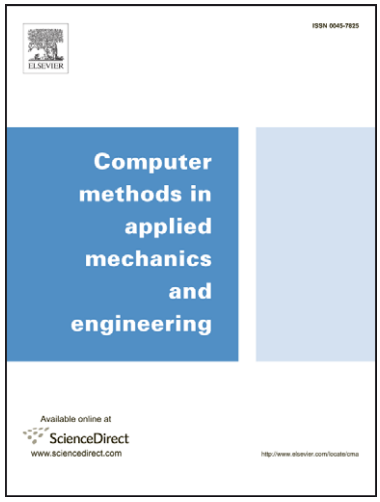

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Multi-element stochastic reduced basis methods

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Abstract

This paper presents multi-element Stochastic Reduced Basis Methods (ME-SRBMs) for solving linear stochastic partial differential equations. In ME-SRBMs, the domain of definition of the random inputs is decomposed into smaller subdomains or random elements. Stochastic Reduced Basis Methods (SRBMs) are employed in each random element to evaluate the response statistics. These elemental statistics are assimilated to compute the overall statistics. The effectiveness of the method is demonstrated by solving the stochastic steady state heat transfer equation on two geometries involving different types of boundary conditions. Numerical studies are conducted to investigate the h-convergence rates of global and local preconditioning strategies.

Key words: stochastic reduced basis methods; stochastic partial differential equations; uncertainty quantification; heat transfer; preconditioning.

1 Introduction

In recent years, stochastic projection schemes based on polynomial chaos (PC) expansions have emerged as a powerful tool to analyze stochastic systems. The fundamentals underpinning this approach is based on the homogeneous chaos theory proposed by Wiener [1] as a generalization of Fourier series expansion. In the original work of Wiener, multi-dimensional Hermite polynomials in terms of a set of Gaussian random variables are used as (orthogonal) basis functions for representing stochastic processes. If the undetermined constants are computed as coefficients of Fourier-Hermite expansions, such expansions converge for any second-order stochastic process in the L_2 sense by the theorem of Cameron and Martin [2]. Ghanem et al [3–8] and others [9,10] have applied

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PC expansions to a wide class of stochastic Partial Differential Equations (PDEs). These results show that unlike perturbation-based methods, good accuracy can attained with PC methods even when the coefficients of variation of the input random variables is large.

Recently, Xiu and Karniadakis [11,12] proposed generalized Polynomial Chaos (gPC) expansions, where basis functions from the Askey family of hypergeometric polynomials are used. It was shown that by employing appropriate basis functions depending on the input (e.g. Legendre polynomials for uniform distribution, Laguerre polynomial functions for exponential distributions, etc.), faster convergence rates can be attained compared to Hermite polynomials for non-Gaussian distributions. The method is not limited only to continuous distributions but is also applicable to discrete distributions (e.g. Charlier Polynomial functions are used in the case of Poisson's distribution).

Le Maître et al. [13,14] introduced *Wiener-Haar expansions* for representing stochastic processes. It has been shown that these basis functions give more accurate representations at discontinuities compared to gPC expansions. Babuška et al. [15] studied the convergence of Galerkin finite element approximations of stochastic elliptic PDEs. A theoretical analysis was presented for the computational costs involved in the Monte-Carlo Simulations (MCS) and the Galerkin projection schemes employing a spectral representation of the solution process. This analysis suggests that for problems with a small number of random variables, the latter method is preferable especially when the accuracy requirement is high. While for problems involving a large number of variables, the Galerkin projection schemes lead to significant increase in the dimensionality of the problem resulting in computational costs comparable to that of MCS.

Stochastic Reduced Basis Methods (SRBMs) were recently introduced in the literature [16,17] to solve linear random algebraic systems of equations arising from discretization of stochastic PDEs. In this approach, the response process is approximated using basis vectors spanning a preconditioned stochastic Krylov subspace. The Bubnov-Galerkin/Petrov-Galerkin projection scheme is used to compute the undetermined coefficients by solving a reduced order deterministic system of equations. Note that the basis vectors in SRBMs are problem dependent – this is in contrast to gPC expansions where the basis functions are chosen solely depending on the input distribution. Numerical studies on linear stochastic discretized PDEs conducted by Sachdeva et al. [18] suggest that SRBMs can be more accurate for a given order of expansion compared to existing approaches where the response process is expanded using PC basis functions. In addition, it has also been observed that the computational cost incurred by SRBMs is orders of magnitude lower than PC projection schemes. SRBMs are hybridized with PC expansions in [19] to deal with non-Gaussian uncertainties.

All the approaches discussed so far spectrally decompose the response process where the accuracy is improved by increasing the order of the expansion of the solution process. This type of improvement is known as p-refinement of the solution process. More recently, Babuška et al. [15], Maître et al. [14], Xiu and Tartakovsky [20], Wan and Karniadakis [21] came up with the idea of decomposing the input random space (h-refinement) in order to decrease the uncertainty in each subdomain, thus resulting in better accuracy in the response statistics. Numerical experiments have shown that h-refinement enhances the stability of stochastic projection schemes when applied to non-linear problems, time dependent problems and problems involving point discontinuities ([21], [22] and [23]).

In the present paper, we propose multi-element Stochastic Reduced Basis Methods (ME-SRBMs) to achieve h-refinement of SRBMs. We note that the desired response statistics are integrals with respect to the input PDFs and hence their accuracy is inversely proportional to the variability in the random space. In addition, as with any Krylov subspace based method, preconditioners have a significant role to play in convergence as can be seen in the following sections. Thus partitioning the space of the random inputs into disjoint elements provides two-pronged benefits, namely, better preconditioners and better cascade of local estimates for the undetermined coefficients. Thus to sum up, the basic idea behind a multi-element formulation is to compare the relative error in variance against a user-defined tolerance and then decompose the random space into a number of random elements or subdomains. The objective of this paper is to extend this concept of decomposing the input random space to SRBMs and to demonstrate the two fold benefits associated with such a formulation.

The remainder of the paper is organized as follows. The next section introduces notation and provides a brief exposition of SRBMs. Then section 3 presents a multi-element reformulation of SRBMs and its theoretical properties. In section 4, the computational aspects of ME-SRBMs including different types of preconditioning strategies and post-processing are discussed. Numerical studies are presented in section 5 on two stochastic steady state heat transfer problems. Section 6 concludes the paper.

2 Stochastic reduced basis methods

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra associated with Ω and $\mathcal{P}$ is a probability measure. Let $\omega \in \Omega$ be a random event. Then any random vector $\boldsymbol{\xi}(\omega) : \Omega \rightarrow \mathbb{R}^d$ is said to be a second order stochastic process if it has a finite second order moment on $(\Omega, \mathcal{F}, \mathcal{P})$. Here d refers to the dimensionality of vector $\boldsymbol{\xi}$. By definition, $L_2(\Omega, \mathcal{F}, \mathcal{P})$ is a

Hilbert space of random variables. Let $\mathbf{u}(\boldsymbol{\xi})$ and $\mathbf{v}(\boldsymbol{\xi})$ be two stochastic vectors, then the inner product or expectation operator $\langle\cdot\rangle$ (sometimes denoted as $\langle\cdot,\cdot\rangle$ or $(\cdot,\cdot)$) is defined as

$$\langle \mathbf{u}^*(\boldsymbol{\xi})\mathbf{v}(\boldsymbol{\xi})\rangle = \int \mathbf{u}^*(\boldsymbol{\xi})\mathbf{v}(\boldsymbol{\xi})w(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},$$

where $\boldsymbol{\xi}$ is a continuous random variable, $w(\boldsymbol{\xi})$ denotes the weight function and the superscript * denotes the complex conjugate transpose.

In the present work, we focus on linear stochastic PDEs of the form

$$\left.\begin{aligned} \nabla\cdot[\kappa(x;\omega)\nabla u(x;\omega)] &= f(x;\omega) \qquad (x;\omega)\in D\times\Omega \\ \mathcal{B}u(x;\omega) &= g(x;\omega) \qquad (x;\omega)\in \partial D\times\Omega \end{aligned}\right\}, \tag{1}$$

where $\kappa(x;\omega)$ and $f(x;\omega)$ are random fields defined on $D\times\Omega$ while $g(x;\omega)$ is a random field defined on $\partial D\times\Omega$. D and ∂D denotes the physical domain and its boundary, respectively. $\mathcal{B}$ is an operator indicating the type of boundary conditions, e.g., Dirichlet, Neumann, Robin or mixed boundary conditions. $u(x;\omega)$ is the solution process defined on $D\times\Omega$ whose statistics are of practical interest.

Discretization of this class of PDEs in the physical space and the random dimensions and subsequent application of the specified boundary conditions result in a linear random algebraic system of equations of the form (Refer [3] for details.)

$$\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\xi}), \tag{2}$$

where $\boldsymbol{\xi}:\Omega\to\mathbb{R}^d$ is a vector of random variables arising from the discretization of the input random fields $\kappa(x;\omega)$, $f(x;\omega)$ and $g(x;\omega)$. Also $\mathbf{K}(\boldsymbol{\xi}):\mathbb{R}^d\to\mathbb{R}^{n\times n}$ and $\mathbf{f}(\boldsymbol{\xi}),\mathbf{u}(\boldsymbol{\xi}):\mathbb{R}^d\to\mathbb{R}^n$, where n is the number of spatial degrees of freedom.

Stochastic reduced basis methods [16,17] approximate the solution of equation Eq. (2) using basis vectors spanning the stochastic Krylov subspace defined as

$$\mathcal{K}_p(\mathbf{K}(\boldsymbol{\xi}),\mathbf{f}(\boldsymbol{\xi})) = \mathrm{span}\left\{\mathbf{f}(\boldsymbol{\xi}),\mathbf{K}(\boldsymbol{\xi})\mathbf{f}(\boldsymbol{\xi}),(\mathbf{K}(\boldsymbol{\xi}))^2\mathbf{f}(\boldsymbol{\xi}),\cdots,(\mathbf{K}(\boldsymbol{\xi}))^{p-1}\mathbf{f}(\boldsymbol{\xi})\right\}.$$

We now state an important theorem from [17] which proves the existence of the solution of Eq. (2) in the stochastic Krylov subspace

Theorem 1. *If the minimal random polynomial of a nonsingular random square matrix* $\mathbf{K}(\boldsymbol{\xi})$ *has a degree* m*, then the solution of the random algebraic system of equations* $\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\xi})$ *lies in the stochastic Krylov subspace*

$\mathcal{K}_m(\mathbf{K}(\boldsymbol{\xi}), \mathbf{f}(\boldsymbol{\xi}))$. *Thus a stochastic reduced basis representation of the solution process* $\mathbf{u}(\boldsymbol{\xi})$ *can be written as*

$$\mathbf{u}(\boldsymbol{\xi}) = \alpha_1 \boldsymbol{\psi}_1(\boldsymbol{\xi}) + \alpha_2 \boldsymbol{\psi}_2(\boldsymbol{\xi}) + \cdots + \alpha_p \boldsymbol{\psi}_m(\boldsymbol{\xi}) = \boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha},$$

where $\boldsymbol{\psi}_i$ *are the basis vectors spanning* $\mathcal{K}_m(\mathbf{K}(\boldsymbol{\xi}), \mathbf{f}(\boldsymbol{\xi}))$.

For large scale problems (with large n), the order of the minimal random polynomial may be comparable to n. To address this issue, we solve the random algebraic system of equations in a preconditioned stochastic Krylov subspace. A deterministic matrix $\mathbf{M}$ is called a *preconditioner* of a stochastic matrix $\mathbf{K}(\boldsymbol{\xi})$, if the order of the minimal random polynomial of the transformed stochastic matrix $\mathbf{MK}(\boldsymbol{\xi})$ is lower than that of the original matrix $\mathbf{K}(\boldsymbol{\xi})$. Hence, we observe accelerated convergence rates with basis vectors spanning a preconditioned stochastic Krylov subspace which in turn is defined as

$$\mathcal{K}_m\left(\mathbf{MK}(\boldsymbol{\xi}), \mathbf{Mf}(\boldsymbol{\xi})\right) = \text{span}\left\{\boldsymbol{\psi}_1(\boldsymbol{\xi}), \boldsymbol{\psi}_2(\boldsymbol{\xi}), \boldsymbol{\psi}_3(\boldsymbol{\xi}), \cdots, \boldsymbol{\psi}_m(\boldsymbol{\xi})\right\},$$

where the stochastic basis vectors $\boldsymbol{\psi}_i$ can be recursively computed as follows

$$\left.\begin{aligned} \boldsymbol{\psi}_1(\boldsymbol{\xi}) &= \mathbf{Mf}(\boldsymbol{\xi}) \\ \boldsymbol{\psi}_2(\boldsymbol{\xi}) &= \mathbf{MK}(\boldsymbol{\xi})\boldsymbol{\psi}_1(\boldsymbol{\xi}) \\ \boldsymbol{\psi}_3(\boldsymbol{\xi}) &= \mathbf{MK}(\boldsymbol{\xi})\boldsymbol{\psi}_2(\boldsymbol{\xi}) \\ &\vdots \\ \boldsymbol{\psi}_m(\boldsymbol{\xi}) &= \mathbf{MK}(\boldsymbol{\xi})\boldsymbol{\psi}_{m-1}(\boldsymbol{\xi}) \end{aligned}\right\}.$$

For numerical stability, the basis vectors spanning the preconditioned stochastic Krylov subspace can be orthogonalised using Arnoldi's procedure [16]. Also note that convergence can be guaranteed as long as the preconditioner $\mathbf{M}$ is invertible. In [16,17], the deterministic matrix $\langle \mathbf{K}(\boldsymbol{\xi}) \rangle^{-1}$ is selected as the preconditioner. To ensure computational efficiency, the Cholesky factors of the deterministic matrix $\langle \mathbf{K}(\boldsymbol{\xi}) \rangle$ are used to compute the basis vectors in a recursive fashion. For a more detailed exposition on the theoretical and computational aspects of SRBMs, see Nair [24].

We conclude this section with a result from [17] which gives the error norm that is minimized by SRBMs employing the Bubnov-Galerkin projection scheme to compute the undetermined coefficients. This result is later invoked in the context of ME-SRBMs.

Theorem 2. *Let* $\widehat{\mathbf{u}}(\boldsymbol{\xi}) = \boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha}$ *be a* p^{th} *order stochastic reduced basis approximation to the solution of* $\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\xi})$, *where* $\mathbf{K}(\boldsymbol{\xi}) \in \mathbb{C}^{n \times n}$ *is a Hermitian positive definite matrix,* $\mathbf{u}(\boldsymbol{\xi}), \mathbf{f}(\boldsymbol{\xi}) \in \mathbb{C}^n$ *are random vectors,* $\boldsymbol{\Psi}(\boldsymbol{\xi}) \in \mathbb{C}^{n \times p}$ *is a matrix of stochastic basis vectors and* $\boldsymbol{\alpha} \in \mathbb{C}^p$ *is a vector of undetermined*

coefficients. If the coefficient vector $\boldsymbol{\alpha}$ *is computed by imposing the Bubnov-Galerkin condition* $\mathbf{K}(\boldsymbol{\xi})\boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha} - \mathbf{f}(\boldsymbol{\xi}) \perp \boldsymbol{\Psi}(\boldsymbol{\xi})$ *which in turn is equivalent to* $\langle \boldsymbol{\Psi}^*(\boldsymbol{\xi})\{\mathbf{K}(\boldsymbol{\xi})\boldsymbol{\Psi}(\boldsymbol{\xi})\boldsymbol{\alpha} - \mathbf{f}(\boldsymbol{\xi})\}\rangle = 0$*, then the* $\mathbf{K}$*-norm of the error defined as*

$$\Delta_p = \langle \{\mathbf{u}(\boldsymbol{\xi}) - \widehat{\mathbf{u}}(\boldsymbol{\xi})\}^* \mathbf{K}(\boldsymbol{\xi})\{\mathbf{u}(\boldsymbol{\xi}) - \widehat{\mathbf{u}}(\boldsymbol{\xi})\}\rangle$$

is minimized.

3 Multi-element stochastic reduced basis methods

In this section, we present a multi-element reformulation of SRBMs based on the ideas developed earlier in the context of PC projection schemes [21]. Let $\boldsymbol{\xi}(\omega) = [\xi_1(\omega), \xi_2(\omega), \ldots, \xi_d(\omega)] : \Omega \to \mathbb{R}^d$ denote a d-dimensional random vector defined on the complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\xi_i \ \forall \ i \in \{1, 2, \ldots, d\}$ are identical, independent distributed (IID) random variables. We also assume that all the components of the vector $\boldsymbol{\xi} : \Omega \to [-1, 1]^d$ are uniform random variables with a constant joint PDF $f_{\boldsymbol{\xi}} = \frac{1}{2^d}$. Define B as the decomposition of $[-1, 1]^d$, such that

$$\left.\begin{aligned} B &= \bigcup_i^N B_i \\ B_i &= [a_1^i, b_1^i) \times [a_2^i, b_2^i) \times \cdots \times [a_d^i, b_d^i) \\ B_i \bigcap B_j &= \emptyset \quad \Leftrightarrow i \neq j, \end{aligned}\right\}$$

where $i, j \in \{1, 2, \ldots, N\}$. Thus B is a partition of $[-1, 1]^d$ into N non-overlapping elements with the property

$$\mathcal{P}(B_i \bigcap B_j) = 0, \tag{3}$$

where $\mathcal{P}$ is the probability measure defined on the Hilbert space $L_2(\Omega, \mathcal{F}, \mathcal{P})$. On each random element define an *indicator random variable* I_j such that

$$I_j = \begin{cases} 1 & \text{when } \boldsymbol{\xi} \in B_j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $I_j^{-1}(1) = \{\boldsymbol{\xi} \mid \boldsymbol{\xi} \in B_j\}$. Then it follows that

$$I_j^{-1}(1) \bigcap I_k^{-1}(1) = \emptyset \quad \Leftrightarrow j \neq k.$$

We define a local random variable on $I_j^{-1}(1)$ as

$$\boldsymbol{\chi}^j = [\chi_1^j, \chi_2^j, \ldots, \chi_d^j] : I_j^{-1}(1) \to B_j \ ,$$

subject to the conditional PDF

$$f_{\boldsymbol{\chi}^j} = \frac{1}{2^d Pr(I_j = 1)} \ ,$$

where $Pr(I_j = 1)$, the probability of I_j being 1 is given by

$$Pr(I_j = 1) = \prod_{i=1}^{d} \frac{b_i^j - a_i^j}{2} \ .$$

The local random variable $\boldsymbol{\chi}^j$ is mapped to a new random variable $\boldsymbol{\xi}^j = g_j(\boldsymbol{\chi}^j) : I_j^{-1}(1) \rightarrow [-1, 1]^d$ with conditional PDF $f_{\boldsymbol{\xi}^j} = \frac{1}{2^d}$ where $g_j(\boldsymbol{\chi}^j)$ is defined such that $\chi_i^j = \frac{b_i^j - a_i^j}{2}\xi_i^j + \frac{b_i^j + a_i^j}{2} \ \ \forall i \in \{1, 2, \ldots, d\}$. This random variable transformation is done so that the PDF is the same over all the elements which in turn results in simpler expressions for the moments of the response.

As a consequence of the above decomposition of the input random space, the global stochastic system of equations defined by Eq. (2) can be rewritten as a decoupled random algebraic system of equations (defined over a subdomain B_k) as follows:

$$\mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k))\mathbf{u}(g_k^{-1}(\boldsymbol{\xi}^k)) = \mathbf{f}(g_k^{-1}(\boldsymbol{\xi}^k)) \quad \forall k \in \{1, 2, \cdots N\}. \tag{4}$$

These local or elemental systems of equations are solved for the response statistics using SRBMs where the basis is chosen from the preconditioned stochastic Krylov subspace defined by

$$\mathcal{K}_p^k \left(\mathbf{M}^k \mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k)), \mathbf{M}^k \mathbf{f}(g_k^{-1}(\boldsymbol{\xi}^k)) \right) = \text{span} \left\{ \boldsymbol{\psi}_1^k, \boldsymbol{\psi}_2^k, \boldsymbol{\psi}_3^k, \cdots, \boldsymbol{\psi}_p^k \right\},$$

where the local stochastic basis vectors $\boldsymbol{\psi}_i^k$ are recursively defined as

$$\begin{aligned}
\boldsymbol{\psi}_1^k(g_k^{-1}(\boldsymbol{\xi}^k)) &= \mathbf{M}^k \mathbf{f}(g_k^{-1}(\boldsymbol{\xi}^k)) \\
\boldsymbol{\psi}_2^k(g_k^{-1}(\boldsymbol{\xi}^k)) &= \mathbf{M}^k \mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k)) \boldsymbol{\psi}_1^k(g_k^{-1}(\boldsymbol{\xi}^k)) \\
\boldsymbol{\psi}_3^k(g_k^{-1}(\boldsymbol{\xi}^k)) &= \mathbf{M}^k \mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k)) \boldsymbol{\psi}_2^k(g_k^{-1}(\boldsymbol{\xi}^k)) \\
&\vdots \\
\boldsymbol{\psi}_p^k(g_k^{-1}(\boldsymbol{\xi}^k)) &= \mathbf{M}^k \mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k)) \boldsymbol{\psi}_{p-1}^k(g_k^{-1}(\boldsymbol{\xi}^k))
\end{aligned}$$

In the above set of equations $\mathbf{M}^k$ represents a global or a local preconditioner. A global preconditioner is defined as

$$\mathbf{M}^k = \langle \mathbf{K}(\boldsymbol{\xi}) \rangle^{-1} \tag{5}$$

and a local preconditioner is defined as

$$\mathbf{M}^k = \langle \mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k)) \rangle^{-1}. \tag{6}$$

Thus the local approximate solution obtained by applying SRBMs to Eq. (4) can be written as

$$\widehat{\mathbf{u}}_k(g_k(\boldsymbol{\xi})) = \boldsymbol{\Psi}^k \boldsymbol{\alpha}^k, \tag{7}$$

where $\boldsymbol{\Psi}^k = [\boldsymbol{\psi}_1^k, \boldsymbol{\psi}_2^k, \cdots, \boldsymbol{\psi}_p^k]$ is a matrix of stochastic basis vectors and $\boldsymbol{\alpha}^k$ are the undetermined coefficients computed by imposing the Galerkin condition. The Bubnov-Galerkin condition is enforced as

$$\epsilon(g_k^{-1}(\boldsymbol{\xi}^k)) = \mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k))\boldsymbol{\Psi}^k\boldsymbol{\alpha}^k - \mathbf{f}(g_k^{-1}(\boldsymbol{\xi}^k)) \perp \boldsymbol{\psi}_j^k, \quad \forall j \in \{1, 2, \cdots, p\}. \tag{8}$$

The above condition results in the following reduced order $p \times p$ system of deterministic equations

$$\langle \boldsymbol{\psi}_j^{k*}(g_k^{-1}(\boldsymbol{\xi}^k))\epsilon(g_k^{-1}(\boldsymbol{\xi}^k))\rangle = 0, \forall j \in \{1, 2, \cdots, p\}. \tag{9}$$

The undetermined coefficients obtained by solving the above system of equations are substituted into Eq. (7) to compute the local statistics. These local statistics are assimilated to compute the global statistics as shown in the next section. Absolute continuity on the elemental boundaries is not required as the statistics are integrations with respect to the conditional PDFs. More precisely, the integrals are zero on the elemental boundary region due to Eq. (3). The approximate global response process $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$ can be expressed in terms of local approximations $\widehat{\mathbf{u}}_j(\boldsymbol{\xi}^j)$ defined on the elemental random space $(I_j^{-1}(1), \mathcal{F} \bigcap I_j^{-1}(1), P(\cdot|\boldsymbol{\xi} \in I_j^{-1}(1)))$ as

$$\widehat{\mathbf{u}}(\boldsymbol{\xi}) = \sum_{j=1}^{N} \widehat{\mathbf{u}}_j(g_j(\boldsymbol{\xi}))I_j. \tag{10}$$

This formulation can also be extended to non-uniform uncertainty models by expanding each of the random components using Legendre-chaos basis functions; see, for example, Wan and Karniadakis [21].

We now present a result on the error norm that is minimized by ME-SRBMs that use the Bubnov-Galerkin projection scheme to evaluate the undetermined coefficients in each random element.

Theorem 3. *Let $\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\xi})$ be a random algebraic system of equations where $\boldsymbol{\xi} : \Omega \to [-1, 1]^d$ is a uniform random vector whose components $\boldsymbol{\xi}_j$ are IID random variables, $\mathbf{K}(\boldsymbol{\xi}) \in \mathbb{C}^{n\times n}$ is a Hermitian positive definite matrix and $\mathbf{u}(\boldsymbol{\xi}), \mathbf{f}(\boldsymbol{\xi}) \in \mathbb{C}^n$ are random vectors. The space of the random vector $\boldsymbol{\xi}$ is decomposed into N disjoint elements where $\boldsymbol{\xi}^k$ denotes the local random variable. If p^{th} order SRBMs are employed locally to compute the approximate solution $\widehat{\mathbf{u}}_k(g_k(\boldsymbol{\xi}))$ for the above system of equations where in turn a Bubnov-Galerkin projection scheme is used to compute the undetermined coefficients, then the global $\mathbf{K}$-norm error given by*

$$\Delta_p = \langle \{\mathbf{u}(\boldsymbol{\xi}) - \widehat{\mathbf{u}}(\boldsymbol{\xi})\}^* \mathbf{K}(\boldsymbol{\xi}) \{\mathbf{u}(\boldsymbol{\xi}) - \widehat{\mathbf{u}}(\boldsymbol{\xi})\}\rangle \tag{11}$$

is minimized.

Proof : Let $\widehat{\mathbf{u}}(\boldsymbol{\xi})$ be the approximated solution process. It can be expressed in terms of local approximate solutions as in Eq. (10). Then the global $\mathbf{K}$-norm error given by Eq. (11) can be rewritten as

$$\Delta_p = \int_B \{\mathbf{u}(\boldsymbol{\xi}) - \sum_{k=1}^{N} \widehat{\mathbf{u}}_k(g_k(\boldsymbol{\xi}))I_k\}^* \mathbf{K}(\boldsymbol{\xi}) \{\mathbf{u}(\boldsymbol{\xi}) - \sum_{k=1}^{N} \widehat{\mathbf{u}}_k(g_k(\boldsymbol{\xi}))I_k\} \left(\frac{1}{2}\right)^d \mathrm{d}\boldsymbol{\xi}.$$

After applying Bayes theorem and the law of total probability on the preceding equation, we have

$$\begin{aligned}
\Delta_p &= \sum_{k=1}^{N} Pr(I_k = 1) \int_{B_k} \mathbf{x}^*(\boldsymbol{\chi}^k)\mathbf{K}(\boldsymbol{\chi}^k)\mathbf{x}(\boldsymbol{\chi}^k) f_{\boldsymbol{\chi}^k} \mathrm{d}\boldsymbol{\chi}^k \\
&= \sum_{k=1}^{N} Pr(I_k = 1) \int_{[-1,1]^d} \mathbf{x}^*(g_k^{-1}(\boldsymbol{\xi}^k))\mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k))\mathbf{x}(g_k^{-1}(\boldsymbol{\xi}^k)) \frac{1}{2^d} \mathrm{d}\boldsymbol{\xi}^k \quad , \qquad (12) \\
&= \sum_{k=1}^{N} \Delta_p^k Pr(I_k = 1)
\end{aligned}$$

where $\mathbf{x}(\boldsymbol{\chi}^k) = \mathbf{u}(\boldsymbol{\chi}^k) - \widehat{\mathbf{u}}_k(g_k(\boldsymbol{\chi}^k))$, Δ_p is the global $\mathbf{K}$-norm error and the local $\mathbf{K}$-norm error is defined as

$$\Delta_p^k = \left\langle \{\mathbf{x}^*(g_k^{-1}(\boldsymbol{\xi}^k))\mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k))\mathbf{x}(g_k^{-1}(\boldsymbol{\xi}^k))\} \right\rangle \qquad (13)$$

in the k^{th} random element for the p^{th} order SRBM. Note that $\mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k))$ is a Hermitian positive definite matrix. Hence $\mathbf{x}^*(g_k^{-1}(\boldsymbol{\xi}^k))\mathbf{K}(g_k^{-1}(\boldsymbol{\xi}^k))\mathbf{x}(g_k^{-1}(\boldsymbol{\xi}^k)) > 0$ for any nonzero $\mathbf{x}(\boldsymbol{\xi}^k) \in \mathbb{C}^n$, which leads to the conclusion that $\Delta_p^k > 0$ for any $k \in \{1, 2, \cdots, N\}$. Also note that according to Theorem 2, if SRBMs in conjunction with the Bubnov-Galerkin projection scheme is applied locally, then the local $\mathbf{K}$-norm error ($\Delta_p^k,\ \forall k \in \{1, 2, \cdots, N\}$) is minimized. Hence from the preceding argument and equation Eq. (12), it follows that the global $\mathbf{K}$-norm error is also minimized.

□

Theorem 3 implies that the $\mathbf{K}$-norm error is a strictly non-increasing function of the number of subdomains (N). On the other hand if the Petrov-Galerkin scheme is used to evaluate the undetermined coefficients locally, we conjecture that the global L_2 norm of the residual error is minimized – this in turn will imply that the residual error norm is a strictly non-increasing function of the number of random elements.

It is to be noted that the Galerkin condition imposed in Eq. (9) is said to be a *weak* Galerkin condition because only the ensemble average of the random functions $\boldsymbol{\psi}_j^{k^*}(g_k^{-1}(\boldsymbol{\xi}^k))\epsilon(g_k^{-1}(\boldsymbol{\xi}^k)), \forall j \in \{1, 2, \cdots, p\}$ are set to zero. In contrast, the *strong* Galerkin scheme applied to the global system of equations

Eq. (2), enforces the above condition more stringently, i.e. the orthogonality of the basis and the residual error is enforced for *each realization* of $\boldsymbol{\xi}$. This is accomplished by expanding the undetermined coefficients in a *global* PC basis and subsequently applying a *weak* Galerkin condition; see Nair [24] for details. In the case of ME-SRBMs, a *piecewise* representation of the undetermined coefficients (defined over local partitions of the random space) is employed. Hence ME-SRBMs can be thought of as a projection scheme that implements a relaxed version of the *strong* Galerkin condition.

4 Computational Aspects

4.1 Preconditioners

Preconditioners enhance the convergence characteristics of Krylov subspace methods. A wide variety of preconditioning strategies have been explored in the literature in the context of solving deterministic systems of equations. For a deterministic system of equations $\mathbf{Ku} = \mathbf{f}$, any matrix $\mathbf{M}$ which closely approximates $\mathbf{K}^{-1}$ and is invertible acts as a good preconditioner. In comparison, for random algebraic systems of equations of the form $\mathbf{K}(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\xi})$, it is sought to construct a deterministic preconditioner $\mathbf{M}$ that is a close approximation to the random matrix $\mathbf{K}(\boldsymbol{\xi})^{-1}$. In other words, the degree of the minimal random polynomial of the matrix $\mathbf{MK}(\boldsymbol{\xi})$ must be lower than that of the matrix $\mathbf{K}(\boldsymbol{\xi})$. Thus an incomplete Cholesky factorization or the inverse of the deterministic matrix $\langle\mathbf{K}(\boldsymbol{\xi})\rangle$ works well in an average sense as a preconditioner for random algebraic systems of equations. In previous studies on SRBMs, the deterministic matrix $\langle\mathbf{K}(\boldsymbol{\xi})\rangle^{-1}$ has been used as a preconditioner to accelerate stochastic Krylov methods with a great degree of success [16,17,19,18,24].

As discussed in the preceding section, in the case of ME-SRBMs, we can employ two types of preconditioners, namely, the global preconditioner given by Eq. (5) or the local preconditioner given by Eq. (6). The motivation behind using a local preconditioner is that it provides a better approximation to $\langle\mathbf{K}(\boldsymbol{\xi})\rangle^{-1}$ as compared to the global preconditioner in any random element. On the other hand global preconditioning is computationally more efficient than local preconditioning since only a single preconditioner has to be constructed. Both local and global preconditioning strategies admit coarse grained and fine grained parallelization. Using a global preconditioner, a substantial gain in execution speeds can be achieved owing to the fact that the preconditioner is computed only once and stored in a shared memory space, which can be accessed by other individual nodes. The major advantage with local preconditioning is that the rates of convergence are orders of magnitudes faster than

with global preconditioners, when the random space is decompsed into smaller random elements with lesser degree of variability.

4.2 Post-processing

We now outline how the subdomain response process approximations obtained using ME-SRBMs can be postprocessed to compute the first two statistical moments of the global response. Using Bayes theorem and the law of total probability, the m^{th} moment of the global approximation can be written as

$$\langle \widehat{\mathbf{u}}^m(\boldsymbol{\xi}) \rangle = \int_B \widehat{\mathbf{u}}^m(\boldsymbol{\xi}) \left(\frac{1}{2}\right)^d \mathrm{d}\boldsymbol{\xi} = \sum_{j=1}^{N} Pr(I_j = 1) \int_{B_j} \widehat{\mathbf{u}}_j^m(\boldsymbol{\xi}^j) \left(\frac{1}{2}\right)^d \mathrm{d}\boldsymbol{\xi}^j. \tag{14}$$

The mean of the response process can hence be written as

$$\boldsymbol{\mu} = \langle \widehat{\mathbf{u}}(\boldsymbol{\xi}) \rangle = \sum_{j=1}^{N} Pr(I_j = 1) \boldsymbol{\mu}^j, \tag{15}$$

where $\boldsymbol{\mu}^j$ is the elemental mean given by $\langle \widehat{\mathbf{u}}_j(\boldsymbol{\xi}^j) \rangle$.

The local covariance of the j^{th} element is given by

$$\begin{aligned} \widehat{\mathbf{u}}_{COV}^j &= \langle (\widehat{\mathbf{u}}_j(\boldsymbol{\xi}^j) - \boldsymbol{\mu}^j)(\widehat{\mathbf{u}}_j(\boldsymbol{\xi}^j) - \boldsymbol{\mu}^j)^* \rangle \\ &= \langle \widehat{\mathbf{u}}_j(\boldsymbol{\xi}^j) \widehat{\mathbf{u}}_j^*(\boldsymbol{\xi}^j) \rangle - \boldsymbol{\mu}^j \boldsymbol{\mu}^{j*}. \end{aligned} \tag{16}$$

It can be noted that

$$\langle \widehat{\mathbf{u}}(\boldsymbol{\xi}) \widehat{\mathbf{u}}^*(\boldsymbol{\xi}) \rangle = \sum_{j=1}^{N} Pr(I_j = 1) \langle \widehat{\mathbf{u}}_j(\boldsymbol{\xi}^j) \widehat{\mathbf{u}}_j^*(\boldsymbol{\xi}^j) \rangle. \tag{17}$$

Hence, the global or the overall response covariance can be computed using Eq. (16) and Eq. (17) as shown below

$$\begin{aligned} \widehat{\mathbf{u}}_{COV} &= \langle (\widehat{\mathbf{u}}(\boldsymbol{\xi}) - \boldsymbol{\mu})(\widehat{\mathbf{u}}(\boldsymbol{\xi}) - \boldsymbol{\mu})^* \rangle \\ &= \langle \widehat{\mathbf{u}}(\boldsymbol{\xi}) \widehat{\mathbf{u}}^*(\boldsymbol{\xi}) \rangle - \boldsymbol{\mu} \boldsymbol{\mu}^* \\ &= \sum_{j=1}^{N} Pr(I_j = 1)(\widehat{\mathbf{u}}_{COV}^j + \boldsymbol{\mu}^j \boldsymbol{\mu}^{j*}) - \boldsymbol{\mu} \boldsymbol{\mu}^*. \end{aligned} \tag{18}$$

5 Numerical Studies

In this section we present numerical studies to illustrate the application of ME-SRBMs to two model problems involving steady state heat transfer in media with random conductivity. Results are presented for uniform uncertainty models which are compared against those obtained using direct MCS with 50,000 samples. Note that we carry out MCS after Karhunen-Loéve (KL) expansion of the input random fields. This is because our primary focus is on the error from ME-SRBMs and hence it makes sense to eliminate the error introduced by the truncation of KL expansion which is well documented. The convergence trends of the response mean and standard deviation, **K**-norm error, and residual error norm are studied. The variables considered are the number of stochastic basis vectors (p-convergence), the degree of decomposition of the random space (h-convergence) and global/local preconditioning strategies.

5.1 Stochastic steady state heat transfer on a square surface

We first consider the stochastic steady state heat equation considered earlier by Xiu and Karniadakis [25]

$$\nabla \cdot [\kappa(x, y; \omega) \nabla u(x, y; \omega)] = f(x, y; \omega) \qquad (x, y) \in [-1, 1] \times [-1, 1]$$

with the boundary conditions

$$u(-1, y; \omega) = 1,\ \frac{\partial u}{\partial x}(1, y; \omega) = 0,\ u(x, -1; \omega) = 0,\ \frac{\partial u}{\partial y}(x, 1; \omega) = 0$$

The conductivity $\kappa(x, y; \omega)$ is a stochastic process with certain distribution and a given correlation function with the mean field being $\bar{\kappa}(x, y; \omega) = 1$. Due to the nonhomogeneous Dirichlet boundary conditions, the stochastic system of equations after the imposition of boundary conditions are of the form given by Eq. (2), where $\mathbf{K}(\boldsymbol{\xi}) : \mathbb{R}^d \to \mathbb{R}^{n \times n}$, $\mathbf{u}(\boldsymbol{\xi}), \mathbf{f}(\boldsymbol{\xi}) : \mathbb{R}^d \to \mathbb{R}^n$ and n denotes the total number of free degrees of freedom.

The input correlation function we adopt for this numerical example is of the form

$$C(r) = \sigma^2 e^{-r/b}, \quad r \geq 0 \tag{19}$$

where r is the Euclidean distance between two points in 2D space, $\sigma = 0.4$ is the input standard deviation and $b = 1$ is the correlation length. In order to reduce the dimensionality of the problem, we apply KL decomposition for random field discretization. Since no analytical eigenvalues and eigenvectors can be obtained for the KL eigenvalue problem for the above mentioned correlation function, we perform numerical KL decomposition. The numerical KL

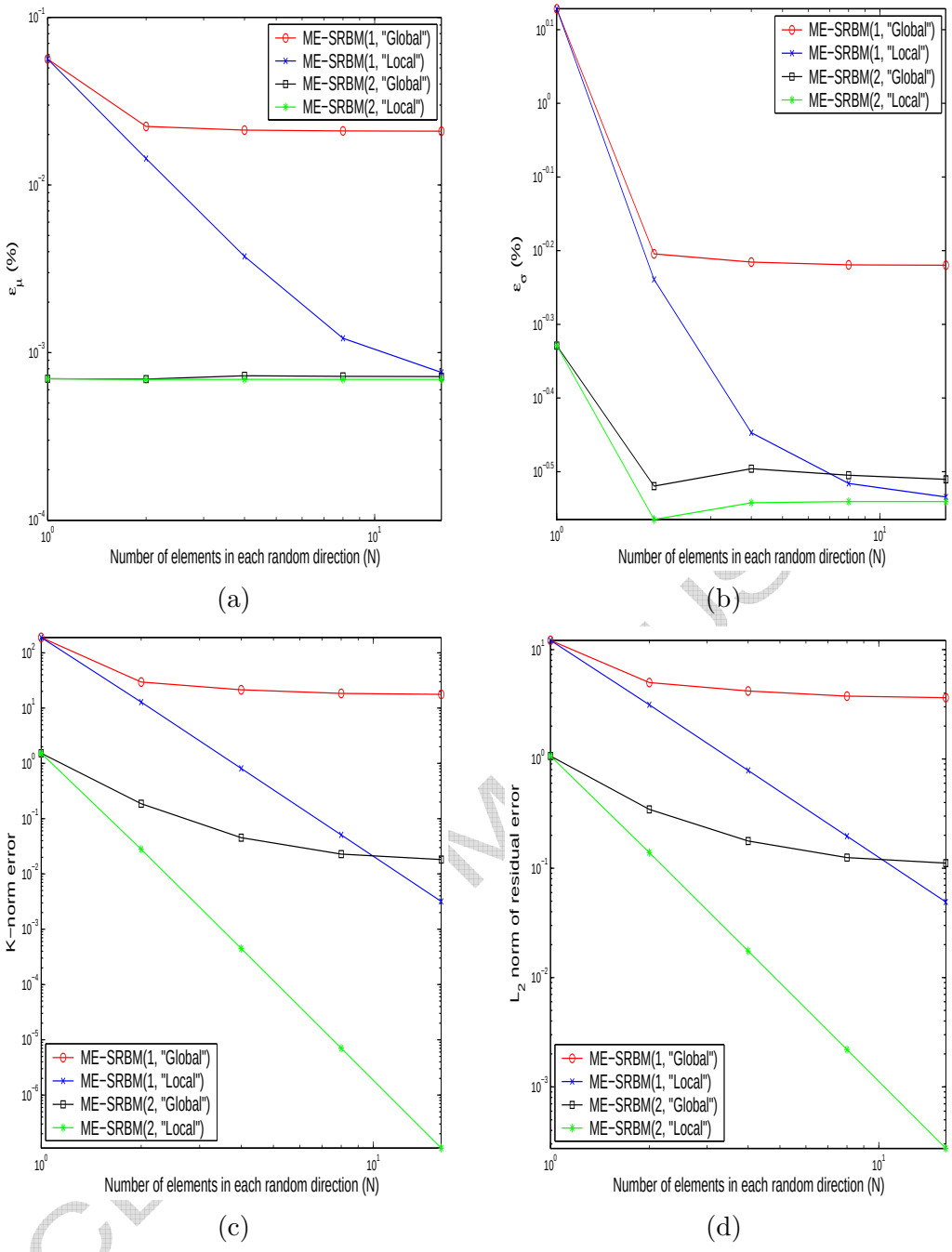


Fig. 1. (a) Percentage norm relative error in mean (ϵ_μ) (b) Percentage norm relative error in standard deviation (ϵ_σ) (c) **K**-norm error and (d) Norm of residual error using upto 2 basis vectors as a function of number of subdomains (N) and different preconditioners when only two terms are retained in the KL expansion.

decomposition is based on the finite element approximation of the KL integral eigenvalue problem, proposed by Keese and the reader is referred to [26] for the discretization error introduced by such an approximation.

Note that in all the subsequent figures, we use the abbreviation ME-SRBM(m,

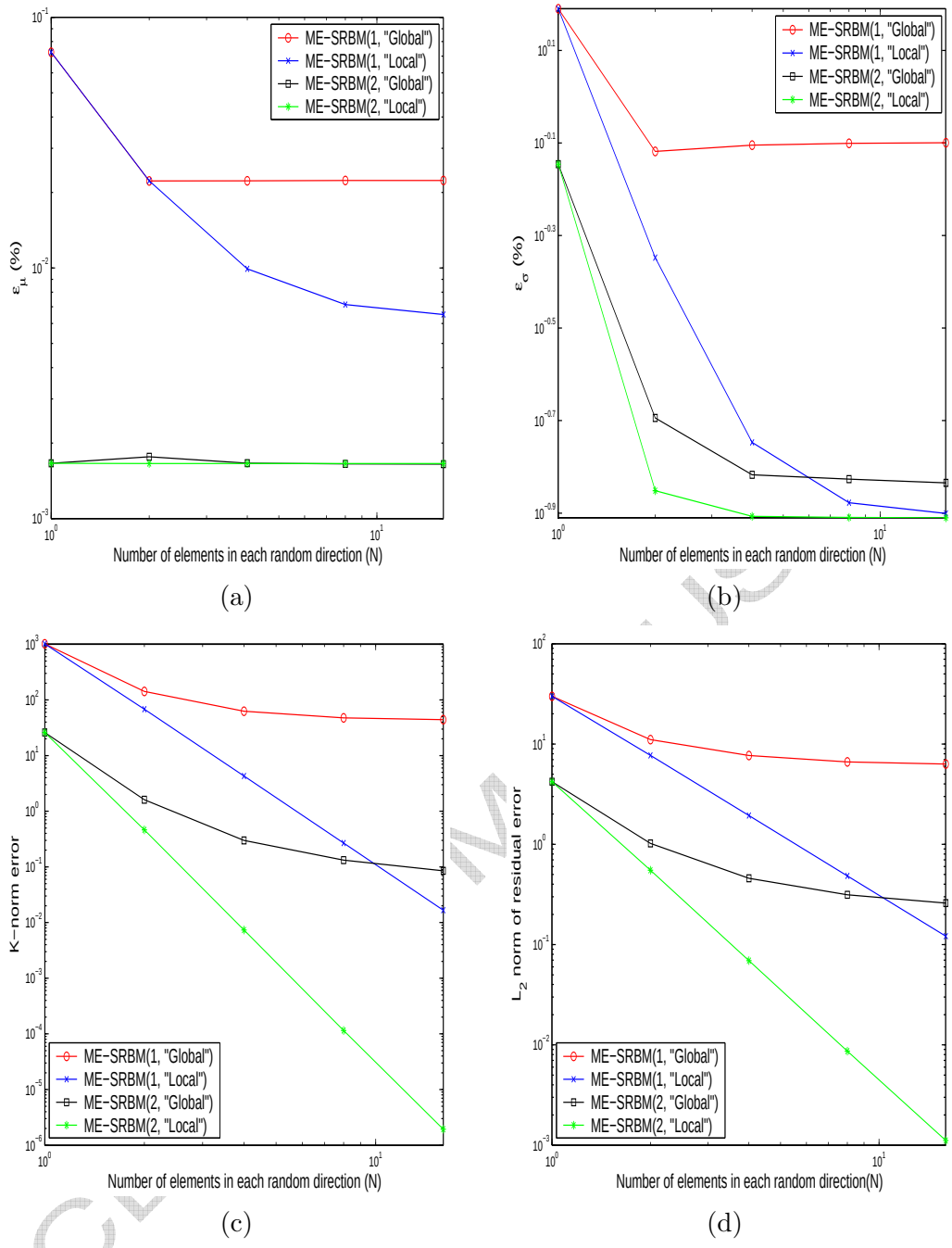


Fig. 2. (a) Percentage norm relative error in mean (ϵ_μ) (b) Percentage norm relative error in standard deviation (ϵ_σ) (c) **K**-norm error and (d) Norm of residual error using upto 2 basis vectors as a function of number of subdomains (N) and different preconditioners when only four terms are retained in the KL expansion.

"*type*") to denote the results obtained using m stochastic basis vectors and a given *type* of preconditioning strategy (global or local). The graphs depicting the convergence show the norm relative error in mean and standard deviation

which are defined as

$$\epsilon_\mu = \frac{\| \boldsymbol{\mu}_{MCS} - \boldsymbol{\mu}_{ME-SRBM} \|_{L_2}}{\| \boldsymbol{\mu}_{MCS} \|_{L_2}} \text{ and } \epsilon_\sigma = \frac{\| \boldsymbol{\sigma}_{MCS} - \boldsymbol{\sigma}_{ME-SRBM} \|_{L_2}}{\| \boldsymbol{\sigma}_{MCS} \|_{L_2}}. \quad (20)$$

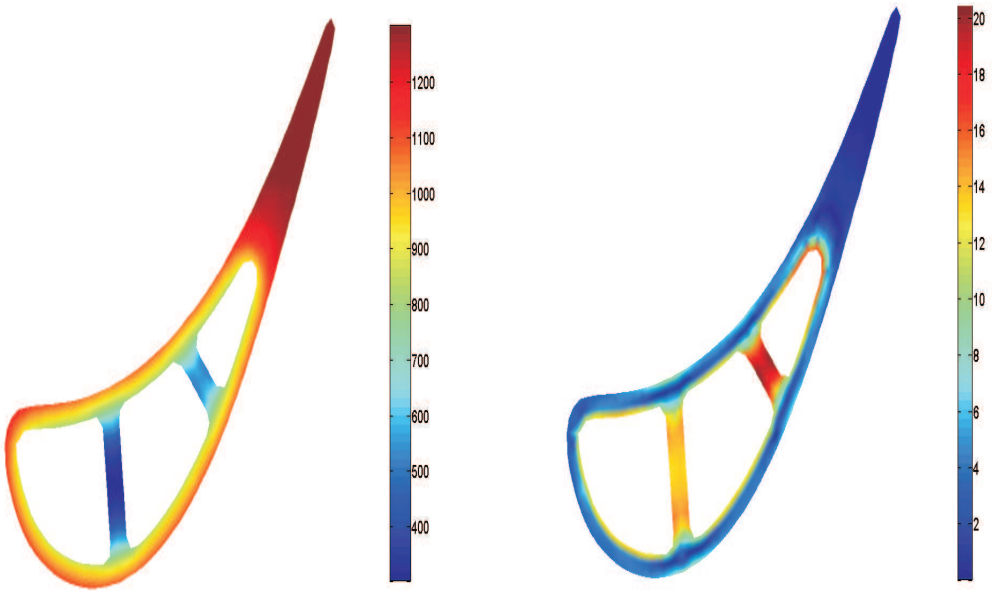
The convergence trends obtained for the case when 2 and 4 terms are retained in the KL expansion of the conductivity are shown in Figures 1 and 2, respectively. It can be seen from the figures that the error in standard deviation decreases (compared to the single element case which corresponds to the standard SRBM formulation) as the number of random elements is increased. Also local preconditioning results in faster convergence of the error in standard deviation as compared to global preconditioning. This trend is not surprising since local preconditioning is expected to numerically result in a lower order minimal random polynomial of the stochastic stiffness matrix in each sub-domain as compared to the global preconditioning strategy. It is also worth noting that that for certain number of random elements (N), the local preconditioning strategy is as good as the strategy employing a higher number of basis vectors with global preconditioning. The above mentioned figures also show a consistent drop in $\mathbf{K}$-norm error and residual error norm as the number of random elements is increased. Accelerated convergence is observed with both metrics for ME-SRBMs with local preconditioning.

Recall that the Galerkin projection scheme imposed here minimizes an energy norm or the $\mathbf{K}$-norm error and its convergence is guaranteed due to Theorem 3. On the other hand, the decrease in residual error may not be strictly monotonic as the number of subdomains is increased for any type of preconditioner used. Similarly no guarantees exist for the strict non-increasing nature of norm relative error in mean and standard deviation. This can be observed with ϵ_μ and ϵ_σ in the above mentioned figures.

Since the method works well for this simple problem, we move onto a more complex geometry with increased number of degrees of freedom. Further, this problem involves Robin boundary conditions instead of the Dirichlet and Neumann boundary conditions applied earlier.

5.2 Stochastic steady state heat transfer on a 2D HP turbine blade

In the high pressure (HP) stage or the first stage of a turbine, the blades are subjected to a high temperature flow due to the hot gases produced by the combustor. To counter the thermal stresses created by this high speed and temperature, the turbine blades are often internally cooled by pumping relatively cool air into the cooling holes. We consider the case when the conductivity of the material of a simplified turbine blade is represented by a random field. Note that the parameter settings for this problem have been adapted from



(a) (b)

Fig. 3. (a) Mean (b) standard deviation of temperature distribution on the turbine blade.

reference [27].

We solve the heat transfer equation defined by Eq. (1) with convective heat transfer boundary conditions. The heat transfer rate is given by

$$\overrightarrow{q} = -\kappa(x, y; \omega)\nabla u(x, y; \omega),$$

where $\kappa(x, y; \omega)$ is the thermal conductivity of the blade material which in turn is a stochastic process with a certain probability distribution and the correlation function defined by Eq. (19) with the mean field being $\bar{\kappa}(x, y; \omega) = 1$ and $u(x, y; \omega)$ is the temperature. The input standard deviation is fixed at $\sigma = 0.2$ and the correlation length (b) is considered to be unity. For the blade surface, the heat flux out of the blade is given by

$$\overrightarrow{q} \cdot \overrightarrow{n} = h_{ext}(u - u_{ext})$$

where $\overrightarrow{q} \cdot \overrightarrow{n}$ is the heat flux out of the blade, $\overrightarrow{n}$ is the unit normal out of the blade surface, h_{ext} is the external convective heat transfer coefficient and u_{ext} is the temperature of the flow external to the blade. Similarly, the heat flux into the cooling passage is given by

$$\overrightarrow{q} \cdot \overrightarrow{n} = h_{int}(u - u_{int})$$

$\overrightarrow{n}$ is still out of the blade i.e., into the cooling passage, h_{int} and u_{int} are the

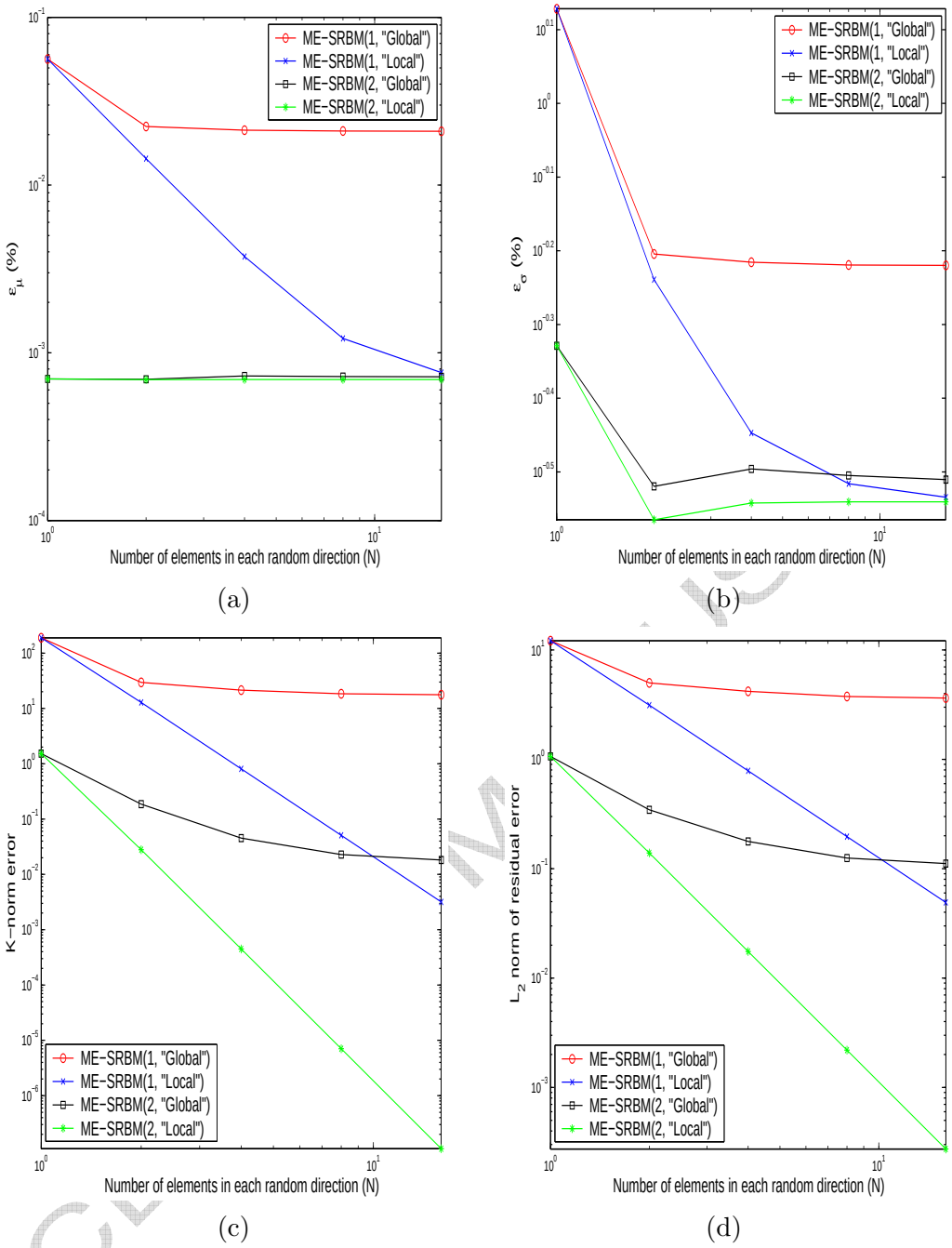


Fig. 4. (a) Percentage norm relative error in mean (ϵ_μ) (b) Percentage norm relative error in standard deviation (ϵ_σ) (c) **K**-norm error and (d) Norm of residual error using upto 2 basis vectors as a function of number of subdomains (N) and different preconditioners when only two terms are retained in the KL expansion.

internal convective heat transfer coefficient and the temperature of the flow in the cooling passage respectively. The dimensions of the blade are normalized using chord length (L) during meshing. Hence $\frac{x}{L}$ and $\frac{y}{L}$ are the coordinates of the grid. The temperatures of flow on the boundaries are $u_{ext} = 1300^\circ\ C$ and $u_{int} = 200^\circ\ C$. The convective heat transfer coefficients in non-dimensional

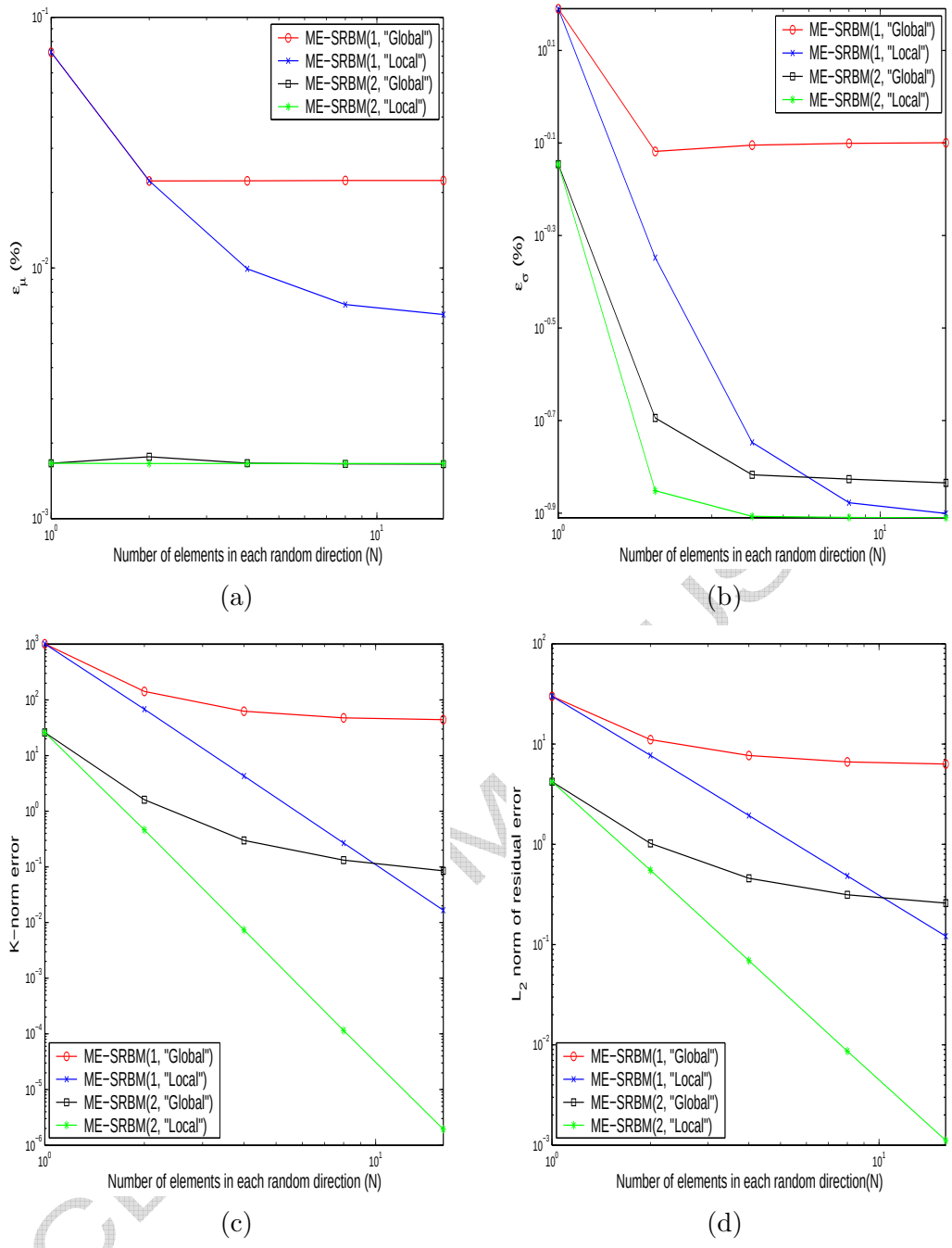


Fig. 5. (a) Percentage norm relative error in mean (ϵ_μ) (b) Percentage norm relative error in standard deviation (ϵ_σ) (c) **K**-norm error and (d) Norm of residual error using upto 2 basis vectors as a function of number of subdomains (N) and different preconditioners when only four terms are retained in the KL expansion.

form are given by

$$\frac{h_{ext}L}{\bar{\kappa}(x,y;\omega)} = 14.0 \ , \frac{h_{int}L}{\bar{\kappa}(x,y;\omega)} = 4.7.$$

The mean and standard deviation of the temperature distribution on the blade obtained using MCS with 50,000 samples are shown in Figure 3. The convergence trends for the case when 2 and 4 terms are retained in the KL expansion of the conductivity field are shown in Figures 4 and 5, respectively. Similar to the observations made in the preceding numerical example, the figures show that the error in standard deviation decreases (as compared to the single-element case) consistently as the number of random elements is increased. Also local preconditioning results in a faster convergence of the error in standard deviation as compared to global preconditioning. The above mentioned figures also show a consistent drop in **K**-norm error and residual error norm as the number of random elements is increased. Accelerated convergence is observed with both metrics when ME-SRBMs are used with local preconditioners. The **K**-norm error monotonically decreases in agreement with the theory for all the cases.

It has already been pointed out that global preconditioning results in a high degree of parallelization and faster speed of execution than an algorithm using local preconditioning as the former algorithm involves only $O(1)$ matrix inversion operations compared to the $O(N^d)$ inversion operations of the latter. But in general local preconditioners result in faster convergence. At this stage, it is not clear as to what preconditioning strategy is more efficient for a given problem and for a given degree of accuracy. In practice, the global preconditioning strategy is computationally efficient for large scale problems with many random variables.

6 Concluding Remarks

In this paper, we proposed a multi-element formulation of stochastic reduced basis projection schemes for solving linear random algebraic systems of equations arising from discretization of stochastic partial differential equations. The objective of this work is to enhance the accuracy of SRBMs for a given order of basis vectors. This has been achieved by decomposing the random space into multiple subdomains and applying SRBMs to local partitions of the random space. The elemental or local statistics are subsequently assimilated to estimate the global statistics.

Numerical studies indicate that ME-SRBMs provide more accurate statistics compared to standard single-element SRBMs.

In contrast to p-refinement, the h-refinement strategies proposed here (ME-SRBMs) admit a large degree of parallelization. Although, ME-SRBMs are conceptually simple and offer improved accuracy, they suffer from the *curse of dimensionality* since N^d local decoupled systems of equations need to be solved

if a random space of d dimensions is decomposed into N subdomains along each dimension. This number rapidly blows into unrealistic proportions for problems with many uncertain variables. It is expected that the incorporation of adaptive domain decomposition schemes will alleviate this problem to some extent.

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