Then $\Phi_{f(U)}(X) = 0 \neq \Phi_U(X)$ and S acts faithfully on X. Let $\rho : S \to \Sigma_{|X|}$ be the corresponding permutation representation of S. Then ρ is injective and since $\Phi_{f(U)}(X) \neq \Phi_U(X)$, it follows that $\rho(U)$ and $\rho(f(U))$ are not conjugate in the symmetric group $\Sigma_{|X|}$. As in [13, Lemma 2.7], the construction used in the proof of [12, Theorem 8.1] gives a class $\zeta \in H^*(\Sigma_{|X|})$ whose image in $\rho(U)$ is non-nilpotent, while its image in $\rho(f(U))$ is zero. If we let $x = \rho^*(\zeta)$, then $\mathrm{res}^S_{E_1}(x) - f^*\mathrm{res}^S_{E_2}(x)$ is not in the nilradical of $H^*(E_1)$, since its image under $\mathrm{res}^{E_1}_U$ is not nilpotent.

It cannot be assumed that there is a map $|\mathcal{L}|_p^\wedge \to (B\Sigma_{|X|})_p^\wedge$ that gives rise to the $\mathcal{F}$-stable S-set X. However, by [9, Proposition 7.8(a)] there exists $k \geq 0$ and a map $\rho' : |\mathcal{L}|_p^\wedge \to (B\Sigma_{p^k|X|})_p^\wedge$ that gives rise to the $\mathcal{F}$-stable S-set $Y = p^kX$. As before, ρ'

defines an embedding of S into $\Sigma_{|Y|}$ and since $\Phi_U(Y) \neq \Phi_{f(U)}(Y)$, it follows that $\rho'(U)$ and $\rho'(f(U))$ are non-conjugate elementary abelian subgroups of $\Sigma_{|Y|}$ of the same rank. Hence there is $\zeta' \in H^*(\Sigma_{|Y|})$ whose image in $\rho'(U)$ is non-nilpotent, while its image in $\rho'(f(U))$ is zero. Now $x' = \rho'^*(\zeta')$ has the property that $\mathrm{res}^S_{E_1}(x') - f^*\mathrm{res}^S_{E_2}(x')$ is not in the nilradical of $H^*(E_1)$ as before. □

We now obtain.

Theorem 6.4. *The restriction maps in cohomology induce natural homeomorphisms*

$$\underset{\mathcal{A}(\mathcal{F})}{\underrightarrow{\mathrm{colim}}}\, X_E(k) \to V_{\mathcal{S}(\mathcal{F})}(k) \to V_{\mathcal{S}'(\mathcal{F})}(k).$$

Proof. The regular representation $\rho : S \to \Sigma_{|S|}$ is obviously $\mathcal{F}$-stable so $\mathcal{S}(\mathcal{F})$ is large. It is also natural because $\mathcal{S}(\mathcal{F})$ is clearly homogeneously generated and closed under the action of the Steenrod algebra. Hence by Theorem 4.3, there exists some category $\mathcal{C}$ of elementary abelian p-subgroups for which

$$\underset{\mathcal{C}}{\underrightarrow{\mathrm{colim}}}\, X_E(k) \to V_{\mathcal{S}(\mathcal{F})}(k)$$

is a homeomorphism. Finally, Lemma 6.3 identifies $\mathcal{C}$ with the category $\mathcal{A}(\mathcal{F})$ defined above.

The subring $\mathcal{S}'(\mathcal{F})$ is also clearly natural. The Chern classes of the regular representation ρ may not lie in $\mathcal{S}'(\mathcal{F})$, but by [9, Proposition 7.8(a)], there exists k so that the Chern classes of $p^k\rho$ lie in $\mathcal{S}'(\mathcal{F})$. The remainder of the argument using Theorem 4.3 and Lemma 6.3 proceeds exactly as for $\mathcal{S}(\mathcal{F})$. □

Corollary 6.5. *The inclusion $\mathcal{S}'(\mathcal{F}) \to \mathcal{S}(\mathcal{F})$ is an inseparable isogeny.*

Proof. Immediate from Theorem 6.4, since this inclusion induces a homeomorphism of varieties. □

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