Supplementary matrial for Robust Iterative Learning Control for Unstable MIMO Systems

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August 29, 2024

Abstract

A summary of the design calculation steps used in Simulation.

1 Problem formulation

Let G^* represent the true, possibly non-linear form of the system to be controlled. This system can be linearised about an operating point to give a model with the state-space form (1), in which $x_k(t) \in \mathbb{R}^n$ represents the system state at time-step t of trial k, and $u_k(t) \in \mathbb{R}^p$ and $y_k(t) \in \mathbb{R}^q$ represent the system input and output respectively.

$$\begin{aligned} x_k(t+1) &= Ax_k(t) + Bu_k(t) \\ y_k(t) &= Cx_k(t) \end{aligned} \tag{1}$$

ILC analysis can be simplified by packaging the state-space model (1) into a single 'lifted' system matrix representing the system dynamics over a full trial. This requires representing signals as 'supervectors' such that

$$\begin{aligned} y_d &= \begin{bmatrix} y_d(1) & \dots & y_d(N) \end{bmatrix}^T \in \mathbb{R}^{qN} \\ y_k &= \begin{bmatrix} y_k(1) & \dots & y_k(N) \end{bmatrix}^T \in \mathbb{R}^{qN} \\ u_k &= \begin{bmatrix} u_k(0) & \dots & u_k(N-1) \end{bmatrix}^T \in \mathbb{R}^{pN} \end{aligned} \tag{2}$$

This allows (1) to be re-written as

$$y_k = \bar{G}u_k + d \tag{3}$$

where $G \in \mathbb{R}^{qN \times pN}$ is given by

$$\bar{G} = \begin{bmatrix} CB & 0 & & \\ CAB & CB & 0 & \\ \vdots & \vdots & & \\ CA^{N-1}B & CA^{N-2}B & \dots & CB \end{bmatrix} \tag{4}$$

and d is the initial condition response

$$d = \begin{bmatrix} Cx_0, CAx_0, \dots, CA_{N-1}x_0 \end{bmatrix}^T \in \Re^{qN}$$

however for simplicity $d = 0$ is assumed throughout.

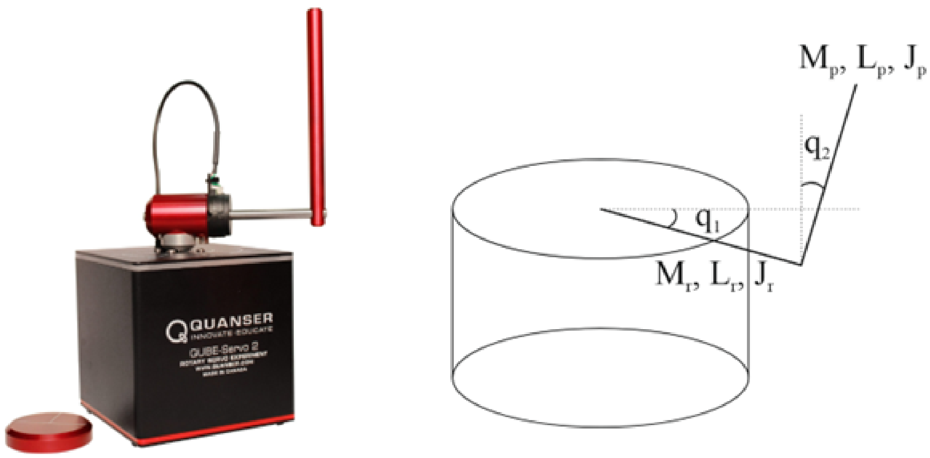

Figure 1: A diagram of the rotary inverted pendulum setup.

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2 Inverted pendulum model

Control algorithms were implemented using MATLAB and Simulink on a simulation of the Quanser QUBE-Servo. The control variable was the input voltage to a rotational servo motor that altered the angle θ of a rotary arm to which the pendulum was attached. This rotation was then used to stabilise the pendulum angle α. A diagram of the setup can be seen in Figure 1.

The full system model is highly non-linear, with equations taking the form

$$\begin{aligned} m_{11}(q_2)\ddot{q}_1 + m_{12}(q_2)\ddot{q}_2 + 2m_2 l_2^2 \sin(q_2)\cos(q_2)\dot{q}_1\dot{q}_2 - m_2 l_2 L_1 \sin(q_2)\dot{q}_2^2 &= \tau \\ m_{21}(q_2)\ddot{q}_1 + m_{22}(q_2)\ddot{q}_2 - m_2 l_2^2 \sin(q_2)\cos(q_2)\dot{q}_1^2 - m_2 g l_2 \sin(q_2) &= 0 \end{aligned} \tag{5}$$

where m_p, m_r, L_p, L_r, J_p, J_r represent the mass, length, and moment of inertia of the pendulum and rotary arm respectively [1]. The applied torque τ is related to the motor voltage V_m via

$$\tau = \frac{k_m(V_m - k_m\dot{\theta})}{R_m} \tag{6}$$

where k_m is the motor torque constant and R_m is its terminal resistance.

These equations can be expressed in the standard form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = F(q)u \tag{7}$$

or

$$\underbrace{\begin{bmatrix} m_{11}(q_2) & m_{12}(q_2) \\ m_{21}(q_2) & m_{22}(q_2) \end{bmatrix}}_{M(q)} \underbrace{\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix}}_{\ddot{q}} + \underbrace{\begin{bmatrix} 2m_2 l_2^2 \sin(q_2)\cos(q_2)\dot{q}_2 & -m_2 l_2 L_1 \sin(q_2)\dot{q}_2 \\ -m_2 l_2^2 \sin(q_2)\cos(q_2)\dot{q}_1 & 0 \end{bmatrix}}_{C(q,\dot{q})} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{q}} + \underbrace{\begin{bmatrix} 0 \\ m_2 g l_2 \sin(q_2) \end{bmatrix}}_{G(q)} = \underbrace{\begin{bmatrix} \tau \\ 0 \end{bmatrix}}_{F(q)u} \tag{8}$$

Equations can be linearised about the operating point at $\alpha = 0$ by using small angle approximations. This allows the system to be expressed in the form

$$\underbrace{\begin{bmatrix} I_1 + m_1 l_1^2 + m_2 L_1^2 & m_2 L_1 l_2 \\ m_2 L_1 l_2 & I_2 + m_2 l_2^2 \end{bmatrix}}_{M} \ddot{q} - \begin{bmatrix} 0 & 0 \\ 0 & m_2 g l_2 \end{bmatrix} q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau \tag{9}$$

so that linear state space equations are given by (1) with $x(t) = [q(t)^\top, \dot{q}(t)^\top]^\top$, $u(t) = V_m$ and

$$A = \begin{bmatrix} 0 & I \\ M^{-1}\begin{bmatrix} 0 & 0 \\ 0 & m_2 g l_2 \end{bmatrix} & 0 \end{bmatrix} - \frac{k_m^2}{R_m}\begin{bmatrix} 0 \\ 0 \\ M^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1}\begin{bmatrix} \frac{k_m}{R_m} \\ 0 \end{bmatrix} \end{bmatrix}, \quad C = \begin{bmatrix} I & 0 \end{bmatrix}^\top \tag{10}$$

These are then used to construct matrix $\bar{G}$ in (4). As outlined in Section 5 of the main paper, feedback controller K took the form of two nested PID controllers, K_1 and K_2. The state-space matrices of the stabilised linearised model P are as follows:

$$A = \begin{bmatrix} 1 & 0 & 1.85e-5 & 0.0016 & -0.0018 & -0.01 & 0 & 0 \\ 0 & 0 & -0.0019 & -0.1644 & -0.1799 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9194 & 0.0806 & 0 & 0 & 0 \\ -0.1791 & -0.3729 & -0.0008 & -0.0714 & 1.0781 & 0.4331 & 0.01 & 0 \\ 0.109 & 0.227 & 0.0005 & 0.0435 & -0.0476 & 0.7399 & 0 & 0.01 \\ -35.8088 & -74.5605 & -0.1606 & -14.2775 & 15.6234 & 86.5892 & 0.9972 & -0.0012 \\ 21.8055 & 45.4029 & 0.0978 & 8.6941 & -9.5137 & -52.0249 & 0.0017 & 1.0043 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.0003 \\ 0.0002 \\ -0.066 \\ 0.0402 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

References

[1] R. Olfati-Saber, “Nonlinear control of underactuated mechanical systems with application to robotics and aerospace vehicles,” Ph.D. dissertation, Massachusetts Institute of Technology, 2001.