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University of Southampton

Faculty of Engineering and Physical Sciences

Institute of Sound and Vibration Research

**Sitting comfort in next-generation cars:
Theory, modelling, and experimentation**

DOI [10.5258/SOTON/T0077](https://doi.org/10.5258/SOTON/T0077)

by

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Thesis for the degree of Doctor of Philosophy

February 2025

University of Southampton

Abstract

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Theory, modelling, and experimentation

Francesco D'Amore

The scientific study of ride quality, and especially of sitting comfort, requires a multidisciplinary approach that is appropriately established within the ambit of human factors and ergonomics. This research project was conducted within such a framework to contribute in paving the way for characterising, understanding, and predicting sitting comfort in next-generation (connected, autonomous, shared, and electric) cars. The contribution was divided into the three strands of theory, modelling, and experimentation, which resulted respectively in three theoretical studies, one modelling study, and one experimental study. Theoretical background and generalisations were reserved for a substantial appendix.

The first theoretical study allows obtaining the solution to the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. The obtained solution results from using the theory of distributions, and it evokes singular distributions, such as 'mysterious' pseudofunctions as well as 'hidden' unit impulses and derivatives of them, all having the origin as singular support.

The second theoretical study allows obtaining a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements; these are the linear spring, the linear viscous damper, the linear inerter (including the rigid mass), and the linear hysteresis damper. In particular, a proper characterisation can be in terms of dynamic stiffness, mechanical impedance, and apparent mass as well as in terms of dynamic compliance, mobility, and accelerance (and their time-domain homologues). The obtained characterisation results from applying the findings of the first theoretical study, and, in the frequency domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null frequency).

The third theoretical study allows obtaining a non-heuristic but application-oriented characterisation of a linear viscoelastic material both in the time domain and in the frequency domain. The obtained characterisation results from adopting the hereditary approach and from applying the findings of both the first theoretical study and the second theoretical study, and, in the time domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null time).

The modelling study allows investigating whether it is possible to calibrate a finite-element model of a seat occupant in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back). The developed occupant model results from applying the findings of the third theoretical study, and it proves having too restrictive a minimum complexity requirement.

[Continued on next page]

Abstract

The experimental study allows establishing whether the concept of sitting configuration is appropriate to characterise a seat–occupant system as a whole with reference to performed activities as well as establishing whether there are significant main effects of, and interactions between, sex, vibration magnitude, and sitting configuration on objective responses of seat–occupant systems to vibration. The obtained experimental results may be used in follow-up studies to validate the occupant model developed in the modelling study.

In the theoretical studies, well-known physical aspects are omitted to focus on unclear mathematical aspects. The findings hold under mildly restrictive conditions (except in the ill-behaved case of the linear hysteresis damper) and contain specification of the input spaces. However, no attempt is made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions are identified for formulations to be mathematically tenable.

The concept of sitting configuration proves appropriate to characterise the seat–occupant system as a whole with reference to performed activities. Indeed, at least as regards in-line transmission of vertical vibration at seat cushion, possible sitting configurations in next-generation cars are associated with distinctive objective seat–occupant responses to vibration. Specifically, when it comes to transmissibility plots, the scale is largely dependent on vibration magnitude, whereas the shape is largely dependent on sitting configuration; it therefore seems appropriate that, in future investigations, the long-standing emphasis on vibration magnitude be curtailed in favour of other factors. Results suggest that not only vibration magnitude but also sitting configuration (and possibly sex) affect sitting comfort in next-generation cars through in-line transmission of vertical vibration at seat cushion. Accordingly, in design and development of seating for next-generation cars, secondary activities and corresponding sitting configurations should be taken into account to optimise not only functionality but also comfort and protection (as well as related affective or emotional attributes).

The findings of this research project suggest that research and development for sitting comfort in next-generation cars benefits from introducing the concept of sitting configuration and from using relatively advanced mathematical methods such as the theory of distributions, which may prove helpful in elaborating a (currently unavailable) theory of human vibration.

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List of accompanying material

Data supporting this research project are openly available from the University of Southampton Institutional Repository at <https://doi.org/10.5258/SOTON/D3382>.

Author's declaration

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Title of thesis: Sitting comfort in next-generation cars:
Theory, modelling, and experimentation

I declare that this thesis and the work presented in it is my own and that it was generated by me as the result of my own original research.

I confirm that:

1. The work presented in this thesis was done wholly or mainly whilst in candidature for a degree of Doctor of Philosophy at the University of Southampton;
2. Where any part of this thesis had been previously submitted for a degree or any other qualification at the University of Southampton or any other institution, this was clearly stated;
3. Where I consulted the work of others, this was always clearly attributed;
4. Where I quoted from the work of others, the source was always given, this thesis being entirely my own work with the exception of such quotations ;
5. I acknowledged all main sources of help;
6. Where the work presented in this thesis was done by me jointly with others, I made clear exactly what was done by others and what I contributed myself;
7. Parts of this thesis were published as:

D'Amore, F. and Qiu, Y. [2017] Whole-body oscillatory motion and road-transport innovations. A human-factors-and-ergonomics perspective within a ride-quality framework IN: Zioupos, P. (ed.) *52nd UK Conference on Human Responses to Vibration (UK HRV 2017)*, Sudbury House, Faringdon, United Kingdom, 5–6 September 2017. Shrivenham, United Kingdom: Cranfield University Press, 1–14.

Author's declaration

D'Amore, F. and Qiu, Y. [2019] Next-generation cars, secondary activities, and sitting configurations. In-line transmission of vertical vibration at seat cushion IN: Taylor, M.D. and Maeda, S. (eds.) *54th UK Conference on Human Responses to Vibration (UK HRV 2019)*, Edinburgh Napier University, Edinburgh, United Kingdom, 24–26 September 2019. Edinburgh, United Kingdom: Edinburgh Napier University, 1–19.

D'Amore, F. and Qiu, Y. [2021] Vibration transmission at seat cushion and sitting comfort in next-generation cars IN: Black, N.L., Neumann, W.P., and Noy, I. (eds.) *21st Congress of the International Ergonomics Association (IEA 2021)*, Virtual, 14–18 June 2021. Cham, Switzerland: Springer, 615–622. Reproduced with permission from Springer Nature Switzerland AG.

Signature:

Date: 2025-02-10

Acknowledgements

The hand is tired. The chisel is heavy. Here I am, about to carve my final marks into this stone. Honestly, I've had enough.

Why did I do it? Because I wanted to learn these things by reading someone else's work in the comfort of my armchair. And yet, no one came forward to carry it out, let alone write it down.

Anyway, I really have to thank Yi for his patience and support. He once urged me to consider him a friend, and well, he meant it. I also want to make public my gratitude to Kazu and NHK Spring Company for sponsoring my work. And how could I not mention Oscar, Weitan, and Mihai? They made my stay in Albion easier to digest, and we'll not lose sight of each other if the pact with the technological devil is worth the price. But most of all, I ought to give credit to Mihaela for keeping the tub afloat. Truth be told, nothing would have been possible if it weren't for her. I won't forget.

The world is not the same place it was when this started. I mean, take cigarettes. I used to hate my father's smoke, but now their smell brings him back to my mind from wherever he is. Yes, a couple of other things have changed too. Damn.

We're not the same either, for that matter, but we're alive and still on the move. As they say where I come from, life is a bite. I dedicate this little big piece to Maria and Arturo.

Time to leave. I make my own a quote from a 2005 short film by Paolo Sorrentino, on the notes of Israel Kamakawiwo'ole.

Can you remember the last time you were driving in your car? Did you notice anything special? The colour of the trees, perhaps. No? Maybe you were too busy thinking about your final destination. Next time, try taking in more during your journey. A bend. Silence inside. An unexpected view. You'll discover that sometimes it's not where you're going this important, but how you get there.

1 Introduction

1.1 Motivation and context

The world of road transport has been rapidly changing in recent years. Meanwhile, academic scholars, management consultants, and business executives have been discussing whether and when a paradigm shift is to take place [Corwin *et al.*, 2016; Hannon *et al.*, 2016, 2019; Bouton *et al.*, 2017; Fulton *et al.*, 2017; Attias, 2017a]. *Connection, automation, sharing, and electrification* are the keywords for next-generation cars. The related processes of innovation originated at different times and, guided by different needs, developed in different ways before starting to converge.

Despite the many technical, technological, economic, ethical, social, cultural, political, and regulative challenges to be faced [Boulanger *et al.*, 2011; Lu *et al.*, 2014; Bagloee *et al.*, 2016; Anderson *et al.*, 2016; Litman, 2017], major advances are expected within the automotive industry, together with new opportunities, new value chains, and new business models [Maitland and Guttman, 2014; KPMG, 2015; Fournier, 2017].

The hype in the media and the speculation amongst insiders have reached incredible peaks; nevertheless, possible developments are not very clear, and it is still a matter of discussion whether the nature of such processes of innovation will prove to be disruptive and revolutionary or rather incremental and evolutionary [Jiang *et al.*, 2015; Attias, 2017b]. However, the changes on the horizon may be so significant to make a paradigm shift possible and a structured approach necessary from all the relevant points of view.

In the belief that the field of study of *human vibration* falls within the ambit of *human factors and ergonomics* (or *HFE*), an HFE perspective is adopted here in an attempt to recognise some possible applicative reverberations of expected automotive innovations. The need for a general framework taking into account ‘hard’ and ‘soft’ features as well as ‘objective’ and ‘subjective’ implications is identified. The comprehensive concept of ride quality is suggested as a good candidate to cope with the complexity of the open problems.

1.1.1 CASE mobility

A *connected and autonomous vehicle* (or *CAV*) [Bimbrow, 2015; Bell *et al.*, 2016; McCarthy *et al.*, 2016; SMMT, 2017; Trustworthy Autonomous Systems Hub *et al.*, 2021] is a *driving automation system* (of one of five different standardised levels of driving automation [SAE International, 2021a])

that is integrated in an ‘ecosystem’ known as *Internet of Things (IoT)* or, more specifically, *Internet of Vehicles (IoV)* [Yang *et al.*, 2014; Lengton *et al.*, 2015] and that exchanges information with entities affecting it through a vehicular communication system known as *vehicle-to-everything* (or *V2X*) or, more specifically, *vehicle-to-vehicle* (or *V2V*) and *vehicle-to-infrastructure* (or *V2I*) [Sprouffske, 2013]. In fact, there is no consensus on whether the processes of connection and automation will eventually converge, and it is unclear whether connected and autonomous vehicles may be part of some future *Intelligent Transport System* (or *ITS*) [Directive 2010/40/EU, 2010; Zmud *et al.*, 2015]; in any case, connected and autonomous vehicles, ideally to be deployed within ‘archipelagos’ of sustainable, functional, and networked *smart cities* [Geoffron, 2017], are expected to change significantly not only the ride experience but also the urban environment [Cuddy *et al.*, 2014].

From a socio-economic point of view, the controversial transition from a product-oriented *traditional economy*, based on individual ownership, to a service-oriented *sharing economy*, based on collective access, has been giving rise to momentous social changes in most countries [Puschmann and Alt, 2016; Frenken and Schor, 2017]. Whilst habitually clear distinctions have been fading, be they between manufacturers and consumers of products or between suppliers and users of services, evolving lifestyle and values scale of today’s young adults have been redefining, perhaps irreversibly, the notions of consumption and use as interpreted by the millennial generation onwards [Godelnik, 2017].

The *shared mobility*, driven by economic and environmental factors [Böcker and Meelen, 2017], is probably the most vital and rapidly evolving expression of sharing economy [Feigon and Murphy, 2016; International Transport Forum, 2016]. Within the age of ‘liquid modernity’, in which Bauman [2012, p. viii] describes “the growing conviction that change is *the only* permanence, and uncertainty *the only* certainty”, private vehicles tend to be regarded less and less as status symbols [Adam, 2015], let alone as totemic representations of individual freedom [D’Costa, 2013], and more and more as mobility tools [Attias, 2017a]. In fact, exercise of individual freedom may progressively result in people just not wanting or even not being able at all to drive whilst taking advantage, by preference, of the attractive flexibility offered by a plethora of new opportunities such as *car-sharing*, *microtransit*, *ride-sharing*, *ride-sourcing*, and *ride-splitting* [Feigon and Murphy, 2016]. Interestingly, some even consider it possible that driving automation systems allow alternative travel providers to capture market share at the expense of established short-haul terrestrial and aerial competitors [Fairs, 2015; Hazan *et al.*, 2016].

Amongst the ongoing processes of innovation, *electrification* was almost certainly the first to be conceived, discussed, and put into practice. Despite the first prototypes dating back to the 1820s,

1 Introduction

electric propulsion reached its heyday only in the early 1900s by virtue of the advantages brought over fuel propulsion in terms of power, startability, and controllability [Brinkman *et al.*, 2012; Matulka, 2014]. After a decline in the 1920s, a renaissance had to wait until the energy crisis of the 1970s and the emergence of the concept of sustainable transport in the 1980s, when the need to diversify energy sources, to moderate resource consumption, to reduce air-pollutant emissions, and, later, to oppose climate change started to gain momentum in public opinion as well as in academic, political, and industrial environments [Brinkman *et al.*, 2012; Matulka, 2014]. Eventually, during the 1990s and the 2000s, the development of automotive electronics and the hybridisation of propulsion systems led to a proliferation of *electric vehicles (EVs)*, which, with respect to electrical grids, may be totally dependent, as in the case of *battery electric vehicles (BEVs)*, totally independent, as in the case of *hybrid electric vehicles (HEVs)*, or partially dependent, as in the case of *plug-in hybrid electric vehicles (PHEVs)* [Kumar and Jain, 2014]. In the future, electric vehicles are expected to integrate with *smart grids* [Mwasilu *et al.*, 2014] consisting of energy-distribution infrastructures evolved to become material substrates for bidirectional information exchange and notably to perform some of the functions of communication, control, and management needed to implement other processes of innovation.

Despite the enthusiasm palpable in some environments, the view is spreading that, in the long term, connection, automation, sharing, and electrification can be sustainable and beneficial only if implemented together in what is known as *CASE mobility* [Hannon *et al.*, 2016, 2019; Bouton *et al.*, 2017; Fulton *et al.*, 2017]. In particular, considering that private vehicles are characterised by very low average values of occupancy rate (of the order of 1.5 persons per private vehicle in the United Kingdom in the years from 2004 to 2008 [European Environment Agency, 2010]) and by very high average values of idle time (approximately 96 % of the total time in the United Kingdom in the years from 2002 to 2008 [Bates and Leibling, 2012]), large-scale adoption of connected and autonomous vehicles is regarded by some as sustainable only in a road-transport scenario dominated by shared electric mobility [Schonberger and Gutmann, 2013; Offer, 2015].

To summarise, it seems useful to quote Attias [2017a, p. 17]:

The traditional concept of a powerful, personal car is being replaced by other types of mobility. The social symbols, priorities and preferences of young people are no longer the same. Real-time access to information has become essential. Mobility is more important than power. People want to remain constantly connected to their environment. Cars can no longer be considered as an end in themselves, but as tools for gaining mobility. These new services contribute to the arrival of new types of mobility that are more relaxed, rational and sustainable, and to the emergence of a new ecosystem.

1.1.2 HFE perspective

Were connected, autonomous, shared, and electric vehicles to become the norm, significant behavioural changes would be to expect amongst users [Jamson *et al.*, 2013; Trommer *et al.*, 2016].

As well as air pollutants, noise and vibration would be emitted to a lesser extent by electric powertrains than by conventional powertrains. However, as a side effect, otherwise overshadowed noise and vibration might emerge because of a 'draining of the swamp' phenomenon [Goetchius, 2011] and, paradoxically, become a more serious issue for road-vehicle users who might have, especially in this respect, higher expectations of electric vehicles.

Being more often passengers rather than drivers, road-vehicle users might engage in new on-board activities, and a more productive and enjoyable use of time would be possible. Working and eating, (self) entertaining and socialising, and even relaxing and sleeping would increasingly enter ride experience, thus providing new opportunities of human-machine and human-human interaction [Laurier and Dant, 2011].

To manage traffic flows and to practise new on-board activities, new (autonomous) driving styles would emerge. Average speed values would be lower, lane changes would be less frequent, and vehicle paths would be optimised [Jamson *et al.*, 2013; Le Vine *et al.*, 2015; Elbanhawi *et al.*, 2015]. Consequently, in the classic compromise between performance and comfort, the balance point would shift from the former to the latter.

Adopting an HFE perspective, new on-board activities and new (autonomous) driving styles imply considering new motion environments and new seating solutions [Veen *et al.*, 2014; Ive *et al.*, 2015] as well as addressing new issues especially in terms of situational awareness, activity interference, and motion sickness [Sivak and Schoettle, 2015; Cunningham and Regan, 2015; Kyriakidis *et al.*, 2017].

To meet vehicle habitability needs that are appropriate to the envisaged road-transport scenario, significant advances in vehicle interiors and, more specifically, in vehicle seats are to be pursued. Indeed, interest in the topic is growing rapidly within academic and industrial communities, to the point that specialised symposia are now organised [Uki Media and Events, 2017], with some insiders predicting that exterior design is to lose importance to the benefit of interior design [Fell, 2017]. Ultimately, if vehicle interiors become living spaces alongside home and office, improving ride experience as a whole may well pass through creating what marketing enthusiasts have already started to promote as *seating experience* [Rychel, 2016; Wysocky, 2016; Dobrian, 2016].

1.1.3 Ride quality and comfort

Either explicitly or implicitly, business models are based on *customer satisfaction*; in International Standard ISO 9000:2015, the latter is defined as the “customer’s perception of the degree to which the customer’s expectations have been fulfilled” [International Organization for Standardization, 2015]. In a road-transport scenario in which comfort is valued more than performance, it is reasonable to expect that ride experience may be strongly influenced by HFE issues. Stated otherwise, to fulfil customer expectations at the core of customer satisfaction, it may be necessary to provide future road-vehicle users with better *ride quality*.

In the context of automotive engineering, the wordings *primary ride*, *secondary ride* (also known as *shake*), and *harshness* normally refer to oscillatory motion of the sprung mass of a vehicle on respectively the ‘lower’ frequency band (ranging from 0 Hz to 5 Hz), the ‘intermediate’ frequency band (ranging from 5 Hz to 30 Hz), and the ‘higher’ frequency band (ranging from 30 Hz to 100 Hz) [SAE International, 2022]. However, in the context of human vibration, the term *ride* normally refers to the “measurable motion environment (including vibration, shock, translational and rotational accelerations) as experienced by people in or on a vehicle” [International Organization for Standardization, 1997a]. On this basis, the derived concept of *ride quality* is defined in International Standard ISO 5805:1997 as the “degree to which the whole subjective experience (including the motion environment and associated factors) of a journey or ensemble of journeys by vehicle is perceived and rated as favourable or unfavourable by passengers or operators” [International Organization for Standardization, 1997a].

In the specified context, the wording ‘ride quality’ has been in use for several decades essentially with reference to the sole *motion environment* [Green, 2012]; nevertheless, with the aforementioned standardised definition, it assumes a significantly broader meaning. Whilst *comfort* can be understood, with reference to the sole motion environment, as *dynamic sitting comfort* and can be defined as the “subjective state of well-being or absence of mechanical disturbance in relation to the induced environment (mechanical vibration or repetitive shock)” [International Organization for Standardization, 1997a], ride quality is to be regarded as a complex psychophysical percept reflecting the whole subjective experience and integrating many subjective human responses related not only to dynamic sitting comfort but also, amongst other things, to *static sitting comfort*, *acoustic comfort*, *visual comfort*, *thermal comfort*, and *olfactory comfort* [International Organization for Standardization, 2006]. Finally, besides ‘hard’ *functional or ergonomic features*, one should not ignore ‘soft’ *affective or emotional features* linked to consumers’ feeling and image of

products (which are also qualified by means of the multifaceted Japanese term *kansei*) [Tanoue *et al.*, 1997].

To summarise, ride quality may be regarded as a standardised and measurable counterpart of some undefined and undetermined idea of overall ‘ride feeling’. Starting from the 1980s, customer expectations in terms of ride quality have grown so significantly within the automotive market that new *noise, vibration, and harshness* (or *NVH*) departments have been created by most manufacturers to conduct specialised research and development focussing on acoustic comfort and dynamic sitting comfort [Mansfield, 2005]. More generally, approaches taking into account a greater number of possibly influential factors should be attempted to understand their main effects and interactions [Oborne, 1978].

That said, neither is it clear what *comfort* actually is from a psychophysical point of view, nor is it understood what relations subsist between it and its alleged opposite *discomfort*; in particular, it is still unknown whether comfort and discomfort should be regarded one as absence of the other or, alternatively, as distinct, possibly multidimensional, psychophysical percepts [Zhang *et al.*, 1996; Looze *et al.*, 2003]. Besides, although standardised definitions can provide useful reference and can help limit misunderstandings in practical situations, relying too much on them may result in overrating current knowledge.

Beyond the psychological and philosophical disputes on the semantic value of the term *comfort*, it is well recognised that *sitting comfort*, both static and dynamic, plays a key role in the wider context of ride quality [Ebe, 1998; Ebe and Griffin, 2000; Mansfield, 2013]. Specifically, at least in static sitting conditions, correlations are known between subjective well-being and surface distribution of contact actions at the seat–occupant interface [Looze *et al.*, 2003; Hiemstra-van Mastrigt *et al.*, 2017]; however, to investigate satisfactorily sitting comfort for automotive applications, structured approaches considering a wide range of potentially influential factors may be needed [Kolich, 2008; Silva *et al.*, 2012].

1.1.4 Comfort and protection

Modern vehicles are part of very complex human–machine systems whose conception and design benefit from a human-oriented approach taking into account, amongst others, requirements of *comfort* and *protection* [Braess and Reichart, 1990]. In research and development, comfort and protection issues have been habitually addressed by applied scientists and engineers who are active in the different and rarely interacting communities of *human factors and ergonomics* and *vehicle safety*. Nevertheless, comfort and protection do overlap; for example, it is now acknowledged that

the design of vehicle seats has implications in terms of both. On the one hand, a vehicle seat can be regarded as (whilst not being reducible to) an *occupant restraint system* with full rights [Adomeit, 1979]; on the other hand, a vehicle seat designed by adopting HFE principles is likely to allow a road-vehicle user to stay in better, and ultimately safer, conditions.

According to International Standard ISO 7862:2004, an *occupant restraint system* is an “arrangement of components which is intended to diminish the risk of injury to the occupant in the event of a vehicle collision by controlling the occupant displacement and acceleration” [International Organization for Standardization, 2004]. As opposed to an *active safety system*, which reduces the risk of injury by reducing the probability of occurrence of the injury, an occupant restraint system is usually regarded as a *passive safety system*, which reduces the risk of injury by reducing the severity of the injury.

With the advent of next-generation cars, the traditional distinction between active and passive safety systems is likely to become less clear [Class *et al.*, 2013; ZF TRW, 2016; Ratingen *et al.*, 2016]; moreover, in a road-transport scenario in which new on-board activities may result in increased interaction with human occupants, the role of occupant restraint systems is likely to become even more relevant. Two guiding principles have already been implemented in advanced occupant restraint systems; specifically, they are *adjustability* (capability of accommodating human occupants having different anthropometric characteristics and taking different body postures) and *predictivity* (capability of being activated early and reversibly using data from environmental sensors, such as radars or cameras, or from active safety systems, such as the electronic stability control and the emergency brake assist). One more guiding principle, *adaptability* (capability of accommodating human occupants engaging in different on-board activities and experiencing different motion environments), is suggested here as worthy of attention.

It is not easy to predict the impact that next-generation cars may have on vehicle safety, especially in terms of risk of injury. On the one hand, lower values of average speed, less frequent lane changes, and optimised vehicle paths should have a beneficial effect. On the other hand, situational awareness and out-of-position condition may become major concerns; indeed, the very concept of out-of-position condition will probably need redefinition. It is worth noting that, at present, vehicle-safety issues of next-generation cars seem to have been investigated by focussing more on possible collisions due to errors in traffic-flow management than on HFE implications.

In addition to vehicle seats themselves, occupant restraint systems in use include safety belts, safety harnesses, airbags, knee bolsters, padded dashboards, collapsible steering columns, laminated

windshields, cargo barriers, anti-intrusion bars, safety cells, and crumple zones [Snyder, 1969; States, 1988]. Amongst all these, *safety belts* and *safety harnesses* are the only ones in continuous and close contact with seated human occupants of vehicles. By fulfilling functions of *primary* (as opposed to *secondary* or *supplemental*) occupant restraint systems, safety belts and safety harnesses have been shown for decades to be key for the sake of vehicle safety [Hodson-Walker, 1970; Johannessen, 1984; Road Safety Observatory, 2013]. Indeed, the necessity of safety belts and safety harnesses is now questioned rather seldom [Adams, 2007; Allsop *et al.*, 2008], and, in fact, their use has long since been mandatory by law in many countries of the world [Weiss *et al.*, 2006] and in a wide range of situations [Glassbrenner *et al.*, 2004]. Understandably, vehicle-safety implications of safety belts and safety harnesses have been investigated quite extensively [Huston, 2001]. Conversely, HFE implications have been considered relatively seldom and, anyhow, more from a *static* point of view [Dejeammes *et al.*, 1984; Balci *et al.*, 2001; Chen *et al.*, 2003] than from a *dynamic* point of view [Wyllie and Griffin, 2007, 2009; Zhao and Schindler, 2014].

1.1.5 Framework

In human vibration, extensive use is made of methods developed for engineering applications; consequently, human vibration is often regarded as a part of vibration engineering. Indeed, human-vibration experts can often be found within engineering faculties in academia and within research and development departments in industry. However, when considering the typical field of application and the necessary cultural background, it seems that human vibration can rightfully be regarded as part of *human factors and ergonomics*, which provides a cultural substrate wide enough to take into account ‘hard’ and ‘soft’ features as well as ‘objective’ and ‘subjective’ implications. Like human factors and ergonomics, human vibration is highly multidisciplinary in nature; indeed, human vibration integrates knowledge and skills from several areas including engineering, industrial design, anthropometry, biomechanics, physiology, medicine, and psychology [Griffin, 1990].

Human responses to vibration in the context of automotive applications aroused some research interest already in the early 1930s [Hirshfield, 1933]. However, as can be shown through targeted search-result analyses within bibliographic databases, the topic emerged of its pioneering niche only in the 1960s. Eventually, it was in the 1990s that the number of related publications started to increase significantly.

In the light of the knowledge gained since the early years, the importance of understanding the relations between *subjective human responses to vibration* (dependent on individuals’ judgement) and *objective human responses to vibration* (independent of individuals’ judgement) is now widely

recognised. Nevertheless, despite the conspicuous amount of research already conducted, still many questions remain to be answered [Sandover, 1986; Dupuis and Zerlett, 1986; Griffin, 1990; Gierke, 1997; Gierke and Brammer, 2002; Mansfield, 2005; Brammer and Peterson, 2009; Rakheja *et al.*, 2020a, 2020b].

From an industrial point of view, careful conception and design of vehicle seats have proved decisive for road-vehicle users, be they drivers or passengers, to be provided with satisfactory ride quality not only in terms of comfort, protection, and other ‘hard’ functional or ergonomic features but also in terms of several ‘soft’ affective or emotional features. Indeed, pertinent design guidelines [Huston and Genaidy, 1997] and technical standards [British Standards Institution, 1987; International Organization for Standardization, 1997b, 2004, 2016] do exist; however, new and updated ones need be developed, ideally on the grounds of scientific evidence rather than of heuristic best practices.

It is well known that vehicle seats have a significant influence on both subjective and objective human responses to vibration [Griffin, 1990]. Indeed, vehicle seat and human occupant together form a *seat–occupant system* that can, and often should, be regarded as a whole.

Ultimately, to bridge the subjective–objective gap, a general framework for the study of human vibration may need be considered within the ambit of human factors and ergonomics. In the specific case of automotive applications, and especially in expectation of next-generation cars, it may be appropriate to focus on the comprehensive concept of ride quality by taking into account human responses related not only to comfort, protection, and other ‘hard’ functional or ergonomic features but also to several ‘soft’ affective or emotional features. Within this framework, a seat–occupant system or, more generally, a *seat–occupant–restraint system* should be regarded as a whole. Hereinafter, whenever an occupant restraint system is present, the wording ‘seat–occupant system’ is to be intended as ‘seat–occupant–restraint system’. In this research project, the study of *sitting comfort* is regarded in a broad sense as the study of both subjective and objective responses of seat–occupant systems aimed at maximising the state of well-being or minimising the lack thereof in the context of automotive applications.

1.2 Automotive human vibration

The discipline of human vibration is focussed on the responses of the *human–environment system* to oscillatory motion. Roughly, oscillatory motion may be either fluctuating (namely, sustained or continuous), in which case reference is made to it as *mechanical vibration* (hereinafter *vibration*), or impulsive (namely, transient or intermittent), in which case reference is made to it as *mechanical shock* (also known as *shock motion*, hereinafter *shock*) [Griffin, 1990; International Organization for Standardization, 2018].

For the sake of clarity, it is worth specifying that human responses to shock are human responses to one or more *shock events* that, in relation to most human reactions, are of short duration (indicatively, less than one second) and low repetition rate (indicatively, less than one per second) [International Organization for Standardization, 1997a; Sandover, 1998; Brammer and Peterson, 2009]. In particular, one of the most typical forms of shock occurring in the field of human vibration, known as *repetitive shock*, is defined as a “series of brief (less than one second) shock motions or bursts of quasi-steady-state vibration affecting man” [International Organization for Standardization, 1997a]. Whilst a shock event implies a certain suddenness and severity, it ought not to be considered as the real-world counterpart of an ideal impulsive event represented mathematically by means of some multiple of the *unit impulse* (also known as *Dirac delta* or *delta distribution*). Indeed, the concept of shock remains distinct from the concept of *impact*, which, as in the case of vehicle collision, implies a significant transfer of momentum [Gierke and Brammer, 2002]. More specifically, within the field of human vibration, *human impact* refers to events resulting in humans being subjected to pain, injury, or significant physiological distress [International Organization for Standardization, 1997a].

Despite belonging to the scientific and technical language with full rights, the terms *vibration*, *shock*, and *impact* are also well established in the common language with necessarily less definite meanings. Accordingly, the meanings of such terms in specialist contexts are to some extent opaque, and the boundaries establishing the transitions between vibration, shock, and impact are in practice rather blurred and largely conventional. A pragmatic, although not optimal, solution to this problem is offered by the sensible use of standardised vocabulary [International Organization for Standardization, 1997a, 2018], to be intended more as a unifying resource than as a collection of rigid prescriptions. It is worth noting that, in fact, the term *vibration* is loosely but frequently used as a shorthand for the wording ‘oscillatory motion’.

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The broad field of human vibration is traditionally partitioned into the subfields of *whole-body vibration* and *segmental vibration* [International Organization for Standardization, 1997a]. On the one hand, *whole-body vibration* (respectively, *whole-body shock*) can be defined as “mechanical vibration” (respectively, shock) “transmitted to the body as a whole, usually through areas of the body (e.g. buttocks, soles of the feet, back) in contact with a supporting contact surface that is vibrating (or subjected to shock motion)” [International Organization for Standardization, 1997a]. In principle, whole-body vibration can be transmitted to the human body either from supporting contact surfaces (in which case it is qualified as *structure-borne*) or from oscillating fluid media (in which case it is qualified as *fluid-borne*) [Shoenberger, 1972]; in practice, as suggested by the previous definition, whole-body vibration is *usually* structure-borne. On the other hand, *segmental vibration* (respectively, *segmental shock*) can be defined as “mechanical vibration” (respectively, shock) “applied or transmitted to a particular segment, area or region of the human body (usually distinguished from a whole-body input) such as the hand-arm system or the head” [International Organization for Standardization, 1997a]. Segmental vibration often occurs in the form of *hand-transmitted vibration* (also known as *hand-arm vibration*) [Griffin, 1990, 1997; International Organization for Standardization, 1997a]. Segmental vibration is only mentioned for completeness and is not discussed further.

Within whole-body vibration, the spatial configuration of the human body and its interaction with the local environment are characterised by specifying *body position*, *body orientation*, *body posture*, *body support*, and *body restraint* [Griffin, 1990; International Organization for Standardization, 1997a].

The wording *body position* refers to the spatial configuration of the human body as regards its gross arrangement with respect to the local environment. Body position is customarily classified as *seated* (or *sitting*), *standing*, and *recumbent* (or *lying*, the term having been deprecated in International Standard ISO 5805:1997 but being still widely used [International Organization for Standardization, 1997a]).

The wording *body orientation* refers to the spatial configuration of the human body as regards its arrangement with respect to a reference direction (for example, the direction of motion in the case of vehicles). Body orientation is customarily classified as *forward facing*, *sideward facing*, and *backward facing*.

The wording *body posture* refers to the spatial configuration of the human body as regards the relative orientation of its parts and as regards minor intentional alterations with respect to a

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specified body position [International Organization for Standardization, 2022a]. Body posture is customarily classified on a case-by-case basis depending on the specified body position. For example, the *seated* body position can match body postures such as *upright*, *forward leaning*, and *backward leaning* (or *reclining*), the *standing* body position can match body postures such as *normal upright*, *erect upright*, and *relaxed upright*, and the *recumbent* body position can match body postures such as *supine* (or *upward facing*), *prone* (or *downward facing*), and *lateral* (or *sideward facing*).

The wording *body support* refers to the interaction between the human body and the local environment as regards weight distribution. Body support is customarily classified depending on whether *buttocks*, *back*, *feet*, *head*, and *arms* are supported or not by respectively *seat cushion* (also known as *seat pan*), *seat back* (also known as *backrest*), *seat floor* (also known as *floor pan* or *footrest* depending on the case), *headrest*, and *armrest*.

The wording *body restraint* refers to the interaction between the human body and the local environment as regards motion restrictions deliberately imposed to diminish the risk of injury to the human occupant. The concepts of body support and body restraint are not mutually exclusive. For example, vehicle seats perform functions of both body support and body restraint, their headrests being head restraints with full rights. In the case of safety belts, body restraint is customarily classified depending on the *number of anchorage points* of the occupant restraint system. On the one hand, ordinary-purpose safety belts can be *two-point* (one lap strap or one shoulder strap) or *three-point* (one lap strap and one shoulder strap); on the other hand, special-purpose safety belts can be *four-point* (one lap strap and two shoulder straps), *five-point* (one lap strap, two shoulder straps, and one crotch strap), *six-point* (one lap strap, two shoulder straps, and two crotch straps), *seven-point* (one lap strap, two shoulder straps, and three crotch straps), and *nine-point* (one lap strap, two doubled shoulder straps, and three crotch straps) [Hodson-Walker, 1970; States, 1988; Huston, 2001; Hilborn, 2015].

To meet habitability needs of next-generation cars, important advances in car interiors and specifically in car seats are to be pursued with targeted applied research. After adopting a conceptual framework in which a seat–occupant system is regarded as a whole, the natural step forward is deploying operational tools that facilitate addressing practical issues. Following this line of thought, the concept of *sitting configuration* is introduced and defined here as *activity-related overall arrangement of a seat–occupant system specified by position, orientation, body posture, body support, and body restraint*. Sitting configuration is intended as a single compendious factor whose purpose is to facilitate the analysis of interactions with factors such as sex and vibration magnitude.

Whilst the standing body position and the recumbent body position are of interest for specific applications respectively in public transport and in patient transport, the seated body position is the most important for the majority of automotive applications. Accordingly, hereinafter, the focus is on automotive applications of *seated whole-body vibration*.

1.2.1 Variables

In general, the many variables of interest for human vibration can be classified as *extrinsic* (which, largely, are neither dependent on nor controllable by the subject) and *intrinsic* (which, largely, are dependent on or controllable by the subject). In the case of seated whole-body vibration, the classification can be detailed as follows [British Standards Institution, 1987; Griffin, 1990].

The *extrinsic variables* are motion-environment features such as input location, direction, frequency band, waveform, severity, and duration of whole-body vibration, seating features such as design, components, position, geometric properties, and physical properties, occupant-restraint-system features such as design, components, position, geometric properties, and physical properties, and non-motion environment features such as segmental vibration, acoustic noise, audible sound, illumination, and microclimate.

The *subject-dependent intrinsic variables*, which cause *inter-subject variability* (or *between-subject variability*), are demographic characteristics such as age, sex, and ethnicity, anthropometric characteristics such as body mass (weight), stature (body height), and body mass index (or BMI), physiological properties, and biomechanical properties. The *body mass (weight)* is the “total mass (weight) of the body”, and the *stature (body height)* is the “vertical distance from the floor to the highest point of the head (vertex)” [International Organization for Standardization, 2017a]; the *body mass index* (or *BMI*) is the quotient of body mass (weight) by the square of stature (body height) [International Organization for Standardization, 2022b].

The *subject-controllable intrinsic variables*, which cause *intra-subject variability* (or *within-subject variability*), are body position, body orientation, body posture, body support, body restraint, and muscular activity.

It is worth emphasising that the classification is not rigid. For example, some motion-environment features and some non-motion-environment features may be partially controllable by the subject under certain circumstances. Besides, variables such as physical conditions, psychological conditions, situational conditions, experiential conditions, and expectational conditions can be regarded,

depending on the case, as subject-dependent or subject-controllable (and hence can cause respectively inter-subject variability or intra-subject variability).

1.2.2 Biodynamic coordinate systems

To analyse human-vibration exposure, *geocentric coordinate systems* (having z-axis lying in the direction of the earth's gravity) are of limited interest. In fact, for this purpose, it is necessary to establish hierarchically inferior *biodynamic coordinate systems* in relation to the human body. Biodynamic coordinate systems are right-handed Cartesian coordinate systems and can be either *anatomical coordinate systems* or *basicentric coordinate systems* [International Organization for Standardization, 1997a, 1997c].

The *anatomical coordinate systems* originate at points within the human body and are orientated with reference to fixed anatomical landmarks (the region of the human body to which the origin of an anatomical coordinate system belongs having to be parenthesised whenever reported). For the sake of accuracy and repeatability, such anatomical landmarks are normally chosen of (inherently segmental) skeletal type to avoid parts of the human body that are ill-defined, freely mobile, or deformable. In the case of seated whole-body vibration, the most common anatomical coordinate system in use can be loosely described as having origin within the *pelvis* and orientation such that x-axis is directed from back to front, y-axis is directed from right to left, and z-axis is directed from bottom to top. For a human body seated on a horizontal flat surface in upright body posture, the described anatomical coordinate system (pelvis) has approximately vertical and upward z-axis.

The *basicentric coordinate systems* originate at points within (or related to) vibrating supporting contact surfaces and are orientated with reference to such contact surfaces. In the case of seated whole-body vibration, the most common basicentric coordinate systems in use can be loosely described as having origin within the three main *input locations* in which vibration is considered to enter the human body and orientation such that, in upright body posture, x-axis, y-axis, and z-axis are orientated like the corresponding axes of the aforementioned anatomical coordinate system (pelvis). The three main input locations are normally assumed to be in prescribed positions at the intersections between the *midsagittal plane* (namely, the ideal anatomical plane passing through the spine and dividing the human body into symmetrical left and right halves) and the interfaces between *seat cushion* and *occupant buttocks* (more specifically, between seat cushion and *ischial tuberosities*, also known as *sitting bones*), between *seat back* and *occupant back*, and between *seat floor* and *occupant feet*.

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Anatomical coordinate systems are of special importance because they move with the human-body segment with respect to which they are defined. Accordingly, they allow specifying, with reference only to fixed anatomical landmarks, the three types of *unidirectional (translational) vibration* and the three types of *single-axis (rotational) vibration* to which a human-body segment can be subjected [International Organization for Standardization, 1997a].

The *unidirectional (translational) vibration* in the direction of the *x*-axis, *y*-axis, and *z*-axis of an anatomical coordinate system is called respectively *x-axis vibration* (also known as *fore-and-aft vibration*, the wording having been deprecated in International Standard ISO 5805:1997 but being still widely used [International Organization for Standardization, 1997a]), *y-axis vibration* (also known as *lateral vibration*, the wording having been deprecated in International Standard ISO 5805:1997 but being still widely used [International Organization for Standardization, 1997a]), and *z-axis vibration* (also known as *vertical vibration*, the wording having been deprecated in International Standard ISO 5805:1997 but being still widely used [International Organization for Standardization, 1997a]).

The *single-axis (rotational) vibration* about the *x*-axis, *y*-axis, and *z*-axis of an anatomical coordinate system is called respectively *roll*, *pitch*, and *yaw*; the corresponding *rotational axes* are sometimes called respectively r_x -axis, r_y -axis, and r_z -axis.

On the one hand, translational vibration in more than one direction of an anatomical coordinate system is called *multidirectional translational vibration*; on the other hand, rotational vibration about more than one axis of an anatomical coordinate system is called *multi-axis rotational vibration*. For the sake of generality, by introducing the concept of *translational vibration axis* alongside that of *rotational vibration axis*, the wording *multi-axis vibration* is often used to include both multidirectional (translational) vibration and multi-axis (rotational) vibration.

For accuracy and repeatability, anatomical coordinate systems are preferred whenever possible; this is the case of some *laboratory experiments*, in which the experimental conditions are strictly controlled and vibration directionality is relatively simple. However, basicentric coordinate systems are preferred whenever coordinate systems originating at the points of measurement are the only viable choice; this is the case of many *field experiments*, in which the experimental conditions are not strictly controlled and vibration directionality is relatively complex. In any case, measurements obtained with respect to basicentric coordinate systems are to be intended as approximation of measurements obtained with respect to anatomical coordinate systems.

1.2.3 Measurement, evaluation, and assessment

In its most complete form, the analysis of human-vibration exposure involves the three successive phases of *measurement or prediction*, *evaluation or judgement*, and *assessment* [Griffin, 1990, 1997, 2007].

The wording *measurement or prediction* refers to a first-level quantitative characterisation of the vibration in terms of objectively measurable *physical quantities*, which are either experimentally measured or mathematically predicted. Measurement or prediction is habitually followed by evaluation or judgement.

The wording *evaluation or judgement* refers to a second-level quantitative or qualitative characterisation of the vibration in terms of conventionally defined *severity parameters*, which are either calculated from objectively measurable physical quantities (possibly taking into account human sensitivity) or obtained from subjectively scalable individual judgements. Evaluation or judgement is normally preceded by measurement or prediction and sometimes followed by assessment.

The wording *assessment* refers to a third-level quantitative or qualitative characterisation of the vibration in terms of predictions about one or more *vibration effects* related to *health*, *activity interference*, *discomfort*, *perception*, and *motion sickness* (also known as *kinetosis*). Assessment is usually, but not always, preceded by measurement or prediction and evaluation or judgement.

Measurement or prediction methods, evaluation or judgement procedures, and assessment criteria are provided in national and international technical standards, both to assist in practical applications and to summarise, on a consensus basis, the state of knowledge at the time of publication [British Standards Institution, 1987; International Organization for Standardization, 1997b, 2017b].

1.2.4 Vibration severity

The concept of *vibration severity* generically refers to some characterisation of *vibration magnitude* (also known as *vibration intensity*) in terms of one or more *severity parameters* (excluding exposure duration) [International Organization for Standardization, 2018]. Such characterisation forms the basis for the phase of evaluation or judgement in the analysis of human-vibration exposure [British Standards Institution, 1987; Griffin, 1990; International Organization for Standardization, 1997b]. Severity parameters are either calculated from objectively measurable physical quantities or

obtained from subjectively scalable individual judgements; hereinafter, only the former case is considered.

Severity parameters calculated from objectively measurable physical quantities are based on measures of *components* of rectilinear or angular kinematic quantities such as *displacement*, *velocity*, or, more often, *acceleration*. Such severity parameters usually take into account human sensitivity (through a preliminary *frequency-weighting* process) as well as multiplicity of vibration axes and input locations (through a postliminary *weighted summation* process); operational details of the calculations depend on the specific vibration effect under consideration. To adapt to different types of vibration waveform, different types of severity parameters can be defined; they can be classified as *instantaneous*, *average*, or *cumulative*.

The *instantaneous severity parameters* are time-localised quantities such as the *peak value* and the *peak-to-peak value* (also known as *peak-to-valley value*) [International Organization for Standardization, 2018]. On the one hand, the peak value is the global maximum value of the absolute value of some frequency-weighted component of kinematic quantity on a specified exposure time interval. On the other hand, the peak-to-peak value is the difference between the global maximum value and the global minimum value of some frequency-weighted component of kinematic quantity on a specified exposure time interval. Instantaneous severity parameters are mainly meant for describing vibrations with ‘simple’ waveforms, which do not have random nature and do not exhibit transient or non-stationary behaviour (and hence do not feature shock events, intermittent periods, or time-variable waveforms); in particular, this is the case of *periodic deterministic vibration*, *quasiperiodic deterministic vibration*, and *almost periodic deterministic vibration* [International Organization for Standardization, 2005, 2018].

The *average severity parameters* are time-averaged quantities such as the *root-mean-square value* (also known as *RMS value*) and the *root-mean-quad value* (also known as *RMQ value*) [Griffin, 1990; Brammer and Peterson, 2009]. On the one hand, the root-mean-square value is the positive square root of the mean value of the square of some frequency-weighted component of kinematic quantity on a specified exposure time interval. On the other hand, the root-mean-quad value is the positive fourth root of the mean value of the fourth power of some frequency-weighted component of kinematic quantity on a specified exposure time interval. More generally, parameters similar to the root-mean-square value and to the root-mean-quad value can be defined with power exponent and root degree taking any pair of identical integer values, the only restriction being the use of even numbers to ensure positivity of the parameters [Brammer and Peterson, 2009]. Average severity parameters are mainly meant for describing vibrations with ‘well-behaved complex’ waveforms,

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which have random nature but do not exhibit transient or non-stationary behaviour (and hence do not feature shock events, intermittent periods, or time-variable waveforms); in particular, this is the case of *stationary random vibration* [International Organization for Standardization, 2005, 2018].

The *cumulative severity parameters* are time-integrated quantities such as the *vibration dose value* (or *VDV*) and the *daily vibration exposure* (or *A(8)*) [International Organization for Standardization, 1997b; Directive 2002/44/EC, 2002]. On the one hand, the vibration dose value is the positive fourth root of the integral of the fourth power of some frequency-weighted component of kinematic quantity on a specified exposure time interval. More generally, parameters similar to the vibration dose value can be defined with power exponent and root degree taking any pair of identical integer values, the only restriction being the use of even numbers to ensure positivity of the parameters [Brammer and Peterson, 2009]. On the other hand, the daily vibration exposure is the energy-equivalent continuous value over an eight-hour duration of the maximum, on three orthogonal axes, of either root-mean-square value or vibration dose value calculated from frequency-weighted components of rectilinear acceleration. Cumulative severity parameters are mainly meant for describing vibrations with ‘ill-behaved complex’ waveforms, which may or may not have random nature but exhibit transient or non-stationary behaviour (and hence feature shock events, intermittent periods, or time-variable waveform); in particular, this is the case of *transient deterministic vibration* and *non-stationary random vibration* [International Organization for Standardization, 2005, 2018].

Some *special severity parameters* are also used for describing vibrations having ‘ill-behaved complex’ waveforms. Special instantaneous severity parameters are time-localised quantities such as the *dynamic response index* (or *DRI*) [Griffin, 1990; Brammer and Peterson, 2009]. The dynamic response index is the ratio of an equivalent static acceleration calculated from some component of kinematic quantity to the standard acceleration due to gravity. Special average severity parameters are time-averaged quantities such as the *maximum transient vibration value* (or *MTVV*) [International Organization for Standardization, 1997b]. The maximum transient vibration value is the maximum value taken, on a specified exposure time interval, by the positive square root of the mean value of the square of some frequency-weighted component of kinematic quantity on a ‘moving’ time interval of ‘short’ duration (normally one second).

Several parameters of quantity dimension one can be defined to provide a rough measure of the complexity of the waveform of a vibration. The most common of them is the *crest factor*, which is defined as the ratio of the peak value to the root-mean-square value. The crest factor is essentially a measure of vibration ‘peakedness’ expressed in terms of the peak value and made of quantity

dimension one by dividing it by a measure of the vibration 'baseline'. The crest factor is associated with a threshold value of 9 by International Standard ISO 2631-1:1997 [International Organization for Standardization, 1997b] and with a threshold value of 6 by British Standard BS 6841-1987 [British Standards Institution, 1987]. For practical applications, both technical standards provide apposite recommendations depending on the specific vibration effect under consideration and depending on whether the crest factor exceeds the threshold value. Furthermore, International Standard ISO 2631-1:1997 provides two different parameters of quantity dimension one as possible replacements for the crest factor. One is defined as the ratio of the root-mean-quad value to the root-mean-square value and has a threshold value of 1.75 associated with it. The other is defined as the ratio of the maximum transient vibration value to the root-mean-square value and has a threshold value of 1.5 associated with it. In either case, the idea is to obtain a more robust version of the crest factor by replacing in its definition the measure of vibration 'peakedness' provided by an instantaneous quantity (namely, the peak value) with the one provided by an average quantity (namely, the root-mean-quad value or the maximum transient vibration value).

1.2.5 Motion environment

Whole-body vibration transmitted at the seat–occupant interface within vehicles is characterised by relatively large displacements within a relatively low frequency band. The primary causes are typically the interaction between road-vehicle wheels and road-surface profiles and, usually to a lesser extent, the operation of road-vehicle powertrain and braking system; whilst depending on the features of the vehicle, all the causes are related to driving style and driving conducts [Shoenberger, 1972; Mansfield, 2013].

In many real-world situations and in many field experiments, whole-body vibration transmitted at the seat–occupant interface within vehicles is characterised by random waveform (often stationary, at least piecewise) and multi-axis directionality. In laboratory experiments, these conditions can be replicated if expensive 'human-rated' vibration generator systems are available [Griffin, 1990; Mansfield, 2005]. However, in many laboratory experiments, it is worth of interest to consider vibrations of simpler waveform (for example, periodic deterministic) or simpler directionality (for example, unidirectional or single-axis) to conduct parametric analyses. Indeed, field experiments and laboratory experiments are complementary; whilst field experiments are necessary to collect data under realistic conditions with vibrations possibly richer in directionality, laboratory experiments are necessary to interpret data under controlled conditions with vibrations possibly richer in frequency content.

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The *frequency content* of whole-body vibration transmitted at the seat–occupant interface within vehicles is customarily considered to be limited to the *frequency band of interest* ranging from 0.5 Hz to 80 Hz, the most important portion of which being the one ranging from 0.5 Hz to 30 Hz and particularly the neighbourhood of 10 Hz [Mansfield, 2005, 2013; Griffin, 2007; Rakheja *et al.*, 2020a; Wang *et al.*, 2020]. The frequency band of interest coincides with most of the three contiguous frequency bands in which oscillatory motion of the sprung mass of a vehicle is described as primary ride, secondary ride, and harshness (ranging, respectively, from 0 Hz to 5 Hz, from 5 Hz to 30 Hz, and from 30 Hz to 100 Hz) [SAE International, 2022]. Specifically, the most important portion of the frequency band of interest virtually coincides with the two contiguous frequency bands in which oscillatory motion of the sprung mass of a vehicle is described as primary ride and secondary ride. Outside the frequency band of interest, motion sickness becomes dominant below 0.5 Hz, whereas segmental vibration and acoustic noise become dominant above 80 Hz; the effects of motion sickness, segmental vibration, and acoustic noise are sufficiently peculiar to deserve separated consideration and therefore are not discussed further.

Depending on the particular sub-band considered within the frequency band of interest, whole-body vibration transmitted at the seat–occupant interface within vehicles can be associated with different causes [Mansfield, 2013; Rahnejat *et al.*, 2014; García-Bonito and Fernández, 2014]. Between 0.5 Hz and 1 Hz, the main cause is the interaction between road-vehicle wheels and *road undulations* (which are the long-wavelength components of road-surface profiles interacting with road-vehicle wheels). Between 1 Hz and 20 Hz, the main cause is the interaction between road-vehicle wheels and *road roughness* (which are the short-wavelength components of road-surface profiles interacting with road-vehicle wheels). Between 20 Hz and 80 Hz, the main cause is the operation of road-vehicle powertrain and braking system.

In applications concerning on-road vehicles, the crest factor typically ranges from 3 to 8 [Griffin, 1990]; correspondingly, vibration magnitude expressed in terms of root-mean-square value of either W_k -weighted or W_b -weighted acceleration typically ranges from 0.2 m s^{-2} to 0.6 m s^{-2} along z-axis at the interface between seat cushion and occupant buttocks (where vibration severity is usually maximum) [Paddan and Griffin, 2002; Mansfield, 2013; Coyte *et al.*, 2016; Rakheja *et al.*, 2020a; Wang *et al.*, 2020]. For comparison, in applications concerning off-road vehicles, the crest factor may occasionally exceed 12 [Griffin, 1990]; correspondingly, vibration magnitude expressed in terms of root-mean-square value of either W_k -weighted or W_b -weighted acceleration may occasionally exceed 1.5 m s^{-2} along z-axis at the interface between seat cushion and occupant buttocks (where

vibration severity is usually maximum) [Paddan and Griffin, 2002; Mansfield, 2013; Rakheja *et al.*, 2020a; Wang *et al.*, 2020].

1.2.6 Whole-body biodynamics and seating dynamics

Within the context of human vibration, *whole-body biodynamics* (hereinafter *biodynamics*) is the “science of the physical, biological and mechanical properties or responses of the body, its tissues, organs, parts and systems, either with reference to impressed forces or motion, or in relation to the body’s own mechanical activity” [Griffin, 1990]. Occasionally, distinction is made between *external biodynamics* and *internal biodynamics* depending on whether applied loading is generated outside or inside the human body [Gierke, 1973; International Organization for Standardization, 1997a].

Biodynamics is focussed on human responses in terms of physical quantities observed by following procedures that are purely objective (except for the possible use of frequency weighting); therefore, outside its scope fall human responses in terms of psychophysical quantities observed by following procedures that are at least partially subjective (such as those employed to quantify vibration effects in relation to discomfort, activity interference, health, perception, and motion sickness) [British Standards Institution, 1987; International Organization for Standardization, 1997b].

It is worth mentioning that the term *biodynamics*, despite being customary in human vibration and despite being adopted in both national and international standards [British Standards Institution, 1987; International Organization for Standardization, 1997a, 1997b], is also used as a synonym for *biodynamic agriculture*, which is a specialised form of sustainable organic agriculture conceived by the Austrian scientist and philosopher Rudolf Joseph Lorenz Steiner (1861–1925) [Paull, 2011; Biodynamic Association, no date]. To avoid confusion, some advocate the use of the more general term *biomechanics*, which was introduced in the 1880s by the German neovitalist philosophers and scientists and which refers to “the study of the structure and function of biological systems by means of the methods of mechanics” [Hatze, 1974; Mansfield, 2005; Sirotkina and Biryukova, 2015]. However, although biodynamics can be etymologically regarded as the dynamics-related subset of biomechanics, in the current practice, the use of the latter term tends to be restricted to statics-related contexts [International Organization for Standardization, 1997a].

Unlike biodynamics, *seating dynamics* does not have an established definition. However, it can be intended as the study of the objective responses of seat–occupant systems to vibration; accordingly, to some extent, seating dynamics can be regarded as a superset of biodynamics. For convenience, the use of the adjective ‘biodynamic’ is extended hereinafter to seat–occupant systems.

1.2.7 Experimentation

General surveys of experimental work in biodynamics and seating dynamics abound in the literature [Gierke, 1973; Hearnshaw, 1974; Lawes, 1974; Rowlands, 1974; Cole, 1978; Sandover, 1982; Fairley, 1986; Dupuis and Zerlett, 1986; Griffin, 1990; Paddan, 1991; Kitazaki, 1994; Boileau, 1995; Ebe, 1998; Boileau *et al.*, 1998; Wu, 1998; Matsumoto, 1999; Wei, 2000; Denninger, 2001; Giacomini, 2002; Nawayseh, 2004; Mansfield, 2005; Abdul Jalil, 2005; Wang, 2006; Huang, 2008; Toward, 2010; Rakheja *et al.*, 2010, 2015, 2020a; Madakashira-Pranesh, 2011; Zheng, 2012; Mandapuram, 2012; Zhang, 2014; Zhou, 2014; Coyte *et al.*, 2016; Liu, 2016; Thirulogasingam, 2016; Yang, 2016; Wu, 2019; Wang *et al.*, 2020; Yin, 2022]. In fact, although experimentation in biodynamics and seating dynamics has been ongoing for decades, the many outcomes that have emerged from it are not always conclusive and consistent; this unwelcome fact is both a cause and an effect of the lack of a *theory* of human vibration. Accordingly, instead of compiling a new list, it seems more useful to focus on what not only is relevant in the present context but also can be considered acquired knowledge.

Whilst a small number of studies have been conducted by performing intrusive measurements inside the human body in clinical settings, the vast majority of studies have been conducted by performing non-intrusive measurements outside the human body in engineering settings. Inter-subject variability and, to a lesser extent (quantifiable in variations of no more than 10 % above or below the central tendency), intra-subject variability have been systematically identified as major sources of dispersion in experimental data. To control for the former, many studies have been conducted by recruiting 6 to 15 participants; nevertheless, to determine statistical distributions, some studies have been conducted by recruiting 60 or more participants. At an early or late stage of each study, experimental results have invariably (though not always innocuously) been averaged in some way to facilitate analysis and representation of the data.

In most studies, the body position has been seated on some rigid or deformable seat, the excitation has been unidirectional (translational) whole-body vibration (typically ‘vertical’ in the direction of the earth’s gravity), and the response has been some (possibly normalised) quantity used to perform a frequency-domain characterisation. In general, there has been a strong emphasis on the

identification of *resonance* phenomena¹. The most important finding, repeatedly and independently confirmed, is that the seated human body exposed to ‘vertical’ whole-body vibration behaves like a rigid mass up to approximately 2 Hz, exhibits a dominant first resonance approximately between 4 Hz and 6 Hz as well as an occasional second resonance approximately between 8 Hz and 16 Hz, and strongly attenuates vibrations above approximately 50 Hz. Unlike the second resonance, the first resonance is marked and clearly visible regardless of the type of frequency-domain characterisation used. It is likely that several modes of vibration are associated with the first resonance and that they involve an overall motion of the torso parallel to the midsagittal plane; however, such modes of vibration are difficult to isolate because of the heavy damping of the human body and, in fact, are still a matter of speculation. The seated human body exposed to ‘vertical’ whole-body vibration has a *nonlinear ‘softening’ behaviour* in the sense that its first resonance frequency decreases with increasing vibration magnitude; this phenomenon is probably due not so much to muscular activity as to the *thixotropy* (namely, time-dependent shear thinning) of soft tissues. Nevertheless, to a first approximation, the seated human body exposed to ‘vertical’ whole-body vibration is often regarded as a system having the property of linearity (which justifies the use of the traditional frequency-domain characterisation). Unfortunately, attempts to establish correlations between subjective and objective human responses to vibration have been disappointing; in fact, it is not yet possible to predict the ones from the others, which is why both still need be obtained directly.

In a seat–occupant system, the seat and the occupant are dynamically coupled in the sense that the dynamic response of one is influenced by the presence of the other; in particular, in conditions other than static or quasi-static, the dynamic response of a seat–occupant system is non-negligibly different from that obtained by replacing the occupant with a rigid mass or some other inert load having the same *sitting mass* (the sitting mass of the occupant of a seat being the fraction, approximately equal to 75 %, of the body mass of the occupant supported by the seat).

¹ In International Standard ISO 2041:2018, a *resonance* is defined as a “state of a system in forced oscillation when any change, however small, in the frequency of excitation causes a decrease in a response of the system”, whereas an *antiresonance* is defined as a “state of a system in forced oscillation at a point when any change, however small, in the frequency of excitation causes an increase in a response at this point” [International Organization for Standardization, 2018]. The definitions do not specify the nature of excitation and response, but, in most cases, these are respectively a dynamic quantity and a kinematic quantity. According to some authors [Giacomin, 2002; Toward, 2010], when excitation and response are instead respectively a kinematic quantity and a dynamic quantity (as is often the case in human vibration), a “state of a system in forced oscillation when any change, however small, in the frequency of excitation causes a decrease in a response of the system” should be said an antiresonance. To some extent, this terminological choice corresponds to physical intuition; however, not only is it far from consolidated and potentially ambiguous but also it implies an artificial and unnecessary discrimination between kinematic and dynamic quantities at the input and output of a system. Accordingly, as is customary and convenient in human vibration, the usual terminology is adopted hereinafter.

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For automotive applications, a seat can be either a *conventional seat* or a *suspension seat* depending on whether it is devoid of or equipped with a dedicated suspension mechanism beneath the seat cushion; both a conventional seat and a suspension seat include some combination of foam, rubber, and metal springing, and both have non-linear behaviour in the sense that they have dynamic responses depending on vibration characteristics (but, unlike a suspension seat, a conventional seat is often regarded as a system having the property of linearity unless vibration magnitude is high). Because of cost and space limitations, only conventional seats are used in cars.

A seat–occupant system exposed to ‘vertical’ whole-body vibration exhibits a resonance between 3 Hz and 4 Hz for a conventional seat and down to 2 Hz or less for a suspension seat. An important threshold is at approximately $2^{1/2}$ times such resonance frequency; both conventional seats and suspension seats behave as amplifiers below and as attenuators above. The resonance frequency and the amplification of vibration at resonance decrease with decreasing seat-back reclination and torso angle; this appears to be due not so much to the variation of the mass in contact with the seat cushion as to the modification of loading distribution and contact area at the interface between seat cushion and occupant buttocks.

1.2.8 Modelling

General surveys of modelling work in biodynamics and seating dynamics abound in the literature [Gierke, 1971; Hearnshaw, 1974; Cole, 1978; Sandover, 1982; Fairley, 1986; Dupuis and Zerlett, 1986; Griffin, 1990; Paddan, 1991; Kitazaki, 1994; Boileau, 1995; Wu, 1998; Matsumoto, 1999; Wei, 2000; Denninger, 2001; Giacomini, 2002; Nawayseh, 2004; Mansfield, 2005; Abdul Jalil, 2005; Wang, 2006; Huang, 2008; Toward, 2010; Madakashira-Pranesh, 2011; Kumbhar *et al.*, 2012; Zheng, 2012; Zhang, 2014; Zhou, 2014; Coyte *et al.*, 2016; Liu, 2016; Thirulogasingam, 2016; Yang, 2016; Wu, 2019; Rakheja *et al.*, 2020b; Yin, 2022]. In fact, although modelling in biodynamics and seating dynamics has been ongoing for decades, the many outcomes that have emerged from it are not always conclusive and consistent; this unwelcome fact is both a cause and an effect of the lack of a theory of human vibration. Accordingly, instead of compiling a new list, it seems more useful to focus on what not only is relevant in the present context but also can be considered acquired knowledge.

There is great scientific and technical interest in the development of *mathematical models* to predict ride quality, and especially sitting comfort, as experienced by seated human occupants in vehicles. Conceptually, one of such mathematical models can be regarded as composed of two cascaded sub-models. An upstream *biodynamic sub-model*, by taking as input the motion-environment features,

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provides as output the *objective responses* of a seat–occupant system coupled with a vehicle when known are the demographic and anthropometric characteristics and the biomechanical properties of the occupant as well as the geometric and mechanical properties of the seat (and possibly of the restraint system) and of the vehicle. A downstream *psychophysical sub-model*, by taking as input the objective responses of the seat–occupant system and possibly the non-motion-environment features, provides as output the *subjective responses* of the seat–occupant system when known are the psychophysical properties of the occupant. Hereinafter, attention is limited to biodynamic models of human bodies possibly coupled with vehicle seats and occupant restraint systems within vehicles.

With respect to *purpose*, biodynamic models can be qualified as *quantitative* (or *input–output*) when they allow predicting objective responses to motion environment through best fit to experimental data, as *mechanistic* when they allow predicting objective responses to motion environment through application of physical principles, and as *effect* when they allow predicting effects of motion environment through recourse to available knowledge base [Griffin, 2001]. With respect to *methodology*, biodynamic models can be qualified as *lumped-element* when they have lumped inertial, elastic, and dissipative properties (and hence can neither reproduce anthropometric characteristics of the human body nor predict surface distribution of contact actions), as *multibody* when they have distributed inertial properties but lumped elastic and dissipative properties (and hence can reproduce anthropometric characteristics of the human body but cannot predict surface distribution of contact actions), and *finite-element* when they have distributed inertial, elastic, and dissipative properties (and hence can both reproduce anthropometric characteristics of the human body and predict surface distribution of contact actions) [Gierke, 1971; Kumbhar *et al.*, 2012; Coyte *et al.*, 2016; Rakheja *et al.*, 2020b]; in fact, with respect to methodology, some biodynamic models can also be qualified as *unconventional* when they are developed by adopting unusual approaches (for example, involving the use of artificial neural networks) [Gohari *et al.*, 2012]. The two classifications are not strict; in fact, there exist biodynamic models that can be qualified as *mixed* in terms of purpose or methodology. Since most biodynamic models are of quantitative nature, the point of view of methodology is adopted hereinafter.

On the one hand, biodynamic models are used in scientific contexts to increase the theoretical knowledge base that is needed, amongst other things, to advance existing modelling methodologies and to develop new ones; on the other hand, biodynamic models are used in technical contexts to assist in solving application problems. Some biodynamic models, typically lumped-element or multibody, can reproduce the anthropometric characteristics and biomechanical properties of the

human body in a rather rough manner (or even cannot reproduce them at all) and are often too simple to provide more than a best fit to experimental data or, at best, prediction in cases rather elementary and far from reality; however, their simplicity becomes an advantage when it comes to using them for interpretation of experimental or numerical data. Other biodynamic models, typically finite-element, can reproduce the anthropometric characteristics and biomechanical properties of the human body in a more faithful manner and are the only ones that allow predicting the surface distribution of contact actions at the seat–occupant interface; however, their complexity implies great development effort and high computational cost. In any case, virtually all biodynamic models still suffer from limitations that make their use difficult to predict objective human responses in realistic conditions.

Operationally, modelling in biodynamics and seating dynamics is made especially difficult by the marked *inhomogeneity*, *anisotropy*, and *nonlinearity* of the human body; amongst these three features, nonlinearity is definitely the most critical in terms of induced mathematical complexity. Each of the three known types of nonlinear behaviour can have important implications. Firstly, *geometric nonlinear behaviour*, often neglected at first approximation under the highly simplifying assumption of infinitesimal displacements and infinitesimal displacement gradients, can be relevant in many real-world cases. Secondly, *material nonlinear behaviour*, characterising many of the biological tissues of the human body, is important especially in the case of high-magnitude vibrations. Thirdly, *contact nonlinear behaviour*, always present and hardly negligible, plays a key role in the interactions exchanged at the seat–occupant interface. Current biodynamic models regard the human body essentially as a *passive* system and allow making predictions in case of vibrations that are mostly of simple directionality and often of *unidirectional or single-axis* type; in some cases, they can take into account nonlinear behaviour.

It is relatively easy to *calibrate* biodynamic models, by means of experimental data, so that they can be used to make reliable predictions in conditions that are close to those of calibration [Dong *et al.*, 2015]. Much more difficult is to *validate* biodynamic models, by means of scientific evidence, so that they can be used to make reliable predictions in a wider range of conditions [Griffin, 2001]. Indeed, given some actual or notional human body, one could try to reproduce in detail its parts, organs, and tissues by employing the most advanced modelling methodologies to attain a high degree of *biofidelity*. However, even if one could develop such a biodynamic model, gather all the experimental data needed for calibration, and access the computational resources required by simulations, it would be at least arduous to take into account inter-subject variability and intra-subject variability through such a ‘brute-force’ approach. The task would be particularly challenging

if, as legitimate and desirable, one aimed at obtaining the parameters of the biodynamic model *a priori* (namely, from the knowledge of anthropometric characteristics and biomechanical properties either experimentally measured or statistically estimated) rather than *a posteriori* (namely, through best-fit algorithms that often operate on parameters of limited anthropometric or biomechanical meaning). Normally, biodynamic models require some calibration process and are rather difficult to validate; for this reason, they are often regarded as a complement to laboratory and field experiments. In fact, current biodynamic models are mostly developed by applying the principles of theories that, whilst having proven satisfactory for most mechanical systems, adapt with some difficulty to biomechanical systems such as the human body. This is a reflection of the fact that a theory of human vibration, on which basis one can make reliable predictions by taking into account inter-subject variability and intra-subject variability, is currently unavailable. Ideally, depending on the specific application, such a theory would assist in identifying what the essential aspects to be reproduced in a biodynamic model are, what the most suitable modelling methodology is, and how complex a biodynamic model should be.

In the envisaged road-transport scenario, reduced emission of noise and vibration from electric propulsion systems combined with new on-board activities and driving styles may result in seated human occupants being subjected to larger displacements and displacement gradients, being more inclined to make voluntary movements, and being exposed to whole-body vibration of reduced magnitude but more complex directionality. Ultimately, inter-subject variability and intra-subject variability seem the greatest challenge in the development of mathematical models meant to assist in predicting ride quality, and especially sitting comfort, as experienced by seated human occupants exposed to whole-body vibration transmitted from and through next-generation cars. It may be helpful that future biodynamic models can regard the human body as an active system, can manage realistic multidirectional or multiaxial vibrations, can take into account all types of nonlinear behaviour, and can represent the surface distribution of contact actions at the seat–occupant interface.

1.3 Knowledge gaps

In experimentation (and, by extension, in mathematical modelling), a seat–occupant system is normally regarded as a *mechanical system excited from the base* either (when in the laboratory) through the vibration table of a vibration generator system or (when on the field) through the floor of a vehicle [Griffin, 1990; International Organization for Standardization, 2016, 2019, 2022c]. On the one hand, the *excitation* is regarded as a predetermined (either acquired or generated) time-domain expression of some kinematic quantity imposed to the vibration table of the vibration generator system or to the floor of the vehicle; on the other hand, the *response* is regarded as a measured time-domain expression of some kinematic or dynamic quantity. The implicit assumption of this approach is that the motion of the vibration table of the vibration generator system or, less realistically, of the floor of the vehicle is independent of the motion of the seat–occupant system attached to it (which means that the vibration table of the vibration generator system or the floor of the vehicle has infinite mechanical impedance or, equivalently, null mobility, as introduced below). At least natively, the excitation quantity is almost always acceleration (be it rectilinear or angular) because, compared with velocity or displacement, it is easier to measure (transducers of acceleration being smaller and lighter than transducers of velocity and displacement [Doebelin, 2004]), easier to process (analogue or digital antidifferentiation of acceleration to obtain velocity and of velocity to obtain displacement being less problematic than analogue or digital differentiation of displacement to obtain velocity and of velocity to obtain acceleration [Doebelin, 2004; Cheng *et al.*, 2007]), and better correlating with psychophysical quantities (subjective human responses to vibration being better explainable as effects of acceleration-related causes than as effects of velocity-related or displacement-related causes except at very low values of frequency [Griffin, 1990; Mansfield, 2005]). Besides, the response quantity is either kinematic, in which case it is normally of the same type as the excitation quantity, or dynamic, in which case it is almost always force (be it force proper or moment of force); unfortunately, compared with acceleration, force is more difficult to measure (transducers of force being larger, heavier, and more in need of signal-conditioning equipment than transducers of acceleration).

The methods typically in use for the study of seat–occupant systems lend themselves well to being employed both in the *time domain* and in the *frequency domain*. This is an interesting feature in that, for many physical phenomena, a time-domain characterisation correspond more naturally to observation, whereas a frequency-domain characterisation provide deeper insight by means of mathematical constructions based on the *Fourier transformation* [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b].

In the frequency domain, a seat–occupant system (and the subsystems of it) having at least approximately the property of linearity are characterised either (for experimentation in the laboratory) in terms of quotients of Fourier transforms of dynamic quantities by Fourier transforms of kinematic quantities or (for experimentation both in the laboratory and on the field) in terms of ratios of Fourier transforms of kinematic quantities to Fourier transforms of kinematic quantities of the same type [Griffin, 1990; International Organization for Standardization, 2018]; such a seat–occupant system (and the subsystems of it) are said to be characterised in the first case in terms of *frequency response* (typically, to determine empirical constitutive relations for the subsystems of the seat–occupant system) and in the second case in terms of *transmissibility* (typically, to quantify the vibration-isolation performance of the seat–occupant system) [Griffin, 1990; International Organization for Standardization, 2018]. A frequency-response characterisation is said to be in terms of *dynamic stiffness*, *mechanical impedance* (also known as *immobility*), or *apparent mass* (also known as *effective mass* or *dynamic mass*) depending on whether the excitation kinematic quantity in the time domain is displacement, velocity, or acceleration [International Organization for Standardization, 2018]. Additionally, a frequency-domain characterisation can be given in terms of quantities that are dual to dynamic stiffness, mechanical impedance, and apparent mass; to them, reference is made as respectively *dynamic compliance* (also known as *receptance*), *mobility* (also known as *mechanical admittance*), and *accelerance* [International Organization for Standardization, 2018]. In particular, the frequency-response characterisation of choice is normally in terms of dynamic stiffness for seats and apparent mass for occupants, the reason being that, in static conditions, a seat and an occupant behave like respectively a linear spring and a rigid mass (in the sense that the dynamic stiffness of a seat and the apparent mass of an occupant normally take non-null finite values for approximately null values of frequency) [Griffin, 1990]. That said, a more inherent (and arguably more relevant) frequency-domain characterisation of a seat–occupant system (and of the subsystems of it) having at least approximately the property of linearity can be given in terms of *modal parameters* by using the methods of *modal analysis* [Ewins, 2000; He and Fu, 2001]; however, despite being promising, such approach is not yet fully established in experimental human vibration [Rützel *et al.*, 2006; Sun *et al.*, 2024]. Finally, it is worth mentioning that a frequency-domain characterisation of a seat–occupant system (and of the subsystems of it) can also be given by using the power characterisation of signals. Under the (physically obvious but mathematically indispensable) assumption of real-valuedness of such a system’s input velocity and output force, the real part of the cross-power spectral density of the ordered pair formed by input velocity and output force (or, equivalently, of the ordered pair formed by output force and input velocity) is said to be the system’s *absorbed-power spectral density* (to which, more properly,

reference should be made as *dissipated-mean-power spectral density*) [Wang, 2006; Wang *et al.*, 2006]; its integral on $\mathbb{R}$, which coincides with the integral on $\mathbb{R}$ of the cross-power spectral density (namely, with the cross-power) of the ordered pair formed by input velocity and output force (or, equivalently, of the ordered pair formed by output force and input velocity), is said to be the system's *absorbed power* (to which, more properly, reference should be made as *dissipated mean power*) [Wang, 2006; Wang *et al.*, 2006]. For human-vibration applications, the approach based on absorbed power attracted some interest in the past because of the appealing physical interpretation it allowed, but, in retrospect, it does not seem to have fulfilled expectations.

Unlike dynamic stiffness and apparent mass, mechanical impedance is defined in terms of an excitation kinematic quantity and a response dynamic quantity that, in the time domain, are power-conjugate quantities (namely, that, in the time domain, have multiplicative product having the quantity dimension of power). Accordingly, mechanical impedance is suitable to establish a mechanical–electrical analogy in which velocity and force correspond to respectively electric current and electric tension [Gardonio and Brennan, 2000, 2002, 2004]; such *mechanical-impedance analogy* (also known as *direct mechanical–electrical analogy*) has the advantage of matching mechanical impedance to electrical impedance (which are physically in direct correspondence) but has the disadvantage of not preserving the topological properties of networks [Gardonio and Brennan, 2000, 2002, 2004]. In fact, it is more common to perform frequency-response characterisations in terms of mobility, which is dual to mechanical impedance and which has the quantity dimension of the multiplicative inverse of mechanical impedance. Indeed, mobility is suitable to establish a mechanical–electrical analogy in which velocity and force correspond to respectively electric tension and electric current [Gardonio and Brennan, 2000, 2002, 2004]; such *mobility analogy* (also known as *inverse mechanical–electrical analogy*) has the disadvantage of matching mobility to electrical impedance (which are physically in inverse correspondence) but has the advantage of preserving the topological properties of networks [Gardonio and Brennan, 2000, 2002, 2004].

1.3.1 Distributional characterisation of two-terminal mechanical elements

In its essence, *lumped-element modelling* is an application of network theory to wave propagation phenomena for which the propagation speed is much greater than the quotient of the characteristic distance by the characteristic time interval [Doebelin, 1980, 1998]. Operationally, it consists in developing a mathematical model in the form of a *discrete system* having the *lumped element* as its building block [Doebelin, 1980, 1998; Karnopp *et al.*, 2012]. Despite having been established within the context of electromagnetism [Wang and Shapiro, 2019], lumped-element modelling has been

employed in disparate fields and eventually integrated into the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b]; in particular, it has been successfully employed in human vibration to describe objective responses of seat–occupant systems in terms of energy flows between subsystems [Doebelin, 1980; Griffin, 1990]. Whilst not the most powerful technique available, lumped-element modelling is still attractive by virtue not only of its simplicity and affordability but also of its capability to provide explicit mathematical solutions even in closed form; in fact, these features make it suitable to conduct parametric analyses as well as to do computer simulations or even hand calculations for both prediction and interpretation of experimental results. Like other techniques, lumped-element modelling lends itself well to being employed both in the time domain and in the frequency domain [Doebelin, 1980, 1998; Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Karnopp *et al.*, 2012].

The *lumped mechanical element* is the building block of a *discrete mechanical system* representing a mathematical model of some real-world mechanical system such as a seat–occupant system [Doebelin, 1980, 1998; Karnopp *et al.*, 2012]. The most common lumped mechanical element is the *two-terminal mechanical element* (or, equivalently, the *one-port mechanical element*), of which there are three basic types having the property of passivity (when characterised in terms of mechanical impedance or mobility) [Doebelin, 1980, 1998; Karnopp *et al.*, 2012]; specifically, these are the *linear spring*, the *linear viscous damper*, and the *linear inerter* (in particular, the *rigid mass*), which idealise respectively elastic properties, dissipative properties, and inertial properties, and to which, in the mobility analogy, correspond respectively the inductor, the resistor, and the capacitor (in particular, the capacitor with an earthed terminal) [Doebelin, 1980, 1998; Hixson, 2010; Karnopp *et al.*, 2012; Makris, 2017, 2018; Smith, 2020].

Whilst excellent in idealising dissipative properties of viscous nature, the linear viscous damper is unsatisfactory when it comes to idealising dissipative properties of structural nature [Crandall, 1970]. Therefore, a fourth basic type of two-terminal mechanical element, the *linear hysteresis damper* (also known as *linear structural damper*), was introduced for the purpose [Crandall, 1970, 1991; Inaudi and Kelly, 1995; Makris, 1997a, 1997b]. Remarkably, a linear spring and a linear hysteresis damper in parallel connection are equivalent to a two-terminal mechanical element (namely, the *linear hysteresis spring*) whose dynamic stiffness (for positive values of frequency) is a complex constant that can be interpreted as a *frequency-domain complex stiffness* having real part and imaginary part that depend only on the properties of respectively the linear spring and the linear hysteresis damper [Makris, 1997a]. Accordingly, the linear hysteresis damper proves useful in

providing heuristic basis for the customary physical interpretation of the real part and imaginary part of the dynamic stiffness of a mechanical system as respectively elasticity-related and dissipation-related; in fact, this physical interpretation can be corroborated within the theory of viscoelasticity. Unlike the other three basic two-terminal mechanical elements, the linear hysteresis damper does not result from the idealisation of some real-world mechanical system; furthermore, as evidenced by the fact that, historically, it was introduced in the frequency domain rather than in the time domain, it is less physical than mathematical in nature. On the one hand, it took some time to realise that the linear hysteresis damper, in its ostensible simplicity, was more complicated than it seemed and definitely ill-behaved [Crandall, 1970, 1991; Inaudi and Kelly, 1995; Makris, 1997a, 1997b]; on the other hand, it took even longer to realise that a proper time-domain definition of it could not be given by means of a differential equation, let alone by invoking the mathematically inconsistent and physically untenable notion of *time-domain complex stiffness*, but rather it needed be given by resorting to the relatively esoteric tool of the *Hilbert transformation* [Crandall, 1970, 1991; Inaudi and Kelly, 1995; Makris, 1997a, 1997b]. Artificiality aside, the deep flaw of the linear hysteresis damper is that it does not have the property of causality [Crandall, 1970, 1991; Inaudi and Kelly, 1995; Makris, 1997a, 1997b]; hence, however paradoxical this may seem, neither has it the property of passivity (when characterised in terms of mechanical impedance or mobility) [Zemnian, 1968, 1972]. Accordingly, despite being suitable for handling physical phenomena of *steady-state vibration* (typically, in the frequency domain), in which any effects of the property of acausality are immaterial, the linear hysteresis damper is at best problematic when it comes to handling physical phenomena of *transient vibration* (typically, in the time domain), in which any effects of the property of acausality can be relevant. Incidentally, it is worth noting that a linear hysteresis damper having the property of causality can be defined [Makris, 1997b]; however, not only does it lack the comparative simplicity of the traditional one but also it is not free from mathematical issues requiring *ad hoc* artifices to be addressed.

A proper characterisation of a real-world mechanical system such as a seat–occupant system by means of a discrete mechanical system requires a proper characterisation of each of the four basic two-terminal mechanical elements both in the time domain and in the frequency domain; in particular, a proper characterisation can be in terms of dynamic stiffness, mechanical impedance, and apparent mass as well as in terms of dynamic compliance, mobility, and accelerance (and their time-domain homologues). The traditional characterisation of linear spring, linear viscous damper, and rigid mass in terms of dynamic stiffness, mechanical impedance, and apparent mass as presented in tabular form by Griffin [1990, p. 361] is shown in **Figure 1**; besides, the traditional characterisation of linear spring, linear viscous damper, and rigid mass in terms of dynamic stiffness,

mechanical impedance, and apparent mass as presented in graphical form by Griffin [1990, p. 362] is shown in **Figure 2**. A frequency-response characterisation in terms of a quantity having denominator factorised by frequency (or angular frequency) cannot be mathematically and physically plausible unless the condition is satisfied that the Fourier transform of the excitation kinematic quantity takes null value at the origin. However, such condition is extremely restrictive as it is equivalent to the excitation kinematic quantity having null integral on $\mathbb{R}$; in fact, this need not be guaranteed even in the special case in which the excitation kinematic quantity has null mean value on $\mathbb{R}$. As is evident, the restrictions are even greater for a frequency-response characterisation in terms of a quantity having denominator factorised by the square of frequency (or angular frequency).

Table 8.2 Dynamic responses of pure masses, dampers and springs ($\omega = 2\pi f$)

Element	Modulus		Phase
Mass, m	Apparent mass	$= m$	a and F in phase
	Mechanical impedance	$= i\omega m$	v lags F by 90°
	Dynamic stiffness	$= -\omega^2 m$	d and F 180° out of phase
Damper, c	Apparent mass	$= c/i\omega$	a leads F by 90°
	Mechanical impedance	$= c$	v and F in phase
	Dynamic stiffness	$= i\omega c$	d lags F by 90°
Spring, k	Apparent mass	$= -k/\omega^2$	a and F 180° out of phase
	Mechanical impedance	$= k/i\omega$	v leads F by 90°
	Dynamic stiffness	$= k$	d and F in phase

Figure 1 Traditional characterisation of linear spring, linear viscous damper, and rigid mass in terms of dynamic stiffness, mechanical impedance, and apparent mass as presented in tabular form by Griffin [1990, p. 361]. From *Handbook of human vibration*, by M.J. Griffin, 1990. Copyright 1990 by Academic Press Limited. The modulus of a complex-valued quantity is by definition a non-negative real-valued quantity; accordingly, the header of the second column of the table is incorrect. k = stiffness of the linear spring; c = viscous damping coefficient of the linear viscous damper; m = mass of the rigid mass; i = imaginary unit; f = frequency; ω = angular frequency; d = displacement; v = velocity; a = acceleration; F = force.

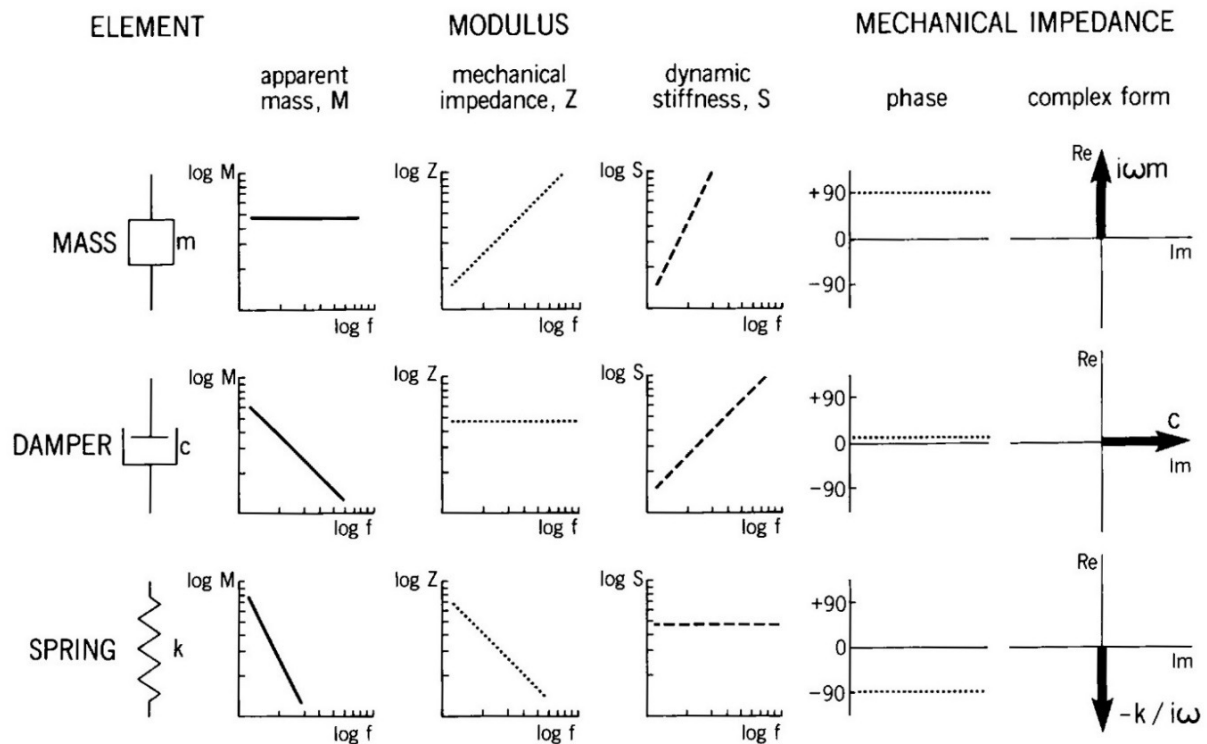


Fig. 8.15 Impedance relationships for masses, springs and dampers.

Figure 2 Traditional characterisation of linear spring, linear viscous damper, and rigid mass in terms of dynamic stiffness, mechanical impedance, and apparent mass as presented in graphical form by Griffin [1990, p. 362]. From *Handbook of human vibration*, by M.J. Griffin, 1990. Copyright 1990 by Academic Press Limited.
 k = stiffness of the linear spring; c = viscous damping coefficient of the linear viscous damper; m = mass of the rigid mass; i = imaginary unit; f = frequency; ω = angular frequency; S = dynamic stiffness; Z = mechanical impedance; M = apparent mass.

For a proper characterisation of each of the four basic two-terminal mechanical elements both in the time domain and in the frequency domain, it is necessary to obtain mathematical expressions of velocity and displacement as antiderivatives of acceleration both in the time domain and in the frequency domain. However, this task is by no means straightforward both because antidifferentiation is not determined uniquely and, most importantly, because the expression in the frequency domain of (possibly iterated) antidifferentiation in the time domain presents many pitfalls. Indeed, the usual procedure of expressing the Fourier transform of velocity as the quotient of the Fourier transform of acceleration by the imaginary unit times 2π times frequency (or, equivalently, by the imaginary unit times angular frequency) cannot be mathematically plausible unless the condition is satisfied that the Fourier transform of acceleration takes null value at the origin. However, such condition is extremely restrictive as it implies that acceleration has null integral on $\mathbb{R}$; in fact, this need not be guaranteed even in the special case in which acceleration has null mean value on $\mathbb{R}$. As is evident, the restrictions are even greater for the usual procedure of expressing the Fourier transform of displacement as the quotient of the Fourier transform of

acceleration by the additive inverse of the square of 2π times frequency (or, equivalently, by the additive inverse of the square of angular frequency).

Despite being a theoretical landmark, the traditional Fourier transformation (namely, the one operating on functions and resulting in functions) is practically constrained by very severe limitations of use; indeed, even simple constant or sinusoidal signals do not have Fourier transforms in the traditional sense [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b]. To overcome these limitations as well as some others, mathematical entities such as the *unit impulse* (also known as *Dirac delta* or *delta distribution*) and its derivatives have been extensively, if loosely, used in engineering and physical sciences since long before (and for long after...) a rigorous theoretical framework was developed for them in mathematics with the *theory of distributions* [Schwartz, 1966]. The most intuitive manner to regard a distribution is as a generalised function. However, distributions differ from functions for two important reasons. Firstly, they need not have point values and actually have an integral nature in the sense that they ‘live’ on the whole $\mathbb{R}$ (or an open subset of it) [Schwartz, 1966]; secondly, despite having the serious disadvantage of being limitedly multipliable (as an undesirable implication of the inherent linearity of the theory of distributions), they have the striking advantage of being unrestrictedly differentiable and antidifferentiable [Schwartz, 1966]. Hence, whilst requiring care to be handled and sometimes having counterintuitive properties, distributions are a precious tool to represent properly a real-world mechanical system such as a seat–occupant system by means of a discrete mechanical system.

In engineering and physical sciences, operations on distributions are usually performed by paying limited attention to mathematical legitimacy. Beyond the more technical mathematical aspects, for which the interest is rightly limited in applied contexts, this practice is problematic because it results in leaving unspecified the set of mathematical entities for which some operation is admissible and, in particular, in leaving unspecified the *input space* of a system (for example, of an antidifferentiator, a two-terminal mechanical element, or the system resulting from their series connection); as it happens, such input space can be remarkably and unexpectedly small even in the most common cases.

Ultimately, a proper characterisation of a real-world mechanical system such as a seat–occupant system by means of a discrete mechanical system requires a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements; these are the linear spring, the linear viscous damper, the linear inerter (including the rigid mass), and the linear hysteresis damper. In particular, a proper characterisation can be in terms

of dynamic stiffness, mechanical impedance, and apparent mass as well as in terms of dynamic compliance, mobility, and accelerance (and their time-domain homologues). The required characterisation need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such characterisation has not been available in the literature. In turn, a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements requires solving the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. The required solution need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such solution has not been available in the literature. Filling these gaps would allow two-terminal mechanical elements to be used in lumped-element modelling on a more solid mathematical basis, which could expand the possibilities for analytical manipulation, improve theoretical understanding, and help to elucidate the behaviour of related frequency-domain characterisations in a neighbourhood of the origin (namely, of null frequency). It is worth noting that replacing the frequency domain with the complex frequency domain (or, equivalently, renouncing the Fourier transformation in favour of the Laplace transformation) would result in a much simpler treatment. However, albeit mathematically well-behaved, quantities ‘living’ in the complex frequency domain are opaque when it comes to physical interpretation; conversely, albeit mathematically ill-behaved, quantities ‘living’ in the frequency domain are transparent when it comes to physical interpretation.

In light of the above, two theoretical studies need be conducted. A *first theoretical study* need be conducted to solve the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. A *second theoretical study* need be conducted to obtain a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements. Well-known physical aspects can be omitted to focus on unclear mathematical aspects. No attempt need be made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions need be sought for formulations to be mathematically tenable.

1.3.2 Distributional characterisation of linear viscoelastic materials

The simplest manner to include energy dissipation in finite-element modelling of seat–occupant systems is via *Rayleigh damping* (also known as *proportional damping*). However, the price to be paid for such simplicity is renouncing any physical interpretation of the damping mechanism. An interesting substitute for Rayleigh damping (or a complement to it) is *viscous damping*; indeed, despite being eminently empirical in nature, viscous damping has a very clear (albeit not always perfectly fitting) physical interpretation. Moreover, by introducing initial and asymptotic parameters in the time domain (or, equivalently, null-frequency and infinite-frequency parameters in the frequency domain), viscous damping advantageously allows decoupling static calibration from dynamic calibration.

In the vast literature concerning the *theory of viscoelasticity* (as well as in that concerning the formally analogous *theory of dielectric relaxation*), the *hereditary approach* has become standard [Gross, 1953; Zemanian, 1973; Flügge, 1975; Pipkin, 1986; Kovařík, 1989, 1995; Beris and Edwards, 1993; Anderssen and Loy, 2002; Ottosen and Ristinmaa, 2005; Hanyga, 2005a, 2005b; Hanyga and Sereďyńska, 2007, 2008; Makris and Kampas, 2009; Teodorescu *et al.*, 2013; Makris, 2019, 2020; Makris and Efthymiou, 2020; Mainardi, 2022]. As for the mathematical methods, extensive use has been made of the *theory of measure*. In contrast, only sporadic use has been made of the *theory of distributions* [Zemanian, 1973; Kovařík, 1989, 1995; Beris and Edwards, 1993; Hanyga, 2005a, 2005b; Hanyga and Sereďyńska, 2007, 2008; Makris and Kampas, 2009; Teodorescu *et al.*, 2013; Makris, 2019, 2020; Makris and Efthymiou, 2020]; the reasons for this lack of interest are not entirely clear, but they are probably related to consolidation of long-standing customs. Even where the theory of distributions has actually been used, either the approach has been heuristic or limited attention has been paid to applications. Furthermore, extensive use has been made of the Laplace transformation; however, to align with the typical point of view of many applications such as finite-element modelling of seat–occupant systems, the Fourier transformation need be used.

Ultimately, a concise but logically self-contained and mathematically rational characterisation, both in the time domain and in the frequency domain, of a *linear viscoelastic material* (shorthand for *non-ageing homogeneous isotropic linear viscoelastic material under isothermal adiabatic conditions*) should be non-heuristic but application-oriented. The required characterisation need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such a characterisation has not been available in the literature. Filling this gap would allow linear viscoelastic materials to be used in finite-element modelling on a more solid mathematical basis, which could expand the possibilities for analytical manipulation, improve theoretical understanding,

and grant both generality and economy of thought. It is worth noting that replacing the frequency domain with the complex frequency domain (or, equivalently, renouncing the Fourier transformation in favour of the Laplace transformation) would result in a much simpler treatment. However, albeit mathematically well-behaved, quantities ‘living’ in the complex frequency domain are opaque when it comes to physical interpretation; conversely, albeit mathematically ill-behaved, quantities ‘living’ in the frequency domain are transparent when it comes to physical interpretation.

In light of the above, a *third theoretical study* need be conducted to obtain a non-heuristic but application-oriented characterisation of a linear viscoelastic material both in the time domain and in the frequency domain. Well-known physical aspects can be omitted to focus on unclear mathematical aspects. No attempt need be made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions need be sought for formulations to be mathematically tenable.

1.3.3 Finite-element analysis of seat occupants

From the point of view of sitting comfort, the most important part of a seat–occupant system is arguably the seat–occupant interface, where any contact interactions are exchanged. Contact pressure and transmissibility at a seat–occupant interface are known to be important determiners of sitting comfort [Griffin, 1990; Looze *et al.*, 2003]; in fact, it is reasonable to assume that they may play an even more important role in connected, autonomous, shared, and electric vehicles, in which sitting configurations will be manifold and seat design will have to enable both primary and secondary activities [Pettersson and Karlsson, 2015; Pfleging and Schmidt, 2015; Ive *et al.*, 2015; Pfleging *et al.*, 2016; Jorlöv *et al.*, 2017; Hecht *et al.*, 2019; Östling and Larsson, 2019; Fitzen *et al.*, 2019].

Ideally, a biodynamic model of a seat occupant for prediction of sitting comfort should capture, without introducing more complexity than strictly necessary, the essential anthropometric characteristics and biomechanical properties of some statistical population; furthermore, it should allow coupling with a wide variety of mechanical models of seats to form biodynamic models of seat–occupant systems. For a biodynamic model of a seat–occupant system to allow representing both contact pressure and transmissibility at a seat–occupant interface, finite-element modelling is the only viable, albeit demanding, technique. Unfortunately, unlike a finite-element model of a seat, a finite-element model of a seat occupant cannot yet be developed *a priori* (namely, from the knowledge of anthropometric characteristics and biomechanical properties either experimentally measured or statistically estimated), but rather it must be developed *a posteriori* (namely, by

applying at some advanced stage best-fit algorithms that often operate on parameters of limited anthropometric or biomechanical meaning). In other words, a finite-element model of a seat occupant requires some calibration process (possibly followed by some validation process). Contact pressure and transmissibility at a seat–occupant interface are the quantities of greatest interest from the point of view of respectively static and dynamic sitting comfort. However, they characterise a seat–occupant system and not a seat occupant alone. To characterise a seat occupant alone, contact pressure at an interface with some standardised (typically rigid) seat and apparent mass may be used instead.

On the one hand, some finite-element models of seat occupants or just of buttocks were developed to investigate static sitting comfort and were calibrated in terms of contact pressure at interfaces with rigid or deformable seats (at seat cushion) [Moes and Horváth, 2002; Verver *et al.*, 2004; Grujicic *et al.*, 2009; Savonnet *et al.*, 2018; Yadav *et al.*, 2021; Alawneh *et al.*, 2022]. On the other hand, some finite-element models of seat occupants were developed to investigate dynamic sitting comfort and were calibrated in terms of apparent mass (of occupant buttocks) [Zheng, 2012; Liu *et al.*, 2015; Gao *et al.*, 2021]. Besides, some finite-element models of seat occupants were developed to investigate both static and dynamic sitting comfort and were calibrated in terms of both contact pressure at interfaces with rigid or deformable seats (at seat cushion) and apparent mass (of occupant buttocks) [Denninger, 2001; Pankoke and Siefert, 2008]; however, it should be noted that the one that is by far the most representative is a commercial product whose internal details are not in the public domain [Pankoke and Siefert, 2008].

Ultimately, for automotive applications, a finite-element model of a seat occupant should be complex enough to allow predicting, possibly in a range of sitting configurations, both contact pressure and transmissibility at an interface with a deformable seat (both at seat cushion and at seat back). To date, such a model has not been available in the literature.

In light of the above, a *modelling study* need be conducted to investigate whether it is possible to *calibrate* a finite-element model of a seat occupant in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back). For the parts defined as deformable bodies, a linear viscoelastic material (shorthand for non-ageing homogeneous isotropic linear viscoelastic material under isothermal adiabatic conditions) is preferable to a simpler linear elastic material (shorthand for non-ageing homogeneous isotropic linear elastic material under isothermal adiabatic conditions) with Rayleigh damping because it has a better physical interpretation and allows decoupling static calibration from dynamic calibration.

1.3.4 Objective responses of seat–occupant systems to vibration

The world of road transport has been rapidly changing in recent years. Indeed, both major advances and new challenges are expected within the automotive industry [Kaas *et al.*, 2016]. The keywords of next-generation cars are *connection*, *automation*, *sharing*, and *electrification* [Hannon *et al.*, 2016, 2019; Bouton *et al.*, 2017]. It is still a matter of discussion whether such innovations will prove truly disruptive and revolutionary or just incremental and evolutionary [Jiang *et al.*, 2015]; in fact, it is even unclear whether they will be welcomed by the public [Nordhoff *et al.*, 2018]. However, at the very least, an automotive paradigm shift is a plausible development.

Were connected, autonomous, shared, and electric vehicles to hit the road, important behavioural changes would spread amongst users [Trommer *et al.*, 2016]. Being more often passengers than drivers, users of next-generation cars might engage not only in *primary activities* such as *travelling* or *driving*, but also in *secondary activities* such as *(self) entertaining*, *socialising*, *relaxing*, *sleeping*, *working*, and *eating* [Pettersson and Karlsson, 2015; Pfleging and Schmidt, 2015; Ive *et al.*, 2015; Pfleging *et al.*, 2016; Jorlöv *et al.*, 2017; Hecht *et al.*, 2019; Östling and Larsson, 2019; Fitzen *et al.*, 2019]. When it comes to customer expectations and related product requirements in such an automotive-industry scenario, performance is likely to be outweighed not only by functionality but also by comfort and protection (as well as related affective or emotional attributes [Tanoue *et al.*, 1997]). In other words, human factors and ergonomics are likely to become key.

We may know ‘what’ people will be doing in next-generation cars, and yet we do not know ‘how’ they will do that. So far, the transport HFE community has addressed the topic mainly within its *organisational* branch (in terms of *traffic management* [Wagner, 2016] and *traffic safety* [Linkov *et al.*, 2019]) and within its *cognitive* branch (in terms of *situational awareness* [Petersen *et al.*, 2019] and *human–machine interaction* [Burnett *et al.*, 2019]); indeed, interest within its *physical* branch (in terms of *comfort* and *protection*) has been limited (albeit with the exception of *motion sickness* [Sivak and Schoettle, 2015; Diels and Bos, 2016]). In particular, *seating issues* specific to next-generation cars remain relatively unexplored from a static point of view (in terms of *occupant packaging* [Reed, 2018; Parida *et al.*, 2018, 2019a, 2019b]) and barely approached from a dynamic point of view (in terms of *vehicle safety* [Boin, 2018] and *human vibration* [Burkhard *et al.*, 2018; Enders *et al.*, 2019]).

As regards human vibration, many experimental studies have been conducted to investigate *objective responses of the seated human body to vibration* [Griffin, 1990; Mansfield, 2005]. On the one hand, working closer to the basic end of the basic–applied research continuum, investigators in

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biodynamics have used rigid or semi-rigid laboratory seats to characterise the occupant alone in the frequency domain (mainly in terms of *apparent mass* [Toward and Griffin, 2009, 2010, 2011a]). On the other hand, working closer to the applied end of the basic–applied research continuum, investigators in *seating dynamics* have used deformable factory seats to characterise both a seat alone in the frequency domain (mainly in terms of *dynamic stiffness* [Wei and Griffin, 1997; Wei, 2000; Zhang *et al.*, 2013]) and a seat–occupant system as a whole in the frequency domain (mainly in terms of *transmissibility* [Wei, 2000; Zhang *et al.*, 2012]). In many cases, attention has been paid to effects of specific factors considered in isolation. The most popular has perhaps been *vibration magnitude*, which has shown unequivocal nonlinear effects [Mansfield and Griffin, 2000; Tufano and Griffin, 2013]. Other favourites have been *seat-back reclination* [Houghton, 2003], *footrest configuration* [Fairley, 1986; Corbridge *et al.*, 1989], and *demographic and anthropometric characteristics* [Toward and Griffin, 2011b]; amongst these, only seat-back reclination has shown clear and consistent effects.

Car seat designers strive to improve dynamic sitting comfort by controlling vibration transmission through a seat–occupant system. When it comes to objective measures, vibration transmission is mostly quantified by using a transmissibility as a response function [International Organization for Standardization, 2018]. The key local feature of a system’s transmissibility is its *principal resonance*, which corresponds to the strongest vibration amplification and often coincides with the *first resonance* (namely, with the resonance occurring at the lowest frequency). A resonance being both principal and first is indicated hereinafter as a *primary resonance*; to characterise it, both *primary-resonance frequency* and *primary-resonance modulus* need be specified. Within the frequency domain, these quantities reveal respectively ‘where’ the strongest vibration amplification occurs and ‘how strong’ the strongest vibration amplification is.

Several studies have been conducted to investigate the effects on a transmissibility of features of motion environment (for example, vibration magnitude [Walton, 2007]), seat (for example, foam properties [Zhang *et al.*, 2015]), occupant (for example, demographic and anthropometric characteristics [Toward and Griffin, 2011b]), and arrangement of seat–occupant system (for example, seat-back reclination and torso angle [Houghton, 2003]). In fact, many are the parameters needed to specify the arrangement of a seat–occupant system; however, as factors within designed experiments, these parameters have usually been examined, with no reference to performed activities, either separately [Corbridge *et al.*, 1989; Houghton, 2003] or by using response functions other than a transmissibility [Nawayseh, 2015].

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Now, some considerations are in order. Firstly, although in real-world scenarios both seat-back reclination and footrest configuration contribute to determining the overall arrangement of a seat–occupant system, they have rarely been subjected to simultaneous experimental manipulation. Secondly, although females represent approximately half of the population, experimental studies have recurrently had samples composed only of males, and, in any case, they have just sporadically had sex as a factor. Thirdly, although main effects are normally superseded by interactions (and, in turn, lower-order interactions are normally superseded by higher-order interactions), *interactions* of any kind have scarcely been investigated.

Ultimately, within the adopted conceptual framework, a seat–occupant system is regarded as a whole, and the concept of *sitting configuration* is central; for the concept of sitting configuration to be deemed fit for purpose, evidence is required that it can be used as a single compendious factor facilitating the analysis of interactions with factors such as sex and vibration magnitude. To date, such evidence has not been available in the literature.

In light of the above, an *experimental study* need be conducted to establish whether the concept of sitting configuration is appropriate to characterise a seat–occupant system as a whole with reference to performed activities as well as to establish whether there are significant main effects of, and interactions between, sex, vibration magnitude, and sitting configuration on objective responses of seat–occupant systems to vibration; objective responses of seat–occupant systems to vibration can be quantified in terms of the fundamental objective response in seating dynamics (namely, *in-line transmission of vertical vibration at seat cushion*). Main effects and interactions of sex, vibration magnitude, and sitting configuration need be considered. Main effects are well known for vibration magnitude [Walton, 2007], unclear for sex [Corbridge *et al.*, 1989; Toward and Griffin, 2011b], and hypothesised for sitting configuration. Little is known about interactions.

1.4 Aims

The scientific study of ride quality, and especially of sitting comfort, requires a multidisciplinary approach that is appropriately established within the ambit of human factors and ergonomics. Ideally, such scientific study needs a comprehensive framework to bridge the objective–subjective gap and to consider not only ‘hard’ functional or ergonomic features but also ‘soft’ affective or emotional features. This research project was conducted within such a framework to contribute in paving the way for characterising, understanding, and predicting sitting comfort in next-generation (connected, autonomous, shared, and electric) cars. The contribution was divided into the three strands of theory, modelling, and experimentation, which resulted respectively in three theoretical studies, one modelling study, and one experimental study. Theoretical background and generalisations were reserved for a substantial appendix.

On the one hand, theory and modelling allow interpreting experimental results and making predictions (in particular, parametric analyses) without resorting to expensive, time-consuming, and potentially dangerous experimental tests. On the other hand, experimentation allows collecting laboratory or field data to develop theories and mathematical models. Indeed, theory, modelling, and experimentation feed each other within a virtuous circle and are all indispensable. However, in both biodynamics and seating dynamics, much experimental work and modelling work but little theoretical work has been done. In fact, a theory of human vibration has yet to be developed; to achieve such a formidable goal, a sustained collective effort will be necessary. In this research project, relatively advanced mathematical methods have been used to take a small step in the indicated direction; their popularisation amongst the community of engineering and physical sciences was deemed useful both scientifically and, to some extent, pedagogically.

Within the theoretical strand of this research project, three theoretical studies were conducted to answer in order the following research questions:

- Q1. What is the solution to the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, to the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain?
- Q2. What is a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements?
- Q3. What is a concise but logically self-contained and mathematically rational characterisation, both in the time domain and in the frequency domain, of a linear viscoelastic material?

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Within the modelling strand of this research project, a modelling study was conducted to answer the following research question:

Q4. Is it possible to calibrate a finite-element model of a seat occupant in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back)?

Within the experimentation strand of this research project, an experimental study was conducted to answer the following research questions:

Q5. Is the concept of sitting configuration appropriate to characterise a seat–occupant system as a whole with reference to performed activities?

Q6. Are there significant main effects of, and interactions between, sex, vibration magnitude, and sitting configuration on objective responses of seat–occupant systems to vibration?

1.5 Outline

Including the present Chapter 1, in which the introduction of this research project is presented, this document is divided into seven chapters and one appendix.

In Chapter 2, the first theoretical study conducted to answer Q1 is described. Specifically, Chapter 2 is focussed on time-domain and frequency-domain antidifferentiation.

In Chapter 3, the second theoretical study conducted to answer Q2 is described. Specifically, Chapter 3 is focussed on the distributional characterisation of two-terminal mechanical elements.

In Chapter 4, the third theoretical study conducted to answer Q3 is described. Specifically, Chapter 4 is focussed on the distributional characterisation of linear viscoelastic materials.

In Chapter 5, the modelling study conducted to answer Q4 is described. Specifically, Chapter 5 is focussed on the finite-element analysis of seat occupants.

In Chapter 6, the experimental study conducted to answer Q5 and Q6 is described. Specifically, Chapter 6 is focussed on objective responses of seat–occupant system to vibration.

In Chapter 7, the conclusion of this research project is drawn.

In Appendix A, theoretical background and generalisations are presented. Specifically, Chapter A.1, Chapter A.2, and Chapter A.3 are focussed on the mathematical framework, Chapter A.4 and Chapter A.5 are focussed on signals, and Chapter A.6 and Chapter A.7 are focussed on systems. In particular, the first six chapters contain a large amount of material synthesised (without proof) from a myriad of very technical mathematical sources some of which not in English; such material is by no means new, but it is reworked with consistency and some originality and is presumably not widely known within the community of engineering and physical sciences. The final chapter is an original contribution, and it can be regarded as a generalised version of Chapter 2 with a greater emphasis on the concept of system.

2 First theoretical study: Time-domain and frequency domain antidifferentiation

2.1 Introduction

As expounded in Section 1.3.1, a proper characterisation of a real-world mechanical system such as a seat–occupant system by means of a discrete mechanical system requires a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements; these are the linear spring, the linear viscous damper, the linear inerter (including the rigid mass), and the linear hysteresis damper. In particular, a proper characterisation can be in terms of dynamic stiffness, mechanical impedance, and apparent mass as well as in terms of dynamic compliance, mobility, and accelerance (and their time-domain homologues). The required characterisation need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such characterisation has not been available in the literature. In turn, a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements requires solving the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. The required solution need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such solution has not been available in the literature. Filling these gaps would allow two-terminal mechanical elements to be used in lumped-element modelling on a more solid mathematical basis, which could expand the possibilities for analytical manipulation, improve theoretical understanding, and help to elucidate the behaviour of related frequency-domain characterisations in a neighbourhood of the origin (namely, of null frequency). It is worth noting that replacing the frequency domain with the complex frequency domain (or, equivalently, renouncing the Fourier transformation in favour of the Laplace transformation) would result in a much simpler treatment. However, albeit mathematically well-behaved, quantities ‘living’ in the complex frequency domain are opaque when it comes to physical interpretation; conversely, albeit mathematically ill-behaved, quantities ‘living’ in the frequency domain are transparent when it comes to physical interpretation.

In light of the above, a *first theoretical study* was conducted to solve the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently,

2 First theoretical study: Time-domain and frequency domain antidifferentiation

the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. Well-known physical aspects were omitted to focus on unclear mathematical aspects. No attempt was made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions were sought for formulations to be mathematically tenable.

In the following, some preliminaries are provided, and then antidifferentiation is discussed in the time domain and in the frequency domain, and an example is provided. For theoretical background and generalisations, reference is made to Appendix A.

2.2 Preliminaries

Hereinafter, when not explicitly clarified, it is understood from the context whether a symbol denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. In particular, we denote by $\mathfrak{U}$ both the *unit step function* (also known as *Heaviside function*) on $\mathbb{R}$ and the *unit step distribution* (also known as *Heaviside distribution*) on $\mathbb{R}$, by sgn both the *signum function* on $\mathbb{R}$ and the *signum distribution* on $\mathbb{R}$, by rect both the *rectangular function* on $\mathbb{R}$ and the *rectangular distribution* on $\mathbb{R}$, by ρ both the *unit ramp function* on $\mathbb{R}$ and the *unit ramp distribution* on $\mathbb{R}$, by sinc both the *sinc function* (also known as *sinus cardinalis function* or *cardinal sine function*) on $\mathbb{R}$ and the *sinc distribution* (also known as *sinus cardinalis distribution* or *cardinal sine distribution*) on $\mathbb{R}$, and by nsinc both the *normalised sinc function* (also known as *normalised sinus cardinalis function* or *normalised cardinal sine function*) on $\mathbb{R}$ and the *normalised sinc distribution* (also known as *normalised sinus cardinalis distribution* or *normalised cardinal sine distribution*) on $\mathbb{R}$.

Let f_{L^0} be a non-locally-integrable function on $\mathbb{R}$ admitting regularisation, and let h_{L^0} be a non-locally-integrable function on $\mathbb{R}$ admitting canonical regularisation. We denote by $\text{Pf}(f_{L^0})$ the pseudofunction on $\mathbb{R}$ generated by f_{L^0} via the Hadamard-finite-part method; besides, we denote by $\text{f.p.}(h_{L^0})$ (when applicable, by $\text{p.v.}(h_{L^0})$) the canonical pseudofunction on $\mathbb{R}$ generated by h_{L^0} via the Hadamard-finite-part method (when applicable, via the Cauchy-principal-value method).

Where applicable, we denote the value at a point of $\mathbb{R}$ of a function on $\mathbb{R}$ or distribution on $\mathbb{R}$ by the symbol of the function on $\mathbb{R}$ or distribution on $\mathbb{R}$ immediately followed by the parenthesised point of $\mathbb{R}$.

By adopting the *Lagrange's notation for differentiation and antidifferentiation*, we denote a derivative (respectively, right-sided antiderivative) of some order by a right superscript containing a parenthesised non-negative (respectively, non-positive) integer constant whose absolute value equals the order of the derivative (respectively, right-sided antiderivative). Hereinafter, unless otherwise stated, a function on $\mathbb{R}$ or distribution on $\mathbb{R}$ to be antidifferentiated is assumed *right-sided* (although, with minor modifications, it can be assumed left-sided). Such assumption is convenient for our applications and is important, because it makes the pertinent input space a convolution algebra. Nevertheless, we emphasise that it is possible to waive it; if this is done, to determine uniquely an antiderivative, the property of right-sidedness need be replaced by the property of vanishing at negative infinity.

2.3 Time domain

Hereinafter, each function on $\mathbb{R}$ or distribution on $\mathbb{R}$ in the time domain is assumed *real-valued*. Such assumption is convenient for our applications but is not crucial.

Let t be *time*. Let a be a right-sided locally integrable function on $\mathbb{R}$ of t . We denote by a_I the *integral on $\mathbb{R}$ of a* defined as

$$a_I := \int_{-\infty}^{+\infty} (a) dt'$$

where

- t' is a bound real variable of the same quantity dimension as t

Furthermore, we denote by a_Q the *quadratic integral on $\mathbb{R}$ of a* defined as

$$a_Q := \int_{-\infty}^{+\infty} (a^2) dt'$$

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], a_Q is the *auto-energy* of a , and, if a_Q is finite, a is a *finite-energy signal*. Each (possibly right-sided) square integrable function on $\mathbb{R}$ has finite quadratic integral on $\mathbb{R}$, and each (possibly right-sided) locally integrable function on $\mathbb{R}$ having finite quadratic integral on $\mathbb{R}$ is a square integrable function on $\mathbb{R}$.

We denote by $\bar{a}$ the *right-sided mean value on $\mathbb{R}$ of a* defined as

$$\begin{aligned} \bar{a} &:= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{t_a}^{t_a+T} (a) dt' \right) \\ &= 2 \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} (a) dt' \right) \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (a) dt' \right) \end{aligned}$$

where

- T is a bound positive real variable of the same quantity dimension as t
- t_a is the left endpoint of the support of a

As is evident, $\bar{a}$ is 2 times the *mean value on $\mathbb{R}$* of a . If $\bar{a}$ is finite, we have

$$a = \bar{a}\mathfrak{G}_{t_a} + \tilde{a}$$

with

$$\begin{cases} \bar{\mathfrak{G}}_{t_a} = 1 \\ \bar{\tilde{a}} = 0 \end{cases}$$

where

- $\mathfrak{G}_{t_a}$ is the t_a -translated $\mathfrak{G}$
- $\bar{\mathfrak{G}}_{t_a}$ is the right-sided mean value on $\mathbb{R}$ of $\mathfrak{G}_{t_a}$
- $\tilde{a}$ is the right-sided centred a
- $\bar{\tilde{a}}$ is the right-sided mean value on $\mathbb{R}$ of $\tilde{a}$

Furthermore, we denote by $\bar{a}_q$ the *right-sided root-mean-square value on $\mathbb{R}$* of a defined as

$$\begin{aligned} \bar{a}_q &:= \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{t_a}^{t_a+T} (a^2) dt' \right) \right)^{\frac{1}{2}} \\ &= 2^{\frac{1}{2}} \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} (a^2) dt' \right) \right)^{\frac{1}{2}} \\ &= \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (a^2) dt' \right) \right)^{\frac{1}{2}} \end{aligned}$$

As is evident, $\bar{a}_q$ is $2^{1/2}$ times the *root-mean-square value on $\mathbb{R}$* of a . In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $(\bar{a}_q)^2$ is 2 times the *auto-power* of a , and, if $(\bar{a}_q)^2$ is finite, a is a *finite-power signal*. A (possibly right-sided) square integrable function on $\mathbb{R}$ has null (possibly right-sided) root-mean-square value on $\mathbb{R}$, but a (possibly right-sided) locally integrable function on $\mathbb{R}$ having null (possibly right-sided) root-mean-square value on $\mathbb{R}$ need not be a (possibly right-sided) square integrable function on $\mathbb{R}$; for example, the function defined almost everywhere on $\mathbb{R}$ by $\left(1 + (t/t_\emptyset)^2\right)^{-1/4} \mathfrak{G}(t)$, $t_\emptyset$ being a non-null real constant of the same quantity dimension as t ,

has null right-sided root-mean-square value on $\mathbb{R}$ but is not a square integrable function on $\mathbb{R}$. If both $\bar{a}$ and $\bar{a}_q$ are finite, we have

$$\begin{aligned} (\bar{a}_q)^2 &= (\bar{a})^2 + 2\bar{a}\bar{\tilde{a}} + (\bar{\tilde{a}}_q)^2 \\ &= (\bar{a})^2 + (\bar{\tilde{a}}_q)^2 \end{aligned}$$

where

– $\bar{\tilde{a}}_q$ is the right-sided root-mean-square value on $\mathbb{R}$ of $\tilde{a}$

Hereinafter, for convenience, we allow the quantities just defined to take values in $\bar{\mathbb{R}}$. We observe that the existence of a_1 implies the existence of $\bar{a}$. Indeed, whenever a_1 exists, we have

$$\begin{aligned} a_1 = -\infty &\Rightarrow \bar{a} \in \bar{\mathbb{R}}_{[-\infty, 0]} \\ a_1 \in \mathbb{R} &\Rightarrow \bar{a} = 0 \\ a_1 = +\infty &\Rightarrow \bar{a} \in \bar{\mathbb{R}}_{[0, +\infty]} \end{aligned} \tag{1}$$

However, the existence of $\bar{a}$ does not imply the existence of a_1 ; for example, the function defined almost everywhere on $\mathbb{R}$ by $\sin(\omega_0 t)\mathfrak{I}(t)$, ω_0 being a real constant of the same quantity dimension as ω (as introduced below), has null right-sided mean value on $\mathbb{R}$ but does not have integral on $\mathbb{R}$. Whenever both a_1 and $\bar{a}$ exist, we have

$$\begin{aligned} \bar{a} \in \bar{\mathbb{R}}_{[-\infty, 0]} &\Rightarrow a_1 = -\infty \\ \bar{a} = 0 &\Rightarrow a_1 \in \bar{\mathbb{R}} \\ \bar{a} \in \bar{\mathbb{R}}_{(0, +\infty]} &\Rightarrow a_1 = +\infty \end{aligned} \tag{2}$$

From Equation (1) and Equation (2), we observe in particular that a null a_1 implies a null $\bar{a}$ but a null $\bar{a}$ does not imply a null a_1 .

We know that, in general, distributions on $\mathbb{R}$ cannot be evaluated against test functions that are not smooth functions on $\mathbb{R}$ and cannot be squared [Schwartz, 1966]. Accordingly, the concepts of integral on $\mathbb{R}$, quadratic integral on $\mathbb{R}$, (possibly right-sided) mean value on $\mathbb{R}$, and (possibly right-sided) root-mean-square value on $\mathbb{R}$ cannot be immediately applied to them. Nevertheless, such concepts can be generalised to be applicable to some distributions on $\mathbb{R}$ and then, by convention, to some others.

A distribution on $\mathbb{R}$ has finite integral on $\mathbb{R}$ if and only if it is an integrable distribution on $\mathbb{R}$ (but a distribution on $\mathbb{R}$ that is not an integrable distribution on $\mathbb{R}$ may have finite improper integral on $\mathbb{R}$). By convention, a non-negative-definite distribution on $\mathbb{R}$ that is not an integrable distribution on $\mathbb{R}$ has integral on $\mathbb{R}$ equalling $+\infty$.

A distribution on $\mathbb{R}$ has finite quadratic integral on $\mathbb{R}$ if and only if it is generable by a square integrable function on $\mathbb{R}$. By convention, a square integrable distribution on $\mathbb{R}$ that is not generable by a square integrable function on $\mathbb{R}$ has quadratic integral on $\mathbb{R}$ equalling $+\infty$.

The mean value on $\mathbb{R}$ (respectively, right-sided mean value on $\mathbb{R}$) of a distribution on $\mathbb{R}$ (respectively, right-sided distribution on $\mathbb{R}$), whenever it exists and is finite, is the value at the origin (respectively, 2 times the value at the origin) of the mean cross-correlation distribution of the ordered pair formed by it and the unit distribution on $\mathbb{R}$.

The root-mean-square value on $\mathbb{R}$ (respectively, right-sided root-mean-square value on $\mathbb{R}$) of a distribution on $\mathbb{R}$ (respectively, right-sided distribution on $\mathbb{R}$), whenever it exists and is finite, is the value at the origin (respectively, $2^{1/2}$ times the value at the origin) of its mean auto-correlation distribution.

2.3.1 Onefold 1st-order antidifferentiation

Let $v \equiv a^{(-1)}$ be the right-sided 1st antiderivative of a . Provisionally, we confine ourselves to the case that a , and thus v , is a right-sided locally integrable function on $\mathbb{R}$. Almost everywhere on $\mathbb{R}$, we have

$$\begin{aligned} v(t) &\equiv a^{(-1)}(t) \\ &:= \int_{-\infty}^t (a) dt' \end{aligned} \tag{3}$$

Because a is right-sided, we observe that the lower limit of integration in Equation (3) is $-\infty$ only formally. Furthermore, because a is a locally integrable function on $\mathbb{R}$, by the Lebesgue's form of the fundamental theorem of calculus [Gordon, 1994] we observe that v exists and is a locally absolutely continuous function on $\mathbb{R}$ with

$$v^{(1)} = a \tag{4}$$

where

2 First theoretical study: Time-domain and frequency domain antidifferentiation

- $v^{(1)}$ is the 1st derivative of v

For any representatives of v and a , Equation (4) holds almost everywhere on $\mathbb{R}$ (and even everywhere on $\mathbb{R}$ if a is a C^0 function on $\mathbb{R}$) [Gordon, 1994].

Almost everywhere on $\mathbb{R}$, by applying elementary results to Equation (3), we have

$$\begin{aligned} v(t) &= \int_{-\infty}^t (a) dt' \\ &= \int_{-\infty}^{+\infty} \left((\tilde{\mathfrak{g}})_t a \right) dt' \\ &= \int_{-\infty}^{+\infty} \left(\mathfrak{g}(\tilde{a})_t \right) dt' \end{aligned}$$

and hence we have

$$\boxed{v = \mathfrak{g} * a} \tag{5}$$

where

- $(\tilde{\mathfrak{g}})_t$ is the t -translated reflected $\mathfrak{g}$
- $(\tilde{a})_t$ is the t -translated reflected a
- $*$ is the *convolution operator*

We observe that both a and $\mathfrak{g}$ are right-sided locally integrable functions on $\mathbb{R}$. Hence, the convolution in Equation (5) is well-defined and commutative, and v is a right-sided locally integrable function on $\mathbb{R}$. Furthermore, we observe that Equation (5) is still valid when $\mathfrak{g}$ is regarded as a right-sided distribution on $\mathbb{R}$ and a , and thus v , is a right-sided distribution on $\mathbb{R}$ [Gelfand and Shilov, 1964; Zemanian, 1965; Schwartz, 1966; Misra and Lavoine, 1986]; also in such case, the convolution in it is well-defined and commutative [Schwartz, 1966].

We now re-express Equation (4) as

$$\delta^{(1)} * v = a \tag{6}$$

where

- δ is the *unit impulse* (also known as *Dirac delta* or *delta distribution*) on $\mathbb{R}$
- $\delta^{(1)}$ is the *unit doublet* on $\mathbb{R}$

By comparing Equation (5) with Equation (6), it is evident that $\mathfrak{V}$ is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are convolutive inverses of $\delta^{(1)}$.

From Equation (4), by replacing ν with the unknown distribution $\left(a^{(-1)}\right)_{\text{gen}}$, we obtain a 1st-order differential equation of constant term $-a$. We have

$$\left(\left(a^{(-1)}\right)_{\text{gen}}\right)^{(1)} = a \quad (7)$$

where

$$- \left(\left(a^{(-1)}\right)_{\text{gen}}\right)^{(1)} \text{ is the 1}^{\text{st}} \text{ derivative of } \left(a^{(-1)}\right)_{\text{gen}}$$

We recognise that ν , as expressed by Equation (5), is a particular solution of Equation (7). Indeed, ν is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of a , which are provided by the general solution of Equation (7) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned} \left(a^{(-1)}\right)_{\text{gen}} &\equiv \int (a) dt' \\ &= \mathfrak{V} * a + \hat{a}^{(-1)} 1_{\mathbb{R}} \\ &= \nu + \hat{a}^{(-1)} 1_{\mathbb{R}} \end{aligned} \quad (8)$$

where

- $\int (a) dt'$ is the 1st indefinite integral of a
- $\hat{a}^{(-1)}$ is an arbitrary complex constant of the same quantity dimension as $a^{(-1)}$
- $1_{\mathbb{R}}$ is the *unit distribution* on $\mathbb{R}$

In agreement with the fact that we assume a real-valued, we assume $\hat{a}^{(-1)}$ real.

We observe that Equation (5) is valid even if a is an integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case ν (which is a bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of a . For example, if we have

$$a = a_0 \text{f.p.} \left(\frac{t_0^3}{t^3} \right)$$

where

- t_0 is a real constant of the same quantity dimension as t
- a_0 is a real constant of the same quantity dimension as a

then we have

$$\begin{aligned} v &= \mathfrak{G} * a \\ &= v_0 \text{f.p.} \left(\frac{t_0^2}{t^2} \right) \end{aligned}$$

with

$$v_0 = -\frac{a_0 t_0}{2}$$

Furthermore, we observe that the convolutive product of a by any finite set of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to 1 is well-defined, commutative, and associative if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if a is an integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither).

2.3.2 Onefold 2nd-order antidifferentiation

Let $x \equiv a^{(-2)}$ be the right-sided 2nd antiderivative of a . Provisionally, we confine ourselves to the case that a , and thus x , is a right-sided locally integrable function on $\mathbb{R}$. Almost everywhere on $\mathbb{R}$, we have

$$\begin{aligned} x(t) &\equiv a^{(-2)}(t) \\ &:= \int_{-\infty}^t \int_{-\infty}^{t_1'} (a) dt_2' dt_1' \end{aligned} \tag{9}$$

where

- t_1' and t_2' are bound real variables of the same quantity dimension as t

Because a is right-sided, we observe that the lower limit of integration in Equation (9) is $-\infty$ only formally. Furthermore, because a is a locally integrable function on $\mathbb{R}$, by the Lebesgue's form of the fundamental theorem of calculus [Gordon, 1994] we observe that x exists and is a locally absolutely continuous function on $\mathbb{R}$ with

$$x^{(2)} = a \quad (10)$$

where

– $x^{(2)}$ is the 2nd derivative of x

For any representatives of x and a , Equation (10) holds almost everywhere on $\mathbb{R}$ (and even everywhere on $\mathbb{R}$ if a is a C^0 function on $\mathbb{R}$) [Gordon, 1994].

Almost everywhere on $\mathbb{R}$, by applying the Cauchy formula for higher-order antidifferentiation [McBride, 1979] to Equation (9), we have

$$\begin{aligned} x &= \int_{-\infty}^t ((t-t')a) dt' \\ &= \int_{-\infty}^{+\infty} ((t-t')(\tilde{\mathfrak{g}})_t a) dt' \\ &= \int_{-\infty}^{+\infty} ((\tilde{\rho})_t a) dt' \\ &= \int_{-\infty}^{+\infty} (\rho(\tilde{a})_t) dt' \end{aligned}$$

and hence we have

$$\boxed{x = \rho * a} \quad (11)$$

where

– $(\tilde{\rho})_t$ is the t -translated reflected ρ

We observe that both a and ρ are right-sided locally integrable functions on $\mathbb{R}$. Hence, the convolution in Equation (11) is well-defined and commutative (as well as associative if ρ is expressed as the convolutive product of $\mathfrak{g}$ by itself), and x is a right-sided locally integrable function on $\mathbb{R}$. Furthermore, we observe that Equation (11) is still valid when ρ is regarded as a right-sided distribution on $\mathbb{R}$ and a , and thus x , is a right-sided distribution on $\mathbb{R}$ [Gelfand and Shilov, 1964; Zemanian, 1965; Schwartz, 1966; Misra and Lavoine, 1986]; also in such case, the convolutions in it are well-defined, commutative, and associative [Schwartz, 1966].

We now re-express Equation (10) as

$$\delta^{(2)} * x = a \quad (12)$$

where

- $\delta^{(2)}$ is the *unit triplet* on $\mathbb{R}$

By comparing Equation (11) with Equation (12), it is evident that ρ is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are convolutive inverses of $\delta^{(2)}$.

From Equation (10), by replacing x with the unknown distribution $\left(a^{(-2)}\right)_{\text{gen}}$, we obtain a 2nd-order differential equation of constant term $-a$. We have

$$\left(\left(a^{(-2)}\right)_{\text{gen}}\right)^{(2)} = a \quad (13)$$

where

- $\left(\left(a^{(-2)}\right)_{\text{gen}}\right)^{(2)}$ is the 2nd derivative of $\left(a^{(-2)}\right)_{\text{gen}}$

We recognise that x , as expressed by Equation (11), is a particular solution of Equation (13). Indeed, x is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are 2nd antiderivatives of a , which are provided by the general solution of Equation (13) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned} \left(a^{(-2)}\right)_{\text{gen}} &\equiv \iint (a) dt'_2 dt'_1 \\ &= \rho * a + \hat{a}^{(-1)} t + \hat{a}^{(-2)} 1_{\mathbb{R}} \\ &= x + \hat{a}^{(-1)} t + \hat{a}^{(-2)} 1_{\mathbb{R}} \end{aligned} \quad (14)$$

where

- $\iint (a) dt'_2 dt'_1$ is the 2nd indefinite integral of a
- $\hat{a}^{(-2)}$ is an arbitrary complex constant of the same quantity dimension as $a^{(-2)}$

In agreement with the fact that we assume a real-valued, we assume $\hat{a}^{(-2)}$ real.

We observe that Equation (11) is valid even if a is a $(+1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case x (which is a (-1) -weighted bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 2nd antiderivatives of a . For example, if we have

$$a = a_0 \text{f.p.} \left(\frac{t_0^3}{t^3} \right)$$

then we have

$$\begin{aligned} x &= \rho * a \\ &= x_0 \text{p.v.} \left(\frac{t_0}{t} \right) \end{aligned}$$

with

$$x_0 = \frac{a_0 t_0^2}{2}$$

Furthermore, we observe that the convolutive product of a by any finite set of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to 2 is well-defined, commutative, and associative if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if a is a $(+1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither).

2.3.3 Twofold 1st-order antidifferentiation

Let $x \equiv v^{(-1)}$ be the right-sided 1st antiderivative of v . Provisionally, we confine ourselves to the case that v , and thus x , is a right-sided locally integrable function on $\mathbb{R}$. Almost everywhere on $\mathbb{R}$, we have

$$\begin{aligned} x(t) &\equiv v^{(-1)}(t) \\ &:= \int_{-\infty}^t (v) dt' \end{aligned} \tag{15}$$

Because v is right-sided, we observe that the lower limit of integration in Equation (15) is $-\infty$ only formally. Furthermore, because v is a locally integrable function on $\mathbb{R}$, by the Lebesgue's form of the fundamental theorem of calculus [Gordon, 1994] we observe that x exists and is a locally absolutely continuous function on $\mathbb{R}$ with

$$x^{(1)} = v \tag{16}$$

where

- $x^{(1)}$ is the 1st derivative of x

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For any representatives of x and v , Equation (16) holds almost everywhere on $\mathbb{R}$ (and even everywhere on $\mathbb{R}$ if v is a C^0 function on $\mathbb{R}$) [Gordon, 1994].

Almost everywhere on $\mathbb{R}$, by applying elementary results to Equation (15), we have

$$\begin{aligned} x(t) &= \int_{-\infty}^t (v) dt' \\ &= \int_{-\infty}^{+\infty} \left((\tilde{\mathfrak{G}})_t v \right) dt' \\ &= \int_{-\infty}^{+\infty} \left(\mathfrak{G}(\tilde{v})_t \right) dt' \end{aligned}$$

and hence we have

$$\boxed{x = \mathfrak{G} * v} \tag{17}$$

where

– $(\tilde{v})_t$ is the t -translated reflected v

We observe that both v and $\mathfrak{G}$ are right-sided locally integrable functions on $\mathbb{R}$. Hence, the convolution in Equation (17) is well-defined and commutative, and x is a right-sided locally integrable function on $\mathbb{R}$. Furthermore, we observe that Equation (17) is still valid when $\mathfrak{G}$ is regarded as a right-sided distribution on $\mathbb{R}$ and v , and thus x , is a right-sided distribution on $\mathbb{R}$ [Gelfand and Shilov, 1964; Zemanian, 1965; Schwartz, 1966; Misra and Lavoine, 1986]; also in such case, the convolution in it is well-defined and commutative [Schwartz, 1966].

We now re-express Equation (16) as

$$\delta^{(1)} * x = v \tag{18}$$

By comparing Equation (17) with Equation (18), it is evident that $\mathfrak{G}$ is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are convolutive inverses of $\delta^{(1)}$.

From Equation (16), by replacing x with the unknown distribution $\left(v^{(-1)} \right)_{\text{gen}}$, we obtain a 1st-order differential equation of constant term $-v$. We have

$$\left(\left(v^{(-1)} \right)_{\text{gen}} \right)^{(1)} = v \tag{19}$$

where

– $\left(\left(v^{(-1)} \right)_{\text{gen}} \right)^{(1)}$ is the 1st derivative of $\left(v^{(-1)} \right)_{\text{gen}}$

We recognise that x , as expressed by Equation (17), is a particular solution of Equation (19). Indeed, x is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of v , which are provided by the general solution of Equation (19) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned} \left(v^{(-1)} \right)_{\text{gen}} &\equiv \int (v) dt' \\ &= \mathfrak{G} * v + \hat{v}^{(-1)} 1_{\mathbb{R}} \\ &= x + \hat{v}^{(-1)} 1_{\mathbb{R}} \end{aligned} \tag{20}$$

where

- $\int (v) dt'$ is the 1st indefinite integral of v
- $\hat{v}^{(-1)}$ is an arbitrary complex constant of the same quantity dimension as $v^{(-1)}$

In agreement with the fact that we assume a , and thus v , real-valued, we assume $\hat{v}^{(-1)}$ real.

We observe that Equation (17) is valid even if a is an integrable distribution on $\mathbb{R}$ with null 0th moment (be it right-sided, one-sided, or neither) or, equivalently, if v is an integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case x (which is a bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of v and 2nd antiderivatives of a . For example, if we have

$$a = a_0 \text{f.p.} \left(\frac{t_0^3}{t^3} \right)$$

or, equivalently, if we have

$$v = v_0 \text{f.p.} \left(\frac{t_0^2}{t^2} \right)$$

with

$$v_0 = -\frac{a_0 t_0}{2}$$

then we have

$$\begin{aligned} x &= \rho * a \\ &= \mathfrak{G} * \nu \\ &= x_0 \text{p.v.} \left(\frac{t_0}{t} \right) \end{aligned}$$

with

$$\begin{aligned} x_0 &= \frac{a_0 t_0^2}{2} \\ &= -v_0 t_0 \end{aligned}$$

Besides, Equation (17) is valid even if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case ν (which is a bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of a , and x (which is a (-1) -weighted bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 2nd antiderivatives of a . It is worth noting that an integrable distribution on $\mathbb{R}$ with null 0th moment need not be a distribution of rapid decay on $\mathbb{R}$ (as shown, for example, by $\text{f.p.}(t_0^2/t^2)$) and a distribution of rapid decay on $\mathbb{R}$ need not be an integrable distribution on $\mathbb{R}$ with null 0th moment (as shown, for example, by the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{-t/\beta} \mathfrak{G}(t)$, e being the *base of natural logarithm* (also known as *Euler number*) and β being a positive real constant of the same quantity dimension as t).

2.4 Frequency domain

Let ω be *angular frequency*. For the following developments, we now assume that a is a right-sided distribution of rapid decay on $\mathbb{R}$. We recall that, for a distribution of rapid decay on $\mathbb{R}$, the moment of any order in $\mathbb{N}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense). In particular, each distribution of rapid decay on $\mathbb{R}$ is an integrable distribution on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966]; accordingly, from Equation (1) and Equation (2) we observe that a_1 is finite and $\bar{a}$ is null.

Because a is a right-sided distribution of rapid decay on $\mathbb{R}$, its Fourier transform is generable by a smooth function of slow growth on $\mathbb{R}$ [Schwartz, 1966]. Let A be the Fourier transform of a .

2.4.1 Onefold 1st-order antidifferentiation

Let v be expressed, from Equation (5), as

$$v = \mathfrak{G} * a \quad (21)$$

Under the assumptions made, we know that the convolution in Equation (21) is well-defined and commutative. In fact, $\mathfrak{G}$ is a right-sided bounded distribution on $\mathbb{R}$, and a is a right-sided distribution of rapid decay on $\mathbb{R}$; hence, v , as the convolutive product of $\mathfrak{G}$ and a , is a right-sided bounded distribution on $\mathbb{R}$, and its Fourier transform is a tempered distribution on $\mathbb{R}$. Accordingly, the convolution Fourier exchange theorem applies in its standard form, and we have

$$\mathcal{F}(\mathfrak{G} * a) = \mathcal{F}(\mathfrak{G}) \cdot \mathcal{F}(a) \quad (22)$$

where

– $\mathcal{F}$ is the *Fourier transformation*

Let $V \equiv A_{(-1)}$ be the Fourier transform of $v \equiv a^{(-1)}$. From Equation (21) and Equation (22), we have

$$\begin{aligned} V &= \mathcal{F}(v) \\ &= \mathcal{F}(\mathfrak{G} * a) \\ &= \mathcal{F}(\mathfrak{G}) \cdot \mathcal{F}(a) \\ &= \Theta \cdot A \end{aligned}$$

and hence

$$\boxed{\begin{aligned} V &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) A \\ &= \text{p.v.} \left(\frac{A}{j\omega} \right) + \pi A(0)\delta \end{aligned}} \quad (23)$$

with [Estrada and Kanwal, 2002]

$$\begin{aligned} A(0) &= \int_{-\infty}^{+\infty} (a) dt' \\ &= \lim_{t \rightarrow +\infty} (v) \end{aligned} \quad (24)$$

where

- j is the *imaginary unit*
- Θ is the Fourier transform of $\mathfrak{g}$
- $\int_{-\infty}^{+\infty} (a) dt'$ is the 0^{th} moment of a

The expression of Θ can be obtained by recalling well-known Fourier-transform pairs [Kanwal, 2004]. We have

$$\begin{aligned} \Theta &\equiv P_0 \\ &= \mathcal{F}(\mathfrak{g}) \\ &= \mathcal{F}\left(\frac{\text{sgn}+1}{2}\right) \\ &= \mathcal{F}\left(\frac{\text{sgn}}{2}\right) + \mathcal{F}\left(\frac{1}{2}\right) \\ &= \text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \end{aligned}$$

Because a is a right-sided distribution of rapid decay on $\mathbb{R}$, A is generable by a smooth function of slow growth on $\mathbb{R}$. Accordingly, in Equation (23), the multiplications of A by both $\text{p.v.}(1/(j\omega))$ and δ are well-defined; furthermore, in Equation (24), $A(0)$ is finite, and so is the integral on $\mathbb{R}$.

Specifically, $A(0)$ equals the integral on $\mathbb{R}$ of a as well as the limit (in general, in the Cesàro sense) of v as t tends to $+\infty$. It is worth noting that $A(0)$ equals the 0^{th} moment of a . In fact, because a is a right-sided distribution of rapid decay on $\mathbb{R}$, its moment of any order in $\mathbb{N}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense).

By applying to Equation (6) the convolution Fourier exchange theorem in its standard form, we have

$$j\omega \cdot V = A \quad (25)$$

By comparing Equation (23) with Equation (25), it is evident that $\text{p.v.}(1/(j\omega)) + \pi\delta$ is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are multiplicative inverses of $j\omega$. Besides, it is worth noting that we have

$$\begin{aligned} \text{p.v.}\left(\frac{1}{j\omega}\right) + \pi\delta &= \lim_{\underline{\sigma} \rightarrow 0^-} \left(\left(\frac{1}{j\underline{s}} \right)_{\underline{s} = \omega + j\underline{\sigma}} \right), \quad \begin{cases} \underline{\sigma} \in \mathbb{R}_{(-\infty, 0)} \\ \underline{s} \in \mathbb{C}_{\text{Im} \in (-\infty, 0)} \end{cases} \\ &= \lim_{\sigma \rightarrow 0^+} \left(\left(\frac{1}{s} \right)_{s = \sigma + j\omega} \right), \quad \begin{cases} \sigma \in \mathbb{R}_{(0, +\infty)} \\ s \in \mathbb{C}_{\text{Re} \in (0, +\infty)} \end{cases} \end{aligned} \quad (26)$$

where

- $s \equiv \sigma + j\omega$ is a free complex variable of real part σ and imaginary part ω
- $\underline{s} \equiv \omega + j\underline{\sigma}$ is a free complex variable of real part ω and imaginary part $\underline{\sigma}$

$1/(j\underline{s})$ and $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$ being respectively the Fourier–Laplace transform of $\mathfrak{G}$ and its region of definition, $1/s$ and $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ being respectively the Laplace transform of $\mathfrak{G}$ and its region of definition, and the limits in Equation (26) being within the distribution space of the tempered distributions on $\mathbb{R}$, which implies that $(1/(j\underline{s}))_{\underline{s} = \omega + j\underline{\sigma}}$ generates a tempered distribution on $\mathbb{R}$ for each fixed $\underline{\sigma}$ in $\mathbb{R}_{(-\infty, 0)}$ and $(1/s)_{s = \sigma + j\omega}$ generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\mathbb{R}_{(0, +\infty)}$.

From Equation (25), by replacing V with the unknown distribution $(A_{(-1)})_{\text{gen}}$, we obtain a 1st-degree algebraic equation of constant term $-A$. We have

$$j\omega(A_{(-1)})_{\text{gen}} = A \quad (27)$$

We recognise that V , as expressed by Equation (23), is a particular solution of Equation (27). Indeed, V is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are quotients of A by $j\omega$, which are provided by the general solution of Equation (27) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned}
 \left(A_{(-1)} \right)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{A}{j\omega} \right) \\
 &= \text{p.v.} \left(\frac{A}{j\omega} \right) + \pi \hat{A} \delta \\
 &= V + 2\pi \hat{a}^{(-1)} \delta
 \end{aligned} \tag{28}$$

where

- $\text{i.q.}(A/j\omega)$ is the indefinite quotient of A by $j\omega$
- $\hat{A}$ is an arbitrary complex constant of the same quantity dimension as A with

$$\hat{A} = A(0) + 2\hat{a}^{(-1)}$$

If we assume $\hat{a}^{(-1)}$ real (in agreement with the fact that we assume a real-valued), then $\hat{A}$ is real. As is evident, Equation (27) and Equation (28) are the Fourier transforms of respectively Equation (7) and Equation (8).

We observe that the localisation of $\text{p.v.}(1/(j\omega))$ at any open subset of $\mathbb{R}$ not containing the origin results in a regular tempered distribution thereon; furthermore, its multiplication by a smooth function of slow growth on $\mathbb{R}$ whose derivative of order 0 vanishes at the origin results in a regular tempered distribution on $\mathbb{R}$. Therefore, from Equation (23), we have

$$\boxed{A(0) = 0 \Leftrightarrow V = \frac{A}{j\omega}} \tag{29}$$

As an implication of Equation (29), a necessary and sufficient condition for $A(0)$ (or, equivalently, the 0th moment of a) to be null is that V , like A , is generable by a smooth function of slow growth on $\mathbb{R}$ (and hence v , like a , is a right-sided distribution of rapid decay on $\mathbb{R}$).

We know that Equation (5) is valid even if a is an integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case v (which is a bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of a ; accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, Equation (23) is still valid too. For example, if we have

$$\begin{aligned} A &= j\frac{\pi}{2}a_0t_0^3|\omega| \\ &= \mathcal{F}\left(a_0\text{f.p.}\left(\frac{t_0^3}{t^3}\right)\right) \end{aligned}$$

then we have

$$\begin{aligned} V &= \Theta \cdot A \\ &= -\pi v_0 t_0^2 |\omega| \\ &= \mathcal{F}\left(v_0\text{f.p.}\left(\frac{t_0^2}{t^2}\right)\right) \end{aligned}$$

with

$$v_0 = -\frac{a_0 t_0}{2}$$

Furthermore, we know that the convolutive product of a by any finite set of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to 1 is well-defined, commutative, and associative if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if a is an integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither); accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, the multiplicative product of the Fourier transform of a by any finite set of Fourier transforms of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to 1 is well-defined, commutative, and associative, and it coincides with the Fourier transform of the convolutive product of a by the corresponding finite set of convolutive powers of $\mathfrak{G}$, if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if a is an integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither).

Finally, we re-express Equation (23) and Equation (25) as

$$\boxed{\mathcal{F}\left(\mathcal{D}^{-1}(a)\right) = \left(\text{p.v.}\left(\frac{1}{j\omega}\right) + \pi\delta\right)\mathcal{F}(a)} \quad (30)$$

$$\boxed{\mathcal{F}\left(\mathcal{D}^1(v)\right) = j\omega\mathcal{F}(v)} \quad (31)$$

where

- $\mathcal{D}^{-1}$ is the *right-sided 1st-order antidifferentiation operator*

– $\mathcal{D}^1$ is the 1st-order differentiation operator

It can be easily verified that we have

$$\boxed{\mathcal{D}^{-1}(\mathcal{F}(a)) = \mathcal{F}\left(\left(\text{p.v.}\left(\frac{1}{-jt}\right) + \pi\delta\right)a\right)} \quad (32)$$

$$\boxed{\mathcal{D}^1(\mathcal{F}(v)) = \mathcal{F}(-jtv)} \quad (33)$$

As is evident, Equation (32) and Equation (33) are the duals to respectively Equation (30) and Equation (31).

2.4.2 Onefold 2nd-order antidifferentiation

Let x be expressed, from Equation (11), as

$$x = \rho * a \quad (34)$$

Under the assumptions made, we know that the convolution in Equation (34) is well-defined and commutative. In fact, ρ is a right-sided (-1) -weighted bounded distribution on $\mathbb{R}$, and a is a right-sided distribution of rapid decay on $\mathbb{R}$; hence, x , as the convolutive product of ρ and a , is a right-sided (-1) -weighted bounded distribution on $\mathbb{R}$, and its Fourier transform is a tempered distribution on $\mathbb{R}$. Accordingly, the convolution Fourier exchange theorem applies in its standard form, and we have

$$\mathcal{F}(\rho * a) = \mathcal{F}(\rho) \cdot \mathcal{F}(a) \quad (35)$$

Let $X \equiv A_{(-2)}$ be the Fourier transform of $x \equiv a^{(-2)}$. From Equation (34) and Equation (35), we have

$$\begin{aligned} X &= \mathcal{F}(x) \\ &= \mathcal{F}(\rho * a) \\ &= \mathcal{F}(\rho) \cdot \mathcal{F}(a) \\ &= P \cdot A \end{aligned} \quad (36)$$

and hence

$$\begin{aligned}
 X &= \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) A \\
 &= \text{f.p.} \left(\frac{A}{-\omega^2} \right) + j\pi A(0)\delta^{(1)} - j\pi A^{(1)}(0)\delta
 \end{aligned}
 \tag{37}$$

with [Estrada and Kanwal, 2002]

$$\left\{ \begin{aligned}
 A(0) &= \int_{-\infty}^{+\infty} (a) dt' \\
 &= \lim_{t \rightarrow +\infty} (v) \\
 jA^{(1)}(0) &= \int_{-\infty}^{+\infty} (t'a) dt' \\
 &= \lim_{t \rightarrow +\infty} ((ta)^{(-1)})
 \end{aligned} \right.
 \tag{38}$$

where

- P is the Fourier transform of p
- $A^{(1)}$ is the 1st derivative of A
- $\int_{-\infty}^{+\infty} (t'a) dt'$ is the 1st moment of a
- $(ta)^{(-1)}$ is the right-sided 1st antiderivative of ta

The expression of P can be obtained by recalling well-known Fourier-transform pairs [Kanwal, 2004] and by applying the multiplication Fourier exchange theorem in its standard form. We have

$$\begin{aligned}
 P &\equiv P_1 \\
 &= \mathcal{F}(p) \\
 &= \mathcal{F}(t\vartheta) \\
 &= \mathcal{F}\left(\frac{t \operatorname{sgn} t}{2}\right) \\
 &= \mathcal{F}\left(t \cdot \frac{\operatorname{sgn}}{2}\right) + \mathcal{F}\left(\frac{t}{2}\right) \\
 &= \frac{1}{2\pi} \left(\mathcal{F}(t) * \mathcal{F}\left(\frac{\operatorname{sgn}}{2}\right) \right) + \mathcal{F}\left(\frac{t}{2}\right) \\
 &= \frac{1}{2\pi} \left(j2\pi\delta^{(1)} * \text{p.v.} \left(\frac{1}{j\omega} \right) \right) + j\pi\delta^{(1)} \\
 &= \text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)}
 \end{aligned}$$

2 First theoretical study: Time-domain and frequency domain antidifferentiation

Because a is a right-sided distribution of rapid decay on $\mathbb{R}$, A and A' are generable by smooth functions of slow growth on $\mathbb{R}$. Accordingly, in Equation (37), the multiplications of A by both $\text{f.p.}\left(\frac{1}{(-\omega^2)}\right)$ and $\delta^{(1)}$ are well-defined; furthermore, in Equation (38), both $A(0)$ and $A^{(1)}(0)$ are finite, and so are both integrals on $\mathbb{R}$. Specifically, $A(0)$ equals the integral on $\mathbb{R}$ of a as well as the limit (in general, in the Cesàro sense) of v as t tends to $+\infty$. It is worth noting that $A(0)$ and $jA^{(1)}(0)$ equal respectively the 0th moment of a and the 1st moment of a . In fact, because a is a right-sided distribution of rapid decay on $\mathbb{R}$, its moment of any order in $\mathbb{N}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense).

By applying to Equation (12) the convolution Fourier exchange theorem in its standard form, we have

$$-\omega^2 \cdot X = A \quad (39)$$

By comparing Equation (37) with Equation (39), it is evident that $\text{f.p.}\left(\frac{1}{(-\omega^2)}\right) + j\pi\delta^{(1)}$ is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are multiplicative inverses of $-\omega^2$. Besides, it is worth noting that we have

$$\begin{aligned} \text{f.p.}\left(\frac{1}{-\omega^2}\right) + j\pi\delta^{(1)} &= \lim_{\underline{\sigma} \rightarrow 0^-} \left(\left(\frac{1}{-\underline{s}^2} \right)_{\underline{s} = \omega + j\underline{\sigma}} \right), \quad \begin{cases} \underline{\sigma} \in \mathbb{R}_{(-\infty, 0)} \\ \underline{s} \in \mathbb{C}_{\text{Im} \in (-\infty, 0)} \end{cases} \\ &= \lim_{\sigma \rightarrow 0^+} \left(\left(\frac{1}{s^2} \right)_{s = \sigma + j\omega} \right), \quad \begin{cases} \sigma \in \mathbb{R}_{(0, +\infty)} \\ s \in \mathbb{C}_{\text{Re} \in (0, +\infty)} \end{cases} \end{aligned} \quad (40)$$

$\frac{1}{(-\underline{s}^2)}$ and $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$ being respectively the Fourier–Laplace transform of ρ and its region of definition, $\frac{1}{s^2}$ and $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ being respectively the Laplace transform of ρ and its region of definition, and the limits in Equation (40) being within the distribution space of the tempered distributions on $\mathbb{R}$, which implies that $\left(\frac{1}{(-\underline{s}^2)}\right)_{\underline{s} = \omega + j\underline{\sigma}}$ generates a tempered distribution on $\mathbb{R}$ for each fixed $\underline{\sigma}$ in $\mathbb{R}_{(-\infty, 0)}$ and $\left(\frac{1}{s^2}\right)_{s = \sigma + j\omega}$ generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\mathbb{R}_{(0, +\infty)}$.

From Equation (39), by replacing X with the unknown distribution $\left(A_{(-2)}\right)_{\text{gen}}$, we obtain a 2nd-degree algebraic equation of constant term $-A$. We have

$$-\omega^2 \left(A_{(-2)} \right)_{\text{gen}} = A \quad (41)$$

We recognise that X , as expressed by Equation (37), is a particular solution of Equation (41). Indeed, X is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are quotients of A by $-\omega^2$, which are provided by the general solution of Equation (41) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned} \left(A_{(-2)} \right)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{A}{-\omega^2} \right) \\ &= \text{f.p.} \left(\frac{A}{-\omega^2} \right) + j\pi \hat{A}^{(1)} - j\pi \hat{A}^{(1)} \delta \\ &= X + j2\pi \hat{a}^{(-1)} \delta^{(1)} + 2\pi \hat{a}^{(-2)} \delta \end{aligned} \quad (42)$$

where

- $\text{i.q.} \left(A / (-\omega^2) \right)$ is the indefinite quotient of A by $-\omega^2$
- $\hat{A}^{(1)}$ is an arbitrary complex constant of the same quantity dimension as $A^{(1)}$ with

$$\hat{A}^{(1)} = A^{(1)}(0) + j2\hat{a}^{(-2)}$$

If we assume $\hat{a}^{(-2)}$ real (in agreement with the fact that we assume a real-valued), then $\hat{A}^{(1)}$ is imaginary. As is evident, Equation (41) and Equation (42) are the Fourier transforms of respectively Equation (13) and Equation (14).

We observe that the localisation of $\text{f.p.} \left(1 / (-\omega^2) \right)$ at any open subset of $\mathbb{R}$ not containing the origin results in a regular tempered distribution thereon; furthermore, its multiplication by a smooth function of slow growth on $\mathbb{R}$ whose derivative of any order in $\mathbb{N}_{[0,1]}$ vanishes at the origin results in a regular tempered distribution on $\mathbb{R}$. Therefore, from Equation (37), we have

$$\boxed{\begin{cases} A(0) = 0 \\ A^{(1)}(0) = 0 \end{cases} \Leftrightarrow X = \frac{A}{-\omega^2}} \quad (43)$$

As an implication of Equation (43), a necessary and sufficient condition for both $A(0)$ and $A^{(1)}(0)$ (or, equivalently, the 0th moment of a and the 1st moment of a) to be null is that X , like A , is generable by a smooth function of slow growth on $\mathbb{R}$ (and hence x , like a , is a right-sided distribution of rapid decay on $\mathbb{R}$).

2 First theoretical study: Time-domain and frequency domain antidifferentiation

We know that Equation (11) is valid even if a is a $(+1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case x (which is a (-1) -weighted bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 2nd antiderivatives of a ; accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, Equation (37) is still valid too. For example, if we have

$$\begin{aligned} A &= j\frac{\pi}{2}a_0t_0^3\omega|\omega| \\ &= \mathcal{F}\left(a_0\text{f.p.}\left(\frac{t_0^3}{t^3}\right)\right) \end{aligned}$$

then we have

$$\begin{aligned} X &= P \cdot A \\ &= -j\pi x_0 t_0 \text{sgn} \\ &= \mathcal{F}\left(x_0\text{p.v.}\left(\frac{t_0}{t}\right)\right) \end{aligned}$$

with

$$x_0 = \frac{a_0 t_0^2}{2}$$

Furthermore, we know that the convolutive product of a by any finite set of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to 2 is well-defined, commutative, and associative if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if a is a $(+1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither);

accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, the multiplicative product of the Fourier transform of a by any finite set of Fourier transforms of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to 2 is well-defined, commutative, and associative, and it coincides with the Fourier transform of the convolutive product of a by the corresponding finite set of convolutive powers of $\mathfrak{G}$, if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if a is a $(+1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither).

Finally, we re-express Equation (37) and Equation (39) as

$$\boxed{\mathcal{F}\left(\mathcal{D}^{-2}(a)\right)=\left(\text{f.p.}\left(\frac{1}{-\omega^2}\right)+j\pi\delta^{(1)}\right)\mathcal{F}(a)} \quad (44)$$

$$\boxed{\mathcal{F}\left(\mathcal{D}^2(x)\right)=-\omega^2\mathcal{F}(x)} \quad (45)$$

where

- $\mathcal{D}^{-2}$ is the *right-sided 2nd-order antidifferentiation operator*
- $\mathcal{D}^2$ is the *2nd-order differentiation operator*

It can be easily verified that we have

$$\boxed{\mathcal{D}^{-2}\left(\mathcal{F}(a)\right)=\mathcal{F}\left(\left(\text{f.p.}\left(\frac{1}{-t^2}\right)-j\pi\delta^{(1)}\right)a\right)} \quad (46)$$

$$\boxed{\mathcal{D}^2\left(\mathcal{F}(x)\right)=\mathcal{F}\left(-t^2x\right)} \quad (47)$$

As is evident, Equation (46) and Equation (47) are the duals to respectively Equation (44) and Equation (45).

2.4.3 Twofold 1st-order antidifferentiation

Let now x be expressed, from Equation (17), as

$$x = \mathfrak{G} * \nu \quad (48)$$

Under the assumptions made, we know that the convolution in Equation (48) is well-defined and commutative. In fact, $\mathfrak{G}$ is a right-sided bounded distribution on $\mathbb{R}$, and ν is a right-sided bounded distribution on $\mathbb{R}$; hence, x , as the convolutive product of $\mathfrak{G}$ and ν , is a right-sided distribution on $\mathbb{R}$ that we know to be a tempered distribution on $\mathbb{R}$, and its Fourier transform is a tempered distribution on $\mathbb{R}$. Accordingly, although the convolution Fourier exchange theorem may not apply in its standard form, it surely applies in its generalised form, and we have

$$\mathcal{F}\left(\mathfrak{G} * \nu\right)=\mathcal{F}\left(\mathfrak{G}\right) \cdot \mathcal{F}\left(\nu\right) \quad (49)$$

Let $X \equiv \mathcal{V}_{(-1)}$ be the Fourier transform of $x \equiv \nu^{(-1)}$. From Equation (48) and Equation (49), we have

$$\begin{aligned}
 X &= \mathcal{F}(x) \\
 &= \mathcal{F}(\mathfrak{g} * v) \\
 &= \mathcal{F}(\mathfrak{g}) \cdot \mathcal{F}(v) \\
 &= \Theta \cdot V
 \end{aligned} \tag{50}$$

and hence

$$X = \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) V \tag{51}$$

We observe that Equation (48), Equation (49), Equation (50), and Equation (51) are equivalent to respectively Equation (34), Equation (35), Equation (36), and Equation (37). Indeed, we have [Reed and Simon, 1975; Shambayati and Zieleźny, 1983; Oberguggenberger, 1992]

$$\begin{aligned}
 \mathcal{F}(\mathfrak{g}) \mathcal{F}(\mathfrak{g}) &\equiv (\mathcal{F}(\mathfrak{g}))^2 \\
 &= \mathcal{F}(\rho)
 \end{aligned}$$

In the special case in which $A(0)$ is null, which occurs if and only if V , like A , is generable by a smooth function of slow growth on $\mathbb{R}$ (and hence v , like a , is a right-sided distribution of rapid decay on $\mathbb{R}$), the convolution Fourier exchange theorem in Equation (49) applies in its standard form. From Equation (48) and Equation (49), we have

$$\begin{aligned}
 A(0) = 0 \quad \Leftrightarrow \quad X &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) V \\
 &= \text{p.v.} \left(\frac{V}{j\omega} \right) + \pi V(0)\delta
 \end{aligned} \tag{52}$$

with [Estrada and Kanwal, 2002]

$$\begin{aligned}
 A(0) = 0 \quad \Leftrightarrow \quad V(0) &= \int_{-\infty}^{+\infty} (v) dt' \\
 &= \lim_{t \rightarrow +\infty} (x)
 \end{aligned} \tag{53}$$

where

$$- \int_{-\infty}^{+\infty} (v) dt' \text{ is the } 0^{\text{th}} \text{ moment of } v$$

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In the special case in which $A(0)$ is null, because v is a right-sided distribution of rapid decay on $\mathbb{R}$, V is generable by a smooth function of slow growth on $\mathbb{R}$. Accordingly, in Equation (52), the multiplications of V by both $\text{p.v.}(1/(j\omega))$ and δ are well-defined; furthermore, in Equation (53), $V(0)$ is finite, and so is the integral on $\mathbb{R}$. Specifically, $V(0)$ equals the integral on $\mathbb{R}$ of v as well as the limit (in general, in the Cesàro sense) of x as t tends to $+\infty$. It is worth noting that $V(0)$ equals the 0^{th} moment of v . In fact, because v is a right-sided distribution of rapid decay on $\mathbb{R}$, its moment of any order in $\mathbb{N}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense).

By applying to Equation (18) the convolution Fourier exchange theorem in its standard form, we have

$$j\omega \cdot X = V \quad (54)$$

By comparing Equation (51) and Equation (52) with Equation (54), it is evident that $\text{p.v.}(1/j\omega) + \pi\delta$ is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are multiplicative inverses of $j\omega$.

From Equation (54), by replacing X with the unknown distribution $(V_{(-1)})_{\text{gen}}$, we obtain a 1st-degree algebraic equation of constant term $-V$. We have

$$j\omega(V_{(-1)})_{\text{gen}} = V \quad (55)$$

We recognise that X , as expressed by Equation (37), is a particular solution of Equation (55). Indeed, X is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are quotients of V by $j\omega$, which are provided by the general solution of Equation (55) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned} (V_{(-1)})_{\text{gen}} &\equiv \text{i.q.}\left(\frac{A}{-\omega^2}\right) \\ &= \text{f.p.}\left(\frac{A}{-\omega^2}\right) + j\pi A(0)\delta^{(1)} - j\pi \hat{A}^{(1)}\delta \\ &= X + 2\pi \hat{v}^{(-1)}\delta \end{aligned} \quad (56)$$

As is evident, Equation (55) and Equation (56) are the Fourier transforms of respectively Equation (19) and Equation (20).

In the special case in which $A(0)$ is null, we recognise that X , as expressed by Equation (52), is a particular solution of Equation (55). Indeed, X is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are quotients of V by $j\omega$, which are provided by the general solution of Equation (55) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned} A(0)=0 &\Leftrightarrow \left(V_{(-1)}\right)_{\text{gen}} \equiv \text{i.q.}\left(\frac{V}{j\omega}\right) \\ &= \text{p.v.}\left(\frac{V}{j\omega}\right) + \pi\hat{V}\delta \\ &= X + 2\pi\hat{v}^{(-1)}\delta \end{aligned} \quad (57)$$

where

- $\text{i.q.}(V/j\omega)$ is the indefinite quotient of V by $j\omega$
- $\hat{V}$ is an arbitrary complex constant of the same quantity dimension as V with

$$\begin{aligned} A(0)=0 &\Leftrightarrow \hat{V} = V(0) + 2\hat{v}^{(-1)} \\ &= -jA^{(1)}(0) + 2\hat{v}^{(-1)} \\ &= -j\hat{A}^{(1)} \end{aligned} \quad (58)$$

In Equation (58), without loss of generality, $\hat{v}^{(-1)}$ is equated to $\hat{a}^{(-2)}$. If we assume $\hat{v}^{(-1)}$ real (in agreement with the fact that we assume a , and thus v , real-valued), then $\hat{V}$ is real. As is evident, Equation (55) and Equation (57) are the Fourier transforms of respectively Equation (19) and Equation (20).

We now show that the fact that both $A(0)$ and $A^{(1)}(0)$ are null is equivalent to the fact that both $A(0)$ and $V(0)$ are null. By recalling the differentiation property of the Fourier transformation, we have

$$\begin{aligned} A^{(1)} &= (j\omega V)^{(1)} \\ &= jV + j\omega V^{(1)} \end{aligned} \quad (59)$$

where

- $(j\omega V)^{(1)}$ is the 1st derivative of $j\omega V$
- $V^{(1)}$ is the 1st derivative of V

Equation (59) can be localised at the origin only in special circumstances; in particular, this is the case when the derivative of any order in $\mathbb{N}_{[0,1]}$ of V is a regular distribution on $\mathbb{R}$. We observe that a necessary and sufficient condition for V (as well as its derivative of any order in $\mathbb{N}$) to be not only regular but also generable by a smooth function of slow growth on $\mathbb{R}$ is that $A(0)$ is null.

Therefore, from Equation (59), we have

$$A(0) = 0 \Leftrightarrow A^{(1)}(0) = jV(0)$$

and hence we conclude that we have

$$\begin{cases} A(0) = 0 \\ A^{(1)}(0) = 0 \end{cases} \Leftrightarrow \begin{cases} A(0) = 0 \\ V(0) = 0 \end{cases}$$

Therefore, we can improve Equation (43) to obtain

$$\boxed{\begin{cases} A(0) = 0 \\ A^{(1)}(0) = 0 \end{cases} \Leftrightarrow \begin{cases} A(0) = 0 \\ V(0) = 0 \end{cases} \Leftrightarrow X = \frac{V}{j\omega} = \frac{A}{-\omega^2}} \quad (60)$$

As an implication of Equation (60), a necessary and sufficient condition for both $A(0)$ and $A^{(1)}(0)$ (or, equivalently, the 0th moment of a and the 1st moment of a) to be null is that both $A(0)$ and $V(0)$ (or, equivalently, the 0th moment of a and the 0th moment of v) are null or, equivalently, that both X and V , like A , are generable by smooth functions of slow growth on $\mathbb{R}$ (and hence both x and v , like a , are right-sided distributions of rapid decay on $\mathbb{R}$).

It is worth noting that, unless $A(0)$ is null, the fact that $A^{(1)}(0)$ is null is not equivalent to the fact that $V(0)$ is null; for example, if we have

$$a = a_0 \mathcal{S}_{\frac{1}{t_g}}(\text{rect})$$

where

- $\mathcal{S}_{1/t_g}$ is the *dilation operator of dilation coefficient* $1/t_g$

then a is a regular distribution of compact support on $\mathbb{R}$ (and hence, *a fortiori*, a regular right-sided distribution of rapid decay on $\mathbb{R}$), and we have

$$\begin{aligned}
 A &= a_0 |t_\theta| \mathcal{S}_{\frac{t_\theta}{2}}(\text{sinc}) \\
 &= a_0 |t_\theta| \mathcal{S}_{\frac{t_\theta}{2\pi}}(\text{nsinc}) \\
 A(0) &= a_0 |t_\theta| \\
 A^{(1)} &= \frac{a_0 t_\theta |t_\theta|}{2} \mathcal{S}_{\frac{t_\theta}{2}}(\text{sinc}^{(1)}) \\
 &= \frac{a_0 t_\theta |t_\theta|}{2\pi} \mathcal{S}_{\frac{t_\theta}{2\pi}}(\text{nsinc}^{(1)}) \\
 A^{(1)}(0) &= 0
 \end{aligned}$$

$$\begin{aligned}
 V &= \text{p.v.} \left(\frac{a_0 |t_\theta| \mathcal{S}_{\frac{t_\theta}{2}}(\text{sinc})}{j\omega} \right) + \pi a_0 \delta \\
 &= \text{p.v.} \left(\frac{a_0 |t_\theta| \mathcal{S}_{\frac{t_\theta}{2\pi}}(\text{nsinc})}{j\omega} \right) + \pi a_0 \delta
 \end{aligned}$$

with

$$\begin{aligned}
 \text{nsinc} &= \mathcal{S}_\pi(\text{sinc}) \\
 \text{nsinc}^{(1)} &= \pi \mathcal{S}_\pi(\text{sinc}^{(1)})
 \end{aligned}$$

where

- $\mathcal{S}_{t_\theta/2}$ is the *dilation operator of dilation coefficient $t_\theta/2$*
- $\mathcal{S}_{t_\theta/(2\pi)}$ is the *dilation operator of dilation coefficient $t_\theta/(2\pi)$*
- $\mathcal{S}_\pi$ is the *dilation operator of dilation factor π*

We know that Equation (17) is valid even if a is an integrable distribution on $\mathbb{R}$ with null 0th moment (be it right-sided, one-sided, or neither) or, equivalently, if v is an integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case x (which is a bounded distribution on

$\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of ν and 2nd antiderivatives of a ; accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, Equation (51) is still valid too. Besides, we know that Equation (17) is valid even if a is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case ν (which is a bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of a , and x (which is a (-1) -weighted bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are 2nd antiderivatives of a ; accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, Equation (51) is still valid too.

2.5 Example

As an example, we consider the case in which we have

$$a = a_0 e^{-t/\beta} \mathfrak{G}$$

The case in which a_0 is null is trivial. Hereinafter, we assume that a_0 is non-null. We observe that a is a right-sided regular distribution of very rapid decay on $\mathbb{R}$.

2.5.1 Time domain

In the time domain, it can be easily verified that we have

$$\begin{aligned} a_{+\infty} &= \lim_{t \rightarrow +\infty} (a) \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_1 &= \int_{-\infty}^{+\infty} (a) dt' \\ &= \operatorname{sgn}(a_0) \int_{-\infty}^{+\infty} (|a|) dt' \\ &= a_0 \beta \end{aligned}$$

$$\begin{aligned} a_Q &= \int_{-\infty}^{+\infty} (a^2) dt' \\ &= \int_{-\infty}^{+\infty} (|a|^2) dt' \\ &= \frac{a_0^2 \beta}{2} \end{aligned}$$

$$\begin{aligned} \bar{a} &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (a) dt' \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \bar{a}_q &= \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (a^2) dt' \right) \right)^{\frac{1}{2}} \\ &= 0 \end{aligned}$$

We observe that the function on $\mathbb{R}$ generating a is both an integrable function on $\mathbb{R}$ and a square integrable function on $\mathbb{R}$. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], it has non-null finite

auto-energy and null auto-power, and hence it is both a finite-energy signal and a finite-power signal.

Furthermore, it can be easily verified that we have

$$\begin{aligned} v &= \mathfrak{G} * a \\ &= a_0 \beta (1 - e^{-t/\beta}) \mathfrak{G} \end{aligned}$$

$$\begin{aligned} v^{(1)} &= a \\ &= a_0 e^{-t/\beta} \mathfrak{G} \end{aligned}$$

$$\begin{aligned} v_{+\infty} &= \lim_{t \rightarrow +\infty} (v) \\ &= a_0 \beta \end{aligned}$$

$$\begin{aligned} v_1 &= \int_{-\infty}^{+\infty} (v) dt' \\ &= \text{sgn}(a_0) \int_{-\infty}^{+\infty} (|v|) dt' \\ &= \text{sgn}(a_0) (+\infty) \end{aligned}$$

$$\begin{aligned} v_Q &= \int_{-\infty}^{+\infty} (v^2) dt' \\ &= \int_{-\infty}^{+\infty} (|v|^2) dt' \\ &= +\infty \end{aligned}$$

$$\begin{aligned} \bar{v} &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (v) dt' \right) \\ &= a_0 \beta \end{aligned}$$

$$\begin{aligned} \bar{v}_q &= \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (v^2) dt' \right) \right)^{\frac{1}{2}} \\ &= a_0 \beta \end{aligned}$$

We observe that the function on $\mathbb{R}$ generating v is neither an integrable function on $\mathbb{R}$ nor a square integrable function on $\mathbb{R}$. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], it has infinite auto-energy and non-null finite auto-power, and hence it is not a finite-energy signal but a finite-power signal.

Finally, it can be easily verified that we have

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$$\begin{aligned}
 x &= \mathfrak{G} * v \\
 &= \rho * a \\
 &= a_0 \beta^2 \left(\frac{t}{\beta} - (1 - e^{-t/\beta}) \right) \mathfrak{G}
 \end{aligned}$$

$$\begin{aligned}
 x^{(1)} &= v \\
 &= a_0 \beta (1 - e^{-t/\beta}) \mathfrak{G}
 \end{aligned}$$

$$\begin{aligned}
 x_{+\infty} &= \lim_{t \rightarrow +\infty} (x) \\
 &= \lim_{t \rightarrow +\infty} \left(a_0 \beta^2 \left(\frac{t}{\beta} - 1 \right) \right) \\
 &= +\infty
 \end{aligned}$$

$$\begin{aligned}
 x_I &= \int_{-\infty}^{+\infty} (x) dt' \\
 &= \text{sgn}(a_0) \int_{-\infty}^{+\infty} (|x|) dt' \\
 &= \text{sgn}(a_0) (+\infty)
 \end{aligned}$$

$$\begin{aligned}
 x_Q &= \int_{-\infty}^{+\infty} (x^2) dt' \\
 &= \int_{-\infty}^{+\infty} (|x|^2) dt' \\
 &= +\infty
 \end{aligned}$$

$$\begin{aligned}
 \bar{x} &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (x) dt' \right) \\
 &= +\infty
 \end{aligned}$$

$$\begin{aligned}
 \bar{x}_q &= \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (x^2) dt' \right) \right)^{\frac{1}{2}} \\
 &= +\infty
 \end{aligned}$$

We observe that the function on $\mathbb{R}$ generating x is neither an integrable function on $\mathbb{R}$ nor a square integrable function on $\mathbb{R}$. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], it has infinite auto energy and infinite auto-power, and hence it is neither a finite-energy signal nor a finite-power signal.

2.5.2 Frequency domain

In the frequency domain, it can be easily verified that we have

$$\begin{aligned} A &= \int_{-\infty}^{+\infty} (ae^{-j\omega t'}) dt' \\ &= \frac{a_0\beta}{1+j\beta\omega} \end{aligned}$$

$$\begin{aligned} A(0) &= \int_{-\infty}^{+\infty} (a) dt' \\ &= \mathcal{V}_{+\infty} \\ &= a_0\beta \end{aligned}$$

$$\begin{aligned} jA^{(1)} &= \int_{-\infty}^{+\infty} ((t'a)e^{-j\omega t'}) dt' \\ &= \frac{a_0\beta^2}{(1+j\beta\omega)^2} \end{aligned}$$

$$\begin{aligned} jA^{(1)}(0) &= \int_{-\infty}^{+\infty} (t'a) dt' \\ &= a_0\beta^2 \end{aligned}$$

$$\begin{aligned} V &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) A \\ &= \text{p.v.} \left(\frac{a_0\beta}{j\omega(1+j\beta\omega)} \right) + \pi a_0\beta\delta \end{aligned}$$

$$\begin{aligned} X &= \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) A \\ &= \text{f.p.} \left(\frac{a_0\beta}{-\omega^2(1+j\beta\omega)} \right) + j\pi a_0\beta\delta^{(1)} - \pi a_0\beta^2\delta \end{aligned}$$

We observe that A is generable by a smooth function of slow growth on $\mathbb{R}$ and that neither $A(0)$ nor $A^{(1)}(0)$ is null. Accordingly, we observe that both V and X are singular tempered distributions on $\mathbb{R}$.

2.6 Conclusion

This first theoretical study allows obtaining the solution to the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. Well-known physical aspects are omitted to focus on unclear mathematical aspects. The obtained solution results from using the theory of distributions, and it evokes singular distributions, such as ‘mysterious’ pseudofunctions as well as ‘hidden’ unit impulses and derivatives of them, all having the origin as singular support. The obtained solution holds under mildly restrictive conditions and contains specification of the input spaces. However, no attempt is made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions are identified for formulations to be mathematically tenable.

Let a be a right-sided distribution of rapid decay on $\mathbb{R}$, $v \equiv a^{(-1)}$ be the right-sided 1st antiderivative of a , and $x \equiv v^{(-1)} \equiv a^{(-2)}$ be the right-sided 1st antiderivative of v as well as the right-sided 2nd antiderivative of a . Let A , V , and X be the Fourier transforms of respectively a , v , and x .

In the time domain, relations hold

$$v = \mathfrak{I} * a$$

$$x = \rho * a$$

$$x = \mathfrak{I} * v$$

In the frequency domain, relations hold

$$\begin{aligned} V &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) A \\ &= \text{p.v.} \left(\frac{A}{j\omega} \right) + \pi A(0)\delta \end{aligned}$$

$$\begin{aligned} X &= \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) A \\ &= \text{f.p.} \left(\frac{A}{-\omega^2} \right) + j\pi A(0)\delta^{(1)} - j\pi A^{(1)}(0)\delta \end{aligned}$$

$$X = \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) V$$

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and, in particular, relations hold

$$A(0)=0 \Leftrightarrow V=\frac{A}{j\omega}$$

$$\begin{cases} A(0)=0 \\ A^{(1)}(0)=0 \end{cases} \Leftrightarrow X=\frac{A}{-\omega^2}$$

$$\begin{aligned} A(0)=0 \Leftrightarrow X &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) V \\ &= \text{p.v.} \left(\frac{V}{j\omega} \right) + \pi V(0)\delta \end{aligned}$$

$$\begin{cases} A(0)=0 \\ A^{(1)}(0)=0 \end{cases} \Leftrightarrow \begin{cases} A(0)=0 \\ V(0)=0 \end{cases} \Leftrightarrow X=\frac{V}{j\omega}=\frac{A}{-\omega^2}$$

In terms of operators, relations hold

$$\mathcal{F}(\mathcal{D}^{-1}(a)) = \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \mathcal{F}(a)$$

$$\mathcal{F}(\mathcal{D}^1(v)) = j\omega \mathcal{F}(v)$$

$$\mathcal{F}(\mathcal{D}^{-2}(a)) = \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \mathcal{F}(a)$$

$$\mathcal{F}(\mathcal{D}^2(x)) = -\omega^2 \mathcal{F}(x)$$

and, dually, relations hold

$$\mathcal{D}^{-1}(\mathcal{F}(a)) = \mathcal{F} \left(\left(\text{p.v.} \left(\frac{1}{-jt} \right) + \pi\delta \right) a \right)$$

$$\mathcal{D}^1(\mathcal{F}(v)) = \mathcal{F}(-jtv)$$

$$\mathcal{D}^{-2}(\mathcal{F}(a)) = \mathcal{F} \left(\left(\text{f.p.} \left(\frac{1}{-t^2} \right) - j\pi\delta^{(1)} \right) a \right)$$

$$\mathcal{D}^2(\mathcal{F}(x)) = \mathcal{F}(-t^2x)$$

3 Second theoretical study: Distributional characterisation of two-terminal mechanical elements

3.1 Introduction

As expounded in Section 1.3.1, a proper characterisation of a real-world mechanical system such as a seat–occupant system by means of a discrete mechanical system requires a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements; these are the linear spring, the linear viscous damper, the linear inerter (including the rigid mass), and the linear hysteresis damper. In particular, a proper characterisation can be in terms of dynamic stiffness, mechanical impedance, and apparent mass as well as in terms of dynamic compliance, mobility, and accelerance (and their time-domain homologues). The required characterisation need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such characterisation has not been available in the literature. In turn, a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements requires solving the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. The required solution need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such solution has not been available in the literature. Filling these gaps would allow two-terminal mechanical elements to be used in lumped-element modelling on a more solid mathematical basis, which could expand the possibilities for analytical manipulation, improve theoretical understanding, and help to elucidate the behaviour of related frequency-domain characterisations in a neighbourhood of the origin (namely, of null frequency). It is worth noting that replacing the frequency domain with the complex frequency domain (or, equivalently, renouncing the Fourier transformation in favour of the Laplace transformation) would result in a much simpler treatment. However, albeit mathematically well-behaved, quantities ‘living’ in the complex frequency domain are opaque when it comes to physical interpretation; conversely, albeit mathematically ill-behaved, quantities ‘living’ in the frequency domain are transparent when it comes to physical interpretation.

In light of the above, a *second theoretical study* was conducted to obtain a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal

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mechanical elements. Well-known physical aspects were omitted to focus on unclear mathematical aspects. No attempt was made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions were sought for formulations to be mathematically tenable.

In the following, some preliminaries are provided, and then the linear spring, the linear viscous damper, the linear inerter (including the rigid mass), and the linear hysteresis damper are discussed. The material in Chapter 2 is assumed known. For theoretical background and generalisations, reference is made to Appendix A.

3.2 Preliminaries

The *lumped mechanical element* is the building block of a *discrete mechanical system*. Hereinafter, each lumped mechanical element is assumed to be *single-degree-of-freedom* and *rectilinear* (although, with minor modifications, it can be assumed to be *angular*); accordingly, each lumped mechanical element is assumed to be a *two-terminal mechanical element* (hereinafter *two-terminal element*) or, equivalently, a *one-port mechanical element* (hereinafter *one-port element*). As a group of reference conventions, we adopt the *motor convention* (also known as *passive sign convention*) [International Electrotechnical Commission, 2018]. Let a , v , and x be respectively the *relative acceleration*, the *relative velocity*, and the *relative displacement* across a generic two-terminal element (namely, the difference of the values of respectively acceleration, velocity, and displacement at the two terminals of the element), and let respectively A , V , and X be their Fourier transforms; furthermore, let f be the *force* through a generic two-terminal element, and let F be its Fourier transform.

For each of the considered two-terminal elements, we define a convolution system by means of an input–output representation in which the input variable is a kinematic quantity (expressing the relative motion across the element in terms of one amongst acceleration, velocity, or displacement) and the output variable is a dynamic quantity (expressing the force through the element); we refer to such convolution system as the *direct simple system* associated with the two-terminal element. With a suitable choice of input space and output space, we can invert the input–output representation of the direct simple system to obtain an input–output representation in which the input variable is a dynamic quantity and the output variable is a kinematic quantity; such input–output representation defines a convolution system that is the inverse system of the direct simple system and to which we refer as *inverse simple system* associated with the two-terminal element.

First by connecting in series the direct simple system (respectively, inverse simple system) and a suitable antiderivator or differentiator, and then by connecting in series the same direct simple system (respectively, inverse simple system) and a different suitable antiderivator or differentiator, we obtain two series-equivalent convolution systems to which we refer as *direct compound systems* (respectively, *inverse compound systems*) associated with the two-terminal element. Like the direct simple system (respectively, inverse simple system) from which they derive, the direct compound systems (respectively, inverse compound systems) are characterised by input–output representations in which the input variables are kinematic quantities (respectively, dynamic quantities) and the output variables are dynamic quantities (respectively, kinematic quantities). However, whilst the direct simple system and the inverse simple system

always form a pair of mutually inverse convolution systems, the direct compound systems and the inverse compound systems need not.

Ultimately, we associate each of the considered two-terminal elements with six convolution systems. By abuse of language, we say that the convolution kernel distributions and the frequency-response distributions of the six convolution systems associated with a two-terminal element are respectively the convolution kernel distributions and the frequency-response distributions of the two-terminal element. As seen below, a convolution system associated with a two-terminal element need not admit δ as an input variable; accordingly, not always can the convolution kernel distribution of any such system be regarded as a unit-impulse response.

To determine any of the convolution kernel distributions or frequency-response distributions of a two-terminal element, we need specify an input variable and an output variable. Unlike frequency-response distributions, convolution kernel distributions of mechanical systems do not have standardised names. To remedy this lack, we use the standardised names of frequency-response distributions also for convolution kernel distributions, and we distinguish them by adding the qualifiers *time-domain* and *frequency-domain*.

When the input variable is a kinematic quantity and the output variable is a dynamic quantity, depending on whether the kinematic quantity is displacement, velocity, or acceleration, we refer to the convolution kernel distribution as respectively *time-domain dynamic stiffness*, *time-domain mechanical impedance*, and *time-domain apparent mass*; we denote such quantities by the symbols g_S , g_Z , and g_M respectively. When the input variable is a dynamic quantity and the output variable is a kinematic quantity, depending on whether the kinematic quantity is displacement, velocity, or acceleration, we refer to the convolution kernel distribution as respectively *time-domain dynamic compliance*, *time-domain mobility*, and *time-domain accelerance*; we denote such quantities by the symbols g_C , g_Y , and g_A respectively.

When the input variable is a kinematic quantity and the output variable is a dynamic quantity, depending on whether the kinematic quantity is displacement, velocity, or acceleration, we refer to the frequency-response distribution as respectively *frequency-domain dynamic stiffness*, *frequency-domain mechanical impedance*, and *frequency-domain apparent mass*; we denote such quantities by the symbols G_S , G_Z , and G_M respectively. When the input variable is a dynamic quantity and the output variable is a kinematic quantity, depending on whether the kinematic quantity is displacement, velocity, or acceleration, we refer to the frequency-response distribution as respectively *frequency-domain dynamic compliance*, *frequency-domain mobility*, and

frequency-domain accelerance; we denote such quantities by the symbols G_C , G_Y , and G_A respectively.

It is worth noting that, for convolution kernel distributions and frequency-response distributions of two-terminal elements, both the qualification as either *direct* or *transfer* and the qualification as either *free* or *blocked* [International Organization for Standardization, 2018] is immaterial.

We know that the property of passivity has a physical interpretation in terms of energy only for a *power-conjugate system*, which is a system whose input variable and output variable are *power-conjugate quantities* (namely, a system for which the multiplicative product of input variable and output variable has the quantity dimension of power). Accordingly, hereinafter, we consider the property of passivity only for the convolution systems that define either the mechanical impedance or the mobility of the considered two-terminal elements.

Hereinafter, given a convolution system with input variable u of Fourier transform U and output variable w of Fourier transform W , we denote by $w_u(u)$ and $W_U(U)$ the input–output representation of the system respectively in the time domain and in the frequency domain; where applicable, we nest such input–output representations.

If a is a right-sided distribution of rapid decay on $\mathbb{R}$ and v is its right-sided 1st antiderivative (which implies that A is generable by a smooth function of slow growth on $\mathbb{R}$ and V is a tempered distribution on $\mathbb{R}$), we have

$$\begin{aligned} a = a_v(v) &\Leftrightarrow A = A_v(V) \\ &= \delta^{(1)} * v &= j\omega \cdot V \end{aligned} \quad (61)$$

$$\begin{aligned} v = v_a(a) &\Leftrightarrow V = V_a(A) \\ &= \mathfrak{I} * a &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot A \end{aligned} \quad (62)$$

On the one hand, Equation (61) provides the input–output representations both in the time domain and in the frequency domain of a 1st-order differentiator of input variable v and output variable a in the time domain and of input variable V and output variable A in the frequency domain. On the other hand, Equation (62) provides the input–output representations both in the time domain and in the frequency domain of a 1st-order antidifferentiator of input variable a and output variable v in the time domain and of input variable A and output variable V in the frequency domain.

3 Second theoretical study: Distributional characterisation of two-terminal mechanical elements

If v is a right-sided distribution of rapid decay on $\mathbb{R}$ and x is its right-sided 1st antiderivative (which implies that V is generable by a smooth function of slow growth on $\mathbb{R}$ and X is a tempered distribution on $\mathbb{R}$), we have

$$\begin{aligned} v = v_x(x) &\Leftrightarrow V = V_x(X) \\ &= \delta^{(1)} * x &= j\omega \cdot X \end{aligned} \quad (63)$$

$$\begin{aligned} x = x_v(v) &\Leftrightarrow X = X_v(V) \\ &= \mathfrak{G} * v &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot V \end{aligned} \quad (64)$$

On the one hand, Equation (63) provides the input–output representations both in the time domain and in the frequency domain of a 1st-order differentiator of input variable x and output variable v in the time domain and of input variable X and output variable V in the frequency domain. On the other hand, Equation (64) provides the input–output representations both in the time domain and in the frequency domain of a 1st-order antidifferentiator of input variable v and output variable x in the time domain and of input variable V and output variable X in the frequency domain.

If a is a right-sided distribution of rapid decay on $\mathbb{R}$ and x is its right-sided 2nd antiderivative (which implies that A is generable by a smooth function of slow growth on $\mathbb{R}$ and X is a tempered distribution on $\mathbb{R}$), we have

$$\begin{aligned} a = a_x(x) &\Leftrightarrow A = A_x(X) \\ &= \delta^{(2)} * x &= -\omega^2 \cdot X \end{aligned} \quad (65)$$

$$\begin{aligned} x = x_a(a) &\Leftrightarrow X = X_A(A) \\ &= \rho * a &= \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \cdot A \end{aligned} \quad (66)$$

On the one hand, Equation (65) provides the input–output representations both in the time domain and in the frequency domain of a 2nd-order differentiator of input variable x and output variable a in the time domain and of input variable X and output variable A in the frequency domain. On the other hand, Equation (66) provides the input–output representations both in the time domain and in the frequency domain of a 2nd-order antidifferentiator of input variable a and output variable x in the time domain and of input variable A and output variable X in the frequency domain.

3.3 Linear spring

The *linear spring* is a two-terminal element usually introduced in the time domain [Doebelin, 1980, 1998; Hixson, 2010; Karnopp *et al.*, 2012; Makris, 2017, 2018; Smith, 2020]. The direct simple system and the inverse simple system associated with the linear spring are defined respectively by

$$\begin{aligned} f = f_x(x) &\Leftrightarrow F = F_X(X) \\ &= k\delta * x &= k1_{\mathbb{R}} \cdot X \end{aligned} \quad (67)$$

and by

$$\begin{aligned} x = x_f(f) &\Leftrightarrow X = X_F(F) \\ &= \frac{\delta}{k} * f &= \frac{1_{\mathbb{R}}}{k} \cdot F \end{aligned} \quad (68)$$

where

- k is a positive real constant of the same quantity dimension as $f x^{-1}$ to which reference is made as *stiffness*

We observe that Equation (67) (respectively, Equation (68)) is well-defined (in the sense that the operations in it are legitimate) if x (respectively, f) is a tempered distribution on $\mathbb{R}$ or, equivalently, if X (respectively, F) is a tempered distribution on $\mathbb{R}$. In these conditions, f (respectively, x) is a tempered distribution on $\mathbb{R}$, and F (respectively, X) is a tempered distribution on $\mathbb{R}$.

3.3.1 Dynamic stiffness and dynamic compliance

The time-domain and frequency-domain dynamic stiffness of the linear spring are respectively the convolution kernel distribution and the frequency-response distribution of the direct simple system associated with the linear spring. We have

$$\left\{ \begin{aligned} f &= f_x(x) \\ &= k\delta * x \\ &= g_s * x \\ g_s &= k\delta \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned} F &= F_X(X) \\ &= k1_{\mathbb{R}} \cdot X \\ &= G_s \cdot X \\ G_s &= k1_{\mathbb{R}} \end{aligned} \right\} \quad (69)$$

We observe that Equation (69) is well-defined (in the sense that the operations in it are legitimate) if x is a tempered distribution on $\mathbb{R}$ (or, equivalently, if X is a tempered distribution on $\mathbb{R}$). In these conditions, f is a tempered distribution on $\mathbb{R}$ (and F is a tempered distribution on $\mathbb{R}$).

The time-domain and frequency-domain dynamic compliance of the linear spring are respectively the convolution kernel distribution and the frequency-response distribution of the inverse simple system associated with the linear spring. We have

$$\left\{ \begin{array}{l} x = x_f(f) \\ = \frac{\delta}{k} * f \\ = g_c * f \\ g_c = \frac{\delta}{k} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} X = X_F(F) \\ = \frac{1_{\mathbb{R}}}{k} \cdot F \\ = G_c \cdot F \\ G_c = \frac{1_{\mathbb{R}}}{k} \end{array} \right. \quad (70)$$

We observe that Equation (70) is well-defined (in the sense that the operations in it are legitimate) if f is a tempered distribution on $\mathbb{R}$ (or, equivalently, if F is a tempered distribution on $\mathbb{R}$). In these conditions, x is a tempered distribution on $\mathbb{R}$ (and X is a tempered distribution on $\mathbb{R}$).

It can be easily verified that the convolution systems defining the dynamic stiffness and the dynamic compliance of the linear spring have the property of causality. Furthermore, such convolution systems are mutually inverse.

3.3.2 Mechanical impedance and mobility

The time-domain and frequency-domain mechanical impedance of the linear spring are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 1st-order antidifferentiator (upstream) and the direct simple system associated with the linear spring (downstream). We have

$$\left\{ \begin{array}{l} f = f_v(v) \\ = f_x(x_v(v)) \\ = k\delta * (\mathfrak{V} * v) \\ = (k\delta * \mathfrak{V}) * v \\ = k\mathfrak{V} * v \\ = g_z * v \\ g_z = k\mathfrak{V} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_v(V) \\ = F_x(X_v(V)) \\ = k1_{\mathbb{R}} \cdot \left(\left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot V \right) \\ = \left(k1_{\mathbb{R}} \cdot \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \right) \cdot V \\ = \left(\text{p.v.} \left(\frac{k}{j\omega} \right) + \pi k\delta \right) \cdot V \\ = G_z \cdot V \\ G_z = \text{p.v.} \left(\frac{k}{j\omega} \right) + \pi k\delta \end{array} \right. \quad (71)$$

We observe that Equation (71) is well-defined (in the sense that the operations in it are legitimate) if v is a right-sided distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if V is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, f is a right-sided tempered distribution on $\mathbb{R}$ (and F is a right-retarded distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the 1st-order antidifferentiator (upstream) and the direct simple system associated with the linear spring (downstream) admits δ as an input variable and is commutative.

The time-domain and frequency-domain mobility of the linear spring are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear spring (upstream) and the 1st-order differentiator (downstream). We have

$$\left\{ \begin{array}{l} v = v_f(f) \\ = v_x(x_f(f)) \\ = \delta^{(1)} * \left(\frac{\delta}{k} * f \right) \\ = \left(\delta^{(1)} * \frac{\delta}{k} \right) * f \\ = \frac{\delta^{(1)}}{k} * f \\ = g_y * f \\ g_y = \frac{\delta^{(1)}}{k} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} V = V_f(F) \\ = V_x(X_f(F)) \\ = j\omega \cdot \left(\frac{1_{\mathbb{R}}}{k} \cdot F \right) \\ = \left(j\omega \cdot \frac{1_{\mathbb{R}}}{k} \right) \cdot F \\ = \frac{j\omega}{k} \cdot F \\ = G_y \cdot F \\ G_y = \frac{j\omega}{k} \end{array} \right. \quad (72)$$

We observe that Equation (72) is well-defined (in the sense that the operations in it are legitimate) if f is a tempered distribution on $\mathbb{R}$ (or, equivalently, if F is a tempered distribution on $\mathbb{R}$). In these conditions, v is a tempered distribution on $\mathbb{R}$ (and V is a tempered distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the inverse simple system associated with the linear spring (upstream) and the 1st-order differentiator (downstream) admits δ as an input variable and is commutative.

It can be easily verified that the convolution systems defining the mechanical impedance and the mobility of the linear spring have the property of causality and passivity. Furthermore, such convolution systems are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the right-sided distributions of rapid decay on $\mathbb{R}$.

3.3.3 Apparent mass and accelerance

The time-domain and frequency-domain apparent mass of the linear spring are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 2nd-order antidifferentiator (upstream) and the direct simple system associated with the linear spring (downstream). We have

$$\left\{ \begin{array}{l} f = f_a(a) \\ = f_x(x_a(a)) \\ = k\delta * (\rho * a) \\ = (k\delta * \rho) * a \\ = k\rho * a \\ = g_M * a \\ g_M = k\rho \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_A(A) \\ = F_X(X_A(A)) \\ = k1_{\mathbb{R}} \cdot \left(\left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \cdot A \right) \\ = \left(k1_{\mathbb{R}} \cdot \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \right) \cdot A \\ = \left(\text{f.p.} \left(\frac{k}{-\omega^2} \right) + j\pi k\delta^{(1)} \right) \cdot A \\ = G_M \cdot A \\ G_M = \text{f.p.} \left(\frac{k}{-\omega^2} \right) + j\pi k\delta^{(1)} \end{array} \right. \quad (73)$$

We observe that Equation (73) is well-defined (in the sense that the operations in it are legitimate) if a is a right-sided distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if A is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, f is a right-sided tempered distribution on $\mathbb{R}$ (and F is a right-retarded distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the 2nd-order antidifferentiator (upstream)

and the direct simple system associated with the linear spring (downstream) admits δ as an input variable and is commutative.

The time-domain and frequency-domain accelerance of the linear spring are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear spring (upstream) and the 2nd-order differentiator (downstream). We have

$$\left\{ \begin{array}{l} a = a_f(f) \\ = a_x(x_f(f)) \\ = \delta^{(2)} * \left(\frac{\delta}{k} * f \right) \\ = \left(\delta^{(2)} * \frac{\delta}{k} \right) * f \\ = \frac{\delta^{(2)}}{k} * f \\ = g_A * f \\ g_A = \frac{\delta^{(2)}}{k} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A = A_F(F) \\ = A_X(X_F(F)) \\ = -\omega^2 \cdot \left(\frac{1_{\mathbb{R}}}{k} \cdot F \right) \\ = \left(-\omega^2 \cdot \frac{1_{\mathbb{R}}}{k} \right) \cdot F \\ = \frac{-\omega^2}{k} \cdot F \\ = G_A \cdot F \\ G_A = \frac{-\omega^2}{k} \end{array} \right. \quad (74)$$

We observe that Equation (74) is well-defined (in the sense that the operations in it are legitimate) if f is a tempered distribution on $\mathbb{R}$ (or, equivalently, if F is a tempered distribution on $\mathbb{R}$). In these conditions, a is a tempered distribution on $\mathbb{R}$ (and A is a tempered distribution on $\mathbb{R}$). Furthermore, we observe that the series connection of the inverse simple system associated with the linear spring (upstream) and the 2nd-order differentiator (downstream) admits δ as an input variable and is commutative.

It can be easily verified that the convolution systems defining the apparent mass and the accelerance of the linear spring have the property of causality. Furthermore, such convolution systems are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the right-sided distributions of rapid decay on $\mathbb{R}$.

3.4 Linear viscous damper

The *linear viscous damper* is a two-terminal element usually introduced in the time domain [Doebelin, 1980, 1998; Hixson, 2010; Karnopp *et al.*, 2012; Makris, 2017, 2018; Smith, 2020]. The direct simple system and the inverse simple system associated with the linear viscous damper are defined respectively by

$$\begin{aligned} f = f_v(v) &\Leftrightarrow F = F_v(V) \\ &= c\delta * v &= c1_{\mathbb{R}} \cdot V \end{aligned} \quad (75)$$

and by

$$\begin{aligned} v = v_f(f) &\Leftrightarrow V = V_F(F) \\ &= \frac{\delta}{c} * f &= \frac{1_{\mathbb{R}}}{c} \cdot F \end{aligned} \quad (76)$$

where

- c is a positive real constant of the same quantity dimension as $f v^{-1}$ to which reference is made as *viscous damping coefficient*

We observe that Equation (75) (respectively, Equation (76)) is well-defined (in the sense that the operations in it are legitimate) if v (respectively, f) is a tempered distribution on $\mathbb{R}$ or, equivalently, if V (respectively, F) is a tempered distribution on $\mathbb{R}$. In these conditions, f (respectively, v) is a tempered distribution on $\mathbb{R}$, and F (respectively, V) is a tempered distribution on $\mathbb{R}$.

3.4.1 Dynamic stiffness and dynamic compliance

The time-domain and frequency-domain dynamic stiffness of the linear viscous damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 1st-order differentiator (upstream) and the direct simple system associated with the linear viscous damper (downstream). We have

$$\left\{ \begin{array}{l} f = f_x(x) \\ = f_v(v_x(x)) \\ = c\delta * (\delta^{(1)} * x) \\ = (c\delta * \delta^{(1)}) * x \\ = c\delta^{(1)} * x \\ = g_s * x \\ g_s = c\delta^{(1)} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_X(X) \\ = F_V(V_X(X)) \\ = c1_{\mathbb{R}} \cdot (j\omega \cdot X) \\ = (c1_{\mathbb{R}} \cdot j\omega) \cdot X \\ = jc\omega \cdot X \\ = G_s \cdot X \\ G_s = jc\omega \end{array} \right. \quad (77)$$

We observe that Equation (77) is well-defined (in the sense that the operations in it are legitimate) if x is a tempered distribution on $\mathbb{R}$ (or, equivalently, if X is a tempered distribution on $\mathbb{R}$). In these conditions, f is a tempered distribution on $\mathbb{R}$ (and F is a tempered distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the 1st-order differentiator (upstream) and the direct simple system associated with the linear viscous damper (downstream) admits δ as an input variable and is commutative.

The time-domain and frequency-domain dynamic compliance of the linear viscous damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear viscous damper (upstream) and the 1st-order antidifferentiator (downstream). We have

$$\left\{ \begin{array}{l} x = x_f(f) \\ = x_v(v_f(f)) \\ = \mathfrak{g} * \left(\frac{\delta}{c} * f \right) \\ = \left(\mathfrak{g} * \frac{\delta}{c} \right) * f \\ = \frac{\mathfrak{g}}{c} * f \\ = g_c * f \\ g_c = \frac{\mathfrak{g}}{c} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} X = X_F(F) \\ = X_V(V_F(F)) \\ = \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot \left(\frac{1_{\mathbb{R}}}{c} \cdot F \right) \\ = \left(\left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot \frac{1_{\mathbb{R}}}{c} \right) \cdot F \\ = \left(\text{p.v.} \left(\frac{1}{jc\omega} \right) + \frac{\pi\delta}{c} \right) \cdot F \\ = G_c \cdot F \\ G_c = \text{p.v.} \left(\frac{1}{jc\omega} \right) + \frac{\pi\delta}{c} \end{array} \right. \quad (78)$$

We observe that Equation (78) is well-defined (in the sense that the operations in it are legitimate) if f is a right-sided distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if F is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, x is a right-sided tempered distribution on $\mathbb{R}$ (and X is a right-retarded distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the inverse simple system associated with

the linear viscous damper (upstream) and the 1st-order antidifferentiator (downstream) admits δ as an input variable and is commutative.

It can be easily verified that the convolution systems defining the dynamic stiffness and the dynamic compliance of the linear viscous damper have the property of causality. Furthermore, such convolution systems are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the right-sided distributions of rapid decay on $\mathbb{R}$.

3.4.2 Mechanical impedance and mobility

The time-domain and frequency-domain mechanical impedance of the linear viscous damper are respectively the convolution kernel distribution and the frequency-response distribution associated with the direct simple system of the linear viscous damper. We have

$$\left\{ \begin{array}{l} f = f_v(v) \\ = c\delta * v \\ = g_z * v \\ g_z = c\delta \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_v(V) \\ = c1_{\mathbb{R}} \cdot V \\ = G_z \cdot V \\ G_z = c1_{\mathbb{R}} \end{array} \right. \quad (79)$$

We observe that Equation (79) is well-defined (in the sense that the operations in it are legitimate) if v is a tempered distribution on $\mathbb{R}$ (or, equivalently, if V is a tempered distribution on $\mathbb{R}$). In these conditions, f is a tempered distribution on $\mathbb{R}$ (and F is a tempered distribution on $\mathbb{R}$).

The time-domain and frequency-domain mobility of the linear viscous damper are respectively the convolution kernel distribution and the frequency-response distribution of the inverse simple system associated with the linear viscous damper. We have

$$\left\{ \begin{array}{l} v = v_f(f) \\ = \frac{\delta}{c} * f \\ = g_y * f \\ g_y = \frac{\delta}{c} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} V = V_f(F) \\ = \frac{1_{\mathbb{R}}}{c} \cdot F \\ = G_y \cdot F \\ G_y = \frac{1_{\mathbb{R}}}{c} \end{array} \right. \quad (80)$$

We observe that Equation (80) is well-defined (in the sense that the operations in it are legitimate) if f is a tempered distribution on $\mathbb{R}$ (or, equivalently, if F is a tempered distribution on $\mathbb{R}$). In these conditions, v is a tempered distribution on $\mathbb{R}$ (and V is a tempered distribution on $\mathbb{R}$).

It can be easily verified that the convolution systems defining the mechanical impedance and the mobility of the linear viscous damper have the property of causality and passivity. Furthermore, such convolution systems are mutually inverse.

3.4.3 Apparent mass and accelerance

The time-domain and frequency-domain apparent mass of the linear viscous damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 1st-order antidifferentiator (upstream) and the direct simple system associated with the linear viscous damper (downstream). We have

$$\left\{ \begin{array}{l} f = f_a(a) \\ = f_v(v_a(a)) \\ = c\delta * (\mathfrak{I} * a) \\ = (c\delta * \mathfrak{I}) * a \\ = c\mathfrak{I} * a \\ = g_M * a \\ g_M = c\mathfrak{I} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_A(A) \\ = F_V(V_A(A)) \\ = c1_{\mathbb{R}} \cdot \left(\left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot A \right) \\ = \left(c1_{\mathbb{R}} \cdot \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \right) \cdot A \\ = \left(\text{p.v.} \left(\frac{c}{j\omega} \right) + \pi c\delta \right) \cdot A \\ = G_M \cdot A \\ G_M = \text{p.v.} \left(\frac{c}{j\omega} \right) + \pi c\delta \end{array} \right. \quad (81)$$

We observe that Equation (81) is well-defined (in the sense that the operations in it are legitimate) if a is a right-sided distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if A is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, f is a right-sided tempered distribution on $\mathbb{R}$ (and F is a right-retarded distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the 1st-order antidifferentiator (upstream) and the direct simple system associated with the linear viscous damper (downstream) admits δ as an input variable and is commutative.

The time-domain and frequency-domain accelerance of the linear viscous damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear viscous damper (upstream) and the 1st-order differentiator (downstream). We have

$$\left\{ \begin{array}{l} a = a_f(f) \\ = a_v(v_f(f)) \\ = \delta^{(1)} * \left(\frac{\delta}{c} * f \right) \\ = \left(\delta^{(1)} * \frac{\delta}{c} \right) * f \\ = \frac{\delta^{(1)}}{c} * f \\ = g_A * f \\ g_A = \frac{\delta^{(1)}}{c} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A = A_F(F) \\ = A_V(V_F(F)) \\ = j\omega \cdot \left(\frac{1_{\mathbb{R}}}{c} \cdot F \right) \\ = \left(j\omega \cdot \frac{1_{\mathbb{R}}}{c} \right) \cdot F \\ = \frac{j\omega}{c} \cdot F \\ = G_A \cdot F \\ G_A = \frac{j\omega}{c} \end{array} \right. \quad (82)$$

We observe that Equation (82) is well-defined (in the sense that the operations in it are legitimate) if f is a tempered distribution on $\mathbb{R}$ (or, equivalently, if F is a tempered distribution on $\mathbb{R}$). In these conditions, a is a tempered distribution on $\mathbb{R}$ (and A is a tempered distribution on $\mathbb{R}$). Furthermore, we observe that the series connection of the inverse simple system associated with the linear viscous damper (upstream) and the 1st-order differentiator (downstream) admits δ as an input variable and is commutative.

It can be easily verified that the convolution systems defining the apparent mass and the accelerance of the linear viscous damper have the property of causality. Furthermore, such convolution systems are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the right-sided distributions of rapid decay on $\mathbb{R}$.

3.5 Linear inerter (including rigid mass)

The *linear inerter* is a two-terminal element usually introduced in the time domain [Makris, 2017, 2018; Smith, 2020]. The direct simple system and the inverse simple system associated with the linear inerter are defined respectively by

$$\begin{aligned} f = f_a(a) &\Leftrightarrow F = F_A(A) \\ &= b\delta * a &= b1_{\mathbb{R}} \cdot A \end{aligned} \quad (83)$$

and by

$$\begin{aligned} a = a_f(f) &\Leftrightarrow A = A_F(F) \\ &= \frac{\delta}{b} * f &= \frac{1_{\mathbb{R}}}{b} \cdot F \end{aligned} \quad (84)$$

where

- b is a positive real constant of the same quantity dimension as fa^{-1} to which reference is made as *inertance*

We observe that Equation (83) (respectively, Equation (84)) is well-defined (in the sense that the operations in it are legitimate) if a (respectively, f) is a tempered distribution on $\mathbb{R}$ or, equivalently, if A (respectively, F) is a tempered distribution on $\mathbb{R}$. In these conditions, f (respectively, a) is a tempered distribution on $\mathbb{R}$, and F (respectively, A) is a tempered distribution on $\mathbb{R}$.

A linear inerter for which one of the two terminals is virtual and connected to a motionless reference point is said to be a *rigid mass* [Doebelin, 1980, 1998; Hixson, 2010; Karnopp *et al.*, 2012; Makris, 2017, 2018; Smith, 2020]; the inertance of such a linear inerter is denoted by m and is said to be the *mass* of the rigid mass. Hereinafter, results obtained for a linear inerter of inertance b hold in particular for a rigid mass of mass m .

3.5.1 Dynamic stiffness and dynamic compliance

The time-domain and frequency-domain dynamic stiffness of the linear inerter are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 2nd-order differentiator (upstream) and the direct simple system associated with the linear inerter (downstream). We have

$$\left\{ \begin{array}{l} f = f_x(x) \\ = f_a(a_x(x)) \\ = b\delta * (\delta^{(2)} * x) \\ = (b\delta * \delta^{(2)}) * x \\ = b\delta^{(2)} * x \\ = g_s * x \\ g_s = b\delta^{(2)} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_X(X) \\ = F_A(A_X(X)) \\ = b1_{\mathbb{R}} \cdot (-\omega^2 \cdot X) \\ = (b1_{\mathbb{R}} \cdot -\omega^2) \cdot X \\ = -b\omega^2 \cdot X \\ = G_s \cdot X \\ G_s = -b\omega^2 \end{array} \right. \quad (85)$$

We observe that Equation (85) is well-defined (in the sense that the operations in it are legitimate) if x is a tempered distribution on $\mathbb{R}$ (or, equivalently, if X is a tempered distribution on $\mathbb{R}$). In these conditions, f is a tempered distribution on $\mathbb{R}$ (and F is a tempered distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the 2nd-order differentiator (upstream) and the direct simple system associated with the linear inerter (downstream) admits δ as an input variable and is commutative.

The time-domain and frequency-domain dynamic compliance of the linear inerter are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear inerter (upstream) and the 2nd-order antidifferentiator (downstream). We have

$$\left\{ \begin{array}{l} x = x_f(f) \\ = x_a(a_f(f)) \\ = \rho * \left(\frac{\delta}{b} * f \right) \\ = \left(\rho * \frac{\delta}{b} \right) * f \\ = \frac{\rho}{b} * f \\ = g_c * f \\ g_c = \frac{\rho}{b} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} X = X_F(F) \\ = X_A(A_F(F)) \\ = \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \cdot \left(\frac{1_{\mathbb{R}}}{b} \cdot F \right) \\ = \left(\left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \cdot \frac{1_{\mathbb{R}}}{b} \right) \cdot F \\ = \left(\text{f.p.} \left(\frac{1}{-b\omega^2} \right) + \frac{j\pi\delta^{(1)}}{b} \right) \cdot F \\ = G_c \cdot F \\ G_c = \text{f.p.} \left(\frac{1}{-b\omega^2} \right) + \frac{j\pi\delta^{(1)}}{b} \end{array} \right. \quad (86)$$

We observe that Equation (86) is well-defined (in the sense that the operations in it are legitimate) if f is a right-sided distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if F is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, x is a right-sided tempered distribution on $\mathbb{R}$ (and X is a right-retarded distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the inverse simple system associated with

the linear inerter (upstream) and the 2nd-order antidifferentiator (downstream) admits δ as an input variable and is commutative.

It can be easily verified that the convolution systems defining the dynamic stiffness and the dynamic compliance of the linear inerter have the property of causality. Furthermore, such convolution systems are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the right-sided distributions of rapid decay on $\mathbb{R}$.

3.5.2 Mechanical impedance and mobility

The time-domain and frequency-domain mechanical impedance of the linear inerter are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 1st-order differentiator (upstream) and the direct simple system associated with the linear inerter (downstream). We have

$$\left\{ \begin{array}{l} f = f_v(v) \\ = f_a(a_v(v)) \\ = b\delta * (\delta^{(1)} * v) \\ = (b\delta * \delta^{(1)}) * v \\ = b\delta^{(1)} * v \\ = g_z * v \\ g_z = b\delta^{(1)} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_v(V) \\ = F_a(A_v(V)) \\ = b1_{\mathbb{R}} \cdot (j\omega \cdot V) \\ = (b1_{\mathbb{R}} \cdot j\omega) \cdot V \\ = jb\omega \cdot V \\ = G_z \cdot V \\ G_z = jb\omega \end{array} \right. \quad (87)$$

We observe that Equation (87) is well-defined (in the sense that the operations in it are legitimate) if v is a tempered distribution on $\mathbb{R}$ (or, equivalently, if V is a tempered distribution on $\mathbb{R}$). In these conditions, f is a tempered distribution on $\mathbb{R}$ (and F is a tempered distribution on $\mathbb{R}$).

Furthermore, we observe that the series connection of the 1st-order differentiator (upstream) and the direct simple system associated with the linear inerter (downstream) admits δ as an input variable and is commutative.

The time-domain and frequency-domain mobility of the linear inerter are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear inerter (upstream) and the 1st-order antidifferentiator (downstream). We have

$$\left\{ \begin{array}{l} v = v_f(f) \\ = v_a(a_f(f)) \\ = \mathfrak{V} * \left(\frac{\delta}{b} * f \right) \\ = \left(\mathfrak{V} * \frac{\delta}{b} \right) * f \\ = \frac{\mathfrak{V}}{b} * f \\ = g_Y * f \\ g_Y = \frac{\mathfrak{V}}{b} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} V = V_F(F) \\ = V_A(A_F(F)) \\ = \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot \left(\frac{1_{\mathbb{R}}}{b} \cdot F \right) \\ = \left(\left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot \frac{1_{\mathbb{R}}}{b} \right) \cdot F \\ = \left(\text{p.v.} \left(\frac{1}{jb\omega} \right) + \frac{\pi\delta}{b} \right) \cdot F \\ = G_Y \cdot F \\ G_Y = \text{p.v.} \left(\frac{1}{jb\omega} \right) + \frac{\pi\delta}{b} \end{array} \right. \quad (88)$$

We observe that Equation (88) is well-defined (in the sense that the operations in it are legitimate) if f is a right-sided distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if F is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, v is a right-sided tempered distribution on $\mathbb{R}$ (and V is a right-retarded distribution on $\mathbb{R}$). Furthermore, we observe that the series connection of the inverse simple system associated with the linear inerter (upstream) and the 1st-order antidifferentiator (downstream) admits δ as an input variable and is commutative.

It can be easily verified that the convolution systems defining the mechanical impedance and the mobility of the linear inerter have the property of causality and passivity. Furthermore, such convolution systems are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the right-sided distributions of rapid decay on $\mathbb{R}$.

3.5.3 Apparent mass and accelerance

The time-domain and frequency-domain apparent mass of the linear inerter are respectively the convolution kernel distribution and the frequency-response distribution associated with the direct simple system of the linear inerter. We have

$$\left\{ \begin{array}{l} f = f_a(a) \\ = b\delta * a \\ = g_M * a \\ g_M = b\delta \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_A(A) \\ = b1_{\mathbb{R}} \cdot A \\ = G_M \cdot A \\ G_M = b1_{\mathbb{R}} \end{array} \right. \quad (89)$$

We observe that Equation (89) is well-defined (in the sense that the operations in it are legitimate) if a is a tempered distribution on $\mathbb{R}$ (or, equivalently, if A is a tempered distribution on $\mathbb{R}$). In these conditions, f is a tempered distribution on $\mathbb{R}$ (and F is a tempered distribution on $\mathbb{R}$).

The time-domain and frequency-domain accelerance of the linear inerter are respectively the convolution kernel distribution and the frequency-response distribution of the inverse simple system associated with the linear inerter. We have

$$\left\{ \begin{array}{l} a = a_f(f) \\ = \frac{\delta}{b} * f \\ = g_A * f \\ g_A = \frac{\delta}{b} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A = A_F(F) \\ = \frac{1_{\mathbb{R}}}{b} \cdot F \\ = G_A \cdot F \\ G_A = \frac{1_{\mathbb{R}}}{b} \end{array} \right. \quad (90)$$

We observe that Equation (90) is well-defined (in the sense that the operations in it are legitimate) if f is a tempered distribution on $\mathbb{R}$ (or, equivalently, if F is a tempered distribution on $\mathbb{R}$). In these conditions, a is a tempered distribution on $\mathbb{R}$ (and A is a tempered distribution on $\mathbb{R}$).

It can be easily verified that the convolution systems defining the apparent mass and the accelerance of the linear inerter have the property of causality. Furthermore, such convolution systems are mutually inverse.

3.6 Linear hysteresis damper

The *linear hysteresis damper* is a two-terminal element usually introduced in the frequency domain [Crandall, 1970, 1991; Inaudi and Kelly, 1995; Makris, 1997a, 1997b]. The direct simple system and the inverse simple system associated with the linear hysteresis damper are defined respectively by

$$\begin{aligned}
 f &= f_x(x) \\
 &= \mathcal{H}(hx) \\
 &= \text{p.v.} \left(-\frac{h}{\pi t} \right) * x \\
 F &= F_X(X) \\
 &= \mathcal{F} \left(\mathcal{H}(hx) \right) \\
 &= jh \operatorname{sgn} \cdot X
 \end{aligned} \quad \Leftrightarrow \quad (91)$$

and by

$$\begin{aligned}
 x &= x_f(f) \\
 &= \mathcal{H} \left(-\frac{f}{h} \right) \\
 &= \text{p.v.} \left(\frac{1}{\pi h t} \right) * f \\
 X &= X_F(F) \\
 &= \mathcal{F} \left(\mathcal{H} \left(-\frac{f}{h} \right) \right) \\
 &= \frac{\operatorname{sgn}}{jh} \cdot F
 \end{aligned} \quad \Leftrightarrow \quad (92)$$

with [Kanwal, 2004]

$$\mathcal{F} \left(\text{p.v.} \left(\frac{1}{t} \right) \right) = -j\pi \operatorname{sgn}$$

where

- $\mathcal{H}$ is the *Hilbert transformation*
- h is a positive real constant of the same quantity dimension as $f x^{-1}$ to which reference is made as *hysteresis damping coefficient*

We observe that Equation (91) (respectively, Equation (92)) is well-defined (in the sense that the operations in it are legitimate) if x (respectively, f) is a (-1) -weighted integrable distribution on $\mathbb{R}$, which implies that X (respectively, F) is a tempered distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$. In these conditions, f (respectively, x) is a tempered distribution on $\mathbb{R}$, and F (respectively, X) is a tempered distribution on $\mathbb{R}$.

It is worth noting that, unlike the direct simple system and the inverse simple system associated with the linear spring, with the linear viscous damper, and with the linear inerter (including the rigid mass), the direct simple system and the inverse simple system associated with the linear hysteresis damper have input–output representations that cannot be re-expressed as differential equations; in fact, each of such input–output representations can only be re-expressed as a *Cauchy-type singular integral equation of the first kind* [Estrada and Kanwal, 2000].

3.6.1 Dynamic stiffness and dynamic compliance

The time-domain and frequency-domain dynamic stiffness of the linear hysteresis damper are respectively the convolution kernel distribution and the frequency-response distribution of the direct simple system associated with the linear hysteresis damper. We have

$$\left\{ \begin{array}{l} f = f_x(x) \\ = \text{p.v.} \left(-\frac{h}{\pi t} \right) * x \\ = g_s * x \\ g_s = \text{p.v.} \left(-\frac{h}{\pi t} \right) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} F = F_X(X) \\ = jh \text{sgn} \cdot X \\ = G_s \cdot X \\ G_s = jh \text{sgn} \end{array} \right. \quad (93)$$

We observe that Equation (93) is well-defined (in the sense that the operations in it are legitimate) if x is a (-1) -weighted integrable distribution on $\mathbb{R}$ (which implies that X is a tempered distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$). In these conditions, f is a tempered distribution on $\mathbb{R}$ (and F is a tempered distribution on $\mathbb{R}$).

The time-domain and frequency-domain dynamic compliance of the linear hysteresis damper are respectively the convolution kernel distribution and the frequency-response distribution of the inverse simple system associated with the linear hysteresis damper. We have

$$\left\{ \begin{array}{l} x = x_f(f) \\ = \text{p.v.} \left(\frac{1}{\pi h t} \right) * f \\ = g_c * f \\ g_c = \text{p.v.} \left(\frac{1}{\pi h t} \right) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} X = X_F(F) \\ = \frac{\text{sgn}}{jh} \cdot F \\ = G_c \cdot F \\ G_c = \frac{\text{sgn}}{jh} \end{array} \right. \quad (94)$$

We observe that Equation (94) is well-defined (in the sense that the operations in it are legitimate) if f is a (-1) -weighted integrable distribution on $\mathbb{R}$ (which implies that F is a tempered

distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$). In these conditions, x is a tempered distribution on $\mathbb{R}$ (and X is a tempered distribution on $\mathbb{R}$).

It can be easily verified that the convolution systems defining the dynamic stiffness and the dynamic compliance of the linear hysteresis damper have the property of acausality. Furthermore, such convolution systems are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the $(-1, +1)$ -weighted integrable distributions on $\mathbb{R}$.

3.6.2 Mechanical impedance and mobility

The time-domain and frequency-domain mechanical impedance of the linear hysteresis damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 1st-order antidifferentiator (upstream) and the direct simple system associated with the linear hysteresis damper (downstream). We have

$$\left\{ \begin{array}{l} f = f_v(v) \\ = f_x(x_v(v)) \\ = \text{p.v.} \left(-\frac{h}{\pi t} \right) * (\mathfrak{G} * v) \\ = \left(\delta^{(1)} * \left(-\frac{h}{\pi} (\ln(|t|) + C1_{\mathbb{R}}) \right) \right) * (\mathfrak{G} * v) \\ = \left(-\frac{h}{\pi} (\ln(|t|) + C1_{\mathbb{R}}) \right) * \left((\delta^{(1)} * \mathfrak{G}) * v \right) \\ = -\frac{h \ln(|t|)}{\pi} * v \\ = g_z * v \\ g_z = -\frac{h \ln(|t|)}{\pi} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_v(V) \\ = F_x(X_v(V)) \\ = jh \text{sgn} \cdot \left(\left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot V \right) \\ = \left(j\omega \cdot \left(-\frac{h}{\pi} \left(-\pi \text{Pf} \left(\frac{1}{|\omega|} \right) - 2\pi(\gamma - C)\delta \right) \right) \right) \cdot \left(\left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot V \right) \\ = \left(-\frac{h}{\pi} \left(-\pi \text{Pf} \left(\frac{1}{|\omega|} \right) - 2\pi(\gamma - C)\delta \right) \right) \cdot \left(j\omega \cdot \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \cdot V \right) \\ = \text{Pf} \left(\frac{h}{|\omega|} \right) \cdot V \\ = G_z \cdot V \\ G_z = \text{Pf} \left(\frac{h}{|\omega|} \right) \end{array} \right. \quad (95)$$

with

$$\mathcal{F}(\ln(|t|)) = -\pi \text{Pf}\left(\frac{1}{|\omega|}\right) - 2\pi\gamma\delta$$

where

- γ is the *Euler–Mascheroni constant*
- C is an arbitrary complex constant of quantity dimension one

We observe that Equation (95) is well-defined (in the sense that the operations in it are legitimate) if v is a right-sided distribution of rapid decay on $\mathbb{R}$ with null 0^{th} moment (or, equivalently, if V is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$ with 0^{th} derivative vanishing at the origin); specifically, this ensures that x is a right-sided distribution of rapid decay on $\mathbb{R}$ (and X is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, f is a square integrable distribution on $\mathbb{R}$ and, in fact, a $p_{(1,+\infty]}$ -integrable distribution on $\mathbb{R}$ for each $p_{(1,+\infty]}$ in $\overline{\mathbb{R}}_{(1,+\infty]}$ (and F is a tempered distribution on $\mathbb{R}$ that is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by a square integrable function on $\mathbb{R}$). Furthermore, we observe that the series connection of the 1^{st} -order antidifferentiator (upstream) and the direct simple system associated with the linear hysteresis damper (downstream) does not admit δ as an input variable and may not be commutative. It is worth noting that, as consequence of $V(0)$ being null, γ and C are immaterial and hence dropped in Equation (95) (the terms in which they appear being cancelled by the operations of multiplication or convolution involving them), which explains why g_z and G_z , as expressed by Equation (95), do not form a Fourier-transform pair (as they would do if γ and C were maintained). It is also worth noting that, as consequence of $V(0)$ being null, F is a regular tempered distribution on $\mathbb{R}$ that is generable by the function defined almost everywhere on $\mathbb{R}$ by $hV/|\omega|$ (but G_z is in any case a singular tempered distribution on $\mathbb{R}$ and, in fact, a singular $p_{(1,+\infty]}$ -integrable distribution on $\mathbb{R}$ for each $p_{(1,+\infty]}$ in $\overline{\mathbb{R}}_{(1,+\infty]}$).

The time-domain and frequency-domain mobility of the linear hysteresis damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear hysteresis damper (upstream) and the 1^{st} -order differentiator (downstream). We have

$$\left\{ \begin{array}{l} v = v_f(f) \\ = v_x(x_f(f)) \\ = \delta^{(1)} * \left(\text{p.v.} \left(\frac{1}{\pi h t} \right) * f \right) \\ = \left(\delta^{(1)} * \text{p.v.} \left(\frac{1}{\pi h t} \right) \right) * f \\ = \text{f.p.} \left(-\frac{1}{\pi h t^2} \right) * f \\ = g_Y * f \\ g_Y = \text{f.p.} \left(-\frac{1}{\pi h t^2} \right) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} V = V_F(F) \\ = V_X(X_F(F)) \\ = j\omega \cdot \left(\frac{\text{sgn}}{jh} \cdot F \right) \\ = \left(j\omega \cdot \frac{\text{sgn}}{jh} \right) \cdot F \\ = \frac{|\omega|}{h} \cdot F \\ = G_Y \cdot F \\ G_Y = \frac{|\omega|}{h} \end{array} \right. \quad (96)$$

with [Kanwal, 2004]

$$\mathcal{F} \left(\text{f.p.} \left(\frac{1}{t^2} \right) \right) = -\pi |\omega|$$

We observe that Equation (96) is well-defined (in the sense that the operations in it are legitimate) if f is a distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if F is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, v is a square integrable distribution on $\mathbb{R}$ and, in fact, a $p_{(1,+\infty]}$ -integrable distribution on $\mathbb{R}$ for each $p_{(1,+\infty]}$ in $\overline{\mathbb{R}}_{(1,+\infty]}$ (and V is a tempered distribution on $\mathbb{R}$ that is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by a square integrable function on $\mathbb{R}$). Furthermore, we observe that the series connection of the inverse simple system associated with the linear hysteresis damper (upstream) and the 1st-order differentiator (downstream) admits δ as an input variable and is commutative.

It can be easily verified that the convolution systems defining the mechanical impedance and the mobility of the linear hysteresis damper have the property of acausality and activity. Furthermore, such convolution systems may not be mutually inverse.

3.6.3 Apparent mass and accelerance

The time-domain and frequency-domain apparent mass of the linear hysteresis damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the 2nd-order antidifferentiator (upstream) and the direct simple system associated with the linear hysteresis damper (downstream). We have

$$\left\{ \begin{array}{l} f = f_a(a) \\ \\ = f_x(x_a(a)) \\ \\ = \text{p.v.} \left(-\frac{h}{\pi t} \right) * (\rho * a) \\ \\ = \left(\delta^{(2)} * \left(-\frac{h}{\pi} (t \ln(|t|) - t + Dt + E1_{\mathbb{R}}) \right) \right) * (\rho * a) \\ \\ = \left(-\frac{h}{\pi} (t \ln(|t|) + (D-1)t + E1_{\mathbb{R}}) \right) * ((\delta^{(2)} * \rho) * a) \\ \\ = -\frac{ht \ln(|t|)}{\pi} * a \\ \\ = g_M * a \\ \\ g_M = -\frac{ht \ln(|t|)}{\pi} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} F = F_A(A) \\ \\ = F_X(X_A(A)) \\ \\ = jh \text{sgn} \cdot \left(\left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi \delta^{(1)} \right) \cdot A \right) \\ \\ = \left(-\omega^2 \cdot \left(-\frac{h}{\pi} \left(j\pi \text{Pf} \left(\frac{1}{\omega|\omega|} \right) + j2\pi(D-\gamma)\delta^{(1)} + 2\pi E\delta \right) \right) \right) \cdot \left(\left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi \delta^{(1)} \right) \cdot A \right) \\ \\ = \left(-\frac{h}{\pi} \left(j\pi \text{Pf} \left(\frac{1}{\omega|\omega|} \right) + j2\pi(D-\gamma)\delta^{(1)} + 2\pi E\delta \right) \right) \cdot \left(\left(-\omega^2 \cdot \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi \delta^{(1)} \right) \right) \cdot A \right) \\ \\ = \text{Pf} \left(\frac{jh}{-\omega|\omega|} \right) \cdot A \\ \\ = G_M \cdot A \\ \\ G_M = \text{Pf} \left(\frac{jh}{-\omega|\omega|} \right) \end{array} \right. \quad (97)$$

where

- D is an arbitrary complex constant of the same quantity dimension as t^{-1}
- E is an arbitrary complex constant of quantity dimension one

The expression of $\mathcal{F}(t \cdot \ln(|t|))$ can be obtained by recalling well-known Fourier-transform pairs [Kanwal, 2004] and by applying the multiplication Fourier exchange theorem in its standard form. We have

$$\begin{aligned}\mathcal{F}(t \cdot \ln(|t|)) &= \frac{1}{2\pi} \left(\mathcal{F}(t) * \mathcal{F}(\ln(|t|)) \right) \\ &= \frac{1}{2\pi} \left(j2\pi\delta^{(1)} * \left(-\pi \text{Pf} \left(\frac{1}{|\omega|} \right) - 2\pi\gamma\delta \right) \right) \\ &= j\pi \text{Pf} \left(\frac{1}{\omega|\omega|} \right) + j2\pi(1-\gamma)\delta^{(1)}\end{aligned}$$

We observe that Equation (97) is well-defined (in the sense that the operations in it are legitimate) if a is a right-sided distribution of rapid decay on $\mathbb{R}$ with null 0^{th} moment and 1^{st} moment (or, equivalently, if A is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$ with 0^{th} derivative and 1^{st} derivative vanishing at the origin); specifically, this ensures that x is a right-sided distribution of rapid decay on $\mathbb{R}$ (and X is a right-retarded distribution on $\mathbb{R}$ that is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, f is a square integrable distribution on $\mathbb{R}$ and, in fact, a $p_{(1,+\infty]}$ -integrable distribution on $\mathbb{R}$ for each $p_{(1,+\infty]}$ in $\overline{\mathbb{R}}_{(1,+\infty]}$ (and F is a tempered distribution on $\mathbb{R}$ that is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by a square integrable function on $\mathbb{R}$). Furthermore, we observe that the series connection of the 2^{nd} -order antidifferentiator (upstream) and the direct simple system associated with the linear hysteresis damper (downstream) does not admit δ as an input variable and may not be commutative. It is worth noting that, as consequence of $A(0)$ and $A'(0)$ being null, γ , D , and E are immaterial and hence dropped in Equation (97) (the terms in which they appear being cancelled by the operations of multiplication or convolution involving them), which explains why g_M and G_M , as expressed by Equation (97), do not form a Fourier-transform pair (as they would do if γ , D , and E were maintained). It is also worth noting that, as consequence of $A(0)$ and $A'(0)$ being null, F is a regular tempered distribution on $\mathbb{R}$ that is

generable by the function defined almost everywhere on $\mathbb{R}$ by $j\hbar A/(-\omega|\omega|)$ (but G_M is in any case a singular tempered distribution on $\mathbb{R}$ and, in fact, a singular integrable distribution on $\mathbb{R}$).

The time-domain and frequency-domain accelerance of the linear hysteresis damper are respectively the convolution kernel distribution and the frequency-response distribution of the series-equivalent system of the inverse simple system associated with the linear hysteresis damper (upstream) and the 2nd-order differentiator (downstream). We have

$$\left\{ \begin{array}{l} a = a_f(f) \\ = a_x(x_f(f)) \\ = \delta^{(2)} * \left(\text{p.v.} \left(\frac{1}{\pi \hbar t} \right) * f \right) \\ = \left(\delta^{(2)} * \text{p.v.} \left(\frac{1}{\pi \hbar t} \right) \right) * f \\ = \text{f.p.} \left(\frac{2}{\pi \hbar t^3} \right) * f \\ = g_A * f \\ g_A = \text{f.p.} \left(\frac{2}{\pi \hbar t^3} \right) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} A = A_F(F) \\ = A_X(X_F(F)) \\ = -\omega^2 \cdot \left(\frac{\text{sgn}}{j\hbar} \cdot F \right) \\ = \left(-\omega^2 \cdot \frac{\text{sgn}}{j\hbar} \right) \cdot F \\ = \frac{-\omega|\omega|}{j\hbar} \cdot F \\ = G_A \cdot F \\ G_A = \frac{-\omega|\omega|}{j\hbar} \end{array} \right. \quad (98)$$

with [Kanwal, 2004]

$$\mathcal{F} \left(\text{f.p.} \left(\frac{1}{t^3} \right) \right) = j \frac{\pi}{2} \omega |\omega|$$

We observe that Equation (98) is well-defined (in the sense that the operations in it are legitimate) if f is a distribution of rapid decay on $\mathbb{R}$ (or, equivalently, if F is generable by a smooth function of slow growth on $\mathbb{R}$). In these conditions, a is a square integrable distribution on $\mathbb{R}$ and, in fact, a $p_{(1,+\infty]}$ -integrable distribution on $\mathbb{R}$ for each $p_{(1,+\infty]}$ in $\overline{\mathbb{R}}_{(1,+\infty]}$ (and A is a tempered distribution on $\mathbb{R}$ that is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by a square integrable function on $\mathbb{R}$). Furthermore, we observe that the series connection of the inverse simple system associated with the linear hysteresis damper (upstream) and the 1st-order differentiator (downstream) admits δ as an input variable and is commutative.

3 Second theoretical study: Distributional characterisation of two-terminal mechanical elements

It can be easily verified that the convolution systems defining the apparent mass and the acceleration of the linear hysteresis damper have the property of acausality. Furthermore, such convolution systems may not be mutually inverse.

3.7 Conclusion

This second theoretical study allows obtaining a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements; these are the linear spring, the linear viscous damper, the linear inerter (including the rigid mass), and the linear hysteresis damper. In particular, a proper characterisation can be in terms of dynamic stiffness, mechanical impedance, and apparent mass as well as in terms of dynamic compliance, mobility, and accelerance (and their time-domain homologues). Well-known physical aspects are omitted to focus on unclear mathematical aspects. The obtained characterisation results from applying the findings of the first theoretical study, and, in the frequency domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null frequency). The obtained characterisation holds under mildly restrictive conditions (except in the ill-behaved case of the linear hysteresis damper) and contains specification of the input spaces. However, no attempt is made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions are identified for formulations to be mathematically tenable.

Dynamic stiffness, mechanical impedance, and apparent mass as well as dynamic compliance, mobility, and accelerance (and their time-domain homologues) of linear spring, linear viscous damper, linear inerter (including rigid mass), and linear hysteresis damper are listed in **Table 1**. The data listed in **Table 1** complement and correct those available elsewhere (for example, the data presented in tabular form by Griffin [1990, p. 361]).

3 Second theoretical study: Distributional characterisation of two-terminal mechanical elements

Table 1 Dynamic stiffness, mechanical impedance, and apparent mass as well as dynamic compliance, mobility, and accelerance (and their time-domain homologues) of linear spring, linear viscous damper, linear inerter (including rigid mass), and linear hysteresis damper.

	Linear spring		Linear viscous damper		Linear inerter (including rigid mass)		Linear hysteresis damper	
	Time domain	Frequency domain	Time domain	Frequency domain	Time domain	Frequency domain	Time domain	Frequency domain
Dynamic stiffness	$k\delta$	$k1_{\mathbb{R}}$	$c\delta^{(1)}$	$j\omega c$	$b\delta^{(2)}$	$-b\omega^2$	$\text{p.v.}\left(-\frac{h}{\pi t}\right)$	$j h \text{sgn}$
Dynamic compliance	$\frac{\delta}{k}$	$\frac{1_{\mathbb{R}}}{k}$	$\frac{\vartheta}{c}$	$\text{p.v.}\left(\frac{1}{j\omega c}\right) + \frac{\pi\delta}{c}$	$\frac{\rho}{b}$	$\text{f.p.}\left(\frac{1}{-b\omega^2}\right) + \frac{j\pi\delta^{(1)}}{b}$	$\text{p.v.}\left(\frac{1}{\pi h t}\right)$	$\frac{\text{sgn}}{j h}$
Mechanical impedance	$k\vartheta$	$\text{p.v.}\left(\frac{k}{j\omega}\right) + \pi k\delta$	$c\delta$	$c1_{\mathbb{R}}$	$b\delta^{(1)}$	$j b \omega$	$-\frac{h \ln(t)}{\pi}$	$\text{Pf}\left(\frac{h}{ \omega }\right)$
Mobility	$\frac{\delta^{(1)}}{k}$	$\frac{j\omega}{k}$	$\frac{\delta}{c}$	$\frac{1_{\mathbb{R}}}{c}$	$\frac{\vartheta}{b}$	$\text{p.v.}\left(\frac{1}{j b \omega}\right) + \frac{\pi\delta}{b}$	$\text{f.p.}\left(-\frac{1}{\pi h t^2}\right)$	$\frac{ \omega }{h}$
Apparent mass	$k\rho$	$\text{f.p.}\left(\frac{k}{-\omega^2}\right) + j\pi k\delta^{(1)}$	$c\vartheta$	$\text{p.v.}\left(\frac{c}{j\omega}\right) + \pi c\delta$	$b\delta$	$b1_{\mathbb{R}}$	$-\frac{h t \ln(t)}{\pi}$	$\text{Pf}\left(\frac{j h}{-\omega \omega }\right)$
Accelerance	$\frac{\delta^{(2)}}{k}$	$\frac{-\omega^2}{k}$	$\frac{\delta^{(1)}}{c}$	$\frac{j\omega}{c}$	$\frac{\delta}{b}$	$\frac{1_{\mathbb{R}}}{b}$	$\text{f.p.}\left(\frac{2}{\pi h t^3}\right)$	$\frac{-\omega \omega }{j h}$
k = stiffness of the linear spring; c = viscous damping coefficient of the linear viscous damper; b (in particular, m) = inertance of the linear inerter (in particular, mass of the rigid mass); h = hysteresis damping coefficient of the linear hysteresis damper; j = imaginary unit; t = time; ω = angular frequency; $1_{\mathbb{R}}$ = unit distribution on $\mathbb{R}$; ϑ = unit step distribution on $\mathbb{R}$; ρ = unit ramp distribution on $\mathbb{R}$; δ = unit impulse on $\mathbb{R}$; $\delta^{(1)}$ = unit doublet on $\mathbb{R}$; $\delta^{(2)}$ = unit triplet on $\mathbb{R}$; f_{L^0} = non-locally-integrable function on $\mathbb{R}$ admitting regularisation; h_{L^0} = non-locally-integrable function on $\mathbb{R}$ admitting canonical regularisation; $\text{Pf}(f_{L^0})$ = pseudofunction on $\mathbb{R}$ generated by f_{L^0} via the Hadamard-finite-part method; $\text{f.p.}(h_{L^0})$ (when applicable, $\text{p.v.}(h_{L^0})$) = canonical pseudofunction on $\mathbb{R}$ generated by h_{L^0} via the Hadamard-finite-part method (when applicable, via the Cauchy-principal-value method).								

4 Third theoretical study: Distributional characterisation of linear viscoelastic materials

4.1 Introduction

As expounded in Section 1.3.2, a concise but logically self-contained and mathematically rational characterisation, both in the time domain and in the frequency domain, of a *linear viscoelastic material* (shorthand for *non-ageing homogeneous isotropic linear viscoelastic material under isothermal adiabatic conditions*) should be non-heuristic but application-oriented. The required characterisation need hold under mildly restrictive conditions and contain specification of the input spaces. To date, such a characterisation has not been available in the literature. Filling this gap would allow linear viscoelastic materials to be used in finite-element modelling on a more solid mathematical basis, which could expand the possibilities for analytical manipulation, improve theoretical understanding, and grant both generality and economy of thought. It is worth noting that replacing the frequency domain with the complex frequency domain (or, equivalently, renouncing the Fourier transformation in favour of the Laplace transformation) would result in a much simpler treatment. However, albeit mathematically well-behaved, quantities ‘living’ in the complex frequency domain are opaque when it comes to physical interpretation; conversely, albeit mathematically ill-behaved, quantities ‘living’ in the frequency domain are transparent when it comes to physical interpretation.

In light of the above, a *third theoretical study* was conducted to obtain a non-heuristic but application-oriented characterisation of a linear viscoelastic material both in the time domain and in the frequency domain. Well-known physical aspects were omitted to focus on unclear mathematical aspects. No attempt was made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions were sought for formulations to be mathematically tenable.

In the following, linear viscoelastic materials are discussed in the time domain and in the frequency domain, and then a generalised Maxwell material is considered as a special case. The material in Chapter 2 and Chapter 3 is assumed known. For theoretical background and generalisations, reference is made to Appendix A.

4.2 Time domain

Relations hold

$$\begin{aligned} \rho^{(2)} = \mathfrak{G}^{(1)} & & \rho^{(1)} = \mathfrak{G}^{(0)} & & \rho \equiv \rho^{(0)} \\ = \delta^{(0)} & \Leftrightarrow & \equiv \mathfrak{G} & \Leftrightarrow & = \mathfrak{G}^{(-1)} \\ \equiv \delta & & = \delta^{(-1)} & & = \delta^{(-2)} \end{aligned}$$

We assume that a linear viscoelastic material is a *Cauchy continuum* (also known as *nonpolar continuum*), as opposed to a *Cosserat continuum* (also known as *polar continuum*), subjected to *infinitesimal displacements* and *infinitesimal displacement gradients*. Let $\{x_1, x_2, x_3\}$ be a right-handed Cartesian coordinate system fixed within the *observational frame of reference* (hereinafter *coordinate system*). By adopting the *tensor index notation of Ricci calculus* (including in particular the *Einstein summation convention for repeated indices*), the constitutive relations at a fixed point of a linear viscoelastic material are

$$\begin{cases} \sigma_{pq,S} = 3\mathcal{K}^{(1)} * \varepsilon_{pq,S} \\ \sigma_{pq,D} = 2\mathcal{S}^{(1)} * \varepsilon_{pq,D} \end{cases} \quad (99)$$

with

$$\begin{cases} \sigma_{pq} = \sigma_{pq,S} + \sigma_{pq,D} \\ = \underbrace{\frac{1}{3}\sigma_{rr}\delta_{pq}}_{\sigma_{pq,S}} + \underbrace{\sigma_{pq} - \frac{1}{3}\sigma_{rr}\delta_{pq}}_{\sigma_{pq,D}} \\ \varepsilon_{pq} = \varepsilon_{pq,S} + \varepsilon_{pq,D} \\ = \underbrace{\frac{1}{3}\varepsilon_{rr}\delta_{pq}}_{\varepsilon_{pq,S}} + \underbrace{\varepsilon_{pq} - \frac{1}{3}\varepsilon_{rr}\delta_{pq}}_{\varepsilon_{pq,D}} \end{cases}$$

where

- p is a free index taking values in the set of the labels of the axes of the coordinate system
- q is a free index taking values in the set of the labels of the axes of the coordinate system
- r is a bound index taking values in the set of the labels of the axes of the coordinate system
- δ_{pq} is the (p, q) -component of the Kronecker tensor with

$$\delta_{pq} = \begin{cases} 1 & , \quad p = q \\ 0 & , \quad p \neq q \end{cases}$$

4 Third theoretical study: Distributional characterisation of linear viscoelastic materials

- σ_{pq} is the (p, q) -component of the stress tensor in the coordinate system
- σ_{rr} is the trace of the stress tensor (also known as first principal invariant of the stress tensor)
- $\sigma_{pq,S}$ is the (p, q) -component of the spherical stress tensor in the coordinate system
- $\sigma_{pq,D}$ is the (p, q) -component of the deviatoric stress tensor in the coordinate system
- ε_{pq} is the (p, q) -component of the infinitesimal strain tensor in the coordinate system
- ε_{rr} is the trace of the infinitesimal strain tensor (also known as first principal invariant of the infinitesimal strain tensor)
- $\varepsilon_{pq,S}$ is the (p, q) -component of the spherical infinitesimal strain tensor in the coordinate system
- $\varepsilon_{pq,D}$ is the (p, q) -component of the deviatoric infinitesimal strain tensor in the coordinate system
- $\mathcal{K}^{(1)}$ is the bulk relaxation kernel
- $\mathcal{G}^{(1)}$ is the shear relaxation kernel

Remarkably, the first and the second of Equation (99) are independent. Equation (99) is well-defined (in the sense that the operations in it are legitimate) if any of the following independent conditions is satisfied:

- Condition A.** $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ are right-sided (in particular, causal) distributions on $\mathbb{R}$ other than the null distribution on $\mathbb{R}$, whereas $\varepsilon_{pq,S}$ and $\varepsilon_{pq,D}$ (and hence $\sigma_{pq,S}$ and $\sigma_{pq,D}$) are right-sided (in particular, causal) distributions on $\mathbb{R}$.
- Condition B.** $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ are integrable distributions on $\mathbb{R}$ other than the null distribution on $\mathbb{R}$, whereas $\varepsilon_{pq,S}$ and $\varepsilon_{pq,D}$ (and hence $\sigma_{pq,S}$ and $\sigma_{pq,D}$) are bounded distributions on $\mathbb{R}$.

To ensure the legitimacy of the mathematical manipulations hereinafter, we assume that any of **Condition A** and **Condition B** is satisfied. If **Condition A** is satisfied, each of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ has one (and necessarily only one) 1st antiderivative that is a right-sided distribution on $\mathbb{R}$ (necessarily other than the null distribution on $\mathbb{R}$). If **Condition B** is satisfied and **Condition A** is not satisfied, we assume that each of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ has one (and necessarily only one) 1st antiderivative that is an integrable distribution on $\mathbb{R}$ (necessarily other than the null distribution on $\mathbb{R}$); such assumption implies that the integrals on $\mathbb{R}$ of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ are null. As seen below, $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ need be

right-sided (in fact, causal) distributions on $\mathbb{R}$ regardless of which of **Condition A** and **Condition B** is satisfied; accordingly, if **Condition B** is satisfied and **Condition A** is not satisfied, the unique 1st antiderivative of each of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ that is an integrable distribution on $\mathbb{R}$ coincides with the unique 1st antiderivative of each of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ that is a right-sided distribution on $\mathbb{R}$. Hereinafter, we denote by $\mathcal{K}$ and $\mathcal{G}$ the right-sided 1st antiderivatives of respectively $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$. Under the assumptions made, Equation (99) can be re-expressed as

$$\left\{ \begin{array}{l} \sigma_{pq,S} = 3\mathcal{K}^{(1)} * \varepsilon_{pq,S} \\ \quad = 3\mathcal{K} * \varepsilon_{pq,S}^{(1)} \\ \sigma_{pq,D} = 2\mathcal{G}^{(1)} * \varepsilon_{pq,D} \\ \quad = 2\mathcal{G} * \varepsilon_{pq,D}^{(1)} \end{array} \right. \quad (100)$$

with

$$\left\{ \begin{array}{l} \sigma_{pq}^{(1)} = \sigma_{pq,S}^{(1)} + \sigma_{pq,D}^{(1)} \\ \quad = \underbrace{\frac{1}{3}\sigma_{rr}^{(1)}\delta_{pq}}_{\sigma_{pq,S}^{(1)}} + \underbrace{\sigma_{pq}^{(1)} - \frac{1}{3}\sigma_{rr}^{(1)}\delta_{pq}}_{\sigma_{pq,D}^{(1)}} \\ \varepsilon_{pq}^{(1)} = \varepsilon_{pq,S}^{(1)} + \varepsilon_{pq,D}^{(1)} \\ \quad = \underbrace{\frac{1}{3}\varepsilon_{rr}^{(1)}\delta_{pq}}_{\varepsilon_{pq,S}^{(1)}} + \underbrace{\varepsilon_{pq}^{(1)} - \frac{1}{3}\varepsilon_{rr}^{(1)}\delta_{pq}}_{\varepsilon_{pq,D}^{(1)}} \end{array} \right.$$

where

- $\varepsilon_{pq}^{(1)}$ is the (p, q) -component of the infinitesimal strain-rate tensor in the coordinate system
- $\varepsilon_{rr}^{(1)}$ is the trace of the infinitesimal strain-rate tensor (also known as first principal invariant of the infinitesimal strain-rate tensor)
- $\varepsilon_{pq,S}^{(1)}$ is the (p, q) -component of the spherical infinitesimal strain-rate tensor in the coordinate system
- $\varepsilon_{pq,D}^{(1)}$ is the (p, q) -component of the deviatoric infinitesimal strain-rate tensor in the coordinate system
- $\mathcal{K}$ is the bulk relaxation modulus
- $\mathcal{G}$ is the shear relaxation modulus

It is worth noting that only in the most basic cases can Equation (99) and Equation (100) be re-expressed as differential equations; indeed, in situations that are more general, it is necessary to resort to (possibly fractional) integro-differential equations.

Relations hold

$$\left\{ \begin{array}{l} \mathcal{K}^{(1)} = \delta^{(1)} * \mathcal{K} \\ \quad = \delta^{(2)} * \mathcal{K}^{(-1)} \\ \mathcal{G}^{(1)} = \delta^{(1)} * \mathcal{G} \\ \quad = \delta^{(2)} * \mathcal{G}^{(-1)} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \mathfrak{G} * \mathcal{K}^{(1)} = \mathcal{K} \\ \quad = \delta^{(1)} * \mathcal{K}^{(-1)} \\ \mathfrak{G} * \mathcal{G}^{(1)} = \mathcal{G} \\ \quad = \delta^{(1)} * \mathcal{G}^{(-1)} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \rho * \mathcal{K}^{(1)} = \mathfrak{G} * \mathcal{K} \\ \quad = \mathcal{K}^{(-1)} \\ \rho * \mathcal{G}^{(1)} = \mathfrak{G} * \mathcal{G} \\ \quad = \mathcal{G}^{(-1)} \end{array} \right\} \quad (101)$$

Remarkably, all the distributions in Equation (101) are right-sided (in fact, causal) distributions on $\mathbb{R}$ (necessarily other than the null distribution on $\mathbb{R}$). Indeed, $\mathcal{K}$ is the only one that is right-sided (in fact, causal) amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of $\mathcal{K}^{(1)}$, and $\mathcal{G}$ is the only one that is right-sided (in fact, causal) amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of $\mathcal{G}^{(1)}$. Furthermore, $\mathcal{K}^{(-1)}$ is the only one that is right-sided (in fact, causal) amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of $\mathcal{K}$ as well as amongst the infinitely many distributions on $\mathbb{R}$ that are 2nd antiderivatives of $\mathcal{K}^{(1)}$, and $\mathcal{G}^{(-1)}$ is the only one that is right-sided (in fact, causal) amongst the infinitely many distributions on $\mathbb{R}$ that are 1st antiderivatives of $\mathcal{G}$ as well as amongst the infinitely many distributions on $\mathbb{R}$ that are 2nd antiderivatives of $\mathcal{G}^{(1)}$.

We know that the property of passivity has a physical interpretation in terms of energy only for a *power-conjugate system*, which is a system whose input variable and output variable are *power-conjugate quantities* (namely, a system for which the multiplicative product of input variable and output variable has the quantity dimension of power). On the one hand, a linear viscoelastic material can be regarded as a set of two independent convolution systems of input variables respectively $\varepsilon_{pq,S}$ and $\varepsilon_{pq,D}$ and (forced) output variables respectively $\sigma_{pq,S}$ and $\sigma_{pq,D}$; such convolution systems are not power-conjugate convolution systems. On the other hand, a linear viscoelastic material can be regarded as a set of two independent convolution systems of input variables respectively $\varepsilon_{pq,S}^{(1)}$ and $\varepsilon_{pq,D}^{(1)}$ and (forced) output variables respectively $\sigma_{pq,S}$ and $\sigma_{pq,D}$; such convolution systems are power-conjugate convolution systems.

In accordance with Equation (100), $3\mathcal{K}^{(1)}$ and $2\mathcal{G}^{(1)}$ can be regarded as respectively $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}$ -unit-impulse excitation and $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}$ -unit-impulse excitation. Besides, $3\mathcal{K}$

and $2\mathcal{G}$ can be regarded as respectively $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}$ -unit-step excitation (or, equivalently, $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}^{(1)}$ -unit-impulse excitation) and $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}$ -unit-step excitation (or, equivalently, $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}^{(1)}$ -unit-impulse excitation). Furthermore, $3\mathcal{K}^{(-1)}$ and $2\mathcal{G}^{(-1)}$ can be regarded as respectively $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}$ -unit-ramp excitation (or, equivalently, $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}^{(1)}$ -unit-step excitation) and $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}$ -unit-ramp excitation (or, equivalently, $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}^{(1)}$ -unit-step excitation). Experimentally, $\mathcal{K}$ and $\mathcal{G}$ can be determined via respectively a *bulk relaxation test* and a *shear relaxation test*, whereas $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ can be determined via respectively a *bulk constant strain-rate test* and a *shear constant strain-rate test*.

A linear viscoelastic material can be defined by specifying $\mathcal{K}$ and $\mathcal{G}$ (or, equivalently, $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ or, equivalently, $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$). However, even leaving aside the restrictions imposed by **Condition A** or **Condition B**, $\mathcal{K}$ and $\mathcal{G}$ (and hence $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ as well as $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$) cannot be arbitrary distributions on $\mathbb{R}$. In fact, for a linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems to have the indispensable property of *passivity* (and hence of *causality*), it is necessary and sufficient that each of $\mathcal{K}$ and $\mathcal{G}$ is a causal bounded distribution on $\mathbb{R}$ that can be expressed as the sum of a non-negative real constant of the same quantity dimension as $t^2\mathcal{K}$ and $t^2\mathcal{G}$ times $\delta^{(1)}$ and a weakly non-negative-definite causal distribution of C -order in $\mathbb{Z}_{[-\infty,0]}$ on $\mathbb{R}$ (a weakly non-negative-definite distribution on $\mathbb{R}$ being a distribution on $\mathbb{R}$ whose conjugate-symmetric part is non-negative-definite). It is worth noting that a distribution of C -order in $\mathbb{Z}_{[-\infty,0]}$ on $\mathbb{R}$ need be a distribution of order in $\mathbb{N}_{[0,2]}$ on $\mathbb{R}$; in particular, for each n_0 in $\mathbb{N}$, $\delta^{(n_0)}$ is a distribution of order n_0 and C -order n_0 on $\mathbb{R}$.

Experimental evidence suggests that a linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems need be such that output variables depend more on the near past than on the distant past of input variables. This requirement is satisfied if the linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems has the property of *fading memory*, which means that each of $\mathcal{K}$ and $\mathcal{G}$ is a (possibly singular) distribution of order 0 on $\mathbb{R}$ the localisation of which at $\mathbb{R}_{(0,+\infty)}$ results in a regular distribution on $\mathbb{R}_{(0,+\infty)}$ that is generable by a smooth function on $\mathbb{R}_{(0,+\infty)}$ whose derivatives of any order in $\mathbb{N}$ are either non-negative (for even order) or non-positive (for odd order); such a smooth function on $\mathbb{R}_{(0,+\infty)}$ is said to

be a *completely monotonic function* (also known as *completely monotone function*) on $\mathbb{R}_{(0,+\infty)}$. If the linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems has the property of *fading memory*, for each n_0 in $\mathbb{N}$, relations hold [Widder, 1941, 1971; Berg and Forst, 1975; Gripenberg *et al.*, 1990; Miller and Samko, 2001; Schilling *et al.*, 2012; Merkle, 2014]

$$\begin{cases} (-1)^{n_0} \mathcal{K}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)}(t) \in \mathbb{R}_{[0,+\infty)} \\ (-1)^{n_0} \mathcal{G}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)}(t) \in \mathbb{R}_{[0,+\infty)} \end{cases}, \quad t \in \mathbb{R}_{(0,+\infty)}$$

with

$$\begin{cases} \lim_{t \rightarrow 0^+} \left((-1)^{n_0} \mathcal{K}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \in \overline{\mathbb{R}}_{[0,+\infty)} \\ \lim_{t \rightarrow 0^+} \left((-1)^{n_0} \mathcal{G}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \in \overline{\mathbb{R}}_{[0,+\infty)} \end{cases}$$

$$\begin{cases} \lim_{t \rightarrow +\infty} \left((-1)^{n_0} \mathcal{K}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \begin{cases} \in \mathbb{R}_{[0,+\infty)} & , \quad n_0 = 0 \\ = 0 & , \quad n_0 \in \mathbb{N}_{(0,+\infty)} \end{cases} \\ \lim_{t \rightarrow +\infty} \left((-1)^{n_0} \mathcal{G}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \begin{cases} \in \mathbb{R}_{[0,+\infty)} & , \quad n_0 = 0 \\ = 0 & , \quad n_0 \in \mathbb{N}_{(0,+\infty)} \end{cases} \end{cases}$$

where

- $\mathcal{K}_{\mathbb{R}_{(0,+\infty)}}$ is the regular distribution on $\mathbb{R}_{(0,+\infty)}$ resulting from the localisation of $\mathcal{K}$ at $\mathbb{R}_{(0,+\infty)}$
- $\mathcal{G}_{\mathbb{R}_{(0,+\infty)}}$ is the regular distribution on $\mathbb{R}_{(0,+\infty)}$ resulting from the localisation of $\mathcal{G}$ at $\mathbb{R}_{(0,+\infty)}$

A derivative of order in $\mathbb{N}$ (in particular, of order 0) of a completely monotonic function on $\mathbb{R}_{(0,+\infty)}$ vanishes at some point of $\mathbb{R}_{(0,+\infty)}$ if and only if it coincides with a constant function on $\mathbb{R}_{(0,+\infty)}$ (in particular, with the null function on $\mathbb{R}_{(0,+\infty)}$). In fact, if the linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems has the property of fading memory and if $\mathcal{K}_{\mathbb{R}_{(0,+\infty)}}$ and $\mathcal{G}_{\mathbb{R}_{(0,+\infty)}}$ are not constant distributions on $\mathbb{R}_{(0,+\infty)}$, each of $\mathcal{K}$ and $\mathcal{G}$ is a (possibly singular) distribution of order 0 on $\mathbb{R}$ the localisation of which at $\mathbb{R}_{(0,+\infty)}$ results in a regular distribution that is generable by a smooth function on $\mathbb{R}_{(0,+\infty)}$ whose derivatives of any order in $\mathbb{N}$ are either positive (for even order) or negative (for odd order); such a smooth function on $\mathbb{R}_{(0,+\infty)}$ is said to be a *strictly completely monotonic function* (also known as *strictly completely monotone function*) on $\mathbb{R}_{(0,+\infty)}$. If the linear viscoelastic material regarded as a pair of independent

power-conjugate convolution systems has the property of *fading memory*, for each n_0 in $\mathbb{N}$, relations hold [Widder, 1941, 1971; Berg and Forst, 1975; Gripenberg *et al.*, 1990; Miller and Samko, 2001; Schilling *et al.*, 2012; Merkle, 2014]

$$\left\{ \begin{array}{l} -\mathcal{K}_{\mathbb{R}_{(0,+\infty)}}^{(1)}(t) \in \mathbb{R}_{(0,+\infty)} \\ -\mathcal{G}_{\mathbb{R}_{(0,+\infty)}}^{(1)}(t) \in \mathbb{R}_{(0,+\infty)} \end{array} \right., \quad t \in \mathbb{R}_{(0,+\infty)} \Leftrightarrow \left\{ \begin{array}{l} (-1)^{n_0} \mathcal{K}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)}(t) \in \mathbb{R}_{(0,+\infty)} \\ (-1)^{n_0} \mathcal{G}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)}(t) \in \mathbb{R}_{(0,+\infty)} \end{array} \right., \quad t \in \mathbb{R}_{(0,+\infty)}$$

with

$$\left\{ \begin{array}{l} \lim_{t \rightarrow 0^+} \left((-1)^{n_0} \mathcal{K}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \in \overline{\mathbb{R}}_{(0,+\infty)} \\ \lim_{t \rightarrow 0^+} \left((-1)^{n_0} \mathcal{G}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \in \overline{\mathbb{R}}_{(0,+\infty)} \end{array} \right.$$

$$\left\{ \begin{array}{l} \lim_{t \rightarrow +\infty} \left((-1)^{n_0} \mathcal{K}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \left\{ \begin{array}{l} \in \mathbb{R}_{[0,+\infty)} \quad , \quad n_0 = 0 \\ = 0 \quad \quad \quad , \quad n_0 \in \mathbb{N}_{(0,+\infty)} \end{array} \right. \\ \lim_{t \rightarrow +\infty} \left((-1)^{n_0} \mathcal{G}_{\mathbb{R}_{(0,+\infty)}}^{(n_0)} \right) \left\{ \begin{array}{l} \in \mathbb{R}_{[0,+\infty)} \quad , \quad n_0 = 0 \\ = 0 \quad \quad \quad , \quad n_0 \in \mathbb{N}_{(0,+\infty)} \end{array} \right. \end{array} \right.$$

Unlike a finite conical combination, a finite linear combination of completely monotonic functions on $\mathbb{R}_{(0,+\infty)}$ (in particular, of strictly completely monotonic functions on $\mathbb{R}_{(0,+\infty)}$) need not be a completely monotonic function on $\mathbb{R}_{(0,+\infty)}$ (in particular, a strictly completely monotonic function on $\mathbb{R}_{(0,+\infty)}$); hence, the function space of the completely monotonic functions on $\mathbb{R}_{(0,+\infty)}$ (in particular, of the strictly completely monotonic functions on $\mathbb{R}_{(0,+\infty)}$) cannot be equipped with the algebraic structure of linear space over either $\mathbb{C}$ or $\mathbb{R}$.

Whilst the property of fading memory implies the property of passivity, the property of passivity does not imply the property of fading memory. For example, a linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems would have the property of passivity but not the property of fading memory if any of $\mathcal{K}$ and $\mathcal{G}$ coincided with the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by a positive real constant of the same quantity dimension as $\mathcal{K}$ and $\mathcal{G}$ times $\cos(\omega_0 t) \mathfrak{I}(t)$.

Hereinafter, a linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems has the property of fading memory. The fact that each of $\mathcal{K}$ and $\mathcal{G}$ need be a distribution of order 0 on $\mathbb{R}$ implies that it may include a term proportional to δ but not a term

proportional to $\delta^{(n_1)}$ with n_1 in $\mathbb{N}_{[1,+\infty)}$ and that, up to addition by a term proportional to δ , each of $\mathcal{H}$ and $\mathcal{G}$ need be a regular distribution on $\mathbb{R}$; in any case, $\mathcal{H}$ or $\mathcal{G}$ including a term proportional to $\delta^{(n_2)}$ with n_2 in $\mathbb{N}_{[2,+\infty)}$ would even be incompatible with the aforementioned necessary and sufficient conditions for a linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems to have the property of passivity. The fact that each of $\mathcal{H}$ and $\mathcal{G}$ need be a distribution of order 0 on $\mathbb{R}$ also implies that it cannot be such that any of $\mathcal{H}_{\mathbb{R}_{(0,+\infty)}}$ and $\mathcal{G}_{\mathbb{R}_{(0,+\infty)}}$ coincides with the distribution on $\mathbb{R}_{(0,+\infty)}$ generated by the function defined almost everywhere on $\mathbb{R}_{(0,+\infty)}$ by a positive real constant of the same quantity dimension as $t^{n_1}\mathcal{H}$ and $t^{n_1}\mathcal{G}$ times t^{-n_1} with n_1 in $\mathbb{N}_{[1,+\infty)}$ (although such distribution on $\mathbb{R}_{(0,+\infty)}$ is generable by a completely monotonic function on $\mathbb{R}_{(0,+\infty)}$); in any case, such a $\mathcal{H}$ or $\mathcal{G}$ would even be incompatible with the aforementioned necessary and sufficient conditions for a linear viscoelastic material regarded as a pair of independent power-conjugate convolution systems to have the property of passivity.

Under the assumptions made, according to the *Bernstein theorem* [Widder, 1941, 1971; Berg and Forst, 1975; Gripenberg *et al.*, 1990; Miller and Samko, 2001; Schilling *et al.*, 2012; Merkle, 2014], relations hold

$$\begin{cases} \mathcal{H} = C_{\mathcal{H}}\delta + \left(\left(\mathcal{L}(\mathcal{U}_{\mathcal{H}}) \right)_{\mathbb{C}_{\operatorname{Re}\in(0,+\infty)} \cap \mathbb{C}_{\operatorname{Im}=0}} \right) \mathfrak{g} \\ \mathcal{G} = C_{\mathcal{G}}\delta + \left(\left(\mathcal{L}(\mathcal{U}_{\mathcal{G}}) \right)_{\mathbb{C}_{\operatorname{Re}\in(0,+\infty)} \cap \mathbb{C}_{\operatorname{Im}=0}} \right) \mathfrak{g} \end{cases} \quad (102)$$

with

$$\begin{cases} (\mathcal{H} - C_{\mathcal{H}}\delta)_{\mathbb{R}_{(0,+\infty)}} = \left(\mathcal{L}(\mathcal{U}_{\mathcal{H}}) \right)_{\mathbb{C}_{\operatorname{Re}\in(0,+\infty)} \cap \mathbb{C}_{\operatorname{Im}=0}} \\ \quad \quad \quad = \mathcal{H}_{\mathbb{R}_{(0,+\infty)}} \\ (\mathcal{G} - C_{\mathcal{G}}\delta)_{\mathbb{R}_{(0,+\infty)}} = \left(\mathcal{L}(\mathcal{U}_{\mathcal{G}}) \right)_{\mathbb{C}_{\operatorname{Re}\in(0,+\infty)} \cap \mathbb{C}_{\operatorname{Im}=0}} \\ \quad \quad \quad = \mathcal{G}_{\mathbb{R}_{(0,+\infty)}} \end{cases}$$

where

- $C_{\mathcal{H}}$ is a non-negative real constant of the same quantity dimension as $t\mathcal{H}$
- $C_{\mathcal{G}}$ is a non-negative real constant of the same quantity dimension as $t\mathcal{G}$
- $\mathcal{L}$ is the *Laplace transformation*

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- $\mathcal{U}_{\mathcal{K}}$ is a non-negative causal distribution of order 0 on $\mathbb{R}$ associated uniquely with $\mathcal{K} - C_{\mathcal{K}}\delta$
- $\mathcal{U}_{\mathcal{G}}$ is a non-negative causal distribution of order 0 on $\mathbb{R}$ associated uniquely with $\mathcal{G} - C_{\mathcal{G}}\delta$
- $\left(\mathcal{L}(\mathcal{U}_{\mathcal{K}})\right)_{\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}}$ is the regular distribution on $\mathbb{R}_{(0, +\infty)}$ generated by the result of the localisation of $\mathcal{L}(\mathcal{U}_{\mathcal{K}})$ at $\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}$
- $\left(\mathcal{L}(\mathcal{U}_{\mathcal{G}})\right)_{\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}}$ is the regular distribution on $\mathbb{R}_{(0, +\infty)}$ generated by the result of the localisation of $\mathcal{L}(\mathcal{U}_{\mathcal{G}})$ at $\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}$
- $\left(\left(\mathcal{L}(\mathcal{U}_{\mathcal{K}})\right)_{\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}}\right)_{\mathfrak{G}}$ is the causal regular distribution on $\mathbb{R}$ generated by the result of the localisation of $\mathcal{L}(\mathcal{U}_{\mathcal{K}})$ at $\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}$ extended to a causal locally integrable function on $\mathbb{R}$
- $\left(\left(\mathcal{L}(\mathcal{U}_{\mathcal{G}})\right)_{\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}}\right)_{\mathfrak{G}}$ is the causal regular distribution on $\mathbb{R}$ generated by the result of the localisation of $\mathcal{L}(\mathcal{U}_{\mathcal{G}})$ at $\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im}=0}$ extended to a causal locally integrable function on $\mathbb{R}$
- $\left(\mathcal{K} - C_{\mathcal{K}}\delta\right)_{\mathbb{R}_{(0, +\infty)}}$ is the regular distribution on $\mathbb{R}_{(0, +\infty)}$ resulting from the localisation of $\mathcal{K} - C_{\mathcal{K}}\delta$ at $\mathbb{R}_{(0, +\infty)}$
- $\left(\mathcal{G} - C_{\mathcal{G}}\delta\right)_{\mathbb{R}_{(0, +\infty)}}$ is the regular distribution on $\mathbb{R}_{(0, +\infty)}$ resulting from the localisation of $\mathcal{G} - C_{\mathcal{G}}\delta$ at $\mathbb{R}_{(0, +\infty)}$

It is understood that the free real variable of $\mathcal{U}_{\mathcal{K}}$ and $\mathcal{U}_{\mathcal{G}}$ is ω . Like any distribution of order 0 on $\mathbb{R}$, each of $\mathcal{U}_{\mathcal{K}}$ and $\mathcal{U}_{\mathcal{G}}$ is generable by a Radon measure on $\mathbb{R}$. Remarkably, $\left(\mathcal{K} - C_{\mathcal{K}}\delta\right)_{\mathbb{R}_{(0, +\infty)}}$ or $\left(\mathcal{G} - C_{\mathcal{G}}\delta\right)_{\mathbb{R}_{(0, +\infty)}}$ is generable by a completely monotonic function on $\mathbb{R}_{(0, +\infty)}$ that can be extended to a causal locally integrable function on $\mathbb{R}$ if and only if respectively $\mathcal{U}_{\mathcal{K}}$ or $\mathcal{U}_{\mathcal{G}}$ is a non-negative causal (-1) -weighted integrable distribution of S -order 0 on $\mathbb{R}$ and has Laplace transform with region of definition including (in particular, coinciding with) $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (a (-1) -weighted integrable distribution of S -order 0 on $\mathbb{R}$ being a (-1) -weighted integrable distribution on $\mathbb{R}$ that is generable by a (-1) -weighted integrable Radon measure on $\mathbb{R}$). In particular, $\left(\mathcal{K} - C_{\mathcal{K}}\delta\right)_{\mathbb{R}_{(0, +\infty)}}$ or

$(\mathcal{G} - C_{\mathcal{G}}\delta)_{\mathbb{R}_{(0,+\infty)}}$ is generable by a completely monotonic function on $\mathbb{R}_{(0,+\infty)}$ that can be extended to a causal locally integrable function on $\mathbb{R}$ and whose limit as t tends to 0^+ exists and is in $\mathbb{R}_{[0,+\infty)}$ if and only if respectively $\mathcal{U}_{\mathcal{H}}$ or $\mathcal{U}_{\mathcal{G}}$ is a non-negative causal integrable distribution of S -order 0 on $\mathbb{R}$ and has Laplace transform with region of definition including (in particular, coinciding with) $\mathbb{C}_{\text{Re} \in (0,+\infty)}$ (an integrable distribution of S -order 0 on $\mathbb{R}$ being an integrable distribution on $\mathbb{R}$ that is generable by an integrable Radon measure on $\mathbb{R}$). Furthermore, $(\mathcal{H} - C_{\mathcal{H}}\delta)_{\mathbb{R}_{(0,+\infty)}}$ or $(\mathcal{G} - C_{\mathcal{G}}\delta)_{\mathbb{R}_{(0,+\infty)}}$ is generable by a completely monotonic function on $\mathbb{R}_{(0,+\infty)}$ that can be extended to a causal integrable function on $\mathbb{R}$ whose limit as t tends to 0^+ exists and is in $\mathbb{R}_{[0,+\infty)}$ only if respectively $\mathcal{U}_{\mathcal{H}}$ or $\mathcal{U}_{\mathcal{G}}$ is a non-negative causal integrable distribution of S -order 0 on $\mathbb{R}$, has Laplace transform with region of definition including (in particular, coinciding with) $\mathbb{C}_{\text{Re} \in (0,+\infty)}$, and does not include a term proportional to δ . In fact, Gripenberg *et al.* [1990, p. 143] state without proof that the latter result holds in the stronger form obtained by replacing the material conditional (namely, only if) with the logical biconditional (namely, if and only if); however, this seems not true. For example, the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by a positive real constant of the same quantity dimension as $t\mathcal{H}$ and $t\mathcal{G}$ times $e^{-\beta_0\omega}(\pi\beta_0|\omega|)^{-1/2}\mathfrak{Y}(\omega)$, β_0 being a positive real constant of the same quantity dimension as t , is a non-negative causal integrable distribution of S -order 0 on $\mathbb{R}$, has Laplace transform with region of definition $\mathbb{C}_{\text{Re} \in (-\beta_0,+\infty)}$, and does not include a term proportional to δ ; however, the result of the localisation of such Laplace transform at $\mathbb{C}_{\text{Re} \in (0,+\infty)} \cap \mathbb{C}_{\text{Im}=0}$ is the function defined almost everywhere on $\mathbb{R}_{(0,+\infty)}$ by a positive real constant of the same quantity dimension as $t\mathcal{H}$ and $t\mathcal{G}$ times $(\beta_0(\beta_0 + |t|))^{-1/2}$, which is a completely monotonic function on $\mathbb{R}_{(0,+\infty)}$ that can be extended to a causal locally integrable function on $\mathbb{R}$, but not to a causal integrable function on $\mathbb{R}$, whose limit as t tends to 0^+ exists and is in $\mathbb{R}_{[0,+\infty)}$.

For simplicity, we assume that $\mathcal{H}^{(1)}$, $\mathcal{H}$, and $\mathcal{H}^{(-1)}$ have the same initial behaviour (respectively, asymptotic behaviour) as respectively $\mathcal{G}^{(1)}$, $\mathcal{G}$, and $\mathcal{G}^{(-1)}$. On the one hand, this means that the limit as t tends to 0^+ of $\mathcal{H}^{(1)}$, $\mathcal{H}$, or $\mathcal{H}^{(-1)}$ is 0 (respectively, $+\infty$) if and only if so is the limit as t tends to 0^+ of respectively $\mathcal{G}^{(1)}$, $\mathcal{G}$, or $\mathcal{G}^{(-1)}$. On the other hand, this means that the limit as t tends to $+\infty$ of $\mathcal{H}^{(1)}$, $\mathcal{H}$, or $\mathcal{H}^{(-1)}$ is 0 (respectively, $+\infty$) if and only if so is the limit as t tends

to $+\infty$ of respectively $\mathcal{G}^{(1)}$, $\mathcal{G}$, or $\mathcal{G}^{(-1)}$. However, with minor modifications, such assumption can be waived.

Under the assumptions made, the limits of $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ as t tends to 0^+ exist and are in $\mathbb{R}_{(0,+\infty)}$, and the limits of $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ as t tends to $+\infty$ (or, equivalently, the integrals on $\mathbb{R}$ of $\mathcal{K}$ and $\mathcal{G}$) exist and are in $\overline{\mathbb{R}}_{(0,+\infty]}$. A linear viscoelastic material is qualified as *fluid* if the limits of $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ as t tends to $+\infty$ exist and are in $\mathbb{R}_{(0,+\infty)}$ (or, equivalently, if $\mathcal{K}$ and $\mathcal{G}$ are integrable distributions on $\mathbb{R}$); otherwise, it is qualified as *solid* (which implies that **Condition A** is satisfied, because a linear viscoelastic material need be a fluid for **Condition B** to be satisfied).

Under the assumptions made, the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to 0^+ exist and are in $\overline{\mathbb{R}}_{[0,+\infty]}$. In fact, the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to 0^+ can be 0 only in the limiting case of (necessarily fluid) linear viscous material (as introduced below). Besides, it seems plausible to conjecture that the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to 0^+ can be $+\infty$ only in the case of solid linear viscoelastic material; to settle the question once and for all, for a fluid linear viscoelastic material, we assume that the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to 0^+ exist and are in $\mathbb{R}_{[0,+\infty)}$. A linear viscoelastic material is said to have *elastic initial response* (also known as *elastic glass response* or, somehow unfortunately, *elastic instantaneous response* or *elastic impact response*) if $\mathcal{K}$ and $\mathcal{G}$ are regular distributions on $\mathbb{R}$ and the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to 0^+ exist are in $\mathbb{R}_{(0,+\infty)}$; otherwise, it is said to have *viscous initial response* (also known as *viscous glass response* or, somehow unfortunately, *viscous instantaneous response* or *viscous impact response*).

Under the assumptions made, the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to $+\infty$ exist and are in $\mathbb{R}_{[0,+\infty)}$; accordingly, for both a solid linear viscoelastic material and a fluid linear viscoelastic material, $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ are integrable distributions on $\mathbb{R}$. The limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to $+\infty$ can be in $\mathbb{R}_{(0,+\infty)}$ only in the case of solid linear viscoelastic material. A linear viscoelastic material is said to have *elastic asymptotic response* (also known as *elastic long-term response* or, somehow unfortunately, *elastic equilibrium response*, *elastic steady-state response*, or *elastic static response*) if the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to $+\infty$ exist and are in $\mathbb{R}_{(0,+\infty)}$; otherwise, it is said to have *viscous asymptotic response* (also known as *viscous long-term response* or, somehow unfortunately, *viscous equilibrium response*, *viscous steady-state response*, or *viscous static response*).

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Under the assumptions made, the limits of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ as t tends to 0^+ exist and are in $\overline{\mathbb{R}}_{[-\infty,0]}$, and the limits of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ as t tends to $+\infty$ exist and are 0. It seems plausible to conjecture that the limits of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ as t tends to 0^+ can be $-\infty$ only in the case of solid linear viscoelastic material; to settle the question once and for all, for a fluid linear viscoelastic material, we assume that the limits of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ as t tends to 0^+ exist and are in $\mathbb{R}_{(-\infty,0]}$.

For a solid linear viscoelastic material, $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ are causal integrable distributions of order in $\mathbb{N}_{[0,1]}$ on $\mathbb{R}$, $\mathcal{K}$ and $\mathcal{G}$ are causal bounded distributions of order 0 on $\mathbb{R}$, and $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ are causal regular tempered distributions on $\mathbb{R}$. For a fluid linear viscoelastic material, $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ are causal integrable distributions of order in $\mathbb{N}_{[0,1]}$ on $\mathbb{R}$ with null integrals on $\mathbb{R}$, $\mathcal{K}$ and $\mathcal{G}$ are causal integrable distributions of order 0 on $\mathbb{R}$, and $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ are causal regular bounded distributions on $\mathbb{R}$. A summary of initial behaviour and asymptotic behaviour of bulk relaxation kernel and shear relaxation kernel as well as of related quantities is provided in **Table 2**.

Table 2 Summary of initial behaviour and asymptotic behaviour of bulk relaxation kernel and shear relaxation kernel as well as of related quantities.

	Solid		Fluid	
	$\lim_{t \rightarrow 0^+}$	$\lim_{t \rightarrow +\infty}$	$\lim_{t \rightarrow 0^+}$	$\lim_{t \rightarrow +\infty}$
$\mathcal{K}^{(1)}, \mathcal{G}^{(1)}$	$\overline{\mathbb{R}}_{[-\infty,0]}$	0	$\mathbb{R}_{(-\infty,0]}$	0
$\mathcal{K}, \mathcal{G}$	$\overline{\mathbb{R}}_{(0,+\infty]}$	$\mathbb{R}_{[0,+\infty)}$	$\mathbb{R}_{[0,+\infty)}$	0
$\mathcal{K}^{(-1)}, \mathcal{G}^{(-1)}$	$\mathbb{R}_{[0,+\infty)}$	$+\infty$	$\mathbb{R}_{[0,+\infty)}$	$\mathbb{R}_{(0,+\infty)}$
t = time; $\mathcal{K}^{(1)}$ = bulk relaxation kernel; $\mathcal{G}^{(1)}$ = shear relaxation kernel; $\mathcal{K}$ = bulk relaxation modulus; $\mathcal{G}$ = shear relaxation modulus; $\mathcal{K}^{(-1)}$ = right-sided 1 st antiderivative of $\mathcal{K}$; $\mathcal{G}^{(-1)}$ = right-sided 1 st antiderivative of $\mathcal{G}$. For a fluid linear viscoelastic material, it is assumed that the limits of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ as t tends to 0^+ as well as the limits of $\mathcal{K}$ and $\mathcal{G}$ as t tends to 0^+ exist and are in $\mathbb{R}_{[0,+\infty)}$.				

On the one hand, a solid linear viscoelastic material can have either elastic initial response or viscous initial response, and it can have either elastic asymptotic response or viscous asymptotic response. On the other hand, a fluid linear viscoelastic material can have either elastic initial response or viscous initial response, but it need have viscous asymptotic response. Examples of bulk relaxation modulus and shear relaxation modulus for the possible types of linear viscoelastic material are provided in **Table 3**.

Table 3 Examples of bulk relaxation modulus and shear relaxation modulus for the possible types of linear viscoelastic material.

		Solid		Fluid	
		Asymptotic response		Asymptotic response	
		Elastic	Viscous	Elastic	Viscous
Initial response	Elastic	$\gamma_2 \mathfrak{G}$	$\gamma_4 \frac{1}{(1+ t /\beta_4)^{1/2}} \mathfrak{G}$	N/A	$\gamma_5 \frac{1}{(1+t/\beta_5)^2} \mathfrak{G}$
		$\gamma_2 \mathfrak{G}$ $+ \gamma_4 \frac{1}{(1+ t /\beta_4)^{1/2}} \mathfrak{G}$			
	Viscous	$\gamma_2 \mathfrak{G}$ $+ \gamma_3 \frac{1}{(t /\beta_3)^{1/2}} \mathfrak{G}$	$\gamma_3 \frac{1}{(t /\beta_3)^{1/2}} \mathfrak{G}$	N/A	$\gamma_1 \beta_1 \delta$
		$\gamma_1 \beta_1 \delta$ $+ \gamma_2 \mathfrak{G}$ $+ \gamma_4 \frac{1}{(1+ t /\beta_4)^{1/2}} \mathfrak{G}$	$\gamma_1 \beta_1 \delta$ $+ \gamma_4 \frac{1}{(1+ t /\beta_4)^{1/2}} \mathfrak{G}$		$\gamma_1 \beta_1 \delta$ $+ \gamma_5 \frac{1}{(1+t/\beta_5)^2} \mathfrak{G}$
t = time; $\mathcal{H}$ = bulk relaxation modulus; $\mathcal{G}$ = shear relaxation modulus; $\beta_1, \beta_2, \beta_3, \beta_4$ = positive real constants of the same quantity dimension as t ; $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ = positive real constants of the same quantity dimension as $\mathcal{H}$ and $\mathcal{G}$; $\mathfrak{G}$ = unit step distribution on $\mathbb{R}$; δ = unit impulse on $\mathbb{R}$. Although mathematically admissible, the listed examples need not be physically realisable.					

For any $\mathcal{H}$ and $\mathcal{G}$ having limits as t tends to 0^+ existing and in $\mathbb{R}_{[0,+\infty)}$, it is convenient to let

$$\begin{cases} \mathcal{H}_0 \equiv \lim_{t \rightarrow 0^+} (\mathcal{H}) \\ \mathcal{G}_0 \equiv \lim_{t \rightarrow 0^+} (\mathcal{G}) \end{cases}$$

where

- $\mathcal{H}_0$ is the *initial bulk relaxation modulus* (also known as *glass bulk relaxation modulus* or, somehow unfortunately, *instantaneous bulk relaxation modulus* or *impact bulk relaxation modulus*)
- $\mathcal{G}_0$ is the *initial shear relaxation modulus* (also known as *glass shear relaxation modulus* or, somehow unfortunately, *instantaneous shear relaxation modulus* or *impact shear relaxation modulus*)

For any $\mathcal{K}$ and $\mathcal{G}$, it is convenient to let

$$\begin{cases} \mathcal{K}_\infty \equiv \lim_{t \rightarrow +\infty} (\mathcal{K}) \\ \mathcal{G}_\infty \equiv \lim_{t \rightarrow +\infty} (\mathcal{G}) \end{cases}$$

where

- $\mathcal{K}_\infty$ is the *asymptotic bulk relaxation modulus* (also known as *long-term bulk relaxation modulus* or, somehow unfortunately, *equilibrium bulk relaxation modulus*, *steady-state bulk relaxation modulus*, or *static bulk relaxation modulus*)
- $\mathcal{G}_\infty$ is the *asymptotic shear relaxation modulus* (also known as *long-term shear relaxation modulus* or, somehow unfortunately, *equilibrium shear relaxation modulus*, *steady-state shear relaxation modulus*, or *static shear relaxation modulus*)

It is worth noting that $\mathcal{K}_\infty$ and $\mathcal{G}_\infty$ equal the integrals on $\mathbb{R}$ of respectively $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$.

For any $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$, it is convenient to let

$$\begin{cases} \mathcal{K}_0^{(-1)} \equiv \lim_{t \rightarrow 0^+} (\mathcal{K}^{(-1)}) \\ \mathcal{G}_0^{(-1)} \equiv \lim_{t \rightarrow 0^+} (\mathcal{G}^{(-1)}) \end{cases}$$

For any $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ having limits as t tends to $+\infty$ existing and in $\mathbb{R}_{[0,+\infty)}$, it is convenient to let

$$\begin{cases} \mathcal{K}_\infty^{(-1)} \equiv \lim_{t \rightarrow +\infty} (\mathcal{K}^{(-1)}) \\ \mathcal{G}_\infty^{(-1)} \equiv \lim_{t \rightarrow +\infty} (\mathcal{G}^{(-1)}) \end{cases}$$

It is worth noting that $\mathcal{K}_\infty^{(-1)}$ and $\mathcal{G}_\infty^{(-1)}$ equal the integrals on $\mathbb{R}$ of respectively $\mathcal{K}$ and $\mathcal{G}$.

Without loss of generality, Equation (102) can be re-expressed uniquely as

$$\begin{cases} \mathcal{K} = C_{\mathcal{K}} \delta + \mathcal{K}_\infty \mathfrak{I} + \mathcal{K} \\ \mathcal{G} = C_{\mathcal{G}} \delta + \mathcal{G}_\infty \mathfrak{I} + \mathcal{G} \end{cases}$$

with

$$\begin{cases} \lim_{t \rightarrow +\infty} (\dot{\mathcal{K}}) = 0 \\ \lim_{t \rightarrow +\infty} (\dot{\mathcal{G}}) = 0 \end{cases}$$

It is worth emphasising that, unlike what the notation may suggest, $\dot{\mathcal{K}}$ and $\dot{\mathcal{G}}$ are not the 1st derivatives of respectively $\mathcal{K}$ and $\mathcal{G}$. Each of $\dot{\mathcal{K}}$ and $\dot{\mathcal{G}}$, if other than the null distribution on $\mathbb{R}$, is a causal regular distribution vanishing at infinity on $\mathbb{R}$ (in particular, for a fluid linear viscoelastic material, a causal regular integrable distribution on $\mathbb{R}$) the result of whose localisation at $\mathbb{R}_{(0,+\infty)}$ is a distribution on $\mathbb{R}_{(0,+\infty)}$ generated by a strictly completely monotonic function on $\mathbb{R}_{(0,+\infty)}$.

If each of $\mathcal{U}_{\mathcal{K}}$ and $\mathcal{U}_{\mathcal{G}}$ is generable by a Radon measure on $\mathbb{R}$ that is atomic, relations hold

$$\begin{cases} \mathcal{K} = C_{\mathcal{K}} \delta + \mathcal{K}_{\infty} \mathfrak{g} + \underbrace{\left(\sum_{h_{\mathcal{K}}=1}^{M_{\mathcal{K}}} \left(\mathcal{K}_{h_{\mathcal{K}}} e^{-\omega_{h_{\mathcal{K}}} t} \right) \right)}_{\dot{\mathcal{K}}} \mathfrak{g} \\ \quad = C_{\mathcal{K}} \delta + \mathcal{K}_{\infty} \mathfrak{g} + \underbrace{\left(\sum_{h_{\mathcal{K}}=1}^{M_{\mathcal{K}}} \left(\mathcal{K}_{h_{\mathcal{K}}} e^{-t/\tau_{h_{\mathcal{K}}}} \right) \right)}_{\dot{\mathcal{K}}} \mathfrak{g} \\ \mathcal{G} = C_{\mathcal{G}} \delta + \mathcal{G}_{\infty} \mathfrak{g} + \underbrace{\left(\sum_{h_{\mathcal{G}}=1}^{M_{\mathcal{G}}} \left(\mathcal{G}_{h_{\mathcal{G}}} e^{-\omega_{h_{\mathcal{G}}} t} \right) \right)}_{\dot{\mathcal{G}}} \mathfrak{g} \\ \quad = C_{\mathcal{G}} \delta + \mathcal{G}_{\infty} \mathfrak{g} + \underbrace{\left(\sum_{h_{\mathcal{G}}=1}^{M_{\mathcal{G}}} \left(\mathcal{G}_{h_{\mathcal{G}}} e^{-t/\tau_{h_{\mathcal{G}}}} \right) \right)}_{\dot{\mathcal{G}}} \mathfrak{g} \end{cases} \quad (103)$$

with

$$\begin{cases} \mathcal{K}_{\infty} \delta + \sum_{h_{\mathcal{K}}=1}^{M_{\mathcal{K}}} \left(\mathcal{K}_{h_{\mathcal{K}}} \delta_{\omega_{h_{\mathcal{K}}}} \right) = \mathcal{U}_{\mathcal{K}} \\ \mathcal{G}_{\infty} \delta + \sum_{h_{\mathcal{G}}=1}^{M_{\mathcal{G}}} \left(\mathcal{G}_{h_{\mathcal{G}}} \delta_{\omega_{h_{\mathcal{G}}}} \right) = \mathcal{U}_{\mathcal{G}} \end{cases}$$

$$\begin{cases} \tau_{h_{\mathcal{K}}} = \frac{1}{\omega_{h_{\mathcal{K}}}} \\ \tau_{h_{\mathcal{G}}} = \frac{1}{\omega_{h_{\mathcal{G}}}} \end{cases}$$

where

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- $M_{\mathcal{K}}$ is an non-null extended natural constant of quantity dimension one
- $M_{\mathcal{G}}$ is an non-null extended natural constant of quantity dimension one
- $h_{\mathcal{K}}$ is a bound index taking values in $\bar{\mathbb{N}}_{[1, M_{\mathcal{K}}]}$
- $h_{\mathcal{G}}$ is a bound index taking values in $\bar{\mathbb{N}}_{[1, M_{\mathcal{G}}]}$
- $\mathcal{K}_{h_{\mathcal{K}}}$ is a non-negative real constant of the same quantity dimension as $\mathcal{K}$ to which reference is made as $h_{\mathcal{K}}^{\text{th}}$ *bulk relaxation coefficient*
- $\mathcal{G}_{h_{\mathcal{G}}}$ is a non-negative real constant of the same quantity dimension as $\mathcal{G}$ to which reference is made as $h_{\mathcal{G}}^{\text{th}}$ *shear relaxation coefficient*
- $\omega_{h_{\mathcal{K}}}$ is a positive real constant of the same quantity dimension as ω to which reference is made as $h_{\mathcal{K}}^{\text{th}}$ *bulk relaxation angular frequency*
- $\omega_{h_{\mathcal{G}}}$ is a positive real constant of the same quantity dimension as ω to which reference is made as $h_{\mathcal{G}}^{\text{th}}$ *shear relaxation angular frequency*
- $\tau_{h_{\mathcal{K}}}$ is a positive real constant of the same quantity dimension as ω^{-1} to which reference is made as $h_{\mathcal{K}}^{\text{th}}$ *bulk relaxation time constant*
- $\tau_{h_{\mathcal{G}}}$ is a positive real constant of the same quantity dimension as ω^{-1} to which reference is made as $h_{\mathcal{G}}^{\text{th}}$ *shear relaxation time constant*
- $\delta_{\omega_{h_{\mathcal{K}}}}$ is the $\omega_{h_{\mathcal{K}}}$ -translated δ
- $\delta_{\omega_{h_{\mathcal{G}}}}$ is the $\omega_{h_{\mathcal{G}}}$ -translated δ

If each of $\mathcal{U}_{\mathcal{K}}$ and $\mathcal{U}_{\mathcal{G}}$ is generable by a Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ (or, equivalently, if each of $\mathcal{U}_{\mathcal{K}}$ and $\mathcal{U}_{\mathcal{G}}$ is a regular distribution on $\mathbb{R}$), relations hold

$$\left\{ \begin{aligned}
 \mathcal{K} &= C_{\mathcal{K}} \delta + \underbrace{\left(\int_0^{+\infty} \left((\mathcal{U}_{\mathcal{K}})_{\mathbb{R}_{(0,+\infty)}} \right) e^{-\omega' t} d\omega' \right)}_{\mathcal{K}'} \mathfrak{g} \\
 &= C_{\mathcal{K}} \delta + \underbrace{\left(\int_0^{+\infty} \left(\mathcal{S}_{\mathcal{K}} e^{-\omega' t} \right) d(\ln(|\omega'|)) \right)}_{\mathcal{K}'} \mathfrak{g} \\
 &= C_{\mathcal{K}} \delta + \underbrace{\left(\int_0^{+\infty} \left(\mathcal{P}_{\mathcal{K}} e^{-t/\tau'} \right) d(\ln(|\tau'|)) \right)}_{\mathcal{K}'} \mathfrak{g} \\
 \mathcal{G} &= C_{\mathcal{G}} \delta + \underbrace{\left(\int_0^{+\infty} \left((\mathcal{U}_{\mathcal{G}})_{\mathbb{R}_{(0,+\infty)}} \right) e^{-\omega' t} d\omega' \right)}_{\mathcal{G}'} \mathfrak{g} \\
 &= C_{\mathcal{G}} \delta + \underbrace{\left(\int_0^{+\infty} \left(\mathcal{S}_{\mathcal{G}} e^{-\omega' t} \right) d(\ln(|\omega'|)) \right)}_{\mathcal{G}'} \mathfrak{g} \\
 &= C_{\mathcal{G}} \delta + \underbrace{\left(\int_0^{+\infty} \left(\mathcal{P}_{\mathcal{G}} e^{-t/\tau'} \right) d(\ln(|\tau'|)) \right)}_{\mathcal{G}'} \mathfrak{g}
 \end{aligned} \right. \quad (104)$$

with

$$\begin{cases} \mathcal{S}_{\mathcal{K}} = \omega' (\mathcal{U}_{\mathcal{K}})_{\mathbb{R}_{(0,+\infty)}} \\ \mathcal{S}_{\mathcal{G}} = \omega' (\mathcal{U}_{\mathcal{G}})_{\mathbb{R}_{(0,+\infty)}} \end{cases}$$

$$\begin{cases} \mathcal{P}_{\mathcal{K}} = \frac{\left((\mathcal{U}_{\mathcal{K}})_{\mathbb{R}_{(0,+\infty)}} \right)_{\omega'=1/\tau'}}{\tau'} \\ \mathcal{P}_{\mathcal{G}} = \frac{\left((\mathcal{U}_{\mathcal{G}})_{\mathbb{R}_{(0,+\infty)}} \right)_{\omega'=1/\tau'}}{\tau'} \end{cases}$$

where

- ω' is a bound real variable of the same quantity dimension as ω
- τ' is a bound real variable of the same quantity dimension as ω^{-1}
- $(\mathcal{U}_{\mathcal{K}})_{\mathbb{R}_{(0,+\infty)}}$ is the regular distribution on $\mathbb{R}_{(0,+\infty)}$ resulting from the localisation of $\mathcal{U}_{\mathcal{K}}$ at $\mathbb{R}_{(0,+\infty)}$
- $(\mathcal{U}_{\mathcal{G}})_{\mathbb{R}_{(0,+\infty)}}$ is the regular distribution on $\mathbb{R}_{(0,+\infty)}$ resulting from the localisation of $\mathcal{U}_{\mathcal{G}}$ at $\mathbb{R}_{(0,+\infty)}$

4 Third theoretical study: Distributional characterisation of linear viscoelastic materials

- $\mathcal{S}_{\mathcal{K}}$ is the *bulk relaxation angular-frequency spectrum*
- $\mathcal{S}_{\mathcal{G}}$ is the *shear relaxation angular-frequency spectrum*
- $\mathcal{R}_{\mathcal{K}}$ is the *bulk relaxation time-constant spectrum*
- $\mathcal{R}_{\mathcal{G}}$ is the *shear relaxation time-constant spectrum*

As an implication of introducing the logarithm in the first and the second of Equation (104), $\mathcal{S}_{\mathcal{K}}$ and $\mathcal{R}_{\mathcal{K}}$ as well as $\mathcal{S}_{\mathcal{G}}$ and $\mathcal{R}_{\mathcal{G}}$ have a logarithmic abscissa scale and the same quantity dimension as $\mathcal{K}$ and $\mathcal{G}$. A linear viscoelastic material defined by Equation (104) need be a fluid.

Because of difficult characterisation and limited applicative interest, we ignore the residual case in which any of $\mathcal{U}_{\mathcal{K}}$ and $\mathcal{U}_{\mathcal{G}}$ is generable by a Radon measure on $\mathbb{R}$ that is continuous but singular with respect to the Lebesgue measure on $\mathbb{R}$ (namely, the case in which any of $\mathcal{U}_{\mathcal{K}}$ and $\mathcal{U}_{\mathcal{G}}$ is generable by a Radon measure on $\mathbb{R}$ that is neither atomic nor absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$).

The limiting case of (necessarily solid) *linear elastic material* (shorthand for *non-ageing homogeneous isotropic linear elastic material under isothermal adiabatic conditions*) corresponds to

$$\begin{cases} \mathcal{K}^{(1)} = K_E \delta \\ \mathcal{G}^{(1)} = G_E \delta \end{cases} \Leftrightarrow \begin{cases} \mathcal{K} = K_E \mathfrak{D} \\ \mathcal{G} = G_E \mathfrak{D} \end{cases} \Leftrightarrow \begin{cases} \mathcal{K}^{(-1)} = K_E \rho \\ \mathcal{G}^{(-1)} = G_E \rho \end{cases}$$

with

$$\begin{cases} \sigma_{pq,S} = 3K_E \varepsilon_{pq,S} \\ \sigma_{pq,D} = 2G_E \varepsilon_{pq,D} \end{cases}$$

$$\begin{cases} \mathcal{K}_0 = \mathcal{K}_{\infty} \\ \quad = K_E \\ \mathcal{G}_0 = \mathcal{G}_{\infty} \\ \quad = G_E \end{cases}$$

where

- K_E is a positive real constant of the same quantity dimension as $\mathcal{K}$ to which reference is made as *bulk modulus* (also known as *modulus of compression*)
- G_E is a positive real constant of the same quantity dimension as $\mathcal{G}$ to which reference is made as *shear modulus* (also known as *modulus of rigidity*)

4 Third theoretical study: Distributional characterisation of linear viscoelastic materials

A linear elastic material can be physically interpreted in terms of a pair of independent linear springs one of stiffness proportional to K_E and the other of stiffness proportional to G_E .

The limiting case of (necessarily fluid) *linear viscous material* (shorthand for *non-ageing homogeneous isotropic linear viscous material under isothermal adiabatic conditions*) corresponds to

$$\begin{cases} \mathcal{K}^{(1)} = \zeta_v \delta^{(1)} \\ \mathcal{G}^{(1)} = \eta_v \delta^{(1)} \end{cases} \Leftrightarrow \begin{cases} \mathcal{K} = \zeta_v \delta \\ \mathcal{G} = \eta_v \delta \end{cases} \Leftrightarrow \begin{cases} \mathcal{K}^{(-1)} = \zeta_v \vartheta \\ \mathcal{G}^{(-1)} = \eta_v \vartheta \end{cases}$$

with

$$\begin{cases} \sigma_{pq,S} = 3\zeta_v \varepsilon_{pq,S}^{(1)} \\ \sigma_{pq,D} = 2\eta_v \varepsilon_{pq,D}^{(1)} \end{cases}$$

$$\begin{cases} \mathcal{K}_0^{(-1)} = \mathcal{K}_\infty^{(-1)} \\ \quad = \zeta_v \\ \mathcal{G}_0^{(-1)} = \mathcal{G}_\infty^{(-1)} \\ \quad = \eta_v \end{cases}$$

where

- ζ_v is a positive real constant of the same quantity dimension as $\mathcal{K}^{(-1)}$ to which reference is made as *bulk viscosity*
- η_v is a positive real constant of the same quantity dimension as $\mathcal{G}^{(-1)}$ to which reference is made as *shear viscosity*

A linear viscous material can be physically interpreted in terms of a pair of independent linear viscous dampers one of viscous damping coefficient proportional to ζ_v and the other of viscous damping coefficient proportional to η_v . When, in the expression of $\sigma_{pq,S}$, the spherical tensor $-p_T \delta_{pq}$ (p_T being the *thermodynamic pressure* that appears in some equation of state) is added to $3\zeta_v \varepsilon_{pq,S}^{(1)}$, the constitutive relations at a fixed point of a linear viscous material become those of a *homogeneous isotropic Newtonian fluid*.

For a solid linear viscoelastic material with elastic initial response, it is convenient to normalise $\mathcal{K}$ and $\mathcal{G}$ by letting

$$\left\{ \begin{array}{l} \kappa^{(1)} \equiv \frac{\mathcal{K}^{(1)}}{\mathcal{K}_0} \\ \mathcal{J}^{(1)} \equiv \frac{\mathcal{G}^{(1)}}{\mathcal{G}_0} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \kappa \equiv \frac{\mathcal{K}}{\mathcal{K}_0} \\ \mathcal{J} \equiv \frac{\mathcal{G}}{\mathcal{G}_0} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \kappa^{(-1)} \equiv \frac{\mathcal{K}^{(-1)}}{\mathcal{K}_0} \\ \mathcal{J}^{(-1)} \equiv \frac{\mathcal{G}^{(-1)}}{\mathcal{G}_0} \end{array} \right\}$$

where

- κ is the *normalised bulk relaxation modulus*
- $\mathcal{J}$ is the *normalised shear relaxation modulus*

For a fluid linear viscoelastic material, it is convenient to normalise $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ by letting

$$\left\{ \begin{array}{l} \kappa^{(1)} \equiv \frac{\mathcal{K}^{(1)}}{\mathcal{K}_\infty^{(-1)}} \\ \mathcal{J}^{(1)} \equiv \frac{\mathcal{G}^{(1)}}{\mathcal{G}_\infty^{(-1)}} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \kappa \equiv \frac{\mathcal{K}}{\mathcal{K}_\infty^{(-1)}} \\ \mathcal{J} \equiv \frac{\mathcal{G}}{\mathcal{G}_\infty^{(-1)}} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \kappa^{(-1)} \equiv \frac{\mathcal{K}^{(-1)}}{\mathcal{K}_\infty^{(-1)}} \\ \mathcal{J}^{(-1)} \equiv \frac{\mathcal{G}^{(-1)}}{\mathcal{G}_\infty^{(-1)}} \end{array} \right\}$$

As expressed by Equation (100), the constitutive relations at a fixed point of a linear viscoelastic material originate from the empirical observation of *stress relaxation* phenomena and are said to be in *stress-relaxation formulation*. If (and only if) both $\mathcal{K}^{(2)}$ and $\mathcal{G}^{(2)}$ (or, equivalently, both $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ or, equivalently, both $\mathcal{K}$ and $\mathcal{G}$) admit right-sided (in fact, causal) convolutive inverses, equivalent constitutive relations that originate from the empirical observation of *strain creep* (also known as *strain retardation*) phenomena and are said to be in *strain-creep formulation* (also known as *strain-retardation formulation*) can be obtained from those in stress-relaxation formulation; remarkably, the stress-relaxation formulation and the strain-creep formulation are dual to each other. The right-sided (in fact, causal) convolutive inverses of $\mathcal{K}^{(2)}$ and $\mathcal{G}^{(2)}$ are denoted by respectively $\mathcal{C}$ and $\mathcal{J}$; reference is made to them as respectively *bulk creep compliance* (also known as *bulk retardation compliance*) and *shear creep compliance* (also known as *shear retardation compliance*). The right-sided (in fact, causal) convolutive inverses of $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ coincide with respectively $\mathcal{C}^{(1)}$ and $\mathcal{J}^{(1)}$; reference is made to them as respectively *bulk creep kernel* (also known as *bulk retardation kernel*) and *shear creep kernel* (also known as *shear retardation kernel*). The right-sided (in fact, causal) convolutive inverses of $\mathcal{K}$ and $\mathcal{G}$ coincide with respectively $\mathcal{C}^{(2)}$ and $\mathcal{J}^{(2)}$. As in the case of stress-relaxation formulation, also in the case of strain-creep formulation a linear viscoelastic material can be regarded as a set of two independent convolution systems of suitable input variables and output variables. Experimentally, $\mathcal{C}$ and $\mathcal{J}$ can be determined via respectively a *bulk creep test* (also known as *bulk retardation test*) and a *shear creep test* (also known as *shear retardation test*). Such developments are beyond the scope of this third theoretical study and are not pursued further.

4.3 Frequency domain

Hereinafter, we assume that $\varepsilon_{pq,S}$ (and hence $\varepsilon_{pq,S}^{(1)}$ and $\sigma_{pq,S}$) and $\varepsilon_{pq,D}$ (and hence $\varepsilon_{pq,D}^{(1)}$ and $\sigma_{pq,D}$) are tempered distributions on $\mathbb{R}$. Under the assumptions made, it is possible to apply to Equation (100) both the Fourier transformation and the generalised form of the convolution Fourier exchange theorem. Relations hold

$$\begin{cases} \mathcal{F}(\sigma_{pq,S}) = 3K \cdot \mathcal{F}(\varepsilon_{pq,S}) \\ \quad = 3\zeta \cdot \mathcal{F}(\varepsilon_{pq,S}^{(1)}) \\ \mathcal{F}(\sigma_{pq,D}) = 2G \cdot \mathcal{F}(\varepsilon_{pq,D}) \\ \quad = 2\eta \cdot \mathcal{F}(\varepsilon_{pq,D}^{(1)}) \end{cases} \quad (105)$$

with

$$\begin{cases} K = \mathcal{F}(\mathcal{K}^{(1)}) \\ G = \mathcal{F}(\mathcal{G}^{(1)}) \end{cases}$$

$$\begin{cases} \zeta = \mathcal{F}(\mathcal{K}) \\ \eta = \mathcal{F}(\mathcal{G}) \end{cases}$$

where

- K is the *dynamic bulk modulus* (also known as *dynamic modulus of compression*)
- G is the *dynamic shear modulus* (also known as *dynamic modulus of rigidity*)
- ζ is the *dynamic bulk viscosity*
- η is the *dynamic shear viscosity*

Remarkably, the first and the second of Equation (105) are independent. Under the assumptions made, K and G are tempered distributions on $\mathbb{R}$ that are generable by C^0 functions of slow growth on $\mathbb{R}$, whereas ζ and η are weakly non-negative distributions on $\mathbb{R}$ (a weakly non-negative distribution on $\mathbb{R}$ being a distribution on $\mathbb{R}$ whose real part is non-negative).

Under the assumptions made, it is possible to apply to Equation (101) both the Fourier transformation and the generalised form of the convolution Fourier exchange theorem. Relations hold

$$\left\{ \begin{array}{l} K = j\omega\zeta \\ \quad = -\omega^2\mu \\ G = j\omega\eta \\ \quad = -\omega^2\lambda \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \left(\text{p.v.}\left(\frac{1}{j\omega}\right) + \pi\delta \right) K = \zeta \\ \quad = j\omega\mu \\ \left(\text{p.v.}\left(\frac{1}{j\omega}\right) + \pi\delta \right) G = \eta \\ \quad = j\omega\lambda \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \left(\text{f.p.}\left(\frac{1}{-\omega^2}\right) + j\pi\delta^{(1)} \right) K = \left(\text{p.v.}\left(\frac{1}{j\omega}\right) + \pi\delta \right) \zeta \\ \quad = \mu \\ \left(\text{f.p.}\left(\frac{1}{-\omega^2}\right) + j\pi\delta^{(1)} \right) G = \left(\text{p.v.}\left(\frac{1}{j\omega}\right) + \pi\delta \right) \eta \\ \quad = \lambda \end{array} \right\} \quad (106)$$

with

$$\left\{ \begin{array}{l} \mu = \mathcal{F}\left(\mathcal{K}^{(-1)}\right) \\ \lambda = \mathcal{F}\left(\mathcal{L}^{(-1)}\right) \end{array} \right.$$

It is worth noting that the complex-frequency-domain homologue of Equation (106) contains neither terms proportional to pseudofunctions nor terms proportional to δ or $\delta^{(1)}$ (in fact, they only contain holomorphic functions on suitable regions of definition). Indeed, as anticipated, quantities ‘living’ in the frequency domain are physically transparent but mathematically ill-behaved, whereas quantities ‘living’ in the complex frequency domain are physically opaque but mathematically well-behaved.

In accordance with Equation (105), because $3\mathcal{K}^{(1)}$ and $2\mathcal{G}^{(1)}$ can be regarded as respectively $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}$ -unit-impulse excitation and $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}$ -unit-impulse excitation, $3K$ and $2G$ can be regarded as the corresponding frequency-response distributions. Besides, because $3\mathcal{K}$ and $2\mathcal{G}$ can be regarded as respectively $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}$ -unit-step excitation (or, equivalently, $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}^{(1)}$ -unit-impulse excitation) and $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}$ -unit-step excitation (or, equivalently, $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}^{(1)}$ -unit-impulse excitation), 3ζ and 2η can be regarded as the corresponding frequency-response distributions. Furthermore, because $3\mathcal{K}^{(-1)}$ and $2\mathcal{G}^{(-1)}$ can be regarded as respectively $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}$ -unit-ramp excitation (or, equivalently, $\sigma_{pq,S}$ response to $\varepsilon_{pq,S}^{(1)}$ -unit-step excitation) and $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}$ -unit-ramp excitation (or, equivalently, $\sigma_{pq,D}$ response to $\varepsilon_{pq,D}^{(1)}$ -unit-step excitation), 3μ and 2λ can be regarded as the corresponding frequency-response distributions. Experimentally, K (as well as ζ and μ) and G (as well as η and λ) can be determined via respectively a *bulk oscillatory test* and a *shear oscillatory test*.

The limiting case of linear elastic material corresponds to

$$\begin{cases} K = K_E 1_{\mathbb{R}} \\ G = G_E 1_{\mathbb{R}} \end{cases} \Leftrightarrow \begin{cases} \zeta = K_E \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \\ \eta = G_E \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \end{cases} \Leftrightarrow \begin{cases} \mu = K_E \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \\ \lambda = G_E \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \end{cases}$$

The limiting case of linear viscous material corresponds to

$$\begin{cases} K = j\zeta_v \omega \\ G = j\eta_v \omega \end{cases} \Leftrightarrow \begin{cases} \zeta = \zeta_v 1_{\mathbb{R}} \\ \eta = \eta_v 1_{\mathbb{R}} \end{cases} \Leftrightarrow \begin{cases} \mu = \zeta_v \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \\ \lambda = \eta_v \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \end{cases}$$

As an implication of the *elastic-viscoelastic correspondence principle*, K and G for a linear viscoelastic material can be regarded as angular-frequency-dependent generalised versions of respectively $K_E 1_{\mathbb{R}}$ and $G_E 1_{\mathbb{R}}$ for a linear elastic material, whereas ζ and η for a linear viscoelastic material can be regarded as angular-frequency-dependent generalised versions of respectively $\zeta_v 1_{\mathbb{R}}$ and $\eta_v 1_{\mathbb{R}}$ for a linear viscous material. To some extent, adopting such approach implies regarding the frequency domain rather than the time domain as the ‘natural’ one in which to express in a familiar manner the constitutive relations at a fixed point of a linear viscoelastic material.

Remarkably, K and G are real-valued for a linear elastic material and imaginary-valued for a linear viscous material, whereas ζ and η are imaginary-valued for a linear elastic material and real-valued for a linear viscous material. For a linear viscoelastic material, this justifies the physical interpretation of the real part of K and G as well as of the imaginary part of ζ and η as related to elasticity (or storage) and of the imaginary part of K and G as well as of the real part of ζ and η as related to dissipation (or loss).

It is convenient to let

$$\begin{cases} K \equiv K' + jK'' \\ \quad = \underbrace{\text{Re}(K)}_{K'} + j \underbrace{\text{Im}(K)}_{K''} \\ G \equiv G' + jG'' \\ \quad = \underbrace{\text{Re}(G)}_{G'} + j \underbrace{\text{Im}(G)}_{G''} \end{cases}$$

$$\left\{ \begin{array}{l} \zeta \equiv \zeta' + j\zeta'' \\ \quad = \underbrace{\operatorname{Re}(\zeta)}_{\zeta'} + j \underbrace{\operatorname{Im}(\zeta)}_{\zeta''} \\ \eta \equiv \eta' + j\eta'' \\ \quad = \underbrace{\operatorname{Re}(\eta)}_{\eta'} + j \underbrace{\operatorname{Im}(\eta)}_{\eta''} \end{array} \right.$$

where

- K' is the *storage dynamic bulk modulus* (also known as *storage dynamic modulus of compression*)
- G' is the *storage dynamic shear modulus* (also known as *storage dynamic modulus of rigidity*)
- K'' is the *loss dynamic bulk modulus* (also known as *loss dynamic modulus of compression*)
- G'' is the *loss dynamic shear modulus* (also known as *loss dynamic modulus of rigidity*)
- ζ' is the *loss dynamic bulk viscosity*
- η' is the *loss dynamic shear viscosity*
- ζ'' is the *storage dynamic bulk viscosity*
- η'' is the *storage dynamic shear viscosity*

For a solid linear viscoelastic material with elastic initial response, $\mathcal{K}$ and $\mathcal{G}$ are regular bounded distributions on $\mathbb{R}$ expressible as respectively $\mathcal{K}_0\vartheta + (\mathcal{K} - \mathcal{K}_0\vartheta)$ and $\mathcal{G}_0\vartheta + (\mathcal{G} - \mathcal{G}_0\vartheta)$ with $\mathcal{K} - \mathcal{K}_0\vartheta$ and $\mathcal{G} - \mathcal{G}_0\vartheta$ regular bounded distributions on $\mathbb{R}$ that are generable by bounded absolutely continuous functions on $\mathbb{R}$; this is the case because $\mathcal{K} - \mathcal{K}_0\vartheta$ and $\mathcal{G} - \mathcal{G}_0\vartheta$ are regular bounded distributions on $\mathbb{R}$ that are generable by functions of bounded variation on $\mathbb{R}$ (in fact, by non-increasing C^0 bounded functions on $\mathbb{R}$) as well as by locally absolutely continuous functions on $\mathbb{R}$ (the fact that $\mathcal{K} - \mathcal{K}_0\vartheta$ and $\mathcal{G} - \mathcal{G}_0\vartheta$ are generable by locally absolutely continuous functions on $\mathbb{R}$ being equivalent to the fact that they have regular distributions on $\mathbb{R}$ as 1st derivatives). In turn, $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$ are singular integrable distributions of order 0 on $\mathbb{R}$ expressible as respectively $\mathcal{K}_0\delta + (\mathcal{K}^{(1)} - \mathcal{K}_0\delta)$ and $\mathcal{G}_0\delta + (\mathcal{G}^{(1)} - \mathcal{G}_0\delta)$ with $\mathcal{K}^{(1)} - \mathcal{K}_0\delta$ and $\mathcal{G}^{(1)} - \mathcal{G}_0\delta$ regular integrable distributions on $\mathbb{R}$ that are generable by integrable functions on $\mathbb{R}$ (namely, regular integrable distributions of S -order 0 on $\mathbb{R}$); this is the case because $\mathcal{K}^{(1)} - \mathcal{K}_0\delta$ and $\mathcal{G}^{(1)} - \mathcal{G}_0\delta$ are 1st derivatives of regular bounded distributions on $\mathbb{R}$ that are generable by bounded absolutely continuous functions on $\mathbb{R}$. Ultimately, as Fourier transforms of respectively $\mathcal{K}^{(1)}$ and $\mathcal{G}^{(1)}$, K and G are regular bounded distributions on $\mathbb{R}$ that are generable by C^0 bounded functions on $\mathbb{R}$,

and, as Fourier transforms of respectively $\mathcal{K}^{(1)} - \mathcal{K}_0\delta$ and $\mathcal{G}^{(1)} - \mathcal{G}_0\delta$, $K - \mathcal{K}_0 1_{\mathbb{R}}$ and $G - \mathcal{G}_0 1_{\mathbb{R}}$ are regular distributions vanishing at infinity on $\mathbb{R}$ that are generable by C^0 functions vanishing at infinity on $\mathbb{R}$. For a solid linear viscoelastic material with elastic initial response, relations hold

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow 0} (K) = \lim_{\omega \rightarrow 0} (K') + j \lim_{\omega \rightarrow 0} (K'') \\ \quad = \underbrace{\mathcal{K}_{\infty}}_{\lim_{\omega \rightarrow 0} (K')} + j \underbrace{0}_{\lim_{\omega \rightarrow 0} (K'')} \\ \quad = \mathcal{K}_{\infty} \\ \lim_{\omega \rightarrow 0} (G) = \lim_{\omega \rightarrow 0} (G') + j \lim_{\omega \rightarrow 0} (G'') \\ \quad = \underbrace{\mathcal{G}_{\infty}}_{\lim_{\omega \rightarrow 0} (G')} + j \underbrace{0}_{\lim_{\omega \rightarrow 0} (G'')} \\ \quad = \mathcal{G}_{\infty} \end{array} \right. \quad (107)$$

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow \pm\infty} (K) = \lim_{\omega \rightarrow \pm\infty} (K') + j \lim_{\omega \rightarrow \pm\infty} (K'') \\ \quad = \underbrace{\mathcal{K}_0}_{\lim_{\omega \rightarrow \pm\infty} (K')} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty} (K'')} \\ \quad = \mathcal{K}_0 \\ \lim_{\omega \rightarrow \pm\infty} (G) = \lim_{\omega \rightarrow \pm\infty} (G') + j \lim_{\omega \rightarrow \pm\infty} (G'') \\ \quad = \underbrace{\mathcal{G}_0}_{\lim_{\omega \rightarrow \pm\infty} (G')} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty} (G'')} \\ \quad = \mathcal{G}_0 \end{array} \right. \quad (108)$$

For a fluid linear viscoelastic material, $\mathcal{K}^{(-1)}$ and $\mathcal{G}^{(-1)}$ are regular bounded distributions on $\mathbb{R}$ expressible as respectively $\mathcal{K}_0^{(-1)}\mathfrak{g} + (\mathcal{K}^{(-1)} - \mathcal{K}_0^{(-1)}\mathfrak{g})$ and $\mathcal{G}_0^{(-1)}\mathfrak{g} + (\mathcal{G}^{(-1)} - \mathcal{G}_0^{(-1)}\mathfrak{g})$ with $\mathcal{K}^{(-1)} - \mathcal{K}_0^{(-1)}\mathfrak{g}$ and $\mathcal{G}^{(-1)} - \mathcal{G}_0^{(-1)}\mathfrak{g}$ regular bounded distributions on $\mathbb{R}$ that are generable by bounded absolutely continuous functions on $\mathbb{R}$; this is the case because $\mathcal{K}^{(-1)} - \mathcal{K}_0^{(-1)}\mathfrak{g}$ and $\mathcal{G}^{(-1)} - \mathcal{G}_0^{(-1)}\mathfrak{g}$ are regular bounded distributions on $\mathbb{R}$ that are generable by functions of bounded variation on $\mathbb{R}$ (in fact, by non-decreasing C^0 bounded functions on $\mathbb{R}$) as well as by locally absolutely continuous functions on $\mathbb{R}$ (the fact that $\mathcal{K}^{(-1)} - \mathcal{K}_0^{(-1)}\mathfrak{g}$ and $\mathcal{G}^{(-1)} - \mathcal{G}_0^{(-1)}\mathfrak{g}$ are generable by locally absolutely continuous functions on $\mathbb{R}$ being equivalent to the fact that they have regular distributions on $\mathbb{R}$ as 1st derivatives). In turn, $\mathcal{K}$ and $\mathcal{G}$ are integrable distributions of order 0 on $\mathbb{R}$ expressible as respectively $\mathcal{K}_0^{(-1)}\delta + (\mathcal{K} - \mathcal{K}_0^{(-1)}\delta)$ and $\mathcal{G}_0^{(-1)}\delta + (\mathcal{G} - \mathcal{G}_0^{(-1)}\delta)$ with $\mathcal{K} - \mathcal{K}_0^{(-1)}\delta$ and $\mathcal{G} - \mathcal{G}_0^{(-1)}\delta$ regular integrable distributions on $\mathbb{R}$ that are generable by integrable functions on $\mathbb{R}$ (namely, regular integrable distributions of $\mathcal{S}$ -order 0 on $\mathbb{R}$); this is the

case because $\mathcal{H} - \mathcal{H}_0^{(-1)}\delta$ and $\mathcal{G} - \mathcal{G}_0^{(-1)}\delta$ are 1st derivatives of regular bounded distributions on $\mathbb{R}$ that are generable by bounded absolutely continuous functions on $\mathbb{R}$. Ultimately, as Fourier transforms of respectively $\mathcal{H}$ and $\mathcal{G}$, ζ and η are regular bounded distributions on $\mathbb{R}$ that are generable by C^0 bounded functions on $\mathbb{R}$, and, as Fourier transforms of respectively $\mathcal{H} - \mathcal{H}_0^{(-1)}\delta$ and $\mathcal{G} - \mathcal{G}_0^{(-1)}\delta$, $\zeta - \mathcal{H}_0^{(-1)}1_{\mathbb{R}}$ and $\eta - \mathcal{G}_0^{(-1)}1_{\mathbb{R}}$ are regular distributions vanishing at infinity on $\mathbb{R}$ that are generable by C^0 functions vanishing at infinity on $\mathbb{R}$. For a fluid linear viscoelastic material, relations hold

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow 0}(\zeta) = \lim_{\omega \rightarrow 0}(\zeta') + j \lim_{\omega \rightarrow 0}(\zeta'') \\ \quad = \underbrace{\mathcal{H}_\infty^{(-1)}}_{\lim_{\omega \rightarrow 0}(\zeta')} + j \underbrace{0}_{\lim_{\omega \rightarrow 0}(\zeta'')} \\ \quad = \mathcal{H}_\infty^{(-1)} \\ \lim_{\omega \rightarrow 0}(\eta) = \lim_{\omega \rightarrow 0}(\eta') + j \lim_{\omega \rightarrow 0}(\eta'') \\ \quad = \underbrace{\mathcal{G}_\infty^{(-1)}}_{\lim_{\omega \rightarrow 0}(\eta')} + j \underbrace{0}_{\lim_{\omega \rightarrow 0}(\eta'')} \\ \quad = \mathcal{G}_\infty^{(-1)} \end{array} \right. \quad (109)$$

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow \pm\infty}(\zeta) = \lim_{\omega \rightarrow \pm\infty}(\zeta') + j \lim_{\omega \rightarrow \pm\infty}(\zeta'') \\ \quad = \underbrace{\mathcal{H}_0^{(-1)}}_{\lim_{\omega \rightarrow \pm\infty}(\zeta')} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty}(\zeta'')} \\ \quad = \mathcal{H}_0^{(-1)} \\ \lim_{\omega \rightarrow \pm\infty}(\eta) = \lim_{\omega \rightarrow \pm\infty}(\eta') + j \lim_{\omega \rightarrow \pm\infty}(\eta'') \\ \quad = \underbrace{\mathcal{G}_0^{(-1)}}_{\lim_{\omega \rightarrow \pm\infty}(\eta')} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty}(\eta'')} \\ \quad = \mathcal{G}_0^{(-1)} \end{array} \right. \quad (110)$$

For a solid linear viscoelastic material with elastic initial response, it is convenient to let

$$\left\{ \begin{array}{l} k \equiv \frac{K}{\mathcal{H}_0} \\ g \equiv \frac{G}{\mathcal{G}_0} \end{array} \right.$$

with

$$\left\{ \begin{array}{l} k = \mathcal{F}(\kappa^{(1)}) \\ g = \mathcal{F}(\mathcal{G}^{(1)}) \end{array} \right.$$

Furthermore, it is convenient to let

$$\left\{ \begin{array}{l} k \equiv k' + jk'' \\ \quad = \underbrace{\operatorname{Re}(k)}_{k'} + j \underbrace{\operatorname{Im}(k)}_{k''} \\ g \equiv g' + jg'' \\ \quad = \underbrace{\operatorname{Re}(g)}_{g'} + j \underbrace{\operatorname{Im}(g)}_{g''} \end{array} \right.$$

where

- k is the *normalised dynamic bulk modulus* (also known as *normalised dynamic modulus of compression*)
- g is the *normalised dynamic shear modulus* (also known as *normalised dynamic modulus of rigidity*)
- k' is the *normalised storage dynamic bulk modulus* (also known as *normalised storage dynamic modulus of compression*)
- g' is the *normalised storage dynamic shear modulus* (also known as *normalised storage dynamic modulus of rigidity*)
- k'' is the *normalised loss dynamic bulk modulus* (also known as *normalised loss dynamic modulus of compression*)
- g'' is the *normalised loss dynamic shear modulus* (also known as *normalised loss dynamic modulus of rigidity*)

Relations hold

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow 0} (k) = \lim_{\omega \rightarrow 0} (k') + j \lim_{\omega \rightarrow 0} (k'') \\ \quad = \frac{\mathcal{K}_{\infty}}{\mathcal{K}_0} + j \underbrace{\lim_{\omega \rightarrow 0} (k'')}_{\lim_{\omega \rightarrow 0} (k'')} \\ \quad = \frac{\mathcal{K}_{\infty}}{\mathcal{K}_0} \\ \lim_{\omega \rightarrow 0} (g) = \lim_{\omega \rightarrow 0} (g') + j \lim_{\omega \rightarrow 0} (g'') \\ \quad = \frac{\mathcal{G}_{\infty}}{\mathcal{G}_0} + j \underbrace{\lim_{\omega \rightarrow 0} (g'')}_{\lim_{\omega \rightarrow 0} (g'')} \\ \quad = \frac{\mathcal{G}_{\infty}}{\mathcal{G}_0} \end{array} \right. \quad (111)$$

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow \pm\infty} (k) = \lim_{\omega \rightarrow \pm\infty} (k') + j \lim_{\omega \rightarrow \pm\infty} (k'') \\ \quad = \underbrace{\frac{\mathcal{K}_0}{\mathcal{K}_0}}_{\lim_{\omega \rightarrow \pm\infty} (k')} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty} (k'')} \\ \quad = 1 \\ \lim_{\omega \rightarrow \pm\infty} (g) = \lim_{\omega \rightarrow \pm\infty} (g') + j \lim_{\omega \rightarrow \pm\infty} (g'') \\ \quad = \underbrace{\frac{\mathcal{G}_0}{\mathcal{G}_0}}_{\lim_{\omega \rightarrow \pm\infty} (g')} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty} (g'')} \\ \quad = 1 \end{array} \right. \quad (112)$$

For a fluid linear viscoelastic material, it is convenient to let

$$\left\{ \begin{array}{l} \zeta_N \equiv \frac{\zeta}{\mathcal{K}_\infty^{(-1)}} \\ \eta_N \equiv \frac{\eta}{\mathcal{G}_\infty^{(-1)}} \end{array} \right.$$

with

$$\left\{ \begin{array}{l} \zeta_N = \mathcal{F}(\kappa) \\ \eta_N = \mathcal{F}(\mathcal{J}) \end{array} \right.$$

Furthermore, it is convenient to let

$$\left\{ \begin{array}{l} \zeta_N \equiv \zeta'_N + j \zeta''_N \\ \quad = \underbrace{\text{Re}(\zeta_N)}_{\zeta'_N} + j \underbrace{\text{Im}(\zeta_N)}_{\zeta''_N} \\ \eta_N \equiv \eta'_N + j \eta''_N \\ \quad = \underbrace{\text{Re}(\eta_N)}_{\eta'_N} + j \underbrace{\text{Im}(\eta_N)}_{\eta''_N} \end{array} \right.$$

where

- ζ_N is the *normalised dynamic bulk viscosity*
- η_N is the *normalised dynamic shear viscosity*
- ζ'_N is the *normalised loss dynamic bulk viscosity*
- η'_N is the *normalised loss dynamic shear viscosity*
- ζ''_N is the *normalised storage dynamic bulk viscosity*
- η''_N is the *normalised storage dynamic shear viscosity*

Relations hold

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow 0} (\zeta_N) = \lim_{\omega \rightarrow 0} (\zeta'_N) + j \lim_{\omega \rightarrow 0} (\zeta''_N) \\ \quad = \underbrace{\frac{\mathcal{H}_\infty^{(-1)}}{\mathcal{H}_\infty^{(-1)}}}_{\lim_{\omega \rightarrow 0} (\zeta'_N)} + j \underbrace{0}_{\lim_{\omega \rightarrow 0} (\zeta''_N)} \\ \quad = 1 \\ \lim_{\omega \rightarrow 0} (\eta_N) = \lim_{\omega \rightarrow 0} (\eta'_N) + j \lim_{\omega \rightarrow 0} (\eta''_N) \\ \quad = \underbrace{\frac{\mathcal{G}_\infty^{(-1)}}{\mathcal{G}_\infty^{(-1)}}}_{\lim_{\omega \rightarrow 0} (\eta'_N)} + j \underbrace{0}_{\lim_{\omega \rightarrow 0} (\eta''_N)} \\ \quad = 1 \end{array} \right. \quad (113)$$

$$\left\{ \begin{array}{l} \lim_{\omega \rightarrow \pm\infty} (\zeta_N) = \lim_{\omega \rightarrow \pm\infty} (\zeta'_N) + j \lim_{\omega \rightarrow \pm\infty} (\zeta''_N) \\ \quad = \underbrace{\frac{\mathcal{H}_0^{(-1)}}{\mathcal{H}_\infty^{(-1)}}}_{\lim_{\omega \rightarrow \pm\infty} (\zeta'_N)} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty} (\zeta''_N)} \\ \quad = \frac{\mathcal{H}_0^{(-1)}}{\mathcal{H}_\infty^{(-1)}} \\ \lim_{\omega \rightarrow \pm\infty} (\eta_N) = \lim_{\omega \rightarrow \pm\infty} (\eta'_N) + j \lim_{\omega \rightarrow \pm\infty} (\eta''_N) \\ \quad = \underbrace{\frac{\mathcal{G}_0^{(-1)}}{\mathcal{G}_\infty^{(-1)}}}_{\lim_{\omega \rightarrow \pm\infty} (\eta'_N)} + j \underbrace{0}_{\lim_{\omega \rightarrow \pm\infty} (\eta''_N)} \\ \quad = \frac{\mathcal{G}_0^{(-1)}}{\mathcal{G}_\infty^{(-1)}} \end{array} \right. \quad (114)$$

The limits as ω tends to 0 expressed by Equation (107) and Equation (109) (as well as by Equation (111) and Equation (113)) and the limits as ω tends to $\pm\infty$ expressed by Equation (108) and Equation (110) (as well as by Equation (112) and Equation (114)) are physically revealing inasmuch as they establish a connection between the time domain and the frequency domain. However, unlike the limits as ω tends to $\pm\infty$, the limits as ω tends to 0 cannot be obtained as straightforwardly and generally as their complex-frequency-domain homologues; indeed, whilst the complex-frequency-domain homologues of the limits as ω tends to 0 can be obtained by applying an Abelian theorem of the final-value type whose validity can be extended to a sufficient extent from the Laplace transformation to the Fourier transformation, the complex-frequency-domain homologues of the limits as ω tends to $\pm\infty$ can be obtained by applying an Abelian theorem of the initial-value type whose validity cannot be extended to a sufficient extent from the Laplace transformation to the Fourier transformation. Then again, as observed, quantities ‘living’ in the

frequency domain are physically transparent but mathematically ill-behaved, whereas quantities ‘living’ in the complex frequency domain are physically opaque but mathematically well-behaved.

Hereinafter, a linear viscoelastic material is assumed a solid linear viscoelastic material with elastic initial response. As a consequence of Equation (107) and Equation (108), $3K + G$ cannot coincide with the null distribution on $\mathbb{R}$ (and hence neither can $1 - 2\nu$ nor $1 + \nu$ as introduced below).

Accordingly, because K and G are regular bounded distributions on $\mathbb{R}$ that are generable by C^0 bounded functions on $\mathbb{R}$, it is possible to let

$$\begin{cases} E \equiv \frac{9KG}{3K + G} \\ \nu \equiv \frac{3K - 2G}{2(3K + G)} \end{cases} \Leftrightarrow \begin{cases} K \equiv \frac{E}{3(1 - 2\nu)} \\ G \equiv \frac{E}{2(1 + \nu)} \end{cases}$$

$$\begin{aligned} \kappa &\equiv \frac{K}{G} \\ &= \frac{2(1 + \nu)}{3(1 - 2\nu)} \end{aligned}$$

with

$$\begin{cases} \lim_{\omega \rightarrow 0} (E) = \frac{9\mathcal{H}_\infty \mathcal{G}_\infty}{3\mathcal{H}_\infty + \mathcal{G}_\infty} \\ \lim_{\omega \rightarrow 0} (\nu) = \frac{3\mathcal{H}_\infty - 2\mathcal{G}_\infty}{2(3\mathcal{H}_\infty + \mathcal{G}_\infty)} \end{cases} \Leftrightarrow \begin{cases} \mathcal{H}_\infty = \frac{\lim_{\omega \rightarrow 0} (E)}{3(1 - 2\lim_{\omega \rightarrow 0} (\nu))} \\ \mathcal{G}_\infty = \frac{\lim_{\omega \rightarrow 0} (E)}{2(1 + \lim_{\omega \rightarrow 0} (\nu))} \end{cases}$$

$$\lim_{\omega \rightarrow 0} \kappa = \frac{\mathcal{H}_\infty}{\mathcal{G}_\infty}$$

$$\begin{cases} \lim_{\omega \rightarrow \pm\infty} (E) = \frac{9\mathcal{H}_0 \mathcal{G}_0}{3\mathcal{H}_0 + \mathcal{G}_0} \\ \lim_{\omega \rightarrow \pm\infty} (\nu) = \frac{3\mathcal{H}_0 - 2\mathcal{G}_0}{2(3\mathcal{H}_0 + \mathcal{G}_0)} \end{cases} \Leftrightarrow \begin{cases} \mathcal{H}_0 = \frac{\lim_{\omega \rightarrow \pm\infty} (E)}{3(1 - 2\lim_{\omega \rightarrow \pm\infty} (\nu))} \\ \mathcal{G}_0 = \frac{\lim_{\omega \rightarrow \pm\infty} (E)}{2(1 + \lim_{\omega \rightarrow \pm\infty} (\nu))} \end{cases}$$

$$\lim_{\omega \rightarrow \pm\infty} \kappa = \frac{\mathcal{H}_0}{\mathcal{G}_0}$$

where

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- E is the *dynamic Young modulus* (also known as *dynamic modulus of elasticity*)
- ν is the *dynamic Poisson number*
- κ is the *bulk-to-shear dynamic-modulus ratio*

It is worth noting that κ is nothing more than a well-behaved proxy for ν . Time-domain homologues of E and ν can be obtained by means of the inverse Fourier transformation, and they can be given physical interpretation in the special case of uniaxial stress state. However, unlike the time-domain homologue of E , the time-domain homologue of ν is not very useful as it is quite difficult both to express mathematically and to determine experimentally [Hilton, 2017].

4.4 Generalised Maxwell material

For convenience, Equation (103) can be re-expressed as

$$\left\{ \begin{array}{l} \mathcal{H} = C_{\mathcal{H}} \delta + \mathcal{H}_{\infty} \vartheta + \left(\sum_{h_{\mathcal{H}}=1}^{M_{\mathcal{H}}} \left(\mathcal{H}_{h_{\mathcal{H}}} e^{-\omega_{h_{\mathcal{H}}} t} \right) \right) \vartheta \\ \quad = C_{\mathcal{H}} \delta + \mathcal{H}_{\infty} \vartheta + \left(\sum_{h_{\mathcal{H}}=1}^{M_{\mathcal{H}}} \left(\mathcal{H}_{h_{\mathcal{H}}} e^{-t/\tau_{h_{\mathcal{H}}}} \right) \right) \vartheta \\ \mathcal{G} = C_{\mathcal{G}} \delta + \mathcal{G}_{\infty} \vartheta + \left(\sum_{h_{\mathcal{G}}=1}^{M_{\mathcal{G}}} \left(\mathcal{G}_{h_{\mathcal{G}}} e^{-\omega_{h_{\mathcal{G}}} t} \right) \right) \vartheta \\ \quad = C_{\mathcal{G}} \delta + \mathcal{G}_{\infty} \vartheta + \left(\sum_{h_{\mathcal{G}}=1}^{M_{\mathcal{G}}} \left(\mathcal{G}_{h_{\mathcal{G}}} e^{-t/\tau_{h_{\mathcal{G}}}} \right) \right) \vartheta \end{array} \right. \quad (115)$$

A linear viscoelastic material for which $\mathcal{H}$ and $\mathcal{G}$ are expressed by Equation (115) with $M_{\mathcal{H}}$ and $M_{\mathcal{G}}$ finite is said to be a *generalised Maxwell material* (also known as *generalised Maxwell–Wiechert material*).

A generalised Maxwell material can be physically interpreted in terms of a pair of independent sets of linear springs and linear viscous dampers (specifically, in terms of a linear spring of stiffness proportional to $\mathcal{H}_{\infty}$ and a linear viscous damper of viscous damping coefficient proportional to $C_{\mathcal{H}}$ in parallel connection with $M_{\mathcal{H}}$ branches the $h_{\mathcal{H}}$ th of which contains a linear spring of stiffness proportional to $\mathcal{H}_{h_{\mathcal{H}}}$ in series connection with a linear viscous damper of viscous damping coefficient proportional to $\mathcal{H}_{h_{\mathcal{H}}} \omega_{h_{\mathcal{H}}}^{-1} \equiv \mathcal{H}_{h_{\mathcal{H}}} \tau_{h_{\mathcal{H}}}$, and, independently, a linear spring of stiffness proportional to $\mathcal{G}_{\infty}$ and a linear viscous damper of viscous damping coefficient proportional to $C_{\mathcal{G}}$ in parallel connection with $M_{\mathcal{G}}$ branches the $h_{\mathcal{G}}$ th of which contains a linear spring of stiffness proportional to $\mathcal{G}_{h_{\mathcal{G}}}$ in series connection with a linear viscous damper of viscous damping coefficient proportional to $\mathcal{G}_{h_{\mathcal{G}}} \omega_{h_{\mathcal{G}}}^{-1} \equiv \mathcal{G}_{h_{\mathcal{G}}} \tau_{h_{\mathcal{G}}}$).

A generalised Maxwell material is a solid (necessarily with elastic asymptotic response) or a fluid (necessarily with viscous asymptotic response) depending on whether $\mathcal{H}_{\infty}$ and $\mathcal{G}_{\infty}$ are respectively non-null or null. Furthermore, a generalised Maxwell material has elastic initial response or viscous initial response depending on whether $C_{\mathcal{H}}$ and $C_{\mathcal{G}}$ are respectively null or non-null. In particular, a generalised Maxwell material is a linear elastic material if and only if all constants other than $\mathcal{H}_{\infty}$

and $\mathcal{G}_\infty$ are null, whereas it is a linear viscous material if and only if all constants other than $C_{\mathcal{H}}$ and $C_{\mathcal{G}}$ are null.

A *generalised Maxwell solid material with elastic initial response* (hereinafter *generalised Maxwell solid material*) can be defined by

$$\left\{ \begin{array}{l} \kappa = \frac{\mathcal{H}_\infty}{\mathcal{H}_0} \mathfrak{G} + \left(\sum_{i_\kappa=1}^{N_\kappa} \left(\kappa_{i_\kappa} e^{-\omega_{i_\kappa} t} \right) \right) \mathfrak{G} \\ = \frac{\mathcal{H}_\infty}{\mathcal{H}_0} \mathfrak{G} + \left(\sum_{i_\kappa=1}^{N_\kappa} \left(\kappa_{i_\kappa} e^{-t/\tau_{i_\kappa}} \right) \right) \mathfrak{G} \\ \mathcal{G} = \frac{\mathcal{G}_\infty}{\mathcal{G}_0} \mathfrak{G} + \left(\sum_{i_\mathcal{G}=1}^{N_\mathcal{G}} \left(\mathcal{G}_{i_\mathcal{G}} e^{-\omega_{i_\mathcal{G}} t} \right) \right) \mathfrak{G} \\ = \frac{\mathcal{G}_\infty}{\mathcal{G}_0} \mathfrak{G} + \left(\sum_{i_\mathcal{G}=1}^{N_\mathcal{G}} \left(\mathcal{G}_{i_\mathcal{G}} e^{-t/\tau_{i_\mathcal{G}}} \right) \right) \mathfrak{G} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \kappa = \left(1 - \sum_{i_\kappa=1}^{N_\kappa} \left(\kappa_{i_\kappa} \left(1 - e^{-\omega_{i_\kappa} t} \right) \right) \right) \mathfrak{G} \\ = \left(1 - \sum_{i_\kappa=1}^{N_\kappa} \left(\kappa_{i_\kappa} \left(1 - e^{-t/\tau_{i_\kappa}} \right) \right) \right) \mathfrak{G} \\ \mathcal{G} = \left(1 - \sum_{i_\mathcal{G}=1}^{N_\mathcal{G}} \left(\mathcal{G}_{i_\mathcal{G}} \left(1 - e^{-\omega_{i_\mathcal{G}} t} \right) \right) \right) \mathfrak{G} \\ = \left(1 - \sum_{i_\mathcal{G}=1}^{N_\mathcal{G}} \left(\mathcal{G}_{i_\mathcal{G}} \left(1 - e^{-t/\tau_{i_\mathcal{G}}} \right) \right) \right) \mathfrak{G} \end{array} \right. \quad (116)$$

with

$$\left\{ \begin{array}{l} \frac{\mathcal{H}_\infty}{\mathcal{H}_0} = 1 - \sum_{i_\kappa=1}^{N_\kappa} \left(\kappa_{i_\kappa} \right) \\ \frac{\mathcal{G}_\infty}{\mathcal{G}_0} = 1 - \sum_{i_\mathcal{G}=1}^{N_\mathcal{G}} \left(\mathcal{G}_{i_\mathcal{G}} \right) \end{array} \right.$$

$$\left\{ \begin{array}{l} \tau_{i_\kappa} = \frac{1}{\omega_{i_\kappa}} \\ \tau_{i_\mathcal{G}} = \frac{1}{\omega_{i_\mathcal{G}}} \end{array} \right.$$

where

- N_κ is a non-null natural constant of quantity dimension one
- $N_\mathcal{G}$ is a non-null natural constant of quantity dimension one
- i_κ is a bound index taking values in $\mathbb{N}_{[1, N_\kappa]}$
- $i_\mathcal{G}$ is a bound index taking values in $\mathbb{N}_{[1, N_\mathcal{G}]}$
- κ_{i_κ} is a non-negative real constant of quantity dimension one to which reference is made as i_κ^{th} *normalised bulk relaxation coefficient*

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- $\mathcal{G}_{i_s}$ is a non-negative real constant of quantity dimension one to which reference is made as i_s^{th} *normalised shear relaxation coefficient*
- ω_{i_ℓ} is a positive real constant of the same quantity dimension as ω to which reference is made as i_ℓ^{th} *bulk relaxation angular frequency*
- ω_{i_s} is a positive real constant of the same quantity dimension as ω to which reference is made as i_s^{th} *shear relaxation angular frequency*
- τ_{i_ℓ} is a positive real constant of the same quantity dimensions as ω^{-1} to which reference is made as i_ℓ^{th} *bulk relaxation time constant*
- τ_{i_s} is a positive real constant of the same quantity dimensions as ω^{-1} to which reference is made as i_s^{th} *shear relaxation time constant*

Equation (116) defines a generalised Maxwell solid material by means of finite sums; nevertheless, by abuse of language, $\mathcal{K}$ and $\mathcal{G}$ are said to be expressed by Equation (116) as respectively a *bulk relaxation Prony series* and a *shear relaxation Prony series*. Accordingly, N_ℓ and N_s are said to be respectively the *number of terms of the bulk relaxation Prony series* and the *number of terms of the shear relaxation Prony series*, whereas the $\mathcal{K}_{i_\ell} e^{-\omega_{i_\ell} t} \equiv \mathcal{K}_{i_\ell} e^{-t/\tau_{i_\ell}}$ with i_ℓ in $\mathbb{N}_{[1, N_\ell]}$ and the $\mathcal{G}_{i_s} e^{-\omega_{i_s} t} \equiv \mathcal{G}_{i_s} e^{-t/\tau_{i_s}}$ with i_s in $\mathbb{N}_{[1, N_s]}$ are said to be respectively the *terms of the bulk relaxation Prony series* and the *terms of the shear relaxation Prony series*.

Equation (116) defines a solid linear viscoelastic material with elastic initial response if and only if

$\sum_{i_\ell=1}^{N_\ell} (\mathcal{K}_{i_\ell})$ is in $\mathbb{R}_{[0,1]}$ and $\sum_{i_s=1}^{N_s} (\mathcal{G}_{i_s})$ is in $\mathbb{R}_{[0,1]}$; in particular, Equation (116) defines a linear elastic

material if and only if $\sum_{i_\ell=1}^{N_\ell} (\mathcal{K}_{i_\ell})$ equals 0 and $\sum_{i_s=1}^{N_s} (\mathcal{G}_{i_s})$ equals 0 (or, equivalently, if and only if all the

$\mathcal{K}_{i_\ell}$ and all the $\mathcal{G}_{i_s}$ are null). As is evident, Equation (116) is incompatible with **Condition B**;

accordingly, we assume that **Condition A** is satisfied.

A generalised Maxwell solid material can be physically interpreted in terms of a pair of independent sets of linear springs and linear viscous dampers (specifically, in terms of a linear spring of stiffness

proportional to $1 - \sum_{i_\ell=1}^{N_\ell} (\mathcal{K}_{i_\ell})$ in parallel connection with N_ℓ branches the i_ℓ^{th} of which contains a

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linear spring of stiffness proportional to κ_{i_ϵ} in series connection with a linear viscous damper of viscous damping coefficient proportional to $\kappa_{i_\epsilon} \omega_{i_\epsilon}^{-1} \equiv \kappa_{i_\epsilon} \tau_{i_\epsilon}$, and, independently, a linear spring of stiffness proportional to $1 - \sum_{i_\rho=1}^{N_\rho} (\mathcal{J}_{i_\rho})$ in parallel connection with N_ρ branches the i_ρ^{th} of which contains a linear spring of stiffness proportional to $\mathcal{J}_{i_\rho}$ in series connection with a linear viscous damper of viscous damping coefficient proportional to $\mathcal{J}_{i_\rho} \omega_{i_\rho}^{-1} \equiv \mathcal{J}_{i_\rho} \tau_{i_\rho}$).

Hereinafter, the simplifying assumptions hold

$$\begin{aligned} N_\kappa &= N_\rho \\ &\equiv N \end{aligned}$$

$$\begin{aligned} i_\kappa &= i_\rho \\ &\equiv i, \quad i \in \mathbb{N}_{[1,N]} \end{aligned}$$

$$\begin{aligned} \omega_{i_\epsilon} &= \omega_{i_\rho} \\ &\equiv \omega_i, \quad i \in \mathbb{N}_{[1,N]} \end{aligned} \Leftrightarrow \begin{aligned} \tau_{i_\epsilon} &= \tau_{i_\rho} \\ &\equiv \tau_i, \quad i \in \mathbb{N}_{[1,N]} \end{aligned}$$

which implies

$$\begin{cases} \kappa_{i_\epsilon} \equiv \kappa_i \\ \mathcal{J}_{i_\rho} \equiv \mathcal{J}_i \end{cases}, \quad i \in \mathbb{N}_{[1,N]}$$

where

- N is a non-null natural constant of quantity dimension one
- i is a bound index taking values in $\mathbb{N}_{[1,N]}$
- ω_i is a positive real constant of the same quantity dimension as ω to which reference is made as i^{th} *relaxation angular frequency*
- τ_i is a positive real constant of the same quantity dimensions as ω^{-1} to which reference is made as i^{th} *relaxation time constant*

Relations hold

$$\left\{ \begin{array}{l} \kappa^{(1)} = \left(\frac{\mathcal{H}_\infty}{\mathcal{H}_0} + \sum_{i=1}^N (\kappa_i) \right) \delta - \left(\sum_{i=1}^N (\kappa_i \omega_i e^{-\omega_i t}) \right) \mathfrak{g} \\ \quad = \left(\frac{\mathcal{H}_\infty}{\mathcal{H}_0} + \sum_{i=1}^N (\kappa_i) \right) \delta - \left(\sum_{i=1}^N \left(\frac{\kappa_i}{\tau_i} e^{-t/\tau_i} \right) \right) \mathfrak{g} \\ \mathscr{J}^{(1)} = \left(\frac{\mathcal{G}_\infty}{\mathcal{G}_0} + \sum_{i=1}^N (\mathscr{J}_i) \right) \delta - \left(\sum_{i=1}^N (\mathscr{J}_i \omega_i e^{-\omega_i t}) \right) \mathfrak{g} \\ \quad = \left(\frac{\mathcal{G}_\infty}{\mathcal{G}_0} + \sum_{i=1}^N (\mathscr{J}_i) \right) \delta - \left(\sum_{i=1}^N \left(\frac{\mathscr{J}_i}{\tau_i} e^{-t/\tau_i} \right) \right) \mathfrak{g} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \kappa^{(1)} = \delta - \left(\sum_{i=1}^N (\kappa_i \omega_i e^{-\omega_i t}) \right) \mathfrak{g} \\ \quad = \delta - \left(\sum_{i=1}^N \left(\frac{\kappa_i}{\tau_i} e^{-t/\tau_i} \right) \right) \mathfrak{g} \\ \mathscr{J}^{(1)} = \delta - \left(\sum_{i=1}^N (\mathscr{J}_i \omega_i e^{-\omega_i t}) \right) \mathfrak{g} \\ \quad = \delta - \left(\sum_{i=1}^N \left(\frac{\mathscr{J}_i}{\tau_i} e^{-t/\tau_i} \right) \right) \mathfrak{g} \end{array} \right.$$

$$\left\{ \begin{array}{l} k = k' + jk'' \\ \quad = 1_{\mathbb{R}} - \underbrace{\sum_{i=1}^N \left(\kappa_i \frac{1}{1 + (\omega/\omega_i)^2} \right)}_{k'} + j \underbrace{\sum_{i=1}^N \left(\kappa_i \frac{\omega/\omega_i}{1 + (\omega/\omega_i)^2} \right)}_{k''} \\ \quad = 1_{\mathbb{R}} - \underbrace{\sum_{i=1}^N \left(\kappa_i \frac{1}{1 + (\tau_i \omega)^2} \right)}_{k'} + j \underbrace{\sum_{i=1}^N \left(\kappa_i \frac{\tau_i \omega}{1 + (\tau_i \omega)^2} \right)}_{k''} \\ g = g' + jg'' \\ \quad = 1_{\mathbb{R}} - \underbrace{\sum_{i=1}^N \left(\mathscr{J}_i \frac{1}{1 + (\omega/\omega_i)^2} \right)}_{g'} + j \underbrace{\sum_{i=1}^N \left(\mathscr{J}_i \frac{\omega/\omega_i}{1 + (\omega/\omega_i)^2} \right)}_{g''} \\ \quad = 1_{\mathbb{R}} - \underbrace{\sum_{i=1}^N \left(\mathscr{J}_i \frac{1}{1 + (\tau_i \omega)^2} \right)}_{g'} + j \underbrace{\sum_{i=1}^N \left(\mathscr{J}_i \frac{\tau_i \omega}{1 + (\tau_i \omega)^2} \right)}_{g''} \end{array} \right.$$

Hereinafter, the simplifying assumption holds

$$\begin{aligned} \kappa_i &= \mathscr{J}_i \\ &\equiv \mathscr{J}_i \quad , \quad i \in \mathbb{N}_{[1,N]} \end{aligned}$$

where

- $\mathscr{J}_i$ is a non-negative real constant of quantity dimension one to which reference is made as i th *normalised relaxation coefficient*

which implies

$$\begin{aligned} \kappa &= \mathscr{J} \\ &\equiv \mathscr{J} \end{aligned}$$

$$\begin{aligned}\frac{\mathcal{H}_\infty}{\mathcal{H}_0} &= \frac{\mathcal{G}_\infty}{\mathcal{G}_0} \\ &= 1 - \sum_{i=1}^N (\mathcal{J}_i)\end{aligned}$$

$$\begin{aligned}\kappa^{(1)} &= \mathcal{J}^{(1)} \\ &\equiv \mathcal{J}^{(1)}\end{aligned}$$

$$\begin{aligned}k &= g \\ &\equiv s\end{aligned}$$

$$\begin{cases} k' = g' \\ \quad \equiv s' \\ k'' = g'' \\ \quad \equiv s'' \end{cases}$$

$$\begin{cases} E = \frac{9\mathcal{H}_0\mathcal{G}_0}{3\mathcal{H}_0 + \mathcal{G}_0} s \\ \quad = \frac{9\mathcal{H}_\infty\mathcal{G}_\infty}{3\mathcal{H}_\infty + \mathcal{G}_\infty} \frac{s}{1 - \sum_{i=1}^N (\mathcal{J}_i)} \\ \nu = \frac{3\mathcal{H}_0 - 2\mathcal{G}_0}{2(3\mathcal{H}_0 + \mathcal{G}_0)} 1_{\mathbb{R}} \\ \quad = \frac{3\mathcal{H}_\infty - 2\mathcal{G}_\infty}{2(3\mathcal{H}_\infty + \mathcal{G}_\infty)} 1_{\mathbb{R}} \end{cases}$$

$$\begin{aligned}\kappa &= \frac{\mathcal{H}_0}{\mathcal{G}_0} 1_{\mathbb{R}} \\ &= \frac{\mathcal{H}_\infty}{\mathcal{G}_\infty} 1_{\mathbb{R}}\end{aligned}$$

$$\begin{cases} \lim_{\omega \rightarrow 0} (E) = \frac{9\mathcal{H}_\infty\mathcal{G}_\infty}{3\mathcal{H}_\infty + \mathcal{G}_\infty} \\ \quad = \frac{9\mathcal{H}_0\mathcal{G}_0}{3\mathcal{H}_0 + \mathcal{G}_0} \left(1 - \sum_{i=1}^N (\mathcal{J}_i) \right) \\ \lim_{\omega \rightarrow 0} (\nu) = \nu \\ \quad = \frac{3\mathcal{H}_\infty - 2\mathcal{G}_\infty}{2(3\mathcal{H}_\infty + \mathcal{G}_\infty)} \\ \quad = \frac{3\mathcal{H}_0 - 2\mathcal{G}_0}{2(3\mathcal{H}_0 + \mathcal{G}_0)} \end{cases} \Leftrightarrow \begin{cases} \mathcal{H}_\infty = \mathcal{H}_0 \left(1 - \sum_{i=1}^N (\mathcal{J}_i) \right) \\ \quad = \frac{\lim_{\omega \rightarrow 0} (E)}{3(1 - 2\nu)} \\ \mathcal{G}_\infty = \mathcal{G}_0 \left(1 - \sum_{i=1}^N (\mathcal{J}_i) \right) \\ \quad = \frac{\lim_{\omega \rightarrow 0} (E)}{2(1 + \nu)} \end{cases}$$

$$\begin{aligned}
 \lim_{\omega \rightarrow 0}(\kappa) &= \kappa \\
 &= \frac{\mathcal{H}_\infty}{\mathcal{G}_\infty} \\
 &= \frac{\mathcal{H}_0}{\mathcal{G}_0}
 \end{aligned}$$

$$\left\{ \begin{aligned}
 \lim_{\omega \rightarrow \pm\infty}(E) &= \frac{9\mathcal{H}_0\mathcal{G}_0}{3\mathcal{H}_0 + \mathcal{G}_0} \\
 &= \frac{9\mathcal{H}_\infty\mathcal{G}_\infty}{3\mathcal{H}_\infty + \mathcal{G}_\infty} \frac{1}{1 - \sum_{i=1}^N(d_i)} \\
 \lim_{\omega \rightarrow 0}(\nu) &= \nu \\
 &= \frac{3\mathcal{H}_0 - 2\mathcal{G}_0}{2(3\mathcal{H}_0 + \mathcal{G}_0)} \\
 &= \frac{3\mathcal{H}_\infty - 2\mathcal{G}_\infty}{2(3\mathcal{H}_\infty + \mathcal{G}_\infty)}
 \end{aligned} \right\} \Leftrightarrow \left\{ \begin{aligned}
 \mathcal{H}_0 &= \mathcal{H}_\infty \frac{1}{1 - \sum_{i=1}^N(d_i)} \\
 &= \frac{\lim_{\omega \rightarrow \pm\infty}(E)}{3(1-2\nu)} \\
 \mathcal{G}_0 &= \mathcal{G}_\infty \frac{1}{1 - \sum_{i=1}^N(d_i)} \\
 &= \frac{\lim_{\omega \rightarrow \pm\infty}(E)}{2(1+\nu)}
 \end{aligned} \right\}$$

$$\begin{aligned}
 \lim_{\omega \rightarrow 0}(\kappa) &= \kappa \\
 &= \frac{\mathcal{H}_0}{\mathcal{G}_0} \\
 &= \frac{\mathcal{H}_\infty}{\mathcal{G}_\infty}
 \end{aligned}$$

where

- d_i is the *normalised relaxation modulus*
- s is the *normalised dynamic modulus*
- s' is the *normalised storage dynamic modulus*
- s'' is the *normalised loss dynamic modulus*

By abuse of language, as in the case of $\mathcal{H}$ and $\mathcal{G}$, d_i is said to be expressed as a *relaxation Prony series*. Accordingly, N is said to be the *number of terms of the relaxation Prony series*, whereas the $d_i e^{-\omega_i t} \equiv d_i e^{-t/\tau_i}$ with i in $\mathbb{N}_{[1,N]}$ are said to be the *terms of the relaxation Prony series*.

Under the simplifying assumptions made, ν and κ are constant distributions on $\mathbb{R}$, and hence reference can be made to them as respectively *constant dynamic Poisson number* and *constant bulk-to-shear dynamic-modulus ratio*.

4.5 Conclusion

This third theoretical study allows obtaining a non-heuristic but application-oriented characterisation of a linear viscoelastic material (shorthand for non-ageing homogeneous isotropic linear viscoelastic material under isothermal adiabatic conditions) both in the time domain and in the frequency domain. Well-known physical aspects are omitted to focus on unclear mathematical aspects. The obtained characterisation results from adopting the hereditary approach and from applying the findings of both the first theoretical study and the second theoretical study, and, in the time domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null time). The obtained characterisation holds under mildly restrictive conditions and contains specification of the input spaces. However, no attempt is made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions are identified for formulations to be mathematically tenable.

A linear viscoelastic material is assumed a Cauchy continuum subjected to infinitesimal displacements and infinitesimal displacement gradients, and it is regarded as a pair of independent power-conjugate convolution systems having the property of fading memory. The hereditary approach is adopted, and the theory of distributions is used.

Bulk relaxation kernel and shear relaxation kernel as well as bulk relaxation modulus and shear relaxation modulus are introduced. A linear viscoelastic material is classified as either solid or fluid; both its initial response and its asymptotic response are classified as either elastic or viscous. Each of the bulk relaxation modulus and the shear relaxation modulus is expressed as the sum of a non-negative real constant of suitable quantity dimension times the unit impulse on $\mathbb{R}$, a non-negative real constant of suitable quantity dimension times the unit step distribution on $\mathbb{R}$, and a causal regular distribution vanishing at infinity on $\mathbb{R}$ (in particular, for a fluid linear viscoelastic material, a causal regular integrable distribution on $\mathbb{R}$) the result of whose localisation at $\mathbb{R}_{(0,+\infty)}$ is a distribution on $\mathbb{R}_{(0,+\infty)}$ generated by a strictly completely monotonic function on $\mathbb{R}_{(0,+\infty)}$.

Dynamic bulk modulus and dynamic shear modulus as well as dynamic bulk viscosity and dynamic shear viscosity are introduced; each of them is expressed as the sum of a storage term and a loss term. Dynamic Young modulus, dynamic Poisson number, and bulk-to-shear dynamic-modulus ratio are introduced. Normalised quantities are introduced.

A linear elastic material (shorthand for non-ageing homogeneous isotropic linear elastic material under isothermal adiabatic conditions) and a linear viscous material (shorthand for non-ageing homogeneous isotropic linear viscous material under isothermal adiabatic conditions) are obtained as limiting cases. A generalised Maxwell material is considered as a special case.

5 Modelling study: Finite-element analysis of seat occupants

5.1 Introduction

As expounded in Section 1.3.3, for automotive applications, a finite-element model of a seat occupant should be complex enough to allow predicting, possibly in a range of sitting configurations, both contact pressure and transmissibility at an interface with a deformable seat (both at seat cushion and at seat back). To date, such a model has not been available in the literature.

In light of the above, a *modelling study* was conducted to investigate whether it was possible to *calibrate* a finite-element model of a seat occupant in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back). For the parts defined as deformable bodies, a linear viscoelastic material (shorthand for non-ageing homogeneous isotropic linear viscoelastic material under isothermal adiabatic conditions) was preferred to a simpler linear elastic material (shorthand for non-ageing homogeneous isotropic linear elastic material under isothermal adiabatic conditions) with Rayleigh damping because it had a better physical interpretation and allowed decoupling static calibration from dynamic calibration.

In the following, the method is described, the results are presented, and a discussion is provided. The material in Chapter 4 is assumed known.

5.2 Method

Several amongst the finite-element models of seat occupants developed to investigate dynamic sitting comfort had torsos defined as sets of interconnected rigid bodies [Zheng, 2012; Liu *et al.*, 2015; Gao *et al.*, 2021]. Accordingly, they were calibrated in terms of apparent mass of occupant buttocks only. For the finite-element model described in this modelling study, the simplification of torso defined as a set of interconnected rigid bodies had to be overcome. Nevertheless, a *minimum complexity* requirement was imposed, and attention was focussed on the usually dominant movements parallel to the *midsagittal plane* of the seat occupant. The *frequency band of interest* was chosen as ranging from 0.75 Hz to 15 Hz in continuity with previous studies [Zheng, 2012; Liu *et al.*, 2015; Gao *et al.*, 2021].

5.2.1 Geometric properties

For this modelling study, a finite-element model of a seat occupant (hereinafter *occupant model*) was developed in Simulia Abaqus/CAE 2020 environment. The geometric properties of the occupant model were adjusted, from those of the finite-element model developed in a modelling study conducted by Liu *et al.* [2015], to approximate those of one of the participants in a laboratory experimental study conducted by Yin [2022] (hereinafter *human subject*). On the one hand, the study conducted by Liu *et al.* [2015] was chosen because it was accompanied by a publicly available set of files in IGES format defining a piecewise geometric model of a seat occupant; on the other hand, the study conducted by Yin [2022] was chosen because it contained results of recent experimental laboratory measurements of apparent mass of seat occupants obtained with a rigid seat having reclined seat back. The human subject was a 32-year-old white male having body mass (weight) 74 kg, stature (body height) 180 cm, and body mass index 22.8 kg m^{-2} ; amongst the participants in the study conducted by Yin [2022], the human subject was chosen as the one whose body mass (weight) and stature (body height) most closely approximated estimates of the 50th-percentile (namely, median) body mass (weight) and stature (body height) of the population of the British men aged 19 to 65, which are respectively 75 kg and 174 cm [Pheasant and Haslegrave, 2006].

A part of the occupant model was classified as internal or external depending on whether it was included or not within some other part of the occupant model. Overall views of the occupant model are shown in **Figure 3**. Detail views of internal parts and external parts of the occupant model compared with pictorial representations of real-world counterparts are shown in **Figure 4**.

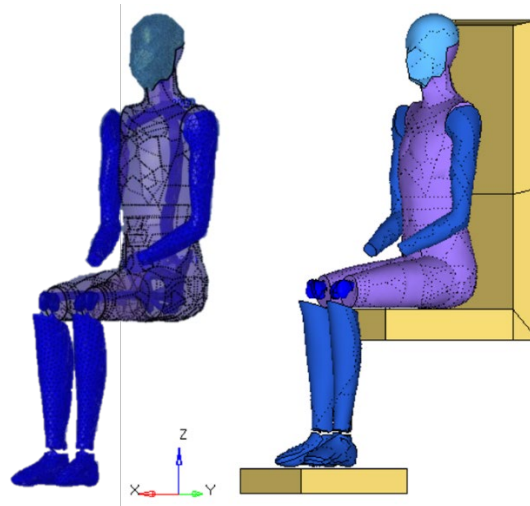


Figure 3 Overall views of the occupant model. **Left:** internal parts and external parts. **Right:** external parts.



Figure 4 Detail views of internal parts and external parts of the occupant model compared with pictorial representations of real-world counterparts. Adapted from [Human skeleton upper body anterior view](#), by A. Meyerson, 2015. Used under [CC BY-SA 3.0](#).

In accordance with the minimum complexity requirement, the rib cage was not represented (and neither were the viscera); accordingly, non-anatomical parts were created to fulfil its structural function. The internal parts of the occupant model were the arm–forearm–assembly supports (hereinafter, by abuse of language, *clavicles*), the *upper spine*, the *middle spine*, the *lower spine*, the *pelvis–coccyx–sacrum assembly*, and the *femurs*. The external parts of the occupant model were the *head*, the *arm–forearm assemblies*, the *legs*, the *feet*, and the *neck–torso–buttocks–thighs assembly*.

The three-part spine of the occupant model was designed to represent approximately the *cervical anterior arch* (also known as *cervical lordosis*), the *thoracic posterior arch* (also known as *thoracic kyphosis*), the *lumbar anterior arch* (also known as *lumbar lordosis*), and the *sacrococcygeal posterior arch* (also known as *sacrococcygeal kyphosis*) whilst maintaining reasonable agreement with published data of length of spinal cord segments [Ko *et al.*, 2004]. The three-part spine of the occupant model was divided to separate the part corresponding to vertebrae connected to true ribs from the part corresponding to vertebrae connected to false ribs (vertebrae connected to true ribs being less flexible than vertebrae connected to false ribs). On the one hand, the upper spine corresponded to the vertebrae from C1 to C7 (namely, to the *cervical spine*); on the other hand, the middle spine corresponded to the vertebrae from T1 to T10 rather than to the vertebrae for T1 to

T12 (namely, rather than to the *thoracic spine*), and the lower spine corresponded to the vertebrae from T11 to L5 rather than to the vertebrae for L1 to L5 (namely, rather than to the *lumbar spine*). Detail views of internal parts of the occupant model compared with pictorial representations of real-world counterparts are shown in **Figure 5**.

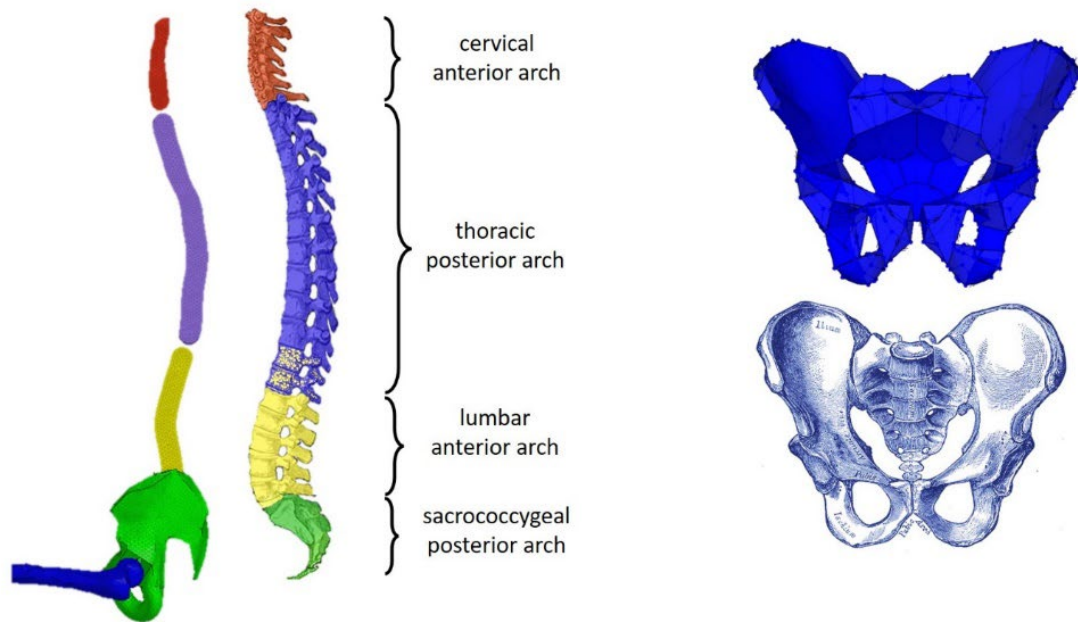


Figure 5 Detail views of internal parts of the occupant model compared with pictorial representations of real-world counterparts. Adapted from *Wikimedia Commons, the free media repository*, 2024. In the public domain. **Left:** Three-part spine. **Right:** pelvis–coccyx–sacrum assembly.

5.2.2 Mechanical properties

In accordance with the minimum complexity requirement, the occupant model was meshed by using linear (as opposed to quadratic) stress–displacement solid elements; they were chosen because, despite their known issues of stiffness and slow numerical convergence with mesh refinement, they had been successfully used in previous studies [Verver *et al.*, 2004; Grujicic *et al.*, 2009; Savonnet *et al.*, 2018; Alawneh *et al.*, 2022]. Specifically, the choice fell on four-node linear tetrahedral stress–displacement solid elements of type C3D4, which could fill the space delimited by a complex surface more efficiently than the alternative eight-node linear hexahedral stress–displacement solid elements of type C3D8 could.

In accordance with the minimum complexity requirement, all parts of the occupant model except the neck–torso–buttocks–thighs assembly were defined as rigid bodies with homogeneous mass density. Besides, the neck–torso–buttocks–thighs assembly was defined as a deformable body with homogeneous mass density. The mesh of the neck–torso–buttocks–thighs assembly had a total of 49438 elements and 10222 nodes.

The mass density of the neck–torso–buttocks–thighs assembly was the same used for the finite-element model developed by Liu *et al.* [2015]. For the other parts of the occupant model, mass density was chosen to match the body mass (weight) of the human subject whilst maintaining reasonable agreement with published data of mass fraction of body segments [Dempster and Gaughran, 1967; Adolphe *et al.*, 2017]. Inertial properties of the parts of the occupant model are listed in **Table 4**.

Table 4 Inertial properties of the parts of the occupant model.

Part	Mass density / (kg m^{-3})	Mass / kg	Mass fraction / %
Clavicle (left)	5392	0.7	0.9
Clavicle (right)	5392	0.7	0.9
Cervical spine	5392	0.3	0.4
Middle spine	5392	1.3	1.7
Lower spine	5392	0.8	1.0
Pelvis–coccyx–sacrum assembly	2480	2.0	2.7
Femur (left)	2480	0.8	1.0
Femur (right)	2480	0.8	1.0
Head	1000	7.0	9.5
Arm–forearm assembly (left)	2480	4.5	6.1
Arm–forearm assembly (right)	2480	4.5	6.1
Leg (left)	2144	4.8	6.5
Leg (right)	2144	4.8	6.5
Foot (left)	2144	1.6	2.1
Foot (right)	2144	1.6	2.1
Neck–torso–buttocks–thighs assembly	1085	37.9	51.2
Total		74	100

In accordance with the minimum complexity requirement, a linear viscoelastic material (specifically, a generalised Maxwell solid material) was assigned to the neck–torso–buttocks–thighs assembly. For such assembly, a linear viscoelastic material was preferred to a simpler linear elastic material with Rayleigh damping because it had a better physical interpretation and allowed decoupling static calibration from dynamic calibration. In accordance with the treatment and simplifying assumptions of Chapter 5 (but, for convenience, using frequency instead of angular frequency), the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly of the occupant model was completely determined by setting number of terms of the relaxation Prony series, normalised relaxation coefficients, relaxation frequencies (or, equivalently, relaxation time constants), null-frequency dynamic Young modulus, and constant dynamic Poisson number (or, equivalently, constant bulk-to-shear dynamic-modulus ratio). The properties of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly of the occupant model were determined so that the normalised loss dynamic modulus was approximately constant on the frequency band of interest, and hence so that the dissipative behaviour was approximately uniform thereon. Properties

of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly of the occupant model are listed in **Table 5**. The normalised dynamic modulus of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly of the occupant model is shown in **Figure 6**. During calibration, it was found beneficial to provide the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly with a limited amount of stiffness-proportional Rayleigh damping; the stiffness-proportional Rayleigh damping coefficient was heuristically set to 1.5×10^{-3} s.

Table 5 Properties of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly of the occupant model.

Term of the relaxation Prony series	Normalised relaxation coefficient / 1	Relaxation frequency / Hz	Relaxation time constant / s
1 st	8.57×10^{-2}	$1.58 \times 10^{+2}$	1.01×10^{-3}
2 nd	8.57×10^{-2}	$5.00 \times 10^{+1}$	3.18×10^{-3}
3 rd	8.57×10^{-2}	$1.58 \times 10^{+1}$	1.01×10^{-2}
4 th	8.57×10^{-2}	5.00×10^0	3.18×10^{-2}
5 th	8.57×10^{-2}	1.58×10^0	1.01×10^{-1}
6 th	8.57×10^{-2}	5.00×10^{-1}	3.18×10^{-1}
7 th	8.57×10^{-2}	1.58×10^{-1}	1.01×10^0
A relaxation time constant is the multiplicative inverse of 2π times the corresponding relaxation frequency.			

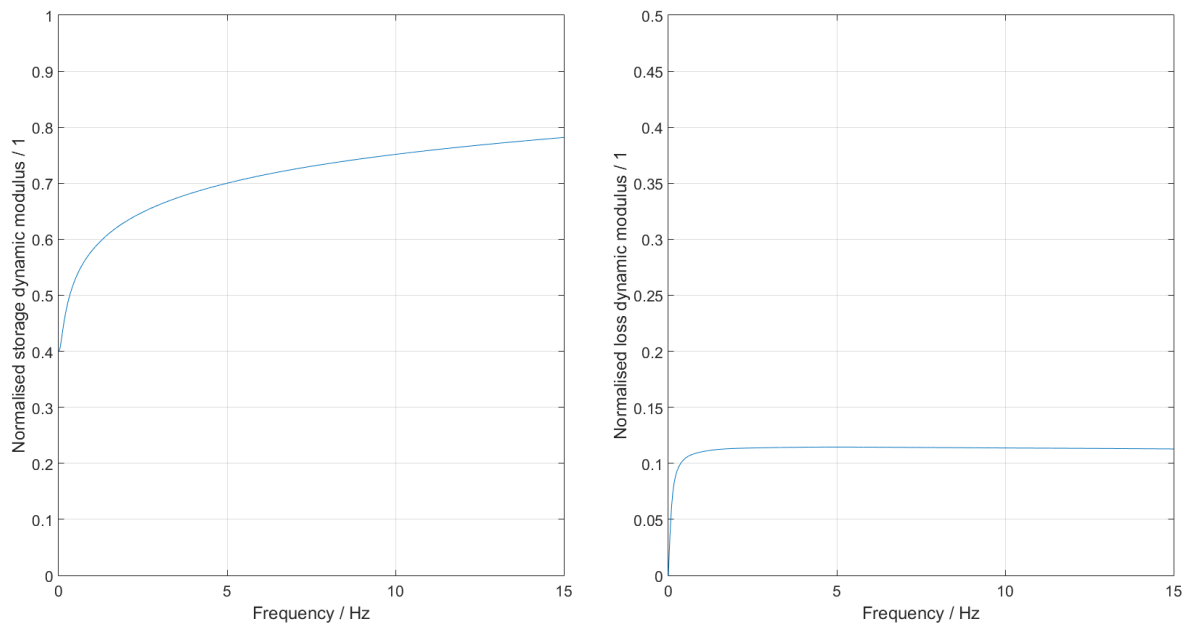


Figure 6 Normalised dynamic modulus of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly of the occupant model. **Left:** normalised storage dynamic modulus vs frequency. **Right:** normalised loss dynamic modulus vs frequency. The normalised storage dynamic modulus and the normalised loss dynamic modulus have respectively even parity and odd parity (because they are respectively the real part and the imaginary part of the Fourier transform of a real-valued integrable distribution on $\mathbb{R}$); accordingly, they are shown only for non-negative frequency.

The null-frequency dynamic Young modulus of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly was initially set to 1 kPa on the basis of a previous study [Liu,

2016] and then adjusted to 9.25 kPa during calibration. Because a human body is mainly composed of water, which is nearly incompressible, the constant dynamic Poisson number of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly was set to 0.475, which corresponds to a constant bulk-to-shear dynamic-modulus ratio approximately equal to 20. Higher constant dynamic Poisson number and constant bulk-to-shear dynamic-modulus ratio were avoided to prevent problems of numerical convergence.

Tie constraints were imposed at the permanent interfaces between neck–torso–buttocks–thighs assembly and clavicles, cervical spine, middle spine, lower spine, pelvis–coccyx–sacrum assembly, femurs, and head to prevent any relative motion. Besides, the parts of the occupant model defined as rigid bodies were interconnected by connector elements to represent anatomical joints by means of (frictionless and massless) linear point mechanical joints. The joints of the occupant model were chosen by seeking a compromise between stability and biofidelity in light of the fact that, from the macroscopic point of view, the main interest lay in movements of the whole occupant model parallel to its midsagittal plane. The joints of the occupant model are shown in **Figure 7**.

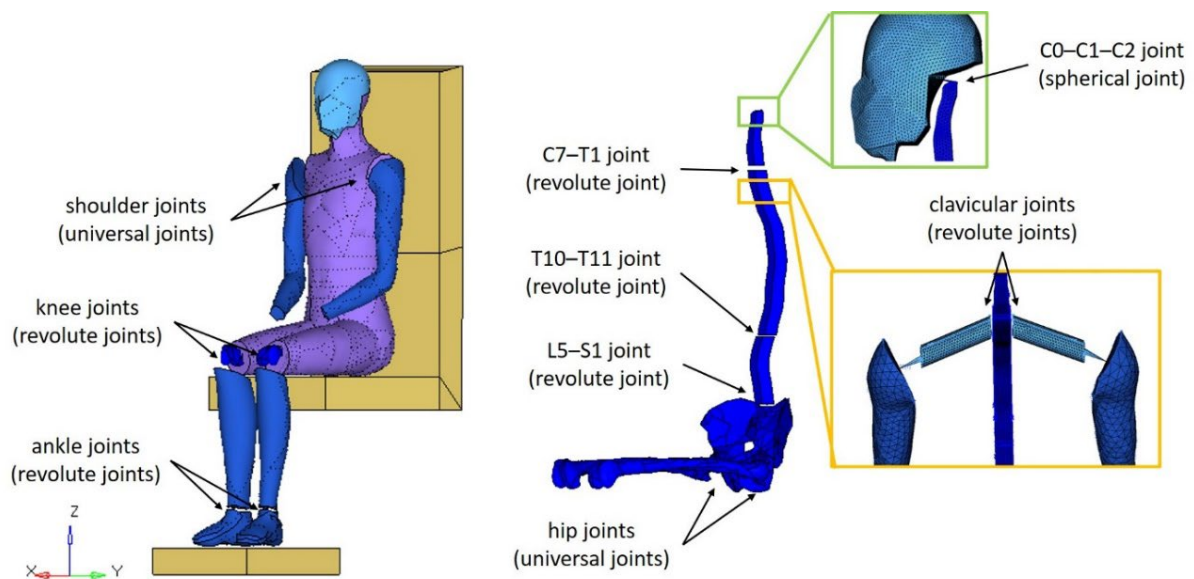


Figure 7 Joints of the occupant model.

The rotational stiffness and the rotational viscous damping coefficient of the joints of the occupant model were initially set to values extrapolated from a previous study [Liu, 2016] and then adjusted during calibration. Mechanical properties of the joints of the occupant model are listed in **Table 6**. In accordance with the minimum complexity requirement, each of the joints having more than one degree of freedom (namely, the spherical joint and each of the universal joints) had the same rotational stiffness and the same rotational viscous damping coefficient at each degree of freedom. The anatomical assembly of a pair of condyloid joints and a pivot joint was represented as a mechanical spherical joint to allow the head moving in a reasonably realistic manner. The anatomical

intervertebral discs were represented as mechanical revolute joints to allow the spine moving only parallel to the midsagittal plane of the occupant model for the benefit of stability. In accordance with the minimum complexity requirement, the non-anatomical clavicular joints were represented by revolute joints to allow the clavicles moving only parallel to the transverse plane of the occupant model. The anatomical ball-and-socket joints were represented as mechanical universal joints to prevent the arm–forearm assemblies and the femurs from rotating about their longitudinal axes for the benefit of stability. The anatomical hinge joints were represented as mechanical revolute joints to allow the knees and the ankles moving in a reasonably realistic manner. Schematic representations of the joints of the occupant model are shown in **Figure 8**.

Table 6 Mechanical properties of the joints of the occupant model.

Joint	Anatomical type	Mechanical type	Rotational stiffness / (N m (rad) ⁻¹)	Rotational viscous damping coefficient / (N m (rad s ⁻¹) ⁻¹)
C0–C1–C2 joint	assembly of two condyloid joints and one pivot joint	spherical joint	0.01	10
C7–T1 joint	intervertebral disc	revolute joint	10	50
T10–T11 joint	intervertebral disc	revolute joint	150	0.1
L5–S1 joint	intervertebral disc	revolute joint	5	0.5
Clavicular joint (left)	N/A	revolute joint	1	1
Clavicular joint (right)	N/A	revolute joint	1	1
Shoulder joint (left)	ball-and-socket joint	universal joint	0.5	0.4
Shoulder joint (right)	ball-and-socket joint	universal joint	0.5	0.4
Hip joint (left)	ball-and-socket joint	universal joint	0.6	0.01
Hip joint (right)	ball-and-socket joint	universal joint	0.6	0.01
Knee joint (left)	hinge joint	revolute joint	2	0.5
Knee joint (right)	hinge joint	revolute joint	2	0.5
Ankle joint (left)	hinge joint	revolute joint	0.08	2
Ankle joint (right)	hinge joint	revolute joint	0.08	2
The C0–C1–C2 joint (also known as craniovertebral junction) consists of the C0–C1 atlanto-occipital joint (having a pair of condyloid joints) and the C1–C2 atlanto-axial joint (having a pivot joint). All joints are (frictionless and massless) linear point mechanical joints. For a joint having more than one degree of freedom (namely, for a spherical joint and a universal joint), the same rotational stiffness and the same rotational viscous damping coefficient apply to each degree of freedom.				

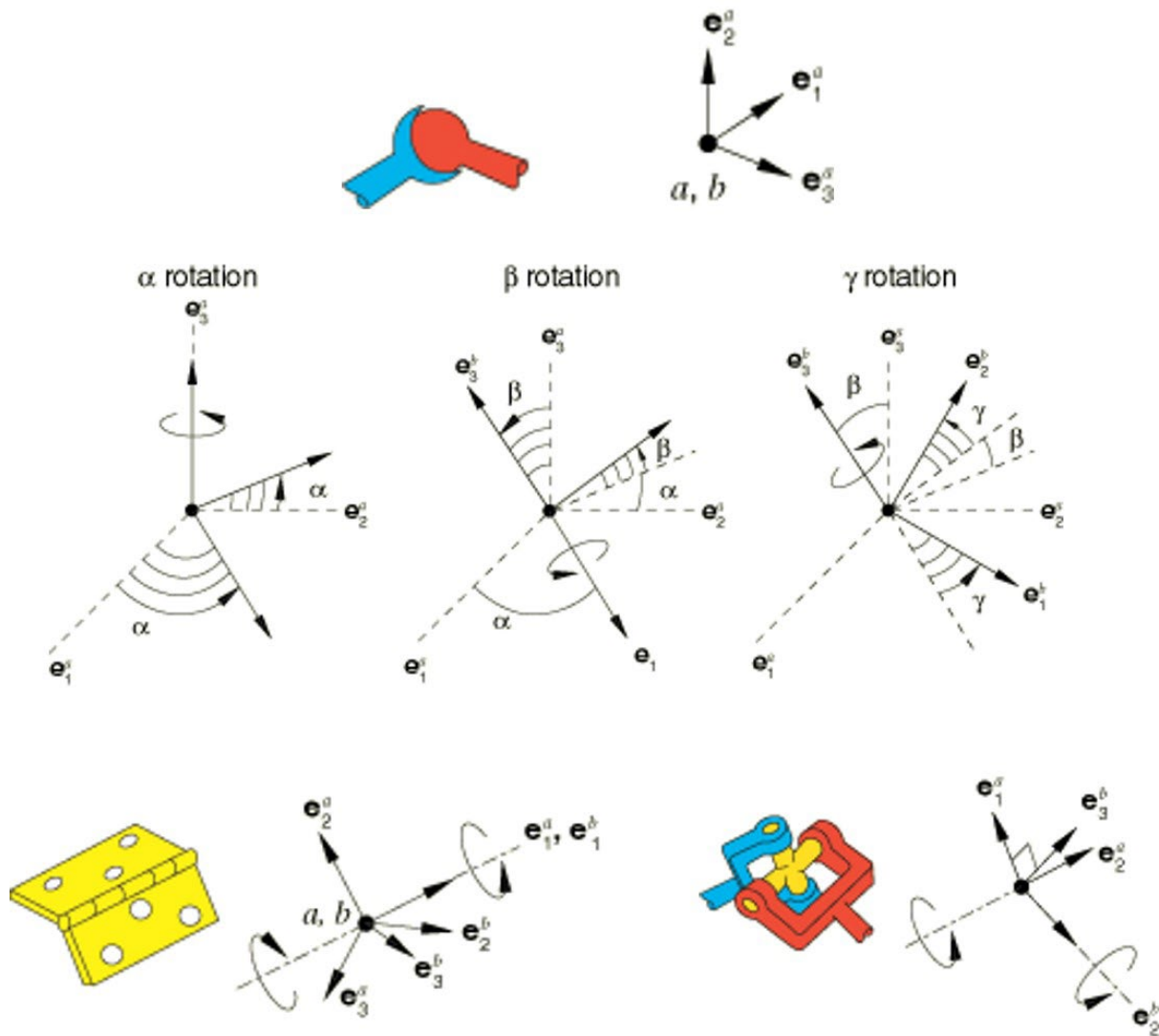


Figure 8 Schematic representations of the joints of the occupant model. From *SIMULIA User Assistance 2020*, by Dassault Systèmes Simulia Corp., 2019. Copyright 2019 by Dassault Systèmes Simulia Corp. **Top:** spherical joints. **Bottom left:** revolute joints. **Bottom right:** universal joint.

5.2.3 Calibration

For calibration, the occupant model was coupled with the finite-element model of a rigid seat–footrest system (hereinafter *seat model*) to obtain the finite-element model of a seat–occupant system (hereinafter *seat–occupant model*). Ideally, calibration should have been performed by considering a range of sitting configurations. However, this was excluded because, in the early stage of development, it would have significantly complicated the calibration process. Furthermore, in some sitting configurations of possible interest, the seat–occupant interface of the seat–occupant model could have been unstable unless some form of occupant restraint system or muscular activity was introduced. The seat model had horizontal flat seat cushion, 20°-reclined flat seat back, and horizontal flat footrest (geometrically disconnected from the rest of the seat model but kinematically bound to it). For the seat model, a reclined flat seat back was preferred to a vertical

flat seat back for the benefit of stability. On the one hand, contact interactions at the seat–occupant interface of the seat–occupant model and contact interactions between parts likely to meet each other were defined by means of the contact-pair algorithm (which is better at handling non-interpenetration but has more limited scope than the alternative general-contact algorithm). On the other hand, contact interactions between parts unlikely to meet each other were defined by means of the general-contact algorithm (which is worse at handling non-interpenetration but has less limited scope than the alternative contact-pair algorithm). At the interface between seat cushion and occupant buttocks and at the interface between seat back and occupant back, the static friction factor and the dynamic friction factor were both adjusted to 0.6 during calibration. At the interface between footrest and occupant feet, the static friction factor and the dynamic friction factor were both adjusted to 0.2 during calibration. In accordance with the minimum complexity requirement, all the other interfaces were frictionless.

The occupant model was calibrated by running a series of two-step calibration simulations with different values of the parameters until a reasonable outcome was obtained. As common for simulations involving significant contact interactions, an explicit dynamic analysis procedure was used for each of the two successive steps of each calibration simulation. For all steps, the time increment was set to 4×10^{-5} s, which was observed to be close to the stable time increment; in other words, a higher time increment would result in problems of numerical convergence.

The first of the two successive steps of each calibration simulation (hereinafter *gravity step*) allowed performing a static calibration. The occupant model was positioned above the seat model in close proximity to but not in contact with it. The seat model was bound to the fixed frame of reference, and the occupant model was subjected to the standard acceleration due to gravity. The arrangement of the seat–occupant model at the beginning of each gravity step is shown in **Figure 9**. The simulated duration of each gravity step was 5 s, which proved enough to reach the condition in which the kinetic energy of the occupant model was virtually null (hereinafter *equilibrium condition*). For the gravity step of each calibration simulation, *sitting configuration* and contact pressure at the seat–occupant interface (hereinafter *contact pressure*) were obtained in equilibrium condition; in the following, both are qualified as *computational*.

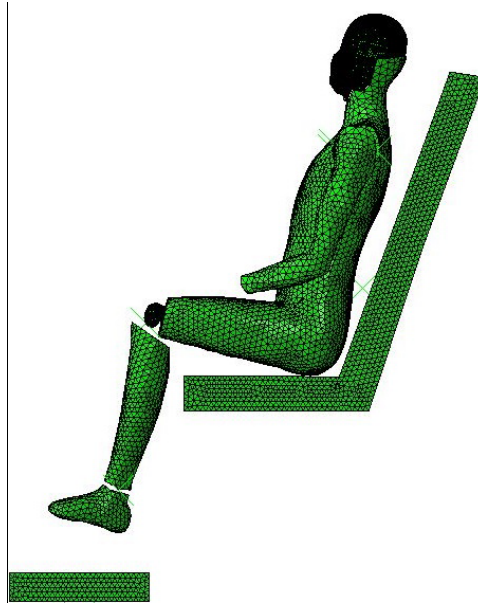


Figure 9 Arrangement of the seat–occupant model at the beginning of each gravity step.

In the equilibrium condition, two basicentric biodynamic coordinate systems were defined in accordance with International Standard ISO 8727:1997 [International Organization for Standardization, 1997c]. Specifically, the $\{x_c, y_c, z_c\}$ coordinate system was located at the interface between seat cushion and occupant buttocks, whereas the $\{x_b, y_b, z_b\}$ coordinate system was located at the interface between seat back and occupant back. For convenience, an auxiliary coordinate system was also defined. Specifically, the $\{x'_b, y'_b, z'_b\}$ coordinate system was located at the interface between seat back and occupant back, its origin being coincident with that of the $\{x_b, y_b, z_b\}$ coordinate system and its axes being parallel to those of the $\{x_c, y_c, z_c\}$ coordinate system. Coordinate systems are shown in **Figure 10**.

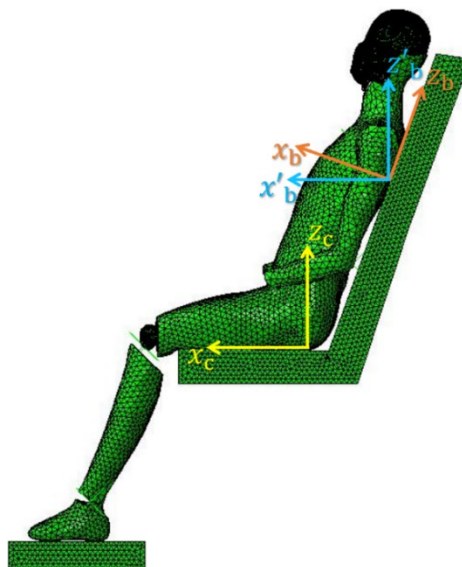


Figure 10 Coordinate systems. $\{x_c, y_c, z_c\}$ = coordinate system at interface between seat cushion and occupant buttocks; $\{x_b, y_b, z_b\}$ = coordinate system at interface between seat back and occupant back; $\{x'_b, y'_b, z'_b\}$ = auxiliary coordinate system at interface between seat back and occupant back.

The second of the two successive steps of each calibration simulation (hereinafter *vibration step*) allowed performing a dynamic calibration. The arrangement of the seat–occupant model at the beginning of each vibration step was the equilibrium condition reached at the end of the previous gravity step. The seat model was unbound from the fixed frame of reference and subjected to a designed excitation (hereinafter the *excitation*), and the occupant model was subjected to the standard acceleration due to gravity. To avoid compromising the equilibrium condition reached at the end of each gravity step, no other changes were made to the seat–occupant model between each gravity step and the subsequent vibration step. The excitation was a whole-body vibration with prescribed acceleration time history used in the study conducted by Yin [2022]. For the rest, the excitation was intended to represent ‘baseline’ conditions with ‘simple’ direction (namely, ‘vertical’ in the direction of the earth’s gravity), ‘wide’ frequency band (namely, ranging from 0.5 Hz to 50 Hz), ‘well-behaved’ waveform (namely, low-crest-factor approximately stationary random with constant auto-power spectral density on the frequency band), ‘relatively modest’ vibration magnitude (namely, 0.35 m s^{-2} in terms of root-mean-square value of unweighted acceleration), and ‘sufficiently long’ simulated duration (namely, 11 s, including an initial taper of 1 s, extracted from the 57 s of the experimental excitation). For the vibration step of a calibration simulation, four pairs of response functions were obtained from input accelerations along the z_c -axis and z'_b -axis and from output forces along the x_c -axis, z_c -axis, x_b -axis, and z_b -axis; because the seat model was rigid, an input acceleration along the z'_b -axis was the same as an input acceleration along the z_c -axis. The response functions were the (complex-valued) $\{z_c, x_c\}$ *apparent mass* (namely, *vertical-to-sagittal-tangent apparent mass at seat cushion*) and the corresponding (real-valued) $\{z_c, x_c\}$ *coherence function* (hereinafter $\{z_c, x_c\}$ *coherence*), the (complex-valued) $\{z_c, z_c\}$ *apparent mass* (namely, *vertical-to-normal apparent mass at seat cushion*) and the corresponding (real-valued) $\{z_c, z_c\}$ *coherence function* (hereinafter $\{z_c, z_c\}$ *coherence*), the (complex-valued) $\{z'_b, x_b\}$ *apparent mass* (namely, *vertical-to-normal apparent mass at seat back*) and the corresponding (real-valued) $\{z'_b, x_b\}$ *coherence function* (hereinafter $\{z'_b, x_b\}$ *coherence*), and the (complex-valued) $\{z'_b, z_b\}$ *apparent mass* (namely, *vertical-to-sagittal-tangent apparent mass at seat back*) and the corresponding (real-valued) $\{z'_b, z_b\}$ *coherence function* (hereinafter $\{z'_b, z_b\}$ *coherence*), all defined on the basis of International Standard ISO 2041:2018 [International Organization for Standardization, 2018]; in the following, all are qualified as *computational*. Both computational apparent mass and computational coherence were calculated, as described in International Standard ISO 18431-1:2005 [International Organization for Standardization, 2005], starting from estimated auto-power spectral density and cross-power spectral density of input accelerations and output forces. In particular, by making the customary assumption of noiseless input accelerations, computational apparent mass was calculated

as a frequency-response function of the first type [Ewins, 2000; He and Fu, 2001; International Organization for Standardization, 2005; Bendat and Piersol, 2010; Coyte *et al.*, 2016].

The static-calibration check consisted in comparing the computational sitting configuration and the computational contact pressure with those obtained from a laboratory measurement session conducted for this modelling study; in the following, the latter are qualified as *experimental*. For such laboratory session, the human subject and a rigid seat having horizontal flat seat cushion, 20°-reclined flat seat back, and horizontal flat footrest were employed. The laboratory measurements of contact pressure were performed by using a Tekscan Body Pressure Measurement System (BPMS) consisting of a Tekscan Seat Sensor 5315 with measuring range from 0 kPa to 34 kPa, a Tekscan Evolution Sensor Handle EH-2, and the Tekscan Research Software (version 7.20C). For calibration of the measuring system, *in situ* checks were performed according to the Tekscan BPMS User Manual Research Software v7.2x (revision O).

The dynamic-calibration check consisted in comparing computational apparent mass modulus, computational apparent mass argument, and computational coherence with those obtained from a vibration test of the study conducted by Yin [2022]; in the following, the latter are qualified as *experimental*. For such vibration test, the excitation, the human subject, and a rigid seat having horizontal flat seat cushion, 20°-reclined flat seat back, and horizontal flat footrest were employed. In principle, the dynamic calibration check could have been performed by comparing the natural modes of vibration obtained computationally from a suitable modal analysis step with those obtained experimentally from a suitable modal analysis test. Such approach would have allowed comparing more naturally the dynamic responses of occupant model and human subject; however, for two reasons, apparent mass was preferred and let serve as a proxy for quantities characterising natural modes of vibration. Firstly, from the computational point of view, a suitable modal analysis step would have required introducing the assumption of linearity at an early stage; secondly, from the experimental point of view, a suitable modal analysis test had not been previously conducted, and it would have been challenging to conduct.

Finally, a terminological note is in order. On the one hand, reference can be made to the $\{z_c, x_c\}$ apparent mass and $\{z_c, x_c\}$ coherence as respectively *cross-axis apparent mass at seat cushion* and *cross-axis coherence at seat cushion*, and reference can be made to the $\{z_c, z_c\}$ apparent mass and $\{z_c, z_c\}$ coherence as respectively *in-line apparent mass at seat cushion* and *in-line coherence at seat cushion*. On the other hand, by abuse of language, reference can be made to the $\{z'_b, x_b\}$ apparent mass and $\{z'_b, x_b\}$ coherence as respectively *cross-axis apparent mass at seat back* and *cross-axis coherence at seat back*, and reference can be made to the $\{z'_b, z_b\}$ apparent mass and $\{z'_b, z_b\}$

coherence as respectively *in-line apparent mass at seat back* and *in-line coherence at seat back*.

However, to obtain response functions at seat back that are truly in-line and cross-axis with respect to the local basicentric biodynamic coordinate system, components of the input acceleration along the z'_b -axis obtained from projections onto the x_b -axis and z_b -axis should have been used instead of the input acceleration along the z'_b -axis itself. This was excluded because it would have resulted not only in introducing unnecessary scale factors but also in questionably treating a unidirectional excitation as multidirectional (with completely mean correlated components).

5.3 Results

Results of sitting configuration and contact pressure and results of apparent mass are presented separately.

5.3.1 Sitting configuration and contact pressure

Contact-pressure data processing was performed in Tekscan Research Software environment. Static-calibration-check outcome in terms of sitting configuration is shown in **Figure 11** (front view) and **Figure 12** (side view). Static-calibration-check outcome in terms of contact pressure is shown in **Figure 13**.

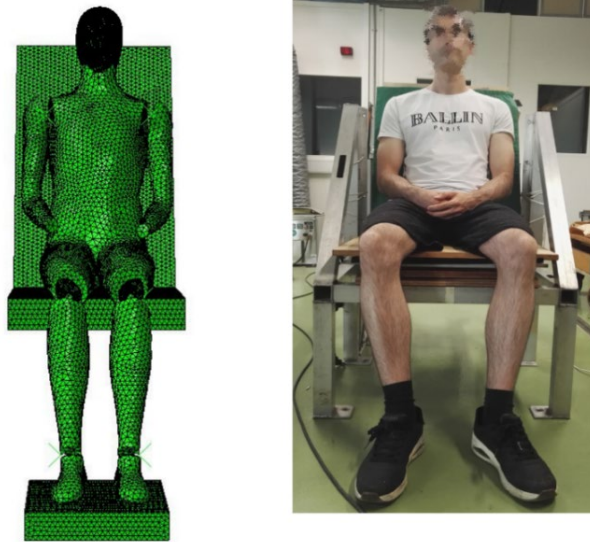


Figure 11 Static-calibration-check outcome in terms of sitting configuration (front view). **Left:** front view of computational sitting configuration. **Right:** front view of experimental sitting configuration.

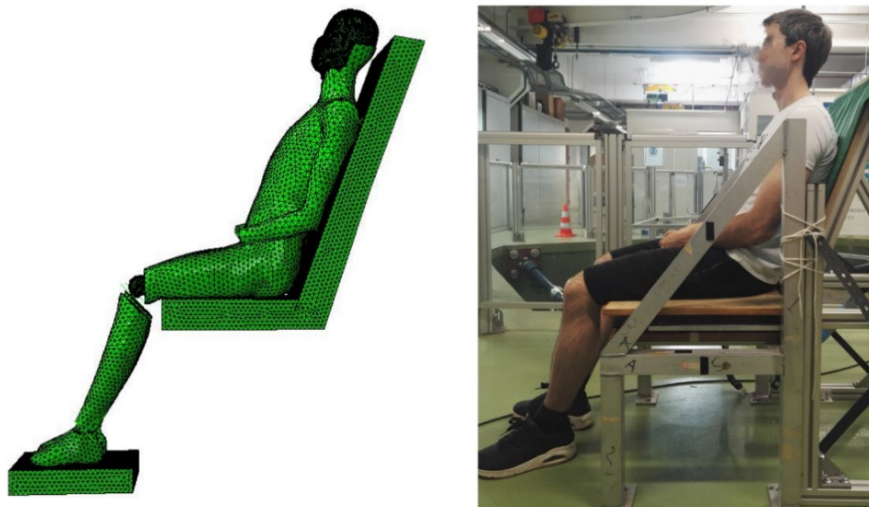


Figure 12 Static-calibration-check outcome in terms of sitting configuration (side view). **Left:** side view of computational sitting configuration. **Right:** side view of experimental sitting configuration.

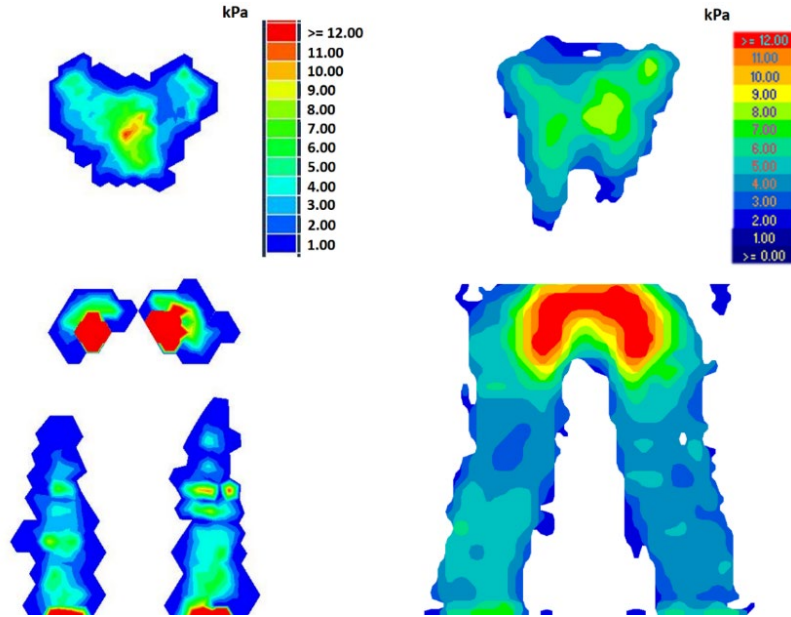


Figure 13 Static-calibration-check outcome in terms of contact pressure. **Left:** computational contact pressure. **Right:** experimental contact pressure.

5.3.2 Apparent mass

Apparent-mass data processing was performed in MATLAB software environment (version 8.5.1.959712) with a frequency resolution of 0.25 Hz. Dynamic-calibration-check outcome in terms of apparent-mass-coherence plots of response functions is shown in **Figure 14**.

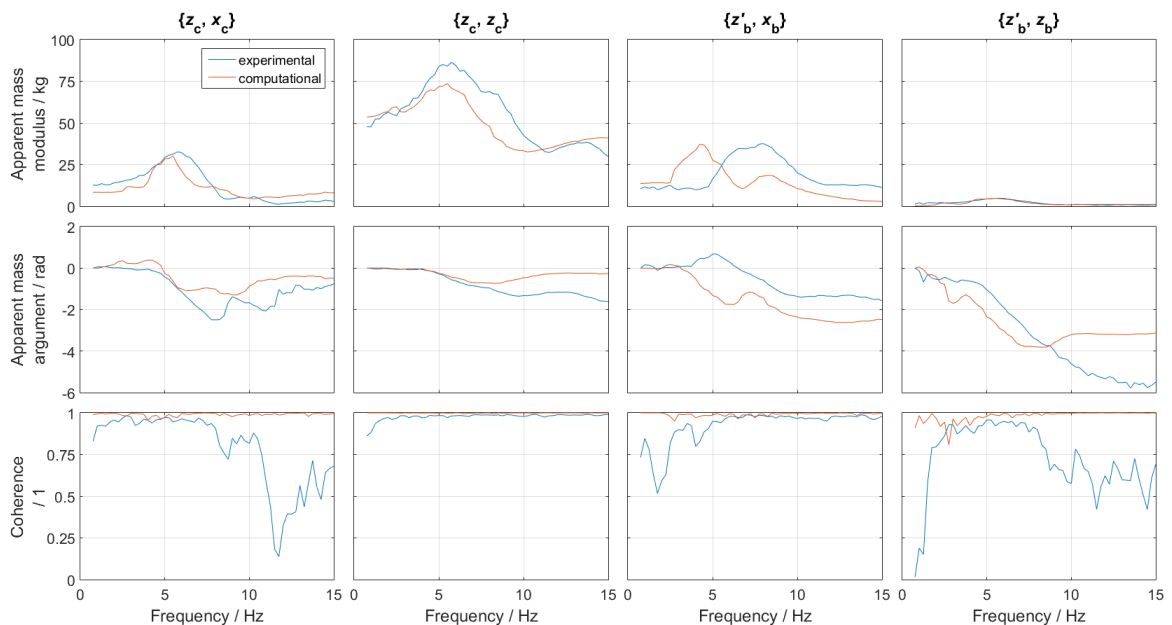


Figure 14 Dynamic-calibration-check outcome in terms of apparent-mass-coherence plots of response functions. **Top row:** apparent mass modulus vs frequency. **Middle row:** apparent mass argument vs frequency. **Bottom row:** coherence vs frequency. $\{z_c, x_c\}$ = vertical-to-sagittal-tangent response function at seat cushion (or cross-axis response function at seat cushion); $\{z_c, z_c\}$ = vertical-to-normal response function at seat cushion (or in-line response function at seat cushion); $\{z'_b, x_b\}$ = vertical-to-normal response function at seat back (or, by abuse of language, cross-axis response function at seat back); $\{z'_b, z_b\}$ = vertical-to-sagittal-tangent response function at seat back (or, by abuse of language, in-line response function at seat back).

5.4 Discussion

Experimental sitting configuration and computational sitting configuration are in good agreement, the only discrepancy of some importance being the spread of the thighs and the distance between the knees, which is greater for the human subject than for the occupant model. This is probably due not so much to a non-optimal choice of rotational stiffness and rotational viscous damping coefficient of the joints (especially the hip joints) as to the passivity of the occupant model, which prevents muscular activity from being represented in any manner. A possible approach to solving the problem is to provide the occupant model with suitable actuators.

Experimental contact pressure and computational contact pressure (in particular, location and magnitude of maxima) are in partial agreement, the main discrepancy being the contact area, which is greater for the human subject than for the occupant model, and the spatial gradient of contact pressure, which is less for the human subject than for the occupant model. This is probably due not so much to imperfect mass distribution as to the linearity of the generalised Maxwell solid material used for the neck–torso–buttocks–thighs assembly (a viscohyperelastic material being known to provide better experimental–computational match [Moes and Horváth, 2002; Verver *et al.*, 2004; Grujicic *et al.*, 2009; Zheng, 2012; Savonnet *et al.*, 2018; Yadav *et al.*, 2021; Alawneh *et al.*, 2022]).

Experimental apparent mass and computational apparent mass are in partial agreement, the main discrepancy being the primary-resonance modulus of $\{z_c, z_c\}$ apparent mass, which is greater for the human subject than for the occupant model, and the primary-resonance frequency of $\{z'_b, x_b\}$ apparent mass, which is greater for the human subject than for the occupant model. In all cases, coherence is high on the whole frequency band of interest, which proves the input–output relation to be linear and noiseless. There are two different scenarios for respectively vertical-to-sagittal-tangent apparent mass and vertical-to-normal apparent mass; somehow unexpectedly, the overall experimental–computational match is better for the former than for the latter. In particular, the good experimental–computational match of $\{z_c, x_c\}$ apparent mass modulus and $\{z'_b, z_b\}$ apparent mass modulus, as well as the fair experimental–computational match of $\{z_c, x_c\}$ apparent mass argument and $\{z'_b, z_b\}$ apparent mass argument, suggests that contact interactions and friction were reasonably represented. The discrepancy in primary-resonance modulus of $\{z_c, z_c\}$ apparent mass (and the conceivably related discrepancy in $\{z_c, z_c\}$ apparent mass argument beyond approximately 7 Hz) may be due to a non-optimal choice of damping parameters or, possibly, to rotational stiffness and rotational viscous damping coefficient of the joints being too diverse from joint to joint (which can result in undesired out-of-phase motion of different parts of the occupant model and ultimately in blunting and coalescence of neighbouring resonances). The discrepancy in

primary-resonance frequency of $\{z'_b, x_b\}$ apparent mass (and, more generally, the unsatisfactory experimental–computational match of $\{z'_b, x_b\}$ apparent mass) is probably a symptom of the fact that the geometric (and possibly mechanical) properties of the occupant model result from oversimplification. A possible approach to solving the problem is to represent the rib cage (and perhaps the viscera), to replace the arm–forearm-assembly supports to which reference is made as clavicles in this modelling study with more realistic structures, and possibly to use an inhomogeneous material for the neck–torso–buttocks–thighs assembly.

5.5 Conclusion

This modelling study allows investigating whether it is possible to calibrate a finite-element model of a seat occupant in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back). The developed occupant model results from applying the findings of the third theoretical study, and it proves having too restrictive a minimum complexity requirement.

This is a first attempt to develop a finite-element model of a seat occupant that can predict, possibly in a range of sitting configurations, both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back). The overall experimental–computational match seems promising, but there is some room for improvement. The experimental–computational match in terms of sitting configuration is good, which makes it plausible that the occupant model can be used in a range of sitting configurations. However, results suggest that the imposed minimum complexity requirement is too restrictive, especially as regards implications on vertical-to-normal apparent mass at seat back.

To improve the experimental–computational match in terms of contact pressure, a (possibly inhomogeneous) viscohyperelastic material and quadratic stress–displacement solid elements should be used for the parts defined as deformable bodies. To improve the experimental–computational match in terms of apparent mass, a higher degree of biofidelity should be pursued in the torso part. For future investigations, use should be made of multidirectional or multiaxial excitations, modal analysis data for calibration or validation, and deformable seat models for evaluation of transmissibility possibly in a range of sitting configurations.

6 Experimental study: Objective responses of seat–occupant systems to vibration

6.1 Introduction

As expounded in Section 1.3.4, within the adopted conceptual framework, a seat–occupant system is regarded as a whole, and the concept of *sitting configuration* is central; for the concept of sitting configuration to be deemed fit for purpose, evidence is required that it can be used as a single compendious factor facilitating the analysis of interactions with factors such as sex and vibration magnitude. To date, such evidence has not been available in the literature.

In light of the above, an *experimental study* was conducted to establish whether the concept of sitting configuration was appropriate to characterise a seat–occupant system as a whole with reference to performed activities as well as to establish whether there were significant main effects of, and interactions between, sex, vibration magnitude, and sitting configuration on objective responses of seat–occupant systems to vibration; objective responses of seat–occupant systems to vibration were quantified in terms of the fundamental objective response in seating dynamics (namely, *in-line transmission of vertical vibration at seat cushion*). Main effects and interactions of sex, vibration magnitude, and sitting configuration were considered. Main effects were well known for vibration magnitude [Walton, 2007], unclear for sex [Corbridge *et al.*, 1989; Toward and Griffin, 2011b], and hypothesised for sitting configuration. Little was known about interactions.

In the following, the method is described, the results are presented, and a discussion is provided.

6.2 Method

By considering findings of published studies, four *sitting configurations* were identified and characterised, both qualitatively and quantitatively, to match one pair of primary activities [Picard and Wisner, 1961; Rebiffé, 1969; Babbs, 1979; Tilley, 1993; Porter and Gyi, 1998; Harrison *et al.*, 2000; Park *et al.*, 2000; Andreoni *et al.*, 2002; Reed *et al.*, 2002; Vogt *et al.*, 2005; Hanson *et al.*, 2006; Kyung and Nussbaum, 2009; Schmidt *et al.*, 2014; Peng *et al.*, 2017] and three pairs of secondary activities [Keegan, 1964; Kamp *et al.*, 2011; Ive *et al.*, 2015; Parida *et al.*, 2018, 2019a, 2019b; Reed *et al.*, 2019]. Sitting configurations are shown in **Figure 15**. They were labelled *(self) entertaining/socialising (ES)*, *relaxing/sleeping (RS)*, *travelling/driving (TD)*, and *working/eating (WE)*. Qualitative and quantitative characteristics of sitting configurations are listed in respectively **Table 7** and **Table 8**.



Figure 15 Sitting configurations. **Top left:** (self) entertaining/socialising (ES). **Top right:** relaxing/sleeping (RS). **Bottom left:** travelling/driving (TD). **Bottom right:** working/eating (WE).

6 Experimental study: Objective responses of seat–occupant systems to vibration

Table 7 Qualitative characteristics of sitting configurations.

	ES	RS	TD	WE
Body posture	crouched	zero-gravity	regular	upright
Seat-back reclination	medium–high	high	medium–low	low
Footrest horizontal position	far	far	far	near
Footrest vertical position	high	medium–low	medium–high	low
Footrest inclination	high	medium	medium–high	null
The <i>zero-gravity body posture</i> is the body posture assumed by a human subject in a microgravity environment [Tengwall <i>et al.</i> , 1982].				

Table 8 Quantitative characteristics of sitting configurations.

	ES	RS	TD	WE
L53 (SgRP to heel)^a	740 mm	770 mm	830 mm	470 mm
H30 (seat height)^b	100 mm	280 mm	230 mm	380 mm
A27 (cushion angle)	14°	13°	14°	15°
A40 (torso angle)	41°	43°	23°	16°
A42 (hip angle)	98°	120°	100°	90°
A44 (knee angle)	106°	128°	136°	91°
A46 (ankle angle)	79°	102°	94°	114°
A48 (floor plane angle)^c	65°	45°	60°	0°
A57 (thigh angle)^d	33°	13°	13°	16°
^a In this experimental study, reference item L53 (SgRP to heel) is to be intended as horizontal distance between H-point (rather than seating reference point SgRP) and heel of shoe HOS (rather than accelerator heel point AHP). ^b In this experimental study, reference item H30 (seat height) is to be intended as vertical distance between H-point (rather than seating reference point SgRP) and heel of shoe HOS (rather than accelerator heel point AHP). ^c Reference item A48 (floor plane angle), which is used for passengers, is equivalent to reference item A47 (shoe plane angle), which is used for drivers. ^d Reference item A57 (thigh angle) was calculated as $A57 = 90^\circ + A40 - A42$. Reference items, as defined in Surface Vehicle Recommended Practice SAE J1100:2009-11 [SAE International, 2009], were measured with an H-point machine HPM-I as described in Surface Vehicle Standard SAE J826:2021-06 [SAE International, 2021b]. Lower-leg and thigh segments were extended to the 50 th percentile.				

The *excitations* were four whole-body vibrations with prescribed acceleration time histories. They were designed to perform an appropriate frequency-domain characterisation on the widest possible portion of the frequency band of interest for automotive applications involving exposure to whole-body vibration (but not motion sickness) [Griffin, 1990; Mansfield, 2005]. In particular, chosen excitations were intended to represent ‘baseline’ conditions with ‘simple’ direction, ‘wide’ frequency band, ‘well-behaved’ waveform, ‘relatively modest’ vibration magnitude, and ‘sufficiently long’ duration. Even if it seemed reasonable to expect passengers travelling aboard next-generation cars to be exposed to more complex motion environments [Burkhard *et al.*, 2018; Enders *et al.*, 2019], baseline conditions were preferred to facilitate not only performing an appropriate

frequency-domain characterisation but also referencing to previous studies. Characteristics of excitations are listed in **Table 9**.

Table 9 Characteristics of excitations.

Type	whole-body vibration with prescribed acceleration time history
Direction	‘vertical’ in the direction of the earth’s gravity (z-axis of laboratory geocentric coordinate system)
Frequency band	ranging from 0.5 Hz to 50 Hz
Waveform	low-crest-factor approximately stationary random with constant auto-power spectral density on the frequency band
Vibration magnitude	{0.28 m s ⁻² , 0.45 m s ⁻² , 0.71 m s ⁻² , 1.12 m s ⁻² }
Duration	60 s
Vibration magnitude is expressed in terms of root-mean-square value of W_k -weighted acceleration [International Organization for Standardization, 1997b] (approximately equal to root-mean-square value of W_b -weighted acceleration [British Standards Institution, 1987]). The corresponding data in terms of root-mean-square value of unweighted acceleration are {0.45 m s ⁻² , 0.72 m s ⁻² , 1.13 m s ⁻² , 1.80 m s ⁻² }.	

Apart from small-scale differences in waveform, the four excitations differed only in *vibration magnitude*; accordingly, they were characterised in terms of it. In principle, vibration magnitude should have been regarded as a continuous quantity; indeed, the four chosen values were equally spaced on a logarithmic scale. However, for the purposes of this experimental study, it was conveniently regarded as an (ordinal) categorical quantity assuming four values labelled *0d28w*, *0d45w*, *0d71w*, and *1d12w*. In terms of approximate indications of likely reactions, these four vibration magnitudes corresponded to the four verbal descriptors *not uncomfortable*, *a little uncomfortable*, *fairly uncomfortable*, and *uncomfortable* respectively [British Standards Institution, 1987; International Organization for Standardization, 1997b]. The four vibration magnitudes were chosen to cover the range that may be expected in automotive applications involving next-generation cars. The use of four values was decided to allow capturing possible non-monotonic trends with some detail.

6.2.1 Participants

This experimental study was approved by the Faculty Ethics Committee of Engineering and the Environment at the University of Southampton (ERGO II submission reference 41143). A *sample* of adult and healthy males and females was selected amongst students and staff members of the University of Southampton to represent the target population of users of next-generation cars. For the formal assessment of eligibility in terms of adulthood and health, a questionnaire was designed in compliance with International Standard ISO 13090-1:1998 [International Organization for Standardization, 1998]. On the one hand, to be regarded as adult, participants needed be 18 years

old or more. On the other hand, to be regarded as healthy (without further medical advice), participants needed be fit to travel in public transport without assistance and to accept the stress of a normal day's work, needed not be suffering any serious illness or injury, needed not be under medical treatment or suffering disability affecting their daily life, and needed not have certain conditions (active disease of the respiratory system, active disease of the digestive system, active disease of the cardiovascular system, active disease of the genitourinary system, active disease of the musculoskeletal system, active or chronic disease or disorder of the nervous system, mental health problems, recent trauma, recent surgical procedures, prosthesis, pregnancy, and breastfeeding).

Neither random nor systematic sampling plan was implemented, but equal numbers of males and females were sought to obtain two *independent groups* of the same size (namely, *male group* and *female group*). The intended *total sample size* of 12 participants was chosen, in the wake of the psychophysical tradition [Rouder and Haaf, 2018], with the aim of detecting 'large' effect sizes. It was anticipated that the *group sample size* of six participants could be small for sex to be included as a factor; nevertheless, inclusion was admitted to document separately and explicitly the results obtained for males and for females, to pilot the chosen research design, and to guide future investigations; the implications of this choice were assessed *a posteriori* by performing a sensitivity-type statistical power analysis. Initially, 14 potential participants were reached haphazardly, either through direct approach (five males and four females) or through advertisement (one male and four females), and were preliminarily screened. Of them, two directly approached females were excluded (one because of a refusal to participate and one because of a scheduling conflict). Subsequently, 12 potential participants (six males and six females) were formally assessed for eligibility and were recruited. They gave their informed consent to participate and received a reimbursement payment. Personal data collected or created about participants were pseudonymised and managed confidentially in compliance with the Data Protection Policy of the University of Southampton as well as with the Data Protection Act 2018 (for the purposes of which the University of Southampton was the data controller). Major demographic and anthropometric characteristics of independent groups are listed in **Table 10**.

Table 10 Major demographic and anthropometric characteristics of independent groups.

	M				
	Minimum	1 st quartile	Median	2 nd quartile	Maximum
Age / y	25	26	28	29	36
Body mass (weight) / kg	66	67	79	81	83
Stature (body height) / cm	172	178	180	182	183
Body mass index / (kg m ⁻²)	21.0	22.2	23.8	24.9	25.6
	F				
	Minimum	1 st quartile	Median	2 nd quartile	Maximum
Age / y	23	24	30	33	42
Body mass (weight) / kg	45	61	68	72	74
Stature (body height) / cm	157	164	167	170	171
Body mass index / (kg m ⁻²)	18.4	21.4	23.6	26.6	27.6
Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. The <i>body mass index</i> (or <i>BMI</i>) is the quotient of body mass (weight) by the square of stature (body height) [International Organization for Standardization, 2022b].					

6.2.2 Measures

Objective responses of seat–occupant system to vibration were measured within a laboratory experiment. Measurements of *rectilinear components of acceleration* were performed, by means of measuring instrumentation complying with International Standard ISO 8041-1:2017 [International Organization for Standardization, 2017b], in the three perpendicular directions of each of two *coordinate systems*. Of the two coordinate systems, one was labelled $\{x_t, y_t, z_t\}$ and was located at the *vibration table*, whereas the other was labelled $\{x_c, y_c, z_c\}$ and was located at *the interface between seat cushion and occupant buttocks*. Experimental arrangement and coordinate systems are shown in **Figure 16**.



Figure 16 Experimental arrangement and coordinate systems. $\{x_t, y_t, z_t\}$ = coordinate system at vibration table; $\{x_c, y_c, z_c\}$ = coordinate system at interface between seat cushion and occupant buttocks.

At the seat–occupant interface, the non-null cushion angle led to the adoption of a *basentric biodynamic coordinate system* [International Organization for Standardization, 1997c]. Hence, unlike the axes of the $\{x_t, y_t, z_t\}$ coordinate system, the axes of the $\{x_c, y_c, z_c\}$ coordinate system were not exactly parallel to those of the geocentric coordinate system of the laboratory (having z -axis lying in the direction of the earth’s gravity). The $\{x_c, y_c, z_c\}$ coordinate system seemed appropriate on the reasonable assumption that psychophysiological responses to vibration depend on motion components in the directions of the axes of biodynamic rather than geocentric coordinate systems [British Standards Institution, 1987; International Organization for Standardization, 1997b]. In any case, by considering the parallelism tolerances specified in British Standard BS 6841:1987 [British Standards Institution, 1987] and in International Standard ISO 2631-1:1997 [International Organization for Standardization, 1997b] (respectively up to 15° and up to 20°), the z_t axis and the z_c axis could be considered parallel to a first approximation.

The seat–occupant system had a longitudinal plane of symmetry passing through the centreline of the seat and coinciding with the sagittal plane of the occupant; besides, the excitations were one-directional and vertical. Accordingly, relevant measurements of acceleration were along the z_t -axis, x_c -axis, and z_c -axis; nevertheless, for verification purposes, measurements of acceleration were also performed along the x_t -axis, y_t -axis, and y_c -axis. In what follows, only the $\{z_t, z_c\}$ input–output pair is considered; this implies regarding the seat–occupant system as a single-input–single-output (SISO) system.

For each test, two *response functions* and two *response variables*, chosen for their theoretical and practical importance, were obtained from the input acceleration along the z_t -axis and from the output acceleration along the z_c -axis. The response functions were the (complex-valued) $\{z_t, z_c\}$ *transmissibility* (also known as *vertical in-line transmissibility at seat cushion*) and the corresponding (real-valued) $\{z_t, z_c\}$ *coherence*, both defined on the basis of International Standard ISO 2041:2018 [International Organization for Standardization, 2018]. For subsequent analyses, $\{z_t, z_c\}$ transmissibility modulus and $\{z_t, z_c\}$ transmissibility argument were calculated. Both $\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence were calculated, as described in International Standard ISO 18431-1:2005 [International Organization for Standardization, 2005], starting from estimated auto-power spectral density and cross-power spectral density of input and output accelerations. In particular, by making the customary assumption of noiseless input accelerations, $\{z_t, z_c\}$ transmissibility was calculated as a frequency-response function of the first type [Ewins, 2000; He and Fu, 2001; International Organization for Standardization, 2005; Bendat and Piersol, 2010; Coyte *et al.*, 2016]. The response variables for statistical analysis were *primary-resonance frequency* of $\{z_t, z_c\}$ transmissibility and *primary-resonance modulus* of $\{z_t, z_c\}$ transmissibility; primary resonance was determined by inspection from the calculated $\{z_t, z_c\}$ transmissibility modulus and $\{z_t, z_c\}$ transmissibility argument.

6.2.3 Research design

A (*balanced*) *three-factor mixed design* with one *two-level between-subjects factor* (*sex*) and two *four-level within-subjects factors* (*vibration magnitude* and *sitting configuration*) was implemented. Accordingly, each of the six participants within each of the two independent groups received 16 *treatments*. The chosen research design allowed adopting all the three fundamental principles of experimental design attributed to the British statistician and geneticist Ronald Aylmer Fisher (1890–1962), which are replication, blocking, and randomisation. To control random measurement error and improve measurement precision, *replication* was systematically implemented: specifically, each of the 16 treatments was replicated three times with each of the 12 participants to obtain 36 tests per treatment. To control the effect of nuisance factors and improve statistical power, *blocking* was implemented in a twofold manner: specifically, independent groups were used as blocks with respect to between-subjects factor and participants were used as blocks with respect to within-subjects factors and replicate. To control systematic measurement error and improve measurement trueness, *randomisation* was completely implemented with respect to within-subjects factors and replicate: specifically, a different random test sequence was used for the 48 tests of each participant. In total, 576 tests were performed. Characteristics of research design are listed in **Table 11**.

Table 11 Characteristics of research design.

	Number	Description
Between-subjects factor	1	sex (two levels: {M, F})
Within-subjects factors	2	vibration magnitude (four levels: {0d28w, 0d45w, 0d71w, 1d12w}) sitting configuration (four levels: {ES, RS, TD, WE})
Independent groups	2	male group (six participants: {P01, P02, P03, P05, P06, P12}) female group (six participants: {P04, P07, P08, P09, P10, P11})
Treatments	16	{0d28w, 0d45w, 0d71w, 1d12w}×{ES, RS, TD, WE}
Participants	12	{P01, ..., P12}
Replicates (tests per treatment and participant)	3	{R1, R2, R3}
Tests per treatment	36	{P01, ..., P12}×{R1, R2, R3}
Tests per participant	48	{0d28w, 0d45w, 0d71w, 1d12w}×{ES, RS, TD, WE}×{R1, R2, R3}
Tests	576	{0d28w, 0d45w, 0d71w, 1d12w}×{ES, RS, TD, WE}×{P01, ..., P12}×{R1, R2, R3}

For inferential statistical analysis, a *three-way mixed-design univariate analysis of variance (ANOVA)* was used. The (*nil*) *null hypothesis* was that experimental manipulation of sex, vibration magnitude, and sitting configuration had neither significant main effects nor significant interactions. Despite requiring more restrictive assumptions than semiparametric and nonparametric homologues, a parametric technique was preferred because of the greater statistical power; the legitimacy of this choice was assessed *a posteriori* both by verifying the assumption of normality for the ANOVA model residuals and by performing a sensitivity-type statistical power analysis.

Criticisms of the procedure of *null hypothesis statistical significance testing* are well known [Nickerson, 2000]. In fact, controversies about statistical inference are perennial. At a higher level, epistemological discussions about paradigms (frequentist as opposed to Bayesian) [Kass, 2011], frameworks (parameter estimation as opposed to hypothesis testing) [Woodson, 1969], and even approaches within the same framework (point as opposed to interval parameter estimation and Fisher's as opposed to Neyman–Pearson's hypothesis testing) [Lehmann, 1993; Amrhein and Greenland, 2022] have persisted for decades. At a lower level, the old frequentist dilemma between parametric and nonparametric techniques [Koch *et al.*, 1980] has been flanked by a more recent debate about possible parametric substitutes for ANOVA [Gelman, 2005; McCulloch, 2005]. Pragmatically, a well-attested technique allowing the analysis of three-way interactions with a relatively easy implementation was deemed appropriate for the purposes of this experimental study. However, considering both the many well-founded reservations about the procedure of null hypothesis statistical significance testing [Nickerson, 2000] and recent authoritative recommendations [Wasserstein *et al.*, 2019], no declarations of 'statistical' significance are issued in

this experimental study. Instead, assessments of ‘practical’ significance of effects are made by complementing p values with measures of *effect size* (eta-squared η^2 , partial eta-squared η_p^2 , and generalised eta-squared η_g^2)² and point estimates with graphical representations of *interval estimates* (within-subjects or between-subjects confidence intervals at 95 % confidence level).

6.2.4 Experimental manipulations

The *vibration generator system* (Six-Axis Motion Simulator of the University of Southampton) was composed of a hydraulic vibration generator with table and a digital control system. To reproduce realistic seat–occupant interactions, a *production reclining car seat* (NHK left-outboard front-row passenger seat for right-hand-drive Subaru Outback) rigidly mounted on the vibration table was used. A *four-piece configurable footrest* made of wood and carpet (handcrafted at the University of Southampton) was mounted on the vibration table in front of the seat.

The *target acceleration signals* for the four excitations were generated in MATLAB software environment (version 8.5.1.959712) by means of the HVLab Human Response to Vibration Toolbox (version 2.0). The corresponding *drive displacement signals* were obtained through an iterative ‘equalisation’ process by means of the vibration generator system.

All *acceleration measurements* were performed according to International Standard ISO 10326-1:2016 [International Organization for Standardization, 2016]. At vibration table, three uniaxial accelerometers (Silicon Design 2260-005) were used; their sensitivity and measuring range were respectively 800 mV g^{-1} and the interval from $-5 g$ to $+5 g$ (g being the standard acceleration due to gravity, namely, 9.80665 m s^{-2}). At the interface between seat cushion and occupant buttocks, one special-purpose triaxial accelerometer (HVLab SITpad-3-10g) was used; its sensitivity and measuring range were respectively 200 mV g^{-1} and the interval from $-10 g$ to $+10 g$. The instrumented seat is shown in **Figure 17**. For calibration of the measuring system, *in situ* checks were performed according to International Standard 8041-1:2017 [International Organization for Standardization, 2017b].

² Interpreting measures of effect size against arbitrary benchmarks is not recommended, and hence it is not done in this experimental study. In principle, it is sensible to use eta-squared η^2 for comparisons of effects within a single study, partial eta-squared η_p^2 for statistical power analyses and for comparisons of effects across studies with the same research design, and generalised eta-squared η_g^2 for meta-analyses and for comparisons of effects across studies with different research design [Lakens, 2013].

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Figure 17 Instrumented seat. **Left:** front view. **Right:** side view.

The *acquisition of acceleration signals* was performed in MATLAB software environment via the HVLab Human Response to Vibration Toolbox at a sampling frequency of 512 Hz. The multichannel data acquisition system consisted of a multifunction I/O device (National Instruments NI USB6211) and a signal conditioning system; in turn, the latter was composed of two eight-channel mobile Micro Analog 2 enclosures (Fylde FE-MM8), two power supply modules (Fylde FE-810-BPS DC), six two-channel bridge transducer amplifier modules (Fylde FE-366-TA), and two auto-zero modules (Fylde FE-366-AZ). The anti-aliasing filters of the multichannel data acquisition system had a cut-off frequency of 100 Hz.

The 48 tests of each participant were performed at the University of Southampton within two *sessions* held on two different days, each including 24 tests and lasting between 2.5 h and 3 h. The long duration of each session was due to the randomisation with respect to sitting configuration, which required long breaks between consecutive tests. Within these breaks, seat-back reclination and footrest configuration would be adjusted by using respectively an inclinometer and a set of reference stickers on the vibration table. Special care was taken to ensure that sitting configurations were reproduced consistently across different tests, both with the same participant and with different participants. Each sitting configuration was visually checked and documented photographically with each participant.

In accordance with the approved *research protocol*, participants would have their demographic data collected and their anthropometric data measured at the beginning of their first session. Before actual tests of each of their two sessions, participants would read an instruction sheet and receive preliminary training for a duration of 5 min. Before each test, participants would be secured with a safety harness and given ready access to an emergency stop control. Within each test, participants

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would be exposed to an excitation for a duration of 60 s whilst sitting relaxed but unmoving and whilst keeping a posture as natural as possible but in complete contact with the seat. Between tests, participants would be given the opportunity to take breaks. As expected, no participant suffered from motion sickness or other malaise throughout sessions.

6.3 Results

The *data processing* was performed in MATLAB software environment with a frequency resolution of 0.25 Hz. Transmissibility-modulus plots, transmissibility-argument plots, and coherence plots of the response functions ($\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence) for all 36 tests per treatment are shown in respectively **Figure 18**, **Figure 19**, and **Figure 20** to provide graphical information about *inter-participant variability*.

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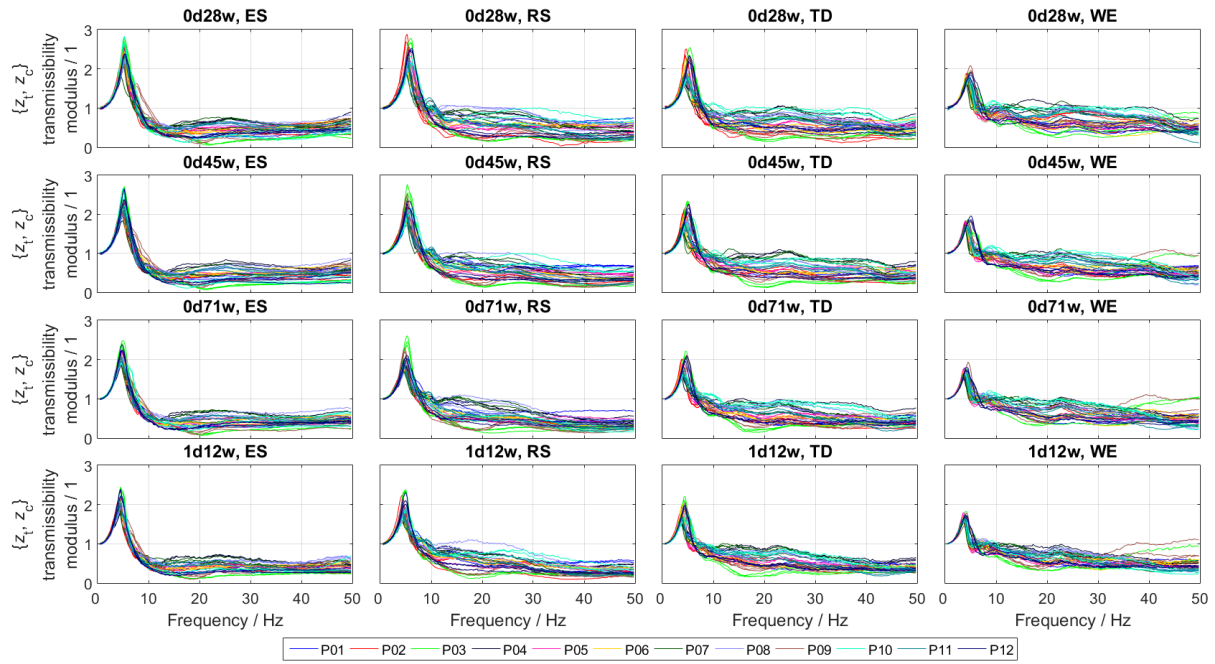


Figure 18 Transmissibility-modulus plots of the response functions ($\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence) for all 36 tests per treatment. **Cells:** $\{z_t, z_c\}$ transmissibility modulus vs frequency. Total sample: 12 participants (six males and six females). Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. Participants: {P01, ..., P12}.

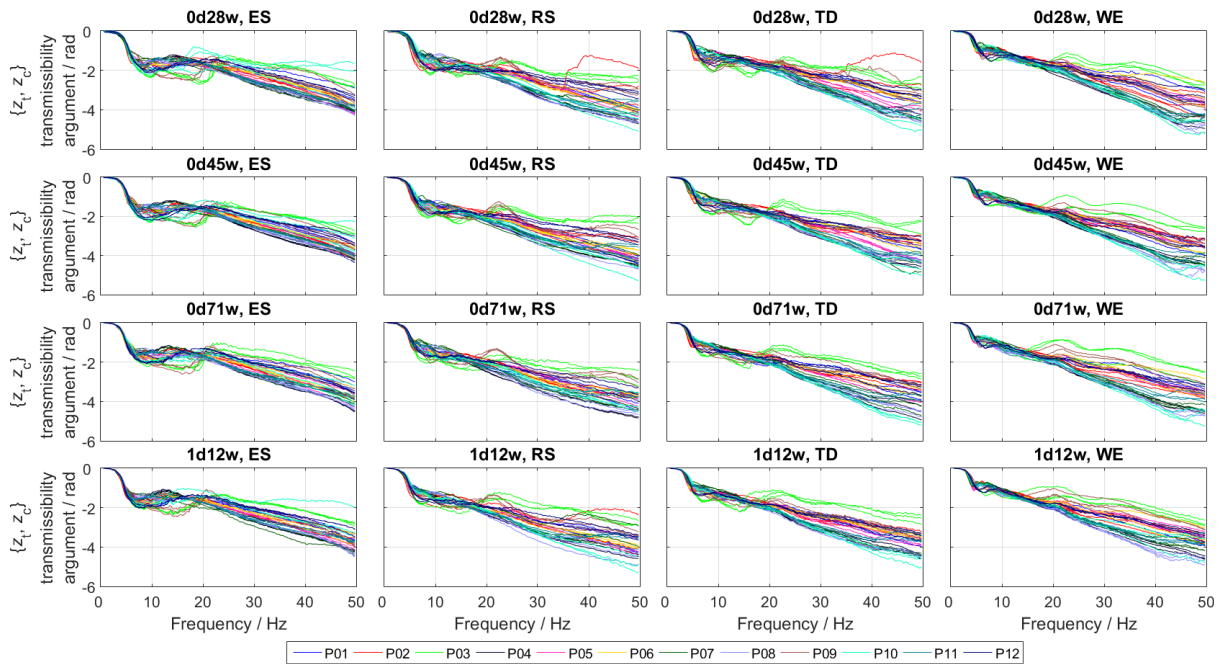


Figure 19 Transmissibility-argument plots of the response functions ($\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence) for all 36 tests per treatment. **Cells:** $\{z_t, z_c\}$ transmissibility argument vs frequency. Total sample: 12 participants (six males and six females). Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. Participants: {P01, ..., P12}.

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Figure 20 Coherence plots of the response functions ($\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence) for all 36 tests per treatment. **Cells:** $\{z_t, z_c\}$ coherence vs frequency. Total sample: 12 participants (six males and six females). Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. Participants: {P01, ..., P12}.

Transmissibility–coherence plots of the grand-median response functions ($\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence) calculated across all 36 tests per treatment are shown in **Figure 21** (parameterised on vibration magnitude and arranged in columns by sitting configuration) and in **Figure 22** (parameterised on sitting configuration and arranged in columns by vibration magnitude) to provide graphical summaries of *frequency-domain analysis*. Transmissibility–coherence plots shown in **Figure 21** and **Figure 22** contain the same information, but they are parameterised and arranged differently for convenience.

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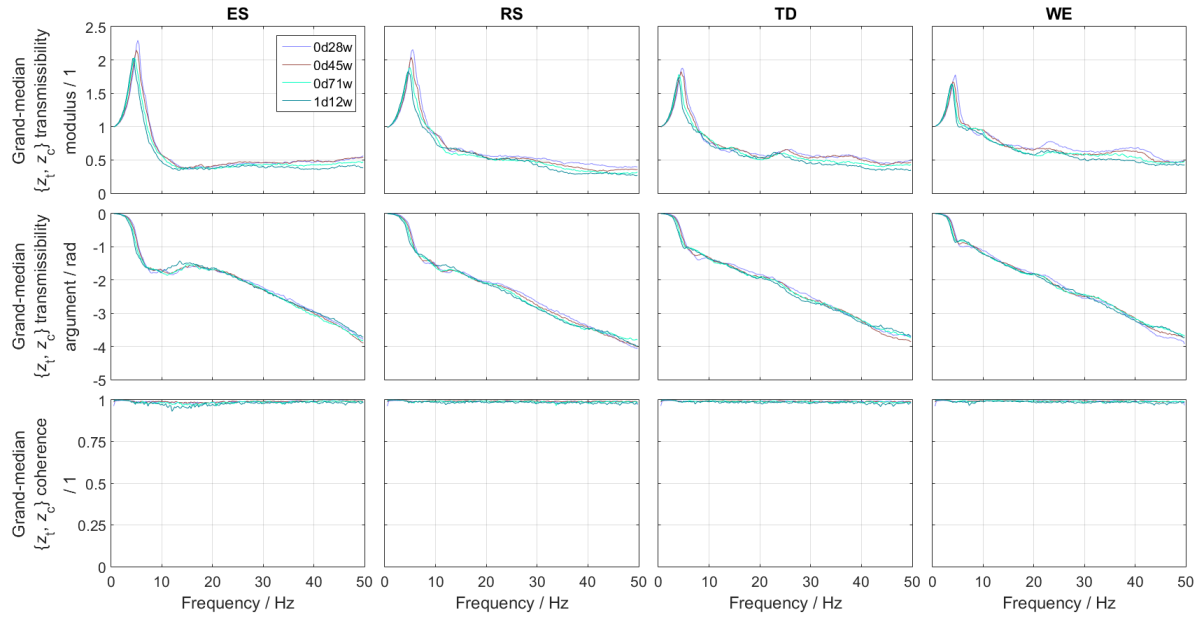


Figure 21 Transmissibility–coherence plots of the grand-median response functions ($\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence) calculated across all 36 tests per treatment (parameterised on vibration magnitude and arranged in columns by sitting configuration). Total sample: 12 participants (six males and six females). Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Top row:** grand-median $\{z_t, z_c\}$ transmissibility modulus vs frequency. **Middle row:** grand-median $\{z_t, z_c\}$ transmissibility argument vs frequency. **Bottom row:** grand-median $\{z_t, z_c\}$ coherence vs frequency. All graphs are parameterised on vibration magnitude and arranged in columns by sitting configuration.

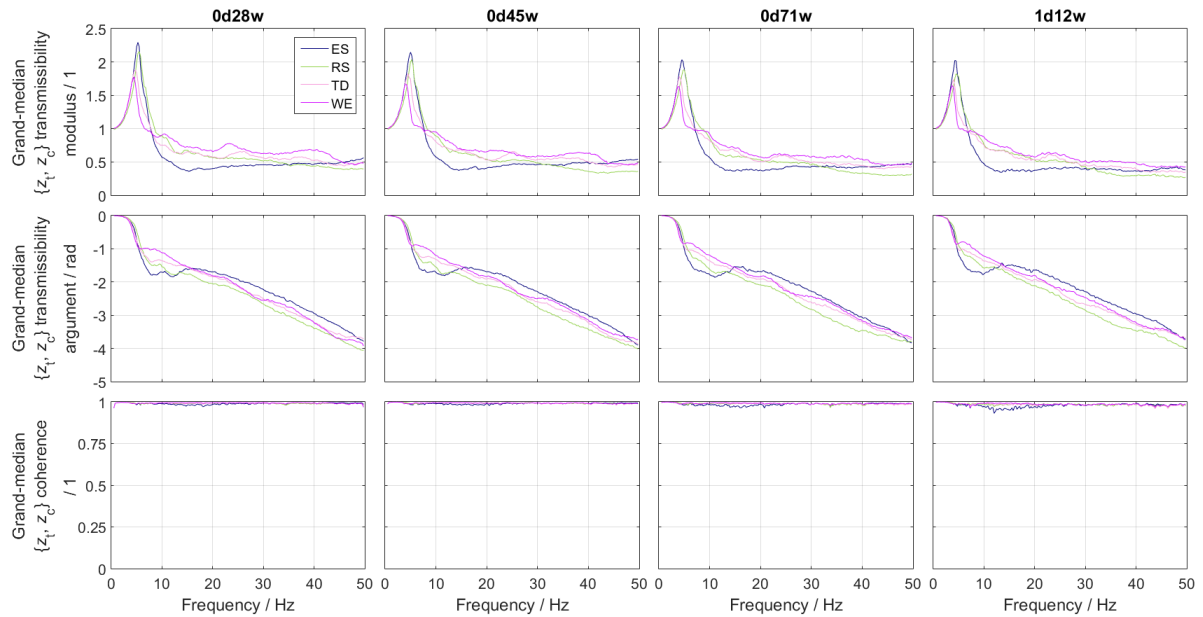


Figure 22 Transmissibility–coherence plots of the grand-median response functions ($\{z_t, z_c\}$ transmissibility and $\{z_t, z_c\}$ coherence) calculated across all 36 tests per treatment (parameterised on sitting configuration and arranged in columns by vibration magnitude). Total sample: 12 participants (six males and six females). Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Top row:** grand-median $\{z_t, z_c\}$ transmissibility modulus vs frequency. **Middle row:** grand-median $\{z_t, z_c\}$ transmissibility argument vs frequency. **Bottom row:** grand-median $\{z_t, z_c\}$ coherence vs frequency. All graphs are parameterised on sitting configuration and arranged in columns by vibration magnitude.

The *statistical analysis* was performed in RStudio integrated development environment for R (version 1.2.5033) running against R software environment (version 3.6.2) [R Core Team, 2019]. In the following, results are reported separately for the two response variables (in order, primary-resonance frequency of $\{z_t, z_c\}$ transmissibility and primary-resonance modulus of $\{z_t, z_c\}$ transmissibility).

6.3.1 Primary-resonance frequency

Median and interquartile range of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates are listed in respectively **Table 12** and **Table 13** to provide information about *central tendency* and *dispersion* across replicates. Median and interquartile range of the response variable calculated across all tests were respectively 4.5 Hz and 1 Hz.

Table 12 Median of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates.

	M															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P01	5.25	5.25	4.50	4.50	5.00	5.00	4.50	4.00	4.50	4.75	4.00	3.75	4.50	4.50	4.00	3.75
P02	5.00	5.25	4.50	4.25	5.00	5.00	4.50	3.75	4.50	4.75	4.00	3.50	4.25	4.50	3.75	3.50
P03	5.25	6.00	5.25	4.50	5.25	5.25	5.00	4.50	4.75	5.25	4.50	4.00	4.50	5.00	4.25	3.75
P05	5.00	5.50	4.50	4.00	5.00	5.25	4.50	4.00	4.75	5.00	4.00	3.75	4.25	4.50	3.75	3.50
P06	5.25	5.25	4.50	4.00	4.75	5.00	4.25	4.00	4.50	4.50	4.00	3.75	4.25	4.25	3.75	3.50
P12	5.25	6.00	5.25	5.00	5.25	5.25	5.25	5.00	4.75	5.25	4.50	4.25	4.50	5.00	4.50	4.00
	F															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P04	5.25	5.50	4.25	4.00	5.00	5.25	4.00	3.75	4.75	5.00	3.75	3.75	4.50	4.50	3.50	3.50
P07	5.00	5.25	4.50	4.00	5.00	5.00	3.75	3.75	4.50	4.50	4.00	3.75	4.25	4.25	3.75	3.50
P08	5.50	5.75	5.25	4.50	5.25	5.25	5.00	4.25	5.00	5.25	4.00	4.00	5.00	5.00	4.25	3.75
P09	5.25	5.25	5.25	5.00	5.00	5.00	5.00	4.50	4.50	4.75	4.50	4.00	4.25	4.50	4.50	4.00
P10	5.25	6.00	4.75	4.50	5.25	6.00	4.50	4.00	4.50	5.00	4.50	4.00	4.50	5.00	4.50	3.75
P11	5.25	5.25	4.50	4.50	5.00	5.00	4.25	4.25	4.50	4.75	4.00	4.00	4.25	4.50	3.75	3.75
Unit of measurement: Hz. Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}.																

Table 13 Interquartile range of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates.

	M															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P01	0.19	0.00	0.19	0.19	0.00	0.19	13.0	0.00	0.00	0.38	0.19	0.38	0.00	0.00	0.00	0.19
P02	0.19	0.19	0.19	0.38	0.00	0.00	0.38	0.19	0.00	0.00	0.19	0.00	0.19	0.19	0.00	0.00
P03	0.19	0.00	0.19	0.19	0.00	0.38	0.00	0.38	0.38	0.00	0.19	0.19	0.00	0.00	0.00	0.19
P05	0.19	0.38	0.38	0.19	0.00	0.00	0.19	0.00	0.19	0.19	0.00	0.38	0.00	0.00	0.00	0.00
P06	0.19	0.00	0.19	0.19	0.19	0.00	0.19	0.19	0.19	0.00	0.00	0.00	0.19	0.19	0.19	0.19
P12	0.19	0.38	0.00	0.38	0.00	0.00	0.19	0.38	0.00	0.19	0.19	0.19	0.00	0.00	0.00	0.00
	F															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P04	0.00	0.19	0.38	0.19	0.19	0.00	0.75	0.19	0.00	0.00	0.19	0.19	0.00	0.19	0.19	0.00
P07	0.38	0.38	0.38	0.00	0.19	0.19	0.00	0.19	0.00	0.38	0.00	0.19	0.38	0.38	0.19	0.19
P08	0.00	0.19	0.75	0.00	0.00	0.00	0.19	0.38	0.19	0.19	0.56	0.00	0.38	0.00	0.19	0.19
P09	0.38	0.19	0.00	0.00	0.56	0.19	0.00	0.56	0.38	0.00	0.19	0.38	0.19	0.00	0.19	0.19
P10	0.00	0.19	0.19	0.19	0.19	0.19	0.75	0.19	0.19	0.19	0.38	0.19	0.19	0.00	0.38	0.19
P11	0.00	0.19	0.00	0.19	0.00	0.00	0.00	0.38	0.19	0.19	0.19	0.38	0.19	0.00	0.00	0.00
Unit of measurement: Hz. Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}.																

The *data set* used for inferential statistical analysis is the one listed in **Table 12**. By virtue of the implemented replication, this data set was deemed robust enough to allow abstaining from detection and treatment of outliers. Indeed, the values of interquartile range listed in **Table 13** were generally small (and often less than the frequency resolution value of 0.25 Hz). Accordingly, there were neither missing data nor deleted cases.

Box plots of the median response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates and all six participants of each independent group are shown in **Figure 23** to provide graphical summaries of *descriptive statistical analysis*³. Box plots shown on the

³ The bottom line of each box (*lower hinge*) corresponds to the *lower quartile* (also known as *first quartile*), which coincides with the 25th percentile. The middle line of each box corresponds to the *median* (also known as *second quartile*), which coincides with the 50th percentile. The top line of each box (*upper hinge*) corresponds to the *upper quartile* (also known as *third quartile*), which coincides with the 75th percentile. The difference between upper quartile and lower quartile is the *interquartile range*. The *lower whisker* extends from the lower hinge to the minimum value at or above the lower fence (which is the value situated 1.5 times the interquartile range below the lower quartile). The *upper whisker* extends from the upper hinge to the maximum value at or below the upper fence (which is the value situated 1.5 times the interquartile range above the upper quartile). Values beyond the lower fence and the upper fence are marked separately.

left side and of the right side of **Figure 23** contain the same information, but they are parameterised and arranged differently for convenience.

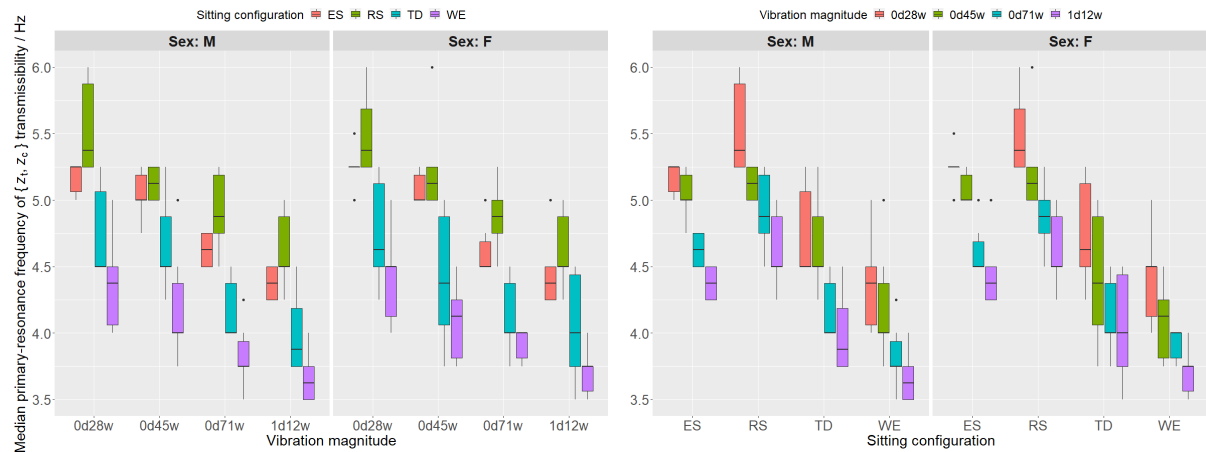


Figure 23 Box plots of the median response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates and all six participants of each independent group. Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Left:** median primary-resonance frequency of $\{z_t, z_c\}$ transmissibility vs vibration magnitude (graphs parameterised on sitting configuration and arranged in columns by sex). **Right:** median primary-resonance frequency of $\{z_t, z_c\}$ transmissibility vs sitting configuration (graphs parameterised on vibration magnitude and arranged in columns by sex).

A three-way mixed-design univariate ANOVA model of the median response variable calculated across all three replicates (hereinafter *response variable*) was fit by using the R package afex (version 0.26-0) [Singmann *et al.*, 2020]. To protect against possible violation of the *sphericity assumption*, conservative Greenhouse–Geisser corrections were applied to eligible statistical degrees of freedom (namely, all except those corresponding to the main effect of sex, which, as a two-level factor, was not affected). To mitigate the *multiple-comparisons problem* inherent in multiway ANOVA [Cramer *et al.*, 2016], *p* values were adjusted using the Holm–Bonferroni method. To verify the *normality assumption* for the ANOVA model residuals, *a posteriori* diagnostic tests were performed. Diagnostic plots of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) are shown in **Figure 24** to provide graphical assessment of the normality assumption for the ANOVA model residuals. The flipped-axes histogram of residuals, the residual-fitted plot, and the quantile-quantile plot of residuals indicate no gross violation of the normality assumption for the ANOVA model residuals.

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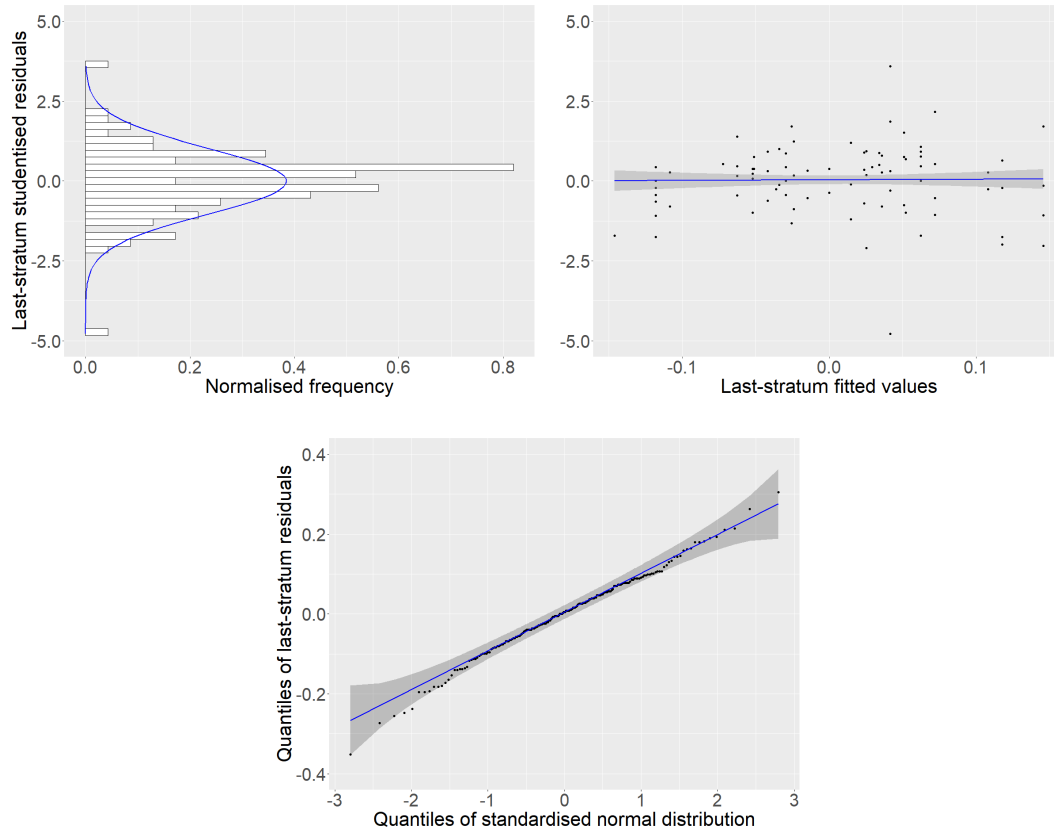


Figure 24 Diagnostic plots of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility). **Top left:** flipped-axes histogram of residuals (last-stratum studentised residuals vs normalised frequency) with equivalent normal-distribution curve. **Top right:** residual-fitted plot (last-stratum studentised residuals vs last-stratum fitted values) with robust-regression line and confidence intervals at 95 % confidence level. **Bottom:** quantile-quantile plot of residuals (quantiles of last-stratum residuals vs quantiles of standardised normal distribution) with robust-regression line and confidence intervals at 95 % confidence level. In the flipped-axes histogram of residuals, frequency is the number of observations per bin.

The ANOVA model of the response variable showed a significant main effect of both vibration magnitude, $F(2.21, 22.11) = 369.54, p < 0.001, \eta^2 = 0.28, \eta_p^2 = 0.97, \eta_G^2 = 0.54$, and sitting configuration, $F(1.98, 19.80) = 82.27, p < 0.001, \eta^2 = 0.48, \eta_p^2 = 0.89, \eta_G^2 = 0.67$, but failed to show a significant main effect of sex, $F(1, 10) < 0.001, p > 0.99, \eta^2 < 0.001, \eta_p^2 < 0.001, \eta_G^2 < 0.001$, and any significant interactions, $p > 0.99, \eta^2 \leq 0.004, \eta_p^2 \leq 0.12, \eta_G^2 \leq 0.02$. Three-way interaction plots of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) with within-subjects confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of sex)⁴ are shown in **Figure 25** to provide graphical summaries of *inferential statistical analysis*. Interaction plots shown on the left side and of the right side of **Figure 25** are equivalent, but they are parameterised and arranged differently for convenience. Two-way

⁴ It is worth noting that overlapping confidence intervals at $1 - \alpha$ confidence level do not preclude a ‘statistically’ significant difference between the corresponding means at a *significance level* less than or equal to α . In fact, non-overlapping confidence-intervals at $1 - \alpha$ confidence level imply a ‘statistically’ significant difference between the corresponding means at a significance level distinctly less than α [Belia *et al.*, 2005].

interaction plots of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) with between-subjects confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of vibration magnitude and sitting configuration) are shown in **Figure 26**. Interaction plots shown on the left side and of the right side of **Figure 26** are equivalent, but they are arranged differently for convenience.

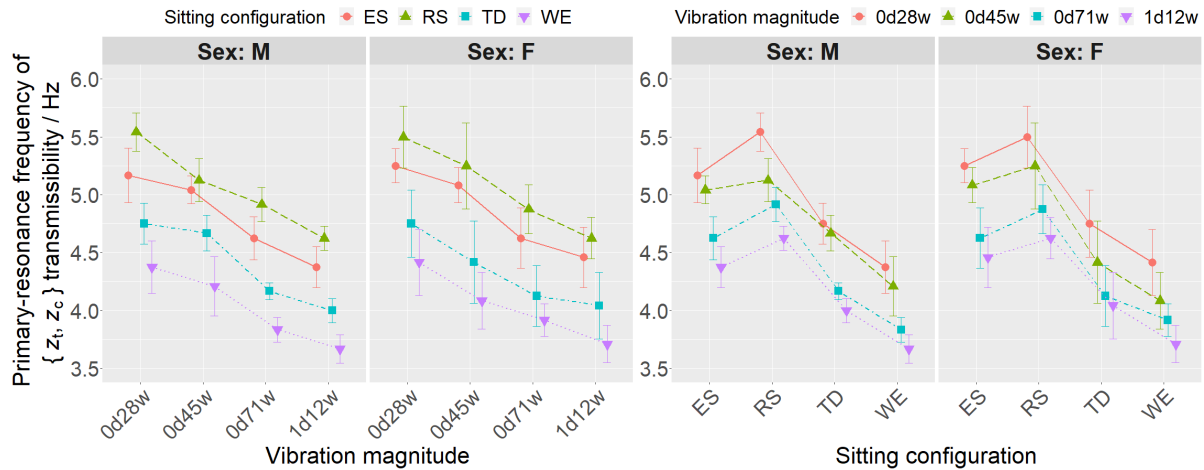


Figure 25 Three-way interaction plots of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) with within-subjects confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of sex). Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Left:** primary-resonance frequency of $\{z_t, z_c\}$ transmissibility vs vibration magnitude (graphs parameterised on sitting configuration and arranged in columns by sex). **Right:** primary-resonance frequency of $\{z_t, z_c\}$ transmissibility vs sitting configuration (graphs parameterised on vibration magnitude and arranged in columns by sex). Data points are horizontally offset for clarity.

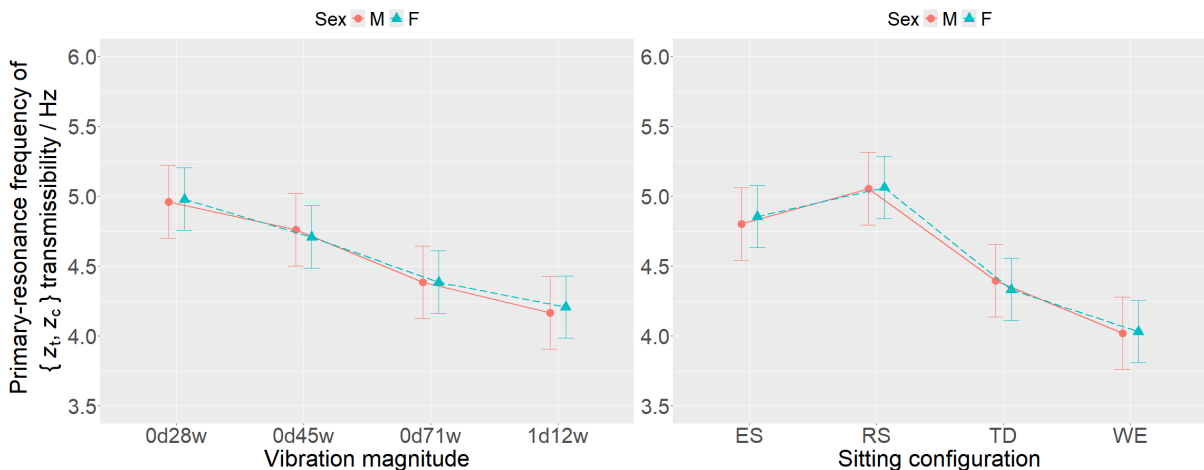


Figure 26 Two-way interaction plots of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) with between-subjects confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of vibration magnitude and sitting configuration). Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Left:** primary-resonance modulus of $\{z_t, z_c\}$ transmissibility vs vibration magnitude (graphs parameterised on sex). **Right:** primary-resonance modulus of $\{z_t, z_c\}$ transmissibility vs sitting configuration (graphs parameterised on sex). Data points are horizontally offset for clarity.

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A sensitivity-type statistical power analysis was performed by focussing on the partial eta-squared measure of effect size (η_p^2) and on the non-centrality parameter of the non-central F -distribution (λ); denoting by N the total sample size, the equivalent relations $\lambda = N \eta_p^2 / (1 - \eta_p^2)$ and $\eta_p^2 = \lambda / (\lambda + N)$ hold [Mayr *et al.*, 2007]. Sensitivity-type statistical power analysis data of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility) are listed in **Table 14** for each predictor by providing observed values (η_p^2 and λ), minimum values ($\eta_p^2_{\min}$ and $\lambda_{\min}$) to meet simultaneously conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$), and corresponding ratios ($\eta_p^2 / \eta_p^2_{\min}$ and $\lambda / \lambda_{\min}$). Values of $\lambda_{\min}$ were obtained with the G*Power software (version 3.1.9.4) [Faul *et al.*, 2007].

Table 14 Sensitivity-type statistical power analysis data of the ANOVA model of the response variable (primary-resonance frequency of $\{z_t, z_c\}$ transmissibility).

Predictor	η_p^2	$\eta_p^2_{\min}$	$\eta_p^2 / \eta_p^2_{\min}$	λ	$\lambda_{\min}$	$\lambda / \lambda_{\min}$
Sex	3.84×10^{-5}	4.46×10^{-1}	8.60×10^{-5}	4.61×10^{-4}	9.67×10^0	4.76×10^{-5}
Vibration magnitude	9.74×10^{-1}	4.90×10^{-1}	1.99×10^0	$4.43 \times 10^{+2}$	$1.15 \times 10^{+1}$	$3.85 \times 10^{+1}$
Sitting configuration	8.92×10^{-1}	4.83×10^{-1}	1.85×10^0	$9.87 \times 10^{+1}$	$1.12 \times 10^{+1}$	8.81×10^0
Sex × Vibration magnitude	1.09×10^{-1}	4.90×10^{-1}	2.23×10^{-1}	$1.47 \times 10^{+1}$	$1.15 \times 10^{+1}$	1.28×10^{-1}
Sex × Sitting configuration	2.13×10^{-2}	4.83×10^{-1}	4.42×10^{-2}	2.62×10^{-1}	$1.12 \times 10^{+1}$	2.33×10^{-2}
Vibration magnitude × Sitting configuration	1.25×10^{-1}	5.06×10^{-1}	2.46×10^{-1}	$1.71 \times 10^{+1}$	$1.23 \times 10^{+1}$	1.39×10^{-1}
Sex × Vibration magnitude × Sitting configuration	1.07×10^{-1}	5.06×10^{-1}	2.12×10^{-1}	$1.44 \times 10^{+1}$	$1.23 \times 10^{+1}$	1.17×10^{-1}
Independent groups: male group (six participants) and female group (six participants). η_p^2 = partial eta-squared; $\eta_p^2_{\min}$ = minimum value of partial eta-squared to meet simultaneously conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$); λ = non-centrality parameter of non-central F -distribution; $\lambda_{\min}$ = minimum value of non-centrality parameter of non-central F -distribution to meet simultaneously conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$).						

6.3.2 Primary-resonance modulus

Median and interquartile range of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates are listed respectively in **Table 15** and **Table 16**

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to provide information about *central tendency* and *dispersion* across replicates. Median and interquartile range of the response variable calculated across all tests were respectively 1.9 and 0.4.

Table 15 Median of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates.

	M															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P01	2.28	1.99	1.91	1.8	2.02	1.82	1.76	1.76	2.08	1.78	1.78	1.74	2.06	1.78	1.69	1.7
P02	2.53	2.69	2.35	1.87	2.38	2.3	2.11	1.75	2.09	2	1.98	1.66	2.12	1.98	1.85	1.65
P03	2.72	2.67	2.36	1.81	2.68	2.55	2.31	1.83	2.42	2.46	2.09	1.67	2.44	2.38	2.11	1.75
P05	2.13	2.19	1.92	1.82	2.09	2.06	1.93	1.8	2.05	1.96	1.88	1.74	1.98	1.88	1.85	1.78
P06	2.34	2.02	2.1	1.82	2.12	1.9	1.95	1.69	2.04	1.79	1.83	1.66	2.06	1.74	1.87	1.68
P12	2.38	2.49	2.22	1.85	2.62	2.26	2.16	1.87	2.25	2.04	2.04	1.72	2.36	2.11	1.94	1.73
	F															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P04	2.11	2.01	1.65	1.84	2.03	1.96	1.65	1.68	1.95	1.8	1.61	1.63	1.79	1.69	1.54	1.6
P07	2.09	2.06	1.84	1.68	2.29	1.99	1.6	1.64	1.94	1.72	1.76	1.66	1.85	1.66	1.58	1.59
P08	2.22	2.17	1.79	1.74	2.09	2.11	1.82	1.69	2.01	1.92	1.66	1.63	1.98	1.92	1.67	1.64
P09	2.15	2.36	2.3	1.96	2.06	2.42	2.02	1.82	2.11	2.21	2.03	1.63	2.03	2.11	1.9	1.67
P10	2.64	2.15	1.74	1.76	2.44	2.12	1.78	1.68	1.92	2.02	1.64	1.63	2.23	1.86	1.67	1.63
P11	2.33	2.14	1.83	1.77	2.12	2.06	1.77	1.67	2.02	1.82	1.68	1.72	2	1.8	1.66	1.69
Unit of measurement: 1. Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}.																

Table 16 Interquartile range of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates.

	M															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P01	0.34	0.092	0.13	0.067	0.087	0.093	1	0.092	0.27	0.1	0.065	0.23	0.16	0.11	0.13	0.039
P02	0.19	0.33	0.12	0.047	0.042	0.16	0.091	0.051	0.0076	0.041	0.068	0.069	0.14	0.21	0.026	0.017
P03	0.094	0.099	0.17	0.2	0.051	0.17	0.08	0.19	0.1	0.18	0.1	0.11	0.032	0.017	0.077	0.19
P05	0.077	0.14	0.11	0.031	0.2	0.035	0.046	0.1	0.14	0.19	0.038	0.092	0.12	0.054	0.078	0.043
P06	0.16	0.11	0.19	0.12	0.22	0.019	0.12	0.012	0.16	0.042	0.08	0.089	0.078	0.033	0.14	0.032
P12	0.012	0.037	0.12	0.088	0.21	0.12	0.14	0.096	0.13	0.063	0.099	0.065	0.13	0.23	0.065	0.041
	F															
	0d28w				0d45w				0d71w				1d12w			
	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE	ES	RS	TD	WE
P04	0.072	0.1	0.13	0.086	0.35	0.054	0.14	0.07	0.083	0.048	0.02	0.054	0.16	0.097	0.06	0.014
P07	0.34	0.13	0.21	0.08	0.26	0.025	0.069	0.048	0.091	0.12	0.13	0.039	0.087	0.16	0.065	0.018
P08	0.081	0.055	0.19	0.083	0.067	0.0074	0.18	0.02	0.085	0.12	0.13	0.066	0.083	0.048	0.14	0.08
P09	0.39	0.25	0.035	0.12	0.22	0.28	0.19	0.12	0.21	0.073	0.18	0.24	0.14	0.11	0.16	0.2
P10	0.45	0.15	0.11	0.016	0.33	0.097	0.13	0.043	0.26	0.026	0.05	0.021	0.086	0.079	0.15	0.044
P11	0.053	0.26	0.026	0.047	0.082	0.039	0.036	0.044	0.088	0.025	0.04	0.12	0.031	0.062	0.052	0.059
Unit of measurement: 1. Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}.																

The *data set* used for inferential statistical analysis is the one listed in **Table 15**. By virtue of the implemented replication, this data set was deemed robust enough to allow abstaining from detection and treatment of outliers. Indeed, the values of interquartile range listed in **Table 16** were generally small. Accordingly, there were neither missing data nor deleted cases.

Box plots of the median response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates and all six participants of each independent group are shown in **Figure 27** to provide graphical summaries of *descriptive statistical analysis* as well as graphical information about inter-participant variability. Box plots shown on the left side and of the right side of **Figure 27** contain the same information, but they are parameterised and arranged differently for convenience.

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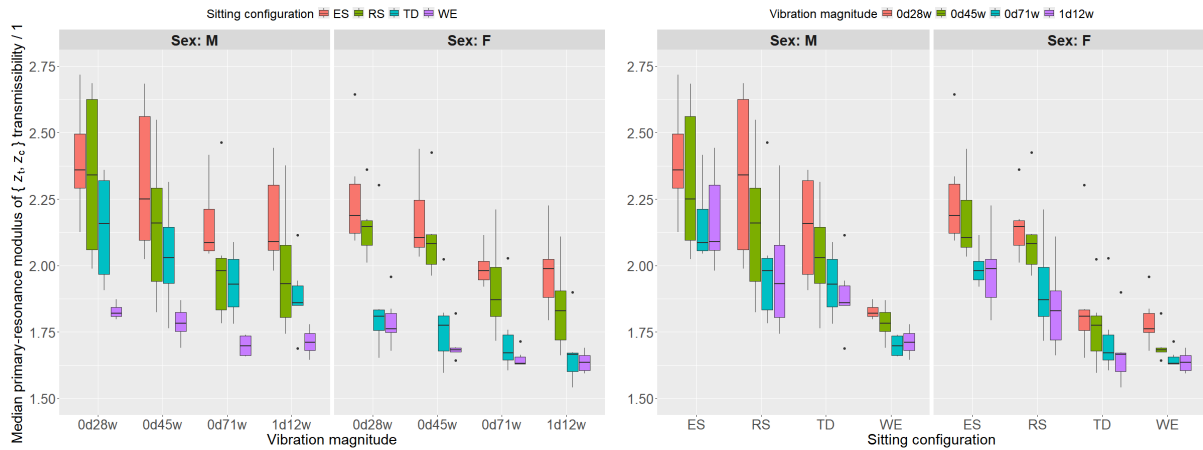


Figure 27 Box plots of the median response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) calculated across all three replicates and all six participants of each independent group. Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Left:** median primary-resonance frequency of $\{z_t, z_c\}$ transmissibility vs vibration magnitude (graphs parameterised on sitting configuration and arranged in columns by sex). **Right:** median primary-resonance frequency of $\{z_t, z_c\}$ transmissibility vs sitting configuration (graphs parameterised on vibration magnitude and arranged in columns by sex).

A *three-way mixed-design univariate ANOVA model* of the median response variable calculated across all three replicates (hereinafter *response variable*) was fit by using the R package *afex* (version 0.26-0) [Singmann *et al.*, 2020]. To protect against possible violation of the *sphericity assumption*, conservative Greenhouse–Geisser corrections were applied to eligible statistical degrees of freedom (namely, all except those corresponding to the main effect of sex, which, as a two-level factor, was not affected). To mitigate the *multiple-comparisons problem* inherent in multiway ANOVA [Cramer *et al.*, 2016], *p* values were adjusted using the Holm–Bonferroni method. To verify the *normality assumption* for the ANOVA model residuals, *a posteriori* diagnostic tests were performed. Diagnostic plots of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) are shown in **Figure 28** to provide graphical assessment of the normality assumption for the ANOVA model residuals. The flipped-axes histogram of residuals, the residual-fitted plot, and the quantile-quantile plot of residuals indicate no gross violation of the normality assumption for the ANOVA model residuals.

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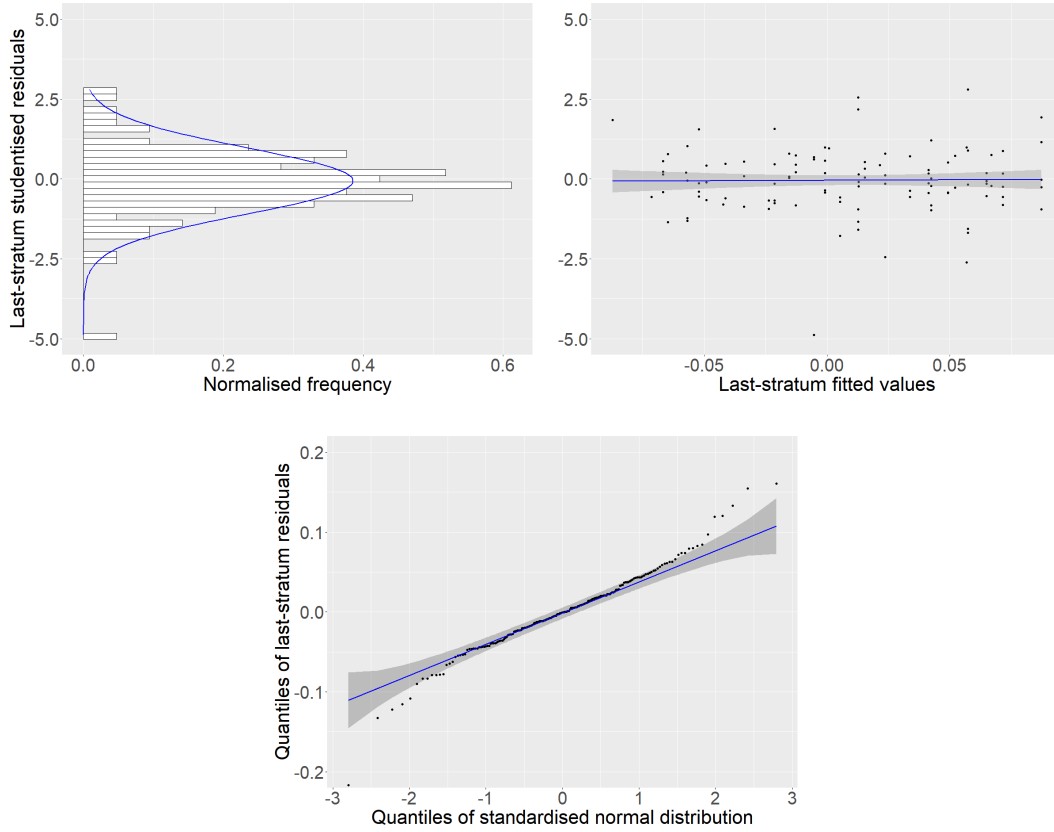


Figure 28 Diagnostic plots of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility). **Top left:** flipped-axes histogram of residuals (last-stratum studentised residuals vs normalised frequency) with equivalent normal-distribution curve. **Top right:** residual-fitted plot (last-stratum studentised residuals vs last-stratum fitted values) with robust-regression line and confidence intervals at 95 % confidence level. **Bottom:** quantile-quantile plot of residuals (quantiles of last-stratum residuals vs quantiles of standardised normal distribution) with robust-regression line and confidence intervals at 95 % confidence level. In the flipped-axes histogram of residuals, frequency is the number of observations per bin.

The ANOVA model of the response variable showed a significant main effect of both vibration magnitude, $F(2.10, 21.01) = 61.98, p < 0.001, \eta^2 = 0.13, \eta_p^2 = 0.86, \eta_G^2 = 0.23$, and sitting configuration, $F(2.10, 20.99) = 44.28, p < 0.001, \eta^2 = 0.43, \eta_p^2 = 0.82, \eta_G^2 = 0.50$, as well as a limited interaction between them, $F(3.49, 34.87) = 3.99, p = 0.06, \eta^2 = 0.01, \eta_p^2 = 0.29, \eta_G^2 = 0.03$, but failed to show a significant main effect of sex, $F(1, 10) = 3.90, p = 0.31, \eta^2 = 0.07, \eta_p^2 = 0.28, \eta_G^2 = 0.17$, and any significant sex-related interactions, $p \geq 0.67, \eta^2 \leq 0.02, \eta_p^2 \leq 0.14, \eta_G^2 \leq 0.04$. Still, for all treatments, the within-group mean value of the response variable was greater for males than for females. Three-way interaction plots of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) with within-subjects confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of sex) are shown in **Figure 29** to provide graphical summaries of *inferential statistical analysis*. Interaction plots shown on the left side and of the right side of **Figure 29** are equivalent, but they are parameterised and arranged differently for convenience. Two-way interaction plots of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) with between-subjects

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confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of vibration magnitude and sitting configuration) are shown in **Figure 30**. Interaction plots shown on the left side and of the right side of **Figure 30** are equivalent, but they are arranged differently for convenience.

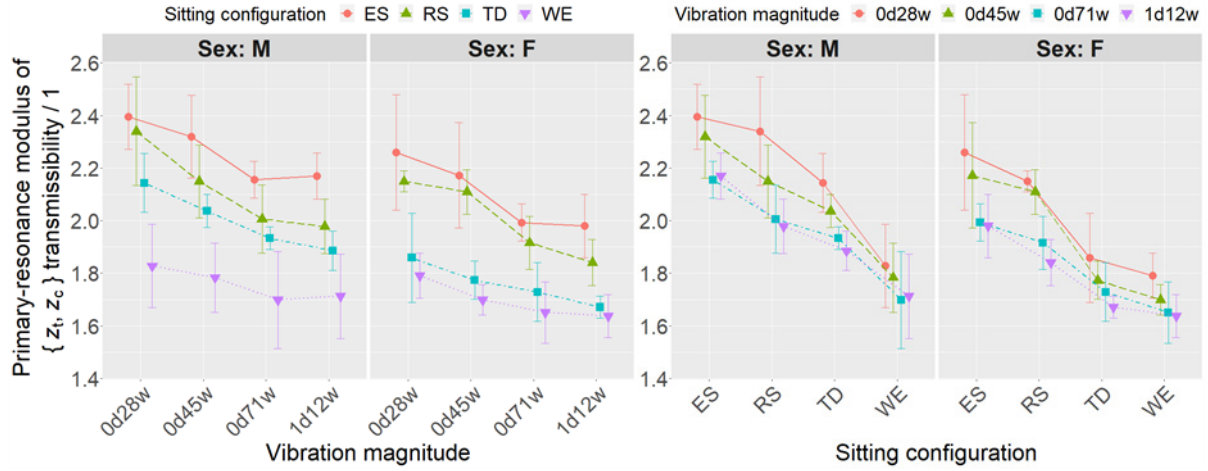


Figure 29 Three-way interaction plots of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) with within-subjects confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of sex). Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Left:** primary-resonance modulus of $\{z_t, z_c\}$ transmissibility vs vibration magnitude (graphs parameterised on sitting configuration and arranged in columns by sex). **Right:** primary-resonance modulus of $\{z_t, z_c\}$ transmissibility vs sitting configuration (graphs parameterised on vibration magnitude and arranged in columns by sex). Data points are horizontally offset for clarity.

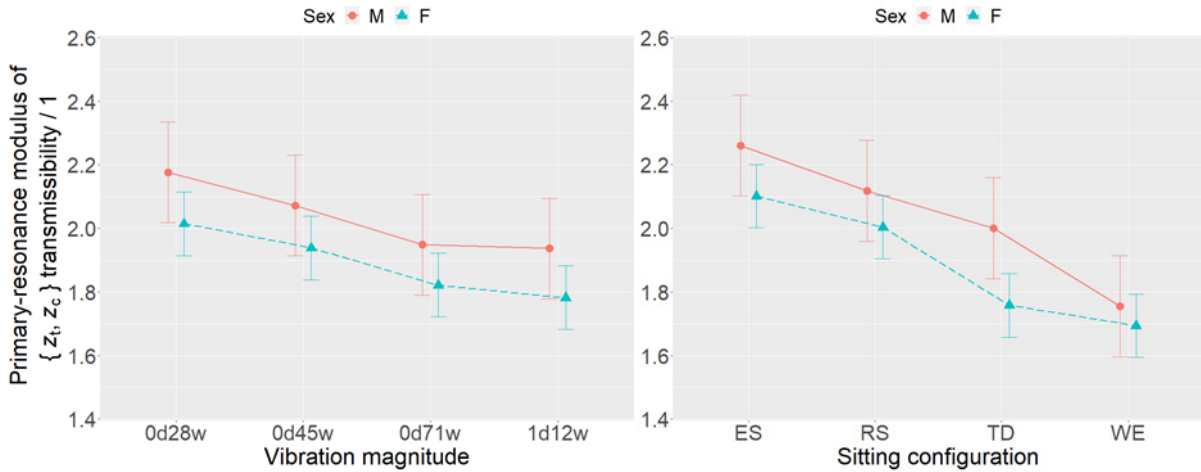


Figure 30 Two-way interaction plots of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) with between-subjects confidence intervals at 95 % confidence level (not to be used for comparisons across different levels of vibration magnitude and sitting configuration). Independent groups: male group (six participants) and female group (six participants). Sex levels: {M, F}. Vibration magnitude levels: {0d28w, 0d45w, 0d71w, 1d12w}. Sitting configuration levels: {ES, RS, TD, WE}. **Left:** primary-resonance modulus of $\{z_t, z_c\}$ transmissibility vs vibration magnitude (graphs parameterised on sex). **Right:** primary-resonance modulus of $\{z_t, z_c\}$ transmissibility vs sitting configuration (graphs parameterised on sex). Data points are horizontally offset for clarity.

A sensitivity-type statistical power analysis was performed by focussing on the partial eta-squared measure of effect size (η_p^2) and on the non-centrality parameter of the non-central F -distribution (λ);

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denoting by N the total sample size, the equivalent relations $\lambda = N \eta_p^2 / (1 - \eta_p^2)$ and $\eta_p^2 = \lambda / (\lambda + N)$ hold [Mayr *et al.*, 2007]. Sensitivity-type statistical power analysis data of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility) are listed in **Table 17** for each predictor by providing observed values (η_p^2 and λ), minimum values ($\eta_{p^2_{\min}}$ and $\lambda_{\min}$) to meet simultaneously conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$), and corresponding ratios ($\eta_p^2 / \eta_{p^2_{\min}}$ and $\lambda / \lambda_{\min}$). Values of $\lambda_{\min}$ were obtained with the G*Power software (version 3.1.9.4) [Faul *et al.*, 2007].

Table 17 Sensitivity-type statistical power analysis data of the ANOVA model of the response variable (primary-resonance modulus of $\{z_t, z_c\}$ transmissibility).

Predictor	η_p^2	$\eta_{p^2_{\min}}$	$\eta_p^2 / \eta_{p^2_{\min}}$	λ	$\lambda_{\min}$	$\lambda / \lambda_{\min}$
Sex	2.81×10^{-1}	4.46×10^{-1}	6.29×10^{-1}	4.68×10^0	9.67×10^0	4.84×10^{-1}
Vibration magnitude	8.61×10^{-1}	4.86×10^{-1}	1.77×10^0	$7.44 \times 10^{+1}$	$1.14 \times 10^{+1}$	6.54×10^0
Sitting configuration	8.16×10^{-1}	4.86×10^{-1}	1.68×10^0	$5.31 \times 10^{+1}$	$1.14 \times 10^{+1}$	4.68×10^0
Sex × Vibration magnitude	3.53×10^{-2}	4.86×10^{-1}	7.25×10^{-2}	4.39×10^{-1}	$1.14 \times 10^{+1}$	3.86×10^{-2}
Sex × Sitting configuration	1.38×10^{-1}	4.86×10^{-1}	2.84×10^{-1}	1.93×10^0	$1.14 \times 10^{+1}$	1.69×10^{-1}
Vibration magnitude × Sitting configuration	2.85×10^{-1}	5.19×10^{-1}	5.49×10^{-1}	4.78×10^0	$1.29 \times 10^{+1}$	3.70×10^{-1}
Sex × Vibration magnitude × Sitting configuration	9.73×10^{-2}	5.19×10^{-1}	1.87×10^{-1}	1.29×10^0	$1.29 \times 10^{+1}$	1.00×10^{-1}
Independent groups: male group (six participants) and female group (six participants). η_p^2 = partial eta-squared; $\eta_{p^2_{\min}}$ = minimum value of partial eta-squared to meet simultaneously conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$); λ = non-centrality parameter of non-central F-distribution; $\lambda_{\min}$ = minimum value of non-centrality parameter of non-central F-distribution to meet simultaneously conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$).						

6.4 Discussion

In quasi-static conditions, near null frequency, the $\{z_t, z_c\}$ transmissibility is almost real-valued; in particular, the modulus is almost unitary (by virtue of input and output axes being almost parallel), and the argument is almost null (by virtue of input and output accelerations being almost in phase). At the low end of the frequency band of interest, the seat–occupant system behaves approximately as a single-degree-of-freedom linear mass–spring–damper system kinematically excited from the base, and a primary resonance can be clearly identified by means of a prominent peak in modulus and a corresponding drop in argument.

Outside a neighbourhood of the origin, $\{z_t, z_c\}$ transmissibility is affected by vibration magnitude in terms of *scaling*. Indeed, with increasing vibration magnitude, a decrease is observed both in primary-resonance frequency and in primary-resonance modulus. This nonlinear ‘*softening*’ *behaviour*, well known from previous studies [Mansfield and Griffin, 2000; Tufano and Griffin, 2013], occurs not only locally but also on the whole frequency band of interest; in particular, with increasing vibration magnitude, both $\{z_t, z_c\}$ transmissibility modulus and (less markedly) $\{z_t, z_c\}$ transmissibility argument scale left and down.

Outside a neighbourhood of the origin, $\{z_t, z_c\}$ transmissibility is affected by sitting configuration in terms of *shaping*. Indeed, different behaviours can be observed at the ends of the frequency band of interest; this allows identifying a pair of *end zones* separated by a *transition zone*. The frequency boundaries of these zones and the overall shape of $\{z_t, z_c\}$ transmissibility modulus and $\{z_t, z_c\}$ transmissibility argument depend on the sitting configuration. By inspection, the transition zone seems to have lower bound approximately at the primary-resonance frequency (having median value 4.5 Hz and interquartile range 1 Hz across all tests) and upper bound approximately at a frequency between 14 Hz and 16 Hz. In accordance with previously reported results, it can be hypothesised that $\{z_t, z_c\}$ transmissibility is controlled in the low-end zone by seat-back reclination [Houghton, 2003] and in the high-end zone by footrest horizontal and vertical position [Fairley, 1986; Corbridge *et al.*, 1989]. In turn, these parameters may exert their influence by affecting loading distribution and contact area at the interface between seat cushion and occupant buttocks and, ultimately, by affecting dynamic stiffness of the seat [Wei and Griffin, 1997; Wei, 2000; Zhang *et al.*, 2013] and apparent mass of the occupant [Toward and Griffin, 2009, 2010]. In terms of the reference items defined in Surface Vehicle Recommended Practice SAE J1100:2009-11 [SAE International, 2009], the most important role is probably played in the low-end zone by A40 (torso angle) and in the high-end zone by the difference of A57 (thigh angle) and A27 (cushion angle).

Remarkably, $\{z_t, z_c\}$ coherence is high on the whole frequency band of interest, which proves the *input–output relation* to be *linear* and *noiseless*. Nevertheless, with increasing vibration magnitude, a slight decrease in $\{z_t, z_c\}$ coherence can be observed on the whole frequency band of interest; this can be explained by the contact at the interface between seat cushion and occupant buttocks becoming less firm. In particular, approximately between 5 Hz and 25 Hz, $\{z_t, z_c\}$ coherence takes slightly (but consistently) lower values for the ES sitting configuration; this can be interpreted as an effect of (*involuntary*) *muscular activity* of the occupant, which is a confounder in the adopted research design.

For *primary-resonance frequency*, median and interquartile range calculated across all tests (respectively 4.5 Hz and 1 Hz) are in agreement with previous observations [Zhang *et al.*, 2012]. Main effects and interactions of the considered factors offer a varied picture. On the one hand, whilst a *significant main effect of vibration magnitude* is unsurprising [Mansfield and Griffin, 2000; Tufano and Griffin, 2013], a *significant main effect of sitting configuration* is a new finding, which suggests that the concept of sitting configuration is appropriate to characterise the seat–occupant system as a whole. On the other hand, whilst *failure to observe a significant main effect of sex* is consistent with previous inconclusive observations [Corbridge *et al.*, 1989; Toward and Griffin, 2011a, 2011b], *failure to observe any significant interactions* cannot be compared with known previous findings; however, if confirmed by further studies, it will have relevant implications in product design and development.

For *primary-resonance modulus*, median and interquartile range calculated across all tests (respectively 1.9 and 0.4) are in agreement with previous observations [Corbridge *et al.*, 1989; Houghton, 2003; Walton, 2007; Toward and Griffin, 2011b; Zhang *et al.*, 2015]. The *significant main effect of vibration magnitude* is well known [Walton, 2007]; indeed, the nonlinear ‘softening’ behaviour of the seat–occupant system is confirmed by primary-resonance modulus decreasing with increasing vibration magnitude. Regarding the *significant main effect* of sitting configuration, there are no known results to be used for direct comparisons. For sitting configurations involving complete contact between thighs and seat cushion (namely, RS, TD, and WE), primary-resonance modulus (as well as primary-resonance frequency) decreases as seat-back reclination and torso angle decrease [Houghton, 2003]. It would be tempting to interpret this result in terms of variation of mass in contact with the seat cushion by referring to a single-degree-of-freedom linear mass–spring–damper system kinematically excited from the base. However, for this explanation to be legitimate, an opposite trend should be observed for primary-resonance modulus. Rather, the dynamic responses of the seat–occupant system (dynamic stiffness of the seat and apparent mass of the occupant) are

likely to change as the weight supported by the seat cushion increases. For the ES sitting configuration, the limited contact between thighs and seat cushion may account for results out of the general trend. The *interaction between vibration magnitude and sitting configuration* is just limited; however, the effect of vibration magnitude seems to decrease as seat-back reclination and torso angle decrease and as the weight supported by the seat cushion increases. A ‘saturation’ of the nonlinear ‘softening’ behaviour may be an explanation. The *failure to observe a significant main effect of sex* (unlike the *failure to observe any significant sex-related interactions*) seems borderline. On the one hand, graphical trends are unambiguous, and measures of effect size are unclearly dismissible; on the other hand, confidence intervals are close, and p value is high (p value before Holm–Bonferroni adjustment being 0.08). A possible explanation for this contradiction is provided by the lower sensitivity of mixed experimental designs to effects of between-subjects factors than to effects of within-subjects factors. In fact, despite inconclusive previous observations [Corbridge *et al.*, 1989; Toward and Griffin, 2011b], a main effect of sex seems reasonable in light of the obtained experimental data. Such effect may be related to body composition and fat distribution; specifically, given that a dynamic response at resonance is normally governed by damping, within-group mean values of primary-resonance modulus greater for males than for females may be explained by males having proportionally more lean mass and by females having proportionally more fat mass (especially at buttocks and thighs, right at the measurement interface) [Karastergiou *et al.*, 2012]. If a single-degree-of-freedom linear mass–spring–damper system kinematically excited from the base is adopted as a reference model, lower damping should correspond not only to higher resonance modulus, but also to higher resonance frequency; if damping is insufficient, though, the variation in resonance frequency may be small and go undetected. That said, it should be acknowledged that demographic and anthropometric characteristics such as age and body mass (weight) are not controlled in the adopted experimental design; hence, their main effects are confounded with the main effect of sex.

For the main effects of vibration magnitude and sitting configuration, the inequality $\lambda > \lambda_{\min}$ (respectively, $\eta_p^2 > \eta_{p^2 \min}^2$) holds, which indicates that conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$) are simultaneously met. Conversely, for the main effect of sex and for the interactions, the inequality $\lambda < \lambda_{\min}$ (respectively, $\eta_p^2 < \eta_{p^2 \min}^2$) holds, which indicates that conventional requirements of significance level ($\alpha = 0.05$) and statistical power ($1 - \beta = 0.80$) are not simultaneously met. However, whilst λ (respectively, η_p^2) takes much smaller values for the main effect of sex and for the interactions than for the main effects of vibration magnitude and sitting configuration, $\lambda_{\min}$ (respectively, $\eta_{p^2 \min}^2$) takes similar values for all predictors; in other words, across predictors, in the face of large differences in observed effect size, essentially

the same is the minimum effect size to meet simultaneously conventional requirements of significance level and statistical power. Given a research design and some requirements of significance level and statistical power, an increase in total sample size results in a decrease in $\lambda_{\min}$ (respectively, $\eta_{\min}^2$); in other words, given a research design and some requirements of significance level and statistical power, an increase in total sample size results in a decrease of the minimum effect size that the corresponding ANOVA model can detect. However, such minimum effect size may become too small to have practical relevance. Ultimately, it is likely that the main effect of sex and the interactions were either absent or significantly minor with respect to the main effects of vibration magnitude and sitting configuration. It should be noted that, since the research design implemented replication, the outcomes of this sensitivity-type statistical power analysis are to be regarded as conservative [Rouder and Haaf, 2018].

6.5 Conclusion

This experimental study allows establishing whether the concept of sitting configuration is appropriate to characterise a seat–occupant system as a whole with reference to performed activities as well as establishing whether there are significant main effects of, and interactions between, sex, vibration magnitude, and sitting configuration on objective responses of seat–occupant systems to vibration. The obtained experimental results may be used in follow-up studies to validate the occupant model developed in the modelling study.

The concept of sitting configuration proves appropriate to characterise the seat–occupant system as a whole with reference to performed activities. Indeed, at least as regards in-line transmission of vertical vibration at seat cushion, possible sitting configurations in next-generation cars are associated with distinctive objective seat–occupant responses to vibration. Specifically, when it comes to transmissibility plots, the scale is largely dependent on vibration magnitude, whereas the shape is largely dependent on sitting configuration; it therefore seems appropriate that, in future investigations, the long-standing emphasis on vibration magnitude be curtailed in favour of other factors. Results suggest that not only vibration magnitude but also sitting configuration (and possibly sex) affect sitting comfort in next-generation cars through in-line transmission of vertical vibration at seat cushion. Accordingly, in design and development of seating for next-generation cars, secondary activities and corresponding sitting configurations should be taken into account to optimise not only functionality but also comfort and protection (as well as related affective or emotional attributes).

By using a larger total sample size, these findings may be generalised to be applicable to the target population of users of next-generation cars. Indeed, the main technical limitation of this experimental study is the total sample size and, in particular, the group sample size. Future investigations should consider more realistic motion environments as well as a wider range of response functions and response variables. The effects of different seat designs should also be investigated.

7 Conclusion

7.1 General discussion

The scientific study of ride quality, and especially of sitting comfort, requires a multidisciplinary approach that is appropriately established within the ambit of human factors and ergonomics. Ideally, such scientific study needs a comprehensive framework to bridge the objective–subjective gap and to consider not only ‘hard’ functional or ergonomic features but also ‘soft’ affective or emotional features. This research project was conducted within such a framework to contribute in paving the way for characterising, understanding, and predicting sitting comfort in next-generation (connected, autonomous, shared, and electric) cars. The contribution was divided into the three strands of theory, modelling, and experimentation, which resulted respectively in three theoretical studies, one modelling study, and one experimental study. Theoretical background and generalisations were reserved for a substantial appendix.

On the one hand, theory and modelling allow interpreting experimental results and making predictions (in particular, parametric analyses) without resorting to expensive, time-consuming, and potentially dangerous experimental tests. On the other hand, experimentation allows collecting laboratory or field data to develop theories and mathematical models. Indeed, theory, modelling, and experimentation feed each other within a virtuous circle and are all indispensable. However, in both biodynamics and seating dynamics, much experimental work and modelling work but little theoretical work has been done. In fact, a theory of human vibration has yet to be developed; to achieve such a formidable goal, a sustained collective effort will be necessary. In this research project, relatively advanced mathematical methods have been used to take a small step in the indicated direction; their popularisation amongst the community of engineering and physical sciences was deemed useful both scientifically and, to some extent, pedagogically.

Within the theoretical strand of this research project, three theoretical studies were conducted to answer in order the following research questions:

- Q1. What is the solution to the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, to the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain?

Q2. What is a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements?

Q3. What is a concise but logically self-contained and mathematically rational characterisation, both in the time domain and in the frequency domain, of a linear viscoelastic material?

Within the modelling strand of this research project, a modelling study was conducted to answer the following research question:

Q4. Is it possible to calibrate a finite-element model of a seat occupant in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back)?

Within the experimentation strand of this research project, an experimental study was conducted to answer the following research questions:

Q5. Is the concept of sitting configuration appropriate to characterise a seat–occupant system as a whole with reference to performed activities?

Q6. Are there significant main effects of, and interactions between, sex, vibration magnitude, and sitting configuration on objective responses of seat–occupant systems to vibration?

The theoretical studies conducted to answer Q1, Q2, and Q3, as described in Chapter 2, Chapter 3, and Chapter 4, allows providing in order the following research answers:

A1. The solution to the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, to the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain results from using the theory of distributions, and it evokes singular distributions, such as ‘mysterious’ pseudofunctions as well as ‘hidden’ unit impulses and derivatives of them, all having the origin as singular support.

A2. A proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements results from applying the findings of the first theoretical study, and, in the frequency domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null frequency).

A3. A concise but logically self-contained and mathematically rational characterisation, both in the time domain and in the frequency domain, of a linear viscoelastic material results from adopting the hereditary approach and applying the findings of both the first theoretical study and the second theoretical study, and, in the time domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null time).

The modelling study conducted to answer Q4, as described in Chapter 5, allows providing the following research answer:

A4. A finite-element model of a seat occupant calibrated in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back) results from applying the findings of the third theoretical study, but it proves having too restrictive a minimum complexity requirement.

The experimental study conducted to answer Q5 and Q6, as described in Chapter 6, allows providing the following research answers:

A5. The concept of sitting configuration is appropriate to characterise a seat–occupant system as a whole with reference to performed activities, and, at least as regards in-line transmission of vertical vibration at seat cushion, possible sitting configurations in next-generation cars are associated with different objective seat–occupant responses to vibration.

A6. On objective responses of seat–occupant systems to vibration, there is a possible main effect of sex, significant main effects of both vibration magnitude and sitting configuration, limited interaction between vibration magnitude and sitting configuration, and no sex-related interactions.

The research answers can be summarised by the following research statement:

Research and development for sitting comfort in next-generation cars benefits from introducing the concept of sitting configuration and from using relatively advanced mathematical methods such as the theory of distributions, which may prove helpful in elaborating a (currently unavailable) theory of human vibration.

7.2 Summary of findings

The findings of each study are presented separately.

7.2.1 First theoretical study

This first theoretical study allows obtaining the solution to the problem of expressing in the frequency domain the (possibly iterated) antidifferentiation in the time domain or, equivalently, the problem of expressing in the time domain the (possibly iterated) antidifferentiation in the frequency domain. Well-known physical aspects are omitted to focus on unclear mathematical aspects. The obtained solution results from using the theory of distributions, and it evokes singular distributions, such as ‘mysterious’ pseudofunctions as well as ‘hidden’ unit impulses and derivatives of them, all having the origin as singular support. The obtained solution holds under mildly restrictive conditions and contains specification of the input spaces. However, no attempt is made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions are identified for formulations to be mathematically tenable.

Let a be a right-sided distribution of rapid decay on $\mathbb{R}$, $v \equiv a^{(-1)}$ be the right-sided 1st antiderivative of a , and $x \equiv v^{(-1)} \equiv a^{(-2)}$ be the right-sided 1st antiderivative of v as well as the right-sided 2nd antiderivative of a . Let A , V , and X be the Fourier transforms of respectively a , v , and x .

In the time domain, relations hold

$$v = \mathfrak{G} * a$$

$$x = \rho * a$$

$$x = \mathfrak{G} * v$$

where

- $*$ is the convolution operator
- $\mathfrak{G}$ is the unit step distribution on $\mathbb{R}$
- ρ is the unit ramp distribution on $\mathbb{R}$

In the frequency domain, relations hold

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$$\begin{aligned} V &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) A \\ &= \text{p.v.} \left(\frac{A}{j\omega} \right) + \pi A(0)\delta \end{aligned}$$

$$\begin{aligned} X &= \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) A \\ &= \text{f.p.} \left(\frac{A}{-\omega^2} \right) + j\pi A(0)\delta^{(1)} - j\pi A^{(1)}(0)\delta \end{aligned}$$

$$X = \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) V$$

and, in particular, relations hold

$$A(0) = 0 \Leftrightarrow V = \frac{A}{j\omega}$$

$$\begin{cases} A(0) = 0 \\ A^{(1)}(0) = 0 \end{cases} \Leftrightarrow X = \frac{A}{-\omega^2}$$

$$\begin{aligned} A(0) = 0 \Leftrightarrow X &= \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) V \\ &= \text{p.v.} \left(\frac{V}{j\omega} \right) + \pi V(0)\delta \end{aligned}$$

$$\begin{cases} A(0) = 0 \\ A^{(1)}(0) = 0 \end{cases} \Leftrightarrow \begin{cases} A(0) = 0 \\ V(0) = 0 \end{cases} \Leftrightarrow X = \frac{V}{j\omega} = \frac{A}{-\omega^2}$$

where

- j is the imaginary unit
- ω is angular frequency
- δ is the unit impulse on $\mathbb{R}$
- $\delta^{(1)}$ is the unit doublet on $\mathbb{R}$
- $\text{p.v.}(1/(j\omega))$ is the canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/(j\omega)$ via the Cauchy-principal-value method
- $\text{f.p.}(1/(-\omega^2))$ is the canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/(-\omega^2)$ via the Hadamard-finite-part method

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- $A^{(1)}$ is the 1st derivative of A

In terms of operators, relations hold

$$\mathcal{F}(\mathcal{D}^{-1}(a)) = \left(\text{p.v.} \left(\frac{1}{j\omega} \right) + \pi\delta \right) \mathcal{F}(a)$$

$$\mathcal{F}(\mathcal{D}^1(v)) = j\omega \mathcal{F}(v)$$

$$\mathcal{F}(\mathcal{D}^{-2}(a)) = \left(\text{f.p.} \left(\frac{1}{-\omega^2} \right) + j\pi\delta^{(1)} \right) \mathcal{F}(a)$$

$$\mathcal{F}(\mathcal{D}^2(x)) = -\omega^2 \mathcal{F}(x)$$

and, dually, relations hold

$$\mathcal{D}^{-1}(\mathcal{F}(a)) = \mathcal{F} \left(\left(\text{p.v.} \left(\frac{1}{-jt} \right) + \pi\delta \right) a \right)$$

$$\mathcal{D}^1(\mathcal{F}(v)) = \mathcal{F}(-jtv)$$

$$\mathcal{D}^{-2}(\mathcal{F}(a)) = \mathcal{F} \left(\left(\text{f.p.} \left(\frac{1}{-t^2} \right) - j\pi\delta^{(1)} \right) a \right)$$

$$\mathcal{D}^2(\mathcal{F}(x)) = \mathcal{F}(-t^2x)$$

where

- t is time
- $\text{p.v.}(1/(-jt))$ is the canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/(-jt)$ via the Cauchy-principal-value method
- $\text{f.p.}(1/(-t^2))$ is the canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/(-t^2)$ via the Hadamard-finite-part method
- $\mathcal{F}$ is the Fourier transformation
- $\mathcal{D}^{-1}$ is the right-sided 1st-order antidifferentiation operator
- $\mathcal{D}^1$ is the 1st-order differentiation operator
- $\mathcal{D}^{-2}$ is the right-sided 2nd-order antidifferentiation operator
- $\mathcal{D}^2$ is the 2nd-order differentiation operator

Generalised relations are in Chapter A.7 of Appendix A.

7.2.2 Second theoretical study

This second theoretical study allows obtaining a proper characterisation, both in the time domain and in the frequency domain, of each of the four basic two-terminal mechanical elements; these are the linear spring, the linear viscous damper, the linear inerter (including the rigid mass), and the linear hysteresis damper. In particular, a proper characterisation can be in terms of dynamic stiffness, mechanical impedance, and apparent mass as well as in terms of dynamic compliance, mobility, and accelerance (and their time-domain homologues). Well-known physical aspects are omitted to focus on unclear mathematical aspects. The obtained characterisation results from applying the findings of the first theoretical study, and, in the frequency domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null frequency). The obtained characterisation holds under mildly restrictive conditions (except in the ill-behaved case of the linear hysteresis damper) and contains specification of the input spaces. However, no attempt is made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions are identified for formulations to be mathematically tenable.

Dynamic stiffness, mechanical impedance, and apparent mass as well as dynamic compliance, mobility, and accelerance (and their time-domain homologues) of linear spring, linear viscous damper, linear inerter (including rigid mass), and linear hysteresis damper are listed in **Table 1** (repeated here for convenience). The data listed in **Table 1** complement and correct those available elsewhere (for example, the data presented in tabular form by Griffin [1990, p. 361]).

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Table 1 Dynamic stiffness, mechanical impedance, and apparent mass as well as dynamic compliance, mobility, and accelerance (and their time-domain homologues) of linear spring, linear viscous damper, linear inerter (including rigid mass), and linear hysteresis damper.

	Linear spring		Linear viscous damper		Linear inerter (including rigid mass)		Linear hysteresis damper	
	Time domain	Frequency domain	Time domain	Frequency domain	Time domain	Frequency domain	Time domain	Frequency domain
Dynamic stiffness	$k\delta$	$k1_{\mathbb{R}}$	$c\delta^{(1)}$	$j\omega$	$b\delta^{(2)}$	$-b\omega^2$	$\text{p.v.}\left(-\frac{h}{\pi t}\right)$	$jh \operatorname{sgn}$
Dynamic compliance	$\frac{\delta}{k}$	$\frac{1_{\mathbb{R}}}{k}$	$\frac{\vartheta}{c}$	$\text{p.v.}\left(\frac{1}{j\omega}\right) + \frac{\pi\delta}{c}$	$\frac{\rho}{b}$	$\text{f.p.}\left(\frac{1}{-b\omega^2}\right) + \frac{j\pi\delta^{(1)}}{b}$	$\text{p.v.}\left(\frac{1}{\pi h t}\right)$	$\frac{\operatorname{sgn}}{jh}$
Mechanical impedance	$k\vartheta$	$\text{p.v.}\left(\frac{k}{j\omega}\right) + \pi k\delta$	$c\delta$	$c1_{\mathbb{R}}$	$b\delta^{(1)}$	$j\omega$	$-\frac{h \ln(t)}{\pi}$	$\text{Pf}\left(\frac{h}{ \omega }\right)$
Mobility	$\frac{\delta^{(1)}}{k}$	$\frac{j\omega}{k}$	$\frac{\delta}{c}$	$\frac{1_{\mathbb{R}}}{c}$	$\frac{\vartheta}{b}$	$\text{p.v.}\left(\frac{1}{jb\omega}\right) + \frac{\pi\delta}{b}$	$\text{f.p.}\left(-\frac{1}{\pi h t^2}\right)$	$\frac{ \omega }{h}$
Apparent mass	$k\rho$	$\text{f.p.}\left(\frac{k}{-\omega^2}\right) + j\pi k\delta^{(1)}$	$c\vartheta$	$\text{p.v.}\left(\frac{c}{j\omega}\right) + \pi c\delta$	$b\delta$	$b1_{\mathbb{R}}$	$-\frac{ht \ln(t)}{\pi}$	$\text{Pf}\left(\frac{jh}{-\omega \omega }\right)$
Accelerance	$\frac{\delta^{(2)}}{k}$	$\frac{-\omega^2}{k}$	$\frac{\delta^{(1)}}{c}$	$\frac{j\omega}{c}$	$\frac{\delta}{b}$	$\frac{1_{\mathbb{R}}}{b}$	$\text{f.p.}\left(\frac{2}{\pi h t^3}\right)$	$\frac{-\omega \omega }{jh}$
k = stiffness of the linear spring; c = viscous damping coefficient of the linear viscous damper; b (in particular, m) = inertance of the linear inerter (in particular, mass of the rigid mass); h = hysteresis damping coefficient of the linear hysteresis damper; j = imaginary unit; t = time; ω = angular frequency; $1_{\mathbb{R}}$ = unit distribution on $\mathbb{R}$; ϑ = unit step distribution on $\mathbb{R}$; ρ = unit ramp distribution on $\mathbb{R}$; δ = unit impulse on $\mathbb{R}$; $\delta^{(1)}$ = unit doublet on $\mathbb{R}$; $\delta^{(2)}$ = unit triplet on $\mathbb{R}$; f_{L^0} = non-locally-integrable function on $\mathbb{R}$ admitting regularisation; h_{L^0} = non-locally-integrable function on $\mathbb{R}$ admitting canonical regularisation; $\text{Pf}(f_{L^0})$ = pseudofunction on $\mathbb{R}$ generated by f_{L^0} via the Hadamard-finite-part method; $\text{f.p.}(h_{L^0})$ (when applicable, $\text{p.v.}(h_{L^0})$) = canonical pseudofunction on $\mathbb{R}$ generated by h_{L^0} via the Hadamard-finite-part method (when applicable, via the Cauchy-principal-value method).								

7.2.3 Third theoretical study

This third theoretical study allows obtaining a non-heuristic but application-oriented characterisation of a linear viscoelastic material (shorthand for non-ageing homogeneous isotropic linear viscoelastic material under isothermal adiabatic conditions) both in the time domain and in the frequency domain. Well-known physical aspects are omitted to focus on unclear mathematical aspects. The obtained characterisation results from adopting the hereditary approach and from applying the findings of both the first theoretical study and the second theoretical study, and, in the time domain, it generally features nonobvious behaviour on a neighbourhood of the origin (namely, of null time). The obtained characterisation holds under mildly restrictive conditions and contains specification of the input spaces. However, no attempt is made to specify the most general (ideally, necessary and sufficient) conditions for results to hold; rather, sets of incremental and reasonably general (sufficient) conditions are identified for formulations to be mathematically tenable.

A linear viscoelastic material is assumed a Cauchy continuum subjected to infinitesimal displacements and infinitesimal displacement gradients, and it is regarded as a pair of independent power-conjugate convolution systems having the property of fading memory. The hereditary approach is adopted, and the theory of distributions is used.

Bulk relaxation kernel and shear relaxation kernel as well as bulk relaxation modulus and shear relaxation modulus are introduced. A linear viscoelastic material is classified as either solid or fluid; both its initial response and its asymptotic response are classified as either elastic or viscous. Each of the bulk relaxation modulus and the shear relaxation modulus is expressed as the sum of a non-negative real constant of suitable quantity dimension times the unit impulse on $\mathbb{R}$, a non-negative real constant of suitable quantity dimension times the unit step distribution on $\mathbb{R}$, and a causal regular distribution vanishing at infinity on $\mathbb{R}$ (in particular, for a fluid linear viscoelastic material, a causal regular integrable distribution on $\mathbb{R}$) the result of whose localisation at $\mathbb{R}_{(0,+\infty)}$ is a distribution on $\mathbb{R}_{(0,+\infty)}$ generated by a strictly completely monotonic function on $\mathbb{R}_{(0,+\infty)}$.

Dynamic bulk modulus and dynamic shear modulus as well as dynamic bulk viscosity and dynamic shear viscosity are introduced; each of them is expressed as the sum of a storage term and a loss term. Dynamic Young modulus, dynamic Poisson number, and bulk-to-shear dynamic-modulus ratio are introduced. Normalised quantities are introduced.

A linear elastic material (shorthand for non-ageing homogeneous isotropic linear elastic material under isothermal adiabatic conditions) and a linear viscous material (shorthand for non-ageing homogeneous isotropic linear viscous material under isothermal adiabatic conditions) are obtained as limiting cases. A generalised Maxwell material is considered as a special case.

7.2.4 Modelling study

This modelling study allows investigating whether it is possible to calibrate a finite-element model of a seat occupant in terms of both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back). The developed occupant model results from applying the findings of the third theoretical study, but it proves having too restrictive a minimum complexity requirement.

This is a first attempt to develop a finite-element model of a seat occupant that can predict, possibly in a range of sitting configurations, both contact pressure at an interface with a rigid seat (not only at seat cushion but also at seat back) and apparent mass (not only of occupant buttocks but also of occupant back). The overall experimental–computational match seems promising, but there is some room for improvement. The experimental–computational match in terms of sitting configuration is good, which makes it plausible that the occupant model can be used in a range of sitting configurations. However, results suggest that the imposed minimum complexity requirement is too restrictive, especially as regards implications on vertical-to-normal apparent mass at seat back.

7.2.5 Experimental study

This experimental study allows establishing whether the concept of sitting configuration is appropriate to characterise a seat–occupant system as a whole with reference to performed activities as well as establishing whether there are significant main effects of, and interactions between, sex, vibration magnitude, and sitting configuration on objective responses of seat–occupant systems to vibration. The obtained experimental results may be used in follow-up studies to validate the occupant model developed in the modelling study.

The concept of sitting configuration proves appropriate to characterise the seat–occupant system as a whole with reference to performed activities. Indeed, at least as regards in-line transmission of vertical vibration at seat cushion, possible sitting configurations in next-generation cars are associated with distinctive objective seat–occupant responses to vibration. Specifically, when it comes to transmissibility plots, the scale is largely dependent on vibration magnitude, whereas the shape is largely dependent on sitting configuration; it therefore seems appropriate that, in future

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investigations, the long-standing emphasis on vibration magnitude be curtailed in favour of other factors. Results suggest that not only vibration magnitude but also sitting configuration (and possibly sex) affect sitting comfort in next-generation cars through in-line transmission of vertical vibration at seat cushion. Accordingly, in design and development of seating for next-generation cars, secondary activities and corresponding sitting configurations should be taken into account to optimise not only functionality but also comfort and protection (as well as related affective or emotional attributes).

7.3 Methodology assessment

The methodology used for the theoretical studies is standard in the community of applied mathematics but not in the community of engineering and physical sciences. In light of the results, it seems appropriate. The main technical limitation is the inherent linearity of the theory of distributions.

The methodology used for the modelling study is standard. In light of the results, it seems appropriate. The main technical limitations are the homogeneity and linearity of the viscoelastic material, the linearity of the stress–displacement solid elements, and the limited biofidelity of the torso part of the occupant model.

The methodology used for the experimental studies is standard. In light of the results, it seems appropriate. The main technical limitation is the total sample size and, in particular, the group sample size.

7.4 Recommendations

For future theoretical studies, fractional and multidimensional differentiation and antidifferentiation, networks of multi-terminal mechanical elements, and ageing and anisotropic linear viscoelastic materials should be considered.

For future modelling studies, multidirectional or multiaxial excitations, calibration or validation via modal analysis data, and deformable seat models for evaluation of transmissibility possibly in a range of sitting configurations should be considered.

For future experimental studies, more realistic motion environments, a wider range of response functions and response variables, and different seat designs should be considered.

Theoretical background and generalisations

Appendix

Abstract (A)

Appendix

Theoretical background and generalisations

In Appendix A, theoretical background and generalisations are presented. Specifically, Chapter A.1, Chapter A.2, and Chapter A.3 are focussed on the mathematical framework, Chapter A.4 and Chapter A.5 are focussed on signals, and Chapter A.6 and Chapter A.7 are focussed on systems. In particular, the first six chapters contain a large amount of material synthesised (without proof) from a myriad of very technical mathematical sources some of which not in English; such material is by no means new, but it is reworked with consistency and some originality and is presumably not widely known within the community of engineering and physical sciences. The final chapter is an original contribution, and it can be regarded as a generalised version of Chapter 2 with a greater emphasis on the concept of system.

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A.1 Mathematical framework: Spaces and maps

A.1.1 Introduction

We highlight keywords in italics either at the first occurrence or in the defining sentence. For common mathematical terms and symbols, we refer to International Standard ISO 80000-2:2019 [International Organization for Standardization, 2019]. For any other symbol, we clarify the meaning at the first occurrence and we prevent different uses from there on.

We assume the *axiom of choice* of the *theory of sets* to hold (namely, we assume that the *Cartesian product* of a nonempty collection of nonempty sets is a nonempty set) [Jacobson, 1989; Schwartz, 1991; Bourbaki, 2004a; Adámek *et al.*, 2004]. We denote by $:=$ the relation of *equality by definition* and by $\equiv$ the relation of *identity* (also known as *universally quantified equality*). We denote by $\subseteq$ (respectively, $\supseteq$) the relation of *inclusion* (respectively, *reverse inclusion*) and by $\subset$ (respectively, $\supset$) the relation of *proper inclusion* (respectively, *proper reverse inclusion*). Let the *power set* of a set be the collection of all its subsets including the *empty set* $\emptyset$ and the set itself [Jacobson, 1985, 1989; Pinter, 1990; Adámek *et al.*, 2004]. Let $\mathbb{C}$ be the *set of complex numbers*, $\mathbb{R}$ be the *set of real numbers*, $\mathbb{Q}$ be the *set of rational numbers*, $\mathbb{Z}$ be the *set of integer numbers*, and $\mathbb{N}$ be the *set of natural numbers* (namely, the set of positive integer numbers and zero). The geometric representations of $\mathbb{C}$ and $\mathbb{R}$ are respectively the *complex plane* and the *real line* [Simmons, 1963; Willard, 1970; Bourbaki, 1989, 1995]. Let $\aleph_0$ be the *first transfinite cardinal number* (namely, the cardinality of $\mathbb{N}$). Let $\bar{\mathbb{R}}$ be the *set of extended real numbers* (namely, the union $\mathbb{R} \cup \{-\infty, +\infty\}$ of $\mathbb{R}$ and the set of *negative infinity* $-\infty$ and *positive infinity* $+\infty$, which are respectively the infimum and the supremum of $\mathbb{R}$ in $\bar{\mathbb{R}}$) [Bourbaki, 1995]; hereinafter, we stipulate that any statement involving *infinity* with no qualifier is separately valid for both negative infinity and positive infinity. Let $\bar{\mathbb{N}}$ be the *set of extended natural numbers* (namely, the union $\mathbb{N} \cup \{+\infty\}$ of $\mathbb{N}$ and $+\infty$, which is the supremum of $\mathbb{N}$ in $\bar{\mathbb{N}}$) [Koch, 2020].

Let j be the *imaginary unit*, and let e be the *base of natural logarithm* (also known as *Euler number*). Let p and p' be either real constants greater than or equal to 1 or $+\infty$, and let p and p' form a pair of *Hölder-conjugate exponents*, which is a pair satisfying the relation [Leoni, 2017]

$$\frac{1}{p} + \frac{1}{p'} = 1$$

A.1 Mathematical framework: Spaces and maps

with the convention that, if one of p and p' equals 1, the other equals $+\infty$. Let q and q' be real constants greater than 1, and let q and q' form a pair of Hölder-conjugate exponents.

Let z , z_1 , and z_2 be integer constants. Let n_0 be a natural constant, and let i_0 be the generic element of the index set consisting of the natural constants in $\mathbb{N}_{[0, n_0]}$. Let the *binomial coefficient*

$\binom{n_0}{i_0}$ be defined by

$$\binom{n_0}{i_0} \equiv \binom{n_0}{n_0 - i_0} \\ := \frac{n_0!}{i_0!(n_0 - i_0)!}$$

Let $n \equiv n_1$ be a natural constant in $\mathbb{N}_{[1, +\infty)}$, and let $i \equiv i_1$ be the generic element of the index set consisting of the natural constants in $\mathbb{N}_{[1, n]} \equiv \mathbb{N}_{[1, n_1]}$. Let n_2 be a natural constant in $\mathbb{N}_{[2, +\infty)}$, and let i_2 be the generic element of the index set consisting of the natural constants in $\mathbb{N}_{[2, n_2]}$. Let $\mathbb{C}^n$ and $\mathbb{R}^n$ be the sets of all the n -tuples of respectively complex constants and real constants.

By abuse of language, we say that a quantity of quantity dimension other than one takes value in a subset of one of (in particular, in one of) $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$, $\bar{\mathbb{R}}$, $\bar{\mathbb{N}}$, $\mathbb{C}^n$, $\mathbb{R}^n$ whenever its numerical value expressed in a suitable unit of measurement takes value therein.

We say that a variable (in particular, an index) in a well-formed formula is either *free* or *bound* depending on whether it does or does not occur within the scope of a quantifier. Hereinafter, unless otherwise stated, a variable (in particular, an index) is a free variable (in particular, a free index). Let ξ be a free real variable of suitable quantity dimension, let ξ_0 be a real constant of the same quantity dimension as ξ , and let $\xi_{\neq 0}$ be a non-null real constant of the same quantity dimension as ξ .

Where applicable, we denote the value at a point of $\mathbb{R}$ of a function on $\mathbb{R}$ or distribution on $\mathbb{R}$ (as introduced below) by the symbol of the function on $\mathbb{R}$ or distribution on $\mathbb{R}$ immediately followed by the parenthesised point of $\mathbb{R}$.

By adopting the *Lagrange's notation for differentiation and antidifferentiation*, we denote a derivative (respectively, right-sided antiderivative) of some order by a right superscript containing a parenthesised non-negative (respectively, non-positive) integer constant whose absolute value equals the order of the derivative (respectively, right-sided antiderivative).

In the following, some information about spaces and maps is summarised.

A.1.2 Spaces

By the term *space*, we refer to a set possibly equipped with some *structure* (namely, a set whose elements may have some arrangement and relations) [Simmons, 1963]. A set on which a structure is defined to obtain a space is said to be the *underlying set* of the space [Bourbaki, 1995]. In some specified context, a certain structure may be induced ‘naturally’ on a given space; we liberally qualify such a structure as *canonical*.

A.1.2.1 Measurable spaces

A *measurable structure* (also known as σ -*algebra*, or σ -*field*, or *tribe*) on a nonempty set is a subcollection of the power set of the nonempty set that is invariant under complementation, is nonempty (or, equivalently, contains any of $\emptyset$ and the nonempty set itself, and hence both of them), and is invariant under countable union (or, equivalently, is invariant under countable intersection) [Schwartz, 1973; Berberian, 1988; Folland, 1999; Ambrosio *et al.*, 2000; Bauer, 2001; Evans and Gariepy, 2015; Leoni, 2017]. A measurable structure on a nonempty set is said to be *coarser*, *weaker*, or *smaller* (respectively, *finer*, *stronger*, or *larger*) than another measurable structure on the same nonempty set if and only if each element of the former (respectively, of the latter) is an element of the latter (respectively, of the former) [Berberian, 1988]; we say that two measurable structures on a nonempty set are *equivalent* if and only if one is both coarser and finer than the other. A *measurable space* is a nonempty set equipped with a measurable structure [Folland, 1999; Bauer, 2001; Leoni, 2017]. An element of the measurable structure of a measurable space is said to be a *measurable subset* of the measurable space [Folland, 1999; Bauer, 2001].

A.1.2.2 Algebraic spaces

An *algebraic structure* on a set is a collection of operations defined on the elements of the set and having specified properties [Simmons, 1963]. An *algebraic space* (also known as *algebraic system*) is a set equipped with an algebraic structure [Simmons, 1963; Jacobson, 1985; Pinter, 1990; Adámek *et al.*, 2004]; it is worth mentioning that, by algebraic structure, it is often intended what we name algebraic space [Jacobson, 1985; Pinter, 1990].

A *group* is a nonempty set equipped with, and invariant under, a binary *group operation* having the properties of associativity, existence (and necessarily uniqueness) of an identity element, and existence (and necessarily uniqueness) of an inverse element for each element [Simmons, 1963; Jacobson, 1985; Pinter, 1990]; the group operation of a group proves to have also the property of

cancellation [Jacobson, 1985; Pinter, 1990]. A *commutative group* (also known as *Abelian group*) is a group whose group operation has the property of commutativity [Simmons, 1963; Jacobson, 1985; Pinter, 1990]. The group having only one element (necessarily coinciding with the identity element) is a commutative group that is said to be the *trivial group*. A group whose group operation is named *addition* (respectively, *multiplication*) is said to be an *additive group* (respectively, a *multiplicative group*) [Simmons, 1963; Jacobson, 1985; Pinter, 1990]. For an additive commutative group, it is convenient to introduce a binary operation named *subtraction* consisting of the addition of a first element and the additive inverse of a second element of an ordered pair [Simmons, 1963]. It is worth noting that, when equipped with the usual operation of addition, $\mathbb{N}$ fails to be a group as it does not have the property of existence (and necessarily uniqueness) of an inverse element for each element (in fact, it is an example of a certain type of algebraic space to which reference is made as *commutative monoid*) [Jacobson, 1985].

A *ring* is an additive commutative group equipped with, and invariant under, a binary operation named *multiplication* having the properties of associativity, left-distributivity over addition, and right-distributivity over addition [Simmons, 1963; Pinter, 1990]. A *unital ring* is a ring whose operation of multiplication has the property of existence (and necessarily uniqueness) of an identity element [Simmons, 1963; Pinter, 1990]. A *commutative ring* is a ring whose operation of multiplication has the property of commutativity (and hence distributivity over addition) [Simmons, 1963; Jacobson, 1985; Pinter, 1990]. The ring having only one element (necessarily coinciding with both the additive identity element and the multiplicative identity element) is a commutative unital ring that is said to be the *trivial ring* and that, as an additive commutative group, is the trivial group [Pinter, 1990]; the additive identity element and the multiplicative identity element of a unital ring coincide if and only if the unital ring is the trivial ring [Pinter, 1990]. An element of a ring is said to be a *divisor of zero* if and only if its left-multiplication or right-multiplication by some element other than the additive identity element results in the additive identity element [Simmons, 1963; Jacobson, 1985; Pinter, 1990]; we refer to a divisor of zero coinciding with the additive identity element as a *trivial divisor of zero*. In a ring without nontrivial divisors of zero, the multiplication of two elements results in the additive identity element if and only if at least one of them is the additive identity element [Pinter, 1990].

A *domain ring* (also known as *domain*) is a nontrivial unital ring without nontrivial divisors of zero [Jacobson, 1985]; equivalently, a domain ring is a nontrivial unital ring whose operation of multiplication has the property of restricted cancellation [Jacobson, 1985]. A *commutative domain ring* (also known as *integral domain*) is a domain ring that, as a ring, is a commutative ring [Pinter,

1990]. When equipped with the usual operations of addition and multiplication, $\mathbb{Z}$ forms the archetypal example of commutative domain ring [Pinter, 1990].

A *division ring* (also known as *skew field*) is a domain ring whose operation of multiplication has the property of existence (and necessarily uniqueness) of an inverse element for each element apart from the additive identity element [Simmons, 1963; Jacobson, 1985]. A *field* is a division ring that, as a ring, is a commutative ring [Simmons, 1963; Jacobson, 1985]; equivalently, a field is an additive commutative group such that the difference of itself and its additive identity element is a multiplicative commutative group whose operation of multiplication has the property of distributivity over addition [Simmons, 1963; Jacobson, 1985]. An element of a field is said to be a *scalar* [Simmons, 1963; Jacobson, 1985]. When equipped with the usual operations of addition and multiplication, each of $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$ forms a field [Simmons, 1963; Pinter, 1990]; hereinafter, we denote by $\mathbb{F}$ a field formed by either $\mathbb{C}$ or $\mathbb{R}$.

A *linear space* (also known as *vector space*) over a field is an additive commutative group equipped with, and invariant under, a binary operation named *multiplication by a scalar* having the properties of existence (and necessarily uniqueness) of an identity element, distributivity over addition of vectors, pseudodistributivity over addition of scalars, and pseudoassociativity (also known as mixed associativity) over multiplication of scalars [Simmons, 1963; Pinter, 1990]. An element of a linear space over a field is said to be a *vector* [Simmons, 1963; Pinter, 1990]. In particular, the additive identity element of a linear space over a field is said to be its *origin* [Simmons, 1963]. The linear space over a field having only one element (necessarily coinciding with the origin) is a linear space over a field that is said to be the *trivial linear space* and that, as an additive commutative group, is the trivial group. As is evident, each field forms a linear space over itself. Hereinafter, each linear space over a field is a linear space over $\mathbb{F}$. In particular, a linear space over $\mathbb{C}$ is said to be a *complex linear space*, and a linear space over $\mathbb{R}$ is said to be a *real linear space* [Simmons, 1963; Horváth, 1966; Treves, 1967]. Given a linear space over $\mathbb{F}$ and two equinumerous ordered sets (namely, two ordered sets having the same cardinality), one of vectors and one of scalars (to which reference is made as *coefficients*), the *linear combination* of the vectors and the coefficients is the vector coinciding with the sum of the products of the corresponding elements of the two equinumerous ordered sets [Horváth, 1966; Pinter, 1990]. An *affine combination* of a set of vectors of a linear space over $\mathbb{F}$ is a linear combination of them with coefficients summing to 1 [Hiriart-Urruty and Lemaréchal, 2001]. A *conical combination* of a set of vectors of a linear space over $\mathbb{F}$ is a linear combination of them with non-negative real coefficients [Hiriart-Urruty and Lemaréchal, 2001]; in particular, a *trivial combination* (respectively, *nontrivial combination*) of a set of vectors of

a linear space over $\mathbb{F}$ is a conical combination of them with all (respectively, not all) coefficients equalling 0. A *convex combination* of a set of vectors of a linear space over $\mathbb{F}$ is a linear combination of them that is both an affine combination and a conical combination [Hiriart-Urruty and Lemaréchal, 2001]. A set of vectors of a linear space over $\mathbb{F}$ is said to be *linearly dependent* if and only if there exists a nontrivial combination of it coinciding with the origin [Horváth, 1966]; a set of vectors of a linear space over $\mathbb{F}$ is said to be *linearly independent* if and only if it is not linearly dependent [Horváth, 1966]. The *linear span* of a set of vectors of a linear space over $\mathbb{F}$ is the subset of the linear space over $\mathbb{F}$, which can itself be canonically made a linear space over $\mathbb{F}$, coinciding with the set of the linear combinations of the set of vectors [Simmons, 1963; Pinter, 1990]. A subset of a linear space over $\mathbb{F}$ is said to be a *convex subset* if and only if any convex combination of any two elements (or, equivalently, of any finitely many elements) of it is an element of it [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970]. When equipped with the usual componentwise operations of addition and multiplication by a scalar, $\mathbb{C}^n$ forms a complex linear space that is said to be a *complex coordinate space* (an element of which is said to be a *complex coordinate n -tuple*), and $\mathbb{R}^n$ forms a real linear space that is said to be a *real coordinate space* (an element of which is said to be a *real coordinate n -tuple*) [Simmons, 1963].

An *algebra* over a field is both a linear space over a field and a ring whose operation of multiplication has the property of homogeneity of degree 1 [Simmons, 1963; Jacobson, 1985]. A *unital algebra* over a field is an algebra over a field that, as a ring, is a unital ring [Simmons, 1963]. A *commutative algebra* over a field is an algebra over a field that, as a ring, is a commutative ring [Simmons, 1963]. The algebra over a field having only one element (necessarily coinciding with both the additive identity element and the multiplicative identity element) is a commutative unital algebra over a field that is said to be the *trivial algebra* and that, as a linear space over a field and as a ring, is respectively the trivial linear space and the trivial ring. A (possibly commutative) *domain algebra* over a field and a (possibly commutative) *division algebra* over a field are algebras over a field that, as rings, are respectively a (possibly commutative) domain ring and a (possibly commutative) division ring [Simmons, 1963]. Hereinafter, each algebra over a field is an algebra over $\mathbb{F}$. In particular, an algebra over $\mathbb{C}$ is said to be a *complex algebra*, and an algebra over $\mathbb{R}$ is said to be a *real algebra* [Simmons, 1963].

A (possibly trivial) *differential ring* is a (possibly trivial) commutative unital ring equipped with, and invariant under, a finite set of unary operations each named *derivation* having the properties of additivity, compliance with the Leibniz rule over multiplication, and mutual commutativity [Kaplansky, 1957; Kolchin, 1973]. A *differential domain ring* and a *differential field* are differential

rings that, as rings, are respectively a commutative domain ring and a field [Kaplansky, 1957; Kolchin, 1973].

A *differential linear space* over a field is a linear space over a field (made a differential field) equipped with, and invariant under, a finite set of unary operations each named *derivation* having the properties of additivity, compliance with the Leibniz rule over multiplication by a scalar, and mutual commutativity [Kolchin, 1973]. Hereinafter, each differential linear space over a field is a differential linear space over $\mathbb{F}$. In particular, a differential linear space over $\mathbb{C}$ is said to be a *complex differential linear space*, and a differential linear space over $\mathbb{R}$ is said to be a *real differential linear space*.

A *differential algebra* over a field is a commutative unital algebra over a field (made a differential field) that, as a commutative unital ring and as a linear space over a field, is respectively a differential ring and a differential linear space over a field [Kolchin, 1973]. A *differential domain algebra* over a field and a *differential division algebra* over a field are differential algebras over a field that, as algebras over a field, are respectively a commutative domain algebra over a field and a commutative division algebra over a field. Hereinafter, each differential algebra over a field is a differential algebra over $\mathbb{F}$. In particular, a differential algebra over $\mathbb{C}$ is said to be a *complex differential algebra*, and a differential algebra over $\mathbb{R}$ is said to be a *real differential algebra*.

A differential ring, differential linear space over a field, or differential algebra over a field is qualified as *ordinary* or *partial* depending on whether it is equipped with one or more than one operations of derivation [Kolchin, 1973]. A *trivial derivation* is an operation of derivation that always results in the additive identity element [Kaplansky, 1957; Kolchin, 1973]; any commutative unital ring, linear space over a field, and commutative unital algebra over a field can be made respectively a differential ring, a differential linear space over a field, and a differential algebra over a field by being equipped with a finite set of trivial derivations [Kaplansky, 1957; Kolchin, 1973]. An element of a differential ring, differential linear space over a field, or differential algebra for which each derivation results in the additive identity element is said to be a *constant* [Kaplansky, 1957; Kolchin, 1973]; in particular, the multiplicative identity element of a differential ring or differential algebra over a field is a constant [Kaplansky, 1957]. A (commutative and associative) composite of n (not necessarily distinct) operations of derivation is said to be an operation of *differentiation of order n* (hereinafter *n^{th} -order differentiation*), and its result is said to be a *derivative of order n* (hereinafter *n^{th} -order derivative*) [Kolchin, 1973]; by definition, an operation of *differentiation of order 0* (hereinafter *0^{th} -order differentiation*) coincides with the identity operation, and its result is said to be a *derivative of order 0* (hereinafter *0^{th} -order derivative*) [Kolchin, 1973]. An n^{th} -order

differentiation and the corresponding n^{th} -order derivative are qualified as *ordinary* or *partial* depending on whether the n operations of derivation composing the n^{th} -order differentiation are distinct or not [Kolchin, 1973].

A.1.2.3 Topological spaces

A *topological structure* (also known as *topology*) on a set is a collection of its subsets that is invariant both under countable union and under finite intersection (and hence contains both $\emptyset$ and the set itself) [Simmons, 1963; Horváth, 1966; Treves, 1967; Bourbaki, 1995; Willard, 1970]. A topological structure on a set is said to be *coarser*, *weaker*, or *smaller* (respectively, *finer*, *stronger*, or *larger*) than another topological structure on the same set if and only if each element of the former (respectively, of the latter) is an element of the latter (respectively, of the former) [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970]; we say that two topological structures on a set are *equivalent* if and only if one is both coarser and finer than the other. A *topological space* is a set equipped with a topological structure [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 1995; Adámek *et al.*, 2004]. An element of a topological space is often said to be a *point* of it [Simmons, 1963; Willard, 1970; Bourbaki, 1995]. We say that $\emptyset$ equipped with its unique topological structure is the *empty topological space*. Hereinafter, unless otherwise stated, $\mathbb{C}^n$ and $\mathbb{R}^n$ are equipped with their canonical topological structures. A *pointed topological space* is a topological space equipped with a *basepoint* (namely, a specified fixed element amongst the elements of the topological space) [Adámek *et al.*, 2004; Perrone, 2024].

An element of the topological structure of a topological space and its complement are said to be respectively an *open subset* and a *closed subset* of the topological space [Simmons, 1963; Willard, 1970]; a subset of a topological space can be either open or closed, neither open nor closed, and both open and closed (namely, *clopen*) [Simmons, 1963; Willard, 1970; Narici and Beckenstein, 2011]. A topological space is said to be *connected* if and only if it cannot be represented as the union of disjoint nonempty open (or, equivalently, closed) subsets [Simmons, 1963; Willard, 1970]; a topological space is connected if and only if there are no subsets of it other than $\emptyset$ and the underlying set that are both open and closed [Simmons, 1963; Willard, 1970]. In particular, to a connected subset of either $\mathbb{R}$ or $\bar{\mathbb{R}}$, reference is made as *interval* [Bourbaki, 1995].

The *interior* of a subset of a topological space is the largest open subset contained in it [Bourbaki, 1995]; a subset of a topological space is open if and only if it coincides with its interior [Bourbaki, 1995]. The *exterior* of a subset of a topological space is the interior of the difference of the topological space and the subset [Bourbaki, 1995]. The *closure* of a subset of a topological space is

the smallest closed subset containing it [Bourbaki, 1995]; a subset of a topological space is closed if and only if it coincides with its closure [Bourbaki, 1995]. The *frontier* (also known as *boundary*) of a subset of a topological space is the closed subset of the topological space obtained as the difference of the closure and the interior of the subset [Bourbaki, 1995]. In particular, to an element of the frontier of an interval of $\mathbb{R}$ or $\bar{\mathbb{R}}$, reference is made as *endpoint* [Bourbaki, 1995]. The interior, the exterior, and the frontier of a subset of a topological space are mutually disjoint, have the topological space as their union, and, whenever all nonempty, form a *partition* of the topological space [Bourbaki, 1995]. A subset of a topological space is said to be an *everywhere dense subset* (also known as *dense subset*) of the topological space if and only if its closure coincides with the topological space, whereas it is said to be a *nowhere dense subset* (also known as *rare subset*) of the topological space if and only if its closure has empty interior [Simmons, 1963]. A closed subset of a topological space is nowhere dense if and only if it is the frontier of an open subset of the topological space (or, equivalently, if and only if its complement is everywhere dense) [Simmons, 1963; Willard, 1970]. A *separable space* is a topological space having a countable everywhere dense subset [Simmons, 1963; Willard, 1970].

A *neighbourhood* of a subset (in particular, of an element) of a topological space is a subset of the topological space containing an open subset of the topological space containing the subset (in particular, the element) [Horváth, 1966; Treves, 1967; Willard, 1970]. A subset of a topological space is a neighbourhood of an element of the topological space if and only if the interior of the subset contains the element [Willard, 1970]; a subset of a topological space is open if and only if it is a neighbourhood of each of its elements (in particular, of itself) [Horváth, 1966]. The collection of all the neighbourhoods of a subset (in particular, of an element) of a topological space is said to be the *neighbourhood system* (also known as *neighbourhood filter* or *complete system of neighbourhoods*) at the subset (in particular, at the element) [Treves, 1967; Willard, 1970]. A collection of neighbourhoods of a subset (in particular, of an element) of a topological space is said to be a *neighbourhood basis* (also known as *neighbourhood base* or *fundamental system of neighbourhoods*) at the subset (in particular, at the element) if and only if each element of the neighbourhood system at the subset (in particular, at the element) contains an element of the neighbourhood basis at the subset (in particular, at the element) [Horváth, 1966; Treves, 1967; Willard, 1970]; the neighbourhood system at a subset (in particular, at an element) of a topological space is a neighbourhood basis thereat [Willard, 1970]. A *Hausdorff space* (also known as *separated space* or T_2 -space) is a topological space for which there exists a pair of disjoint neighbourhoods of any pair of distinct elements [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970]. A collection of subsets of a topological space is said to be *locally finite* if and only if each element of the topological

space has a neighbourhood that intersects only finitely many subsets of the collection [Willard, 1970; Bourbaki, 1995]; a finite collection of subsets of a topological space is locally finite [Bourbaki, 1995]. An element of a subset of a topological space is said to be an *isolated point* of the subset if and only if a neighbourhood of it contains no element of the subset other than itself [Simmons, 1963; Bourbaki, 1995]; in particular, an element of a topological space is an isolated point of the topological space if and only if it is an open subset of the topological space [Willard, 1970; Bourbaki, 1995]. A subset of a topological space is said to be a *discrete subset* of the topological space if and only if its elements are (necessarily countably many) isolated points of the subset [Willard, 1970]. An element of a topological space is said to be an *accumulation point* (also known as a *cluster point*) of a subset of the topological space if and only if each neighbourhood of it contains an element of the subset other than itself [Simmons, 1963; Willard, 1970]. An element of a subset of a topological space is either an isolated point or an accumulation point of the subset [Simmons, 1963]; in fact, given a closed subset of a topological space, the set of its isolated points and the set of its accumulation points are mutually disjoint, have the closed subset as their union, and, if both nonempty, form a partition of the closed subset [Simmons, 1963]. A subset of a topological space is closed (in particular, closed and discrete) if and only if it contains all its accumulation points (in particular, if and only if it has no accumulation point) [Simmons, 1973].

A *cover* (also known as *covering*) of a set is a collection of sets whose union contains the set [Bourbaki, 2004a]. An *open cover* (respectively, a *closed cover*) of a subset of a topological space is a cover of the subset consisting of open subsets (respectively, closed subsets) of the topological space [Bourbaki, 1995]. A topological space is said to be a *compact space* if and only if it is a Hausdorff space and each open cover of it contains a finite cover [Horváth, 1966; Treves, 1967; Bourbaki, 1995]. A subset of a topological space is said to be a *compact subset* if and only if it can be canonically made a Hausdorff space and each cover of it by open subsets of the topological space contains a finite cover [Horváth, 1966; Treves, 1967; Bourbaki, 1995]. A subset of a Hausdorff space (in particular, of a compact space) is compact only if (in particular, if and only if) it is closed [Horváth, 1966; Treves, 1967; Bourbaki, 1995]. A *neighbourhood of infinity* of a nonempty topological space is a subset of it containing the complement of a closed and compact subset of it [ProofWiki, 2017]; in particular, a *neighbourhood of negative infinity* (respectively, *neighbourhood of positive infinity*) of $\mathbb{R}$ is an interval of $\mathbb{R}$ having as endpoints $-\infty$ and a point of $\mathbb{R}$ (respectively, a point of $\mathbb{R}$ and $+\infty$) [ProofWiki, 2017]. A *relatively compact subset* of a topological space is a subset that is contained in a compact subset [Horváth, 1966; Bourbaki, 1995]; a compact subset of a topological space is relatively compact. A subset of a Hausdorff space (in particular, of a compact space) is relatively compact if and only if its closure is compact (in particular, is always relatively compact)

[Horváth, 1966; Bourbaki, 1995]. A *locally compact space* is a Hausdorff space whose each element has a compact neighbourhood [Horváth, 1966; Treves, 1967; Bourbaki, 1995]; a compact space is locally compact [Bourbaki, 1995; Willard, 1970]. In particular, $\mathbb{C}^n$ and $\mathbb{R}^n$ are locally compact spaces [Willard, 1970].

A.1.2.4 Borel-measurable spaces

The *Borel-measurable structure* (also known as *Borel σ -algebra*, *Borel σ -field*, or *Borel tribe*) on a topological space is the coarsest measurable structure on the underlying set of the topological space that contains each open subset (or, equivalently, each closed subset) of the topological space [Berberian, 1988; Folland, 1999; Bauer, 2001; Leoni, 2017]. A *Borel-measurable space* is a topological space equipped with its Borel-measurable structure [Berberian, 1988; Bauer, 2001]. An element of the Borel-measurable structure of a Borel-measurable space is said to be a *Borel-measurable subset* of the Borel-measurable space [Berberian, 1988; Folland, 1999; Ambrosio *et al.*, 2000; Bauer, 2001]. A *Hausdorff Borel-measurable space* is a Borel-measurable space that, as a topological space, is a Hausdorff space. Hereinafter, unless otherwise stated, $\mathbb{C}^n$ and $\mathbb{R}^n$ are equipped with their Borel-measurable structures.

A.1.2.5 Topological algebraic spaces

A *compatible topological structure* on an algebraic space is a topological structure on it for which the operations of the algebraic structure (including, as appropriate, additive and multiplicative inverses) are jointly continuous as bilinear maps (as specified below) [Warner, 1989, 1993; Balachandran, 2000]. A *topological algebraic space* is an algebraic space equipped with a compatible topological structure [Warner, 1989, 1993; Balachandran, 2000].

A (possibly commutative or trivial) *topological group*, a (possibly unital, commutative, or trivial) *topological ring*, a (possibly commutative) *topological domain ring* (also known as *topological domain*), a *topological division ring* (also known as *topological skew field*), and a *topological field* are respectively a (possibly commutative or trivial) group, a (possibly unital, commutative, or trivial) ring, a (possibly commutative) domain ring, a division ring, and a field equipped with a compatible topological structure [Warner, 1989, 1993; Balachandran, 2000].

A (possibly trivial) *topological linear space* (also known as *topological vector space*) over a field is a linear space over a field (made a topological field) equipped with a compatible topological structure [Warner, 1989, 1993; Balachandran, 2000]. Hereinafter, each topological linear space over a field is a

topological linear space over $\mathbb{F}$. In particular, a topological linear space over $\mathbb{C}$ is said to be a *complex topological linear space*, and a topological linear space over $\mathbb{R}$ is said to be a *real topological linear space* [Horváth, 1966; Treves, 1967]. A *locally convex topological linear space* over $\mathbb{F}$ is a topological linear space over $\mathbb{F}$ at each element of which there is a neighbourhood basis consisting of convex subsets of it [Horváth, 1966; Treves, 1967; Willard, 1970]; locally convex topological linear spaces over $\mathbb{F}$ are by far the most important and common topological linear spaces [Horváth, 1966; Treves, 1967]. A *Hausdorff locally convex topological linear space* over $\mathbb{F}$ is a locally convex topological linear space over $\mathbb{F}$ that, as a topological space, is a Hausdorff space [Treves, 1967]. A *bounded subset* of a topological linear space over $\mathbb{F}$ is a subset contained in a subset whose elements are the elements of any neighbourhood of the origin multiplied by some sufficiently large scalar [Horváth, 1966; Treves, 1967; Bourbaki, 2003; Narici and Beckenstein, 2011]. We refer to a subset of a topological linear space over $\mathbb{F}$ that is not bounded as an *unbounded subset* of the topological linear space over $\mathbb{F}$. The closure of a bounded subset of a topological linear space over $\mathbb{F}$ is bounded [Horváth, 1966; Treves, 1967; Narici and Beckenstein, 2011]. Furthermore, a relatively compact subset (in particular, a compact subset) of a topological linear space over $\mathbb{F}$ is bounded [Horváth, 1966; Treves, 1967; Narici and Beckenstein, 2011]. A subset of $\mathbb{C}^n$ or $\mathbb{R}^n$ is relatively compact (in particular, compact) if and only if it is bounded (in particular, closed and bounded) [Horváth, 1966; Bourbaki, 1989, 1995]. The absolute value of the difference of the two endpoints of a bounded interval of $\mathbb{R}$ is said to be its *length* [Bourbaki, 1995]; by convention, the length of an unbounded interval of $\mathbb{R}$ or $\bar{\mathbb{R}}$ is $+\infty$.

A (possibly unital, commutative, or trivial) *topological algebra* over a field is a (possibly unital, commutative, or trivial) algebra over a field (made a topological field) equipped with a compatible topological structure [Warner, 1989, 1993; Balachandran, 2000]. A (possibly unital, commutative, or trivial) *weak topological algebra* over a field is a (possibly unital, commutative, or trivial) algebra over a field (made a topological field) equipped with a topological structure for which all the operations of the algebraic structure (including additive and multiplicative inverses) but the operation of multiplication are jointly continuous (as specified below), whereas the operation of multiplication is separately continuous (as specified below) [Balachandran, 2000]. A (possibly commutative or weak) *topological domain algebra* over a field and a (possibly commutative or weak) *topological division algebra* over a field are topological algebras over a field that, as algebras over a field, are respectively a (possibly commutative) domain algebra over a field and a (possibly commutative) division algebra over a field [Warner, 1989, 1993; Balachandran, 2000]. Hereinafter, each (possibly weak) topological algebra over a field is a (possibly weak) topological algebra over $\mathbb{F}$. In particular, a (possibly weak) topological algebra over $\mathbb{C}$ is said to be a *complex* (possibly weak)

topological algebra, and a (possibly weak) topological algebra over $\mathbb{R}$ is said to be a *real* (possibly weak) *topological algebra* [Warner, 1989, 1993; Balachandran, 2000]. A (possibly weak) topological algebra over $\mathbb{F}$ is said to be *locally convex* if and only if, as a topological linear space over $\mathbb{F}$, it is locally convex [Warner, 1989, 1993; Balachandran, 2000].

A.1.3 Maps

A *category* consists of a class of *objects* and a class of *morphisms* equipped with a partial binary operation of *composition* having the properties of associativity and existence (and necessarily uniqueness) of an identity morphism for each object [Jacobson, 1989; Adámek *et al.*, 2004; Frerick and Sieg, 2010]. Morphisms are defined between ordered pairs of objects, to the first and the second of which reference is made as respectively *domain* (or *source*) and *codomain* (or *target*) [Jacobson, 1989; Adámek *et al.*, 2004; Frerick and Sieg, 2010]; for technical convenience and without loss of generality, it is possible to impose the condition that sets of morphisms between distinct ordered pairs of objects are disjoint [Jacobson, 1989; Adámek *et al.*, 2004; Frerick and Sieg, 2010]. The operation of composition is defined for ordered pairs of morphisms if and only if the codomain of the first coincides with the domain of the second [Jacobson, 1989; Adámek *et al.*, 2004; Frerick and Sieg, 2010]; we adopt the usual convention that composition operands have right-to-left order (which means that, in the composition of an ordered pair of morphisms, the first morphism of the ordered pair follows the second morphism of the ordered pair).

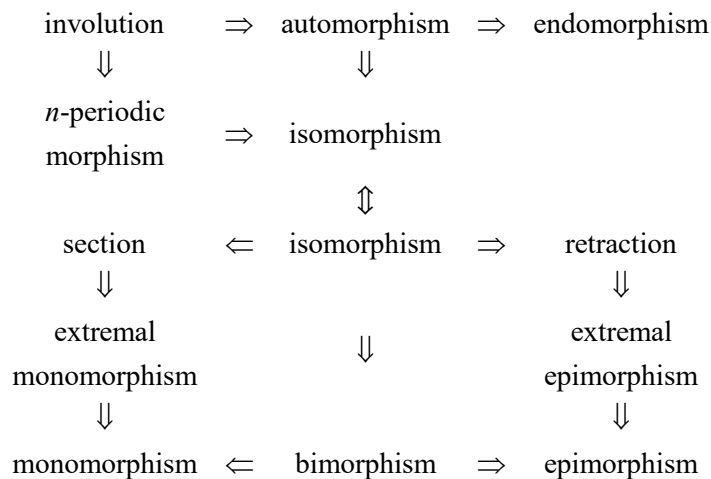
By the term *map* (also known as *mapping*, *transformation*, or *operator*), we refer to a rule, possibly structure-preserving, that assigns to each element of a space acting as domain one element (in the case of a *single-valued map*) or several elements (in the case of a *multi-valued map*) of a generally different space acting as codomain. We reserve the term *transformation* for integral transformations (as introduced below) and the term *operator* for symbols denoting mathematical operations. Hereinafter, unless otherwise stated, a map is a single-valued map. In particular, a structure-preserving map between algebraic spaces is said to be a *homomorphism* [Jacobson, 1985, 1989; Pinter, 1990]. In some specified context, a certain map may be induced ‘naturally’ between two given spaces; we liberally qualify such a map as *canonical*.

A.1.3.1 Constructs

A *construct* is a category whose objects are spaces and whose morphisms are maps between them [Adámek *et al.*, 2004; Frerick and Sieg, 2010]; in a construct, each object is associated with an underlying set, and each morphism between objects is associated with an *underlying map* between underlying sets [Adámek *et al.*, 2004; Frerick and Sieg, 2010]. Hereinafter, unless otherwise stated, a category is a construct; accordingly, each object is a space, and each morphism is a map.

An *endomorphism* is a morphism from an object to itself, a *monomorphism* (also known as *monic morphism*) is a left-cancellable morphism, an *extremal monomorphism* is a monomorphism

satisfying the extremal condition (as introduced below), a *section* (also known as *coretraction* or *split monomorphism*) is a left-invertible morphism (and necessarily a left-invertible extremal monomorphism), an *epimorphism* (also known as *epic morphism*) is a right-cancellable morphism, an *extremal epimorphism* is an epimorphism satisfying the extremal condition, a *retraction* (also known as *cosection* or *split epimorphism*) is a right-invertible morphism (and necessarily a right-invertible extremal epimorphism), a *bimorphism* is a morphism that is both a monomorphism and an epimorphism, an *isomorphism* is a bimorphism that is both a section and a retraction (or, equivalently, a bimorphism that is both a monomorphism and an epimorphism one of which is extremal), and an *automorphism* is an endomorphism that is also an isomorphism [Adámek *et al.*, 2004; Frerick and Sieg, 2010]. An *n-periodic morphism* is an isomorphism for which n is the smallest positive integer constant such that the n^{th} iterate of the isomorphism (namely, the n^{th} composite of the isomorphism with itself) coincides with the identity morphism [Encyclopedia of Mathematics, 2020]; in particular, a 2-periodic morphism is an automorphism to which reference is made as *involution* [Encyclopedia of Mathematics, 2020; nLab authors, 2024]. A monomorphism satisfies the extremal condition if and only if, whenever it coincides with the composite of an epimorphism and a morphism, the epimorphism is an isomorphism [Adámek *et al.*, 2004]; dually, an epimorphism satisfies the extremal condition if and only if, whenever it coincides with the composite of a morphism and a monomorphism, the monomorphism is an isomorphism [Adámek *et al.*, 2004]. Apart from extremal monomorphisms and extremal epimorphisms, all the aforementioned morphisms are invariant under composition [Adámek *et al.*, 2004; Frerick and Sieg, 2010]. The left-inverse (respectively, right-inverse) morphism of a section (respectively, of a retraction) is a generally non-unique retraction (respectively, section) [Adámek *et al.*, 2004; Frerick and Sieg, 2010]; however, the inverse morphism of an isomorphism is a unique isomorphism [Adámek *et al.*, 2004; Frerick and Sieg, 2010]. Amongst the aforementioned morphisms, relations hold [Adámek *et al.*, 2004]



For constructs, one can introduce a concept of *embedding morphism* (also known as *imbedding morphism*) from a *subobject* into an object (to which we refer as *superobject*) and a dual concept of *identification morphism* (also known as *quotient morphism*) from an object (to which we refer as *base object*) into its *quotient object* under a structure-preserving equivalence relation on its underlying set [Adámek *et al.*, 2004]. Relatedly, one can introduce the concepts of *image* (also known as *direct image*) and *preimage* (also known as *inverse image*) of a subobject under a morphism [Adámek *et al.*, 2004]; in particular, given a morphism, we refer to the subobject of its codomain that is the image of its domain as its *range*. An embedding morphism is defined as a morphism that satisfies a certain condition (namely, being an *initial morphism*) and whose underlying map is injective (as specified below) [Adámek *et al.*, 2004]; dually, an identification morphism is defined as a morphism that satisfies a certain condition (namely, being a *final morphism*) and whose underlying map is surjective (as specified below) [Adámek *et al.*, 2004]. Both embedding morphisms and identification morphisms are invariant under composition [Adámek *et al.*, 2004]. Each section (respectively, retraction) is an embedding morphism (respectively, identification morphism), and each embedding morphism (respectively, identification morphism) is a monomorphism (respectively, an epimorphism) [Adámek *et al.*, 2004]. In fact, a morphism whose underlying map is injective (respectively, surjective) is a monomorphism (respectively, an epimorphism), but, in contrast, the underlying map of a monomorphism (respectively, of an epimorphism) need not be injective (respectively, surjective) [Adámek *et al.*, 2004; Frerick and Sieg, 2010]; furthermore, a morphism whose underlying map is bijective is a bimorphism, but, in contrast, the underlying map of a bimorphism need not be bijective [Adámek *et al.*, 2004; Frerick and Sieg, 2010]. For all the constructs considered hereinafter, the embedding morphisms (respectively, identification morphisms) coincide with the extremal monomorphisms (respectively, extremal epimorphisms) [Brümmer, 1984; Adámek *et al.*, 2004; Lowen *et al.*, 2009].

A.1.3.2 Classification

We now introduce several properties that maps between spaces may have. Clearly, a property characterising some maps between spaces of certain types also characterises some maps between spaces of the same types some or all of which are equipped with additional structures.

A *sequence* in a space is a map from $\mathbb{N}$ into the space [Horváth, 1966; Willard, 1970; Bourbaki, 1995]. In a topological space, it is possible to introduce the concept of *convergence* of a sequence in the topological space to a *limit point* (hereinafter *limit*) [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970].

A *form* (also known as *functional*) is a map from a space into $\mathbb{F}$ [Horváth, 1966; Treves, 1967].

An *injective map* (also known as *one-to-one map* or *injection*) between two sets is a map under which each element of the codomain is the image of at most one element of the domain [Horváth, 1966]. A *surjective map* (also known as *onto map* or *surjection*) between two sets is a map under which each element of the codomain is the image of at least one element of the domain [Horváth, 1966]. A *bijective map* (also known as *invertible map* or *bijection*) between two sets is a map under which each element of the codomain is the image of precisely one element of the domain [Horváth, 1966].

A *measurable map* between two measurable spaces is a map under which the preimage of each measurable subset of the codomain is a measurable subset of the domain [Berberian, 1988; Bauer, 2001; Leoni, 2017]. A *bimeasurable map* between two measurable spaces is a bijective measurable map whose underlying map has measurable inverse [Berberian, 1988].

A *linear map* (respectively, an *antilinear map*) between two linear spaces over $\mathbb{F}$ is a map having the properties of additivity and homogeneity of degree 1 (respectively, conjugate homogeneity) [Horváth, 1966; Treves, 1967]; an antilinear map between two real linear spaces is a linear map [Horváth, 1966; Treves, 1967]. We refer to a map between two linear spaces over $\mathbb{F}$ that is not linear as a *nonlinear map*. In particular, a linear map (respectively, an antilinear map) from a linear space over $\mathbb{F}$ into $\mathbb{F}$ with $\mathbb{F}$ regarded as a linear space over itself is said to be a *linear form* (respectively, an *antilinear form*) on the linear space over $\mathbb{F}$ [Horváth, 1966; Treves, 1967]. Each linear or antilinear map between two linear spaces over $\mathbb{F}$ is origin-preserving (but, in contrast, an origin-preserving map between two linear spaces over $\mathbb{F}$ need not be linear or antilinear) [Horváth, 1966; Treves, 1967]. Linear maps are the homomorphisms between linear spaces over $\mathbb{F}$ [Jacobson, 1985, 1989; Pinter, 1990]. An *affine map* between two linear spaces over $\mathbb{F}$ is a map under which the image of a vector with the subtraction of the image of the origin is the image of an *associated linear map* between the same two linear spaces over $\mathbb{F}$ [Väisälä, 2003]; as is evident, an affine map between two linear spaces over $\mathbb{F}$ is a linear map if and only if it is origin-preserving. We refer to the image of the origin of a linear space over $\mathbb{F}$ under an affine map from the linear space over $\mathbb{F}$ into a linear space over $\mathbb{F}$ as the *translation vector* of the affine map; furthermore, we refer to an affine map between two linear spaces over $\mathbb{F}$ whose associated linear map is the identity map as a *translation map*.

A *bilinear map* (respectively, *sesquilinear map*) between an ordered pair of linear spaces over $\mathbb{F}$ and a linear space over $\mathbb{F}$ is a map having the properties of linearity in the first argument and linearity

(respectively, antilinearity) in the second argument [Horváth, 1966; Treves, 1967]; a sesquilinear map between an ordered pair of real linear spaces and a real linear space is a bilinear map [Horváth, 1966; Treves, 1967]. It is worth mentioning that, in physics, a sesquilinear map is often defined to have the properties of antilinearity in the first argument and linearity in the second argument. In particular, a bilinear map (respectively, sesquilinear map) from an ordered pair of linear spaces over $\mathbb{F}$ into $\mathbb{F}$ with $\mathbb{F}$ regarded as a linear space over itself is said to be a *bilinear form* (respectively, *sesquilinear form*) on the ordered pair of linear spaces over $\mathbb{F}$ [Horváth, 1966; Treves, 1967]. A sesquilinear form on the ordered pair formed by a linear space over $\mathbb{F}$ and itself having the property of conjugate symmetry (also known as Hermitian symmetry) is said to be a *Hermitian form* [Treves, 1967; Bourbaki, 2003]; a sesquilinear form on the ordered pair formed by a linear space over $\mathbb{F}$ and itself has the property of real-definiteness (in particular, non-negative-definiteness) if and only if (in particular, only if) it is a Hermitian form [Treves, 1967; Bourbaki, 2003]. Importantly, a non-negative-definite Hermitian form on a linear space over $\mathbb{F}$ satisfies the *Cauchy–Schwarz inequality*, according to which the square of the modulus of the image in $\mathbb{F}$ of an ordered pair of elements of the linear space over $\mathbb{F}$ under the Hermitian form is less than or equal to the multiplicative product of the images in $\mathbb{F}$ of the two ordered pairs consisting of each element of the linear space over $\mathbb{F}$ and itself under the Hermitian form [Treves, 1967; Bourbaki, 2003].

A *continuous map* between two topological spaces is a map under which the preimage of each open subset of the codomain is an open subset of the domain [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970]; equivalently, a continuous map between two topological spaces is a map under which the preimage of each closed subset of the codomain is a closed subset of the domain [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970]. As is evident, the finer (respectively, the coarser) the topological structure of a topological space is, the more (respectively, the less) continuous maps there are from it into some other topological space, and the less (respectively, the more) continuous maps there are from some other topological space into it. We refer to a map between two topological spaces that is not continuous as a *discontinuous map*.

A *relatively open map* (respectively, *relatively closed map*) between two topological spaces is a map under which the image of each open subset (respectively, closed subset) of the domain is an open subset (respectively, a closed subset) of the range [Kelley and Namioka, 1963; Horváth, 1966; Narici and Beckenstein, 2011]. An *open map* (respectively, a *closed map*) between two topological spaces is a map under which the image of each open subset (respectively, closed subset) of the domain is an open subset (respectively, a closed subset) of the codomain [Simmons, 1963; Willard, 1970]. A map between two topological spaces is open (respectively, closed) if and only if it is relatively open (respectively, relatively closed) and its range is an open subset (respectively, a closed subset) of its

codomain; in particular, a surjective map between two topological spaces is open (respectively, closed) if and only if it is relatively open (respectively, relatively closed). A (continuous or discontinuous) map between two topological spaces can be either open or closed, neither open nor closed, and both open and closed [Willard, 1970]. A bijective (continuous or discontinuous) map between two topological spaces is open (or, equivalently, closed) if and only if its underlying map has continuous inverse [Willard, 1970; Bourbaki, 1995]. A *bicontinuous map* (also known as *homeomorphism*) between two topological spaces is a bijective continuous map whose underlying map has continuous inverse (both the underlying map and its inverse being necessarily open and closed) [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970]. A *map vanishing at infinity* from a locally compact space into a pointed topological space is a map such that for each neighbourhood of the basepoint of the codomain there is a neighbourhood of infinity of the domain whose image in the codomain under the map is in the neighbourhood of the basepoint [nLab authors, 2019].

A *Borel-measurable map* between two Borel-measurable spaces is a map under which the preimage of each Borel-measurable subset of the codomain is a Borel-measurable subset of the domain [Berberian, 1988; Folland, 1999; Bauer, 2001; Leoni, 2017]. A continuous map between two Borel-measurable spaces is a Borel-measurable map (but, in contrast, a Borel-measurable map between two Borel-measurable spaces need not be a continuous map) [Berberian, 1988; Folland, 1999; Bauer, 2001; Leoni, 2017].

A *locally bounded map* between two topological linear spaces over $\mathbb{F}$ is a map under which the image of each bounded subset of the domain is a bounded subset of the codomain [Treves, 1967; Narici and Beckenstein, 2011]. We refer to a map between two topological linear spaces over $\mathbb{F}$ that is not locally bounded as a *non-locally-bounded map*. A continuous linear map between two topological linear spaces over $\mathbb{F}$ is a locally bounded map [Treves, 1967; Bourbaki, 2003; Narici and Beckenstein, 2011]. A locally convex topological linear space over $\mathbb{F}$ is said to be a *bornological space* if and only if a linear map from it into a locally convex topological linear space over $\mathbb{F}$ is continuous when and only when it is locally bounded [Horváth, 1966; Bourbaki, 2003; Narici and Beckenstein, 2011]; in particular, each locally convex topological linear space over $\mathbb{F}$ that is metrisable (as specified below) is a bornological space [Horváth, 1966; Bourbaki, 2003; Narici and Beckenstein, 2011]. It is worth mentioning that the property of local boundedness can be introduced more generally with respect to maps between bornological sets, a *bornological set* (also known as *bounded structure*) being a set equipped with a certain *bornological structure* (also known as *bornology*) [Hogbe-Nlend, 1971, 1977; Bourbaki, 2003; Narici and Beckenstein, 2011]; indeed, both a topological space and a topological linear space over $\mathbb{F}$ can be canonically made a bornological set

[Hogbe-Nlend, 1971, 1977; Narici and Beckenstein, 2011]. For linear spaces over $\mathbb{F}$, there exists a remarkable duality between the concepts of topological structure and bornological structure [Hogbe-Nlend, 1971, 1977]. A *uniformly continuous map* between two topological linear spaces over $\mathbb{F}$ is a (necessarily continuous) map such that for each neighbourhood of the origin of the codomain there is a neighbourhood of the origin of the domain for which, given any two elements of the domain whose difference is in the neighbourhood of the origin, the difference of their images in the codomain under the map is in the neighbourhood of the origin [Horváth, 1966; Treves, 1967]; a continuous linear map between two topological linear spaces over $\mathbb{F}$ is a uniformly continuous map [Horváth, 1966; Treves, 1967; Bourbaki, 2003]. A *uniformly bicontinuous map* (also known as *uniform homeomorphism*) between two topological linear spaces over $\mathbb{F}$ is a bijective uniformly continuous map whose underlying map has uniformly continuous inverse (both the underlying map and its inverse being necessarily open and closed) [Willard, 1970; Bourbaki, 1995]. It is worth mentioning that the property of uniform continuity can be introduced more generally with respect to maps between uniform spaces, a *uniform space* being a set equipped with a certain *uniform structure* (also known as *uniformity*) [Willard, 1970; Bourbaki, 1995, 2003]; indeed, both a metric space (as introduced below) and a topological linear space over $\mathbb{F}$ can be canonically made a uniform space, and a uniform space can be canonically made a topological space [Willard, 1970; Bourbaki, 1995, 2003]. A *separately continuous map* between an ordered pair of topological linear spaces over $\mathbb{F}$ and a topological linear space over $\mathbb{F}$ is a bilinear map such that the linear maps obtained from it by fixing either of the arguments at any possible value are continuous [Horváth, 1966; Treves, 1967]. A *hypocontinuous map* between an ordered pair of topological linear spaces over $\mathbb{F}$ and a topological linear space over $\mathbb{F}$ is a bilinear map being a separately continuous map and having a certain additional property [Horváth, 1966; Treves, 1967]. A *jointly continuous map* between an ordered pair of topological linear spaces over $\mathbb{F}$ and a topological linear space over $\mathbb{F}$ is a (necessarily hypocontinuous) bilinear map that is continuous as a map between the Cartesian product (equipped with a suitable topological structure) of the underlying sets of the ordered pair of topological linear spaces over $\mathbb{F}$ and the topological space of the topological linear space over $\mathbb{F}$ [Horváth, 1966; Treves, 1967]. In many cases, a separately continuous map that is not a jointly continuous map is in fact a hypocontinuous map [Horváth, 1966; Treves, 1967].

Measurable spaces, measurable maps, and bimeasurable maps are respectively the objects, the morphisms, and the isomorphisms of a construct denoted by **Meas** [Brümmer, 1984; Berberian, 1988; Perrone, 2024]. Objects, morphisms, monomorphisms, extremal monomorphisms, epimorphisms, extremal epimorphisms, bismorphisms, and isomorphisms of selected constructs are listed in **Table A.1** [Horváth, 1966; Brümmer, 1984; Warner, 1993; Adámek *et al.*, 2004; Lowen *et al.*, 2009; Frerick and Sieg, 2010].

A.1 Mathematical framework: Spaces and maps

Table A.1 Objects, morphisms, monomorphisms, extremal monomorphisms, epimorphisms, extremal epimorphisms, bismorphisms, and isomorphisms of selected constructs.

Construct	Objects	Morphisms	Monomorphisms	Extremal monomorphisms	Epimorphisms	Extremal epimorphisms	Bismorphisms	Isomorphisms
Set	sets	maps	injective maps	injective maps	surjective maps	surjective maps	bijjective maps	bijjective maps
Vect_{$\mathbb{F}$}	linear spaces over $\mathbb{F}$	linear maps	injective linear maps	injective maps	surjective linear maps	surjective maps	bijjective linear maps	bijjective linear maps
Top	topological spaces	continuous maps	injective continuous maps	relatively open injective continuous maps	surjective continuous maps	relatively open surjective continuous maps	bijjective continuous maps	bicontinuous maps
TopVect_{$\mathbb{F}$}	topological linear spaces over $\mathbb{F}$	continuous linear maps	injective continuous linear maps	relatively open injective continuous linear maps	surjective continuous linear maps	relatively open surjective continuous linear maps	bijjective continuous linear maps	bicontinuous linear maps
Relatively open surjective (continuous or discontinuous, linear or nonlinear) maps coincide with open surjective (continuous or discontinuous, linear or nonlinear) maps. A relatively open morphism of TopVect_{$\mathbb{F}$} is said to be a <i>topological homomorphism</i> (also known as <i>strict morphism</i>) [Horváth, 1966; Warner, 1993; Bourbaki, 1995; Frerick and Sieg, 2010]. For Set, Vect_{$\mathbb{F}$}, Top, and TopVect_{$\mathbb{F}$}, an embedding morphism (respectively, identification morphism) is not only an extremal monomorphism (respectively, extremal epimorphism) but also a certain type of extremal monomorphism (respectively, extremal epimorphism) to which reference is made as <i>strict monomorphism</i> (respectively, <i>strict epimorphism</i>) and, in turn, a certain type of strict monomorphism (respectively, strict epimorphism) to which reference is made as <i>regular monomorphism</i> (respectively, <i>regular epimorphism</i>) [Brümmer, 1984; Warner, 1993; Adámek <i>et al.</i>, 2004; Lowen <i>et al.</i>, 2009; Frerick and Sieg, 2010]. For Set and Vect_{$\mathbb{F}$}, an embedding morphism (respectively, identification morphism) is also a section (respectively, retraction).								

A.1.3.3 Measures

An *outer measure* (also known as *exterior measure*) on a nonempty set is a map from the power set of the nonempty set into $\overline{\mathbb{R}}$ having the properties of non-negativity, empty-set nullity, countable subadditivity (also known as σ -subadditivity), and monotonicity (the property of monotonicity being a consequence of the others when the property of countable subadditivity is interpreted in a strong sense [Evans and Gariepy, 2015]) [Folland, 1999; Bauer, 2001; Leoni, 2017]. The image in $\overline{\mathbb{R}}$ of a subset of a nonempty set under an outer measure on it is said to be the outer measure of the subset [Folland, 1999; Bauer, 2001; Leoni, 2017]. A subset of a nonempty set on which an outer measure is defined is said to be *measurable with respect to the outer measure on the nonempty set* if and only if it satisfies the *Carathéodory's criterion* (namely, if and only if the outer measure of an arbitrary subset of the nonempty set coincides with the sum of the outer measure of its intersection with the subset of the nonempty set and the outer measure of its intersection with the complement of the subset of the nonempty set) [Folland, 1999; Bauer, 2001; Evans and Gariepy, 2015; Leoni, 2017]. The collection of the subsets of a nonempty set that are measurable with respect to the same outer measure on it is a measurable structure on the nonempty set [Folland, 1999; Bauer, 2001; Evans and Gariepy, 2015; Leoni, 2017]. In particular, the *Lebesgue outer measure* on $\mathbb{R}$ is an outer measure on $\mathbb{R}$ under which the image (to which we refer as the Lebesgue outer measure) of a subset of $\mathbb{R}$ is the infimum of the sum of the lengths of the elements of an open cover of the subset of $\mathbb{R}$ as the open cover varies on $\mathbb{R}$ [Gordon, 1994; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]. A subset of $\mathbb{R}$ is said to be *Lebesgue-measurable* if and only if it is measurable with respect to the Lebesgue outer measure on $\mathbb{R}$ [Gordon, 1994; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]. The collection of the Lebesgue-measurable subsets of $\mathbb{R}$ is a measurable structure on $\mathbb{R}$, to which we refer as the *Lebesgue-measurable structure* of $\mathbb{R}$ [Gordon, 1994; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]. The Lebesgue-measurable structure of $\mathbb{R}$ contains the Borel-measurable structure of $\mathbb{R}$ [Folland, 1999]; in fact, each Lebesgue-measurable subset of $\mathbb{R}$ results from the union of a Borel-measurable subset of $\mathbb{R}$ and a Lebesgue-null subset of $\mathbb{R}$ (as introduced below) [Folland, 1999]. A *Lebesgue-measurable map* on $\mathbb{R}$ is a measurable map from the measurable space obtained by equipping $\mathbb{R}$ with its Lebesgue-measurable structure into the measurable space obtained by equipping $\mathbb{F}$ with its Borel-measurable structure [Folland, 1999].

A *measure* on a measurable space is a map from the measurable structure of the measurable space into $\overline{\mathbb{R}}$ having the properties of non-negativity, empty-set nullity, countable additivity (also known

as σ -additivity), and monotonicity (the property of monotonicity being a consequence of the others) [Folland, 1999; Ambrosio *et al.*, 2000; Bauer, 2001; Leoni, 2017]. The image in $\bar{\mathbb{R}}$ of a measurable subset of a measurable space under a measure on it is said to be the measure of the measurable subset [Folland, 1999; Ambrosio *et al.*, 2000; Bauer, 2001; Leoni, 2017]. In particular, the *Dirac measure* at an element of the underlying set of a measurable space is the measure on the measurable space under which a measurable subset has either unit measure or null measure depending on whether the measurable subset does or does not contain the element [Folland, 1999; Ambrosio *et al.*, 2000; Bauer, 2001]. A *measure space* is a measurable space equipped with a measure [Folland, 1999; Bauer, 2001; Leoni, 2017]. A measure space whose underlying nonempty set has finite measure (in particular, unit measure) is said to be a *finite measure space* (in particular, *probability space*), and its measure is said to be a *finite measure* (in particular, *probability measure*) [Folland, 1999; Bauer, 2001; Ambrosio *et al.*, 2000]. A subset (respectively, the complement of a subset) of a measure space that is contained in a measurable subset of it having null measure is said to be a *null subset* (respectively, *full subset*) of the measure space [Folland, 1999; Ambrosio *et al.*, 2000]. A property holds *almost everywhere* on the underlying set of a measure space if it holds *everywhere* on it (namely, for all its elements) except for a null subset of it [Folland, 1999]. Let there be given a nonempty set and a measurable structure on it formed by the collection of its subsets that are measurable with respect to an outer measure; the images in $\bar{\mathbb{R}}$ of the elements of the measurable structure under the outer measure define a map that proves to be a measure on the measurable space obtained by equipping the nonempty set with the measurable structure on it [Folland, 1999; Bauer, 2001; Leoni, 2017]. In particular, the *Lebesgue measure* on the measurable space obtained by equipping $\mathbb{R}$ with its Lebesgue-measurable structure is a measure on it under which the image (to which we refer as the Lebesgue measure) of a Lebesgue-measurable subset of $\mathbb{R}$ is its Lebesgue outer measure [Gordon, 1994; Ambrosio *et al.*, 2000; Leoni, 2017]. A subset (respectively, the complement of a subset) of $\mathbb{R}$ that is contained in a Lebesgue-measurable subset of $\mathbb{R}$ having null Lebesgue measure is said to be a *Lebesgue-null subset* of $\mathbb{R}$ (respectively, a *Lebesgue-full subset* of $\mathbb{R}$) [Folland, 1999; Ambrosio *et al.*, 2000]. A property holds almost everywhere on $\mathbb{R}$ if it holds everywhere on $\mathbb{R}$ except for a Lebesgue-null subset of $\mathbb{R}$ [Folland, 1999].

A *Borel measure* is a measure on a Borel-measurable space [Schwartz, 1973; Folland, 1999; Ambrosio *et al.*, 2000; Leoni, 2017]. A *Borel measure space* is a Borel-measurable space equipped with a Borel measure [Bauer, 2001]. A Borel measure space for which each element of the underlying nonempty set has a Borel-measurable neighbourhood (in particular, an open neighbourhood) that has finite measure is said to be a *locally finite Borel measure space*, and its

Borel measure is said to be a *locally finite Borel measure* [Schwartz, 1973; König, 1997; Bauer, 2001]. A Borel measure space that, as a topological space, is a Hausdorff space (respectively, locally compact space) is locally finite only if (respectively, if and only if) each compact subset of the Borel-measurable space has finite Borel measure [König, 1997; Bauer, 2001]. A Borel measure on a Hausdorff Borel-measurable space is said to be *inner regular* (respectively, *outer regular*) if and only if the Borel measure of each Borel-measurable subset of the Hausdorff Borel-measurable space is the supremum (respectively, infimum) of the Borel measures of the compact subsets (respectively, open supersets) of the Borel-measurable subset [Schwartz, 1973; Bauer, 2001]. A Borel measure on a Hausdorff Borel-measurable space is said to be *regular* if and only if it is both inner regular and outer regular [Bauer, 2001]; in particular, each inner regular finite Borel measure on a Hausdorff Borel-measurable space is regular [Bauer, 2001]. An inner regular locally finite Borel measure on a Hausdorff Borel-measurable space is said to be a *non-negative real-valued Radon measure* (also known as *non-negative signed Radon measure*) on the Hausdorff Borel-measurable space [Schwartz, 1973; Bauer, 2001].

A.1.3.4 Metrics, norms, and inner multiplications

A *metric* on a set is a map from the Cartesian square of the set into $\mathbb{R}$ having the properties of non-negativity, identity of indiscernibles, symmetry, and triangle inequality [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. The image in $\mathbb{R}$ of a pair of elements of a set under a metric on it is said to be the *distance* between the elements [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. A set equipped with a metric is said to be a *metric space* [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. The *diameter* of a subset of a metric space is the supremum of the distances between its elements [Simmons, 1963; Willard, 1970; Bourbaki, 1989]; a subset of a metric space is said to be *metrically bounded* or *metrically unbounded* depending on whether its diameter is finite or infinite [Simmons, 1963; Bourbaki, 1989, 2003; Narici and Beckenstein, 2011]. A metric on a set induces a canonical topological structure on it [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]; inversely, a topological space (and, *a fortiori*, a topological linear space over $\mathbb{F}$) is said to be *metrisable* if and only if its topological structure can be induced by some metric on it [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. Therefore, a metric space can be canonically made a metrisable topological space [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. Each metrisable topological space is a Hausdorff space [Horváth, 1966; Willard, 1970]. A distance-preserving bijective map between two metric spaces is said to be an *isometry* [Horváth, 1966; Willard, 1970; Bourbaki, 1989]; an isometry between two metric spaces equipped with their induced canonical topological structures is a uniformly bicontinuous map (but,

in contrast, a uniformly bicontinuous map between two metric spaces equipped with their induced canonical topological structures need not be an isometry) [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. A *Cauchy sequence* in a metric space is a sequence in which all but finitely many elements have a mutual distance less than an arbitrarily small fixed value [Simmons, 1963; Horváth, 1966; Willard, 1970]. Each sequence in a metric space that converges to an element of the metric space is a Cauchy sequence, but, in contrast, a Cauchy sequence in a metric space need not converge to an element of the metric space [Simmons, 1963; Horváth, 1966; Willard, 1970]. A *complete metric space* is a metric space whose each Cauchy sequence converges to an element of it [Simmons, 1963; Horváth, 1966; Willard, 1970]. It is worth mentioning that the property of completeness can be introduced more generally with respect to uniform spaces [Willard, 1970; Bourbaki, 1995].

A *norm* on a linear space over $\mathbb{F}$ is a map from the linear space over $\mathbb{F}$ into $\mathbb{R}$ having the properties of absolute homogeneity of degree 1 (and hence nullity on the origin), positivity on the complement of the origin (and hence non-negativity), and subadditivity [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. The image in $\mathbb{R}$ of a vector of a linear space over $\mathbb{F}$ under a norm on it is said to be the *magnitude* (also known, by abuse of language, as *norm*) of the vector [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. A linear space over $\mathbb{F}$ equipped with a norm is said to be a *normed linear space* over $\mathbb{F}$ [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]. A norm on a linear space over $\mathbb{F}$ induces a canonical metric and hence a canonical topological structure on it [Simmons, 1963; Horváth, 1966; Willard, 1970; Bourbaki, 1989]; inversely, a topological linear space over $\mathbb{F}$ is said to be *normable* if and only if its topological structure can be induced by some norm on it [Horváth, 1966; Treves, 1967]. Therefore, a normed linear space over $\mathbb{F}$ can be canonically made both a normable topological linear space over $\mathbb{F}$ and a metrisable topological space [Horváth, 1966; Treves, 1967]. A subset of a metrisable (in particular, normable) topological linear space over $\mathbb{F}$ is a bounded subset only if (in particular, if and only if) it is a metrically bounded subset of the metric space canonically obtained from the topological linear space over $\mathbb{F}$ [Bourbaki, 2003; Narici and Beckenstein, 2011]. A map between two normed linear spaces over $\mathbb{F}$ is a locally bounded linear map (or, equivalently, a continuous linear map) if and only if the magnitude of the image of each element of the domain is less than or equal to the multiplicative product of some positive real constant by the magnitude of the element of the domain [Bourbaki, 1989]; the smallest such a constant is said to be the *magnitude* (also known, by abuse of language, as *norm*) of the locally bounded linear map [Bourbaki, 1989]. A magnitude-preserving bijective linear (respectively, antilinear) map between two normed linear spaces over $\mathbb{F}$ is said to be a *linear isometry* (respectively, an *antilinear isometry*) [Horváth, 1966; Treves, 1967; Narici and

Beckenstein, 2011]. A bijective linear or antilinear map between two normed linear spaces over $\mathbb{F}$ is magnitude-preserving if and only if it is distance-preserving with respect to the induced canonical metrics; besides, a bijective map between two normed linear spaces over $\mathbb{R}$ that is distance-preserving with respect to the induced canonical metrics is an affine map and, if also origin-preserving, is a norm-preserving linear map (but the result need not hold for a bijective linear or antilinear map between two normed linear spaces over $\mathbb{C}$) [Väisälä, 2003; Narici and Beckenstein, 2011]. A *Banach space* over $\mathbb{F}$ is a normed linear space over $\mathbb{F}$ that can be canonically made a complete metric space [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003].

An *inner multiplication* (also known, by abuse of language, as *inner product*) on a linear space over $\mathbb{F}$ is a Hermitian form on the linear space over $\mathbb{F}$ having the property of positive-definiteness (or, equivalently, non-negative-definiteness and non-degeneracy) [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003]. The image in $\mathbb{F}$ of an ordered pair of vectors of a linear space over $\mathbb{F}$ under an inner multiplication on it is said to be the *inner product* of the ordered pair of vectors [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970]. A linear space over $\mathbb{F}$ equipped with an inner multiplication is said to be an *inner product space* (also known as a *Hausdorff pre-Hilbert space*) over $\mathbb{F}$ [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003]. A linear space over $\mathbb{F}$ equipped with a (possibly degenerate) non-negative-definite Hermitian form on it is said to be a *pre-Hilbert space* over $\mathbb{F}$ [Treves, 1967; Bourbaki, 2003]; a pre-Hilbert space over $\mathbb{F}$ is an inner product space if and only if it can be canonically made a Hausdorff space [Treves, 1967; Bourbaki, 2003]. An inner multiplication on a linear space over $\mathbb{F}$ induces a canonical norm and hence both a canonical metric and a topological structure on it [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003]; inversely, a topological linear space over $\mathbb{F}$ is said to be *innermultiplicationable* (or, by abuse of language, *innerproductable*) if and only if its topological structure can be induced by some inner multiplication on it [Lambert, 1968]. Therefore, an inner product space over $\mathbb{F}$ can be canonically made both an innerproductable (and, *a fortiori*, normable) topological linear space over $\mathbb{F}$ as well as a metrisable topological space [Simmons, 1963; Horváth, 1966; Treves, 1967; Lambert, 1968; Willard, 1970; Bourbaki, 2003]. Furthermore, a normed linear space over $\mathbb{F}$ can be canonically made an innerproductable (and, *a fortiori*, normable) topological linear space over $\mathbb{F}$ as well as a metrisable topological space if and only if the *parallelogram identity* (also known as *parallelogram law*) holds for any pair of elements of it (namely, if and only if the sum of the squares of the magnitudes of any pair of elements of it equals half the sum of the squares of the magnitudes of the sum and the difference of the two elements) [Simmons, 1963; Horváth, 1966; Treves, 1967; Lambert, 1968].

According to the Cauchy–Schwarz inequality, the modulus of the inner product of an ordered pair of elements of an inner product space over $\mathbb{F}$ is less than or equal to the multiplicative product of the magnitudes of the two elements with respect to the induced canonical norm [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003]. An inner-product-preserving bijective linear map between two inner product spaces over $\mathbb{F}$ is said to be a *unitary map* [Simmons, 1963; Bourbaki, 2003; Encyclopedia of Mathematics, 2014]. A bijective map between two inner product spaces over $\mathbb{F}$ is inner-product-preserving (and necessarily linear) if and only if it is magnitude-preserving with respect to the induced canonical norms (or, equivalently, if and only if it is distance-preserving with respect to the induced canonical metrics) [Simmons, 1963; Horváth, 1966; Treves, 1967]. A *Hilbert space* over $\mathbb{F}$ is an inner product space over $\mathbb{F}$ that can be canonically made a Banach space over $\mathbb{F}$ [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003].

Two vectors of an inner product space over $\mathbb{F}$ are said to be *orthogonal* if and only if any of (and hence both) the ordered pairs formed by them has null inner product [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003]; a vector of an inner product space over $\mathbb{F}$ is orthogonal to itself if and only if it coincides with the origin [Simmons, 1963; Horváth, 1966; Bourbaki, 2003]. A set of vectors of an inner product space over $\mathbb{F}$ that are mutually orthogonal and have unit magnitude with respect to the induced canonical norm is said to be an *orthonormal system* (also known as *orthonormal set* or *orthonormal family*) [Simmons, 1963; Horváth, 1966; Treves, 1967; Willard, 1970; Bourbaki, 2003]. Each orthonormal system of an inner product space over $\mathbb{F}$ is contained in a maximal orthonormal system, and each inner product space over $\mathbb{F}$ has a maximal orthonormal system (be it countable or uncountable) [Horváth, 1966; Gudder, 1974]. Any two maximal orthonormal systems of an inner product space over $\mathbb{F}$ are equinumerous [Gudder, 1974]; the cardinality of any maximal orthonormal system of an inner product space over $\mathbb{F}$ is said to be its *dimension* [Gudder, 1974]. An *orthonormal basis* (also known as *complete orthonormal system*, *complete orthonormal set*, or *complete orthonormal family*) of an inner product space over $\mathbb{F}$ is an orthonormal system whose linear span is an everywhere dense subset of the inner product space over $\mathbb{F}$ [Treves, 1967; Gudder, 1974; Bourbaki, 2003]; an inner product space over $\mathbb{F}$ need not have an orthonormal basis (be it countable or uncountable) [Gudder, 1974; Bourbaki, 2003]. Each vector of an inner product space over $\mathbb{F}$ having an orthonormal basis can be expressed uniquely as a possibly infinite but convergent linear combination of a countable subset of the orthonormal basis, each coefficient of the linear combination being the inner product of the ordered pair formed by the corresponding vector of the orthonormal basis and the vector [Gudder, 1974; Bourbaki, 2003]; by abuse of language, such coefficients are said to be the *coordinates* of the vector with respect to the

orthonormal basis [Bourbaki, 2003]. Equivalently, for each vector of an inner product space over $\mathbb{F}$ having an orthonormal basis, the square of the magnitude with respect to the induced canonical norm equals the sum of the squares of the moduli of the coordinates with respect to the orthonormal basis [Bourbaki, 2003]. An inner product space over $\mathbb{F}$ is a Hilbert space over $\mathbb{F}$ if and only if each maximal orthonormal system of it is an orthonormal basis [Horváth, 1966; Gudder, 1974; Bourbaki, 2003]. Accordingly, each orthonormal system of a Hilbert space over $\mathbb{F}$ is contained in an orthonormal basis, and each Hilbert space over $\mathbb{F}$ has an orthonormal basis (be it countable or uncountable) [Treves, 1967; Gudder, 1974; Bourbaki, 2003]. Any two orthonormal bases of a Hilbert space over $\mathbb{F}$ are equinumerous, and they can be transformed into each other with an automorphism [Bourbaki, 2003]; the cardinality of any orthonormal basis of a Hilbert space over $\mathbb{F}$ is said to be its *Hilbert dimension* [Horváth, 1966; Bourbaki, 2003]. A Hilbert space over $\mathbb{F}$ has Hilbert dimension not exceeding $\aleph_0$ (namely, has a countable orthonormal basis) if and only if it is a separable space [Treves, 1967; Gudder, 1974]; in contrast, an inner product space over $\mathbb{F}$ may have dimension not exceeding $\aleph_0$ (namely, may have a countable maximal orthonormal system) without being a separable space [Gudder, 1974]. An orthonormal system of a Hilbert space over $\mathbb{F}$ is an orthonormal basis if and only if any of (and hence both) the ordered pairs formed by each vector of the orthonormal system and any vector of the Hilbert space over $\mathbb{F}$ has null inner product when and only when the vector is the origin [Bourbaki, 2003]; in particular, an element of a Hilbert space over $\mathbb{F}$ has null coordinates with respect to an orthonormal basis if and only if it is the origin [Bourbaki, 2003].

A.1.3.5 Duality

Let $\mathcal{V}$ be a linear space over $\mathbb{F}$. The set of the linear forms (respectively, antilinear forms) on $\mathcal{V}$ can be canonically made a linear space over $\mathbb{F}$, which we denote by $\mathcal{V}^*$ (respectively, by $\bar{\mathcal{V}}^*$) [Horváth, 1966; Treves, 1967; Bourbaki, 2003]; we say that $\mathcal{V}^*$ (respectively, $\bar{\mathcal{V}}^*$) is the *algebraic dual space* (respectively, *algebraic antidual space*) of $\mathcal{V}$ and that $\mathcal{V}$ is the *algebraic predual space* of $\mathcal{V}^*$ (respectively, the *algebraic preantidual space* of $\bar{\mathcal{V}}^*$) [Horváth, 1966; Treves, 1967; Bourbaki, 2003]. Let $\mathcal{V}$ be now equipped with a compatible topological structure so as to be made a topological linear space over $\mathbb{F}$. The set of the continuous linear forms (respectively, continuous antilinear forms) on $\mathcal{V}$ can be canonically made a linear space over $\mathbb{F}$, which we denote by $\mathcal{V}'$ (respectively, $\bar{\mathcal{V}}'$) [Horváth, 1966; Treves, 1967; Bourbaki, 2003]; we say that $\mathcal{V}'$ (respectively, $\bar{\mathcal{V}}'$) is the *topological dual space* (respectively, *topological antidual space*) of $\mathcal{V}$ and that $\mathcal{V}$ is the *topological predual space* (respectively, *topological preantidual space*) of $\bar{\mathcal{V}}'$ [Horváth, 1966; Treves, 1967;

Bourbaki, 2003]. The canonical maps from $\mathcal{V}^*$ into $\overline{\mathcal{V}}^*$ and from $\mathcal{V}'$ into $\overline{\mathcal{V}}'$ are bijective antilinear maps [Treves, 1967]. We have [Horváth, 1966; Treves, 1967; Bourbaki, 2003]

$$\mathcal{V}^* \supseteq \mathcal{V}'$$

$$\overline{\mathcal{V}}^* \supseteq \overline{\mathcal{V}}'$$

Let $\mathcal{V}_{\text{LC}}$ be a locally convex topological linear space over $\mathbb{F}$, and let $\mathcal{V}'_{\text{LC}}$ be its topological dual space; we denote by $\mathcal{V}''_{\text{LC}}$ the algebraic dual space of $\mathcal{V}'_{\text{LC}}$, and we say that it is the *algebraic bidual space* of $\mathcal{V}_{\text{LC}}$ [Bourbaki, 2003]. Let $\mathcal{V}'_{\text{LC}}$ be now equipped with a certain *strong topological structure* so as to be made a locally convex topological linear space over $\mathbb{F}$ [Bourbaki, 2003]; we denote by $\mathcal{V}''_{\text{LC}}$ the topological dual space of $\mathcal{V}'_{\text{LC}}$, and we say that it is the *topological bidual space* of $\mathcal{V}_{\text{LC}}$ [Bourbaki, 2003]. We have [Bourbaki, 2003]

$$\mathcal{V}''_{\text{LC}} \supseteq \mathcal{V}''_{\text{LC}} \supseteq \mathcal{V}_{\text{LC}}$$

Now, $\mathcal{V}_{\text{LC}}$ is said to be a *semireflexive space* if and only if the canonical map from $\mathcal{V}_{\text{LC}}$ into $\mathcal{V}''_{\text{LC}}$ is linear and bijective (namely, is an isomorphism of $\mathbf{Vect}_{\mathbb{F}}$) [Bourbaki, 2003]. As a topological space, a semireflexive space is a Hausdorff space [Bourbaki, 2003]. Let $\mathcal{V}'_{\text{LC}}$ be now equipped with the strong topological structure so as to be made a locally convex topological linear space over $\mathbb{F}$ [Bourbaki, 2003]; then, $\mathcal{V}_{\text{LC}}$ is said to be a *reflexive space* if and only if the canonical map from $\mathcal{V}_{\text{LC}}$ into $\mathcal{V}''_{\text{LC}}$ is linear and bicontinuous (namely, is an isomorphism of $\mathbf{TopVect}_{\mathbb{F}}$) [Bourbaki, 2003]. A reflexive space is a semireflexive space [Bourbaki, 2003]. When equipped with the strong topological structure, the topological dual space of a reflexive space is a reflexive space [Bourbaki, 2003]; in contrast, when equipped with the strong topological structure, the topological dual space of a semireflexive space need not be a semireflexive space [Kelley and Namioka, 1963].

Let $\mathcal{V}_{\text{IP}}$ be an inner product space over $\mathbb{F}$, and let $\mathcal{V}'_{\text{IP}}$ (respectively, $\overline{\mathcal{V}}'_{\text{IP}}$) be its topological dual space (respectively, its topological antidual space). Let $\mathcal{V}_{\text{IP}}$ and $\mathcal{V}'_{\text{IP}}$ (respectively, $\overline{\mathcal{V}}'_{\text{IP}}$) be canonically made normed linear spaces over $\mathbb{F}$ [Treves, 1967]. Remarkably, $\mathcal{V}_{\text{IP}}$ is a Hilbert space over $\mathbb{F}$ if and only if the canonical map from $\mathcal{V}_{\text{IP}}$ into $\mathcal{V}'_{\text{IP}}$ (respectively, $\overline{\mathcal{V}}'_{\text{IP}}$) is an antilinear (respectively, a linear) isometry, in which case $\mathcal{V}'_{\text{IP}}$ (respectively, $\overline{\mathcal{V}}'_{\text{IP}}$) can be canonically made a Hilbert space over $\mathbb{F}$ and hence the canonical map from $\mathcal{V}_{\text{IP}}$ into $\mathcal{V}'_{\text{IP}}$ (respectively, $\overline{\mathcal{V}}'_{\text{IP}}$) is in fact a unitary map [Horváth, 1966; Treves, 1967; Bourbaki, 2003]. Furthermore, each Hilbert space over $\mathbb{F}$ is a reflexive space and can be assimilated to its own topological dual space (respectively, topological antidual space) via the aforementioned antilinear (respectively, linear) isometry, which is also a unitary map [Horváth, 1966; Treves, 1967; Bourbaki, 2003].

A.2 Mathematical framework: Functions, Radon measures, and distributions

A.2.1 Introduction

Topological issues are of great importance for the theoretical investigation of spaces and maps. However, their details are very technical and mostly dispensable for our purposes. Accordingly, we stipulate that each undermentioned space is equipped with a suitable topological structure; in this spirit, instead of embedding morphisms and identification morphisms between spaces, we consider inclusion relations between their underlying sets. Such inclusion relations may involve underlying sets of spaces assimilated to some other spaces via suitable maps; as is customary, we do not distinguish between (and we equate by means of the symbol $\equiv$) two spaces that can be assimilated to each other. Hereinafter, unless otherwise stated, spaces and maps are respectively complex topological linear spaces and morphisms between them, and the term *origin* refers to $\mathbb{F}$.

In the following, some information about functions, Radon measures, and distributions is summarised.

A.2.2 Functions

We say that a map from $\mathbb{F}$ (respectively, a subset of $\mathbb{F}$) into $\mathbb{F}$ is a *pointwise function* on $\mathbb{F}$. In particular, we say that a map from $\mathbb{C}$ (respectively, a subset of $\mathbb{C}$) into $\mathbb{C}$ is a *complex-valued pointwise function of a free complex variable* (hereinafter *complex pointwise function*) on $\mathbb{C}$ (respectively, a subset of $\mathbb{C}$), a map from $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$) into $\mathbb{C}$ is a *complex-valued pointwise function of a free real variable* (hereinafter *complex-valued pointwise function*) on $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$), and a map from $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$) into $\mathbb{R}$ is a *real-valued pointwise function of a free real variable* (hereinafter *real-valued pointwise function*) on $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$). A pointwise function on $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$) is said to be *defined almost everywhere* on $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$) if and only if it is defined everywhere on $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$) except for a Lebesgue-null subset of $\mathbb{R}$ (respectively, a subset of $\mathbb{R}$) [Bourbaki, 2004b]. Hereinafter, unless otherwise stated, a pointwise function is a complex-valued pointwise function defined almost everywhere on $\mathbb{R}$. We occasionally consider the *localisation* (also known as *restriction*) of such a pointwise function on $\mathbb{R}$ at a Lebesgue-measurable subset of $\mathbb{R}$, which results in a pointwise function thereon.

An *essential function* (hereinafter *function*) on $\mathbb{R}$ is an equivalence class of pointwise functions defined everywhere on $\mathbb{R}$ with respect to the equivalence relation of *almost-everywhere equality* on $\mathbb{R}$ [Horváth, 1966; Treves, 1967; Poletti, 2015]; we say that any of such pointwise functions on $\mathbb{R}$ is a *representative* of the function on $\mathbb{R}$. Besides, each pointwise function defined almost everywhere (in particular, everywhere) on $\mathbb{R}$ is associated with one and only one function on $\mathbb{R}$ [Bourbaki, 2004b]; we say that the latter is defined almost everywhere on $\mathbb{R}$ by the former. With respect to the equivalence relation of almost-everywhere equality on $\mathbb{R}$, we say that an equivalence class of complex-valued pointwise functions on $\mathbb{R}$ is a *complex-valued function* on $\mathbb{R}$ and an equivalence class of real-valued pointwise functions on $\mathbb{R}$ is a *real-valued function* on $\mathbb{R}$. A real-valued function on $\mathbb{R}$ is said to be *non-negative* (respectively, *non-positive*) if and only if it is an equivalence class of non-negative (respectively, non-positive) real-valued pointwise functions on $\mathbb{R}$. A function on $\mathbb{R}$ that is not real-valued cannot be non-negative (respectively, non-positive). However, we say that a function on $\mathbb{R}$ is *weakly non-negative* (respectively, *weakly non-positive*) if and only if its real part is non-negative (respectively, non-positive); as is evident, a function on $\mathbb{R}$ is non-negative (respectively, non-positive) only if it is weakly non-negative (respectively, weakly non-positive). Hereinafter, unless otherwise stated, a function is a complex-valued function defined almost everywhere on $\mathbb{R}$. We occasionally consider the localisation of such a function on $\mathbb{R}$ at a Lebesgue-measurable subset of $\mathbb{R}$, which results in a function thereon.

Let γ be a complex-valued function on $\mathbb{R}$. Now, γ_{RV} is a *real-valued function* on $\mathbb{R}$ and γ_{IV} is an *imaginary-valued function* on $\mathbb{R}$ if and only if we have

$$\mathcal{C}(\gamma_{\text{RV}}) = \mathcal{I}(\gamma_{\text{RV}})$$

$$\mathcal{C}(\gamma_{\text{IV}}) = -\mathcal{I}(\gamma_{\text{IV}})$$

with

$$\begin{aligned}\mathcal{C} &= \mathcal{P} \circ \mathcal{R} \\ &= \mathcal{R} \circ \mathcal{P}\end{aligned}$$

$$\mathcal{C} \circ \mathcal{C} = \mathcal{I}$$

where

- $\circ$ is the *composition operator* (the usual convention that composition operands have right-to-left order being adopted)
- $\mathcal{I}$ is the *identity transformation*
- $\mathcal{C}$ is the *complex-conjugation involution*
- $\mathcal{R}$ is the *reflection involution*
- $\mathcal{P}$ is the (commutative) *composite involution of $\mathcal{R}$ and $\mathcal{C}$*

Then, γ can be expressed uniquely as

$$\gamma = \gamma_{\text{RV}} + \gamma_{\text{IV}}$$

with

$$\begin{cases} \gamma_{\text{RV}} = \frac{1}{2}(\mathcal{I} + \mathcal{C})\gamma \\ \gamma_{\text{IV}} = \frac{1}{2}(\mathcal{I} - \mathcal{C})\gamma \end{cases}$$

and we say that γ_{RV} and γ_{IV} are respectively the *real-valued part* and the *imaginary-valued part* of γ . Furthermore, we say that the γ_{RV} and γ_{IV}/j are respectively the *real part* and the *imaginary part* of γ .

Besides, γ_{E} is an *even function* on $\mathbb{R}$ and γ_{O} is an *odd function* on $\mathbb{R}$ if and only if we have

$$\mathcal{R}(\gamma_{\text{E}}) = \mathcal{I}(\gamma_{\text{E}})$$

$$\mathcal{R}(\gamma_o) = -\mathcal{I}(\gamma_o)$$

with

$$\begin{aligned}\mathcal{R} &= \mathcal{P} \circ \mathcal{C} \\ &= \mathcal{C} \circ \mathcal{P}\end{aligned}$$

$$\mathcal{R} \circ \mathcal{R} = \mathcal{I}$$

Then, γ can be expressed uniquely as

$$\gamma = \gamma_E + \gamma_o$$

with

$$\begin{cases} \gamma_E = \frac{1}{2}(\mathcal{I} + \mathcal{R})\gamma \\ \gamma_o = \frac{1}{2}(\mathcal{I} - \mathcal{R})\gamma \end{cases}$$

and we say that γ_E and γ_o are respectively the *even part* and the *odd part* of ψ .

Finally, γ_{CS} is a *conjugate-symmetric function* on $\mathbb{R}$ and γ_{CA} is a *conjugate-antisymmetric function* on $\mathbb{R}$ if and only if we have

$$\mathcal{P}(\gamma_{CS}) = \mathcal{I}(\gamma_{CS})$$

$$\mathcal{P}(\gamma_{CA}) = -\mathcal{I}(\gamma_{CA})$$

with

$$\begin{aligned}\mathcal{P} &= \mathcal{R} \circ \mathcal{C} \\ &= \mathcal{C} \circ \mathcal{R}\end{aligned}$$

$$\mathcal{P} \circ \mathcal{P} = \mathcal{I}$$

Then, γ can be expressed uniquely as

$$\gamma = \gamma_{CS} + \gamma_{CA}$$

with

$$\begin{cases} \gamma_{\text{CS}} = \frac{1}{2}(\mathcal{I} + \mathcal{P})\gamma \\ \gamma_{\text{CA}} = \frac{1}{2}(\mathcal{I} - \mathcal{P})\gamma \end{cases}$$

and we say that γ_{CS} and γ_{CA} are respectively the *conjugate-symmetric part* and the *conjugate-antisymmetric part* of γ .

A function on $\mathbb{R}$ is real-valued (respectively, imaginary-valued) if and only if its conjugate-symmetric part coincides with its even part (respectively, odd part) and its conjugate-antisymmetric part coincides with its odd part (respectively, even part). A function on $\mathbb{R}$ is even (respectively, odd) if and only if its conjugate-symmetric part coincides with its real-valued part (respectively, imaginary-valued part) and its conjugate-antisymmetric part coincides with its imaginary-valued part (respectively, real-valued part). A function on $\mathbb{R}$ is conjugate-symmetric (respectively, conjugate-antisymmetric) if and only if its real-valued part coincides with its even part (respectively, odd part) and its imaginary-valued part coincides with its odd part (respectively, even part).

We refer to the multiplicative product of a complex-valued function on $\mathbb{R}$ by its complex conjugate (which is itself a non-negative real-valued function on $\mathbb{R}$) as the *square modulus* of the complex-valued function on $\mathbb{R}$. Besides, we refer to the square root of the square modulus of a complex-valued function on $\mathbb{R}$ (which is itself a non-negative real-valued function on $\mathbb{R}$) as the *modulus* of the complex-valued function on $\mathbb{R}$.

As is customary in the *theory of measure* and in *functional analysis*, we regard a *function space* as a topological linear space whose elements are functions [Horváth, 1966; Treves, 1967; Poletti, 2015]. A function space canonically inherits the usual concepts and operations with which the corresponding space whose elements are pointwise functions is equipped [Poletti, 2015]; furthermore, such concepts and operations are well-defined in the sense that they are independent of representatives of functions [Poletti, 2015]. The adopted framework is particularly convenient, and in fact natural, when it comes to introducing the operation of integration and, in particular, the concept of *Lebesgue integral* on a Lebesgue-measurable subset of $\mathbb{R}$ (in particular, on $\mathbb{R}$) with respect to the Lebesgue measure on $\mathbb{R}$ (which, whenever well-defined, is said to be *absolutely convergent* or *unconditionally convergent*) [Schwartz, 1993b]. The Lebesgue integral on a Lebesgue-measurable subset of $\mathbb{R}$ (in particular, on $\mathbb{R}$) with respect to the Lebesgue measure on $\mathbb{R}$ is well-defined for each non-negative real-valued Lebesgue-measurable function on $\mathbb{R}$ [Folland, 1999; Bauer, 2001; Poletti, 2005]; furthermore, it is well-defined for each (either real-valued or complex-valued) Lebesgue-measurable function on $\mathbb{R}$ whose modulus has finite Lebesgue integral on the Lebesgue-measurable subset of

$\mathbb{R}$ (in particular, on $\mathbb{R}$) with respect to the Lebesgue measure on $\mathbb{R}$ [Folland, 1999; Poletti, 2005]. Given such a Lebesgue-measurable function on $\mathbb{R}$, we say that the Lebesgue integral on the Lebesgue-measurable subset of $\mathbb{R}$ (in particular, on $\mathbb{R}$) with respect to the Lebesgue measure on $\mathbb{R}$ of the result of its localisation thereat is its *integral* thereon. Furthermore, if the Lebesgue-measurable subset of $\mathbb{R}$ is bounded, we say that the quotient of its integral thereon by the Lebesgue measure of the Lebesgue-measurable subset of $\mathbb{R}$ is its *mean value* thereon; besides, whenever well-defined, we say that the square root of the quotient of the integral of its square modulus thereon by the Lebesgue measure of the Lebesgue-measurable subset of $\mathbb{R}$ is its *root-mean-square value* thereon. Within the adopted framework, it is also possible to introduce the weaker concept of *improper Lebesgue integral* on a Lebesgue-measurable subset of $\mathbb{R}$ (in particular, on $\mathbb{R}$) with respect to the Lebesgue measure on $\mathbb{R}$ (which, whenever well-defined, is said to be *conditionally convergent*) [Schwartz, 1993b]. It is worth mentioning that the Lebesgue integral with respect to the Lebesgue measure on $\mathbb{R}$ can be regarded as a special case of *Lebesgue–Stieltjes integral* with respect to a *Lebesgue–Stieltjes measure* on $\mathbb{R}$, which is a Radon measure on $\mathbb{R}$ associated with a function of locally bounded variation on $\mathbb{R}$, as introduced below (a Lebesgue–Stieltjes measure on $\mathbb{R}$ reducing to the Lebesgue measure on $\mathbb{R}$ when the function of locally bounded variation on $\mathbb{R}$ with which it is associated as a Radon measure on $\mathbb{R}$ is the function defined almost everywhere on $\mathbb{R}$ by ξ) [Widder, 1941; Edwards, 1965; Benedetto, 1997; Bauer, 2001; Benedetto and Czaja, 2009]. Hereinafter, unless otherwise stated, a function is a complex-valued Lebesgue-measurable function on $\mathbb{R}$.

A function on $\mathbb{R}$ has at most one representative that is *continuous* (in particular, *uniformly continuous*), to which reference is made as *continuous representative* (in particular, *uniformly continuous representative*) [Treves, 1967; Poletti, 2015]; we refer to a function on $\mathbb{R}$ admitting a continuous representative (in particular, uniformly continuous representative) as a C^0 function on $\mathbb{R}$ (in particular, *uniformly continuous function* on $\mathbb{R}$). We say that the *value* of a C^0 function on $\mathbb{R}$ at a point of $\mathbb{R}$ is the value of its continuous representative thereat. We refer to a C^0 function on an open subset of $\mathbb{R}$ whose continuous representative is a pointwise function identically equal to a complex constant (in particular, 0 and 1) on the open subset of $\mathbb{R}$ as a *constant function* (in particular, the *null function* and the *unit function*) on the open subset of $\mathbb{R}$. We denote by $0_{\mathbb{R}}$ the null function on $\mathbb{R}$ and by $1_{\mathbb{R}}$ the unit function on $\mathbb{R}$. For our purposes, there is no need to introduce a concept of complex essential function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) and hence of representative and continuous representative of it; in fact, hereinafter, each complex pointwise function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) is continuous. Accordingly,

hereinafter, we refer to a complex pointwise function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) as a *complex function* on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$).

The *support of a function* on $\mathbb{R}$ is the smallest closed subset of $\mathbb{R}$ outside of which the function on $\mathbb{R}$ vanishes (or, equivalently, the complement of the largest open subset of $\mathbb{R}$ on which the function on $\mathbb{R}$ vanishes) [Schwartz, 1966; Estrada and Kanwal, 2002; Kanwal, 2004]. We qualify as *one-sided* (either *right-sided* or *left-sided*) a function on $\mathbb{R}$ with support bounded from one side (either from the left or from the right) [Zemanian, 1965a]; hereinafter, when mentioning a one-sided function on $\mathbb{R}$, we intend a function on $\mathbb{R}$ that is arbitrarily either right-sided or left-sided (but is consistently either right-sided or left-sided at each occurrence in a sentence). In particular, we qualify as either *causal* or *anticausal* a function on $\mathbb{R}$ with support bounded from one side (either from the left or from the right) at the origin [Kwakernaak and Sivan, 1991]. A function on $\mathbb{R}$ can be expressed non-uniquely as the sum of a (possibly anticausal) *left-sided part* and a (possibly causal) *right-sided part* [Zemanian, 1965a].

The operations of differentiation and antidifferentiation are standardly introduced for functions [Schwartz, 1992, 1993a, 1993b]. A one-sided function on $\mathbb{R}$ admits at most one *one-sided n^{th} antiderivative* [Schwartz, 1993b]. By definition, the 0^{th} *derivative* and the 0^{th} *antiderivative* of a function on $\mathbb{R}$ coincide with the function on $\mathbb{R}$ itself. We stipulate that the *one-sided z^{th} generalised derivative* of a one-sided function on $\mathbb{R}$ coincides with its *one-sided fractional derivative of order z* [McBride, 1979]; specifically, the one-sided z^{th} generalised derivative of a one-sided function on $\mathbb{R}$ is the one-sided $|z|^{\text{th}}$ antiderivative of the one-sided function on $\mathbb{R}$ (when z is in $\mathbb{Z}_{(-\infty,1]}$), the one-sided function on $\mathbb{R}$ itself (when z equals 0), or the z^{th} derivative of the one-sided function on $\mathbb{R}$ (when z is in $\mathbb{Z}_{[1,+\infty)}$).

A (necessarily continuous) complex function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) is a *holomorphic function* thereon (namely, it has a, necessarily continuous, 1^{st} complex derivative thereon) if and only if it is a *smooth complex function* thereon (namely, it has a, necessarily continuous, complex derivative of any order in $\mathbb{N}$ thereon) or, equivalently, if and only if it is an *analytic function* thereon, which means that it has *Taylor series* about any point of the domain converging absolutely to the function everywhere on some connected open neighbourhood of the point [Ghizzetti and Ossicini, 1971; Stein and Shakarchi, 2003; Poletti, 2005]. In particular, a holomorphic function on $\mathbb{C}$ (also known as *entire function*) is said to be of *exponential type* if and only if its modulus satisfies a certain majorisation condition on a neighbourhood of infinity [Barros-Neto, 1973; Requicha, 1980; Stein and Shakarchi, 2003]. Hereinafter, unless otherwise

stated, a holomorphic function is a holomorphic function on $\mathbb{C}$. We occasionally consider the localisation of such a holomorphic function at a straight line (or a subset of it) parallel to either $\mathbb{C}_{\text{Im}=0}$ or $\mathbb{C}_{\text{Re}=0}$, which results in a function thereon. A holomorphic function on an open subset of $\mathbb{C}$ (in particular, a holomorphic function on $\mathbb{C}$) is real-valued only if it is a constant real-valued holomorphic function thereon [Stein and Shakarchi, 2003]. A holomorphic function on an open subset of $\mathbb{C}$ (in particular, a holomorphic function on a connected open subset of $\mathbb{C}$ such as $\mathbb{C}$ itself) either vanishes only at isolated points of the domain or vanishes at each point of countably many connected open subsets (in particular, at each point) of the domain [Zemanian, 1965a; Beltrami and Wohlers, 1966; Stein and Shakarchi, 2003; Poletti, 2005]. Remarkably, a holomorphic function on $\mathbb{C}$, or, more generally, a holomorphic function on a connected open subset of $\mathbb{C}$ containing $\mathbb{C}_{\text{Im}=0}$ (respectively, $\mathbb{C}_{\text{Re}=0}$), can be regarded as the unique such holomorphic function resulting from the *extension* of the function on $\mathbb{R}$ that coincides with the holomorphic function localised at $\mathbb{C}_{\text{Im}=0}$ (respectively, $\mathbb{C}_{\text{Re}=0}$), such extension being operationally performed by replacing the free real variable with a suitable free complex variable [Beltrami and Wohlers, 1966; Ghizzetti and Ossicini, 1971; Stein and Shakarchi, 2003; Poletti, 2005]; in fact, a holomorphic function on a connected open subset of $\mathbb{C}$ can be extended uniquely to a holomorphic function on a connected open superset of the connected open subset of $\mathbb{C}$ (in particular, to a holomorphic function on $\mathbb{C}$) [Beltrami and Wohlers, 1966; Ghizzetti and Ossicini, 1971; Stein and Shakarchi, 2003; Poletti, 2005].

A *meromorphic function* on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) is a complex function on $\mathbb{C}$ (respectively, on the open subset of $\mathbb{C}$) that is a holomorphic function on the difference of $\mathbb{C}$ (respectively, of the open subset of $\mathbb{C}$) and a discrete closed subset of $\mathbb{C}$ (respectively, of the open subset of $\mathbb{C}$) at which its multiplicative inverse vanishes (each of such isolated points being by definition a *pole* of the meromorphic function and a *zero* of its multiplicative inverse) [Ghizzetti and Ossicini, 1971; Stein and Shakarchi, 2003]. A meromorphic function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) has *Laurent series* about any of its poles converging absolutely to the function everywhere on the difference of some connected open neighbourhood of the pole and the pole itself [Ghizzetti and Ossicini, 1971; Stein and Shakarchi, 2003]. In particular, a *rational function* on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$), which is the quotient of two holomorphic functions thereon with the divisor not identically equal to 0, is a meromorphic function on $\mathbb{C}$ (respectively, on the open subset of $\mathbb{C}$) [Ghizzetti and Ossicini, 1971; Stein and Shakarchi, 2003].

By adopting as codomain a Hausdorff locally convex topological linear space over $\mathbb{F}$, the concepts of holomorphic function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) and meromorphic function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) can be generalised to introduce the concepts of

vector-valued holomorphic function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) and *vector-valued meromorphic function* on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) [Ortner and Wagner, 2013, 2015]; in particular, when the codomain is a suitable distribution space (as introduced below), a vector-valued holomorphic function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) and a vector-valued meromorphic function on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) are said to be respectively a *distribution-valued holomorphic function* on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) and a *distribution-valued meromorphic function* on $\mathbb{C}$ (respectively, on an open subset of $\mathbb{C}$) [Carmichael and Mitrović, 1989; Ortner and Wagner, 2013, 2015].

The chains of inclusion relations hold [Zemanian, 1965a; Schwartz, 1966; Horváth, 1966; Treves, 1967; Carton-Lebrun, 1995, 2005; Estrada and Kanwal, 2002; Larcher, 2013; Ortner and Wagner, 2013]

$$\begin{array}{lcl}
 L^0(\mathbb{R}) & & \\
 \cup & & \\
 L^p_{\text{loc}}(\mathbb{R}) & \supset & L^p(\mathbb{R}) \\
 \cup & & \\
 \mathcal{E}(\mathbb{R}) & & \\
 \cup & & \\
 \mathcal{O}_{\text{M}}(\mathbb{R}) & & \\
 \cup & & \\
 \mathcal{O}_{\text{C}}(\mathbb{R}) & \supset & \mathcal{D}_{L^p}(\mathbb{R}) \\
 \equiv & & \cup \\
 \mathcal{O}_{\text{C}}(\mathbb{R}) \equiv \mathcal{O}_{\text{C}}(\mathbb{R}) \supset \mathcal{D}_{L^\infty}(\mathbb{R}) \supset \dot{\mathcal{B}}(\mathbb{R}) \supset \mathcal{D}_{L^q}(\mathbb{R}) \supset & & \mathcal{D}_{L^1}(\mathbb{R}) \\
 & & \cup \\
 & & \mathcal{D}_{L^\infty,+1}(\mathbb{R}) \supset \dot{\mathcal{B}}_{+1}(\mathbb{R}) \supset \mathcal{D}_{L^q,+1}(\mathbb{R}) \supset \mathcal{D}_{L^1,+1}(\mathbb{R}) \\
 & & \cup \\
 & & \mathcal{D}_{L^\infty,n_2}(\mathbb{R}) \supset \dot{\mathcal{B}}_{n_2}(\mathbb{R}) \supset \mathcal{D}_{L^q,n_2}(\mathbb{R}) \supset \mathcal{D}_{L^1,n_2}(\mathbb{R}) \\
 & & \cup \\
 \mathcal{K}(\mathbb{R}) & \supset & \mathcal{S}(\mathbb{R}) \supset \mathcal{D}(\mathbb{R}) \\
 & & \cup \\
 \exp \mathcal{O}_{\text{C}}(\mathbb{R}) & \supset & \exp \mathcal{S}(\mathbb{R})
 \end{array}$$

where

- $L^0(\mathbb{R})$ is the function space of the *Lebesgue-measurable functions* on $\mathbb{R}$
- $L^p_{\text{loc}}(\mathbb{R})$ is the function space of the *locally p -integrable functions* on $\mathbb{R}$ (also known as *locally Lebesgue functions of exponent p on $\mathbb{R}$*)
- $L^p(\mathbb{R})$ is the function space of the *p -integrable functions* on $\mathbb{R}$ (also known as *Lebesgue functions of exponent p on $\mathbb{R}$*)
- $\mathcal{E}(\mathbb{R})$ is the function space of the *smooth functions* on $\mathbb{R}$ (also known as *smooth functions of arbitrary growth on $\mathbb{R}$, functions of differentiability class C^∞ on $\mathbb{R}$, or infinitely differentiable functions on $\mathbb{R}$*)
- $\mathcal{O}_M(\mathbb{R})$ is the function space of the *smooth functions of slow growth* on $\mathbb{R}$ (also known as *smooth slowly increasing functions on $\mathbb{R}$*)
- $\mathcal{O}_C(\mathbb{R})$ is the function space of the *smooth functions of very slow growth* on $\mathbb{R}$ (also known as *smooth very slowly increasing functions on $\mathbb{R}$*)
- $\exp \mathcal{O}_C(\mathbb{R})$ is the function space of the *smooth functions of very slow growth of exponential type* on $\mathbb{R}$ (also known as *Paley–Wiener functions on $\mathbb{R}$*)
- $\mathcal{K}(\mathbb{R})$ is the function space of the *smooth functions of polynomial growth* on $\mathbb{R}$ (also known as *Grossmann–Loupias–Stein symbols on $\mathbb{R}$ or GLS symbols on $\mathbb{R}$*)
- $\mathcal{D}_{L^p}(\mathbb{R})$ is the function space of the *smooth p -integrable functions* on $\mathbb{R}$
- $\dot{\mathcal{B}}(\mathbb{R})$ is the function space of the *smooth functions vanishing at infinity* on $\mathbb{R}$
- $\mathcal{D}_{L^p, n}(\mathbb{R})$ is the function space of the *(n) -weighted smooth p -integrable functions* on $\mathbb{R}$
- $\dot{\mathcal{B}}_n(\mathbb{R})$ is the function space of the *(n) -weighted smooth functions vanishing at infinity* on $\mathbb{R}$
- $\mathcal{S}(\mathbb{R})$ is the function space of the *smooth functions of rapid decay* on $\mathbb{R}$ (also known as *smooth rapidly decreasing functions on $\mathbb{R}$ or Schwartz functions on $\mathbb{R}$*)
- $\exp \mathcal{S}(\mathbb{R})$ is the function space of the *smooth functions of rapid decay of exponential type* on $\mathbb{R}$
- $\mathcal{D}(\mathbb{R})$ is the function space of the *smooth functions of compact support* on $\mathbb{R}$ (also known, when real-valued, as *bump functions on $\mathbb{R}$*)

We warn that, for n even, the function spaces denoted here by $\dot{\mathcal{B}}(\mathbb{R})$ and $\dot{\mathcal{B}}_n(\mathbb{R})$ are denoted elsewhere by respectively $\mathcal{S}_0(\mathbb{R}) \equiv \mathcal{B}_0(\mathbb{R})$ and $\mathcal{S}_k(\mathbb{R})$ with $k \equiv n/2$ [Horváth, 1966; Ortner and Wagner, 2014]. Furthermore, we warn that the function spaces denoted here by $\exp \mathcal{O}_c(\mathbb{R})$ and $\exp \mathcal{S}(\mathbb{R})$ are denoted elsewhere by respectively $PW(\mathbb{R})$ [García et al., 1998] and $\mathcal{Z}(\mathbb{R})$ [Gelfand and Shilov, 1964, 1968; Zemanian, 1965a; Beltrami and Wohlers, 1966; Misra and Lavoine, 1986; Carmichael and Mitrović, 1989; Hoskins and Sousa Pinto, 2005].

A.2.2.1 Lebesgue functions

The *essential supremum* on a Lebesgue-measurable subset of $\mathbb{R}$ of a real-valued function on $\mathbb{R}$ is either the smallest real constant whose multiplicative product by the continuous representative of $1_{\mathbb{R}}$ localised thereat is almost everywhere thereon greater than any representative of the real-valued function on $\mathbb{R}$ localised thereat or, if such smallest real constant does not exist, $+\infty$ [Treves, 1967; Leoni, 2017]. Given a Lebesgue-measurable function on $\mathbb{R}$, the p^{th} root of the integral of the p^{th} power (when p is in $\mathbb{R}_{[1,+\infty)}$) or the essential supremum (when p equals $+\infty$) of its modulus on a Lebesgue-measurable subset of $\mathbb{R}$ may be either finite or infinite [Treves, 1967; Leoni, 2017].

A locally p -integrable function on $\mathbb{R}$ is a Lebesgue-measurable function on $\mathbb{R}$ (whose each representative is necessarily finite almost everywhere on $\mathbb{R}$) for which the aforementioned p^{th} power (when p is in $\mathbb{R}_{[1,+\infty)}$) or essential supremum (when p equals $+\infty$) is finite on each compact subset of $\mathbb{R}$ [Treves, 1967; Bourbaki, 2004b; Leoni, 2017]. In particular, when p equals 1, 2, and $+\infty$, a locally p -integrable function on $\mathbb{R}$ is said to be respectively a *locally integrable function* on $\mathbb{R}$ (also known as *locally absolutely integrable function* on $\mathbb{R}$), a *locally square integrable function* on $\mathbb{R}$ (also known as *locally quadratically integrable function* on $\mathbb{R}$), and a *locally essentially bounded function* on $\mathbb{R}$ (hereinafter *locally bounded function* on $\mathbb{R}$). For any p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$ with p_2 greater than p_1 , we have [Treves, 1967; Folland, 1999; Leoni, 2017]

$$L_{\text{loc}}^{p_1}(\mathbb{R}) \supset L_{\text{loc}}^{p_2}(\mathbb{R})$$

For a locally integrable function on $\mathbb{R}$ (and, *a fortiori*, for a locally p -integrable function on $\mathbb{R}$), the operation of *weak differentiation* is introduced by generalising the ordinary differentiation [Leoni, 2017]. The n_0^{th} weak derivative of a locally integrable function on $\mathbb{R}$ is unique whenever it exists as

a locally integrable function on $\mathbb{R}$ (and, more generally, whenever it does not exist as a locally integrable function on $\mathbb{R}$, but it does as either a Radon measure on $\mathbb{R}$ or a distribution on $\mathbb{R}$, as introduced below) [Leoni, 2017].

A p -integrable function on $\mathbb{R}$ is a Lebesgue-measurable function on $\mathbb{R}$ for which the aforementioned p^{th} power (when p is in $\mathbb{R}_{[1,+\infty)}$) or essential supremum (when p equals $+\infty$) is finite on $\mathbb{R}$ [Treves, 1967]. Because the aforementioned p^{th} power (when p is in $\mathbb{R}_{[1,+\infty)}$) or essential supremum (when p equals $+\infty$) is a norm on $L^p(\mathbb{R})$, we refer to it as L^p norm [Treves, 1967]; we say that the image in $\mathbb{R}$ of an element of $L^p(\mathbb{R})$ under the L^p norm is its L^p magnitude. In particular, when p equals 1, 2, and $+\infty$, a p -integrable function on $\mathbb{R}$ is said to be respectively an *integrable function* on $\mathbb{R}$ (also known as *absolutely integrable function* on $\mathbb{R}$), a *square integrable function* on $\mathbb{R}$ (also known as *quadratically integrable function* on $\mathbb{R}$), and an *essentially bounded function* on $\mathbb{R}$ (hereinafter *bounded function* on $\mathbb{R}$). Each p -integrable function on $\mathbb{R}$ can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of p -integrable functions on $\mathbb{R}$ selectable as C^0 bounded functions on $\mathbb{R}$ (and, when p is in $\mathbb{R}_{[1,+\infty)}$, even as C^0 functions vanishing at infinity on $\mathbb{R}$, as introduced below) [Schwartz, 1966]. For any p_1 in $\overline{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\overline{\mathbb{R}}_{[1,+\infty]}$ with p_2 greater than p_1 and $q_{(p_1,p_2)}$ in $\mathbb{R}_{(p_1,p_2)}$, denoting by $L^{p_1}(\mathbb{R}) + L^{p_2}(\mathbb{R})$ the function space of the sums of p_1 -integrable functions on $\mathbb{R}$ and p_2 -integrable functions on $\mathbb{R}$, we have [Folland, 1999]

$$\begin{cases} L^{p_1}(\mathbb{R}) \not\supset L^{p_2}(\mathbb{R}) \\ L^{p_2}(\mathbb{R}) \not\supset L^{p_1}(\mathbb{R}) \\ (L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})) \neq \emptyset \end{cases}$$

$$(L^{p_1}(\mathbb{R}) + L^{p_2}(\mathbb{R})) \supset L^{q_{(p_1,p_2)}}(\mathbb{R}) \supset (L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R}))$$

and hence, in particular,

$$(L^1(\mathbb{R}) + L^\infty(\mathbb{R})) \supset L^p(\mathbb{R}) \supset (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$$

Accordingly, $L^{p_1}(\mathbb{R})$ and $L^{p_2}(\mathbb{R})$ are intersecting (like $L^{p_1}_{\text{loc}}(\mathbb{R})$ and $L^{p_2}_{\text{loc}}(\mathbb{R})$) and incomparable with respect to inclusion (unlike $L^{p_1}_{\text{loc}}(\mathbb{R})$ and $L^{p_2}_{\text{loc}}(\mathbb{R})$). An *Euler diagram* representing the relations between selected function spaces of p -integrable functions on $\mathbb{R}$ is shown in **Figure A.1**.

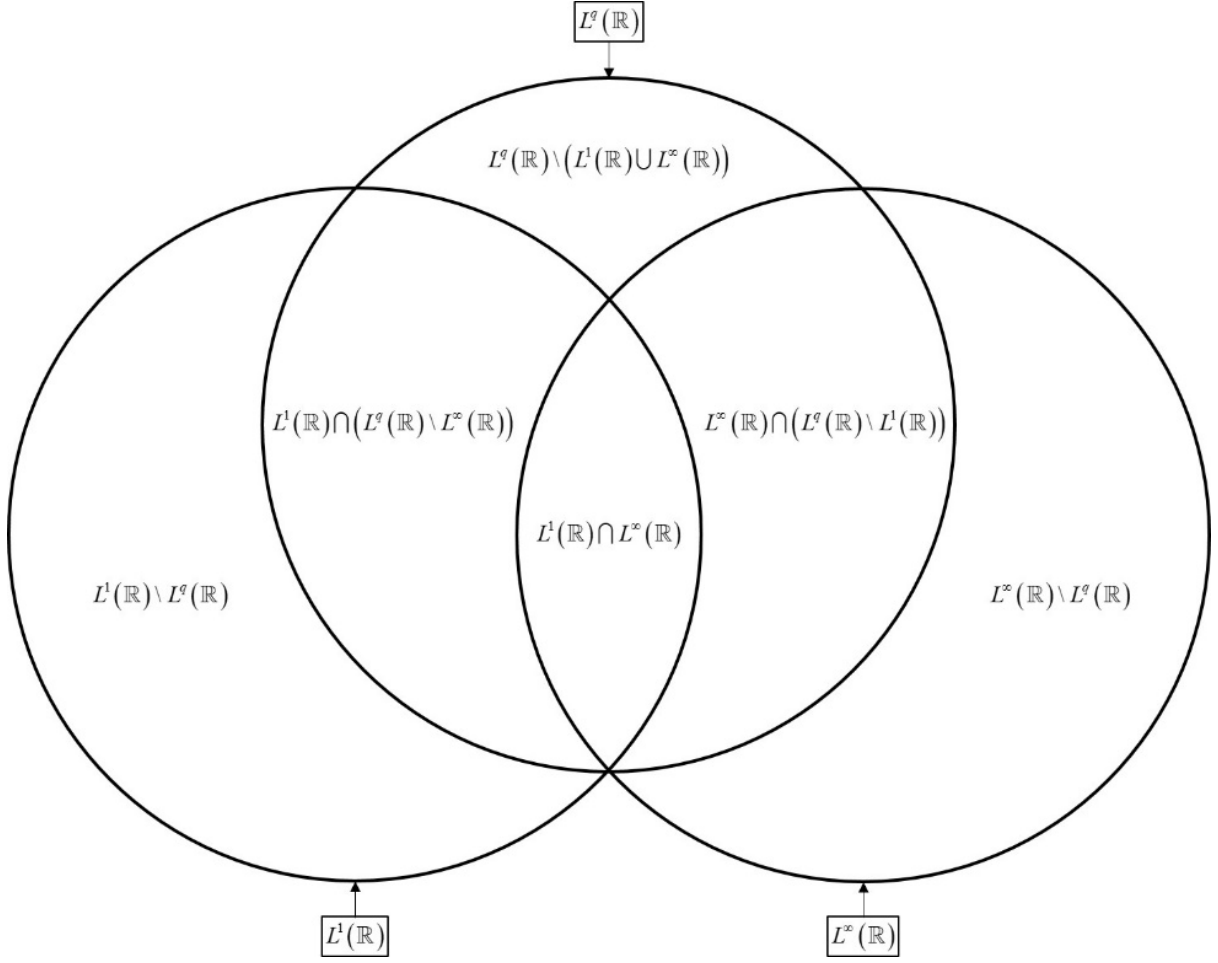


Figure A.1 Euler diagram representing the relations between selected function spaces of p -integrable functions on $\mathbb{R}$. $L^1(\mathbb{R})$ = function space of the integrable functions on $\mathbb{R}$; $L^q(\mathbb{R})$ = function space of the q -integrable functions on $\mathbb{R}$ with q in $\mathbb{R}_{(1,+\infty)}$; $L^\infty(\mathbb{R})$ = function space of the bounded functions on $\mathbb{R}$.

For any p_1 in $\overline{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\overline{\mathbb{R}}_{[1,+\infty]}$ with p_2 greater than p_1 , denoting by $\mathcal{P}_{L^{p_1}}(\mathbb{R})$ the function space of the multiplicative products of polynomial functions on $\mathbb{R}$ by p_1 -integrable functions on $\mathbb{R}$, we have [Poletti, 2015]

$$\mathcal{P}_{L^{p_1}}(\mathbb{R}) \supset \mathcal{P}_{L^{p_2}}(\mathbb{R})$$

The multiplication of two functions on $\mathbb{R}$ that are elements of respectively $L^{p_1}(\mathbb{R})$ and $L^{p_2}(\mathbb{R})$ with p_1 in $\overline{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\overline{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2$ is in $\mathbb{R}_{[0,1]}$ is well-defined and commutative, and it results in a function on $\mathbb{R}$ that is an element of $L^{p_{12}}(\mathbb{R})$ with p_{12} in $\overline{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{12}$ equals $1/p_1 + 1/p_2$ [Poletti, 2015; Leoni, 2017].

The operation of *convolution* is standardly introduced for functions [Schwartz, 1993b]. The convolution of two functions on $\mathbb{R}$ that are elements of respectively $L^{p_1}(\mathbb{R})$ and $L^{p_2}(\mathbb{R})$ with p_1 in

$\bar{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2 - 1$ is in $\mathbb{R}_{[0,1]}$ is well-defined and commutative, and it results in a function on $\mathbb{R}$ that is an element of $L^{p_{12}}(\mathbb{R})$ with p_{12} in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{12}$ equals $1/p_1 + 1/p_2 - 1$ [Schwartz, 1966; Treves, 1967; Poletti, 2015]; when p_{12} equals $+\infty$ (in particular, when p_1 equals 1 and p_2 equals $+\infty$, when p_1 equals $+\infty$ and p_2 equals 1, and when both p_1 and p_2 equal 2), such element of $L^{p_{12}}(\mathbb{R})$ is a C^0 (in fact, uniformly continuous) bounded function on $\mathbb{R}$ (in particular, both p_1 and p_2 equal 2, a C^0 function vanishing at infinity on $\mathbb{R}$, as introduced below) [Schwartz, 1966]. Furthermore, the convolution of three functions on $\mathbb{R}$ that are elements of respectively $L^{p_1}(\mathbb{R})$, $L^{p_2}(\mathbb{R})$, and $L^{p_3}(\mathbb{R})$ with p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$, p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$, and p_3 in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2 + 1/p_3 - 2$ is in $\mathbb{R}_{[0,1]}$ as well as one the equivalent additional conditions that $1/p_1 + 1/p_2 - 1$ is in $\mathbb{R}_{[0,1]}$, $1/p_2 + 1/p_3 - 1$ is in $\mathbb{R}_{[0,1]}$, or $1/p_1 + 1/p_3 - 1$ is in $\mathbb{R}_{[0,1]}$ is well-defined, commutative, and associative, and it results in a function on $\mathbb{R}$ that is an element of $L^{p_{123}}(\mathbb{R})$ with p_{123} in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{123}$ equals $1/p_1 + 1/p_2 + 1/p_3 - 2$ [Poletti, 2015].

Importantly, $L^p(\mathbb{R})$ (in particular, $L^2(\mathbb{R})$) can be canonically made a Banach space over $\mathbb{F}$ (in particular, a Hilbert space over $\mathbb{F}$) [Treves, 1967; Folland, 1999]. In fact, $L^2(\mathbb{R})$ is the only function space of p -integrable functions on $\mathbb{R}$ that can be made a Hilbert space over $\mathbb{F}$ [Stein and Shakarchi, 2011]. The inner product of an ordered pair of square integrable functions on $\mathbb{R}$ is defined as the integral on $\mathbb{R}$ of the product of the first by the complex conjugate of the second [Treves, 1967]. In particular, the inner product of a square integrable function on $\mathbb{R}$ and itself equals the square of its L^2 magnitude [Treves, 1967]. Furthermore, both $L^{p_1}(\mathbb{R}) + L^{p_2}(\mathbb{R})$ and $L^{p_1}(\mathbb{R}) \cap L^{p_2}(\mathbb{R})$ can be canonically made Banach spaces over $\mathbb{F}$ [Liu and Wang, 1968].

Let $p_{[1,+\infty]}$ in $\mathbb{R}_{[1,+\infty]}$ and $p'_{(1,+\infty]}$ in $\bar{\mathbb{R}}_{(1,+\infty]}$ form a pair of Hölder-conjugate exponents. We denote by $(L^{p_{[1,+\infty)}}(\mathbb{R}))'$ the topological dual space of $L^{p_{[1,+\infty)}}(\mathbb{R})$. We have [Treves, 1967; Folland, 1999; Leoni, 2017]

$$(L^{p_{[1,+\infty)}}(\mathbb{R}))' \equiv L^{p'_{(1,+\infty]}}(\mathbb{R})$$

Let $p_{[1,+\infty),1}$ in $\mathbb{R}_{[1,+\infty)}$ and $p'_{(1,+\infty),1}$ in $\overline{\mathbb{R}}_{(1,+\infty)}$ (respectively, q_1 in $\mathbb{R}_{(1,+\infty)}$ and q'_1 in $\mathbb{R}_{(1,+\infty)}$) as well as $p_{[1,+\infty),2}$ in $\mathbb{R}_{[1,+\infty)}$ and $p'_{(1,+\infty),2}$ in $\overline{\mathbb{R}}_{(1,+\infty)}$ (respectively, q_2 in $\mathbb{R}_{(1,+\infty)}$ and q'_2 in $\mathbb{R}_{(1,+\infty)}$) form two pairs of Hölder-conjugate exponents. We denote by $\left(L^{p_{[1,+\infty),1}}(\mathbb{R}) + L^{p_{[1,+\infty),2}}(\mathbb{R}) \right)'$ and $\left(L^{p_{[1,+\infty),1}}(\mathbb{R}) \cap L^{p_{[1,+\infty),2}}(\mathbb{R}) \right)'$ the topological dual spaces of respectively $L^{p_{[1,+\infty),1}}(\mathbb{R}) + L^{p_{[1,+\infty),2}}(\mathbb{R})$ and $L^{p_{[1,+\infty),1}}(\mathbb{R}) \cap L^{p_{[1,+\infty),2}}(\mathbb{R})$. We have [Liu and Wang, 1968]

$$\left(L^{p_{[1,+\infty),1}}(\mathbb{R}) + L^{p_{[1,+\infty),2}}(\mathbb{R}) \right)' \equiv \left(L^{p'_{(1,+\infty),1}}(\mathbb{R}) \cap L^{p'_{(1,+\infty),2}}(\mathbb{R}) \right)$$

$$\left(L^{p_{[1,+\infty),1}}(\mathbb{R}) \cap L^{p_{[1,+\infty),2}}(\mathbb{R}) \right)' \equiv \left(L^{p'_{(1,+\infty),1}}(\mathbb{R}) + L^{p'_{(1,+\infty),2}}(\mathbb{R}) \right)$$

Whilst $L^q(\mathbb{R})$ is a reflexive space (and hence so is its topological dual space), $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are not [Treves, 1967; Folland, 1999; Leoni, 2017]. Indeed, on the one hand, the topological dual space of $L^1(\mathbb{R})$ is $L^\infty(\mathbb{R})$ [Treves, 1967; Folland, 1999; Leoni, 2017]; on the other hand, the topological dual space of $L^\infty(\mathbb{R})$ is not a function space [Leoni, 2017]. Furthermore, whilst both $L^{q_1}(\mathbb{R}) + L^{q_2}(\mathbb{R})$ and $L^{q_1}(\mathbb{R}) \cap L^{q_2}(\mathbb{R})$ are reflexive spaces (and hence so are their topological dual spaces), $L^{p_{[1,+\infty),1}}(\mathbb{R}) + L^{p_{[1,+\infty),2}}(\mathbb{R})$ and $L^{p_{[1,+\infty),1}}(\mathbb{R}) \cap L^{p_{[1,+\infty),2}}(\mathbb{R})$ with $p_{[1,+\infty),1}$ equalling 1 or $p_{[1,+\infty),2}$ equalling 1 are not [Liu and Wang, 1968].

We now introduce a useful type of bounded function on $\mathbb{R}$. We have [Wiener, 1930, 1988; Schwartz, 1966; Amerio and Prouse, 1971; Levitan and Zhikov, 1982; Bass, 1984; Corduneanu, 1989, 2009; Katznelson, 2004; Partington, 2004; Andres *et al.*, 2006; Baake and Grimm, 2013; Moody and Strungaru, 2017]

$$\left(L^\infty(\mathbb{R}) \cap UC(\mathbb{R}) \right) \supset AP(\mathbb{R})$$

where

- $UC(\mathbb{R})$ is the function space of the *uniformly continuous functions* on $\mathbb{R}$
- $AP(\mathbb{R})$ is the function space of the *uniformly almost periodic functions* on $\mathbb{R}$ (also known as *Bohr almost periodic functions* on $\mathbb{R}$, *Bochner almost periodic functions* on $\mathbb{R}$, or *strongly almost periodic functions* on $\mathbb{R}$)

A uniformly almost periodic function on $\mathbb{R}$ is a uniformly continuous bounded function on $\mathbb{R}$ to which a sequence of complex trigonometric polynomial functions on $\mathbb{R}$ converges uniformly on $\mathbb{R}$ (an equivalent definition being obtainable by invoking the concept of *almost period*) [Amerio and Prouse, 1971; Levitan and Zhikov, 1982; Bass, 1984; Bertrandias *et al.*, 1987; Corduneanu, 1989, 2009; Katznelson, 2004; Partington, 2004; Andres *et al.*, 2006; Baake and Grimm, 2013; Moody and Strungaru, 2017]. In particular, each trigonometric polynomial function on $\mathbb{R}$ is a uniformly almost periodic function on $\mathbb{R}$ [Amerio and Prouse, 1971; Levitan and Zhikov, 1982; Bass, 1984; Bertrandias *et al.*, 1987; Corduneanu, 1989, 2009; Katznelson, 2004; Partington, 2004; Andres *et al.*, 2006; Baake and Grimm, 2013; Moody and Strungaru, 2017]. A uniformly almost periodic function on $\mathbb{R}$ vanishes at negative infinity (respectively, positive infinity) if and only if it coincides with $0_{\mathbb{R}}$ [Ciampa *et al.*, 2019]; accordingly, the intersection of $AP(\mathbb{R})$ and $\dot{\mathcal{B}}^0(\mathbb{R})$ (as introduced below) contains only $0_{\mathbb{R}}$. Each uniformly almost periodic function on $\mathbb{R}$ has one and only one *Fourier–Bohr series* (also known as *generalised Fourier series*) associated with it, and two uniformly almost periodic functions on $\mathbb{R}$ coincide if and only if so do their Fourier–Bohr series [Wiener, 1930, 1988; Amerio and Prouse, 1971; Levitan and Zhikov, 1982; Bass, 1984; Corduneanu, 1989, 2009; Lagarias, 2000; Katznelson, 2004; Andres *et al.*, 2006; Baake and Grimm, 2013; Moody and Strungaru, 2017]. In fact, a uniformly almost periodic function on $\mathbb{R}$ has a *Fourier–Bohr series representation* (also known as *generalised Fourier series representation*), which means that it can be expressed as a possibly infinite but convergent (in the mean) linear combination of countably many exponential functions on $\mathbb{R}$ with 1st-degree monomial imaginary arguments [Wiener, 1930, 1988; Amerio and Prouse, 1971; Levitan and Zhikov, 1982; Bass, 1984; Corduneanu, 1989, 2009; Lagarias, 2000; Katznelson, 2004; Andres *et al.*, 2006; Baake and Grimm, 2013; Moody and Strungaru, 2017]; the coefficients of the linear combination and the imaginary parts of the coefficients of the arguments of the exponential functions on $\mathbb{R}$ are said to be respectively the *Fourier–Bohr coefficients* (also known as *generalised Fourier coefficients*) and the *Fourier–Bohr exponents* (also known as *generalised Fourier exponents*) of the uniformly almost periodic function on $\mathbb{R}$ [Amerio and Prouse, 1971; Levitan and Zhikov, 1982; Bass, 1984; Corduneanu, 1989, 2009; Lagarias, 2000; Katznelson, 2004; Andres *et al.*, 2006; Baake and Grimm, 2013; Moody and Strungaru, 2017]. Without loss of generality, we assume that each uniformly almost periodic function on $\mathbb{R}$ has a null Fourier–Bohr exponent; the corresponding Fourier–Bohr coefficient, possibly null, is the *mean value* on $\mathbb{R}$ of the uniformly almost periodic function on $\mathbb{R}$ [Amerio and Prouse, 1971; Levitan and Zhikov, 1982; Bass, 1984; Corduneanu, 1989, 2009; Lagarias, 2000; Katznelson, 2004; Andres *et al.*, 2006; Baake and Grimm, 2013; Moody and Strungaru, 2017]. A uniformly almost periodic function on $\mathbb{R}$ has countably many pairwise incommensurable non-null Fourier–Bohr exponents [Levitan and Zhikov, 1982; Corduneanu, 1989,

2009; Baake and Grimm, 2013]; a uniformly almost periodic function on $\mathbb{R}$ having finitely many (in particular, no) pairwise incommensurable non-null Fourier–Bohr exponents is said to be a C^0 *quasiperiodic function* on $\mathbb{R}$ (in particular, a C^0 *periodic function* on $\mathbb{R}$) [Levitan and Zhikov, 1982; Corduneanu, 1989, 2009; Baake and Grimm, 2013]. In fact, for a C^0 periodic function on $\mathbb{R}$, the Fourier–Bohr series, the Fourier–Bohr coefficients, and the Fourier–Bohr exponents are respectively the *Fourier series*, the *Fourier coefficients*, and the *Fourier exponents*, and the moduli of the non-null Fourier–Bohr exponents are integer multiples of their minimum, 2π times the multiplicative inverse of which being the *period* of the C^0 periodic function on $\mathbb{R}$ [Wiener, 1930, 1988; Amerio and Prouse, 1971; Bass, 1984; Corduneanu, 1989, 2009; Richards and Youn, 1990; Katznelson, 2004; Baake and Grimm, 2013]. The function space of the C^0 periodic functions on $\mathbb{R}$ is not invariant under addition; hence, unlike the function space of the uniformly almost periodic functions on $\mathbb{R}$ and the function space of the C^0 quasiperiodic functions on $\mathbb{R}$, it has the serious pathology that it cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$.

Because of their pathological behaviour, $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ (the topological dual space of the former being the latter, and both being canonically makeable Banach spaces over $\mathbb{F}$) are often conveniently replaced by respectively a subspace of $L^1(\mathbb{R})$ and a superspace of $L^\infty(\mathbb{R})$ (the topological dual space of the former being a quotient space of the latter, and both being canonically makeable Banach spaces over $\mathbb{F}$). Indeed, we have [Fefferman and Stein, 1972; Bourdaud, 2002; Bourdaud *et al.*, 2002; Dafni, 2002; Ortner, 2004; Chang and Sadosky, 2006; Garnett, 2007; Korenovskii, 2007; Stein and Shakarchi, 2011]

$$L^1(\mathbb{R}) \supset H^1(\mathbb{R})$$

$$BMO(\mathbb{R}) \supset (L^\infty(\mathbb{R})/\mathbb{C})$$

$$BMO(\mathbb{R}) \supset VMO(\mathbb{R}) \supset ((UC(\mathbb{R})/\mathbb{C}) \cap BMO(\mathbb{R}))$$

where

- $H^1(\mathbb{R})$ is the function space of the *real-variable Hardy functions of exponent 1* on $\mathbb{R}$
- $BMO(\mathbb{R})$ is the quotient space of the function space of the *functions of bounded mean oscillation* on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant

- $VMO(\mathbb{R})$ is the quotient space of the function space of the *functions of vanishing mean oscillation* on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant
- $L^\infty(\mathbb{R})/\mathbb{C}$ is the quotient space of the function space of the bounded functions on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant
- $UC(\mathbb{R})/\mathbb{C}$ is the quotient space of the function space of the uniformly continuous functions on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant

Denoting by $(H^1(\mathbb{R}))'$ the topological dual space of $(H^1(\mathbb{R}))'$, we have [Fefferman and Stein, 1972; Bourdaud, 2002; Dafni, 2002; Ortner, 2004; Chang and Sadosky, 2006; Garnett, 2007; Stein and Shakarchi, 2011]

$$(H^1(\mathbb{R}))' \equiv BMO(\mathbb{R})$$

A real-variable Hardy function of exponent 1 on $\mathbb{R}$ is an integrable function on $\mathbb{R}$ whose Hilbert transform (as introduced below) is an integrable function on $\mathbb{R}$ [Fefferman and Stein, 1972; Ortner, 2004; Garnett, 2007]. Equivalently, a real-variable Hardy function of exponent 1 on $\mathbb{R}$ is an integrable function on $\mathbb{R}$ that is the Hilbert transform (as introduced below) of an integrable function on $\mathbb{R}$ [Chang and Sadosky, 2006].

The *oscillation* (also known as *absolute deviation*) of a locally integrable function on $\mathbb{R}$ on a compact subset of $\mathbb{R}$ is the modulus of the difference of the locally integrable function on $\mathbb{R}$ localised at the interior of the compact subset of $\mathbb{R}$ and its mean value thereon [Chang and Sadosky, 2006; Korenovskii, 2007]. Furthermore, the *mean oscillation* on a compact subset of $\mathbb{R}$ of a locally integrable function on $\mathbb{R}$ is the mean value on the compact subset of $\mathbb{R}$ of its oscillation [Chang and Sadosky, 2006; Korenovskii, 2007]. Given a locally integrable function on $\mathbb{R}$, the supremum of its mean oscillation on a compact subset of $\mathbb{R}$ as the compact subset of $\mathbb{R}$ varies on $\mathbb{R}$ is null if and only if the locally integrable function on $\mathbb{R}$ is a constant function on $\mathbb{R}$ [Fefferman and Stein, 1972; Bourdaud, 2002; Bourdaud *et al.*, 2002; Dafni, 2002; Chang and Sadosky, 2006; Garnett, 2007; Korenovskii, 2007].

A function of bounded mean oscillation on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ for which the supremum of the mean oscillation on a compact subset of $\mathbb{R}$ as the compact subset of $\mathbb{R}$ varies on $\mathbb{R}$ is finite [Fefferman and Stein, 1972; Bourdaud, 2002; Bourdaud *et al.*, 2002; Dafni, 2002; Chang

and Sadosky, 2006; Garnett, 2007; Korenovskii, 2007]. Because the aforementioned supremum is a norm on $BMO(\mathbb{R})$, we refer to it as *BMO norm* [Fefferman and Stein, 1972; Bourdaud, 2002; Bourdaud *et al.*, 2002; Dafni, 2002; Chang and Sadosky, 2006; Garnett, 2007; Korenovskii, 2007]; we say that the image in $\mathbb{R}$ of a an element of $BMO(\mathbb{R})$ under the *BMO norm* is its *BMO magnitude*.

A function of bounded mean oscillation on $\mathbb{R}$ is an element of $L_{loc}^{p(1,+\infty)}(\mathbb{R})$, but, in contrast, an element of $L_{loc}^{p(1,+\infty)}(\mathbb{R})$ need not be a function of bounded mean oscillation on $\mathbb{R}$ [Korenovskii, 2007].

A function of bounded mean oscillation on $\mathbb{R}$ need not be an element of $L_{loc}^{\infty}(\mathbb{R})$, and an element of $L_{loc}^{\infty}(\mathbb{R})$ need not be a function of bounded mean oscillation on $\mathbb{R}$ [Korenovskii, 2007].

Furthermore, the modulus of a function of bounded mean oscillation on $\mathbb{R}$ is a function of bounded mean oscillation on $\mathbb{R}$, but, in contrast, a function on $\mathbb{R}$ whose modulus is a function of bounded mean oscillation on $\mathbb{R}$ need not be a function of bounded mean oscillation on $\mathbb{R}$ [Korenovskii, 2007].

A function of vanishing mean oscillation on $\mathbb{R}$ is a function of bounded mean oscillation on $\mathbb{R}$ for which the limit of the mean oscillation on each compact subset of $\mathbb{R}$ tends to 0 as the Lebesgue measure of the compact subset of $\mathbb{R}$ tends to 0 [Bourdaud, 2002; Bourdaud *et al.*, 2002; Dafni, 2002; Garnett, 2007]. Remarkably, $VMO(\mathbb{R})$ is the closure of $(UC(\mathbb{R})/\mathbb{C}) \cap BMO(\mathbb{R})$ within $BMO(\mathbb{R})$ [Bourdaud, 2002; Dafni, 2002; Garnett, 2007].

The standard example of element of $BMO(\mathbb{R}) \setminus (VMO(\mathbb{R}) \cup (L^{\infty}(\mathbb{R})/\mathbb{C}(\mathbb{R})))$ is the equivalence class of the functions defined almost everywhere on $\mathbb{R}$ by $\ln(|\xi|)$ with respect to the equivalence relation of equality up to addition by an arbitrary complex constant [Bourdaud *et al.*, 2002]; remarkably, the equivalence class of the functions on $\mathbb{R}$ coinciding with $\ln(|\xi|)$ on $\mathbb{R}_{(0,+\infty)}$ (respectively, $\mathbb{R}_{(-\infty,0)}$) and with 0 on $\mathbb{R}_{(-\infty,0)}$ (respectively, $\mathbb{R}_{(0,+\infty)}$) with respect to the equivalence relation of equality up to addition by an arbitrary complex constant is not an element of $BMO(\mathbb{R})$ [Chang and Sadosky, 2006; Garnett, 2007]. An example of element of $VMO(\mathbb{R}) \cap (L^{\infty}(\mathbb{R})/\mathbb{C}(\mathbb{R}))$ is the equivalence class of the functions defined almost everywhere on $\mathbb{R}$ by $e^{i\xi}$ with respect to the equivalence relation of equality up to addition by an arbitrary complex constant [Dafni, 2002].

Hereinafter, we occasionally need consider function spaces obtained by generalising $L^p(\mathbb{R})$. We have [Keshava Murthy and Unni, 1974; Kuznetsova, 2006; Wagner, 2014; Sağır and Duyar, 2019]

$$L_z^p(\mathbb{R}) \supset L_{z+1}^p(\mathbb{R})$$

with

$$L_0^p(\mathbb{R}) \equiv L^p(\mathbb{R})$$

where

- $L_z^p(\mathbb{R})$ is the function space of the (z) -weighted p -integrable functions on $\mathbb{R}$ (also known as (z) -weighted Lebesgue functions of exponent p on $\mathbb{R}$)

A (z) -weighted p -integrable function on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ whose multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{z/2}$ is a p -integrable function on $\mathbb{R}$ [Keshava Murthy and Unni, 1974; Kuznetsova, 2006; Wagner, 2014; Sağır and Duyar, 2019]. Equivalently, a (z) -weighted p -integrable function on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ that can be expressed as the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-z/2}$ by a p -integrable function on $\mathbb{R}$ [Keshava Murthy and Unni, 1974; Kuznetsova, 2006; Wagner, 2014; Sağır and Duyar, 2019]. In particular, when p equals 1, 2, and $+\infty$, a (z) -weighted p -integrable function on $\mathbb{R}$ is said to be respectively a (z) -weighted integrable function on $\mathbb{R}$ (also known as (z) -weighted absolutely integrable function on $\mathbb{R}$), a (z) -weighted square integrable function on $\mathbb{R}$ (also known as (z) -weighted quadratically integrable function on $\mathbb{R}$), and a (z) -weighted essentially bounded function on $\mathbb{R}$ (hereinafter (z) -weighted bounded function on $\mathbb{R}$).

We denote by $(L_z^{p(1,+\infty)}(\mathbb{R}))'$ the topological dual space of $L_z^{p(1,+\infty)}(\mathbb{R})$. We have [Keshava Murthy and Unni, 1974; Kuznetsova, 2006; Sağır and Duyar, 2019]

$$(L_z^{p(1,+\infty)}(\mathbb{R}))' \equiv L_{-z}^{p'(1,+\infty)}(\mathbb{R})$$

Whilst $L_z^q(\mathbb{R})$ is a reflexive space (and hence so is its topological dual space), $L_z^1(\mathbb{R})$ and $L_z^\infty(\mathbb{R})$ are not [Keshava Murthy and Unni, 1974; Kuznetsova, 2006; Sağır and Duyar, 2019]. Indeed, on the one hand, the topological dual space of $L_z^1(\mathbb{R})$ is $L_z^\infty(\mathbb{R})$ [Keshava Murthy and Unni, 1974; Sağır and Duyar, 2019]; on the other hand, the topological dual space of $L_z^\infty(\mathbb{R})$ is not a function space.

It is worth noting that both the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{z/2}$ and the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-z/2}$ are positive real-valued smooth functions on $\mathbb{R}$ (as introduced below); their purpose is to control the behaviour of the weighted function on a neighbourhood of infinity, where they are of the same order of magnitude as respectively $|\xi|^z$ and $|\xi|^{-z}$.

Finally, we mention that, for any $p_{(0,1)}$ in $\mathbb{R}_{(0,1)}$, it is possible to introduce the function space

$L_{\text{loc}}^{p_{(0,1)}}(\mathbb{R})$ of the *locally $p_{(0,1)}$ -integrable functions* on $\mathbb{R}$ (also known as *locally Lebesgue functions of exponent $p_{(0,1)}$ on $\mathbb{R}$*) and the function space $L^{p_{(0,1)}}(\mathbb{R})$ of the *$p_{(0,1)}$ -integrable functions* on $\mathbb{R}$ (also known as *locally Lebesgue functions of exponent $p_{(0,1)}$ on $\mathbb{R}$*) [Day, 1940; Folland, 1999]. However, such function spaces have undesirable properties that make them of limited interest [Day, 1940; Folland, 1999].

We now introduce for functions the non-commutative operation of *correlation* (which is strictly connected with the commutative operation of convolution) and the related concept of *non-negative-definiteness*; incidentally, we mention that a concept of *non-positive-definiteness* can also be introduced, albeit in several non-equivalent manners [Berg and Forst, 1975; Schilling *et al.*, 2012]. Whenever well-defined, the convolutive product of a first locally integrable function on $\mathbb{R}$ by the reflected complex conjugate (or, equivalently, by the complex conjugate reflected) of a second locally integrable function on $\mathbb{R}$ is said to be the *cross-correlation function* of the ordered pair of locally integrable functions on $\mathbb{R}$ [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Benedetto, 1997; King, 2009b]; in particular, whenever well-defined, the convolutive product of a locally integrable function on $\mathbb{R}$ by its reflected complex conjugate (or, equivalently, by its complex conjugate reflected) is said to be the *auto-correlation function* of the locally integrable function on $\mathbb{R}$ [Richards and Youn, 1990; Oppenheim *et al.*, 1997; Benedetto, 1997; King, 2009b]. In agreement with the fact that correlation is not commutative, the cross-correlation function of an ordered pair of locally integrable functions on $\mathbb{R}$, whenever well-defined, coincides with the reflected complex conjugate (or, equivalently, the complex conjugate reflected) of the cross-correlation function of the reverse ordered pair of locally integrable functions on $\mathbb{R}$ [Benedetto, 1997]. The auto-correlation function of any smooth function of compact support on $\mathbb{R}$ (in fact, of any smooth function of rapid decay on $\mathbb{R}$) is well-defined [Schwartz, 1966]. A locally integrable function on $\mathbb{R}$ that is a C^0 function on a neighbourhood of the origin is said to be *non-negative-definite* (or *positive-semidefinite*, or *of non-negative type*) if and only if the integration on $\mathbb{R}$ of its multiplicative product

by the auto-correlation function of any smooth function of compact support on $\mathbb{R}$ (in fact, of any smooth function of rapid decay on $\mathbb{R}$) results in a finite non-negative real value [Gelfand and Vilenkin, 1964; Schwartz, 1966; Berg and Forst, 1975; Bass, 1984; Vladimirov, 2002]. Each non-negative-definite locally integrable function on $\mathbb{R}$ that is a C^0 function on a neighbourhood of the origin is a conjugate-symmetric element of $L^\infty(\mathbb{R}) \cap UC(\mathbb{R})$ whose continuous representative has modulus less than or equal to the value at the origin (which is necessarily in $\mathbb{R}_{[0,+\infty)}$) and whose mean value on $\mathbb{R}$ exists and is finite [Gelfand and Vilenkin, 1964; Schwartz, 1966; Berg and Forst, 1975; Bass, 1984; Vladimirov, 2002; Katznelson, 2004; Schilling *et al.*, 2012; Moody and Strungaru, 2017]; accordingly, we refer to such a function on $\mathbb{R}$ as a *non-negative-definite C^0 function* on $\mathbb{R}$. A non-negative-definite C^0 function on $\mathbb{R}$ is a non-negative-definite C^0 periodic function on $\mathbb{R}$ of period ξ_p if and only if its value at the origin equals its value at ξ_p but not its value at any other point in $\mathbb{R}_{(0,\xi_p)}$ [Bass, 1984]. Unlike a finite conical combination, a finite linear combination of non-negative-definite C^0 functions on $\mathbb{R}$ (respectively, of non-negative real-valued functions on $\mathbb{R}$) need not be a non-negative-definite C^0 function on $\mathbb{R}$ (respectively, a non-negative real-valued function on $\mathbb{R}$) [Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Gripenberg *et al.*, 1990; Moody and Strungaru, 2017]; hence, the function space of the non-negative-definite C^0 functions on $\mathbb{R}$ (respectively, of the non-negative real-valued functions on $\mathbb{R}$) cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$. A function on $\mathbb{R}$ that is not conjugate-symmetric (in particular, that is either causal or anticausal) cannot be non-negative-definite. However, we say that a function on $\mathbb{R}$ is *weakly non-negative-definite* (or *weakly positive-semidefinite*, or *of weak non-negative type*) if and only if its conjugate-symmetric part is non-negative-definite [Berg and Forst, 1975; Staffans, 1976; Gripenberg *et al.*, 1990]; as is evident, a function on $\mathbb{R}$ is non-negative-definite only if it is weakly non-negative-definite. A real-valued function on $\mathbb{R}$ is weakly non-negative-definite if and only if the integration on $\mathbb{R}$ of its multiplicative product by the auto-correlation function of any real-valued smooth function of compact support on $\mathbb{R}$ (in fact, of any real-valued smooth function of rapid decay on $\mathbb{R}$) results in a finite non-negative real value [Berg and Forst, 1975].

A.2.2.2 Smooth functions

A smooth function on $\mathbb{R}$ is a Lebesgue-measurable function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is continuous on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967].

A smooth function of slow growth on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ has modulus majorised on a neighbourhood of infinity by $|\xi|$ to the power of a real exponent dependent on the derivative order and possibly increasing with it [Schwartz, 1966; Horváth, 1966; Treves, 1967].

A smooth function of very slow growth on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ has modulus majorised on a neighbourhood of infinity by $|\xi|$ to the power of a real exponent independent of the derivative order [Grothendieck, 1955; Horváth, 1966; Estrada and Kanwal, 2002].

A smooth function of very slow growth of exponential type on $\mathbb{R}$ is a smooth function of very slow growth on $\mathbb{R}$ that can be extended to a holomorphic function of exponential type on $\mathbb{C}$ [Schwartz, 1966]. As a consequence of the fact that a holomorphic function on $\mathbb{C}$ either vanishes only at isolated points of $\mathbb{C}$ or vanishes at each point of $\mathbb{C}$, the intersection of $\exp \mathcal{O}_c(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ contains only $0_{\mathbb{R}}$, and no subspace of $\mathcal{O}_c(\mathbb{R})$ that is a superspace of $\mathcal{D}(\mathbb{R})$ can be a subspace of $\exp \mathcal{O}_c(\mathbb{R})$.

A smooth function of polynomial growth on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ has modulus majorised on a neighbourhood of infinity by $|\xi|$ to the power of a real exponent dependent on the derivative order and decreasing with it so that the sum of the real exponent and the derivative order is constant [Estrada and Kanwal, 2002]. In particular, each polynomial function on $\mathbb{R}$ is a smooth function of polynomial growth on $\mathbb{R}$ [Estrada and Kanwal, 2002].

A smooth p -integrable function on $\mathbb{R}$ (respectively, a smooth function vanishing at infinity on $\mathbb{R}$) is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is a p -integrable function on $\mathbb{R}$ (respectively, whose each derivative of any order in $\mathbb{N}$ vanishes at infinity) [Schwartz, 1966; Horváth, 1966]. In particular, when p equals 1, 2, and $+\infty$, a smooth p -integrable function on $\mathbb{R}$ is said to be respectively a *smooth integrable function* on $\mathbb{R}$ (also known as *smooth absolutely integrable function* on $\mathbb{R}$), a *smooth square integrable function* on $\mathbb{R}$ (also known as a *smooth quadratically integrable function* on $\mathbb{R}$), and a *smooth essentially bounded function* on $\mathbb{R}$ (hereinafter *smooth bounded function* on $\mathbb{R}$). For any p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$ with p_2 greater than p_1 , we have [Schwartz, 1966]

$$\mathcal{D}_{L^{p_2}}(\mathbb{R}) \supset \mathcal{D}_{L^{p_1}}(\mathbb{R})$$

We now introduce a useful type of smooth bounded function on $\mathbb{R}$. We have [Schwartz, 1966]

$$\mathcal{D}_{L^\infty}(\mathbb{R}) \supset \mathcal{D}_{AP}(\mathbb{R})$$

where

- $\mathcal{D}_{AP}(\mathbb{R})$ is the function space of the *smooth uniformly almost periodic functions* on $\mathbb{R}$ (also known as *smooth Bohr almost periodic functions* on $\mathbb{R}$, *smooth Bochner almost periodic functions* on $\mathbb{R}$, or *smooth strongly almost periodic functions* on $\mathbb{R}$)

A smooth uniformly almost periodic function on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is a uniformly almost periodic function on $\mathbb{R}$ [Schwartz, 1966; Zaidman, 2011]. A smooth uniformly almost periodic function on $\mathbb{R}$ vanishes at negative infinity (respectively, positive infinity) if and only if it coincides with $0_{\mathbb{R}}$ [Ciampa *et al.*, 2019]; accordingly, the intersection of $AP(\mathbb{R})$ and $\dot{\mathcal{B}}(\mathbb{R})$ contains only $0_{\mathbb{R}}$.

Furthermore, we have [Fefferman and Stein, 1972; Bourdaud, 2002; Bourdaud *et al.*, 2002; Dafni, 2002; Ortner, 2004; Chang and Sadosky, 2006; Garnett, 2007; Korenovskii, 2007; Stein and Shakarchi, 2011]

$$\begin{array}{ccc} L^1(\mathbb{R}) & \supset & H^1(\mathbb{R}) \\ \cup & & \cup \\ \mathcal{D}_{L^1}(\mathbb{R}) & \supset & \mathcal{D}_{H^1}(\mathbb{R}) \end{array}$$

$$\begin{array}{ccc} BMO(\mathbb{R}) & \supset & (L^\infty(\mathbb{R})/\mathbb{C}) \\ \text{III} & & \\ BMO(\mathbb{R}) & \supset & VMO(\mathbb{R}) \supset ((UC(\mathbb{R})/\mathbb{C}) \cap BMO(\mathbb{R})) \\ \cup & & \cup \\ (UC(\mathbb{R})/\mathbb{C}) & \supset & BMO^\infty(\mathbb{R}) \supset VMO^\infty(\mathbb{R}) \end{array}$$

$$\mathcal{D}_{H^1}(\mathbb{R}) \equiv (\mathcal{D}_{L^1}(\mathbb{R}) \cap H^1(\mathbb{R}))$$

where

- $\mathcal{D}_{H^1}(\mathbb{R})$ is the function space of the *smooth real-variable Hardy functions of exponent 1* on $\mathbb{R}$

- $BMO^\infty(\mathbb{R})$ is the quotient space of the function space of the *smooth functions of bounded mean oscillation* on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant
- $VMO^\infty(\mathbb{R})$ is the quotient space of the function space of the *smooth functions of vanishing mean oscillation* on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant

A smooth real-variable Hardy function of exponent 1 on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is a real-variable Hardy function of exponent 1 on $\mathbb{R}$ [Ortner, 2004].

A smooth function of bounded mean oscillation on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is a function of bounded mean oscillation on $\mathbb{R}$ [Bourdaud, 2002].

A smooth function of vanishing mean oscillation on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is a function of vanishing mean oscillation on $\mathbb{R}$ [Ortner, 2004].

Remarkably, $VMO(\mathbb{R})$ is not only the closure of $(UC(\mathbb{R})/\mathbb{C}) \cap BMO(\mathbb{R})$ within $BMO(\mathbb{R})$ but also the closure of $BMO^\infty(\mathbb{R})$ within $BMO(\mathbb{R})$ [Bourdaud, 2002; Dafni, 2002; Garnett, 2007].

An (n) -weighted smooth p -integrable function on $\mathbb{R}$ (respectively, an (n) -weighted smooth function vanishing at infinity on $\mathbb{R}$) is a smooth function on $\mathbb{R}$ whose multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{n/2}$ (or, equivalently, by an n^{th} -degree polynomial function on $\mathbb{R}$) is a smooth p -integrable function on $\mathbb{R}$ (respectively, is a smooth function vanishing at infinity on $\mathbb{R}$) [Horváth, 1966; Carton-Lebrun, 1995, 2005; Ortner and Wagner, 2013]. Equivalently, an (n) -weighted smooth p -integrable function on $\mathbb{R}$ (respectively, an (n) -weighted smooth function vanishing at infinity on $\mathbb{R}$) is a smooth function on $\mathbb{R}$ that can be expressed as the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-n/2}$ by a smooth p -integrable function on $\mathbb{R}$ (respectively, by a smooth function vanishing at infinity on $\mathbb{R}$) [Horváth, 1966; Carton-Lebrun, 1995, 2005; Ortner and Wagner, 2013]. In particular, when p equals 1, 2, and $+\infty$, an (n) -weighted smooth p -integrable function on $\mathbb{R}$ is said to be respectively an (n) -weighted smooth integrable function on $\mathbb{R}$ (also known as (n) -weighted smooth absolutely integrable function on $\mathbb{R}$), an (n) -weighted smooth square integrable function on $\mathbb{R}$ (also known as (n) -weighted smooth quadratically integrable function on

$\mathbb{R}$), and an (n) -weighted smooth essentially bounded function on $\mathbb{R}$ (hereinafter (n) -weighted smooth bounded function on $\mathbb{R}$). For any p_1 in $\overline{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\overline{\mathbb{R}}_{[1,+\infty]}$ with p_2 greater than p_1 , we have [Ortner and Wagner, 2013]

$$\mathcal{D}_{L^{p_2},n}(\mathbb{R}) \supset \mathcal{D}_{L^{p_1},n}(\mathbb{R})$$

Furthermore, we have [Horváth, 1966; Ortner and Wagner, 2013]

$$\mathcal{D}_{L^p}(\mathbb{R}) \supset \mathcal{D}_{L^p,n}(\mathbb{R}) \supset \mathcal{D}_{L^p,n+1}(\mathbb{R})$$

$$\dot{\mathcal{B}}(\mathbb{R}) \supset \dot{\mathcal{B}}_n(\mathbb{R}) \supset \dot{\mathcal{B}}_{n+1}(\mathbb{R})$$

Hereinafter, we occasionally need consider function spaces obtained by generalising $\mathcal{D}_{L^p,n}(\mathbb{R})$ (respectively, $\dot{\mathcal{B}}_n(\mathbb{R})$). The chain of inclusion relations holds [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]

$$\mathcal{D}_{L^p,z_1,z_2}(\mathbb{R}) \supset \dot{\mathcal{B}}_{z_1,z_2}(\mathbb{R}) \supset \mathcal{D}_{L^q,z_1,z_2}(\mathbb{R}) \supset \mathcal{D}_{L^1,z_1,z_2}(\mathbb{R})$$

with

$$\mathcal{D}_{L^p,z_1,0}(\mathbb{R}) \equiv \mathcal{D}_{L^p,z_1}(\mathbb{R})$$

$$\mathcal{D}_{L^p,0,0}(\mathbb{R}) \equiv \mathcal{D}_{L^p}(\mathbb{R})$$

$$\dot{\mathcal{B}}_{z_1,0}(\mathbb{R}) \equiv \dot{\mathcal{B}}_{z_1}(\mathbb{R})$$

$$\dot{\mathcal{B}}_{0,0}(\mathbb{R}) \equiv \dot{\mathcal{B}}(\mathbb{R})$$

where

- $\mathcal{D}_{L^p,z_1,z_2}(\mathbb{R})$ is the function space of the (z_1, z_2) -weighted smooth p -integrable functions on $\mathbb{R}$
- $\dot{\mathcal{B}}_{z_1,z_2}(\mathbb{R})$ is the function space of the (z_1, z_2) -weighted smooth functions vanishing at infinity on $\mathbb{R}$

We warn that, for $|z_1|$ even, the function spaces denoted here by $\dot{\mathcal{B}}_{z_1}(\mathbb{R})$ are denoted elsewhere by $\mathcal{S}_k(\mathbb{R})$ with $k \equiv z_1/2$ [Horváth, 1966; Ortner and Wagner, 2014].

A (z_1, z_2) -weighted smooth p -integrable function on $\mathbb{R}$ (respectively, a (z_1, z_2) -weighted smooth function vanishing at infinity on $\mathbb{R}$) is a smooth function on $\mathbb{R}$ whose multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{z_1/2} (\ln(e + \xi^2))^{z_2}$ is a smooth p -integrable function on $\mathbb{R}$ (respectively, a smooth function vanishing at infinity on $\mathbb{R}$) [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]. Equivalently, a (z_1, z_2) -weighted smooth p -integrable function on $\mathbb{R}$ (respectively, a (z_1, z_2) -weighted smooth function vanishing at infinity on $\mathbb{R}$) is a smooth function on $\mathbb{R}$ that can be expressed as the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-z_1/2} (\ln(e + \xi^2))^{-z_2}$ by a smooth p -integrable function on $\mathbb{R}$ (respectively, by a smooth function vanishing at infinity on $\mathbb{R}$) [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]. In particular, when p equals 1, 2, and $+\infty$, a (z_1, z_2) -weighted smooth p -integrable function on $\mathbb{R}$ is said to be respectively a (z_1, z_2) -weighted smooth integrable function on $\mathbb{R}$ (also known as (z_1, z_2) -weighted smooth absolutely integrable function on $\mathbb{R}$), a (z_1, z_2) -weighted smooth square integrable function on $\mathbb{R}$ (also known as (z_1, z_2) -weighted smooth quadratically integrable function on $\mathbb{R}$), and a (z_1, z_2) -weighted smooth essentially bounded function on $\mathbb{R}$ (hereinafter (z_1, z_2) -weighted smooth bounded function on $\mathbb{R}$). For any p_1 in $\overline{\mathbb{R}}_{[1, +\infty]}$ and p_2 in $\overline{\mathbb{R}}_{[1, +\infty]}$ with p_2 greater than p_1 , and for any z_1 in $\mathbb{Z}$ and z_2 in $\mathbb{Z}$, we have [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]

$$\mathcal{D}_{L^{p_2}, z_1, z_2}(\mathbb{R}) \supset \mathcal{D}_{L^{p_1}, z_1, z_2}(\mathbb{R})$$

Furthermore, we have [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]

$$\mathcal{D}_{L^p, z_1, z_2}(\mathbb{R}) \supset \mathcal{D}_{L^p, z_1+1, z_2}(\mathbb{R})$$

$$\mathcal{D}_{L^p, z_1, z_2}(\mathbb{R}) \supset \mathcal{D}_{L^p, z_1, z_2+1}(\mathbb{R})$$

$$\dot{\mathcal{B}}_{z_1, z_2}(\mathbb{R}) \supset \dot{\mathcal{B}}_{z_1+1, z_2}(\mathbb{R})$$

$$\dot{\mathcal{B}}_{z_1, z_2}(\mathbb{R}) \supset \dot{\mathcal{B}}_{z_1, z_2+1}(\mathbb{R})$$

It is worth noting that both the weight function defined almost everywhere on $\mathbb{R}$ by

$(1 + \xi^2)^{z_1/2} (\ln(e + \xi^2))^{z_2}$ and the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by

$(1 + \xi^2)^{-z_1/2} \left(\ln(e + \xi^2) \right)^{-z_2}$ are positive real-valued smooth functions on $\mathbb{R}$; their purpose is to control the behaviour of the weighted function on a neighbourhood of infinity, where they are of the same order of magnitude as respectively $|\xi|^{z_1} \left(\ln(|\xi|) \right)^{z_2}$ and $|\xi|^{-z_1} \left(\ln(|\xi|) \right)^{-z_2}$.

A smooth function of rapid decay on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ vanishes at infinity more rapidly than $|\xi|$ to the power of any real exponent [Schwartz, 1966; Horváth, 1966; Treves, 1967]. An important example of element of $\mathcal{S}(\mathbb{R})$ is the *Gaussian function*, which is the function defined almost everywhere on $\mathbb{R}$ by $e^{-\xi^2}$.

A smooth function of rapid decay of exponential type on $\mathbb{R}$ is a smooth function of rapid decay on $\mathbb{R}$ that can be extended to a holomorphic function of exponential type on $\mathbb{C}$ [Schwartz, 1966]. As a consequence of the fact that a holomorphic function on $\mathbb{C}$ either vanishes only at isolated points of $\mathbb{C}$ or vanishes at each point of $\mathbb{C}$, the intersection of $\exp \mathcal{S}(\mathbb{R})$ and $\mathcal{D}(\mathbb{R})$ contains only $0_{\mathbb{R}}$, and no subspace of $\mathcal{S}(\mathbb{R})$ that is a superspace of $\mathcal{D}(\mathbb{R})$ can be a subspace of $\exp \mathcal{S}(\mathbb{R})$ [Zemanian, 1965a; Beltrami and Wohlers, 1966; Misra and Lavoine, 1986].

A smooth function of compact support on $\mathbb{R}$ is a smooth function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ has compact support [Schwartz, 1966; Horváth, 1966; Treves, 1967]. The fact that the function space of the smooth functions of compact support on $\mathbb{R}$ contains elements other than $0_{\mathbb{R}}$ is a non-obvious classical result [Schwartz, 1966].

A.2.2.3 Continuously differentiable functions

Hereinafter, we occasionally need consider some locally integrable functions on $\mathbb{R}$ that, despite being not necessarily smooth functions on $\mathbb{R}$, have the property characterising the elements of one of the function spaces above qualified as smooth; for example, a locally integrable function of slow growth on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ that need not be a smooth function on $\mathbb{R}$ but has modulus majorised on a neighbourhood of infinity by $|\xi|$ to the power of a real constant exponent.

More specifically, we now discuss function spaces whose elements, despite being not necessarily smooth functions on $\mathbb{R}$, are smooth in a weaker sense. Let $\mathcal{D}^F(\mathbb{R})$ be the function space obtained by equipping the set of the smooth functions of compact support on $\mathbb{R}$ with the coarsest topological structure for which all the inclusions $\mathcal{D}^{n_0}(\mathbb{R}) \supset \mathcal{D}(\mathbb{R})$ underlie embedding morphisms [Horváth, 1966; Barros-Neto, 1973]. The chains of inclusion relations hold [Horváth, 1966]

$$\begin{array}{ccccc}
L^p_{\text{loc}}(\mathbb{R}) & \supset & \mathcal{E}^{n_0}(\mathbb{R}) & \supset & \mathcal{E}(\mathbb{R}) \\
& & \cup & & \cup \\
& & \dot{\mathcal{B}}^{n_0}(\mathbb{R}) & \supset & \dot{\mathcal{B}}(\mathbb{R}) \\
& & \cup & & \cup \\
& & \dot{\mathcal{B}}^{n_0}_n(\mathbb{R}) & \supset & \dot{\mathcal{B}}_n(\mathbb{R}) \\
& & \cup & & \cup \\
& & \mathcal{D}^{n_0}(\mathbb{R}) & \supset & \mathcal{D}^F(\mathbb{R}) \supseteq \mathcal{D}(\mathbb{R})
\end{array}$$

where

- $\mathcal{E}^{n_0}(\mathbb{R})$ is the function space of the C^{n_0} functions on $\mathbb{R}$ (also known as *functions of differentiability class C^{n_0} on $\mathbb{R}$* or *n_0 -time continuously differentiable functions on $\mathbb{R}$*)
- $\dot{\mathcal{B}}^{n_0}(\mathbb{R})$ is the function space of the C^{n_0} functions vanishing at infinity on $\mathbb{R}$
- $\dot{\mathcal{B}}^{n_0}_n(\mathbb{R})$ is the function space of the (n) -weighted C^{n_0} functions vanishing at infinity on $\mathbb{R}$
- $\mathcal{D}^{n_0}(\mathbb{R})$ is the function space of the C^{n_0} functions of compact support on $\mathbb{R}$

We warn that, for n even, the function spaces denoted here by $\dot{\mathcal{B}}^{n_0}$ and $\dot{\mathcal{B}}^{n_0}_n(\mathbb{R})$ are denoted elsewhere by respectively $\mathcal{S}^m_0(\mathbb{R}) \equiv \mathcal{B}^m_0(\mathbb{R})$ and $\mathcal{S}^m_k(\mathbb{R})$ with $k \equiv n/2$ and $m \equiv n_0$ [Horváth, 1966; Ortner and Wagner, 2014].

A C^{n_0} function on $\mathbb{R}$ is a Lebesgue-measurable function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}_{[0, n_0]}$ is continuous on $\mathbb{R}$. We have [Horváth, 1966]

$$\mathcal{E}^{n_0}(\mathbb{R}) \supset \mathcal{E}^{n_0+1}(\mathbb{R})$$

For a C^{n_0} function on $\mathbb{R}$, the n_0^{th} weak derivative coincides with the n_0^{th} derivative [Ambrosio et al., 2000].

A C^{n_0} function vanishing at infinity on $\mathbb{R}$ is a C^{n_0} function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}_{[0, n_0]}$ vanishes at infinity [Schwartz, 1954a; Horváth, 1966]. We have [Horváth, 1966]

$$\dot{\mathcal{B}}^{n_0}(\mathbb{R}) \supset \dot{\mathcal{B}}^{n_0+1}(\mathbb{R})$$

and, for any $p_{[1, +\infty)}$ in $\mathbb{R}_{[1, +\infty)}$, we have

$$UC(\mathbb{R}) \supset \dot{\mathcal{B}}^0 \supset \left(L^{p_{[1, +\infty)}}(\mathbb{R}) \cap UC(\mathbb{R}) \right)$$

An (n) -weighted C^{n_0} function vanishing at infinity on $\mathbb{R}$ is a C^{n_0} function on $\mathbb{R}$ whose multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{n/2}$ (or, equivalently, by an n^{th} -degree polynomial function on $\mathbb{R}$) is a C^{n_0} function vanishing at infinity on $\mathbb{R}$ [Horváth, 1966]. Equivalently, an (n) -weighted C^{n_0} function vanishing at infinity on $\mathbb{R}$ is a C^{n_0} function on $\mathbb{R}$ that can be expressed as the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-n/2}$ by a C^{n_0} function vanishing at infinity on $\mathbb{R}$ [Horváth, 1966]. We have

$$\dot{\mathcal{B}}_n^{n_0}(\mathbb{R}) \supset \dot{\mathcal{B}}_n^{n_0+1}(\mathbb{R})$$

A C^{n_0} function of compact support on $\mathbb{R}$ is a C^{n_0} function on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}_{[0, n_0]}$ has compact support [Schwartz, 1966; Horváth, 1966; Treves, 1967]. We have [Horváth, 1966]

$$\mathcal{D}^{n_0}(\mathbb{R}) \supset \mathcal{D}^{n_0+1}(\mathbb{R})$$

Finally, by adopting as codomain a Hausdorff locally convex topological linear space over $\mathbb{F}$, the concepts of smooth function on $\mathbb{R}$ and C^{n_0} function on $\mathbb{R}$ can be generalised to introduce the concepts of *vector-valued smooth function* on $\mathbb{R}$ and *vector-valued C^{n_0} function* on $\mathbb{R}$ [Schwartz, 1955, 1957, 1958; Ortner and Wagner, 2015]; in particular, when the codomain is a suitable distribution space (as introduced below), a vector-valued smooth function on $\mathbb{R}$ and a vector-valued C^{n_0} function on $\mathbb{R}$ are said to be respectively a *distribution-valued smooth function* on $\mathbb{R}$ and a *distribution-valued C^{n_0} function* on $\mathbb{R}$ [Schwartz, 1955, 1957, 1958; Łojasiewicz, 1958; Zemanian, 1965a; Ortner and Wagner, 2015].

A.2.3 Radon measures

The chain of inclusion relations holds [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967; Ambrosio *et al.*, 2000; Ortner and Wagner, 2013; Evans and Gariepy, 2015; Leoni, 2017]

$$\mathcal{M}_{\text{loc}}^1(\mathbb{R}) \supset \mathcal{M}_{L_n}^1(\mathbb{R}) \supset \mathcal{M}^1(\mathbb{R}) \supset \mathcal{M}_c(\mathbb{R})$$

where

- $\mathcal{M}_{\text{loc}}^1(\mathbb{R})$ is the Radon-measure space of the *Radon measures* on $\mathbb{R}$
- $\mathcal{M}_{L_n}^1(\mathbb{R})$ is the Radon-measure space of the $(-n)$ -weighted integrable Radon measures on $\mathbb{R}$
- $\mathcal{M}^1(\mathbb{R})$ is the Radon-measure space of the *integrable Radon measures* on $\mathbb{R}$ (also known as *finite Radon measures* on $\mathbb{R}$ or, somehow unfortunately, *bounded Radon measures* on $\mathbb{R}$)
- $\mathcal{M}_c(\mathbb{R})$ is the Radon-measure space of the *Radon measures of compact support* on $\mathbb{R}$

We warn that, to the countable union for all n in $\mathbb{N}$ of the Radon-measure spaces of the $(-n)$ -weighted integrable Radon measures on $\mathbb{R}$, reference is made in the literature, somehow unfortunately, as Radon-measure space of the *Radon measures of slow growth* on $\mathbb{R}$ (or *slowly increasing Radon measures* on $\mathbb{R}$) [Schwartz, 1966; Argabright and Gil de Lamadrid, 1974; Baake and Grimm, 2013; Baake and Strungaru, 2023].

A.2.3.1 Characterisation

A Radon measure on $\mathbb{R}$ is a map from the collection of the relatively compact Borel-measurable subsets of $\mathbb{R}$ into $\mathbb{F}$ having the properties of countable additivity, inner regularity, and local finiteness (the last two properties requiring some technical adjustments to reconcile the concept of Radon measure on $\mathbb{R}$ with the concept of non-negative real-valued Radon measure on a Hausdorff Borel-measurable space) [Schwartz, 1973]. We say that the image in $\mathbb{F}$ of a relatively compact Borel-measurable subset of $\mathbb{R}$ (in particular, of $\mathbb{R}$) under a Radon measure on $\mathbb{R}$ is the *integral* on the relatively compact Borel-measurable subset of $\mathbb{R}$ (in particular, on $\mathbb{R}$) of the Radon measure on $\mathbb{R}$. As a consequence of its definition, a Radon measure on $\mathbb{R}$ is locally integrable on $\mathbb{R}$ in the sense that it is finite on each compact subset of $\mathbb{R}$ or, equivalently, that it has finite integral on each compact subset of $\mathbb{R}$ [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967; Ambrosio *et al.*, 2000; Ortner and Wagner,

2013; Evans and Gariepy, 2015; Leoni, 2017]. The multiplication of a Radon measure on $\mathbb{R}$ by any finite set of C^0 functions on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a Radon measure on $\mathbb{R}$ [Schwartz, 1966; Bourbaki, 2004b]. The *support of a Radon measure* on $\mathbb{R}$ is the smallest closed subset of $\mathbb{R}$ outside of which the Radon measure on $\mathbb{R}$ vanishes (or, equivalently, the complement of the largest open subset of $\mathbb{R}$ on which the Radon measure on $\mathbb{R}$ vanishes) [Schwartz, 1966, 1973]. Furthermore, the *modulus* (also known as *total variation*) of a Radon measure on $\mathbb{R}$ (which is itself a non-negative real-valued Radon measure on $\mathbb{R}$) is a map from the collection of the relatively compact Borel-measurable subsets of $\mathbb{R}$ into $\mathbb{R}_{[0,+\infty)}$ under which the image of a relatively compact Borel-measurable subset of $\mathbb{R}$ (in particular, of $\mathbb{R}$), to which we refer as the modulus of the Radon measure thereon, is the supremum of the sum of the moduli of the integrals of the Radon measure on $\mathbb{R}$ on the elements of a partition of the relatively compact Borel-measurable subset of $\mathbb{R}$ as the partition varies on it [Folland, 1999; Ambrosio *et al.*, 2000].

It is possible to distinguish between a *complex-valued Radon measure* on $\mathbb{R}$ and a *real-valued Radon measure* (also known as *signed Radon measure*) on $\mathbb{R}$ (and, in turn, between a *non-negative real-valued Radon measure* on $\mathbb{R}$ and a *non-positive real-valued Radon measure* on $\mathbb{R}$) [Schwartz, 1973]. A Radon measure on $\mathbb{R}$ is a real-valued Radon measure on $\mathbb{R}$ if and only if it can be expressed as the difference of two non-negative real-valued Radon measures on $\mathbb{R}$ [Schwartz, 1966; Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Leoni, 2017]. A Radon measure on $\mathbb{R}$ that is not real-valued cannot be non-negative (respectively, non-positive). However, we say that a Radon measure on $\mathbb{R}$ is *weakly non-negative* (respectively, *weakly non-positive*) if and only if its real part is non-negative (respectively, non-positive); as is evident, a Radon measure on $\mathbb{R}$ is non-negative (respectively, non-positive) only if it is weakly non-negative (respectively, weakly non-positive). Hereinafter, unless otherwise stated, a Radon measure is a complex-valued Radon measure on $\mathbb{R}$.

It is possible to introduce for Radon measures the concepts of *even parity* and *odd parity*, *conjugate symmetry* (also known as *Hermitian symmetry*) and *conjugate antisymmetry* (also known as *Hermitian antisymmetry*), *homogeneity*, and *almost periodicity* (in particular, *periodicity*), related to which are the concepts of *almost period* (in particular, *period*) and *mean value* on $\mathbb{R}$ [Berg and Forst, 1975; Gil de Lamadrid and Argabright, 1990; Moody and Strungaru, 2017; Spindeler and Strungaru, 2019; Strungaru, 2020].

An integrable Radon measure on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ that is finite on $\mathbb{R}$ or, equivalently, that has finite integral on $\mathbb{R}$ [Schwartz, 1966, 1973; Horváth, 1966; Ambrosio *et al.*, 2000; Ortner and Wagner, 2013; Evans and Gariepy, 2015; Leoni, 2017]. Because the modulus is a norm on

$\mathcal{M}^1(\mathbb{R})$, we refer to it as *modulus norm* when it equips $\mathcal{M}^1(\mathbb{R})$ [Bourbaki, 2004b]. A Radon measure on $\mathbb{R}$ is a real-valued integrable Radon measure on $\mathbb{R}$ if and only if it can be expressed as the difference of two non-negative real-valued integrable Radon measures on $\mathbb{R}$ [Schwartz, 1966; Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Leoni, 2017].

A $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ whose multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-n/2}$ is an integrable Radon measure on $\mathbb{R}$ [Schwartz, 1966; Kabanava, 2008; Ortner and Wagner, 2013; Baake and Strungaru, 2023]. Equivalently, a $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ that can be expressed as the multiplicative product of an integrable Radon measure on $\mathbb{R}$ by the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{n/2}$ (or, equivalently, by an n^{th} -degree polynomial function on $\mathbb{R}$) [Schwartz, 1966; Kabanava, 2008; Ortner and Wagner, 2013; Baake and Strungaru, 2023]. An element of $\mathcal{P}_L(\mathbb{R})$ generates an element (absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, as specified below) of any $\mathcal{M}_{L_n}^1(\mathbb{R})$ with n greater than some minimum value. We have

$$\mathcal{M}_{L_{n-1}}^1(\mathbb{R}) \supset \mathcal{M}_{L_n}^1(\mathbb{R})$$

A Radon measure of compact support on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ that has compact support [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967]. A special case of Radon measure of compact support on $\mathbb{R}$ is a *point-supported Radon measure* on $\mathbb{R}$ (namely, a Radon measure on $\mathbb{R}$ having support in a point of $\mathbb{R}$).

We mention in passing that other Radon-measure spaces can be found in the literature. Let us consider the Radon-measure space of the Radon measures on $\mathbb{R}$, which is homologous to the distribution space of the distributions on $\mathbb{R}$ (as introduced below). A subspace of the latter Radon-measure space is the Radon-measure space of the *tempered Radon measures* on $\mathbb{R}$ (also known as *temperate Radon measures* on $\mathbb{R}$), which is denoted by $\mathcal{M}_t(\mathbb{R})$ and is homologous to the distribution space of the tempered distributions on $\mathbb{R}$ (as introduced below) [Schwartz, 1966; Argabright and Gil de Lamadrid, 1974; Lagarias, 2000; Kabanava, 2008; Baake and Grimm, 2013; Strungaru and Terauds, 2016; Moody and Strungaru, 2017; Spindeler and Strungaru, 2019; Strungaru, 2020; Baake and Strungaru, 2023]. A subspace of the latter Radon-measure space is the Radon-measure space of the Radon measures of slow growth on $\mathbb{R}$, which is denoted by $\mathcal{M}_{L_\infty}^1(\mathbb{R})$ and is homologous to the countable union for all n in $\mathbb{N}$ of the distribution spaces of the

$(-n)$ -weighted integrable distributions on $\mathbb{R}$ (as introduced below) [Schwartz, 1966; Argabright and Gil de Lamadrid, 1974; Baake and Grimm, 2013; Baake and Strungaru, 2023]. A subspace of the latter Radon-measure space is the Radon-measure space of the *translation-bounded Radon measures* on $\mathbb{R}$ (also known as *shift-bounded Radon measures* on $\mathbb{R}$), which is denoted by $\mathcal{M}^\infty(\mathbb{R})$ and is homologous to the distribution space of the bounded distributions on $\mathbb{R}$ (as introduced below) [Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Gil de Lamadrid and Argabright, 1990; Lagarias, 2000; Baake and Grimm, 2013; Strungaru and Terauds, 2016; Moody and Strungaru, 2017; Spindeler and Strungaru, 2019; Strungaru, 2020]. A subspace of the latter Radon-measure space is the Radon-measure space of the *Radon measures vanishing at infinity* on $\mathbb{R}$, which is denoted by $\mathcal{M}_0(\mathbb{R})$ and is homologous to the distribution space of the distributions vanishing at infinity on $\mathbb{R}$ (as introduced below) [Berg and Forst, 1975; Spindeler and Strungaru, 2019]. A subspace of the latter Radon-measure space is the Radon-measure space of the integrable Radon measures on $\mathbb{R}$, which is denoted by $\mathcal{M}^1(\mathbb{R})$ and is homologous to the distribution space of the integrable distributions on $\mathbb{R}$ (as introduced below) [Berg and Forst, 1975; Spindeler and Strungaru, 2019]. A subspace of the latter Radon-measure space is the Radon-measure space of the Radon measures of compact support on $\mathbb{R}$, which is denoted by $\mathcal{M}_c(\mathbb{R})$ and is homologous to the distribution space of the distributions of compact support on $\mathbb{R}$ (as introduced below) [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967]. Importantly, although an element of a Radon-measure space generates an element of the homologous distribution space (as introduced below), an element of a distribution space (as introduced below) that is generable by a Radon measure on $\mathbb{R}$ need not be generable by an element of the homologous Radon-measure space (exceptions being any element of the distribution space of the distributions on $\mathbb{R}$ and any element of the distribution space of the distributions of compact support on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Bijma, 1999; Thomas, 2000]). Finally, a terminological note is in order. To the Radon-measure space of the translation-bounded Radon measures on $\mathbb{R}$, reference could legitimately be made as the Radon-measure space of the bounded Radon measures on $\mathbb{R}$ if it were not for the fact that this would cause confusion with the Radon-measure space of the integrable Radon measures on $\mathbb{R}$ (to which, somehow unfortunately, reference is often made precisely as the Radon-measure space of the bounded Radon measures on $\mathbb{R}$).

The Lebesgue measure on $\mathbb{R}$ and a Dirac measure at a point of $\mathbb{R}$ are Radon measures on $\mathbb{R}$ [Schwartz, 1966, 1973; Treves, 1967; Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013; Leoni, 2017]. A Radon measure on $\mathbb{R}$ is said to be *atomic*, or *pure-point*, (in particular, *discrete*) if and only if it is a possibly infinite but convergent (in norm) linear combination of (necessarily countably many) Dirac measures at the points of a subset of $\mathbb{R}$ of which countably

many (in particular, finitely many) are contained within any compact subset of $\mathbb{R}$, the partial sum of the moduli of the coefficients of the linear combination being finite for any possibly infinite set of terms corresponding to Dirac measures at the points of a subset of $\mathbb{R}$ included in a compact subset of $\mathbb{R}$ [Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013; Leoni, 2017; Strungaru, 2020]. A Radon measure on $\mathbb{R}$ is atomic if (but not only if) its support is a discrete closed subset of $\mathbb{R}$ [Benedetto, 1997; Bourbaki, 2004b]; for example, an atomic Radon measure on $\mathbb{R}$ that is an infinite but convergent (in norm) linear combination of (necessarily countably many) Dirac measures at the points of $\mathbb{Q}$ has $\mathbb{R}$ itself as support [Benedetto, 1997]. Nevertheless, a Radon measure on $\mathbb{R}$ is discrete if and only if its support is a discrete closed subset of $\mathbb{R}$ [Bourbaki, 2004b]. An integrable Radon measure on $\mathbb{R}$ is atomic (in particular, discrete) if and only if it is a possibly infinite but convergent (in norm) linear combination of (necessarily countably many) Dirac measures at the points of a subset of $\mathbb{R}$ of which only countably many (in particular, finitely many) are contained within any compact subset of $\mathbb{R}$, the sum of the moduli of the coefficients of the linear combination being finite [Benedetto, 1997; Folland, 1999; Ambrosio *et al.*, 2000]. We warn that, in the literature, by discrete Radon measure on $\mathbb{R}$ is sometimes meant what is meant here by atomic Radon measure on $\mathbb{R}$ [Gil de Lamadrid and Argabright, 1990] or integrable atomic Radon measure on $\mathbb{R}$ [Benedetto, 1997; Folland, 1999; Katznelson, 2004]. A Radon measure on $\mathbb{R}$ is said to be *continuous* (or *non-atomic*, or *diffuse*) if and only if its modulus vanishes on each singleton subset of $\mathbb{R}$ [Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013; Leoni, 2017]. Each Radon measure on $\mathbb{R}$ (in particular, integrable Radon measure on $\mathbb{R}$) is either atomic or continuous, and it can be expressed uniquely as the sum of an *atomic part* (in particular, *discrete part*) and a *continuous part* (also known as *non-atomic part* or *diffuse part*) [Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013].

A Radon measure on $\mathbb{R}$ is said to be *singular* with respect to the Lebesgue measure on $\mathbb{R}$ if and only if its modulus vanishes on the complement of a Lebesgue-null subset of $\mathbb{R}$ [Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013; Evans and Gariepy, 2015; Leoni, 2017]. A Radon measure on $\mathbb{R}$ is said to be *absolutely continuous* with respect to the Lebesgue measure on $\mathbb{R}$ if and only if its modulus vanishes on each Lebesgue-null subset of $\mathbb{R}$ [Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013; Evans and Gariepy, 2015; Leoni, 2017]. With respect to the Lebesgue measure on $\mathbb{R}$, each Radon measure on $\mathbb{R}$ is either singular or absolutely continuous, and it can be expressed uniquely as the sum of a *singular part* and an *absolutely continuous part* [Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013; Evans and Gariepy, 2015; Leoni, 2017]. According to the *Radon-Nikodym theorem*, the integral on a compact subset of $\mathbb{R}$ of a Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ is the

integral on the compact subset of $\mathbb{R}$ of a locally integrable function on $\mathbb{R}$ that is said to be the *density* (also known as *Radon-Nikodym derivative*) of the Radon measure on $\mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$ [Schwartz, 1966; Folland, 1999; Ambrosio *et al.*, 2000; Bourbaki, 2004b; Baake and Grimm, 2013; Evans and Gariepy, 2015; Leoni, 2017].

A Radon measure on $\mathbb{R}$ that is atomic (and hence is not continuous) is singular with respect to the Lebesgue measure on $\mathbb{R}$, and a Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ is continuous (and hence is not atomic) [Folland, 1999; Ambrosio *et al.*, 2000; Baake and Grimm, 2013]. According to the *Lebesgue decomposition theorem*, with respect to the Lebesgue measure on $\mathbb{R}$, each Radon measure on $\mathbb{R}$ (in particular, integrable Radon measure on $\mathbb{R}$) can be expressed uniquely as the sum of an *atomic part*, a *singular continuous part*, and an *absolutely continuous part* [Folland, 1999; Ambrosio *et al.*, 2000; Baake and Grimm, 2013].

As for functions, the concept of *non-negative-definiteness* can be introduced for Radon measures [Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Staffans, 1976; Gil de Lamadrid and Argabright, 1990; Gripenberg *et al.*, 1990; Lagarias, 2000; Baake and Grimm, 2013; Moody and Strungaru, 2017]. A Radon measure on $\mathbb{R}$ is said to be *non-negative-definite* (or *positive-semidefinite*, or *of non-negative type*) if and only if the integration on $\mathbb{R}$ of its multiplicative product by the auto-correlation function of any smooth function of compact support on $\mathbb{R}$ (in fact, of any smooth function of rapid decay on $\mathbb{R}$) results in a finite non-negative real value [Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Staffans, 1976; Gripenberg *et al.*, 1990; Baake and Grimm, 2013; Moody and Strungaru, 2017]. Unlike a finite conical combination, a finite linear combination of non-negative-definite Radon measures on $\mathbb{R}$ (respectively, of non-negative real-valued Radon measures on $\mathbb{R}$) need not be a non-negative-definite Radon measure on $\mathbb{R}$ (respectively, a non-negative real-valued Radon measure on $\mathbb{R}$) [Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Gripenberg *et al.*, 1990; Baake and Grimm, 2013; Moody and Strungaru, 2017]; hence, the Radon-measure space of the non-negative-definite Radon measures on $\mathbb{R}$ (respectively, of the non-negative real-valued Radon measures on $\mathbb{R}$) cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$. Each non-negative-definite C^0 function on $\mathbb{R}$ generates a non-negative-definite Radon measure on $\mathbb{R}$ [Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Staffans, 1976; Gripenberg *et al.*, 1990]. Furthermore, each non-negative-definite non-negative real-valued Radon measure on $\mathbb{R}$ is a conjugate-symmetric element of $\mathcal{M}^\infty(\mathbb{R})$ [Berg and Forst, 1975; Argabright and Gil de Lamadrid, 1974; Lagarias, 2000; Baake and Grimm, 2013; Moody and Strungaru, 2017]; more generally, each non-negative-definite Radon measure on $\mathbb{R}$ generates a conjugate-symmetric element of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (as introduced below) [Schwartz, 1966; Argabright and Gil

de Lamadrid, 1974; Berg and Forst, 1975; Staffans, 1976; Vladimirov, 2002; Strungaru and Terauds, 2016]. A Radon measure on $\mathbb{R}$ that is not conjugate-symmetric (in particular, that is either causal or anticausal) cannot be non-negative-definite. However, we say that a Radon measure on $\mathbb{R}$ is *weakly non-negative-definite* (or *weakly positive-semidefinite*, or of *weak non-negative type*) if and only if its conjugate-symmetric part is non-negative-definite [Berg and Forst, 1975; Staffans, 1976; Gripenberg *et al.*, 1990]; as is evident, a Radon measure on $\mathbb{R}$ is non-negative-definite only if it is weakly non-negative-definite. A real-valued Radon measure on $\mathbb{R}$ is weakly non-negative-definite if and only if the integration on $\mathbb{R}$ of its multiplicative product by the auto-correlation function of any real-valued smooth function of compact support on $\mathbb{R}$ (in fact, of any real-valued smooth function of rapid decay on $\mathbb{R}$) results in a finite non-negative real value [Berg and Forst, 1975].

A.2.3.2 Associated functions

Each Radon measure on $\mathbb{R}$ is generable by the 1st weak derivative of some locally integrable function on $\mathbb{R}$ (in fact, by the 1st weak derivative of a function of locally bounded variation on $\mathbb{R}$, as introduced below) [Schwartz, 1962, 1966; Horváth, 1966; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]; we say that the Radon measure on $\mathbb{R}$ is associated with such locally integrable function on $\mathbb{R}$. In particular, if the Radon measure on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, the 1st weak derivative of the locally integrable function on $\mathbb{R}$ with which the Radon measure on $\mathbb{R}$ is associated is a locally integrable function on $\mathbb{R}$ that coincides with the density of the Radon measure on $\mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$ [Schwartz, 1966; Ambrosio *et al.*, 2000; Leoni, 2017]. Accordingly, each Radon measure on $\mathbb{R}$ (in particular, integrable Radon measure on $\mathbb{R}$) that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ is generable by a locally integrable function on $\mathbb{R}$ (in particular, by an integrable function on $\mathbb{R}$), which coincides with the density of the Radon measure on $\mathbb{R}$ with respect to the Lebesgue measure on $\mathbb{R}$ [Schwartz, 1966]; inversely, each locally integrable function on $\mathbb{R}$ (in particular, integrable function on $\mathbb{R}$) generates a Radon measure on $\mathbb{R}$ (in particular, an integrable Radon measure on $\mathbb{R}$) that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, of which it is the density with respect to the Lebesgue measure on $\mathbb{R}$ [Schwartz, 1966].

To clarify the matter, we now introduce some function spaces whose characterisations are related to the concept of Radon measure. The chains of inclusion relations hold [Schwartz, 1966; Treves, 1967; Benedetto, 1997; Ambrosio *et al.*, 2000; Taylor, 2006; Evans and Gariepy, 2015; Leoni, 2017; Aslan and Duman, 2020]

$$\begin{array}{ccccccc}
 L^1_{\text{loc}}(\mathbb{R}) & & & & & & \\
 \cup & & & & & & \\
 L^\infty_{\text{loc}}(\mathbb{R}) & \supset & BV_{\text{loc}}(\mathbb{R}) & \supset & AC_{\text{loc}}(\mathbb{R}) \equiv W^{1,1}_{\text{loc}}(\mathbb{R}) \supset W^{n_2,1}_{\text{loc}}(\mathbb{R}) \\
 & & & & \cup & & \\
 & & \cup & (BV(\mathbb{R}) \cap AC_{\text{loc}}(\mathbb{R})) & AC(\mathbb{R}) & & \\
 & & & \text{III} & \cup & \cup & \cup \\
 L^\infty(\mathbb{R}) & \supset & BV(\mathbb{R}) \supset (BV(\mathbb{R}) \cap AC(\mathbb{R})) \equiv AC_{L^\infty}(\mathbb{R}) & & & & \\
 \cup & & \cup & \cup & \cup & & \\
 (L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})) \supset (L^1(\mathbb{R}) \cap BV(\mathbb{R})) \equiv BV_{L^1}(\mathbb{R}) \supset (BV_{L^1}(\mathbb{R}) \cap AC_{L^\infty}(\mathbb{R})) \equiv AC_{L^1}(\mathbb{R}) \equiv W^{1,1}(\mathbb{R}) \supset W^{n_2,1}(\mathbb{R}) \\
 & & & & & & \cup \\
 & & & & & & \mathcal{D}_{L^1}(\mathbb{R})
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{E}^0(\mathbb{R}) \supset AC_{\text{loc}}(\mathbb{R}) \supset \mathcal{E}^1(\mathbb{R}) \\
 \cup \quad \quad \cup \\
 UC(\mathbb{R}) \supset AC(\mathbb{R}) \\
 \cup \quad \quad \cup \\
 \dot{\mathcal{B}}^0 \supset AC_{L^1}(\mathbb{R})
 \end{array}$$

where

- $W_{\text{loc}}^{n_0,p}(\mathbb{R})$ is the function space of the *locally Sobolev functions of order n_0 and exponent p* on $\mathbb{R}$
- $W^{n_0,p}(\mathbb{R})$ is the function space of the *Sobolev functions of order n_0 and exponent p* on $\mathbb{R}$
- $BV_{\text{loc}}(\mathbb{R})$ is the function space of the *functions of locally bounded variation* on $\mathbb{R}$
- $BV(\mathbb{R})$ is the function space of the *functions of bounded variation* on $\mathbb{R}$
- $BV_{L^1}(\mathbb{R})$ is the function space of the *integrable functions of bounded variation* on $\mathbb{R}$
- $AC_{\text{loc}}(\mathbb{R})$ is the function space of the *locally absolutely continuous functions* on $\mathbb{R}$
- $AC(\mathbb{R})$ is the function space of the *absolutely continuous functions* on $\mathbb{R}$
- $AC_{L^\infty}(\mathbb{R})$ is the function space of the *bounded absolutely continuous functions* on $\mathbb{R}$
- $AC_{L^1}(\mathbb{R})$ is the function space of the *integrable absolutely continuous functions* on $\mathbb{R}$

A locally Sobolev function of order n_0 and exponent p on $\mathbb{R}$ is a locally p -integrable function on $\mathbb{R}$ whose each weak derivative of any order in $\mathbb{N}_{[0,n_0]}$ is a locally p -integrable function on $\mathbb{R}$ [Treves, 1967; Leoni, 2017]. We have [Ambrosio *et al.*, 2000; Taylor, 2006; Leoni, 2017]

$$W_{\text{loc}}^{0,p}(\mathbb{R}) \equiv L_{\text{loc}}^p(\mathbb{R})$$

$$W_{\text{loc}}^{n_0,p}(\mathbb{R}) \supset W_{\text{loc}}^{n_0+1,p}(\mathbb{R})$$

$$(\mathcal{E}^0(\mathbb{R}) \cap L_{\text{loc}}^\infty(\mathbb{R})) \supset W_{\text{loc}}^{n_0,p}(\mathbb{R})$$

A Sobolev function of order n_0 and exponent p on $\mathbb{R}$ is a p -integrable function on $\mathbb{R}$ whose each weak derivative of any order in $\mathbb{N}_{[0,n_0]}$ is a p -integrable function on $\mathbb{R}$ [Treves, 1967; Leoni, 2017]. Importantly, $W^{n_0,p}(\mathbb{R})$ (in particular, $W^{n_0,2}(\mathbb{R})$) can be canonically made a Banach space over $\mathbb{F}$ (in particular, a Hilbert space over $\mathbb{F}$), $W^{n_0,q}(\mathbb{R})$ being a reflexive space [Treves, 1967; Ambrosio *et al.*, 2000; Leoni, 2017]. We have [Ambrosio *et al.*, 2000; Taylor, 2006; Leoni, 2017]

$$W^{0,p}(\mathbb{R}) \equiv L^p(\mathbb{R})$$

$$W^{n_0,p}(\mathbb{R}) \supset W^{n_0+1,p}(\mathbb{R})$$

$$\left(\mathcal{E}^0(\mathbb{R}) \cap L^\infty(\mathbb{R})\right) \supset W^{n_0,p}(\mathbb{R})$$

A function of locally bounded variation on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ whose 1st weak derivative generates a Radon measure on $\mathbb{R}$ [Schwartz, 1962, 1966; Horváth, 1966; Benedetto, 1997; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]; in fact, the 1st weak derivative of each function of locally bounded variation on $\mathbb{R}$ generates a Radon measure on $\mathbb{R}$, and each Radon measure on $\mathbb{R}$ is generable by the 1st weak derivative of a function of locally bounded variation on $\mathbb{R}$ [Schwartz, 1962, 1966; Horváth, 1966; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]. Each representative of a function of locally bounded variation on $\mathbb{R}$ is differentiable almost everywhere on $\mathbb{R}$ [Leoni, 2017]. Furthermore, a function of locally bounded variation on $\mathbb{R}$ is a real-valued function of locally bounded variation on $\mathbb{R}$ if and only if it can be expressed as the difference of two non-decreasing (or, equivalently, non-increasing) real-valued locally bounded functions on $\mathbb{R}$, and, in fact, $BV_{\text{loc}}(\mathbb{R})$ is the smallest function space that contains all the *monotonic* (or *monotone*, namely, either non-increasing or non-decreasing) elements of $L^\infty_{\text{loc}}(\mathbb{R})$ [Ambrosio *et al.*, 2000; Benedetto and Czaja, 2009; Leoni, 2017]. In particular, the 1st weak derivative of each non-decreasing (respectively, non-increasing) real-valued locally bounded function on $\mathbb{R}$ generates a non-negative (respectively, non-positive) real-valued Radon measure on $\mathbb{R}$, and each non-negative (respectively, non-positive) real-valued Radon measure on $\mathbb{R}$ is generable by the 1st weak derivative of a non-decreasing (respectively, non-increasing) real-valued locally bounded function on $\mathbb{R}$ [Schwartz, 1966]. In loose terms, a function of locally bounded variation on $\mathbb{R}$ is a locally bounded function on $\mathbb{R}$ that has graph with arc length not too large (and hence that does not oscillate too rapidly) on any compact subset of $\mathbb{R}$ [Benedetto, 1997; Ambrosio *et al.*, 2000; Benedetto and Czaja, 2009; Evans and Gariepy, 2015; Leoni, 2017].

A function of bounded variation on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ whose 1st weak derivative generates an integrable Radon measure on $\mathbb{R}$ [Taylor, 2006; Benedetto, 1997; Benedetto and Czaja, 2009; Leoni, 2017]; in fact, the 1st weak derivative of each function of bounded variation on $\mathbb{R}$ generates an integrable Radon measure on $\mathbb{R}$, and each integrable Radon measure on $\mathbb{R}$ is generable by the 1st weak derivative of a function of bounded variation on $\mathbb{R}$ [Taylor, 2006; Benedetto and Czaja, 2009; Leoni, 2017]. Furthermore, a function of bounded variation on $\mathbb{R}$ is a real-valued function of bounded variation on $\mathbb{R}$ if and only if it can be expressed as the difference of two non-decreasing (or, equivalently, non-increasing) real-valued bounded functions on $\mathbb{R}$, and, in fact, $BV(\mathbb{R})$ is the smallest function space that contains all the monotonic elements of $L^\infty(\mathbb{R})$ [Benedetto and Czaja, 2009; Leoni, 2017]. In particular, the 1st weak derivative of each

non-decreasing (respectively, non-increasing) real-valued bounded function on $\mathbb{R}$ generates a non-negative (respectively, non-positive) real-valued integrable Radon measure on $\mathbb{R}$, and each non-negative (respectively, non-positive) real-valued integrable Radon measure on $\mathbb{R}$ is generable by the 1st weak derivative of a non-decreasing (respectively, non-increasing) real-valued bounded function on $\mathbb{R}$ [Benedetto and Czaja, 2009]. In loose terms, a function of bounded variation on $\mathbb{R}$ is a bounded function on $\mathbb{R}$ that has graph with arc length not too large (and hence that does not oscillate too rapidly) on $\mathbb{R}$.

An integrable function of bounded variation on $\mathbb{R}$ is an integrable function on $\mathbb{R}$ whose 1st weak derivative generates an integrable Radon measure on $\mathbb{R}$ (necessarily with null integral on $\mathbb{R}$) [Schwartz, 1966; Horváth, 1966; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]; in fact, the 1st weak derivative of each integrable function of bounded variation on $\mathbb{R}$ generates an integrable Radon measure on $\mathbb{R}$ with null integral on $\mathbb{R}$, and each integrable Radon measure on $\mathbb{R}$ with null integral on $\mathbb{R}$ is generable by the 1st weak derivative of an integrable function of bounded variation on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]. In loose terms, an integrable function of bounded variation on $\mathbb{R}$ is an integrable function on $\mathbb{R}$ that has graph with arc length not too large (and hence that does not oscillate too rapidly) on $\mathbb{R}$.

A locally absolutely continuous function on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ whose 1st weak derivative is a locally integrable function on $\mathbb{R}$ or, equivalently, whose 1st weak derivative generates a Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ [Schwartz, 1966; Ambrosio *et al.*, 2000; Leoni, 2017]; in fact, the 1st weak derivative of each locally absolutely continuous function on $\mathbb{R}$ generates a Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, and each Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ is generable by the 1st weak derivative of a locally absolutely continuous function on $\mathbb{R}$ [Schwartz, 1966; Ambrosio *et al.*, 2000; Leoni, 2017].

We omit the characterisation of an absolutely continuous function on $\mathbb{R}$, which is technical and dispensable for our purposes [Taylor, 2006; Leoni, 2017].

A bounded absolutely continuous function on $\mathbb{R}$ is a locally integrable function on $\mathbb{R}$ whose 1st weak derivative is an integrable function on $\mathbb{R}$ or, equivalently, whose 1st weak derivative generates an integrable Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ [Taylor, 2006; Leoni, 2017]; in fact, the 1st weak derivative of each bounded absolutely continuous function on $\mathbb{R}$ generates an integrable Radon measure on $\mathbb{R}$ that is absolutely continuous

with respect to the Lebesgue measure on $\mathbb{R}$, and each integrable Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ is generable by the 1st weak derivative of a bounded absolutely continuous function on $\mathbb{R}$ [Taylor, 2006; Leoni, 2017].

An integrable absolutely continuous function on $\mathbb{R}$ is an integrable function on $\mathbb{R}$ whose 1st weak derivative is an integrable function on $\mathbb{R}$ (necessarily with null integral on $\mathbb{R}$) or, equivalently, whose 1st weak derivative generates an integrable Radon measure on $\mathbb{R}$ (necessarily with null integral on $\mathbb{R}$) that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ [Schwartz, 1966; Ambrosio *et al.*, 2000; Leoni, 2017]; in fact, the 1st weak derivative of each absolutely continuous function on $\mathbb{R}$ generates an integrable Radon measure on $\mathbb{R}$ with null integral on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, and each integrable Radon measure on $\mathbb{R}$ with null integral on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ is generable by the 1st weak derivative of an absolutely continuous function on $\mathbb{R}$ [Schwartz, 1966; Ambrosio *et al.*, 2000; Leoni, 2017].

Let φ be a locally integrable function on $\mathbb{R}$ of ξ . Remarkably, φ is a locally absolutely continuous function on $\mathbb{R}$ if and only if it is a C^0 function on $\mathbb{R}$, is a function of locally bounded variation on $\mathbb{R}$ (or, equivalently, has continuous representative that is differentiable almost everywhere on $\mathbb{R}$ and has 1st weak derivative that is a locally integrable function on $\mathbb{R}$), and satisfies the Lebesgue's form of the *fundamental theorem of calculus* expressed by the relation

$$\varphi(\xi) - \varphi(\xi_0) = \int_{\xi_0}^{\xi} (\varphi^{(1)}) d\xi'$$

where

- $\varphi^{(1)}$ is the 1st weak derivative of φ
- ξ' is a bound real variable of the same quantity dimension as ξ

(or, equivalently, has the *Luzin N property*, namely, its representatives map Lebesgue-null subsets of $\mathbb{R}$ into Lebesgue-null subsets of $\mathbb{R}$) [Gordon, 1994; Ambrosio *et al.*, 2000; Leoni, 2017]. As a standard counterexample, the function on $\mathbb{R}$ coinciding with $0_{\mathbb{R}}$ on $\mathbb{R}_{(-\infty, 0)}$, with the *Cantor ternary function* (also known as *Devil's staircase*) on $\mathbb{R}_{(0, 1)}$, and with $1_{\mathbb{R}}$ on $\mathbb{R}_{(1, +\infty)}$ is a non-decreasing C^0 (in fact, uniformly continuous) function of locally bounded variation on $\mathbb{R}$ that does not have the Luzin N property, and hence it is not a locally absolutely continuous function on $\mathbb{R}$ [Ambrosio *et al.*, 2000; Leoni, 2017].

Let φ_1 and φ_2 be two locally integrable functions on $\mathbb{R}$ of ξ . Remarkably, φ_1 and φ_2 are locally absolutely continuous functions on $\mathbb{R}$ only if they have continuous representatives that satisfy the *integration by parts theorem* expressed by the relation [Leoni, 2017]

$$\varphi_1(\xi)\varphi_2(\xi) - \varphi_1(\xi_0)\varphi_2(\xi_0) = \int_{\xi_0}^{\xi} (\varphi_1 \varphi_2^{(1)}) d\xi' + \int_{\xi_0}^{\xi} (\varphi_1^{(1)} \varphi_2) d\xi'$$

where

- $\varphi_1^{(1)}$ is the 1st weak derivative of φ_1
- $\varphi_2^{(1)}$ is the 1st weak derivative of φ_2

According to the Lebesgue decomposition theorem, each function of locally bounded variation on $\mathbb{R}$ (in particular, function of bounded variation on $\mathbb{R}$) can be expressed uniquely up to addition by complex constants as the sum of a piecewise-constant function of locally bounded variation on $\mathbb{R}$ (in particular, piecewise-constant function of bounded variation on $\mathbb{R}$) that is not a C^0 function on $\mathbb{R}$, a C^0 function of locally bounded variation on $\mathbb{R}$ (in particular, C^0 function of bounded variation on $\mathbb{R}$) that is not a locally absolutely continuous function on $\mathbb{R}$ (in particular, not a bounded absolutely continuous function on $\mathbb{R}$) and hence does not have the Luzin N property, and a locally absolutely continuous function on $\mathbb{R}$ (in particular, a bounded absolutely continuous function on $\mathbb{R}$) [Benedetto, 1997; Taylor, 2006; Benedetto and Czaja, 2009; Leoni, 2017]; furthermore, each function of integrable bounded variation on $\mathbb{R}$ can be expressed uniquely as the sum of a piecewise-constant function of integrable bounded variation on $\mathbb{R}$ that is not a C^0 function on $\mathbb{R}$, a C^0 function of integrable bounded variation on $\mathbb{R}$ that is not an integrable absolutely continuous function on $\mathbb{R}$ and hence does not have the Luzin N property, and an integrable absolutely continuous function on $\mathbb{R}$ [Ambrosio *et al.*, 2000]. The three terms of the aforementioned sum are said to be respectively the *jump part*, the *singular continuous part* (also known as *Cantor part*), and the *absolutely continuous part* of the function of locally bounded variation on $\mathbb{R}$ (in particular, function of bounded variation on $\mathbb{R}$, possibly integrable) [Ambrosio *et al.*, 2000; Leoni, 2017]; correspondingly, with respect to the Lebesgue measure on $\mathbb{R}$, the atomic part, the singular continuous part, and the absolutely continuous part of each Radon measure on $\mathbb{R}$ (in particular, integrable Radon measure on $\mathbb{R}$, possibly with null integral on $\mathbb{R}$) is generable by the 1st weak derivatives of respectively the jump part, the singular continuous part, and the absolutely continuous part of a function of locally bounded variation on $\mathbb{R}$ (in particular, function of bounded variation on $\mathbb{R}$, possibly integrable) [Benedetto, 1997; Folland, 1999; Ambrosio *et al.*, 2000; Benedetto and Czaja, 2009; Leoni, 2017].

A.2.4 Distributions

Within the *Schwartz–Sobolev theory of distributions*, we adopt the *classical approach* (also known as *functional approach*), other possibilities including the *sequential approach* and the *differential approach* (also known as *axiomatic approach*) [Sikorski, 1961; Schwartz, 1966; Lützen, 1982; Hoskins and Sousa Pinto, 2005]. Reference is often made to distributions as *generalised functions*, but such wording is also used for mathematical entities that include all or some distributions as special cases (such as respectively *hyperfunctions* and *ultradistributions*) as well as for mathematical entities that are unrelated to distributions [Hoskins and Sousa Pinto, 2005]. To avoid suggesting the legitimacy of pointwise operations for distributions, we do not adopt the *functional notation*; in other words, we do not place the parenthesised symbol of a free real variable immediately after the symbol of a distribution (neither, for that matter, do we place the parenthesised symbol of a free real variable or free complex variable immediately after the symbol of a function). All manipulations involving distributions are expressed in *symbolic notation*; for the operational details underlying such symbolic notation, we refer to the cited literature.

Let $\mathcal{T}(\mathbb{R})$ be any of $\mathcal{E}(\mathbb{R})$, $\mathcal{O}_M(\mathbb{R})$, $\mathcal{O}_C(\mathbb{R})$, $\mathcal{K}(\mathbb{R})$, $\dot{\mathcal{B}}(\mathbb{R})$, $\mathcal{D}_{L^{q'}}(\mathbb{R})$, $\mathcal{D}_{L^1}(\mathbb{R})$, $\dot{\mathcal{B}}_{+1}(\mathbb{R})$, $\mathcal{D}_{L^{q'},+1}(\mathbb{R})$, $\mathcal{D}_{L^1,+1}(\mathbb{R})$, $\dot{\mathcal{B}}_{n_2}(\mathbb{R})$, $\mathcal{D}_{L^{q'},n_2}(\mathbb{R})$, $\mathcal{D}_{L^1,n_2}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, and $\mathcal{D}(\mathbb{R})$. An element of $\mathcal{T}(\mathbb{R})$ is said to be a *test function* on $\mathbb{R}$, and $\mathcal{T}(\mathbb{R})$ itself is said to be a *test function space* [Schwartz, 1966; Horváth, 1966; Treves, 1967; Carton-Lebrun, 1995, 2005; Estrada and Kanwal, 2002; Larcher, 2013]. The chains of inclusion relations hold [Schwartz, 1954a, 1966; Horváth, 1966; Treves, 1967; Carton-Lebrun, 1995, 2005; Estrada and Kanwal, 2002; Larcher, 2013; Ortner and Wagner, 2013]

$$\begin{array}{c}
 \mathcal{E}(\mathbb{R}) \\
 \cup \\
 \mathcal{O}_M(\mathbb{R}) \\
 \cup \\
 \mathcal{O}_C(\mathbb{R}) \supset \dot{\mathcal{B}}(\mathbb{R}) \supset \mathcal{D}_{L^{q'}}(\mathbb{R}) \supset \mathcal{D}_{L^1}(\mathbb{R}) \\
 \cup \qquad \qquad \qquad \cup \\
 \dot{\mathcal{B}}_{+1}(\mathbb{R}) \supset \mathcal{D}_{L^{q'},+1}(\mathbb{R}) \supset \mathcal{D}_{L^1,+1}(\mathbb{R}) \\
 \cup \qquad \qquad \qquad \cup \\
 \dot{\mathcal{B}}_{n_2}(\mathbb{R}) \supset \mathcal{D}_{L^{q'},n_2}(\mathbb{R}) \supset \mathcal{D}_{L^1,n_2}(\mathbb{R}) \\
 \cup \\
 \mathcal{K}(\mathbb{R}) \qquad \qquad \qquad \supset \mathcal{S}(\mathbb{R}) \supset \mathcal{D}(\mathbb{R})
 \end{array}$$

Let $(\mathcal{T}(\mathbb{R}))'$ be the topological dual space of $\mathcal{T}(\mathbb{R})$; specifically, let $(\mathcal{T}(\mathbb{R}))'$ be any of $(\mathcal{E}(\mathbb{R}))' \equiv \mathcal{E}'(\mathbb{R})$, $(\mathcal{O}_M(\mathbb{R}))' \equiv \mathcal{O}'_M(\mathbb{R})$, $(\mathcal{O}_C(\mathbb{R}))' \equiv \mathcal{O}'_C(\mathbb{R})$, $(\mathcal{K}(\mathbb{R}))' \equiv \mathcal{K}'(\mathbb{R})$, $(\dot{\mathcal{B}}(\mathbb{R}))' \equiv \mathcal{D}'_L(\mathbb{R})$, $(\mathcal{D}_{L^q}(\mathbb{R}))' \equiv \mathcal{D}'_{L^q}(\mathbb{R})$, $(\mathcal{D}_{L^1}(\mathbb{R}))' \equiv \mathcal{D}'_{L^\infty}(\mathbb{R})$, $(\dot{\mathcal{B}}_{+1}(\mathbb{R}))' \equiv \mathcal{D}'_{L^1,-1}(\mathbb{R})$, $(\mathcal{D}_{L^q,+1}(\mathbb{R}))' \equiv \mathcal{D}'_{L^q,-1}(\mathbb{R})$, $(\mathcal{D}_{L^1,+1}(\mathbb{R}))' \equiv \mathcal{D}'_{L^\infty,-1}(\mathbb{R})$, $(\dot{\mathcal{B}}_{n_2}(\mathbb{R}))' \equiv \mathcal{D}'_{L^1,-n_2}(\mathbb{R})$, $(\mathcal{D}_{L^q,n_2}(\mathbb{R}))' \equiv \mathcal{D}'_{L^q,-n_2}(\mathbb{R})$, $(\mathcal{D}_{L^1,n_2}(\mathbb{R}))' \equiv \mathcal{D}'_{L^\infty,-n_2}(\mathbb{R})$, $(\mathcal{S}(\mathbb{R}))' \equiv \mathcal{S}'(\mathbb{R})$, and $(\mathcal{D}(\mathbb{R}))' \equiv \mathcal{D}'(\mathbb{R})$ [Schwartz, 1966; Ortner and Wagner, 2013]; care must be taken not to confuse $(\mathcal{D}_{L^1}(\mathbb{R}))'$ with $\mathcal{D}'_{L^1}(\mathbb{R})$ and $(\mathcal{D}_{L^q}(\mathbb{R}))'$ with $\mathcal{D}'_{L^q}(\mathbb{R})$. An element of $(\mathcal{T}(\mathbb{R}))'$ is said to be a *distribution* on $\mathbb{R}$, and $(\mathcal{T}(\mathbb{R}))'$ itself is said to be a *distribution space* [Schwartz, 1966; Horváth, 1966; Treves, 1967; Carton-Lebrun, 1995, 2005; Estrada and Kanwal, 2002; Larcher, 2013; Ortner and Wagner, 2013]. Given that $\dot{\mathcal{B}}(\mathbb{R})$ is the closure of $\mathcal{D}(\mathbb{R})$ within $\mathcal{D}_{L^\infty}(\mathbb{R})$, let $\dot{\mathcal{B}}'(\mathbb{R})$ (not to be confused with $(\dot{\mathcal{B}}(\mathbb{R}))' \equiv \mathcal{D}'_L(\mathbb{R})$) be the closure of $\mathcal{E}'(\mathbb{R})$ within $\mathcal{D}'_{L^\infty}(\mathbb{R})$ [Schwartz, 1966; Bijma, 1999]. By abuse of language, we say that the free real variable of the elements of the test function space on which a distribution on $\mathbb{R}$ is defined is also the free real variable of the distribution on $\mathbb{R}$. The chains of inclusion relations hold [Schwartz, 1962, 1966; Horváth, 1966; Treves, 1967; Carton-Lebrun, 1995, 2005; Estrada and Kanwal, 2002; Álvarez *et al.*, 2007; Larcher, 2013; Ortner and Wagner, 2013]

$$\begin{array}{ccccccc}
 \mathcal{D}'(\mathbb{R}) & \supset & \mathcal{S}'(\mathbb{R}) & & \supset & & \mathcal{K}'(\mathbb{R}) \\
 & & \cup & & & & \\
 \mathcal{D}'_{L^\infty, -n_2}(\mathbb{R}) & \supset & \dot{\mathcal{B}}'_{-n_2}(\mathbb{R}) & \supset & \mathcal{D}'_{L^q, -n_2}(\mathbb{R}) & \supset & \mathcal{D}'_{L^1, -n_2}(\mathbb{R}) \\
 & & \cup & & \cup & & \cup \\
 & & \mathcal{D}'_{L^\infty, -1}(\mathbb{R}) & \supset & \dot{\mathcal{B}}'_{-1}(\mathbb{R}) & \supset & \mathcal{D}'_{L^q, -1}(\mathbb{R}) \\
 & & & & \cup & & \cup \\
 & & & & \mathcal{D}'_{L^\infty}(\mathbb{R}) & \supset & \dot{\mathcal{B}}'(\mathbb{R}) & \supset & \mathcal{D}'_{L^q}(\mathbb{R}) & \supset & \mathcal{D}'_{L^1}(\mathbb{R}) & \supset & \mathcal{O}'_c(\mathbb{R}) \\
 & & & & & & & & \cup & & \cup & & \cup \\
 & & & & & & & & \mathcal{O}'_M(\mathbb{R}) & & & & \cup \\
 & & & & & & & & \mathcal{E}'(\mathbb{R}) & & & &
 \end{array}$$

where

- $\mathcal{D}'(\mathbb{R})$ is the distribution space of the *distributions* on $\mathbb{R}$ (also known as *distributions of arbitrary growth* on $\mathbb{R}$ or *Schwartz distributions* on $\mathbb{R}$)
- $\mathcal{S}'(\mathbb{R})$ is the distribution space of the *tempered distributions* on $\mathbb{R}$ (also known as *temperate distributions* on $\mathbb{R}$, *distributions of slow growth* on $\mathbb{R}$, or *slowly increasing distributions* on $\mathbb{R}$)
- $\mathcal{D}'_{L^p, -n}(\mathbb{R})$ is the distribution space of the $(-n)$ -weighted p -integrable distributions on $\mathbb{R}$
- $\dot{\mathcal{B}}'_{-n}(\mathbb{R})$ is the distribution space of the $(-n)$ -weighted distributions vanishing at infinity on $\mathbb{R}$
- $\mathcal{D}'_{L^p}(\mathbb{R})$ is the distribution space of the p -integrable distributions on $\mathbb{R}$
- $\dot{\mathcal{B}}'(\mathbb{R})$ is the distribution space of the distributions vanishing at infinity on $\mathbb{R}$
- $\mathcal{K}'(\mathbb{R})$ is the distribution space of the asymptotically small distributions on $\mathbb{R}$
- $\mathcal{O}'_C(\mathbb{R})$ is the distribution space of the distributions of rapid decay on $\mathbb{R}$ (also known as *rapidly decreasing distributions* on $\mathbb{R}$)
- $\mathcal{O}'_M(\mathbb{R})$ is the distribution space of the distributions of very rapid decay on $\mathbb{R}$ (also known as *very rapidly decreasing distributions* on $\mathbb{R}$)
- $\mathcal{E}'(\mathbb{R})$ is the distribution space of the distributions of compact support on $\mathbb{R}$

We warn that, for n even, the distribution spaces denoted here by $\mathcal{D}'_{L^1, -n}(\mathbb{R})$ and $\mathcal{D}'_{L^1}(\mathbb{R})$ are denoted elsewhere by respectively $\mathcal{S}'_k(\mathbb{R})$ with $k \equiv n/2$ and $\mathcal{S}'_0(\mathbb{R}) \equiv \mathcal{B}'_0(\mathbb{R})$ [Horváth, 1966; Ortner and Wagner, 2014].

Whilst $\mathcal{E}(\mathbb{R})$, $\mathcal{O}_M(\mathbb{R})$, $\mathcal{O}_C(\mathbb{R})$, $\mathcal{D}_{L^{q'}}(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$, and $\mathcal{D}(\mathbb{R})$ are reflexive spaces (and hence so are their topological dual spaces), $\mathcal{D}_{L^\infty}(\mathbb{R})$, $\dot{\mathcal{B}}(\mathbb{R})$, and $\mathcal{D}_{L^1}(\mathbb{R})$ are not [Schwartz, 1966; Horváth, 1966]. Indeed, on the one hand, the topological dual space of $\dot{\mathcal{B}}(\mathbb{R})$ is $\mathcal{D}'_{L^1}(\mathbb{R})$, the topological dual space of $\mathcal{D}'_{L^1}(\mathbb{R})$ is $\mathcal{D}_{L^\infty}(\mathbb{R})$, and the topological dual space of $\mathcal{D}_{L^\infty}(\mathbb{R})$ is not a distribution space (however, by being equipped with a suitable topological structure, $\mathcal{D}_{L^\infty}(\mathbb{R})$ can be made a semireflexive space having $\mathcal{D}'_{L^1}(\mathbb{R})$ as topological dual space) [Schwartz, 1954a, 1955, 1957, 1966]; on the other hand, the topological dual space of $\dot{\mathcal{B}}'(\mathbb{R})$ is $\mathcal{D}_{L^1}(\mathbb{R})$, the topological dual space of

$\mathcal{D}_{L^1}(\mathbb{R})$ is $\mathcal{D}'_{L^\infty}(\mathbb{R})$, and the topological dual space of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ is not a function space (however, by being equipped with a suitable topological structure, $\mathcal{D}'_{L^\infty}(\mathbb{R})$ can be made a semireflexive space having $\mathcal{D}_{L^1}(\mathbb{R})$ as topological dual space) [Schwartz, 1966].

Also an element of $\mathcal{D}^F(\mathbb{R})$ is a test function on $\mathbb{R}$, and $\mathcal{D}^F(\mathbb{R})$ itself is a test function space [Horváth, 1966; Barros-Neto, 1973]. A test function on $\mathbb{R}$ need not be a smooth function on $\mathbb{R}$ though. In particular, let $\mathcal{T}^{n_0}(\mathbb{R})$ be any of $\mathcal{E}^{n_0}(\mathbb{R})$, $\dot{\mathcal{B}}^{n_0}(\mathbb{R})$, $\dot{\mathcal{B}}_n^{n_0}(\mathbb{R})$, and $\mathcal{D}^{n_0}(\mathbb{R})$. An element of $\mathcal{T}^{n_0}(\mathbb{R})$ is a test function on $\mathbb{R}$, and $\mathcal{T}^{n_0}(\mathbb{R})$ itself is a test function space [Schwartz, 1954a, 1966; Horváth, 1966; Treves, 1967]. The chains of inclusion relations hold [Schwartz, 1954a, 1966; Horváth, 1966; Treves, 1967]

$$\begin{array}{ccc} \mathcal{E}^{n_0}(\mathbb{R}) & \supset & \mathcal{E}(\mathbb{R}) \\ \cup & & \cup \\ \dot{\mathcal{B}}^{n_0}(\mathbb{R}) & \supset & \dot{\mathcal{B}}(\mathbb{R}) \\ \cup & & \cup \\ \dot{\mathcal{B}}_n^{n_0}(\mathbb{R}) & \supset & \dot{\mathcal{B}}_n(\mathbb{R}) \\ \cup & & \cup \\ \mathcal{D}^{n_0}(\mathbb{R}) & \supset & \mathcal{D}^F(\mathbb{R}) \supseteq \mathcal{D}(\mathbb{R}) \end{array}$$

Let $(\mathcal{D}^F(\mathbb{R}))' \equiv \mathcal{D}'^F(\mathbb{R})$ be the topological dual space of $\mathcal{D}^F(\mathbb{R})$. An element of $\mathcal{D}'^F(\mathbb{R})$ is a distribution on $\mathbb{R}$, and $\mathcal{D}'^F(\mathbb{R})$ itself is a distribution space [Horváth, 1966; Barros-Neto, 1973]. Let $(\mathcal{T}^{n_0}(\mathbb{R}))'$ be the topological dual space of $\mathcal{T}^{n_0}(\mathbb{R})$; specifically, let $(\mathcal{T}^{n_0}(\mathbb{R}))'$ be any of $(\mathcal{E}^{n_0}(\mathbb{R}))' \equiv \mathcal{E}'^{n_0}(\mathbb{R})$, $(\dot{\mathcal{B}}^{n_0}(\mathbb{R}))' \equiv \mathcal{D}'^{n_0}_{L^1}(\mathbb{R})$, $(\dot{\mathcal{B}}_n^{n_0}(\mathbb{R}))' \equiv \mathcal{D}'^{n_0}_{L^1, -n}(\mathbb{R})$, and $(\mathcal{D}^{n_0}(\mathbb{R}))' \equiv \mathcal{D}'^{n_0}(\mathbb{R})$. An element of $(\mathcal{T}^{n_0}(\mathbb{R}))'$ is a distribution on $\mathbb{R}$, and $(\mathcal{T}^{n_0}(\mathbb{R}))'$ itself is a distribution space [Schwartz, 1954a, 1966; Horváth, 1966; Treves, 1967]. The chains of inclusion relations hold [Schwartz, 1954a, 1966; Horváth, 1966; Treves, 1967]

$$\begin{array}{ccccccc} \mathcal{D}'(\mathbb{R}) & \supset & \mathcal{D}'^F(\mathbb{R}) & \supset & \mathcal{D}'_{L^1, -n}(\mathbb{R}) & \supset & \mathcal{D}'_{L^1}(\mathbb{R}) \supset \mathcal{E}'(\mathbb{R}) \\ \cup & & \cup & & \cup & & \cup \\ \mathcal{D}'^{n_0}(\mathbb{R}) & \supset & \mathcal{D}'^{n_0}_{L^1, -n}(\mathbb{R}) & \supset & \mathcal{D}'^{n_0}_{L^1}(\mathbb{R}) & \supset & \mathcal{E}'^{n_0}(\mathbb{R}) \end{array}$$

We warn that, for n even, the distribution spaces denoted here by $\mathcal{D}'_{L',-n}(\mathbb{R})$ and $\mathcal{D}'_{L'}(\mathbb{R})$ are denoted elsewhere by respectively $\mathcal{S}'_k(\mathbb{R})$ and $\mathcal{S}'_0(\mathbb{R}) \equiv \mathcal{B}'_0(\mathbb{R})$ with $k \equiv n/2$ and $m \equiv n_0$ [Horváth, 1966; Ortner and Wagner, 2014].

A.2.4.1 Operations

The *evaluation* of a distribution on $\mathbb{R}$ against a suitable test function on $\mathbb{R}$, whenever well-defined, results in a value in $\mathbb{C}$ both *in the ordinary sense* and *in the Cesàro sense*, (the value in the ordinary sense, whenever it exists, being the same as the one in the Cesàro sense) [Schwartz, 1966; Estrada and Kanwal, 2002]. In particular, the n_0^{th} *moment* of a distribution on $\mathbb{R}$ is the result of its evaluation, either in the ordinary sense or in the Cesàro sense, against the test function ξ^{n_0} (or, equivalently, the result of the evaluation of its multiplicative product by ξ^{n_0} against $1_{\mathbb{R}}$) [Estrada and Kanwal, 2002]. The *localisation* (also known as *restriction*) of a distribution on $\mathbb{R}$ at an open subset of $\mathbb{R}$, always well-defined, results in a distribution on the open subset of $\mathbb{R}$ [Treves, 1967]. The distribution on the open subset of $\mathbb{R}$ can be extended to a distribution on $\mathbb{R}$ vanishing on the complement of the closure of the open subset of $\mathbb{R}$; however, unless the open subset of $\mathbb{R}$ coincides with $\mathbb{R}$ itself, such distribution on $\mathbb{R}$ is determined only up to addition by a distribution on $\mathbb{R}$ whose support is the frontier of the open subset of $\mathbb{R}$ [Wexler, 1966]. The localisation of a distribution on $\mathbb{R}$ at a point of $\mathbb{R}$, whenever well-defined, results in a complex constant [Łojasiewicz, 1957; Treves, 1967]. On an open subset of $\mathbb{R}$, it is possible to specify the conditions for a function to generate a Radon measure or a distribution and for a Radon measure to generate a distribution [Schwartz, 1966; Horváth, 1966; Treves, 1967]; inversely, on an open subset of $\mathbb{R}$, it is possible to specify the conditions for a distribution to be generable by a Radon measure or a function and for a Radon measure to be generable by a function [Schwartz, 1966; Horváth, 1966; Treves, 1967]. Furthermore, via a suitable embedding morphism, a function space can be embedded into a Radon-measure space or a distribution space, and a Radon-measure space can be embedded into a distribution space [Schwartz, 1966; Horváth, 1966; Treves, 1967]; inversely, via a suitable identification morphism, a distribution space can be identified with a Radon-measure space or a function space, and a Radon-measure space can be identified with a function space [Schwartz, 1966; Horváth, 1966; Treves, 1967]. A distribution on $\mathbb{R}$ is classified as either *regular* or *singular* depending on whether it is generable or not by a locally integrable function on $\mathbb{R}$ [Kanwal, 2004]. We refer to a regular distribution on an open subset of $\mathbb{R}$ generated by a constant function (in particular, the null function and the unit function) on the open subset of $\mathbb{R}$ as a *constant distribution* (in particular, the *null distribution* and the *unit distribution*) on the open subset

of $\mathbb{R}$. We denote by $0_{\mathbb{R}}$ not only the null function on $\mathbb{R}$ but also the null distribution on $\mathbb{R}$, and we denote by $1_{\mathbb{R}}$ not only the unit function on $\mathbb{R}$ but also the unit distribution on $\mathbb{R}$. Hereinafter, when not explicitly clarified, it is understood from the context whether any of the symbols $0_{\mathbb{R}}$ and $1_{\mathbb{R}}$ denote a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. A distribution on $\mathbb{R}$ is localisable at each point of an open subset of $\mathbb{R}$ at a neighbourhood of which its localisation results in a distribution generated by a C^0 function thereon [Łojasiewicz, 1957; Misra and Lavoine, 1986]; in contrast, a distribution on $\mathbb{R}$ that is localisable at a point of an open subset of $\mathbb{R}$ need not have its localisation at a neighbourhood of the point of $\mathbb{R}$ result in a distribution generated by a C^0 function thereon [Misra and Lavoine, 1986]. The most important singular distribution on $\mathbb{R}$ is the *unit impulse* (also known as *Dirac delta* or *delta distribution*) on $\mathbb{R}$, which is denoted by δ and is the continuous linear form under which the image of a test function is its value at the origin [Schwartz, 1966]; importantly, δ is an element of all the aforementioned distribution spaces, and, although it is not generable by a locally integrable function on $\mathbb{R}$, it is generable by a Radon measure on $\mathbb{R}$ [Schwartz, 1966]. Hereinafter, we denote all operations by the usual symbols, but we intend them in the distributional sense; specifically, unless otherwise stated, an operation involving functions or Radon measures is intended as an operation on the distributions generated by them.

The relation of *equality* (as well as the related concept of *vanishing*) on an open subset of $\mathbb{R}$ and the operations of *addition* and *multiplication by a scalar* are naturally introduced for distributions [Zemanian, 1965a; Schwartz, 1966; Hörmander, 1990; Estrada and Kanwal, 2002].

The *support of a distribution* on $\mathbb{R}$ is the smallest closed subset of $\mathbb{R}$ outside of which the distribution on $\mathbb{R}$ vanishes (or, equivalently, the complement of the largest open subset of $\mathbb{R}$ on which the distribution on $\mathbb{R}$ vanishes) [Estrada and Kanwal, 2002; Kanwal, 2004]. We qualify as *one-sided* (either *right-sided* or *left-sided*) a distribution on $\mathbb{R}$ with support bounded from one side (either from the left or from the right) [Zemanian, 1965a]; hereinafter, when mentioning a one-sided distribution on $\mathbb{R}$, we intend a distribution on $\mathbb{R}$ that is arbitrarily either right-sided or left-sided (but is consistently either right-sided or left-sided at each occurrence in a sentence). In particular, we qualify as either *causal* or *anticausal* a distribution on $\mathbb{R}$ with support bounded from one side (either from the left or from the right) at the origin [Kwakernaak and Sivan, 1991]. A distribution on $\mathbb{R}$ can be expressed non-uniquely as the sum of a *left-sided part* and a *right-sided part* [Zemanian, 1965a]. In particular, we have [Schwartz, 1966]

$$\mathcal{D}'_{\mathbb{R}}(\mathbb{R}) \equiv (\mathcal{D}_{\mathbb{L}}(\mathbb{R}))'$$

$$\mathcal{D}'_L(\mathbb{R}) \equiv (\mathcal{D}_R(\mathbb{R}))'$$

$$\begin{aligned}\mathcal{S}'_R(\mathbb{R}) &\equiv (\mathcal{S}_L(\mathbb{R}))' \\ &\equiv (\mathcal{S}'(\mathbb{R}) \cap \mathcal{D}'_R(\mathbb{R}))\end{aligned}$$

$$\begin{aligned}\mathcal{S}'_L(\mathbb{R}) &\equiv (\mathcal{S}_R(\mathbb{R}))' \\ &\equiv (\mathcal{S}'(\mathbb{R}) \cap \mathcal{D}'_L(\mathbb{R}))\end{aligned}$$

where

- $\mathcal{D}_R(\mathbb{R})$ is the function space of the *right-sided smooth functions of compact support* on $\mathbb{R}$ (whose topological dual space we denote by $(\mathcal{D}_R(\mathbb{R}))'$)
- $\mathcal{D}_L(\mathbb{R})$ is the function space of the *left-sided smooth functions of compact support* on $\mathbb{R}$ (whose topological dual space we denote by $(\mathcal{D}_L(\mathbb{R}))'$)
- $\mathcal{D}'_R(\mathbb{R})$ is the distribution space of the *right-sided distributions* on $\mathbb{R}$
- $\mathcal{D}'_L(\mathbb{R})$ is the distribution space of the *left-sided distributions* on $\mathbb{R}$
- $\mathcal{S}_R(\mathbb{R})$ is the function space of the *right-sided smooth functions of rapid decay* on $\mathbb{R}$ (whose topological dual space we denote by $(\mathcal{S}_R(\mathbb{R}))'$)
- $\mathcal{S}_L(\mathbb{R})$ is the function space of the *left-sided smooth functions of rapid decay* on $\mathbb{R}$ (whose topological dual space we denote by $(\mathcal{S}_L(\mathbb{R}))'$)
- $\mathcal{S}'_R(\mathbb{R})$ is the distribution space of the *right-sided tempered distributions* on $\mathbb{R}$
- $\mathcal{S}'_L(\mathbb{R})$ is the distribution space of the *left-sided tempered distributions* on $\mathbb{R}$

The *singular support* of a distribution on $\mathbb{R}$ is the smallest closed subset of $\mathbb{R}$ outside of which the distribution is generable by a smooth function (or, equivalently, the complement of the largest open subset of $\mathbb{R}$ on which the distribution is generable by a smooth function) [Reed and Simon, 1975; Hörmander, 1990; Estrada and Kanwal, 2002; Kanwal, 2004]. The singular support of a distribution on $\mathbb{R}$ is a subset of the support of the distribution on $\mathbb{R}$ [Reed and Simon, 1975]. It is worth mentioning that the concept of singular support of a distribution on $\mathbb{R}$ can be usefully generalised (namely, by introducing the concept of *wave front set* of a distribution on $\mathbb{R}$) [Reed and Simon, 1975; Hörmander, 1990; Oberguggenberger, 1986, 1992; Friedlander and Joshi, 1998].

By introducing the operation of *complex conjugation*, it is possible to distinguish between a *complex-valued distribution* on $\mathbb{R}$ and a *real-valued distribution* on $\mathbb{R}$ [Gelfand and Shilov, 1964; Schwartz, 1966]. Hereinafter, unless otherwise stated, a distribution is a complex-valued distribution on $\mathbb{R}$. An operation of change of free real variable cannot be introduced in full generality for distributions [Albertoni and Cugiani, 1951, 1953; Hoskins and Sousa Pinto, 2005]. However, it is possible to introduce for distributions the operation of *real-valued affine change of free real variable*, from which it is possible to introduce the concepts of *even parity* and *odd parity*, *conjugate symmetry* (also known as *Hermitian symmetry*) and *conjugate antisymmetry* (also known as *Hermitian antisymmetry*), *homogeneity*, and *almost periodicity* (in particular, *periodicity*), related to which are the concepts of *almost period* (in particular, *period*) and *mean value* on $\mathbb{R}$ [Albertoni and Cugiani, 1951, 1953; Gelfand and Shilov, 1964; Zemanian, 1965a; Schwartz, 1966; Pfaffelhuber, 1975; Jones, 1982; Burkill and Rennie, 1983; Bertrandias *et al.*, 1987; Hörmander, 1990; Friedlander and Joshi, 1998; Lagarias, 2000; Vladimirov, 2002; Estrada and Kanwal, 2002; Kanwal, 2004; Corduneanu, 2009; Zaidman, 2011; Strungaru and Terauds, 2016; Strungaru, 2020]. It is worth noting that the operation of real-valued affine change of free real variable can be regarded as the (non-commutative) composite of the operations of *horizontal translation* (hereinafter *translation*) by a *translation vector* in $\mathbb{R}$ of suitable quantity dimension and *horizontal dilation* (hereinafter *dilation*) by a *dilation coefficient* of suitable quantity dimension (respectively, by a *dilation factor*) in $\mathbb{R}_{(-\infty,0)\cup(0,+\infty)}$; in particular, when the dilation coefficient (respectively, dilation factor) is -1 , the operation of dilation becomes the operation of *horizontal reflection about the origin* (hereinafter *reflection*).

Let ψ be a complex-valued distribution on $\mathbb{R}$. Now, ψ_{RV} is a *real-valued distribution* on $\mathbb{R}$ and ψ_{IV} is an *imaginary-valued distribution* on $\mathbb{R}$ if and only if we have

$$\mathcal{C}(\psi_{\text{RV}}) = \mathcal{I}(\psi_{\text{RV}})$$

$$\mathcal{C}(\psi_{\text{IV}}) = -\mathcal{I}(\psi_{\text{IV}})$$

with

$$\begin{aligned}\mathcal{C} &= \mathcal{P} \circ \mathcal{R} \\ &= \mathcal{R} \circ \mathcal{P}\end{aligned}$$

$$\mathcal{C} \circ \mathcal{C} = \mathcal{I}$$

Then, ψ can be expressed uniquely as

$$\psi = \psi_{\text{RV}} + \psi_{\text{IV}}$$

with

$$\begin{cases} \psi_{\text{RV}} = \frac{1}{2}(\mathcal{I} + \mathcal{C})\psi \\ \psi_{\text{IV}} = \frac{1}{2}(\mathcal{I} - \mathcal{C})\psi \end{cases}$$

and we say that ψ_{RV} and ψ_{IV} are respectively the *real-valued part* and the *imaginary-valued part* of ψ . Furthermore, we say that the ψ_{RV} and ψ_{IV}/j are respectively the *real part* and the *imaginary part* of ψ .

Besides, ψ_{E} is an *even distribution* on $\mathbb{R}$ and ψ_{O} is an *odd distribution* on $\mathbb{R}$ if and only if we have

$$\mathcal{R}(\psi_{\text{E}}) = \mathcal{I}(\psi_{\text{E}})$$

$$\mathcal{R}(\psi_{\text{O}}) = -\mathcal{I}(\psi_{\text{O}})$$

with

$$\begin{aligned} \mathcal{R} &= \mathcal{P} \circ \mathcal{C} \\ &= \mathcal{C} \circ \mathcal{P} \end{aligned}$$

$$\mathcal{R} \circ \mathcal{R} = \mathcal{I}$$

Then, ψ can be expressed uniquely as

$$\psi = \psi_{\text{E}} + \psi_{\text{O}}$$

with

$$\begin{cases} \psi_{\text{E}} = \frac{1}{2}(\mathcal{I} + \mathcal{R})\psi \\ \psi_{\text{O}} = \frac{1}{2}(\mathcal{I} - \mathcal{R})\psi \end{cases}$$

and we say that ψ_{E} and ψ_{O} are respectively the *even part* and the *odd part* of ψ .

Finally, ψ_{CS} is a *conjugate-symmetric distribution* on $\mathbb{R}$ and ψ_{CA} is a *conjugate-antisymmetric distribution* on $\mathbb{R}$ if and only if we have

$$\mathcal{P}(\psi_{\text{CS}}) = \mathcal{I}(\psi_{\text{CS}})$$

$$\mathcal{P}(\psi_{\text{CA}}) = -\mathcal{I}(\psi_{\text{CA}})$$

with

$$\begin{aligned}\mathcal{P} &= \mathcal{R} \circ \mathcal{C} \\ &= \mathcal{C} \circ \mathcal{R}\end{aligned}$$

$$\mathcal{P} \circ \mathcal{P} = \mathcal{I}$$

Then, ψ can be expressed uniquely as

$$\psi = \psi_{\text{CS}} + \psi_{\text{CA}}$$

with

$$\begin{cases} \psi_{\text{CS}} = \frac{1}{2}(\mathcal{I} + \mathcal{P})\psi \\ \psi_{\text{CA}} = \frac{1}{2}(\mathcal{I} - \mathcal{P})\psi \end{cases}$$

and we say that ψ_{CS} and ψ_{CA} are respectively the *conjugate-symmetric part* and the *conjugate-antisymmetric part* of ψ .

A distribution on $\mathbb{R}$ is real-valued (respectively, imaginary-valued) if and only if its conjugate-symmetric part coincides with its even part (respectively, odd part) and its conjugate-antisymmetric part coincides with its odd part (respectively, even part). A distribution on $\mathbb{R}$ is even (respectively, odd) if and only if its conjugate-symmetric part coincides with its real-valued part (respectively, imaginary-valued part) and its conjugate-antisymmetric part coincides with its imaginary-valued part (respectively, real-valued part). A distribution on $\mathbb{R}$ is conjugate-symmetric (respectively, conjugate-antisymmetric) if and only if its real-valued part coincides with its even part (respectively, odd part) and its imaginary-valued part coincides with its odd part (respectively, even part).

The *limit* at a point of $\bar{\mathbb{R}}$ (both one-sided and two-sided) and the order relation of *order of magnitude* on a neighbourhood of a point of $\bar{\mathbb{R}}$ are also introduced for distributions in the ordinary sense [Łojasiewicz, 1957; Misra and Lavoine, 1986; Takači, 1990]; in particular, the *limit at infinity* and the order relation of *order of magnitude at infinity* are also introduced for distributions in the Cesàro sense [Estrada and Kanwal, 2002]. If the two-sided limit at a point of $\mathbb{R}$ of a distribution on $\mathbb{R}$ exists, the distribution on $\mathbb{R}$ can be expressed as the sum of a distribution on $\mathbb{R}$ whose

localisation at the point of $\mathbb{R}$ results in the value of such limit and a finite linear combination of derivatives of order in $\mathbb{N}$ of δ each of which translated by the same translation vector so as to have supports in the point of $\mathbb{R}$ (whereat such derivatives have null two-sided limit but is not localisable) [Łojasiewicz, 1957; Misra and Lavoine, 1986; Estrada and Kanwal, 2002]; accordingly, whilst a distribution on $\mathbb{R}$ that is localisable at point of $\mathbb{R}$ has two-sided limit thereat coinciding with the result of the localisation, a distribution on $\mathbb{R}$ that has two-sided limit at point of $\mathbb{R}$ need not be localisable thereat.

The operations of *differentiation* and *antidifferentiation* are easily introduced for distributions [Schwartz, 1966].

The operation of differentiation is introduced in such a manner that each distribution on $\mathbb{R}$ admits an n^{th} *derivative* being itself a distribution on $\mathbb{R}$ [Schwartz, 1966]. In fact, when equipped with such operation of differentiation, all the aforementioned distribution spaces become ordinary differential linear spaces over $\mathbb{F}$ ($\mathbb{F}$ being made an ordinary differential field whose operation of derivation is the trivial derivation). In particular, if the n^{th} weak derivative of a locally integrable function on $\mathbb{R}$ exists, it generates a distribution on $\mathbb{R}$ that coincides with the n^{th} derivative of the regular distribution on $\mathbb{R}$ generated by the locally integrable function on $\mathbb{R}$ [Schwartz, 1966; Leoni, 2017]. The distribution space of the distributions on $\mathbb{R}$ is the smallest enlargement of the function space of the locally integrable functions on $\mathbb{R}$ that is invariant under differentiation [Halperin, 1952]. A distribution on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is localisable at a point of $\mathbb{R}$ is said to be *smooth* at that point of $\mathbb{R}$ [Estrada and Kanwal, 2002]; in particular, a distribution on $\mathbb{R}$ is smooth at a point of $\mathbb{R}$ if its localisation at a neighbourhood of the point of $\mathbb{R}$ results in a distribution generated by a smooth function thereon [Łojasiewicz, 1957].

The operation of antidifferentiation is introduced in such a manner that each distribution on $\mathbb{R}$ admits infinitely many n^{th} *antiderivatives* being themselves distributions on $\mathbb{R}$ (the difference of any two of which being a tempered distribution on $\mathbb{R}$ generated by a polynomial function of degree less than or equal to $n - 1$ on $\mathbb{R}$) [Schwartz, 1966; Barros-Neto, 1973]. The n^{th} derivative of any n^{th} antiderivative of a distribution on $\mathbb{R}$ coincides with the distribution on $\mathbb{R}$ itself [Zemanian, 1965a]. We refer to the set of the infinitely many n^{th} antiderivatives of a distribution on $\mathbb{R}$ as its n^{th} *indefinite integral*. A one-sided distribution on $\mathbb{R}$ admits one and only one *one-sided* n^{th} *antiderivative* [Schwartz, 1966]. By definition, the 0^{th} *derivative* and the 0^{th} *antiderivative* of a distribution on $\mathbb{R}$ coincide with the distribution on $\mathbb{R}$ itself. We stipulate that the *one-sided* z^{th} *generalised derivative* of a one-sided distribution on $\mathbb{R}$ coincides with its *one-sided fractional derivative of order* z [McBride, 1979]; specifically, the one-sided z^{th} generalised derivative of a

one-sided distribution on $\mathbb{R}$ is the one-sided $|z|$ th antiderivative of the one-sided distribution on $\mathbb{R}$ (when z is in $\mathbb{Z}_{(-\infty,1]}$), the one-sided distribution on $\mathbb{R}$ itself (when z equals 0), or the z th derivative of the one-sided distribution on $\mathbb{R}$ (when z is in $\mathbb{Z}_{[1,+\infty)}$).

The operation of *definite integration* (hereinafter *integration*) on an open subset of $\mathbb{R}$ (in particular, on $\mathbb{R}$) is also conveniently introduced for distributions [Schwartz, 1954a, 1957, 1966; Sikorski, 1961; Musielak, 1963; Horváth, 1966; Wawak, 1987, 1988; Richards and Youn, 1990; Takači, 1990; Kierat and Skórnik, 2001; Estrada and Kanwal, 2002].

In general, the operation of *multiplication* and the operation of *convolution* are not well-defined for distributions, and, despite being both commutative by definition, they are not associative and do not determine unique results [Schwartz, 1966]. In fact, within the standard theory of distributions, it is not possible for always well-defined operations of multiplication and convolution to coexist with an always well-defined operation of differentiation and with a mathematical entity such as δ [Schwartz, 1954b; Oberguggenberger, 1992; Hoskins and Sousa Pinto, 2005; Franssens, 2013]; in other words, in general, a distribution space equipped with an operation of multiplication or convolution acting as the operation of multiplication of the algebraic structure can be made neither a complex differential algebra nor a complex topological algebra. We warn that several non-equivalent definitions are available in the literature for both multiplication and convolution of distributions (the definitions that are more general including the definitions that are less general as special cases) [Hirata and Ogata, 1958; Shiraishi, 1959; Hirata and Shiraishi, 1960; Schwartz, 1954c, 1957, 1958, 1962, 1966; Reed and Simon, 1975; Dierolf and Voigt, 1978; Kamiński, 1982; Dierolf, 1984; Oberguggenberger, 1986, 1992; Vladimirov, 2002; Ortner, 2004, 2011]. Hereinafter, unless otherwise stated, we adhere to the definitions of multiplication and convolution provided by Schwartz [1954c, 1957, 1958, 1962, 1966].

The multiplication of a distribution on $\mathbb{R}$ (respectively, a distribution of order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$, as introduced below) by any finite set of smooth functions on $\mathbb{R}$ (respectively, any finite set of C^{n_0} functions on $\mathbb{R}$) is always well-defined, commutative, and associative, and it results in a distribution on $\mathbb{R}$ (respectively, a distribution of order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$) [Schwartz, 1966; Horváth, 1966; Oberguggenberger, 1992; Estrada and Kanwal, 2002]; however, the multiplication of a distribution on $\mathbb{R}$ by a distribution on $\mathbb{R}$ is well-defined provided that special conditions are satisfied (in particular, that the singular supports of the two distributions are disjoint or even, less restrictively, that a certain condition on the wave front sets of the two distributions is satisfied) [Schwartz, 1966; Horváth, 1966; Reed and Simon, 1975; Hörmander, 1990; Oberguggenberger, 1986, 1992; Estrada

and Kanwal, 2002]. Furthermore, the multiplication of a distribution on $\mathbb{R}$ (respectively, a distribution of order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$, as introduced below) by any finite set of smooth functions on $\mathbb{R}$ (respectively, any finite set of C^{n_0+n} functions on $\mathbb{R}$) complies with the Leibniz rule over multiplication for derivatives of order in $\mathbb{N}$ (respectively, $\mathbb{N}_{[0,n]}$) [Schwartz, 1966]. When the repeated multiplication of a distribution by itself is well-defined, the operation of *multiplicative exponentiation* can also be introduced; by definition, the 0^{th} *multiplicative power* and the 1^{st} *multiplicative power* of a distribution on $\mathbb{R}$ coincide with respectively $1_{\mathbb{R}}$ and the distribution on $\mathbb{R}$ itself. A *multiplicative inverse* of a distribution on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose multiplicative product by the distribution on $\mathbb{R}$ itself coincides with $1_{\mathbb{R}}$ [Schwartz, 1966; Misra and Lavoine, 1986]; we say that a distribution on $\mathbb{R}$ admitting a multiplicative inverse is a *multiplication-invertible distribution* on $\mathbb{R}$. In general, a multiplication-invertible distribution on $\mathbb{R}$ admits infinitely many multiplicative inverses (the difference of any two of which being a distribution on $\mathbb{R}$ whose multiplicative product by the multiplication-invertible distribution on $\mathbb{R}$ itself coincides with $0_{\mathbb{R}}$) [Schwartz, 1966; Misra and Lavoine, 1986]. In particular, each distribution on $\mathbb{R}$ generated by a smooth function on $\mathbb{R}$ having countably many zeros of finite multiplicity is multiplication-invertible on $\mathbb{R}$ [Schwartz, 1966; Misra and Lavoine, 1986]; furthermore, the multiplicative inverse is unique if the smooth function on $\mathbb{R}$ has no zeros [Schwartz, 1966; Misra and Lavoine, 1986]. Relatedly, the operation of *division by a function* is introduced in such a manner that the division of a distribution on $\mathbb{R}$ by a smooth function on $\mathbb{R}$ having countably many zeros of finite multiplicity admits infinitely many distributions on $\mathbb{R}$ that are its *quotients* (the difference of any two of which being a finite linear combination of derivatives of order in $\mathbb{N}$ of δ each of which translated by some translation vector so as to have support in each zero of the smooth function on $\mathbb{R}$) [Schwartz, 1966; Misra and Lavoine, 1986]. We refer to the set of the infinitely many quotients of a distribution on $\mathbb{R}$ by such a smooth function on $\mathbb{R}$ as the *indefinite quotient* of the distribution on $\mathbb{R}$ by the smooth function on $\mathbb{R}$.

The convolution of a distribution on $\mathbb{R}$ by any finite set of distributions of compact support on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a distribution on $\mathbb{R}$ [Schwartz, 1966]; however, the convolution of a distribution on $\mathbb{R}$ by a distribution on $\mathbb{R}$ is well-defined provided that special conditions are satisfied (in particular, that the intersection of the support of one of the two distributions on $\mathbb{R}$ and the set of the differences of the elements of any compact subset of $\mathbb{R}$ and the support of the other distribution on $\mathbb{R}$ is compact) [Schwartz, 1966]. When the repeated convolution of a distribution by itself is well-defined, the operation of *convolutive exponentiation* can also be introduced; by definition, the 0^{th} *convolutive power* and the 1^{st}

convolutive power of a distribution on $\mathbb{R}$ coincide with respectively δ and the distribution on $\mathbb{R}$ itself. A *convolutive inverse* of a distribution on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose convolutive product by the distribution on $\mathbb{R}$ itself coincides with δ [Zemanian, 1965a; Schwartz, 1966; Poletti, 2005]; we say that a distribution admitting a convolutive inverse is a *convolution-invertible distribution*. In general, a convolution-invertible distribution admits infinitely many convolutive inverses (the difference of any two of which being a distribution whose convolutive product by the convolution-invertible distribution itself coincides with $0_{\mathbb{R}}$) [Zemanian, 1965a; Schwartz, 1966; Poletti, 2005]. In particular, each finite linear combination of derivatives of any order in $\mathbb{N}$ of δ that does not coincide with $0_{\mathbb{R}}$ is convolution-invertible, the convolutive inverse being unique if required to be one-sided [Zemanian, 1965a; Schwartz, 1966; Poletti, 2005]. The convolution of any finite set of one-sided distributions on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a one-sided distribution on $\mathbb{R}$ that coincides with $0_{\mathbb{R}}$ if and only if at least one of the convolved distributions is $0_{\mathbb{R}}$ [Schwartz, 1966]. In fact, a distribution space of one-sided distributions on $\mathbb{R}$ is a *convolution algebra* (namely, a complex algebra in which the operation of multiplication of the algebraic structure is the operation of convolution) [Zemanian, 1965a; Misra and Lavoine, 1986; Vladimirov, 2002]. Furthermore, both $\mathcal{D}'_{\mathbb{R}}(\mathbb{R})$ and $\mathcal{D}'_{\mathbb{L}}(\mathbb{R})$ are complex commutative weak topological domain algebras, whereas their intersection $\mathcal{E}'(\mathbb{R})$ is even a complex commutative topological domain algebra [Zemanian, 1965a; Schwartz, 1966; Misra and Lavoine, 1986; Vladimirov, 2002; Larcher, 2013; Sasane, 2020]. However, none of $\mathcal{D}'_{\mathbb{R}}(\mathbb{R})$, $\mathcal{D}'_{\mathbb{L}}(\mathbb{R})$, and $\mathcal{E}'(\mathbb{R})$ is a complex commutative (possibly weak) topological division algebra; indeed, a necessary and sufficient condition for a convolution equation whose coefficient and constant term are both elements of one of $\mathcal{D}'_{\mathbb{R}}(\mathbb{R})$, $\mathcal{D}'_{\mathbb{L}}(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$ to have as a solution an element of respectively $\mathcal{D}'_{\mathbb{R}}(\mathbb{R})$, $\mathcal{D}'_{\mathbb{L}}(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$ (and for this solution to be unique by virtue of the absence of nontrivial divisors of zero in a convolution algebra) is that the coefficient of the convolution equation is convolution-invertible in respectively $\mathcal{D}'_{\mathbb{R}}(\mathbb{R})$, $\mathcal{D}'_{\mathbb{L}}(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$ [Zemanian, 1965a; Schwartz, 1966; Poletti, 2005]. Amongst the infinitely many convolutive inverses of a convolution-invertible one-sided distribution on $\mathbb{R}$, only one is a one-sided distribution on $\mathbb{R}$ [Zemanian, 1965a; Schwartz, 1966; Poletti, 2005]; we say that it is the *one-sided convolutive inverse* of such one-sided distribution on $\mathbb{R}$. It is worth mentioning that there are other convolution algebras amongst distributions spaces. In particular, $\mathcal{S}'_{\mathbb{R}}(\mathbb{R})$, $\mathcal{S}'_{\mathbb{L}}(\mathbb{R})$, and $\mathcal{O}'_{\mathbb{M}}(\mathbb{R})$ are complex commutative weak topological domain algebras, whereas $\mathcal{O}'_{\mathbb{C}}(\mathbb{R})$ is even a complex commutative topological domain algebra [Treves, 1967; Misra and Lavoine, 1986; Oberguggenberger, 1992; Vladimirov, 2002; Poletti, 2005]; furthermore, $\mathcal{D}'_{\mathbb{L}^1}(\mathbb{R})$ is a

complex commutative topological unital algebra with nontrivial divisors of zero [González and Negrín, 1995; Larcher, 2013].

Let $\mathcal{W}(\mathbb{R})$ be a distribution space. We denote by $(\mathcal{W}(\mathbb{R}))_{\mathcal{M}}$ the function space of the *multipliers* of $\mathcal{W}(\mathbb{R})$, which is the function space of the smooth functions on $\mathbb{R}$ whose multiplicative products by any element of $\mathcal{W}(\mathbb{R})$ are well-defined and are elements of $\mathcal{W}(\mathbb{R})$ [Larcher, 2013]. Besides, we denote by $(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}$ the distribution space of the *convolutors* of $\mathcal{W}(\mathbb{R})$, which is the distribution space of the distributions on $\mathbb{R}$ whose convolutive products by any element of $\mathcal{W}(\mathbb{R})$ are well-defined and are elements of $\mathcal{W}(\mathbb{R})$ [Larcher, 2013]. Furthermore, we denote by $(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}^*$ the *convolution dual space* (also known as *c-dual space*) of $\mathcal{W}(\mathbb{R})$, which is the distribution space of the distributions on $\mathbb{R}$ whose convolutive products by any element of $\mathcal{W}(\mathbb{R})$ are well-defined [Yoshinaga and Ogata, 1958; Hirata, 1958; Dierolf, 1984; Larcher, 2013; Bargetz *et al.*, 2017; Kleiner and Hilfer, 2022]. Besides, we denote by $(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}^{**}$ the *convolution perfection space* (also known as *c-bidual space*) of $\mathcal{W}(\mathbb{R})$, which is the convolution dual of the convolution dual of $\mathcal{W}(\mathbb{R})$ [Yoshinaga and Ogata, 1958; Kleiner and Hilfer, 2022]. Finally, $\mathcal{W}(\mathbb{R})$ is said to be a *convolution perfect space* (also known as *c-reflexive space*) if and only if it coincides with $(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}^{**}$ [Yoshinaga and Ogata, 1958; Kleiner and Hilfer, 2022]. Function spaces of multipliers, distribution spaces of convolutors, and convolution dual spaces of selected distribution spaces are listed in **Table A.2** [Schwartz, 1966; Yoshinaga and Ogata, 1958; Larcher, 2013].

Table A.2 Function spaces of multipliers, distribution spaces of convolutors, and convolution dual spaces of selected distribution spaces.

$\mathcal{W}(\mathbb{R})$	$(\mathcal{W}(\mathbb{R}))_{\mathcal{M}}$	$(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}$	$(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}^*$
$\mathcal{E}^\wedge(\mathbb{R}), \mathcal{E}'(\mathbb{R})$	$\mathcal{E}(\mathbb{R})$	$\mathcal{E}'(\mathbb{R})$	$\mathcal{E}'(\mathbb{R}), \mathcal{D}'(\mathbb{R})$
$\mathcal{O}_{\mathcal{M}}^\wedge(\mathbb{R}), \mathcal{O}_{\mathcal{M}}'(\mathbb{R})$	$\mathcal{O}_{\mathcal{M}}(\mathbb{R})$	$\mathcal{O}_{\mathcal{M}}'(\mathbb{R})$	$\mathcal{O}_{\mathcal{C}}'(\mathbb{R}), \mathcal{S}'(\mathbb{R})$
$\mathcal{O}_{\mathcal{C}}^\wedge(\mathbb{R}), \mathcal{O}_{\mathcal{C}}'(\mathbb{R})$	$\mathcal{O}_{\mathcal{C}}(\mathbb{R})$	$\mathcal{O}_{\mathcal{C}}'(\mathbb{R})$	$\mathcal{O}_{\mathcal{C}}'(\mathbb{R}), \mathcal{S}'(\mathbb{R})$
$\mathcal{D}_{L^\infty}^\wedge(\mathbb{R})$	$\mathcal{D}_{L^\infty}(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R})$
$\dot{\mathcal{B}}^\wedge(\mathbb{R}), \mathcal{D}_{L'}'(\mathbb{R})$	$\mathcal{D}_{L^\infty}(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R}), \mathcal{D}_{L^\infty}'(\mathbb{R})$
$\mathcal{D}_{L'}^\wedge(\mathbb{R}), \mathcal{D}_{L'}'(\mathbb{R})$	$\mathcal{D}_{L^\infty}(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R}), \mathcal{D}_{L'}'(\mathbb{R})$
$\dot{\mathcal{B}}'(\mathbb{R})$	$\mathcal{D}_{L^\infty}(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R})$
$\mathcal{D}_{L'}^\wedge(\mathbb{R}), \mathcal{D}_{L^\infty}'(\mathbb{R})$	$\mathcal{D}_{L^\infty}(\mathbb{R})$	$\mathcal{D}_{L'}'(\mathbb{R})$	$\mathcal{D}_{L^\infty}'(\mathbb{R}), \mathcal{D}_{L'}'(\mathbb{R})$
$\mathcal{S}^\wedge(\mathbb{R}), \mathcal{S}'(\mathbb{R})$	$\mathcal{O}_{\mathcal{M}}(\mathbb{R})$	$\mathcal{O}_{\mathcal{C}}'(\mathbb{R})$	$\mathcal{S}'(\mathbb{R}), \mathcal{O}_{\mathcal{C}}'(\mathbb{R})$
$\mathcal{D}_{\mathcal{R}}^\wedge(\mathbb{R}), \mathcal{D}_{\mathcal{L}}'(\mathbb{R})$	$\mathcal{E}(\mathbb{R})$	$\mathcal{D}_{\mathcal{R}}'(\mathbb{R}), \mathcal{D}_{\mathcal{L}}'(\mathbb{R})$	$\mathcal{D}_{\mathcal{R}}'(\mathbb{R}), \mathcal{D}_{\mathcal{L}}'(\mathbb{R})$
$\mathcal{D}_{\mathcal{L}}^\wedge(\mathbb{R}), \mathcal{D}_{\mathcal{R}}'(\mathbb{R})$	$\mathcal{E}(\mathbb{R})$	$\mathcal{D}_{\mathcal{L}}'(\mathbb{R}), \mathcal{D}_{\mathcal{R}}'(\mathbb{R})$	$\mathcal{D}_{\mathcal{L}}'(\mathbb{R}), \mathcal{D}_{\mathcal{R}}'(\mathbb{R})$
$\mathcal{D}^\wedge(\mathbb{R}), \mathcal{D}'(\mathbb{R})$	$\mathcal{E}(\mathbb{R})$	$\mathcal{E}'(\mathbb{R})$	$\mathcal{D}'(\mathbb{R}), \mathcal{E}'(\mathbb{R})$

$\mathcal{T}_0(\mathbb{R})$ = test function space; $\mathcal{T}_0^\wedge(\mathbb{R})$ = distribution space of the distributions on $\mathbb{R}$ generated by the elements of $\mathcal{T}_0(\mathbb{R})$; $(\mathcal{T}_0(\mathbb{R}))'$ = topological dual space of $\mathcal{T}_0(\mathbb{R})$; $\mathcal{W}(\mathbb{R})$ = any of $\mathcal{T}_0^\wedge(\mathbb{R})$ and $(\mathcal{T}_0(\mathbb{R}))'$; $(\mathcal{W}(\mathbb{R}))_{\mathcal{M}}$ = function space of the multipliers of $\mathcal{W}(\mathbb{R})$; $(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}$ = distribution space of the convolutors of $\mathcal{W}(\mathbb{R})$; $(\mathcal{W}(\mathbb{R}))_{\mathcal{C}}^*$ = convolution dual space of $\mathcal{W}(\mathbb{R})$. Two distribution spaces separated by a comma in the third (respectively, fourth) column of a row are, in order, the distribution spaces of the convolutors (respectively, the convolution dual spaces) of the two distribution spaces in the first cell on the same row.

In particular, we observe that $\mathcal{O}_{\mathcal{M}}(\mathbb{R})$ coincides with the function space of the multipliers of $\mathcal{S}'(\mathbb{R})$ and $\mathcal{O}_{\mathcal{C}}'(\mathbb{R})$ coincides with the distribution space of the convolutors of $\mathcal{S}'(\mathbb{R})$ [Horváth, 1966; Larcher, 2013]. On the one hand, the multiplication of a tempered distribution on $\mathbb{R}$ by any finite set of smooth functions of slow growth on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a tempered distribution on $\mathbb{R}$ [Schwartz, 1966]. On the other hand, the convolution of a tempered distribution on $\mathbb{R}$ by any finite set of distributions of rapid decay on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a tempered distribution on $\mathbb{R}$ [Schwartz, 1966]. We also observe that $\mathcal{D}'(\mathbb{R})$, $\mathcal{D}_{\mathcal{R}}'(\mathbb{R})$, $\mathcal{D}_{\mathcal{L}}'(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, $\mathcal{O}_{\mathcal{C}}'(\mathbb{R})$, and $\mathcal{E}'(\mathbb{R})$ are convolution perfect spaces.

A real-valued distribution on $\mathbb{R}$ is said to be *non-negative* (respectively, *non-positive*) if and only if its evaluation against any non-negative real-valued smooth function of compact support on $\mathbb{R}$ results in a finite non-negative real value (respectively, finite non-positive real value) [Gelfand and Vilenkin, 1964; Schwartz, 1966; Richards and Youn, 1990]. Each non-negative real-valued locally integrable function on $\mathbb{R}$ generates a non-negative real-valued Radon measure on $\mathbb{R}$, and each

non-negative real-valued Radon measure on $\mathbb{R}$ generates a non-negative real-valued distribution on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Schwartz, 1966; Richards and Youn, 1990]; in fact, a real-valued distribution on $\mathbb{R}$ is non-negative if and only if it is generable by a non-negative real-valued Radon measure on $\mathbb{R}$ (but a non-negative real-valued Radon measure on $\mathbb{R}$ need not be generable by a non-negative real-valued locally integrable function on $\mathbb{R}$) [Gelfand and Vilenkin, 1964; Schwartz, 1966; Richards and Youn, 1990]. A distribution on $\mathbb{R}$ that is not real-valued (and hence that is not generable by a Radon measure on $\mathbb{R}$) cannot be non-negative (respectively, non-positive). However, we say that a distribution on $\mathbb{R}$ is *weakly non-negative* (respectively, *weakly non-positive*) if and only if its real part is non-negative (respectively, non-positive); as is evident, a distribution on $\mathbb{R}$ is non-negative (respectively, non-positive) only if it is weakly non-negative (respectively, weakly non-positive).

We now introduce for distributions the non-commutative operation of *correlation* (which is strictly connected with the commutative operation of convolution) and the related concept of *non-negative-definiteness*. Whenever well-defined, the convolutive product of a first distribution on $\mathbb{R}$ by the reflected complex conjugate (or, equivalently, by the complex conjugate reflected) of a second distribution on $\mathbb{R}$ is said to be the *cross-correlation distribution* of the ordered pair of distributions on $\mathbb{R}$ [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Benedetto, 1997; King, 2009b]; in particular, whenever well-defined, the convolutive product of a distribution on $\mathbb{R}$ by its reflected complex conjugate (or, equivalently, by its complex conjugate reflected) is said to be the *auto-correlation distribution* of the distribution on $\mathbb{R}$ [Richards and Youn, 1990; Oppenheim *et al.*, 1997; Benedetto, 1997; King, 2009b]. In agreement with the fact that correlation is not commutative, the cross-correlation distribution of an ordered pair of distributions on $\mathbb{R}$, whenever well-defined, coincides with the reflected complex conjugate (or, equivalently, the complex conjugate reflected) of the cross-correlation distribution of the reverse ordered pair of distributions on $\mathbb{R}$ [Benedetto, 1997]. A distribution on $\mathbb{R}$ is said to be *non-negative-definite* (or *positive-semidefinite*, or *of non-negative type*) if and only if its evaluation against the auto-correlation function of any smooth function of compact support on $\mathbb{R}$ (in fact, of any smooth function of rapid decay on $\mathbb{R}$) results in a finite non-negative real value [Gelfand and Vilenkin, 1964; Schwartz, 1966; Richards and Youn, 1990; Vladimirov, 2002]; a distribution on $\mathbb{R}$ is non-negative-definite if and only if its evaluation against any non-negative-definite smooth function of compact support on $\mathbb{R}$ (in fact, against any non-negative-definite smooth function of rapid decay on $\mathbb{R}$) results in a finite non-negative real value [Schwartz, 1966]. Furthermore, each non-negative-definite C^0 function on $\mathbb{R}$ generates a non-negative-definite Radon measure on $\mathbb{R}$, and each non-negative-definite Radon measure on $\mathbb{R}$ generates a non-negative-definite distribution

on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Schwartz, 1966; Berg and Forst, 1975; Staffans, 1976]; in fact, each non-negative-definite distribution on $\mathbb{R}$ whose localisation at a neighbourhood of the origin results in a C^0 function thereon is generable by a non-negative-definite C^0 function on $\mathbb{R}$ [Schwartz, 1966]. In particular, a positive real constant times $(-1)^{n_0} \delta^{(2n_0)}$ ($\delta^{(2n_0)}$ being the $2n_0^{\text{th}}$ derivative of δ) is a non-negative-definite distribution on $\mathbb{R}$ [Vladimirov, 2002]. Unlike a finite conical combination, a finite linear combination of non-negative-definite distributions on $\mathbb{R}$ (respectively, of non-negative real-valued distributions on $\mathbb{R}$) need not be a non-negative-definite distribution on $\mathbb{R}$ (respectively, a non-negative real-valued distribution on $\mathbb{R}$) [Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Gripenberg *et al.*, 1990; Moody and Strungaru, 2017]; hence, the distribution space of the non-negative-definite distribution on $\mathbb{R}$ (respectively, of the non-negative real-valued distributions on $\mathbb{R}$) cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$. Each non-negative-definite distribution on $\mathbb{R}$ is a conjugate-symmetric element of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ [Schwartz, 1966; Vladimirov, 2002; Strungaru and Terauds, 2016]. A distribution on $\mathbb{R}$ that is not conjugate-symmetric (in particular, that is either causal or anticausal) cannot be non-negative-definite. However, we say that a distribution on $\mathbb{R}$ is *weakly non-negative-definite* (or *weakly positive-semidefinite*, or *of weak non-negative type*) if and only if its conjugate-symmetric part is non-negative-definite [Berg and Forst, 1975; Staffans, 1976; Gripenberg *et al.*, 1990]; as is evident, a distribution on $\mathbb{R}$ is non-negative-definite only if it is weakly non-negative-definite. A real-valued distribution on $\mathbb{R}$ is weakly non-negative-definite if and only if its evaluation against the auto-correlation function of any real-valued smooth function of compact support on $\mathbb{R}$ (in fact, of any real-valued smooth function of rapid decay on $\mathbb{R}$) results in a finite non-negative real value [Berg and Forst, 1975].

A.2.4.2 Characterisation

A distribution on $\mathbb{R}$ can be expressed (the expression proving non-unique) as a possibly infinite but convergent and locally finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by C^0 functions of compact support on $\mathbb{R}$ (the support of each of such distributions on $\mathbb{R}$ being contained in an arbitrary neighbourhood of the support of the distribution on $\mathbb{R}$) [Schwartz, 1966]; the fact that such linear combination is locally finite implies that, on each relatively compact open subset of $\mathbb{R}$, all but finitely many of its terms vanish [Schwartz, 1966]. Accordingly, the localisation of a distribution on $\mathbb{R}$ at a relatively compact open subset of $\mathbb{R}$ is a derivative of order in $\mathbb{N}$ of a distribution generated by a C^0 function thereon [Schwartz, 1966].

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a tempered distribution on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by C^0 functions of slow growth on $\mathbb{R}$ (in fact, even as a derivative of order in $\mathbb{N}$ of a distribution on $\mathbb{R}$ generated by a C^0 function of slow growth on $\mathbb{R}$) [Schwartz, 1966]. In particular, each locally integrable function of slow growth on $\mathbb{R}$ generates a regular tempered distribution on $\mathbb{R}$, but, in contrast, a regular tempered distribution on $\mathbb{R}$ need not be generable by a locally integrable function of slow growth on $\mathbb{R}$ [Estrada and Kanwal, 2002]; for example, the function defined almost everywhere on $\mathbb{R}$ by e^{je^ξ} is a locally integrable function of slow growth on $\mathbb{R}$ generating a regular tempered distribution on $\mathbb{R}$ whose each derivative of any order in $\mathbb{N}$ is a regular tempered distribution on $\mathbb{R}$ that is not generable by a locally integrable function of slow growth on $\mathbb{R}$ [Reed and Simon, 1975; Estrada and Kanwal, 2002]. Each distribution on $\mathbb{R}$ that is homogeneous of some degree in $\mathbb{C}$ is a tempered distribution on $\mathbb{R}$ [Vladimirov, 2002]. The distribution space of the tempered distributions on $\mathbb{R}$ is the smallest enlargement of the function space of the integrable functions on $\mathbb{R}$ that is invariant both under differentiation and under multiplication by polynomial functions on $\mathbb{R}$ [Hörmander, 1990; Benedetto, 1997].

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a p -integrable distribution on $\mathbb{R}$ (respectively, a distribution vanishing at infinity on $\mathbb{R}$) is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by p -integrable functions on $\mathbb{R}$ (respectively, by C^0 functions vanishing at infinity on $\mathbb{R}$), such p -integrable functions on $\mathbb{R}$ being selectable as C^0 bounded functions on $\mathbb{R}$ (and, when p is in $\mathbb{R}_{[1,+\infty)}$, even as C^0 functions vanishing at infinity on $\mathbb{R}$) [Schwartz, 1966]. In particular, when p equals 1, 2, and $+\infty$, a p -integrable distribution on $\mathbb{R}$ is said to be respectively an *integrable distribution* on $\mathbb{R}$, a *square integrable distribution* on $\mathbb{R}$, and a *bounded distribution* on $\mathbb{R}$. It is worth noting that each bounded function on $\mathbb{R}$ generates a regular bounded distribution on $\mathbb{R}$, but, in contrast, a regular bounded distribution on $\mathbb{R}$ need not be generable by a bounded function on $\mathbb{R}$; for example, the function defined almost everywhere on $\mathbb{R}$ by $|\xi|^{-1/2}$ is an element of $L^1_{\text{loc}}(\mathbb{R}) \setminus L^2_{\text{loc}}(\mathbb{R})$ generating a regular element of $\mathcal{D}'_{L^\infty}(\mathbb{R})$. For any p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$ with p_2 greater than p_1 , we have [Schwartz, 1966]

$$\mathcal{D}'_{L^{p_2}}(\mathbb{R}) \supset \mathcal{D}'_{L^{p_1}}(\mathbb{R})$$

The concept of distribution vanishing at infinity on $\mathbb{R}$ can be usefully generalised [Ciampa *et al.*, 2019]. We have

$$\dot{\mathcal{B}}'(\mathbb{R}) = \dot{\mathcal{B}}'_{-\infty}(\mathbb{R}) \cap \dot{\mathcal{B}}'_{+\infty}(\mathbb{R})$$

where

- $\dot{\mathcal{B}}'_{-\infty}(\mathbb{R})$ is the distribution space of the *distributions vanishing at negative infinity* on $\mathbb{R}$
- $\dot{\mathcal{B}}'_{+\infty}(\mathbb{R})$ is the distribution space of the *distributions vanishing at positive infinity* on $\mathbb{R}$

The multiplication of a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_1}}(\mathbb{R})$ by a function that is an element of $\mathcal{D}_{L^{p_2}}(\mathbb{R})$ with p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2$ is in $\mathbb{R}_{[0,1]}$ is well-defined and commutative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_{12}}}(\mathbb{R})$ with p_{12} in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{12}$ equals $1/p_1 + 1/p_2$ [Schwartz, 1966]. In particular, the multiplication of a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{AP}(\mathbb{R})$ (as introduced below) by a function that is an element of $\mathcal{D}_{AP}(\mathbb{R})$ is well-defined and commutative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{AP}(\mathbb{R})$ [Schwartz, 1966].

The convolution of two distributions on $\mathbb{R}$ that are elements of respectively $\mathcal{D}'_{L^{p_1}}(\mathbb{R})$ and $\mathcal{D}'_{L^{p_2}}(\mathbb{R})$ with p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2 - 1$ is in $\mathbb{R}_{[0,1]}$ is well-defined and commutative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_{12}}}(\mathbb{R})$ with p_{12} in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{12}$ equals $1/p_1 + 1/p_2 - 1$ [Schwartz, 1966]. In particular, the convolution of two distributions on $\mathbb{R}$ that are elements of respectively $\mathcal{D}'_{AP}(\mathbb{R})$ (as introduced below) and $\mathcal{D}'_{L^1}(\mathbb{R})$ is well-defined and commutative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{AP}(\mathbb{R})$ [Schwartz, 1966]. Furthermore, the convolution of three distributions on $\mathbb{R}$ that are elements of respectively $\mathcal{D}'_{L^{p_1}}(\mathbb{R})$, $\mathcal{D}'_{L^{p_2}}(\mathbb{R})$, and $\mathcal{D}'_{L^{p_3}}(\mathbb{R})$ with p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$, p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$, and p_3 in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2 + 1/p_3 - 2$ is in $\mathbb{R}_{[0,1]}$ as well as one the equivalent additional conditions that $1/p_1 + 1/p_2 - 1$ is in $\mathbb{R}_{[0,1]}$, $1/p_2 + 1/p_3 - 1$ is in $\mathbb{R}_{[0,1]}$, or $1/p_1 + 1/p_3 - 1$ is in $\mathbb{R}_{[0,1]}$ is well-defined, commutative, and associative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_{123}}}(\mathbb{R})$ with p_{123} in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{123}$ equals $1/p_1 + 1/p_2 + 1/p_3 - 2$ [Čížek and Jelínek, 1996; Ortner, 2004; Ciampa *et al.*, 2008].

Whenever the 0^{th} moment of a distribution on $\mathbb{R}$ is well-defined in the Cesàro sense (but not necessarily in the ordinary sense) and is finite, it can be taken as the *improper integral* on $\mathbb{R}$ of the distribution on $\mathbb{R}$ [Sikorski, 1961; Musielak, 1963; Wawak, 1987, 1988; Takači, 1990; Estrada and Kanwal, 2002]. In particular, whenever the 0^{th} moment of a distribution on $\mathbb{R}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite, it can be taken as the *integral* on $\mathbb{R}$ of the distribution on $\mathbb{R}$ [Schwartz, 1954a, 1957, 1966; Horváth, 1966; Bijma, 1999; Thomas, 2000; Kierat and Skórník, 2001; Estrada and Kanwal, 2002; Dijk, 2013; Mincheva-Kamińska, 2015; Beffa, 2024]; in other words, the integral on $\mathbb{R}$ of an integrable distribution on $\mathbb{R}$ coincides with the result of its evaluation in the ordinary sense against $1_{\mathbb{R}}$ (such evaluation being possible because each integrable distribution on $\mathbb{R}$ can be regarded as a continuous linear form not only on $\dot{\mathcal{B}}(\mathbb{R})$ but also on $\mathcal{D}_{L^\infty}(\mathbb{R})$) [Schwartz, 1954a, 1957, 1966; Horváth, 1966; Bijma, 1999; Thomas, 2000; Kierat and Skórník, 2001; Dijk, 2013; Mincheva-Kamińska, 2015; Beffa, 2024]. Accordingly, a distribution on $\mathbb{R}$ has finite integral on $\mathbb{R}$ if and only if it is an integrable distribution on $\mathbb{R}$ (but a distribution on $\mathbb{R}$ that is not an integrable distribution on $\mathbb{R}$ may have finite improper integral on $\mathbb{R}$). By convention, a non-negative-definite distribution on $\mathbb{R}$ that is not an integrable distribution on $\mathbb{R}$ has integral on $\mathbb{R}$ equalling $+\infty$ [Schwartz, 1966]. A necessary and sufficient condition for a distribution to be an integrable distribution on $\mathbb{R}$ is that its 0^{th} moment is well-defined in the ordinary sense (and hence in the Cesàro sense), is finite (the 0^{th} moment in the ordinary sense being the same as the one in the Cesàro sense), and coincides with its integral on $\mathbb{R}$ (as well as with its improper integral on $\mathbb{R}$) [Schwartz, 1954a, 1957, 1966; Horváth, 1966; Bijma, 1999; Thomas, 2000; Kierat and Skórník, 2001; Estrada and Kanwal, 2002; Dijk, 2013; Beffa, 2024]. An integrable distribution on $\mathbb{R}$ has finite integral on $\mathbb{R}$, and each derivative of any order in $\mathbb{N}$ of an integrable distribution on $\mathbb{R}$ is an integrable distribution on $\mathbb{R}$ with null integral on $\mathbb{R}$ [Musiak, 1963]. The integral on $\mathbb{R}$ of an integrable distribution on $\mathbb{R}$ does not depend on the specific expression of it as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by integrable functions on $\mathbb{R}$ [Schwartz, 1954a; Musielak, 1963; Oberguggenberger, 1992; Kierat and Skórník, 2001; Ortner and Wagner, 2015]; in fact, the integral on $\mathbb{R}$ of an integrable distribution on $\mathbb{R}$ equals the integral on $\mathbb{R}$ of the 0^{th} -derivative term of its expression as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by integrable functions on $\mathbb{R}$ [Musiak, 1963; Oberguggenberger, 1992].

We now introduce a useful type of bounded distribution on $\mathbb{R}$. We have [Schwartz, 1966; Corduneanu, 2009; Strungaru and Terauds, 2016]

$$\mathcal{D}'_{L^\infty}(\mathbb{R}) \supset \mathcal{D}'_{AP}(\mathbb{R})$$

where

- $\mathcal{D}'_{AP}(\mathbb{R})$ is the distribution space of the *uniformly almost periodic distributions* on $\mathbb{R}$ (also known as *Bohr almost periodic distributions* on $\mathbb{R}$, *Bochner almost periodic distributions* on $\mathbb{R}$, or *strongly almost periodic distributions* on $\mathbb{R}$)

Care must be taken not to confuse $(\mathcal{D}_{AP}(\mathbb{R}))'$ with $\mathcal{D}'_{AP}(\mathbb{R})$.

A uniformly almost periodic distribution on $\mathbb{R}$ is a bounded distribution on $\mathbb{R}$ such that the translation of it by the translation vector ξ_0 results in a distribution-valued uniformly almost periodic function on $\mathbb{R}$ of the free real variable ξ_0 (an equivalent definition being obtainable by invoking the concept of *almost period*) [Schwartz, 1966; Corduneanu, 2009]. A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a uniformly almost periodic distribution on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by uniformly almost periodic functions on $\mathbb{R}$ [Schwartz, 1966; Burkill and Rennie, 1983; Corduneanu, 2009; Zaidman, 2011]. A uniformly almost periodic distribution on $\mathbb{R}$ vanishes at negative infinity (respectively, positive infinity) if and only if it coincides with $0_{\mathbb{R}}$ [Schwartz, 1966; Ciampa *et al.*, 2019]; accordingly, the intersection of $\mathcal{D}'_{AP}(\mathbb{R})$ and $\dot{\mathcal{B}}'(\mathbb{R})$ contains only $0_{\mathbb{R}}$. Each uniformly almost periodic distribution on $\mathbb{R}$ has one and only one *Fourier–Bohr series* (also known as *generalised Fourier series*) associated with it, and two uniformly almost periodic distributions on $\mathbb{R}$ coincide if and only if so do their Fourier–Bohr series [Schwartz, 1966; Burkill and Rennie, 1983; Lagarias, 2000; Corduneanu, 2009]. In fact, each uniformly almost periodic distribution on $\mathbb{R}$ has a *Fourier–Bohr series representation* (also known as *generalised Fourier series representation*), which means that it can be expressed as a possibly infinite but convergent (in $\mathcal{D}'_{L^\infty}(\mathbb{R})$) linear combination of countably many distributions on $\mathbb{R}$ generated by exponential functions on $\mathbb{R}$ with 1st-degree monomial imaginary arguments [Schwartz, 1966; Burkill and Rennie, 1983; Lagarias, 2000; Corduneanu, 2009]; the coefficients of the linear combination and the imaginary parts of the coefficients of the arguments of the exponential functions on $\mathbb{R}$ are said to be respectively the *Fourier–Bohr coefficients* (also known as *generalised Fourier coefficients*) and the *Fourier–Bohr exponents* (also known as *generalised Fourier exponents*) of the uniformly almost periodic distribution on $\mathbb{R}$ [Schwartz, 1966; Burkill and Rennie, 1983; Lagarias, 2000; Corduneanu, 2009]. Without loss of generality, we assume that each uniformly almost periodic distribution on $\mathbb{R}$ has a null Fourier–Bohr exponent; the corresponding Fourier–Bohr coefficient, possibly null, is the *mean value* on $\mathbb{R}$ of the uniformly almost periodic distribution on $\mathbb{R}$ [Schwartz, 1966; Burkill and

Rennie, 1983; Corduneanu, 2009; Zaidman, 2011]. A uniformly almost periodic distribution on $\mathbb{R}$ has countably many pairwise incommensurable non-null Fourier–Bohr exponents [Levitan and Zhikov, 1982; Corduneanu, 1989, 2009; Baake and Grimm, 2013]; a uniformly almost periodic distribution on $\mathbb{R}$ having finitely many (in particular, no) pairwise incommensurable non-null Fourier–Bohr exponents is said to be a *quasiperiodic distribution* on $\mathbb{R}$ (in particular, a *periodic distribution* on $\mathbb{R}$) [Levitan and Zhikov, 1982; Corduneanu, 1989, 2009; Baake and Grimm, 2013]. In fact, for a periodic distribution on $\mathbb{R}$, the Fourier–Bohr series, the Fourier–Bohr coefficients, and the Fourier–Bohr exponents are respectively the *Fourier series*, the *Fourier coefficients*, and the *Fourier exponents*, and the moduli of the non-null Fourier–Bohr exponents are integer multiples of their minimum, 2π times the multiplicative inverse of which being the *period* of the periodic distribution on $\mathbb{R}$ [Lighthill, 1958; Zemanian, 1965a; Schwartz, 1966; Jones, 1982; Richards and Youn, 1990; Friedlander and Joshi, 1998; Vladimirov, 2002; Estrada and Kanwal, 2002; Kanwal, 2004; King, 2009a; Duistermaat and Kolk, 2010]. The distribution space of the periodic distributions on $\mathbb{R}$ is not invariant under addition; hence, unlike the distribution space of the uniformly almost periodic distributions on $\mathbb{R}$ and the distribution space of the quasiperiodic distributions on $\mathbb{R}$, it has the serious pathology that it cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$.

Let $(VMO^\infty(\mathbb{R}))' \equiv \mathcal{D}'_{H^1}(\mathbb{R})$ and $(\mathcal{D}_{H^1}(\mathbb{R}))' \equiv \mathcal{D}'_{BMO}(\mathbb{R})$ be the topological dual spaces of respectively $VMO^\infty(\mathbb{R})$ and $\mathcal{D}_{H^1}(\mathbb{R})$. We have [Ortner, 2004]

$$\mathcal{D}'_{BMO}(\mathbb{R}) \supset (\mathcal{D}'_{L^\infty}(\mathbb{R})/\mathbb{C})$$

$$\mathcal{D}'_{L^1}(\mathbb{R}) \supset \mathcal{D}'_{H^1}(\mathbb{R})$$

where

- $\mathcal{D}'_{BMO}(\mathbb{R})$ is the distribution space of the *distributions of bounded mean oscillation* on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant
- $\mathcal{D}'_{H^1}(\mathbb{R})$ is the distribution space of the *real-variable Hardy distributions of exponent 1* on $\mathbb{R}$
- $\mathcal{D}'_{L^\infty}(\mathbb{R})/\mathbb{C}$ is the quotient space of the distribution space of the bounded distributions on $\mathbb{R}$ under the equivalence relation of equality up to addition by an arbitrary complex constant

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of bounded mean oscillation on $\mathbb{R}$ is that it can be expressed as the sum of a distribution on $\mathbb{R}$ generated by a function of bounded mean oscillation on $\mathbb{R}$ and a bounded distribution on $\mathbb{R}$ [Ortner, 2004].

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a real-variable Hardy distribution of exponent 1 on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by real-variable Hardy functions of exponent 1 on $\mathbb{R}$ [Ortner, 2004]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a real-variable Hardy distribution of exponent 1 on $\mathbb{R}$ is that it is an integrable distribution on $\mathbb{R}$ whose Hilbert transform (as introduced below) is an integrable distribution on $\mathbb{R}$ [Ortner, 2004]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a real-variable Hardy distribution of exponent 1 on $\mathbb{R}$ is that it is an integrable distribution on $\mathbb{R}$ that is the Hilbert transform (as introduced below) of an integrable distribution on $\mathbb{R}$.

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-n)$ -weighted p -integrable distribution on $\mathbb{R}$ (respectively, a $(-n)$ -weighted distribution vanishing at infinity on $\mathbb{R}$) is that its multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-n/2}$ is a p -integrable distribution on $\mathbb{R}$ (respectively, a distribution vanishing at infinity on $\mathbb{R}$) [Schwartz, 1962; Carton-Lebrun, 1995, 2005; Ortner and Wagner, 2013]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-n)$ -weighted p -integrable distribution on $\mathbb{R}$ (respectively, a $(-n)$ -weighted distribution vanishing at infinity on $\mathbb{R}$) is that it can be expressed as the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{n/2}$ (or, equivalently, of an n^{th} -degree polynomial function on $\mathbb{R}$) by a p -integrable distribution on $\mathbb{R}$ (respectively, by a distribution vanishing at infinity on $\mathbb{R}$) [Schwartz, 1962; Carton-Lebrun, 1995, 2005; Ortner and Wagner, 2013]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-n)$ -weighted p -integrable distribution on $\mathbb{R}$ (respectively, a $(-n)$ -weighted distribution vanishing at infinity on $\mathbb{R}$) is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by multiplicative products of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{n/2}$ (or, equivalently, of an n^{th} -degree polynomial function on $\mathbb{R}$) by p -integrable functions on $\mathbb{R}$ (respectively, by C^0 functions vanishing at infinity on $\mathbb{R}$)

[Wagner, 2014]; it is worth noting that the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{n/2}$ (or, equivalently, of an n^{th} -degree polynomial function on $\mathbb{R}$) by a p -integrable function on $\mathbb{R}$ is a $(-n)$ -weighted p -integrable function on $\mathbb{R}$. In particular, when p equals 1, 2, and $+\infty$, a $(-n)$ -weighted p -integrable distribution on $\mathbb{R}$ is said to be respectively a $(-n)$ -weighted integrable distribution on $\mathbb{R}$, a $(-n)$ -weighted square integrable distribution on $\mathbb{R}$ (also known as $(-n)$ -weighted quadratically integrable distribution on $\mathbb{R}$), and a $(-n)$ -weighted bounded distribution on $\mathbb{R}$. Each function of bounded mean oscillation on $\mathbb{R}$ generates a (-2) -weighted integrable distribution on $\mathbb{R}$ [Fefferman and Stein, 1972; Garnett, 2007]. Furthermore, an element of $\mathcal{P}_p(\mathbb{R})$ generates a regular element of any $\mathcal{D}'_{L^p, -n}(\mathbb{R})$ with n greater than some minimum value. For any p_1 in $\bar{\mathbb{R}}_{[1, +\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1, +\infty]}$ with p_2 greater than p_1 , we have [Ortner and Wagner, 2013]

$$\mathcal{D}'_{L^{p_2}, -n}(\mathbb{R}) \supset \mathcal{D}'_{L^{p_1}, -n}(\mathbb{R})$$

Furthermore, we have [Ortner and Wagner, 2013]

$$\mathcal{D}'_{L^p, -n-1}(\mathbb{R}) \supset \mathcal{D}'_{L^p, -n}(\mathbb{R}) \supset \mathcal{D}'_{L^p}(\mathbb{R})$$

$$\dot{\mathcal{B}}'_{-n-1}(\mathbb{R}) \supset \dot{\mathcal{B}}'_{-n}(\mathbb{R}) \supset \dot{\mathcal{B}}'(\mathbb{R})$$

Hereinafter, we occasionally need consider distribution spaces obtained by generalising $\mathcal{D}'_{L^p, -n}(\mathbb{R})$

(respectively, $\dot{\mathcal{B}}'_{-n}(\mathbb{R})$). Let $(\dot{\mathcal{B}}'_{z_1, z_2}(\mathbb{R}))' \equiv \mathcal{D}'_{L^1, -z_1, -z_2}(\mathbb{R})$, $(\mathcal{D}_{L^{q'}, z_1, z_2}(\mathbb{R}))' \equiv \mathcal{D}'_{L^q, -z_1, -z_2}(\mathbb{R})$, and

$(\mathcal{D}_{L^1, z_1, z_2}(\mathbb{R}))' \equiv \mathcal{D}'_{L^\infty, -z_1, -z_2}(\mathbb{R})$ be the topological dual spaces of respectively $\dot{\mathcal{B}}'_{z_1, z_2}(\mathbb{R})$, $\mathcal{D}_{L^{q'}, z_1, z_2}(\mathbb{R})$,

and $\mathcal{D}_{L^1, z_1, z_2}(\mathbb{R})$. The chain of inclusion relations holds [Guzmán-Partida *et al.*, 2011; Ortner and

Wagner, 2013]

$$\mathcal{D}'_{L^p, -z_1, -z_2}(\mathbb{R}) \supset \dot{\mathcal{B}}'_{-z_1, -z_2}(\mathbb{R}) \supset \mathcal{D}'_{L^q, -z_1, -z_2}(\mathbb{R}) \supset \mathcal{D}'_{L^1, -z_1, -z_2}(\mathbb{R})$$

with

$$\mathcal{D}'_{L^p, -z_1, 0}(\mathbb{R}) \equiv \mathcal{D}'_{L^p, -z_1}(\mathbb{R})$$

$$\mathcal{D}'_{L^p,0,0}(\mathbb{R}) \equiv \mathcal{D}'_{L^p}(\mathbb{R})$$

$$\dot{\mathcal{B}}'_{z_1,0}(\mathbb{R}) \equiv \dot{\mathcal{B}}'_{z_1}(\mathbb{R})$$

$$\dot{\mathcal{B}}'_{0,0}(\mathbb{R}) \equiv \dot{\mathcal{B}}'(\mathbb{R})$$

where

- $\mathcal{D}'_{L^p,-z_1,-z_2}(\mathbb{R})$ is the distribution space of the $(-z_1, -z_2)$ -weighted p -integrable distributions on $\mathbb{R}$
- $\dot{\mathcal{B}}'_{-z_1,-z_2}(\mathbb{R})$ is the distribution space of the $(-z_1, -z_2)$ -weighted distributions vanishing at infinity on $\mathbb{R}$

We warn that, for $|z_1|$ even, the distribution spaces denoted here by $\mathcal{D}'_{L^1,-z_1}(\mathbb{R})$ are denoted elsewhere by $\mathcal{S}'_k(\mathbb{R})$ with $k \equiv z_1/2$ [Horváth, 1966; Ortner and Wagner, 2014].

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-z_1, -z_2)$ -weighted p -integrable distribution on $\mathbb{R}$ (respectively, a $(-z_1, -z_2)$ -weighted distribution vanishing at infinity on $\mathbb{R}$) is that its multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{-z_1/2} (\ln(e + \xi^2))^{-z_2}$ is a p -integrable distribution on $\mathbb{R}$ (respectively, a distribution vanishing at infinity on $\mathbb{R}$) [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-z_1, -z_2)$ -weighted p -integrable distribution on $\mathbb{R}$ (respectively, a $(-z_1, -z_2)$ -weighted distribution vanishing at infinity on $\mathbb{R}$) is that it can be expressed as the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{z_1/2} (\ln(e + \xi^2))^{z_2}$ by a p -integrable distribution on $\mathbb{R}$ (respectively, by a distribution vanishing at infinity on $\mathbb{R}$) [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-z_1, -z_2)$ -weighted p -integrable distribution on $\mathbb{R}$ (respectively, a $(-z_1, -z_2)$ -weighted distribution vanishing at infinity on $\mathbb{R}$) is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by multiplicative products of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{z_1/2} (\ln(e + \xi^2))^{z_2}$ by p -integrable functions on $\mathbb{R}$ (respectively, by C^0 functions vanishing at infinity on $\mathbb{R}$) [Wagner, 2014]. In particular, when p equals 1, 2, and $+\infty$, a

$(-z_1, -z_2)$ -weighted p -integrable distribution on $\mathbb{R}$ is said to be respectively a $(-z_1, -z_2)$ -weighted integrable distribution on $\mathbb{R}$, a $(-z_1, -z_2)$ -weighted square integrable distribution on $\mathbb{R}$ (also known as $(-z_1, -z_2)$ -weighted quadratically integrable distribution on $\mathbb{R}$), and a $(-z_1, -z_2)$ -weighted bounded distribution on $\mathbb{R}$. For any p_1 in $\overline{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\overline{\mathbb{R}}_{[1,+\infty]}$ with p_2 greater than p_1 , we have [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]

$$\mathcal{D}'_{L^{p_2}, -z_1, -z_2}(\mathbb{R}) \supset \mathcal{D}'_{L^{p_1}, -z_1, -z_2}(\mathbb{R})$$

Furthermore, we have [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013]

$$\mathcal{D}'_{L^p, -z_1-1, -z_2}(\mathbb{R}) \supset \mathcal{D}'_{L^p, -z_1, -z_2}(\mathbb{R})$$

$$\mathcal{D}'_{L^p, -z_1, -z_2-1}(\mathbb{R}) \supset \mathcal{D}'_{L^p, -z_1, -z_2}(\mathbb{R})$$

$$\dot{\mathcal{B}}'_{-z_1-1, -z_2}(\mathbb{R}) \supset \dot{\mathcal{B}}'_{-z_1, -z_2}(\mathbb{R})$$

$$\dot{\mathcal{B}}'_{-z_1, -z_2-1}(\mathbb{R}) \supset \dot{\mathcal{B}}'_{-z_1, -z_2}(\mathbb{R})$$

The multiplication of a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_1}, z_{11}, z_{12}}(\mathbb{R})$ by a function that is an element of $\mathcal{D}_{L^{p_2}, z_{21}, z_{22}}(\mathbb{R})$ with p_1 in $\overline{\mathbb{R}}_{[1,+\infty]}$, z_{11} in $\mathbb{Z}$, z_{12} in $\mathbb{Z}$, p_2 in $\overline{\mathbb{R}}_{[1,+\infty]}$, z_{21} in $\mathbb{Z}$, and z_{22} in $\mathbb{Z}$ satisfying the condition that $1/p_1 + 1/p_2$ is in $\mathbb{R}_{[0,1]}$ is well-defined and commutative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_{12}}, z_{121}, z_{122}}(\mathbb{R})$ with p_{12} in $\overline{\mathbb{R}}_{[1,+\infty]}$, z_{121} in $\mathbb{Z}$, and z_{122} in $\mathbb{Z}$ satisfying the conditions that $1/p_{12}$ equals $1/p_1 + 1/p_2$, z_{121} equals $z_{11} + z_{21}$, and z_{122} equals $z_{12} + z_{22}$ [Ortner and Wagner, 2013].

The convolution of two distributions on $\mathbb{R}$ that are elements of respectively $\mathcal{D}'_{L^{p_1}, z_{11}, z_{12}}(\mathbb{R})$ and $\mathcal{D}'_{L^{p_2}, z_{21}, z_{22}}(\mathbb{R})$ with p_1 in $\overline{\mathbb{R}}_{[1,+\infty]}$, z_{11} in $\mathbb{Z}$, z_{12} in $\mathbb{Z}$, p_2 in $\overline{\mathbb{R}}_{[1,+\infty]}$, z_{21} in $\mathbb{Z}$, and z_{22} in $\mathbb{Z}$ satisfying the conditions that $1/p_1 + 1/p_2 - 1$ is in $\mathbb{R}_{[0,1]}$, $z_{11} + z_{21}$ is in $\mathbb{N}$, and $z_{12} + z_{22}$ is in $\mathbb{N}$ is well-defined and commutative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_{12}}, z_{121}, z_{122}}(\mathbb{R})$ with p_{12} in $\overline{\mathbb{R}}_{[1,+\infty]}$, z_{121} in $\mathbb{Z}$, and z_{122} in $\mathbb{Z}$ satisfying the conditions that $1/p_{12}$ equals $1/p_1 + 1/p_2 - 1$, z_{121} equals the minimum of z_{11} and z_{21} , and z_{122} equals the minimum of z_{12} and z_{22} [Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013; Wagner, 2014].

The *moment asymptotic expansion* of a distribution on $\mathbb{R}$ is a particular expression of the limit within $\mathcal{D}'(\mathbb{R})$ (in fact, within $\mathcal{S}'(\mathbb{R})$) of the distribution on $\mathbb{R}$ dilated by a dilation coefficient of suitable quantity dimension (respectively, by a dilation factor) in $\mathbb{R}_{(0,+\infty)}$ as the dilation coefficient (respectively, dilation factor) tends to $+\infty$ [Estrada and Kanwal, 2002]; in a certain sense, it describes the behaviour of the distribution on $\mathbb{R}$ on a neighbourhood of infinity [Estrada and Kanwal, 2002]. Whenever well-defined, the moment asymptotic expansion of a distribution on $\mathbb{R}$ coincides with the inverse Fourier transform (as introduced below) of the *Maclaurin series* of the Fourier transform (as introduced below) of the distribution on $\mathbb{R}$ [Estrada and Kanwal, 2002; Kannai, 2003]. A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be an asymptotically small distribution on $\mathbb{R}$ is that its moment asymptotic expansion is well-defined [Estrada and Kanwal, 2002]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be an asymptotically small distribution on $\mathbb{R}$ is that it is of rapid decay at infinity in the Cesàro sense (namely, that it vanishes at infinity in the Cesàro sense more rapidly than any distribution on $\mathbb{R}$ generated by the multiplicative inverse of a polynomial function on $\mathbb{R}$ or, equivalently, that, on a neighbourhood of infinity, it is of order of magnitude in the Cesàro sense inferior to any distribution on $\mathbb{R}$ generated by the multiplicative inverse of a polynomial function on $\mathbb{R}$) [Estrada and Kanwal, 2002]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be an asymptotically small distribution on $\mathbb{R}$ is that its moment of any order in $\mathbb{N}$ is well-defined in the Cesàro sense (but not necessarily in the ordinary sense) and is finite [Estrada and Kanwal, 2002]. We observe that no inclusion relation exists between $\mathcal{K}'(\mathbb{R})$ and $\mathcal{D}'_{L^1}(\mathbb{R})$. An example of element of $\mathcal{K}'(\mathbb{R}) \cap \mathcal{D}'_{L^1}(\mathbb{R})$ is any element of $\mathcal{O}'_C(\mathbb{R})$. An example of element of $\mathcal{D}'_{L^1}(\mathbb{R}) \setminus \mathcal{K}'(\mathbb{R})$ is f.p. $(1/\xi^2)$ (as introduced below). Examples of elements of $\mathcal{K}'(\mathbb{R}) \setminus \mathcal{D}'_{L^1}(\mathbb{R})$ are the distribution on $\mathbb{R}$ generated by the *sinc function* (also known as *sinus cardinalis function* or *cardinal sine function*) sinc , which is defined almost everywhere on $\mathbb{R}$ by [Borwein et al., 2010]

$$\text{sinc}(\xi) := \frac{\sin(\xi)}{\xi}$$

as well as the distribution on $\mathbb{R}$ generated by the *normalised sinc function* (also known as *normalised sinus cardinalis function* or *normalised cardinal sine function*) nsinc , which is defined almost everywhere on $\mathbb{R}$ by

$$\text{nsinc}(\xi) := \frac{\sin(\pi\xi)}{\pi\xi}$$

We observe that sinc and nsinc are elements of $\left(L^{p_{(1,+\infty)}}(\mathbb{R}) \cap \dot{\mathcal{B}}(\mathbb{R})\right) \setminus L^1(\mathbb{R})$ generating elements of $\left(\mathcal{D}'_{L^{p_{(1,+\infty)}}}(\mathbb{R}) \cap \mathcal{K}'(\mathbb{R})\right) \setminus \mathcal{D}'_{L^1}(\mathbb{R})$ with $p_{(1,+\infty)}$ in $\bar{\mathbb{R}}_{(1,+\infty]}$ [Borwein *et al.*, 2010]. In particular, the continuous representatives of sinc and of its 1st derivative $\text{sinc}^{(1)}$ are expressed as

$$\text{sinc}(\xi) = \begin{cases} \frac{\sin(\xi)}{\xi} & , \quad \xi \in \mathbb{R}_{(-\infty,0) \cup (0,+\infty)} \\ 1 & , \quad \xi = 0 \end{cases}$$

$$\text{sinc}^{(1)}(\xi) = \begin{cases} \frac{\cos(\xi) - \text{sinc}(\xi)}{\xi} & , \quad \xi \in \mathbb{R}_{(-\infty,0) \cup (0,+\infty)} \\ 0 & , \quad \xi = 0 \end{cases}$$

and the continuous representatives of nsinc and of its 1st derivative $\text{nsinc}^{(1)}$ are expressed as

$$\text{nsinc}(\xi) = \begin{cases} \frac{\sin(\pi\xi)}{\pi\xi} & , \quad \xi \in \mathbb{R}_{(-\infty,0) \cup (0,+\infty)} \\ 1 & , \quad \xi = 0 \end{cases}$$

$$\text{nsinc}^{(1)}(\xi) = \begin{cases} \frac{\cos(\pi\xi) - \text{nsinc}(\pi\xi)}{\xi} & , \quad \xi \in \mathbb{R}_{(-\infty,0) \cup (0,+\infty)} \\ 0 & , \quad \xi = 0 \end{cases}$$

We refer to the regular distributions on $\mathbb{R}$ generated by sinc and nsinc , which we also denote by sinc and nsinc , as respectively the *sinc distribution* on $\mathbb{R}$ and the *normalised sinc distribution* on $\mathbb{R}$. Hereinafter, when not explicitly clarified, it is understood from the context whether any of the symbols sinc and nsinc denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. We have

$$\text{nsinc} = \mathcal{S}_{\pi}(\text{sinc})$$

$$\text{nsinc}' = \pi \mathcal{S}_{\pi}(\text{sinc}^{(1)})$$

where

- $\mathcal{S}_{\xi_{\theta}}$ is the *dilation operator of dilation coefficient* (possibly, *dilation operator of dilation factor*) ξ_{θ}

Although the distributions on $\mathbb{R}$ generated by sinc and nsinc are not integrable distributions on $\mathbb{R}$ (which is confirmed by the fact that their Fourier transforms, as introduced below, are not generable by C^0 functions of slow growth on $\mathbb{R}$), they are asymptotically small distributions on $\mathbb{R}$ [Estrada and Kanwal, 2002]. Accordingly, although their moments of any order in $\mathbb{N}$ are not well-defined in the ordinary sense, they are well-defined in the Cesàro sense and are finite; in particular, although their improper integrals on $\mathbb{R}$ cannot be calculated as Lebesgue integrals with respect to the Lebesgue measure on $\mathbb{R}$, they can be calculated as improper Lebesgue integrals with respect to the Lebesgue measure on $\mathbb{R}$ (and they result in respectively π , which is twice the *Dirichlet integral*, and 1) [Wawak, 1987; Schwartz, 1993b; Benedetto, 1997; Estrada and Kanwal, 2000; Ortner and Wagner, 2013; Borwein *et al.*, 2010].

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of rapid decay on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by C^0 functions on $\mathbb{R}$ that vanish at infinity more rapidly than the multiplicative inverse of a polynomial function of arbitrary degree on $\mathbb{R}$ (the number of terms of the linear combination generally increasing as the arbitrary degree increases) [Schwartz, 1966; Treves, 1967]. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of rapid decay on $\mathbb{R}$ is that its multiplicative product by a polynomial function of arbitrary degree on $\mathbb{R}$ is an integrable distribution on $\mathbb{R}$ (or, equivalently, a distribution of rapid decay on $\mathbb{R}$) [Barros-Neto, 1973; Bijma, 1999; Thomas, 2000; Dijk, 2013; Beffa, 2024]. Accordingly, for a distribution of rapid decay on $\mathbb{R}$, the moment of any order in $\mathbb{N}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense).

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of very rapid decay on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by C^0 functions of rapid decay on $\mathbb{R}$ [Grothendieck, 1955; Schwartz, 1973].

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of compact support on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by C^0 functions of compact support on $\mathbb{R}$ (the support of each of such distributions being contained in an arbitrary neighbourhood of the

support of the distribution of compact support on $\mathbb{R}$) [Schwartz, 1966]. A special case of distribution of compact support on $\mathbb{R}$ is a *point-supported distribution* on $\mathbb{R}$ (namely, a distribution of compact support on $\mathbb{R}$ having support in a point of $\mathbb{R}$). A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a point-supported distribution on $\mathbb{R}$ is that it admits a unique decomposition as a finite linear combination of derivatives of order in $\mathbb{N}$ of δ each of which translated by the same translation vector [Zemanian, 1965a; Schwartz, 1966]. It can be easily verified that each set of derivatives of order in $\mathbb{N}$ of δ each of which translated by some translation vector is linearly independent if the order in $\mathbb{N}$ of each such derivative is different from the order in $\mathbb{N}$ of any other.

As is evident, the n_0^{th} derivative of an element of $\mathcal{T}(\mathbb{R})$ is an element of $\mathcal{T}(\mathbb{R})$, and the n_0^{th} derivative of an element of $(\mathcal{T}(\mathbb{R}))'$ is an element of $(\mathcal{T}(\mathbb{R}))'$ (but the n_0^{th} derivative of an element of $\mathcal{T}^{n_0}(\mathbb{R})$ need not be an element of $\mathcal{T}^{n_0}(\mathbb{R})$, and the n_0^{th} derivative of an element of $(\mathcal{T}^{n_0}(\mathbb{R}))'$ need not be an element of $(\mathcal{T}^{n_0}(\mathbb{R}))'$). In contrast, the n_0^{th} antiderivatives of an element of $\mathcal{T}(\mathbb{R})$ need not be elements of $\mathcal{T}(\mathbb{R})$, and the n_0^{th} antiderivatives of an element of $(\mathcal{T}(\mathbb{R}))'$ need not be elements of $(\mathcal{T}(\mathbb{R}))'$ (just as the n_0^{th} antiderivatives of an element of $\mathcal{T}^{n_0}(\mathbb{R})$ need not be elements of $\mathcal{T}^{n_0}(\mathbb{R})$, and the n_0^{th} antiderivatives of an element of $(\mathcal{T}^{n_0}(\mathbb{R}))'$ need not be elements of $(\mathcal{T}^{n_0}(\mathbb{R}))'$). A necessary and sufficient condition for an element of any of $\mathcal{K}'(\mathbb{R})$, $\mathcal{O}'_C(\mathbb{R})$, $\mathcal{O}'_M(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$ to have one (and necessarily only one) n_0^{th} antiderivative in the same distribution space is that such element has null moment of any order in $\mathbb{N}_{[0, n_0-1]}$ in the Cesàro sense (for an element of $\mathcal{O}'_C(\mathbb{R})$, $\mathcal{O}'_M(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$, the moment of any order in $\mathbb{N}_{[0, n_0-1]}$ in the ordinary sense being the same as the one in the Cesàro sense) [Estrada and Kanwal, 2002]. In particular, because the limit at infinity in the Cesàro sense of each periodic distribution on $\mathbb{R}$ equals its mean value on $\mathbb{R}$, each periodic distribution on $\mathbb{R}$ with null mean value on $\mathbb{R}$ is an element of $\mathcal{K}'(\mathbb{R})$ and has null moment of any order in $\mathbb{N}$ in the Cesàro sense [Estrada and Kanwal, 2002]; therefore, both the n_0^{th} derivative and one (and only one) of the n_0^{th} antiderivatives of a periodic distribution on $\mathbb{R}$ with null mean value on $\mathbb{R}$ are periodic distributions on $\mathbb{R}$ with null mean values on $\mathbb{R}$ [Estrada and Kanwal, 2002].

We know that an element of $\mathcal{O}'_C(\mathbb{R})$ is an element of $\mathcal{K}'(\mathbb{R})$, but, in contrast, an element of $\mathcal{K}'(\mathbb{R})$ need not be an element of $\mathcal{O}'_C(\mathbb{R})$; for example, the function defined almost everywhere on $\mathbb{R}$ by

$e^{j\xi}$ is an element of $(\mathcal{D}_{L^\infty}(\mathbb{R}) \setminus \dot{\mathcal{B}}(\mathbb{R})) \cap (\mathcal{O}_C(\mathbb{R}) \setminus \mathcal{K}(\mathbb{R}))$ generating an element of $(\mathcal{D}'_{L^\infty}(\mathbb{R}) \setminus \dot{\mathcal{B}}'(\mathbb{R})) \cap (\mathcal{K}'(\mathbb{R}) \setminus \mathcal{O}'_C(\mathbb{R}))$ [Estrada and Kanwal, 2002]. We observe that referring to the elements of $\mathcal{O}'_C(\mathbb{R})$ as distributions of rapid decay on $\mathbb{R}$ is meaningful only in the distributional sense. In fact, although an element of $\mathcal{S}(\mathbb{R})$ generates a regular element of $\mathcal{O}'_C(\mathbb{R})$, a regular element of $\mathcal{O}'_C(\mathbb{R})$ need not be generable by an element of $\mathcal{S}(\mathbb{R})$ [Schwartz, 1966; Treves, 1967; Estrada and Kanwal, 2002]; for example, the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$ generates a regular element of $\mathcal{O}'_C(\mathbb{R})$ but, being an element of $\mathcal{O}_M(\mathbb{R}) \setminus \mathcal{O}_C(\mathbb{R})$, is not an element of $\mathcal{S}(\mathbb{R})$. Remarkably, functions that do not vanish at infinity but oscillate fast enough on a neighbourhood of infinity also generate regular elements of $\mathcal{O}'_C(\mathbb{R})$ [Schwartz, 1966; Treves, 1967; Estrada and Kanwal, 2002]; for example, the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$ is an element of $\mathcal{O}_M(\mathbb{R}) \setminus \mathcal{O}_C(\mathbb{R})$ generating an element of $\mathcal{O}'_C(\mathbb{R}) \setminus \mathcal{O}'_M(\mathbb{R})$ [Bargetz and Ortner, 2014]. Finally, denoting by $\mathcal{T}^\wedge(\mathbb{R})$, $\mathcal{D}^\wedge_{L^p, z_1, z_2}(\mathbb{R})$, and $\dot{\mathcal{B}}^\wedge_{z_1, z_2}(\mathbb{R})$ the distribution spaces of the distributions on $\mathbb{R}$ generated by the elements of respectively $\mathcal{T}(\mathbb{R})$, $\mathcal{D}_{L^p, z_1, z_2}(\mathbb{R})$, and $\dot{\mathcal{B}}_{z_1, z_2}(\mathbb{R})$, the chains of inclusion relations hold [Schwartz, 1966; Ortner and Wagner, 2013; Bargetz and Ortner, 2013]

A.2 Mathematical framework: Functions, Radon measures, and distributions

$$\begin{array}{cccccccccccccccc}
 \mathcal{D}'(\mathbb{R}) & \supset & \mathcal{S}'(\mathbb{R}) & & \supset & \mathcal{D}'_{L^\infty, z_1, z_2}(\mathbb{R}) & \supset & \dot{\mathcal{B}}'_{z_1, z_2}(\mathbb{R}) & \supset & \mathcal{D}'_{L^q, z_1, z_2}(\mathbb{R}) & \supset & \mathcal{D}'_{L^1, z_1, z_2}(\mathbb{R}) & \supset & \mathcal{O}'_c(\mathbb{R}) & \supset & \mathcal{O}'_M(\mathbb{R}) & \supset & \mathcal{E}'(\mathbb{R}) \\
 \cup & & \cup & & & \cup & & \cup & & \cup & & \cup & & & & \cup & & \cup \\
 \mathcal{E}^\wedge(\mathbb{R}) & \supset & \mathcal{O}^\wedge_M(\mathbb{R}) & \supset & \mathcal{O}^\wedge_c(\mathbb{R}) & \supset & \mathcal{D}^\wedge_{L^\infty, z_1, z_2}(\mathbb{R}) & \supset & \dot{\mathcal{B}}^\wedge_{z_1, z_2}(\mathbb{R}) & \supset & \mathcal{D}^\wedge_{L^q, z_1, z_2}(\mathbb{R}) & \supset & \mathcal{D}^\wedge_{L^1, z_1, z_2}(\mathbb{R}) & \supset & \mathcal{S}^\wedge(\mathbb{R}) & \supset & \mathcal{D}^\wedge(\mathbb{R})
 \end{array}$$

A.2.4.3 Regularisation

We refer to a Lebesgue-measurable function on $\mathbb{R}$ that is not a locally integrable function on $\mathbb{R}$ as a *non-locally-integrable function* on $\mathbb{R}$. A non-locally-integrable function on $\mathbb{R}$ is said to have an *algebraic singularity* at a point of $\mathbb{R}$ if and only if its multiplicative product by some polynomial function on $\mathbb{R}$ is a locally integrable function on a neighbourhood of that point of $\mathbb{R}$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Furthermore, a non-locally-integrable function on $\mathbb{R}$ is said to admit *regularisation* if and only if it is a locally integrable function on $\mathbb{R}$ up to the presence of countably many algebraic singularities (any relatively compact subset of $\mathbb{R}$ containing only a finite subset of them) [Gelfand and Shilov, 1964; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. In particular, any locally integrable function on $\mathbb{R}$ admits regularisation [Gelfand and Shilov, 1964; Estrada and Kanwal, 1989, 2002; Kanwal, 2004].

A non-locally-integrable function on $\mathbb{R}$ admitting regularisation generates not only a regular distribution on each open subset of $\mathbb{R}$ containing none of its points of algebraic singularity but also, via a specified *regularisation method* of *ad hoc* nature, a singular distribution on $\mathbb{R}$ [Schwartz, 1966; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Such a singular distribution on $\mathbb{R}$ is said to be a *pseudofunction* on $\mathbb{R}$ generated by the non-locally-integrable function on $\mathbb{R}$ via the specified regularisation method [Lavoine, 1959, 1963; Schwartz, 1966; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. A non-locally-integrable function on $\mathbb{R}$ admitting regularisation generates different pseudofunctions on $\mathbb{R}$ via different regularisation methods [Estrada and Kanwal, 1989, 2002; Kanwal, 2004]; specifically, the difference of any two of such pseudofunctions on $\mathbb{R}$ is a finite linear combination of derivatives of order in $\mathbb{N}$ of δ each of which translated by some translation vector so as to have supports in the points of algebraic singularity of the non-locally-integrable function on $\mathbb{R}$ [Estrada and Kanwal, 1989, 2002; Kanwal, 2004].

The most important regularisation methods are the *Hadamard-finite-part method* and the *Cauchy-principal-value method*, the latter being a special case of the former [Lavoine, 1959, 1963; Estrada and Kanwal, 1989, 2000, 2002; Kanwal, 2004]. Specifically, if a non-locally-integrable function on $\mathbb{R}$ admitting regularisation generates a pseudofunction on $\mathbb{R}$ via the Cauchy-principal-value method, then it also generates a pseudofunction on $\mathbb{R}$ via the Hadamard-finite-part method, and the two pseudofunctions on $\mathbb{R}$ coincide [Estrada and Kanwal 2000]; in contrast, if a non-locally-integrable function on $\mathbb{R}$ admitting regularisation generates a pseudofunction on $\mathbb{R}$ via the Hadamard-finite-part method, then it need not generate a pseudofunction on $\mathbb{R}$ via the Cauchy-principal-value method [Estrada and Kanwal 2000]. In

particular, any locally integrable function on $\mathbb{R}$ generates one and only one pseudofunction on $\mathbb{R}$ (in fact, one and only one regular distribution on $\mathbb{R}$) both via the Cauchy-principal-value method and via the Hadamard-finite-part method.

A non-locally-integrable function on $\mathbb{R}$ admitting regularisation is said to admit *canonical regularisation* and to generate a *canonical pseudofunction* on $\mathbb{R}$ via a specified regularisation method if and only if the following conditions are satisfied (the regularisation method being assumed as fixed) [Gelfand and Shilov, 1964; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]:

- The sum of the non-locally-integrable function on $\mathbb{R}$ and an arbitrary non-locally-integrable function on $\mathbb{R}$ admitting regularisation generates a pseudofunction on $\mathbb{R}$ equalling the sum of the pseudofunction on $\mathbb{R}$ generated by the non-locally-integrable function on $\mathbb{R}$ and the pseudofunction on $\mathbb{R}$ generated by the arbitrary non-locally-integrable function on $\mathbb{R}$.
- The multiplicative product of the non-locally-integrable function on $\mathbb{R}$ by an arbitrary smooth function on $\mathbb{R}$ generates a pseudofunction on $\mathbb{R}$ equalling the multiplicative product of the pseudofunction on $\mathbb{R}$ generated by the non-locally-integrable function on $\mathbb{R}$ by the arbitrary smooth function on $\mathbb{R}$.
- The 1st derivative of the non-locally-integrable function on $\mathbb{R}$ (of which the 1st weak derivative does not exist) generates a pseudofunction on $\mathbb{R}$ equalling the 1st derivative of the pseudofunction on $\mathbb{R}$ generated by the non-locally-integrable function on $\mathbb{R}$.

It can be easily verified that, if such conditions are satisfied, so are generalised versions of them involving more than one arbitrary non-locally-integrable function on $\mathbb{R}$, more than one arbitrary smooth function on $\mathbb{R}$, and derivatives of order greater than 1.

Let f_{L^0} be a non-locally-integrable function on $\mathbb{R}$ admitting regularisation, and let h_{L^0} be a non-locally-integrable function on $\mathbb{R}$ admitting canonical regularisation. Hereinafter, we denote by $\text{Pf}(f_{L^0})$ the pseudofunction on $\mathbb{R}$ generated by f_{L^0} via the Hadamard-finite-part method; besides, we denote by $\text{f.p.}(h_{L^0})$ (when applicable, by $\text{p.v.}(h_{L^0})$) the canonical pseudofunction on $\mathbb{R}$ generated by h_{L^0} via the Hadamard-finite-part method (when applicable, via the Cauchy-principal-value method).

Finally, a terminological note is in order. If well-defined, the convolution of two distributions on $\mathbb{R}$ one of which is generated by a test function on $\mathbb{R}$ is sometimes said to be a *regularisation* in that it results in a distribution on $\mathbb{R}$ that is generable by a test function on $\mathbb{R}$ [Schwartz, 1966; Bargetz *et al.*, 2017]; the result of such regularisation is said to be the *regularised* of the distribution on $\mathbb{R}$ by the distribution on $\mathbb{R}$ generated by the test function on $\mathbb{R}$, which, in turn, is said to be the

regulator of the distribution on $\mathbb{R}$ [Schwartz, 1966; Bargetz *et al.*, 2017]. A regulator is said to be a *mollifier* if and only if, in addition to being an element of $\mathcal{D}^\wedge(\mathbb{R})$, it has unit integral on $\mathbb{R}$ and the limit within $\mathcal{D}'(\mathbb{R})$ of its dilated by a dilation coefficient of suitable quantity dimension (respectively, by a dilation factor) in $\mathbb{R}_{(0,+\infty)}$ times the dilation coefficient (respectively, dilation factor) as the dilation coefficient (respectively, dilation factor) tends to $+\infty$ is δ , which implies that its dilated by any dilation coefficient of suitable quantity dimension (respectively, by any dilation factor) in $\mathbb{R}_{(0,+\infty)}$ times the dilation coefficient (respectively, dilation factor) has unit integral on $\mathbb{R}$ as δ has. In particular, a distribution on $\mathbb{R}$ is an element of $\mathcal{D}'(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, $\mathcal{D}'_{L^p, z_1, z_2}(\mathbb{R})$, $\dot{\mathcal{B}}'_{z_1, z_2}(\mathbb{R})$, $\mathcal{D}'_{AP}(\mathbb{R})$, $\mathcal{O}'_C(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$ if and only if its regularised by any element of $\mathcal{D}^\wedge(\mathbb{R})$ is an element of respectively $\mathcal{E}^\wedge(\mathbb{R})$, $\mathcal{O}^\wedge_C(\mathbb{R})$, $\mathcal{D}^\wedge_{L^p, z_1, z_2}(\mathbb{R})$, $\dot{\mathcal{B}}^\wedge_{z_1, z_2}(\mathbb{R})$, $\mathcal{D}^\wedge_{AP}(\mathbb{R})$, $\mathcal{S}^\wedge(\mathbb{R})$, or $\mathcal{D}^\wedge(\mathbb{R})$ (or, equivalently, if and only if its regularised by any element of $\mathcal{D}^\wedge(\mathbb{R})$ is an element of respectively $\mathcal{D}'(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, $\mathcal{D}'_{L^p, z_1, z_2}(\mathbb{R})$, $\dot{\mathcal{B}}'_{z_1, z_2}(\mathbb{R})$, $\mathcal{D}'_{AP}(\mathbb{R})$, $\mathcal{O}'_C(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$), $\mathcal{D}^\wedge_{AP}(\mathbb{R})$ being the distribution space of the distributions on $\mathbb{R}$ generated by the elements of $\mathcal{D}_{AP}(\mathbb{R})$ [Schwartz, 1966; Zaidman, 2011; Ortner and Wagner, 2013; Strungaru and Terauds, 2016; Bargetz *et al.*, 2017; Ciampa *et al.*, 2019]. That said, to avoid confusion with a different (albeit related [Kanwal, 2004]) concept, we do not use the term *regularisation* and its by-products in this sense hereinafter.

A.2.4.4 Orders of singularity, *H*-singularity, and *S*-singularity

We now introduce the concept of *order of singularity* (hereinafter *order*) of a distribution on $\mathbb{R}$, the concept of *order of H-singularity* (hereinafter *H-order*) of a distribution on $\mathbb{R}$, and the concept of *order of S-singularity* (also known as *summability order*, hereinafter *S-order*) of a $(-n)$ -weighted integrable distribution on $\mathbb{R}$.

An element of $\mathcal{D}'(\mathbb{R}) \setminus \mathcal{D}'^f(\mathbb{R})$ is said to be a *distribution of infinite order* on $\mathbb{R}$, and it is a distribution on $\mathbb{R}$ whose each antiderivative of any order in $\mathbb{N}$ is not generable by a Radon measure on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Barros-Neto, 1973]. The standard example of distribution of infinite order on $\mathbb{R}$ is an infinite but convergent linear combination of derivatives of indefinitely increasing order in $\mathbb{N}$ of δ each of which translated by a translation vector proportional to the derivative order [Halperin, 1952; Schwartz, 1966; Horváth, 1966; Treves, 1967; Barros-Neto, 1973; Hoskins and Sousa Pinto, 2005].

An element of $\mathcal{D}'^F(\mathbb{R})$ is said to be a *distribution of finite order* on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967]. A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of finite order on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by Radon measures on $\mathbb{R}$ (in fact, even as a derivative of order in $\mathbb{N}$ of a distribution on $\mathbb{R}$ generated by a Radon measure on $\mathbb{R}$) or, equivalently, as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by functions of locally bounded variation on $\mathbb{R}$ (in fact, even as a derivative of order in $\mathbb{N}$ of a distribution on $\mathbb{R}$ generated by a function of locally bounded variation on $\mathbb{R}$) [Schwartz, 1966; Horváth, 1966; Treves, 1967]. Remarkably, each tempered distribution on $\mathbb{R}$ is a distribution of finite order on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Barros-Neto, 1973]. Hereinafter, each subspace of $\mathcal{D}'(\mathbb{R})$ is also a subspace of $\mathcal{D}'^F(\mathbb{R})$.

We say that an element of $\mathcal{D}'^{n_0}(\mathbb{R})$ is a *distribution of order in $\mathbb{N}_{[0, n_0]}$* on $\mathbb{R}$. We have [Zemanian, 1965b; Schwartz, 1966; Horváth, 1966; Treves, 1967]

$$\mathcal{D}'^{n_0+1}(\mathbb{R}) \supset \mathcal{D}'^{n_0}(\mathbb{R})$$

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}_{[0, n_0]}$ of distributions on $\mathbb{R}$ generated by Radon measures on $\mathbb{R}$ (in fact, even as a derivative of order in $\mathbb{N}_{[0, n_0]}$ of a distribution on $\mathbb{R}$ generated by a Radon measure on $\mathbb{R}$) or, equivalently, as a finite linear combination of derivatives of order in $\mathbb{N}_{[0, n_0+1]}$ of distributions on $\mathbb{R}$ generated by functions of locally bounded variation on $\mathbb{R}$ (in fact, even as a derivative of order in $\mathbb{N}_{[0, n_0+1]}$ of a distribution on $\mathbb{R}$ generated by a function of locally bounded variation on $\mathbb{R}$) [Schwartz, 1966; Horváth, 1966; Treves, 1967]. The concept of distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ can be generalised to introduce the concept of *distribution of H -order in $\mathbb{R}_{[0^+, (n_0+\alpha)^+]}$* on $\mathbb{R}$ with α in $\mathbb{R}_{(0,1)}$ and $(n_0 + \alpha)^+$ in $\mathbb{R}_{(n_0+\alpha, +\infty)}$ [Mišur and Palle, 2019]; a distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ is a distribution of H -order in $\mathbb{R}_{[0^+, n_0^+]}$ on $\mathbb{R}$ [Mišur and Palle, 2019].

A *distribution of order n_0* on $\mathbb{R}$ is either an element of $\mathcal{D}'^{n_0}(\mathbb{R}) \setminus \mathcal{D}'^{n_0-1}(\mathbb{R})$ (when n_0 is in $\mathbb{N}_{[1, +\infty)}$) or an element of $\mathcal{D}'^0(\mathbb{R})$ (when n_0 equals 0) [Horváth, 1966; Treves, 1967]. The n^{th} derivative of a

distribution of order n_0 on $\mathbb{R}$ is a distribution of order in $\mathbb{N}_{[0, n_0+n]}$ on $\mathbb{R}$ (in particular, when n_0 is in $\mathbb{N}_{[1, +\infty)}$, the n^{th} derivative of a distribution of order n_0 on $\mathbb{R}$ is a distribution of order $n_0 + n$ on $\mathbb{R}$) [Schwartz, 1966]; furthermore, the n^{th} antiderivatives of a distribution of order n_0 on $\mathbb{R}$ are distributions of order the maximum of 0 and $n_0 - n$ on $\mathbb{R}$. When localised at an arbitrary relatively compact open subset of $\mathbb{R}$, each distribution on $\mathbb{R}$ is thereon a distribution of order n_0 [Schwartz, 1966; Treves, 1967]; accordingly, we can loosely say that each distribution on $\mathbb{R}$ (in particular, each distribution of infinite order on $\mathbb{R}$) is locally a distribution of finite order. In particular, a Radon measure on $\mathbb{R}$ generates a distribution of order 0 on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]; inversely, a distribution of order 0 on $\mathbb{R}$ is generable by a Radon measure on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]. Furthermore, a locally integrable function on $\mathbb{R}$ generates a Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, generates a regular distribution of order 0 on $\mathbb{R}$ [Schwartz, 1966]; inversely, a regular distribution of order 0 on $\mathbb{R}$ is generable by a Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, is generable by a locally integrable function on $\mathbb{R}$ [Schwartz, 1966]. For example, the unit function on $\mathbb{R}$ generates the Radon measure on $\mathbb{R}$ whose integral on a relatively compact Borel-measurable subset of $\mathbb{R}$ equals the Lebesgue measure of the relatively compact Borel-measurable subset of $\mathbb{R}$, which, in turn, generates the unit distribution on $\mathbb{R}$ [Horváth, 1966; Treves, 1967; Benedetto, 1997]. The concept of distribution of order n_0 on $\mathbb{R}$ can be generalised to introduce the concept of *distribution of H -order $(n_0 + \alpha)^+$ on $\mathbb{R}$ with α in $\mathbb{R}_{[0,1]}$ and $(n_0 + \alpha)^+$ in $\mathbb{R}_{(n_0 + \alpha, +\infty)}$* [Mišur and Palle, 2019].

We say that an element of $\mathcal{D}'_{L^1}(\mathbb{R}) \cap \mathcal{D}''^{n_0}(\mathbb{R})$ is an *integrable distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$* and that an element of $\mathcal{D}''^{n_0}_{L^1}(\mathbb{R})$ is an *integrable distribution of S -order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$* . We have [Schwartz, 1954a, 1966; Horváth, 1966; Treves, 1967; Bijma, 1999; Thomas, 2000; Dijk, 2013; Beffa, 2024]

$$\begin{aligned} (\mathcal{D}'_{L^1}(\mathbb{R}) \cap \mathcal{D}''^{n_0+1}(\mathbb{R})) &\supset (\mathcal{D}'_{L^1}(\mathbb{R}) \cap \mathcal{D}''^{n_0}(\mathbb{R})) \\ \cup &\cup \\ \mathcal{D}''^{n_0+1}_{L^1}(\mathbb{R}) &\supset \mathcal{D}''^{n_0}_{L^1}(\mathbb{R}) \end{aligned}$$

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be an integrable distribution of S -order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}_{[0,n_0]}$ of distributions on $\mathbb{R}$ generated by integrable Radon measures on $\mathbb{R}$ or, equivalently, as a finite linear combination of derivatives of order in $\mathbb{N}_{[0,n_0+1]}$ of distributions on $\mathbb{R}$ generated by functions of bounded variation on $\mathbb{R}$ [Schwartz, 1954a; Horváth, 1966; Bijma, 1999; Thomas, 2000; Dijk, 2013; Beffa, 2024]. The integral on $\mathbb{R}$ of an integrable distribution of S -order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$ does not depend on the specific expression of it as a finite linear combination of derivatives of order in $\mathbb{N}_{[0,n_0]}$ of distributions on $\mathbb{R}$ generated by integrable Radon measures on $\mathbb{R}$ [Schwartz, 1954a; Musielak, 1963; Kierat and Skórnik, 2001]; in fact, the integral on $\mathbb{R}$ of an integrable distribution of S -order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$ equals the integral on $\mathbb{R}$ of the 0th-derivative term of its expression as a finite linear combination of derivatives of order in $\mathbb{N}_{[0,n_0]}$ of distributions on $\mathbb{R}$ generated by integrable Radon measures on $\mathbb{R}$ [Kierat and Skórnik, 2001].

An *integrable distribution of order n_0* on $\mathbb{R}$ is either an element of $\mathcal{D}'_L(\mathbb{R}) \cap (\mathcal{D}'^{n_0}(\mathbb{R}) \setminus \mathcal{D}'^{n_0-1}(\mathbb{R}))$ (when n_0 is in $\mathbb{N}_{[1,+\infty)}$) or an element of $\mathcal{D}'_L(\mathbb{R}) \cap \mathcal{D}'^0(\mathbb{R})$ (when n_0 equals 0), and an *integrable distribution of S -order n_0* on $\mathbb{R}$ is either an element of $\mathcal{D}'^{n_0}_L(\mathbb{R}) \setminus \mathcal{D}'^{n_0-1}_L(\mathbb{R})$ (when n_0 is in $\mathbb{N}_{[1,+\infty)}$) or an element of $\mathcal{D}'^0_L(\mathbb{R})$ (when n_0 equals 0). The n th derivative of an integrable distribution of S -order n_0 on $\mathbb{R}$ is an integrable distribution of S -order in $\mathbb{N}_{[0,n_0+n]}$ on $\mathbb{R}$ [Bijma, 1999; Thomas, 2000; Dijk, 2013]. When localised at an arbitrary relatively compact open subset of $\mathbb{R}$, each distribution on $\mathbb{R}$ is thereon an integrable distribution of S -order n_0 [Dijk, 2013]; accordingly, we can loosely say that each distribution on $\mathbb{R}$ (in particular, each distribution of infinite order on $\mathbb{R}$) is locally an integrable distribution. In particular, an integrable Radon measure on $\mathbb{R}$ generates an integrable distribution of S -order 0 on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]; inversely, an integrable distribution of S -order 0 on $\mathbb{R}$ is generable by an integrable Radon measure on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Ambrosio *et al.*, 2000; Evans and Gariepy, 2015; Leoni, 2017]. Furthermore, an integrable function on $\mathbb{R}$ generates an integrable Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, generates a regular integrable distribution of S -order 0 on $\mathbb{R}$; inversely, a regular integrable distribution of S -order 0 on $\mathbb{R}$ is generable by an integrable Radon measure on $\mathbb{R}$ that is absolutely continuous with

respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, is generable by an integrable function on $\mathbb{R}$. However, although an integrable Radon measure on $\mathbb{R}$ (respectively, an integrable function on $\mathbb{R}$) generates an integrable distribution of order 0 on $\mathbb{R}$ (respectively, a regular integrable distribution on $\mathbb{R}$), an integrable distribution of order 0 on $\mathbb{R}$ (respectively, a regular integrable distribution on $\mathbb{R}$) need not be generable by an integrable Radon measure on $\mathbb{R}$ (respectively, an integrable function on $\mathbb{R}$). For example, the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$ is not only an element of $\mathcal{O}_M(\mathbb{R}) \setminus \mathcal{O}_C(\mathbb{R})$ generating an element of $\mathcal{O}'_C(\mathbb{R}) \setminus \mathcal{O}'_M(\mathbb{R})$ [Bargetz and Ortner, 2014] but also an element of $(L^\infty(\mathbb{R}) \setminus (L^1(\mathbb{R}) \cup L^2(\mathbb{R}))) \cap (\mathcal{E}^0(\mathbb{R}) \setminus UC(\mathbb{R}))$ generating both an element of $\mathcal{M}_{loc}^1(\mathbb{R}) \setminus \mathcal{M}^1(\mathbb{R})$ and an element of $(\mathcal{D}'_{L^1}(\mathbb{R}) \setminus \mathcal{D}'_L(\mathbb{R})) \cap \mathcal{D}'^0(\mathbb{R})$ [Bijma, 1999; Thomas, 2000]; indeed, although its integral on $\mathbb{R}$ cannot be calculated as a Lebesgue integral with respect to the Lebesgue measure on $\mathbb{R}$, it can be calculated as an improper Lebesgue integral with respect to the Lebesgue measure on $\mathbb{R}$ (and it results in $(\pi/2)^{1/2}(1+j)$, which is twice the *complex Fresnel integral*) [Schwartz, 1993b; Benedetto, 1997; Ortner and Wagner, 2013, 2015]. Incidentally, the sum of the Dirac measure at the origin and the Radon measure on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$ is an element of $\mathcal{M}_{loc}^1(\mathbb{R}) \setminus \mathcal{M}^1(\mathbb{R})$ generating an element of $(\mathcal{D}'_{L^1}(\mathbb{R}) \setminus \mathcal{D}'_L(\mathbb{R})) \cap \mathcal{D}'^0(\mathbb{R})$.

We say that an element of $\mathcal{D}'_{L^1, -n}(\mathbb{R}) \cap \mathcal{D}'^{n_0}(\mathbb{R})$ is a $(-n)$ -weighted integrable distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ and that an element of $\mathcal{D}'^{n_0}_{L^1, -n}(\mathbb{R})$ is a $(-n)$ -weighted integrable distribution of S -order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$. We have

$$\begin{aligned} (\mathcal{D}'_{L^1, -n}(\mathbb{R}) \cap \mathcal{D}'^{n_0+1}(\mathbb{R})) &\supset (\mathcal{D}'_{L^1, -n}(\mathbb{R}) \cap \mathcal{D}'^{n_0}(\mathbb{R})) \\ \cup &\quad \cup \\ \mathcal{D}'^{n_0+1}_{L^1, -n}(\mathbb{R}) &\supset \mathcal{D}'^{n_0}_{L^1, -n}(\mathbb{R}) \end{aligned}$$

with

$$\mathcal{D}'^{n_0}_{L^1, -n-1}(\mathbb{R}) \supset \mathcal{D}'^{n_0}_{L^1, -n}(\mathbb{R}) \supset \mathcal{D}'^{n_0}_{L^1}(\mathbb{R})$$

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-n)$ -weighted integrable distribution of S -order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ is that its multiplicative product by the weight function defined almost everywhere on $\mathbb{R}$ by $(1+\xi^2)^{-n/2}$ is an integrable distribution of S -order in $\mathbb{N}_{[0, n_0]}$ on

$\mathbb{R}$. Equivalently, a necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a $(-n)$ -weighted integrable distribution of S -order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$ is that it can be expressed as the multiplicative product of the reciprocal weight function defined almost everywhere on $\mathbb{R}$ by $(1 + \xi^2)^{n/2}$ (or, equivalently, of an n^{th} -degree polynomial function on $\mathbb{R}$) by an integrable distribution of S -order in $\mathbb{N}_{[0,n_0]}$ on $\mathbb{R}$.

A $(-n)$ -weighted integrable distribution of order n_0 on $\mathbb{R}$ is either an element of $\mathcal{D}'_{L,-n}(\mathbb{R}) \cap (\mathcal{D}'^{n_0}(\mathbb{R}) \setminus \mathcal{D}'^{n_0-1}(\mathbb{R}))$ (when n_0 is in $\mathbb{N}_{[1,+\infty)}$) or an element of $\mathcal{D}'_{L,-n}(\mathbb{R}) \cap \mathcal{D}'^0(\mathbb{R})$ (when n_0 equals 0), and a $(-n)$ -weighted integrable distribution of S -order n_0 on $\mathbb{R}$ is either an element of $\mathcal{D}'^{n_0}_{L,-n}(\mathbb{R}) \setminus \mathcal{D}'^{n_0-1}_{L,-n}(\mathbb{R})$ (when n_0 is in $\mathbb{N}_{[1,+\infty)}$) or an element of $\mathcal{D}'^0_{L,-n}(\mathbb{R})$ (when n_0 equals 0). The n^{th} derivative of a $(-n)$ -weighted integrable distribution of S -order n_0 on $\mathbb{R}$ is a $(-n)$ -weighted integrable distribution of S -order in $\mathbb{N}_{[0,n_0+n]}$ on $\mathbb{R}$. In particular, a $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$ generates a $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$; inversely, a $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$ is generable by a $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$. Furthermore, a $(-n)$ -weighted integrable function on $\mathbb{R}$ generates a $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, generates a regular $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$; inversely, a regular $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$ is generable by a $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, is generable by a $(-n)$ -weighted integrable function on $\mathbb{R}$. However, although a $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$ (respectively, a $(-n)$ -weighted integrable function on $\mathbb{R}$) generates a $(-n)$ -weighted integrable distribution of order 0 on $\mathbb{R}$ (respectively, a regular $(-n)$ -weighted integrable distribution on $\mathbb{R}$), a $(-n)$ -weighted integrable distribution of order 0 on $\mathbb{R}$ (respectively, a regular $(-n)$ -weighted integrable distribution on $\mathbb{R}$) need not be generable by a $(-n)$ -weighted integrable Radon measure on $\mathbb{R}$ (respectively, a $(-n)$ -weighted integrable function on $\mathbb{R}$). Furthermore, an element of $\mathcal{P}_L(\mathbb{R})$ generates a regular element of any $\mathcal{D}'^0_{L,-n}(\mathbb{R})$ with n greater than some minimum value. Finally, a tempered distribution of order 0 on $\mathbb{R}$ is a $(-n)$ -weighted integrable distribution (necessarily of order 0) on $\mathbb{R}$ if and only if it is generable by

a Radon measure whose modulus is a $(-n)$ -weighted integrable distribution (necessarily of order 0) on $\mathbb{R}$ or, equivalently, if and only if it is generable by a Radon measure whose modulus is a tempered distribution (necessarily of order 0) on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Schwartz, 1966; Argabright and Gil de Lamadrid, 1974; Richards and Youn, 1990; Kabanava, 2008; Baake and Strungaru, 2023].

We say that an element of $\mathcal{E}'(\mathbb{R}) \cap \mathcal{D}'^{n_0}(\mathbb{R})$ is a *distribution of compact support of order in $\mathbb{N}_{[0, n_0]}$* on $\mathbb{R}$ and that an element of $\mathcal{E}'^{n_0}(\mathbb{R})$ is a *distribution of compact support of S -order in $\mathbb{N}_{[0, n_0]}$* on $\mathbb{R}$ (namely, a distribution of compact support on $\mathbb{R}$ that, as an integrable distribution on $\mathbb{R}$, is an integrable distribution of S -order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$). We have [Schwartz, 1966; Horváth, 1966; Treves, 1967; Bijma, 1999; Thomas, 2000]

$$\begin{array}{ccc} (\mathcal{E}'(\mathbb{R}) \cap \mathcal{D}'^{n_0+1}(\mathbb{R})) & \supset & (\mathcal{E}'(\mathbb{R}) \cap \mathcal{D}'^{n_0}(\mathbb{R})) \\ \text{III} & & \text{III} \\ \mathcal{E}'^{n_0+1}(\mathbb{R}) & \supset & \mathcal{E}'^{n_0}(\mathbb{R}) \end{array}$$

A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of compact support of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of derivatives of order in $\mathbb{N}_{[0, n_0]}$ of distributions on $\mathbb{R}$ generated by Radon measures of compact support on $\mathbb{R}$ (the support of each of such distributions being contained in an arbitrary neighbourhood of the support of the distribution of compact support on $\mathbb{R}$) [Schwartz, 1966; Horváth, 1966]. A special case of distribution of compact support of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ is a point-supported distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ (namely, a distribution of compact support of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ having support in a point of $\mathbb{R}$). A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a point-supported distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ is that it admits a unique decomposition as a finite linear combination of derivatives of order in $\mathbb{N}_{[0, n_0]}$ of δ each of which translated by the same translation vector [Zemanian, 1965a; Schwartz, 1966].

A *distribution of compact support of order n_0* on $\mathbb{R}$ or, equivalently, a *distribution of compact support of S -order n_0* on $\mathbb{R}$ (namely, a distribution of compact support on $\mathbb{R}$ that, as an integrable distribution on $\mathbb{R}$, is an integrable distribution of S -order n_0 on $\mathbb{R}$) is either an element of

$$(\mathcal{E}'(\mathbb{R}) \cap (\mathcal{D}'^{n_0}(\mathbb{R}) \setminus \mathcal{D}'^{n_0-1}(\mathbb{R}))) \equiv (\mathcal{E}'^{n_0}(\mathbb{R}) \setminus \mathcal{E}'^{n_0-1}(\mathbb{R})) \text{ (when } n_0 \in \mathbb{N}_{[1, +\infty)}) \text{ or an element of}$$

$(\mathcal{E}'(\mathbb{R}) \cap \mathcal{D}'^0(\mathbb{R})) \equiv \mathcal{E}'^{n_0}(\mathbb{R})$ (when $n_0 = 0$). In particular, a Radon measure of compact support on $\mathbb{R}$ generates a distribution of compact support of order 0 on $\mathbb{R}$ [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967]; inversely, a distribution of compact support of order 0 on $\mathbb{R}$ is generable by a Radon measure of compact support on $\mathbb{R}$ [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967]. Furthermore, a locally integrable function of compact support on $\mathbb{R}$ generates a Radon measure of compact support on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, generates a regular distribution of compact support of order 0 on $\mathbb{R}$ [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967]; inversely, a regular distribution of compact support of order 0 on $\mathbb{R}$ is generable by a Radon measure of compact support on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, which, in turn, is generable by a locally integrable function of compact support on $\mathbb{R}$ [Schwartz, 1966, 1973; Horváth, 1966; Treves, 1967].

A.2.4.5 Orders of L -singularity, D -singularity, and C -singularity

We now introduce the concepts of *order of L -singularity* (hereinafter L -order) of a distribution on $\mathbb{R}$, the concept of *order of D -singularity* (also known as *differential order*, hereinafter D -order) of a distribution on $\mathbb{R}$, and the concept of *order of C -singularity* (hereinafter C -order) of a distribution on $\mathbb{R}$. The concepts of L -order and D -order are very similar in spirit to the concept of order. The concept of C -order is a slightly modified, and to some extent improved, version of the concept of D -order.

As we know, a distribution of infinite order on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose each antiderivative of any order in $\mathbb{N}$ is not generable by a Radon measure on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967]. A distribution of order in $\mathbb{N}_{[0, n_0]}$ on $\mathbb{R}$ is a distribution on $\mathbb{R}$ that can be expressed as a derivative of order in $\mathbb{N}_{[0, n_0]}$ of a distribution on $\mathbb{R}$ generated by a Radon measure on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967]; in particular, a distribution of order n_0 on $\mathbb{R}$ is a distribution on $\mathbb{R}$ that can be expressed as the n_0^{th} derivative of a distribution on $\mathbb{R}$ generated by a Radon measure on $\mathbb{R}$ but, when n_0 is in $\mathbb{N}_{[1, +\infty)}$, cannot be expressed as the $(n_0 - 1)^{\text{th}}$ derivative of a distribution on $\mathbb{R}$ generated by a Radon measure on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967]. The n^{th} derivative of a distribution of order n_0 on $\mathbb{R}$ is a distribution of order in $\mathbb{N}_{[0, n_0 + n]}$ on $\mathbb{R}$ (in particular, when n_0 is in $\mathbb{N}_{[1, +\infty)}$, the n^{th} derivative of a distribution of order n_0 on $\mathbb{R}$ is a distribution of order $n_0 + n$ on $\mathbb{R}$); furthermore, the n^{th} antiderivatives of a distribution of order n_0 on $\mathbb{R}$ are distributions of order the maximum of 0 and $n_0 - n$ on $\mathbb{R}$.

A *distribution of infinite L -order* on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose each antiderivative of any order in $\mathbb{N}$ is not generable by a locally integrable function on $\mathbb{R}$ [Halperin, 1952; Kanwal, 2004]; accordingly, the distributions of infinite L -order on $\mathbb{R}$ are all and only the distributions of infinite order on $\mathbb{R}$ [Halperin, 1952; Kanwal, 2004]. A *distribution of L -order in $\mathbb{N}_{[0,n_0]}$* on $\mathbb{R}$ is a distribution on $\mathbb{R}$ that can be expressed as a derivative of order in $\mathbb{N}_{[0,n_0]}$ of a distribution on $\mathbb{R}$ generated by a locally integrable function on $\mathbb{R}$ [Halperin, 1952; Kanwal, 2004]; in particular, a *distribution of L -order n_0* on $\mathbb{R}$ is a distribution on $\mathbb{R}$ that can be expressed as the n_0^{th} derivative of a distribution on $\mathbb{R}$ generated by a locally integrable function on $\mathbb{R}$ but, when n_0 is in $\mathbb{N}_{[1,+\infty)}$, cannot be expressed as the $(n_0 - 1)^{\text{th}}$ derivative of a distribution on $\mathbb{R}$ generated by a locally integrable function on $\mathbb{R}$ [Halperin, 1952; Kanwal, 2004]. The n^{th} derivative of a distribution of L -order n_0 on $\mathbb{R}$ is a distribution of L -order in $\mathbb{N}_{[0,n_0+n]}$ on $\mathbb{R}$ (in particular, when n_0 is in $\mathbb{N}_{[1,+\infty)}$, the n^{th} derivative of a distribution of L -order n_0 on $\mathbb{R}$ is a distribution of L -order $n_0 + n$ on $\mathbb{R}$); furthermore, the n^{th} antiderivatives of a distribution of L -order n_0 on $\mathbb{R}$ are distributions of L -order the maximum of 0 and $n_0 - n$ on $\mathbb{R}$. A distribution of L -order n on $\mathbb{R}$ is a distribution of order in $\mathbb{N}_{[n-1,n]}$ on $\mathbb{R}$. A distribution of L -order 0 on $\mathbb{R}$ is a regular distribution of order 0 on $\mathbb{R}$; in fact, the distributions of L -order 0 on $\mathbb{R}$ are all and only the regular distributions on $\mathbb{R}$.

A *distribution of infinite D -order* on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose each antiderivative of any order in $\mathbb{N}$ is not generable by a C^0 function on $\mathbb{R}$ [König and Zemanian, 1965; Zemanian, 1965a, 1968a; Hoskins and Sousa Pinto, 2005]; accordingly, the distributions of infinite D -order on $\mathbb{R}$ are all and only the distributions of infinite order on $\mathbb{R}$ [König and Zemanian, 1965; Zemanian, 1965a, 1968a; Hoskins and Sousa Pinto, 2005]. A *distribution of D -order in $\mathbb{N}_{[0,n_0]}$* on $\mathbb{R}$ is a distribution on $\mathbb{R}$ that can be expressed as a derivative of order in $\mathbb{N}_{[0,n_0]}$ of a distribution on $\mathbb{R}$ generated by a C^0 function on $\mathbb{R}$ [König and Zemanian, 1965; Zemanian, 1965a, 1968a; Hoskins and Sousa Pinto, 2005]; in particular, a *distribution of D -order n_0* on $\mathbb{R}$ is a distribution on $\mathbb{R}$ that can be expressed as the n_0^{th} derivative of a distribution on $\mathbb{R}$ generated by a C^0 function on $\mathbb{R}$ but, when n_0 is in $\mathbb{N}_{[1,+\infty)}$, cannot be expressed as the $(n_0 - 1)^{\text{th}}$ derivative of a distribution on $\mathbb{R}$ generated by a C^0 function on $\mathbb{R}$ [König and Zemanian, 1965; Zemanian, 1965a, 1968a; Hoskins and Sousa Pinto, 2005]. The n^{th} derivative of a distribution of D -order n_0 on $\mathbb{R}$ is a distribution of D -order in $\mathbb{N}_{[0,n_0+n]}$ on $\mathbb{R}$ (in particular, when n_0 is in $\mathbb{N}_{[1,+\infty)}$, the n^{th} derivative of a distribution of D -order n_0 on $\mathbb{R}$ is a distribution of D -order $n_0 + n$ on $\mathbb{R}$); furthermore, the n^{th} antiderivatives of a

distribution of D -order n_0 on $\mathbb{R}$ are distributions of D -order the maximum of 0 and $n_0 - n$ on $\mathbb{R}$. A distribution of D -order n_2 on $\mathbb{R}$ is a distribution of order in $\mathbb{N}_{[n_2-2, n_2]}$ on $\mathbb{R}$. A distribution of D -order 1 on $\mathbb{R}$ is a distribution of order in $\mathbb{N}_{[0,1]}$ on $\mathbb{R}$. A distribution of D -order 0 on $\mathbb{R}$ is a regular distribution of order 0 on $\mathbb{R}$; in fact, the distributions of D -order 0 on $\mathbb{R}$ are all and only the regular distributions on $\mathbb{R}$ that are generable by C^0 functions on $\mathbb{R}$.

We have [Zemanian, 1965a]

$$\begin{array}{ccccccc}
 \mathcal{D}'(\mathbb{R}) & \supset & \mathcal{D}'^F(\mathbb{R}) & \supset & \mathcal{D}'^{C(z)}(\mathbb{R}) & \supset & \mathcal{D}'^{C(-\infty)}(\mathbb{R}) \equiv \mathcal{E}(\mathbb{R}) \\
 \cup & & & & \cup & & \\
 \mathcal{D}'(\mathbb{R}) \setminus \mathcal{D}'^F(\mathbb{R}) & & & & \mathcal{D}'^{C(z)}(\mathbb{R}) \setminus \mathcal{D}'^{C(z-1)}(\mathbb{R}) & & \\
 \text{III} & & & & & & \\
 \mathcal{D}'^{C(+\infty)}(\mathbb{R}) & & & & & &
 \end{array}$$

where

- $\mathcal{D}'^{C(+\infty)}(\mathbb{R})$ the distribution space of the *distributions of C -order $+\infty$* on $\mathbb{R}$
- $\mathcal{D}'^{C(z)}(\mathbb{R})$ is the distribution space of the *distributions of C -order in $\mathbb{Z}_{[-\infty, z]}$* on $\mathbb{R}$
- $\mathcal{D}'^{C(z)}(\mathbb{R}) \setminus \mathcal{D}'^{C(z-1)}(\mathbb{R})$ is the distribution space of the *distributions of C -order z* on $\mathbb{R}$
- $\mathcal{D}'^{C(-\infty)}(\mathbb{R})$ the distribution space of the *distributions of C -order $-\infty$* on $\mathbb{R}$

A distribution of C -order $+\infty$ on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose each generalised derivative of any order in $\mathbb{Z}$ is not generable by a C^0 function on $\mathbb{R}$ [Zemanian, 1965a]; accordingly, a distribution of C -order $+\infty$ on $\mathbb{R}$ is a distribution of infinite order on $\mathbb{R}$ [Zemanian, 1965a].

A distribution of C -order in $\mathbb{Z}_{[-\infty, z]}$ on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose $(-z-2)^{\text{th}}$ generalised derivative (and hence whose each generalised derivative of any order in $\mathbb{Z}_{(-\infty, -z-2]}$) is generable by a C^0 function on $\mathbb{R}$ [Zemanian, 1965a]. A necessary and sufficient condition for a distribution on $\mathbb{R}$ to be a distribution of C -order in $\mathbb{Z}_{[-\infty, z]}$ on $\mathbb{R}$ is that it can be expressed (the expression proving non-unique) as a finite linear combination of generalised derivatives of order in $\mathbb{Z}_{(-\infty, z+2]}$ of distributions on $\mathbb{R}$ generated by C^0 functions on $\mathbb{R}$ (in fact, even as a generalised derivative of order in $\mathbb{Z}_{(-\infty, z+2]}$ of a distribution on $\mathbb{R}$ generated by a C^0 function on $\mathbb{R}$). In particular, the distribution space of the distributions of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ is a subspace of the distribution

space of the distributions of L -order in $\mathbb{N}_{[0,1]}$ on $\mathbb{R}$, and it includes all the distributions of L -order 0 on $\mathbb{R}$ as well as some distributions of L -order 1 on $\mathbb{R}$. We have

$$\mathcal{D}'^{C(z+1)}(\mathbb{R}) \supset \mathcal{D}'^{C(z)}(\mathbb{R})$$

A distribution of C -order z on $\mathbb{R}$ is an element of $\mathcal{D}'^{C(z)}(\mathbb{R}) \setminus \mathcal{D}'^{C(z-1)}(\mathbb{R})$. The distributions of C -order z on $\mathbb{R}$ are all and only the distributions of C -order the maximum of 0 and $z+2$ on $\mathbb{R}$. For any $z_1 \in \mathbb{Z}$, the z_1^{th} generalised derivative of a distribution of C -order z on $\mathbb{R}$ is a distribution of C -order $z+z_1$ on $\mathbb{R}$ [Zemanian, 1965a]. A distribution of C -order $-n_2$ on $\mathbb{R}$ is a distribution of order 0 on $\mathbb{R}$ that is generable by an element of $\mathcal{E}^{n_2-2}(\mathbb{R}) \setminus \mathcal{E}^{n_2-1}(\mathbb{R})$ [Zemanian, 1965a]. A distribution of C -order n_0-1 on $\mathbb{R}$ is a distribution of order in $\mathbb{N}_{[0,n_0+1]}$ on $\mathbb{R}$ [Zemanian, 1965a].

A distribution of C -order $-\infty$ on $\mathbb{R}$ is a distribution on $\mathbb{R}$ whose each generalised derivative of any order in $\mathbb{Z}$ is generable by a C^0 function on $\mathbb{R}$ [Zemanian, 1965a]; accordingly, a distribution of C -order $-\infty$ on $\mathbb{R}$ is a distribution on $\mathbb{R}$ generated by a smooth function on $\mathbb{R}$ [Zemanian, 1965a].

A.2.4.6 Derivatives of regular distributions

To explore the demarcation between regular distributions and singular distributions, we now consider the distribution space of the *distributions of L -order in $\mathbb{N}_{[0,1]}$* on $\mathbb{R}$, which includes all the distributions of order 0 on $\mathbb{R}$ (as well as some distributions of order 1 on $\mathbb{R}$) and all the distributions of C -order in $\mathbb{Z}_{[-\infty,-1]}$ on $\mathbb{R}$ (as well as some distributions of C -order 0 on $\mathbb{R}$). The distributions of L -order in $\mathbb{N}_{[0,1]}$ on $\mathbb{R}$ are all and only the 1st derivatives of the regular distributions on $\mathbb{R}$. An Euler diagram representing a partition of the function space of the locally integrable functions on $\mathbb{R}$ into convenient subspaces is shown in **Figure A.2**.

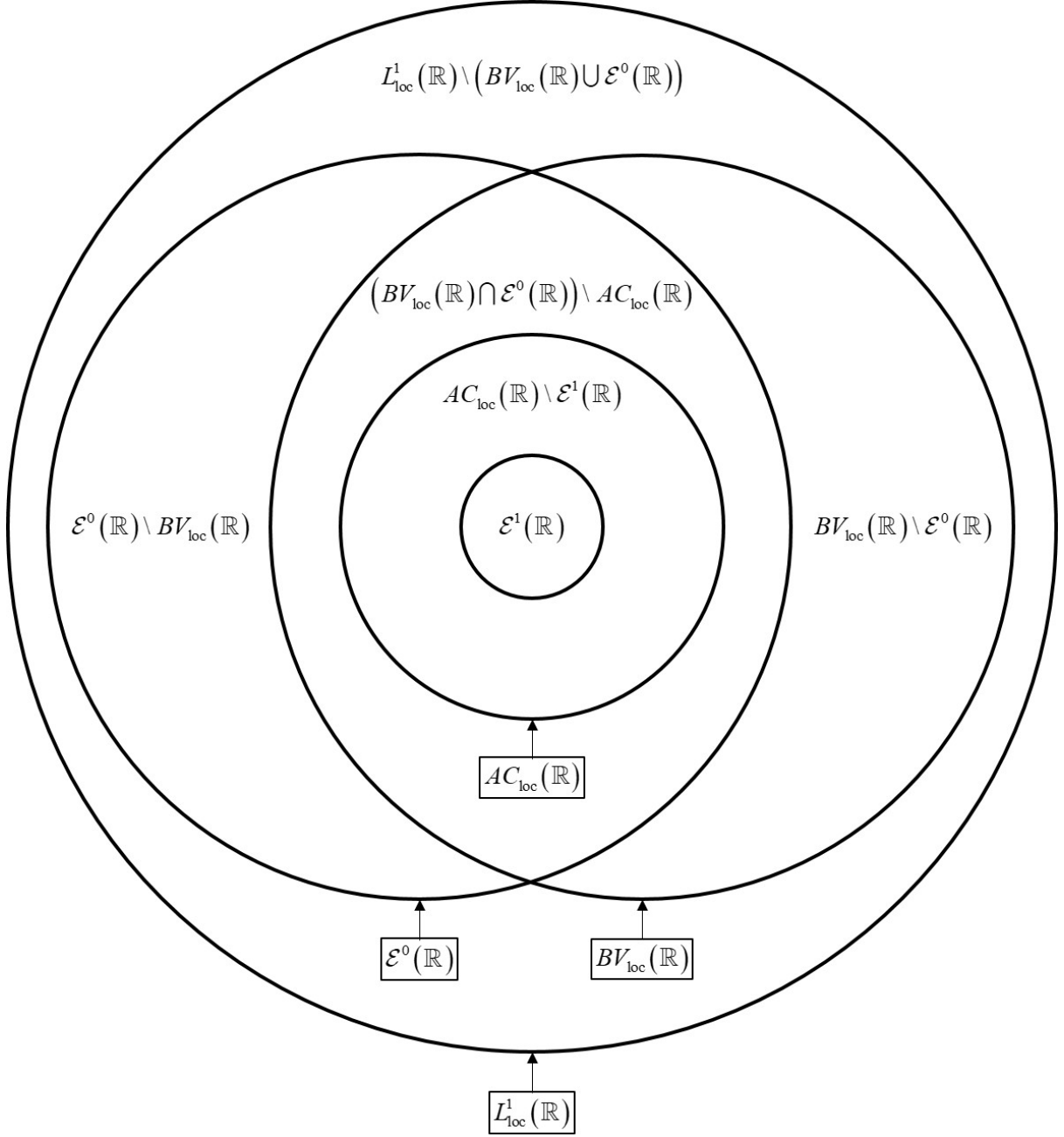


Figure A.2 Euler diagram representing a partition of the function space of the locally integrable functions on $\mathbb{R}$ into convenient subspaces. $L^1_{\text{loc}}(\mathbb{R})$ = function space of the locally integrable functions on $\mathbb{R}$; $\mathcal{E}^0(\mathbb{R})$ = function space of the C^0 functions on $\mathbb{R}$; $BV_{\text{loc}}(\mathbb{R})$ = function space of the functions of locally bounded variation on $\mathbb{R}$; $AC_{\text{loc}}(\mathbb{R})$ = function space of the locally absolutely continuous functions on $\mathbb{R}$; $\mathcal{E}^1(\mathbb{R})$ = function space of the C^1 functions on $\mathbb{R}$.

The 1st derivative of a distribution on $\mathbb{R}$ generated by an element of $L^1_{\text{loc}}(\mathbb{R}) \setminus (BV_{\text{loc}}(\mathbb{R}) \cup \mathcal{E}^0(\mathbb{R}))$ (such as the function defined almost everywhere on $\mathbb{R}$ by $\ln(|\xi|)$) is a distribution on $\mathbb{R}$ that is generable neither by a C^0 function on $\mathbb{R}$, nor by a locally integrable function on $\mathbb{R}$, nor by a Radon measure on $\mathbb{R}$; specifically, it is a singular distribution of order 1, of L -order 1, of D -order 2, and of C -order 0 on $\mathbb{R}$.

The 1st derivative of a distribution on $\mathbb{R}$ generated by an element of $\mathcal{E}^0(\mathbb{R}) \setminus BV_{\text{loc}}(\mathbb{R})$ (such as the *Weierstrass function* [Hardy, 1916] or the function defined almost everywhere on $\mathbb{R}$ by $\xi \sin(1/\xi)$, which, in fact, are elements of $UC(\mathbb{R}) \setminus BV_{\text{loc}}(\mathbb{R})$) is a distribution on $\mathbb{R}$ that is generable neither by a C^0 function on $\mathbb{R}$, nor by a locally integrable function on $\mathbb{R}$, nor by a Radon measure on $\mathbb{R}$; specifically, it is a singular distribution of order 1, of L -order 1, of D -order 1, and of C -order -1 on $\mathbb{R}$. The 2nd derivative of the distribution on $\mathbb{R}$ generated by the Weierstrass function is a singular distribution of order 2, of L -order 2, of D -order 2, and of C -order 0 on $\mathbb{R}$; accordingly, it is an example of distribution of C -order 0 on $\mathbb{R}$ that is not a distribution of L -order in $N_{[0,1]}$ on $\mathbb{R}$.

The 1st derivative of a distribution on $\mathbb{R}$ generated by an element of $BV_{\text{loc}}(\mathbb{R}) \setminus \mathcal{E}^0(\mathbb{R})$ (such as the unit step function on $\mathbb{R}$, as introduced below) is a distribution on $\mathbb{R}$ that is generable neither by a C^0 function on $\mathbb{R}$ nor by a locally integrable function on $\mathbb{R}$ but is generable by a Radon measure on $\mathbb{R}$; specifically, it is a singular distribution of order 0, of L -order 1, of D -order 2, and of C -order 0 on $\mathbb{R}$. The 2nd derivative of the distribution on $\mathbb{R}$ generated by the unit step function on $\mathbb{R}$ (as introduced below) is a singular distribution of order 1, of L -order 2, of D -order 3, and of C -order 1 on $\mathbb{R}$; accordingly, it is an example of distribution of order 1 on $\mathbb{R}$ that is not a distribution of L -order in $N_{[0,1]}$ on $\mathbb{R}$.

The 1st derivative of a distribution on $\mathbb{R}$ generated by an element of $(BV_{\text{loc}}(\mathbb{R}) \cap \mathcal{E}^0(\mathbb{R})) \setminus AC_{\text{loc}}(\mathbb{R})$ (such as the function on $\mathbb{R}$ coinciding with $0_{\mathbb{R}}$ on $\mathbb{R}_{(-\infty,0)}$, with the Cantor ternary function on $\mathbb{R}_{(0,1)}$, and with $1_{\mathbb{R}}$ on $\mathbb{R}_{(1,+\infty)}$ [Ambrosio *et al.*, 2000; Leoni, 2017]) is a distribution on $\mathbb{R}$ that is generable neither by a C^0 function on $\mathbb{R}$ nor by a locally integrable function on $\mathbb{R}$ but is generable by a Radon measure on $\mathbb{R}$; specifically, it is a singular distribution of order 0, of L -order 1, of D -order 1, and of C -order -1 on $\mathbb{R}$.

The 1st derivative of a distribution on $\mathbb{R}$ generated by an element of $AC_{\text{loc}}(\mathbb{R}) \setminus \mathcal{E}^1(\mathbb{R})$ (such as the function defined almost everywhere on $\mathbb{R}$ by $|\xi|$) is a distribution on $\mathbb{R}$ that is not generable by a C^0 function on $\mathbb{R}$ but is generable both by a locally integrable function on $\mathbb{R}$ and by a Radon measure on $\mathbb{R}$; specifically, it is a regular distribution of order 0, of L -order 0, of D -order 1, and of C -order -1 on $\mathbb{R}$.

A.2 Mathematical framework: Functions, Radon measures, and distributions

The 1st derivative of a distribution on $\mathbb{R}$ generated by an element of $\mathcal{E}'(\mathbb{R})$ (such as the function defined almost everywhere on $\mathbb{R}$ by ξ) is a distribution on $\mathbb{R}$ that is generable by a C^0 function on $\mathbb{R}$, by a locally integrable function on $\mathbb{R}$, and by a Radon measure on $\mathbb{R}$; specifically, it is a regular distribution of order 0, of L -order 0, of D -order 0, and of C -order in $\mathcal{Z}_{[-\infty, -2]}$ on $\mathbb{R}$.

A.3 Mathematical framework: Compendium of notable distributions

A.3.1 Introduction

We denote by γ_{ξ_0} , $\tilde{\gamma}$, and $(\tilde{\gamma})_{\xi_0}$ respectively the ξ_0 -translated γ , the reflected γ , and the ξ_0 -translated reflected γ (which coincides with the ξ_0 -translated $\tilde{\gamma}$). For any representative of γ , almost everywhere on $\mathbb{R}$ we have

$$\gamma_{\xi_0}(\xi) = \gamma(\xi - \xi_0)$$

$$\tilde{\gamma}(\xi) = \gamma(-\xi)$$

$$\begin{aligned} (\tilde{\gamma})_{\xi_0}(\xi) &= \tilde{\gamma}(\xi - \xi_0) \\ &= \gamma_{-\xi_0}(-\xi) \\ &= \gamma(\xi_0 - \xi) \end{aligned}$$

We denote by ψ_{ξ_0} , $\tilde{\psi}$, and $(\tilde{\psi})_{\xi_0}$ respectively the ξ_0 -translated ψ , the reflected ψ , and the ξ_0 -translated reflected ψ (which coincides with the ξ_0 -translated $\tilde{\psi}$).

In the following, some information about notable regular distributions and singular distributions is summarised.

A.3.2 Regular distributions

We now introduce some regular distributions that are necessary for our purposes.

A.3.2.1 Unit step, signum, and boxcar

We define almost everywhere on $\mathbb{R}$ the *unit step function* (also known as *Heaviside function*) $\mathfrak{U}$ by

$$\begin{aligned}\mathfrak{U}(\xi) &\equiv \rho_0(\xi) \\ &:= \begin{cases} 0 & , \quad \xi \in \mathbb{R}_{(-\infty, 0)} \\ 1 & , \quad \xi \in \mathbb{R}_{(0, +\infty)} \end{cases}\end{aligned}$$

We refer to the regular bounded distribution on $\mathbb{R}$ generated by $\mathfrak{U}$, which we also denote by $\mathfrak{U}$, as the *unit step distribution* (also known as *Heaviside distribution*) on $\mathbb{R}$. Hereinafter, when not explicitly clarified, it is understood from the context whether the symbol $\mathfrak{U}$ denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. We have [Poletti, 2005]

$$\mathfrak{U} \equiv \mathfrak{U}^{*1}$$

where

- $*$ is the *convolution operator*
- $\mathfrak{U}^{*1}$ is the 1st convolutive power of $\mathfrak{U}$

We define almost everywhere on $\mathbb{R}$ the *signum function* sgn by

$$\text{sgn}(\xi) := \begin{cases} -1 & , \quad \xi \in \mathbb{R}_{(-\infty, 0)} \\ +1 & , \quad \xi \in \mathbb{R}_{(0, +\infty)} \end{cases}$$

We refer to the regular bounded distribution on $\mathbb{R}$ generated by sgn , which we also denote by sgn , as the *signum distribution* on $\mathbb{R}$. Hereinafter, when not explicitly clarified, it is understood from the context whether the symbol sgn denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. We have [Lighthill, 1958; Jones, 1982]

$$\mathfrak{U} + \check{\mathfrak{U}} = 1_{\mathbb{R}}$$

$$\mathfrak{U} - \check{\mathfrak{U}} = \text{sgn}$$

where

- $\check{\mathfrak{U}}$ is the reflected $\mathfrak{U}$

Importantly, an *ad hoc* operation of multiplication having a distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ and a function of locally bounded variation on $\mathbb{R}$ as operands can be defined [Schwartz, 1962; Nussenzveig, 1972]. In particular, by applying such *ad hoc* operation of multiplication, we have

$$\begin{aligned} \mathfrak{G} \mathfrak{G} &\equiv \mathfrak{G}^2 \\ &= \mathfrak{G} \end{aligned} \tag{A.1}$$

$$\begin{aligned} \text{sgn} \text{sgn} &\equiv \text{sgn}^2 \\ &= 1_{\mathbb{R}} \end{aligned} \tag{A.2}$$

In Equation (A.1) and Equation (A.2), each of $\mathfrak{G} \cdot \mathfrak{G}$ and $\text{sgn} \cdot \text{sgn}$ can be interpreted as the multiplicative product of a distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ by a function of locally bounded variation on $\mathbb{R}$. It is worth emphasising that, despite being well-defined *per se*, the multiplications in Equation (A.1) and Equation (A.2), like in general the multiplication of arbitrary distributions, are prone to inconsistency when composed with other operations (in particular, with differentiation) [Hoskins and Sousa Pinto, 2005].

We define almost everywhere on $\mathbb{R}$ the *boxcar function* Π_{ξ_1, ξ_2} with support $\mathbb{R}_{[\xi_1, \xi_2]}$ (in particular, the *rectangular function* rect) by

$$\Pi_{\xi_1, \xi_2}(\xi) := \begin{cases} 0 & , \quad \xi \in \mathbb{R}_{(-\infty, \xi_1) \cup (\xi_2, +\infty)} \\ 1 & , \quad \xi \in \mathbb{R}_{(\xi_1, \xi_2)} \end{cases}$$

$$\begin{aligned} \text{rect}(\xi) &\equiv \Pi_{-\frac{1}{2}, +\frac{1}{2}}(\xi) \\ &:= \begin{cases} 0 & , \quad \xi \in \mathbb{R}_{(-\infty, -\frac{1}{2}) \cup (+\frac{1}{2}, +\infty)} \\ 1 & , \quad \xi \in \mathbb{R}_{(-\frac{1}{2}, +\frac{1}{2})} \end{cases} \end{aligned}$$

where

- ξ_1 and ξ_2 are real constants of the same quantity dimension as ξ with ξ_2 greater than ξ_1

We refer to the regular bounded distribution on $\mathbb{R}$ generated by Π_{ξ_1, ξ_2} (in particular, by rect), which we also denote by Π_{ξ_1, ξ_2} (in particular, by rect), as the *boxcar distribution* on $\mathbb{R}$ with support $\mathbb{R}_{[\xi_1, \xi_2]}$ (in particular, as the *rectangular distribution* on $\mathbb{R}$). Hereinafter, when not explicitly clarified, it is understood from the context whether the symbol Π_{ξ_1, ξ_2} (in particular, rect) denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. We have

$$\Pi_{\xi_1, \xi_2} = \mathfrak{g}_{\xi_1} - \mathfrak{g}_{\xi_2}$$

where

- $\mathfrak{g}_{\xi_1}$ is the ξ_1 -translated $\mathfrak{g}$
- $\mathfrak{g}_{\xi_2}$ is the ξ_2 -translated $\mathfrak{g}$

A.3.2.2 Unit ramp and absolute value

We define almost everywhere on $\mathbb{R}$ the *unit ramp function* ρ by

$$\begin{aligned} \rho(\xi) &\equiv \rho_1(\xi) \\ &:= \xi \mathfrak{g}(\xi) \end{aligned}$$

We refer to the regular tempered distribution on $\mathbb{R}$ generated by ρ , which we also denote by ρ , as the *unit ramp distribution* on $\mathbb{R}$. Hereinafter, when not explicitly clarified, it is understood from the context whether the symbol ρ denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. We have [Poletti, 2005]

$$\begin{aligned} \rho &= \xi \mathfrak{g} \\ &= \mathfrak{g} * \mathfrak{g} \\ &\equiv \mathfrak{g}^{*2} \\ \\ \rho^{(1)} &= \mathfrak{g} \end{aligned}$$

where

- $\mathfrak{g}^{*2}$ is the 2nd convolutive power of $\mathfrak{g}$
- $\rho^{(1)}$ is the 1st derivative of ρ

We define almost everywhere on $\mathbb{R}$ the *absolute value function* abs by

$$\begin{aligned} \text{abs}(\xi) &\equiv |\xi| \\ &:= \begin{cases} -\xi & , \quad \xi \in \mathbb{R}_{(-\infty, 0)} \\ +\xi & , \quad \xi \in \mathbb{R}_{(0, +\infty)} \end{cases} \end{aligned}$$

We refer to the regular tempered distribution on $\mathbb{R}$ generated by abs , which we also denote by abs , as the *absolute value distribution* on $\mathbb{R}$. Hereinafter, when not explicitly clarified, it is understood from the context whether the symbol abs denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. We have [Lighthill, 1958; Jones, 1982]

$$\rho + \tilde{\rho} = \text{abs}$$

$$\rho - \check{\rho} = \xi$$

$$|\xi| = \xi \operatorname{sgn}$$

$$|\xi|^{(1)} = \operatorname{sgn}$$

where

- $\check{\rho}$ is the reflected ρ
- $|\xi|^{(1)}$ is the 1st derivative of $|\xi|$

A.3.2.3 Unit $(n-1)^{\text{th}}$ -power ramp

We define almost everywhere on $\mathbb{R}$ the *unit $(n-1)^{\text{th}}$ -power ramp function* ρ_{n-1} by

$$\rho_{n-1}(\xi) := \xi^{n-1} \mathfrak{g}(\xi)$$

We refer to the regular tempered distribution on $\mathbb{R}$ generated by ρ_{n-1} , which we also denote by ρ_{n-1} , as the *unit $(n-1)^{\text{th}}$ -power ramp distribution* on $\mathbb{R}$. Hereinafter, when not explicitly clarified, it is understood from the context whether the symbol ρ_{n-1} denotes a function on $\mathbb{R}$ or a distribution on $\mathbb{R}$. We have [Poletti, 2005]

$$\begin{aligned} \frac{1}{(n-1)!} \rho_{n-1} &= \frac{1}{(n-1)!} \xi^{n-1} \mathfrak{g} \\ &= \underbrace{(\mathfrak{g} * \mathfrak{g} * \dots * \mathfrak{g})}_n \\ &\equiv \mathfrak{g}^{*n} \end{aligned}$$

$$\frac{1}{(n-1)!} \rho_{n-1}^{(n-1)} = \mathfrak{g}$$

where

- $\mathfrak{g}^{*n}$ is the n^{th} convolutive power of $\mathfrak{g}$
- $\rho_{n-1}^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of ρ_{n-1}

A.3.3 Singular distributions

We now introduce some singular distributions that are necessary for our purposes.

A.3.3.1 Unit impulse and 1st-order pseudofunctions

The *unit impulse* (also known as *Dirac delta* or *delta distribution*) on $\mathbb{R}$, denoted by δ , is the 1st derivative of $\mathfrak{G}$ [Schwartz, 1966]. Specifically, δ is a point-supported distribution on $\mathbb{R}$ (of order 0, of L -order 1, of D -order 2, of C -order 0, homogeneous of degree -1 , and even) with [Schwartz, 1966; Kanwal, 2004]

$$\begin{aligned}\delta &\equiv \delta^{(0)} \\ &\equiv (\psi_{\mathcal{D}'})^{*0} \\ &= \mathfrak{G}^{(1)}\end{aligned}$$

with

$$\gamma_{C^0(0)}\delta = \gamma_{C^0(0)}(0)\delta$$

$$\psi_{\mathcal{D}'} * \delta = \psi_{\mathcal{D}'}$$

where

- $\psi_{\mathcal{D}'}$ is a distribution on $\mathbb{R}$
- $(\psi_{\mathcal{D}'})^{*0}$ is the 0th convolutive power of $\psi_{\mathcal{D}'}$
- $\mathfrak{G}^{(1)}$ is the 1st derivative of $\mathfrak{G}$
- $\gamma_{C^0(0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^0 function on a neighbourhood of the origin

We also have [Schwartz, 1966; Kanwal, 2004]

$$\begin{aligned}\delta_{\xi_0} &\equiv \delta_{\xi_0}^{(0)} \\ &= \delta_{\xi_0} * \delta \\ &= (\mathfrak{G}_{\xi_0})^{(1)} \\ &= (\mathfrak{G}^{(1)})_{\xi_0} \\ &\equiv \mathfrak{G}_{\xi_0}^{(1)}\end{aligned}$$

$$\begin{aligned}\mathcal{S}_{\xi_0}(\delta) &= \frac{\text{sgn}(\xi_0)}{\xi_0} \delta \\ &= \frac{1}{|\xi_0|} \delta\end{aligned}$$

with

$$\gamma_{C^0(\xi_0)} \delta_{\xi_0} = \gamma_{C^0(\xi_0)}(\xi_0) \delta_{\xi_0}$$

$$\psi_{D'} * \delta_{\xi_0} = (\psi_{D'})_{\xi_0}$$

where

- δ_{ξ_0} is the ξ_0 -translated δ
- $\mathfrak{g}_{\xi_0}$ is the ξ_0 -translated $\mathfrak{g}$
- $(\mathfrak{g}_{\xi_0})^{(1)}$ is the 1st derivative of $\mathfrak{g}_{\xi_0}$
- $(\mathfrak{g}^{(1)})_{\xi_0}$ is the ξ_0 -translated $\mathfrak{g}^{(1)}$
- $\gamma_{C^0(\xi_0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^0 function on a neighbourhood of ξ_0
- $(\psi_{D'})_{\xi_0}$ is the ξ_0 -translated $\psi_{D'}$

Unlike its derivative of any order in $\mathbb{N}_{[1,+\infty)}$, δ can be generated by a Radon measure on $\mathbb{R}$ (whose integral on a relatively compact Borel-measurable subset of $\mathbb{R}$ equals the image of the relatively compact Borel-measurable subset of $\mathbb{R}$ under the Dirac measure at the origin on $\mathbb{R}$) [Schwartz, 1966].

Let $\gamma_{C^1(-\infty, \xi_0)}$ (respectively, $\gamma_{C^1(\xi_0, +\infty)}$) be a C^1 function on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) whose derivative of order 0 has limit as ξ tends to ξ_0^- (respectively, ξ_0^+) existing and finite; let $\gamma_{C^1(-\infty, \xi_0)}^{(1)}$ (respectively, $\gamma_{C^1(\xi_0, +\infty)}^{(1)}$) be the 1st derivative of $\gamma_{C^1(-\infty, \xi_0)}$ (respectively, $\gamma_{C^1(\xi_0, +\infty)}$). Let $\psi_{C^1(-\infty, \xi_0)}$ (respectively, $\psi_{C^1(\xi_0, +\infty)}$) be the unique regular distribution on $\mathbb{R}$ whose localisation at $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) results in the regular distribution on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) generated by $\gamma_{C^1(-\infty, \xi_0)}$ (respectively, $\gamma_{C^1(\xi_0, +\infty)}$) and whose localisation at $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$) results in the null distribution on $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$); let $\psi_{C^1(-\infty, \xi_0)}^{[1]}$ (respectively, $\psi_{C^1(\xi_0, +\infty)}^{[1]}$) be the unique regular distribution on $\mathbb{R}$ whose localisation at $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) results in the

regular distribution on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) generated by $\gamma_{C^1(-\infty, \xi_0)}^{(1)}$ (respectively, $\gamma_{C^1(\xi_0, +\infty)}^{(1)}$) and whose localisation at $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$) results in the null distribution on $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$). We have [Schwartz, 1966; Estrada and Kanwal, 2002; Kanwal, 2004]

$$\begin{aligned} \left(\psi_{C^1(-\infty, \xi_0)} + \psi_{C^1(\xi_0, +\infty)} \right)^{(1)} &= \psi_{C^1(-\infty, \xi_0)}^{[1]} + \psi_{C^1(\xi_0, +\infty)}^{[1]} \\ &+ \left(\lim_{\xi \rightarrow \xi_0^+} \gamma_{C^1(\xi_0, +\infty)} - \lim_{\xi \rightarrow \xi_0^-} \gamma_{C^1(-\infty, \xi_0)} \right) \delta_{\xi_0} \end{aligned} \quad (\text{A.3})$$

where

$$- \left(\psi_{C^1(-\infty, \xi_0)} + \psi_{C^1(\xi_0, +\infty)} \right)^{(1)} \text{ is the 1}^{\text{st}} \text{ derivative of } \psi_{C^1(-\infty, \xi_0)} + \psi_{C^1(\xi_0, +\infty)}$$

We say that $\psi_{C^1(-\infty, \xi_0)} + \psi_{C^1(\xi_0, +\infty)}$ is a C^1 distribution on $\mathbb{R}$ up to the presence of a 0^{th} -order jump discontinuity at ξ_0 . Equation (A.3) can be easily generalised to be applicable to C^1 distributions on $\mathbb{R}$ up to the presence of countably many 0^{th} -order jump discontinuities.

The distributions $\text{Pf}(\mathfrak{G}/\xi)$ and $\text{Pf}(\check{\mathfrak{G}}/\xi)$ are the non-canonical pseudofunctions on $\mathbb{R}$ generated by the functions defined almost everywhere on $\mathbb{R}$ by respectively $\mathfrak{G}/\xi$ and $\check{\mathfrak{G}}/\xi$ via the Hadamard-finite-part method [Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Specifically, $\text{Pf}(\mathfrak{G}/\xi)$ and $\text{Pf}(\check{\mathfrak{G}}/\xi)$ are singular tempered distributions on $\mathbb{R}$ (of order 1, of L -order 1, of D -order 2, of C -order 0, non-homogeneous, and respectively causal and anticausal) with

$$\text{Pf}\left(\frac{\mathfrak{G}}{\xi}\right) = \left((\ln(|\xi|)) \mathfrak{G} \right)^{(1)} \quad (\text{A.4})$$

$$\text{Pf}\left(\frac{\check{\mathfrak{G}}}{\xi}\right) = \left((\ln(|\xi|)) \check{\mathfrak{G}} \right)^{(1)} \quad (\text{A.5})$$

where

$$\begin{aligned} - \left((\ln(|\xi|)) \mathfrak{G} \right)^{(1)} &\text{ is the 1}^{\text{st}} \text{ derivative of } (\ln(|\xi|)) \mathfrak{G} \\ - \left((\ln(|\xi|)) \check{\mathfrak{G}} \right)^{(1)} &\text{ is the 1}^{\text{st}} \text{ derivative of } (\ln(|\xi|)) \check{\mathfrak{G}} \end{aligned}$$

The distribution $\text{p.v.}(1/\xi)$ is the canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/\xi$ via the Cauchy-principal-value method [Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Specifically, $\text{p.v.}(1/\xi)$ is a singular $p_{(1,+\infty]}$ -integrable distribution on $\mathbb{R}$ for each $p_{(1,+\infty]}$ in $\overline{\mathbb{R}}_{(1,+\infty]}$ [Kierat and Skórnik, 1996; King, 2009a] (of order 1, of H -order 0^+ [Mišur and Palle, 2019], of L -order 1, of D -order 2, of C -order 0, homogeneous of degree -1 , and odd) with

$$\begin{aligned} \text{p.v.}\left(\frac{1}{\xi}\right) &\equiv \text{f.p.}\left(\frac{1}{\xi}\right) \equiv \text{Pf}\left(\frac{1}{\xi}\right) \\ &= \text{Pf}\left(\frac{\mathfrak{G}}{\xi}\right) + \text{Pf}\left(\frac{\check{\mathfrak{G}}}{\xi}\right) \\ &= \left((\ln(|\xi|))(\mathfrak{G} + \check{\mathfrak{G}})\right)^{(1)} \\ &= \left(\ln(|\xi|)\right)^{(1)} \end{aligned} \tag{A.6}$$

where

- $\left((\ln(|\xi|))(\mathfrak{G} + \check{\mathfrak{G}})\right)^{(1)}$ is the 1st derivative of $(\ln(|\xi|))(\mathfrak{G} + \check{\mathfrak{G}})$
- $\left(\ln(|\xi|)\right)^{(1)}$ is the 1st derivative of $\ln(|\xi|)$

In Equation (A.4), Equation (A.5), and Equation (A.6), each of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ can be interpreted as the multiplicative product of a distribution of C -order in $\mathbb{Z}_{[-\infty,-1]}$ on $\mathbb{R}$ by a function of locally bounded variation on $\mathbb{R}$; alternatively, without implying any multiplications involving distributions, each of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ can be just interpreted as the regular distribution on $\mathbb{R}$ generated by the homographic locally integrable function on $\mathbb{R}$ (in which case the multiplication sign does not denote an operator but is merely a conventional symbol). Unlike $\ln(|\xi|)$, none of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ is generated by a function of bounded mean oscillation on $\mathbb{R}$ [Chang and Sadoski, 2006; Garnett, 2007].

We observe that the functions defined almost everywhere on $\mathbb{R}$ by $\mathfrak{G}/\xi$, $\check{\mathfrak{G}}/\xi$, and their sum $1/\xi$ are not locally integrable functions on $\mathbb{R}$ (although so are the functions defined almost everywhere on $\mathbb{R}$ by $(\ln(|\xi|))\mathfrak{G}$, $(\ln(|\xi|))\check{\mathfrak{G}}$, and their sum $\ln(|\xi|)$). Furthermore, *per se*, the functions defined almost everywhere on $\mathbb{R}$ by both $\mathfrak{G}/\xi$ and $\check{\mathfrak{G}}/\xi$ generate only non-canonical pseudofunctions on

$\mathbb{R}$ such as $\text{Pf}\left(\frac{1}{\xi}\right)$ and $\text{Pf}\left(\frac{1}{\xi}\right)$; nevertheless, the function defined almost everywhere on $\mathbb{R}$ by their sum $1/\xi$ generates a canonical pseudofunction on $\mathbb{R}$ such as $\text{p.v.}\left(\frac{1}{\xi}\right)$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002]. The relation holds [Jones, 1982]

$$\xi \text{p.v.}\left(\frac{1}{\xi}\right) = 1_{\mathbb{R}}$$

More generally, we have

$$\gamma_{C^1(0)} \text{p.v.}\left(\frac{1}{\xi}\right) = \text{p.v.}\left(\frac{\gamma_{C^1(0)}}{\xi}\right)$$

$$\gamma_{C^1(0)}(0) = 0 \Leftrightarrow \gamma_{C^1(0)} \text{p.v.}\left(\frac{1}{\xi}\right) = \frac{\gamma_{C^1(0)}}{\xi}$$

where

- $\gamma_{C^1(0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^1 function on a neighbourhood of the origin

Furthermore, $\text{p.v.}\left(\frac{1}{\xi}\right)$ and δ form a linearly independent set, and the set of all their linear combinations coincides with the set of all tempered distributions homogeneous of degree -1 on $\mathbb{R}$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002].

The distribution $\text{Pf}\left(\frac{1}{|\xi|}\right)$ is the non-canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/|\xi|$ via the Hadamard-finite-part method [Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Specifically, $\text{Pf}\left(\frac{1}{|\xi|}\right)$ is a singular $p_{(1,+\infty]}$ -integrable distribution on $\mathbb{R}$ for each $p_{(1,+\infty]}$ in $\overline{\mathbb{R}}_{(1,+\infty]}$ [Kierat and Skórnik, 1996; Ortner, 2011] (of order 1, of L -order 1, of D -order 2, of C -order 0, non-homogeneous, and even) with

$$\begin{aligned}
\text{Pf}\left(\frac{1}{|\xi|}\right) &\equiv \text{Pf}\left(\frac{\text{sgn}}{\xi}\right) \\
&= \text{Pf}\left(\frac{\vartheta}{\xi}\right) - \text{Pf}\left(\frac{\check{\vartheta}}{\xi}\right) \\
&= \left((\ln(|\xi|))(\vartheta - \check{\vartheta})\right)^{(1)} \\
&= \left((\ln(|\xi|))\text{sgn}\right)^{(1)}
\end{aligned} \tag{A.7}$$

where

- $\left((\ln(|\xi|))(\vartheta - \check{\vartheta})\right)^{(1)}$ is the 1st derivative of $(\ln(|\xi|))(\vartheta - \check{\vartheta})$
- $\left((\ln(|\xi|))\text{sgn}\right)^{(1)}$ is the 1st derivative of $(\ln(|\xi|))\text{sgn}$

In Equation (A.7), each of $(\ln(|\xi|))\vartheta$, $(\ln(|\xi|))\check{\vartheta}$, and $(\ln(|\xi|))\text{sgn}$ can be interpreted as the multiplicative product of a distribution of C -order in $\mathbb{Z}_{[\infty,-1]}$ on $\mathbb{R}$ by a function of locally bounded variation on $\mathbb{R}$; alternatively, without implying any multiplications involving distributions, each of $(\ln(|\xi|))\vartheta$, $(\ln(|\xi|))\check{\vartheta}$, and $(\ln(|\xi|))\text{sgn}$ can be just interpreted as the regular distribution on $\mathbb{R}$ generated by the homographic locally integrable function on $\mathbb{R}$ (in which case the multiplication sign does not denote an operator but is merely a conventional symbol). Unlike $\ln(|\xi|)$, none of $(\ln(|\xi|))\vartheta$, $(\ln(|\xi|))\check{\vartheta}$, and $(\ln(|\xi|))\text{sgn}$ is generated by a function of bounded mean oscillation on $\mathbb{R}$ [Chang and Sadoski, 2006; Garnett, 2007].

We observe that the functions defined almost everywhere on $\mathbb{R}$ by ϑ/ξ , $\check{\vartheta}/\xi$, and their difference $1/|\xi|$ are not locally integrable functions on $\mathbb{R}$ (although so are the functions defined almost everywhere on $\mathbb{R}$ by $(\ln(|\xi|))\vartheta$, $(\ln(|\xi|))\check{\vartheta}$, and their difference $(\ln(|\xi|))\text{sgn}$). Furthermore, *per se*, the functions defined almost everywhere on $\mathbb{R}$ by both ϑ/ξ and $\check{\vartheta}/\xi$ generate only non-canonical pseudofunctions on $\mathbb{R}$ such as $\text{Pf}(\vartheta/\xi)$ and $\text{Pf}(\check{\vartheta}/\xi)$; likewise, the function defined almost everywhere on $\mathbb{R}$ by their difference $1/|\xi|$ generates only non-canonical pseudofunctions on $\mathbb{R}$ such as $\text{Pf}(1/|\xi|)$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002]. The relation holds [Jones, 1982]

$$\xi \text{Pf}\left(\frac{1}{|\xi|}\right) = \text{sgn}$$

More generally, we have

$$\gamma_{C^l(0)} \text{Pf} \left(\frac{1}{|\xi|} \right) = \text{Pf} \left(\frac{\gamma_{C^l(0)}}{|\xi|} \right)$$

$$\gamma_{C^l(0)}(0) = 0 \Leftrightarrow \gamma_{C^l(0)} \text{Pf} \left(\frac{1}{|\xi|} \right) = \frac{\gamma_{C^l(0)}}{|\xi|}$$

Let ζ_1 be a known distribution on $\mathbb{R}$. The 1st-degree algebraic equation

$$\xi \chi_1 = \zeta_1 \tag{A.8}$$

of constant term $-\zeta_1$ in the unknown distribution χ_1 is satisfied by infinitely many distributions on $\mathbb{R}$ that are quotients of ζ_1 by ξ [Schwartz, 1966; Jones, 1982]. In particular, if ζ_1 is a regular distribution on $\mathbb{R}$, the general solution $(\chi_1)_{\text{gen}}$ of Equation (A.8) is the singular distribution on $\mathbb{R}$ given by [Misra and Lavoine, 1986]

$$\begin{aligned} (\chi_1)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{\zeta_1}{\xi} \right) \\ &= \text{Pf} \left(\frac{\zeta_1}{\xi} \right) + B_{1,0} \delta \end{aligned} \tag{A.9}$$

where

- i.q. (ζ_1/ξ) is the indefinite quotient of ζ_1 by ξ
- $B_{1,0}$ is an arbitrary complex constant of the same quantity dimension as ζ_1

For $\zeta_1 = 0_{\mathbb{R}}$, Equation (A.8) and Equation (A.9) become

$$\xi \chi_1 = 0_{\mathbb{R}} \tag{A.10}$$

$$\begin{aligned} (\chi_1)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{0_{\mathbb{R}}}{\xi} \right) \\ &= B_{1,0} \delta \end{aligned}$$

where

- i.q. $(0_{\mathbb{R}}/\xi)$ is the indefinite quotient of $0_{\mathbb{R}}$ by ξ

Equation (A.10) can be regarded as the homogeneous equation associated with Equation (A.8). Accordingly, the general solution $(\chi_1)_{\text{gen}}$ of Equation (A.8) can be regarded as the sum of the particular solution $\text{Pf}(\zeta_1/\xi)$ of Equation (A.8) and the general solution $B_{1,0}\delta$ of Equation (A.10).

For $\zeta_1 = 1_{\mathbb{R}}$, Equation (A.8) and Equation (A.9) become

$$\begin{aligned}\xi\chi_1 &= 1_{\mathbb{R}} \\ (\chi_1)_{\text{gen}} &\equiv \text{i.q.}\left(\frac{1_{\mathbb{R}}}{\xi}\right) \\ &= \text{p.v.}\left(\frac{1}{\xi}\right) + B_{1,0}\delta\end{aligned}$$

where

– $\text{i.q.}(1_{\mathbb{R}}/\xi)$ is the indefinite quotient of $1_{\mathbb{R}}$ by ξ

For $\zeta_1 = \text{sgn}$, Equation (A.8) and Equation (A.9) become

$$\begin{aligned}\xi\chi_1 &= \text{sgn} \\ (\chi_1)_{\text{gen}} &\equiv \text{i.q.}\left(\frac{\text{sgn}}{\xi}\right) \\ &= \text{Pf}\left(\frac{1}{|\xi|}\right) + B_{1,0}\delta\end{aligned}$$

where

– $\text{i.q.}(\text{sgn}/\xi)$ is the indefinite quotient of sgn by ξ

A.3.3.2 Unit doublet and 2nd-order pseudofunctions

The *unit doublet* on $\mathbb{R}$, denoted by $\delta^{(1)}$, is the 2nd derivative of $\mathfrak{g}$ [Schwartz, 1966]. Specifically, $\delta^{(1)}$ is a point-supported distribution on $\mathbb{R}$ (of order 1, of L -order 2, of D -order 3, of C -order 1, homogeneous of degree -2 , and odd) with [Schwartz, 1966; Kanwal, 2004]

$$\begin{aligned}\delta^{(1)} &= \left(\delta^{(1)}\right)^{*1} \\ &= \mathfrak{g}^{(2)}\end{aligned}$$

with

$$\gamma_{C^1(0)} \delta^{(1)} = \gamma_{C^1(0)}(0) \delta^{(1)} - \gamma_{C^1(0)}^{(1)}(0) \delta$$

$$\psi_{D'} * \delta^{(1)} = \psi_{D'}^{(1)}$$

where

- $\left(\delta^{(1)}\right)^{*1}$ is the 1st convolutive power of $\delta^{(1)}$
- $\mathfrak{g}^{(2)}$ is the 2nd derivative of $\mathfrak{g}$
- $\gamma_{C^1(0)}^{(1)}$ is the 1st derivative of $\gamma_{C^1(0)}$ on a neighbourhood of the origin
- $\psi_{D'}^{(1)}$ is the 1st derivative of $\psi_{D'}$

We also have [Schwartz, 1966; Kanwal, 2004]

$$\begin{aligned} \delta_{\xi_0}^{(1)} &= \left(\delta_{\xi_0}\right)^{(1)} \\ &= \left(\delta^{(1)}\right)_{\xi_0} \\ &= \delta_{\xi_0} * \delta^{(1)} \\ &= \left(\mathfrak{g}_{\xi_0}\right)^{(2)} \\ &= \left(\mathfrak{g}^{(2)}\right)_{\xi_0} \\ &\equiv \mathfrak{g}_{\xi_0}^{(2)} \end{aligned}$$

$$\begin{aligned} \mathcal{S}_{\xi_{\emptyset}} \left(\delta^{(1)}\right) &= \frac{\text{sgn}\left(\xi_{\emptyset}\right)}{\xi_{\emptyset}^2} \delta^{(1)} \\ &= \frac{1}{\xi_{\emptyset} \left|\xi_{\emptyset}\right|} \delta^{(1)} \end{aligned}$$

with

$$\gamma_{C^1(\xi_0)} \delta_{\xi_0}^{(1)} = \gamma_{C^1(\xi_0)}(\xi_0) \delta_{\xi_0}^{(1)} - \gamma_{C^1(\xi_0)}^{(1)}(\xi_0) \delta_{\xi_0}$$

$$\begin{aligned} \psi_{D'} * \delta_{\xi_0}^{(1)} &\equiv \left(\psi_{D'}\right)_{\xi_0}^{(1)} \\ &= \left(\left(\psi_{D'}\right)_{\xi_0}\right)^{(1)} \\ &= \left(\psi_{D'}^{(1)}\right)_{\xi_0} \end{aligned}$$

where

- $(\delta_{\xi_0})^{(1)}$ is the 1st derivative of δ_{ξ_0}
- $(\delta^{(1)})_{\xi_0}$ is the ξ_0 -translated $\delta^{(1)}$
- $(\vartheta_{\xi_0})^{(2)}$ is the 2nd derivative of ϑ_{ξ_0}
- $(\vartheta^{(2)})_{\xi_0}$ is the ξ_0 -translated $\vartheta^{(2)}$
- $\gamma_{C^1(\xi_0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^1 function on a neighbourhood of ξ_0
- $\gamma_{C^1(\xi_0)}^{(1)}$ is the 1st derivative of $\gamma_{C^1(\xi_0)}$ on a neighbourhood of ξ_0
- $((\psi_{D'})_{\xi_0})^{(1)}$ is the 1st derivative of $(\psi_{D'})_{\xi_0}$
- $(\psi_{D'}^{(1)})_{\xi_0}$ is the ξ_0 -translated $\psi_{D'}^{(1)}$

Let $\gamma_{C^2(-\infty, \xi_0)}$ (respectively, $\gamma_{C^2(\xi_0, +\infty)}$) be a C^2 function on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) whose derivative of any order in $\mathbb{N}_{[0,1]}$ has limit as ξ tends to ξ_0^- (respectively, ξ_0^+) existing and finite; let $\gamma_{C^2(-\infty, \xi_0)}^{(2)}$ (respectively, $\gamma_{C^2(\xi_0, +\infty)}^{(2)}$) be the 2nd derivative of $\gamma_{C^2(-\infty, \xi_0)}$ (respectively, $\gamma_{C^2(\xi_0, +\infty)}$). Let $\psi_{C^2(-\infty, \xi_0)}$ (respectively, $\psi_{C^2(\xi_0, +\infty)}$) be the unique regular distribution on $\mathbb{R}$ whose localisation at $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) results in the regular distribution on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) generated by $\gamma_{C^2(-\infty, \xi_0)}$ (respectively, $\gamma_{C^2(\xi_0, +\infty)}$) and whose localisation at $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$) results in the null distribution on $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$); let $\psi_{C^2(-\infty, \xi_0)}^{[2]}$ (respectively, $\psi_{C^2(\xi_0, +\infty)}^{[2]}$) be the unique regular distribution on $\mathbb{R}$ whose localisation at $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) results in the regular distribution on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) generated by $\gamma_{C^2(-\infty, \xi_0)}^{(2)}$ (respectively, $\gamma_{C^2(\xi_0, +\infty)}^{(2)}$) and whose localisation at $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$) results in the null distribution on $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$). We have [Schwartz, 1966; Estrada and Kanwal, 2002; Kanwal, 2004]

$$\begin{aligned}
 \left(\psi_{C^2(-\infty, \xi_0)} + \psi_{C^2(\xi_0, +\infty)} \right)^{(2)} &= \psi_{C^2(-\infty, \xi_0)}^{[2]} + \psi_{C^2(\xi_0, +\infty)}^{[2]} \\
 &+ \left(\lim_{\xi \rightarrow \xi_0^+} \gamma_{C^2(\xi_0, +\infty)} - \lim_{\xi \rightarrow \xi_0^-} \gamma_{C^2(-\infty, \xi_0)} \right) \delta_{\xi_0}^{(1)} \\
 &+ \left(\lim_{\xi \rightarrow \xi_0^+} \gamma_{C^2(\xi_0, +\infty)}^{(1)} - \lim_{\xi \rightarrow \xi_0^-} \gamma_{C^2(-\infty, \xi_0)}^{(1)} \right) \delta_{\xi_0}
 \end{aligned} \tag{A.11}$$

where

- $\left(\psi_{C^2(-\infty, \xi_0)} + \psi_{C^2(\xi_0, +\infty)}\right)^{(2)}$ is the 2nd derivative of $\psi_{C^2(-\infty, \xi_0)} + \psi_{C^2(\xi_0, +\infty)}$
- $\gamma_{C^2(-\infty, \xi_0)}^{(1)}$ is the 1st derivative of $\gamma_{C^2(-\infty, \xi_0)}$
- $\gamma_{C^2(\xi_0, +\infty)}^{(1)}$ is the 1st derivative of $\gamma_{C^2(\xi_0, +\infty)}$

We say that $\psi_{C^2(-\infty, \xi_0)} + \psi_{C^2(\xi_0, +\infty)}$ is a C^2 *distribution* on $\mathbb{R}$ up to the presence of a 1st-order *jump discontinuity* at ξ_0 . Equation (A.11) can be easily generalised to be applicable to C^2 distributions on $\mathbb{R}$ up to the presence of countably many 1st-order jump discontinuities.

The distributions $\text{Pf}\left(\mathfrak{G}/\xi^2\right)$ and $\text{Pf}\left(\check{\mathfrak{G}}/\xi^2\right)$ are the non-canonical pseudofunctions on $\mathbb{R}$ generated by the functions defined almost everywhere on $\mathbb{R}$ by respectively $\mathfrak{G}/\xi^2$ and $\check{\mathfrak{G}}/\xi^2$ via the Hadamard-finite-part method [Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Specifically, $\text{Pf}\left(\mathfrak{G}/\xi^2\right)$ and $\text{Pf}\left(\check{\mathfrak{G}}/\xi^2\right)$ are singular tempered distributions on $\mathbb{R}$ (of order 2, of L -order 2, of D -order 3, of C -order 1, non-homogeneous, and respectively causal and anticausal) with

$$\begin{aligned}\text{Pf}\left(\frac{\mathfrak{G}}{\xi^2}\right) &= -\left(\left((\ln(|\xi|))\mathfrak{G}\right)^{(2)} + \delta^{(1)}\right) \\ &= -\left(\text{Pf}\left(\frac{\mathfrak{G}}{\xi}\right) + \delta\right)^{(1)}\end{aligned}\tag{A.12}$$

$$\begin{aligned}\text{Pf}\left(\frac{\check{\mathfrak{G}}}{\xi^2}\right) &= -\left(\left((\ln(|\xi|))\check{\mathfrak{G}}\right)^{(2)} - \delta^{(1)}\right) \\ &= -\left(\text{Pf}\left(\frac{\check{\mathfrak{G}}}{\xi}\right) - \delta\right)^{(1)}\end{aligned}\tag{A.13}$$

where

- $\left((\ln(|\xi|))\mathfrak{G}\right)^{(2)}$ is the 2nd derivative of $(\ln(|\xi|))\mathfrak{G}$
- $\left(\text{Pf}\left(\mathfrak{G}/\xi\right) + \delta\right)^{(1)}$ is the 1st derivative of $\text{Pf}\left(\mathfrak{G}/\xi\right) + \delta$
- $\left((\ln(|\xi|))\check{\mathfrak{G}}\right)^{(2)}$ is the 2nd derivative of $(\ln(|\xi|))\check{\mathfrak{G}}$
- $\left(\text{Pf}\left(\check{\mathfrak{G}}/\xi\right) - \delta\right)^{(1)}$ is the 1st derivative of $\text{Pf}\left(\check{\mathfrak{G}}/\xi\right) - \delta$

The distribution $\text{f.p.}(1/\xi^2)$ is the canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/\xi^2$ via the Hadamard-finite-part method [Lighthill, 1958; Gelfand and

Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004].

Specifically, $\text{f.p.}(1/\xi^2)$ is a singular integrable distribution on $\mathbb{R}$ [Kierat and Skórnik, 1996; King, 2009a] (of order 2, of H -order 1^+ [Mišur and Palle, 2019], of L -order 2, of D -order 3, of C -order 1, homogeneous of degree -2 and even) with

$$\begin{aligned} \text{f.p.}\left(\frac{1}{\xi^2}\right) &\equiv \text{Pf}\left(\frac{1}{\xi^2}\right) \\ &= \text{Pf}\left(\frac{\mathfrak{G}}{\xi^2}\right) + \text{Pf}\left(\frac{\check{\mathfrak{G}}}{\xi^2}\right) \\ &= -\left((\ln(|\xi|))(\mathfrak{G} + \check{\mathfrak{G}})\right)^{(2)} \\ &= -\left(\ln(|\xi|)\right)^{(2)} \\ &= -\left(\text{p.v.}\left(\frac{1}{\xi}\right)\right)^{(1)} \end{aligned} \tag{A.14}$$

where

- $\left((\ln(|\xi|))(\mathfrak{G} + \check{\mathfrak{G}})\right)^{(2)}$ is the 2nd derivative of $(\ln(|\xi|))(\mathfrak{G} + \check{\mathfrak{G}})$
- $\left(\ln(|\xi|)\right)^{(2)}$ is the 2nd derivative of $\ln(|\xi|)$
- $\left(\text{p.v.}(1/\xi)\right)^{(1)}$ is the 1st derivative of $\text{p.v.}(1/\xi)$

In Equation (A.12), Equation (A.13), and Equation (A.14), each of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ can be interpreted as the multiplicative product of a distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ by a function of locally bounded variation on $\mathbb{R}$; alternatively, without implying any multiplications involving distributions, each of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ can be just interpreted as the regular distribution on $\mathbb{R}$ generated by the homographic locally integrable function on $\mathbb{R}$ (in which case the multiplication sign does not denote an operator but is merely a conventional symbol). Unlike $\ln(|\xi|)$, none of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ is generated by a function of bounded mean oscillation on $\mathbb{R}$ [Chang and Sadoski, 2006; Garnett, 2007]).

We observe that the functions defined almost everywhere on $\mathbb{R}$ by $\mathfrak{G}/\xi^2$, $\check{\mathfrak{G}}/\xi^2$, and their sum $1/\xi^2$ are not locally integrable functions on $\mathbb{R}$ (although so are the functions defined almost everywhere on $\mathbb{R}$ by $(\ln(|\xi|))\mathfrak{G}$, $(\ln(|\xi|))\check{\mathfrak{G}}$, and their sum $\ln(|\xi|)$). Furthermore, *per se*, the functions defined almost everywhere on $\mathbb{R}$ by both $\mathfrak{G}/\xi^2$ and $\check{\mathfrak{G}}/\xi^2$ generate only non-canonical

pseudofunctions on $\mathbb{R}$ such as $\text{Pf}\left(\eta/\xi^2\right)$ and $\text{Pf}\left(\tilde{\eta}/\xi^2\right)$; nevertheless, the function defined almost everywhere on $\mathbb{R}$ by their sum $1/\xi^2$ generates a canonical pseudofunction on $\mathbb{R}$ such as $\text{f.p.}\left(1/\xi^2\right)$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002]. From Equation (A.12) and Equation (A.13), we observe that the 1st derivatives of $\text{Pf}\left(\eta/\xi\right)$ and $\text{Pf}\left(\tilde{\eta}/\xi\right)$ are not respectively $-\text{Pf}\left(\eta/\xi^2\right)$ and $-\text{Pf}\left(\tilde{\eta}/\xi^2\right)$ (as it would be were $\text{Pf}\left(\eta/\xi\right)$ and $\text{Pf}\left(\tilde{\eta}/\xi\right)$ canonical pseudofunctions on $\mathbb{R}$); nevertheless, from Equation (A.14), we observe that the 1st derivative of $\text{p.v.}\left(1/\xi\right)$ is $-\text{f.p.}\left(1/\xi^2\right)$ (as it necessarily is being $\text{p.v.}\left(1/\xi\right)$ a canonical pseudofunction on $\mathbb{R}$). The relations hold [Jones, 1982]

$$\xi \text{f.p.}\left(\frac{1}{\xi^2}\right) = \text{p.v.}\left(\frac{1}{\xi}\right)$$

$$\xi^2 \text{f.p.}\left(\frac{1}{\xi^2}\right) = 1_{\mathbb{R}}$$

More generally, we have

$$\gamma_{C^2(0)} \text{f.p.}\left(\frac{1}{\xi^2}\right) = \text{f.p.}\left(\frac{\gamma_{C^2(0)}}{\xi^2}\right)$$

$$\begin{cases} \gamma_{C^2(0)}(0) = 0 \\ \gamma_{C^2(0)}^{(1)}(0) = 0 \end{cases} \Leftrightarrow \gamma_{C^2(0)} \text{f.p.}\left(\frac{1}{\xi^2}\right) = \frac{\gamma_{C^2(0)}}{\xi^2}$$

where

- $\gamma_{C^2(0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^2 function on a neighbourhood of the origin
- $\gamma_{C^2(0)}^{(1)}$ is the 1st derivative of $\gamma_{C^2(0)}$ on a neighbourhood of the origin

Furthermore, $\text{f.p.}\left(1/\xi^2\right)$ and $\delta^{(1)}$ form a linearly independent set, and the set of all their linear combinations coincides with the set of all tempered distributions homogeneous of degree -2 on $\mathbb{R}$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002].

The distribution $\text{Pf}\left(1/(\xi|\xi|)\right)$ is the non-canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/(\xi|\xi|)$ via the Hadamard-finite-part method [Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal,

2004]. Specifically, $\text{Pf}\left(\frac{1}{\xi|\xi|}\right)$ is a singular integrable distribution on $\mathbb{R}$ [Kierat and Skórnik, 1996; Ortner, 2011] (of order 2, of L -order 2, of D -order 3, of C -order 1, non-homogeneous, and odd) with

$$\begin{aligned}
 \text{Pf}\left(\frac{1}{\xi|\xi|}\right) &\equiv \text{Pf}\left(\frac{\text{sgn}}{\xi^2}\right) \\
 &= \text{Pf}\left(\frac{\mathfrak{G}}{\xi^2}\right) - \text{Pf}\left(\frac{\check{\mathfrak{G}}}{\xi^2}\right) \\
 &= -\left(\left((\ln(|\xi|))(\mathfrak{G} - \check{\mathfrak{G}})\right)^{(2)} + 2\delta^{(1)}\right) \\
 &= -\left(\left((\ln(|\xi|))\text{sgn}\right)^{(2)} + 2\delta^{(1)}\right) \\
 &= -\left(\text{Pf}\left(\frac{1}{|\xi|}\right) + 2\delta\right)^{(1)}
 \end{aligned} \tag{A.15}$$

where

- $\left((\ln(|\xi|))(\mathfrak{G} - \check{\mathfrak{G}})\right)^{(2)}$ is the 2nd derivative of $(\ln(|\xi|))(\mathfrak{G} - \check{\mathfrak{G}})$
- $\left((\ln(|\xi|))\text{sgn}\right)^{(2)}$ is the 2nd derivative of $(\ln(|\xi|))\text{sgn}$
- $\left(\text{Pf}\left(\frac{1}{|\xi|}\right) + 2\delta\right)^{(1)}$ is the 1st derivative of $\text{Pf}\left(\frac{1}{|\xi|}\right) + 2\delta$

In Equation (A.15), each of $(\ln(|\xi|))\mathfrak{G}$, $(\ln(|\xi|))\check{\mathfrak{G}}$, and $(\ln(|\xi|))\text{sgn}$ can be interpreted as the multiplicative product of a distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ by a function of locally bounded variation on $\mathbb{R}$; alternatively, without implying any multiplications involving distributions, each of $(\ln(|\xi|))\mathfrak{G}$, $(\ln(|\xi|))\check{\mathfrak{G}}$, and $(\ln(|\xi|))\text{sgn}$ can be just interpreted as the regular distribution on $\mathbb{R}$ generated by the homographic locally integrable function on $\mathbb{R}$ (in which case the multiplication sign does not denote an operator but is merely a conventional symbol). Unlike $\ln(|\xi|)$, none of $(\ln(|\xi|))\mathfrak{G}$, $(\ln(|\xi|))\check{\mathfrak{G}}$, and $(\ln(|\xi|))\text{sgn}$ is generated by a function of bounded mean oscillation on $\mathbb{R}$ [Chang and Sadoski, 2006; Garnett, 2007].

We observe that the functions defined almost everywhere on $\mathbb{R}$ by $\mathfrak{G}/\xi^2$, $\check{\mathfrak{G}}/\xi^2$, and their difference $1/(\xi|\xi|)$ are not locally integrable functions on $\mathbb{R}$ (although so are the functions defined almost everywhere on $\mathbb{R}$ by $(\ln(|\xi|))\mathfrak{G}$, $(\ln(|\xi|))\check{\mathfrak{G}}$, and their difference $(\ln(|\xi|))\text{sgn}$).

Furthermore, *per se*, the functions defined almost everywhere on $\mathbb{R}$ by both $\mathfrak{Y}/\xi^2$ and $\check{\mathfrak{Y}}/\xi^2$ generate only non-canonical pseudofunctions on $\mathbb{R}$ such as $\text{Pf}\left(\mathfrak{Y}/\xi^2\right)$ and $\text{Pf}\left(\check{\mathfrak{Y}}/\xi^2\right)$; likewise, the functions defined almost everywhere on $\mathbb{R}$ by their difference $1/(\xi|\xi|)$ generates only non-canonical pseudofunctions on $\mathbb{R}$ such as $\text{Pf}\left(1/(\xi|\xi|)\right)$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002]. From Equation (A.12) and Equation (A.13), we know that the 1st derivatives of $\text{Pf}\left(\mathfrak{Y}/\xi\right)$ and $\text{Pf}\left(\check{\mathfrak{Y}}/\xi\right)$ are not respectively $-\text{Pf}\left(\mathfrak{Y}/\xi^2\right)$ and $-\text{Pf}\left(\check{\mathfrak{Y}}/\xi^2\right)$ (as it would be were $\text{Pf}\left(\mathfrak{Y}/\xi\right)$ and $\text{Pf}\left(\check{\mathfrak{Y}}/\xi\right)$ canonical pseudofunctions on $\mathbb{R}$); likewise, from Equation (A.15), we observe that the 1st derivative of $\text{Pf}\left(1/|\xi|\right)$ is not $-\text{Pf}\left(1/(\xi|\xi|)\right)$ (as it would be were $\text{Pf}\left(1/|\xi|\right)$ a canonical pseudofunction on $\mathbb{R}$). The relations hold [Jones, 1982]

$$\xi \text{Pf}\left(\frac{1}{\xi|\xi|}\right) = \text{Pf}\left(\frac{1}{|\xi|}\right)$$

$$\xi^2 \text{Pf}\left(\frac{1}{\xi|\xi|}\right) = \text{sgn}$$

More generally, we have

$$\gamma_{C^2(0)} \text{Pf}\left(\frac{1}{\xi|\xi|}\right) = \text{Pf}\left(\frac{\gamma_{C^2(0)}}{\xi|\xi|}\right)$$

$$\begin{cases} \gamma_{C^2(0)}(0) = 0 \\ \gamma_{C^2(0)}^{(1)}(0) = 0 \end{cases} \Leftrightarrow \gamma_{C^2(0)} \text{Pf}\left(\frac{1}{\xi|\xi|}\right) = \frac{\gamma_{C^2(0)}}{\xi|\xi|}$$

Let ζ_2 be a known distribution on $\mathbb{R}$. The 2nd-degree algebraic equation

$$\xi^2 \chi_2 = \zeta_2 \tag{A.16}$$

of constant term $-\zeta_2$ in the unknown distribution χ_2 is satisfied by infinitely many distributions on $\mathbb{R}$ that are quotients of ζ_2 by ξ^2 [Schwartz, 1966; Jones, 1982]. In particular, if ζ_2 is a regular distribution on $\mathbb{R}$, the general solution $(\chi_2)_{\text{gen}}$ of Equation (A.16) is the singular distribution on $\mathbb{R}$ given by [Misra and Lavoine, 1986]

$$\begin{aligned}
 (\chi_2)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{\zeta_2}{\xi^2} \right) \\
 &= \text{Pf} \left(\frac{\zeta_2}{\xi^2} \right) + B_{2,0} \delta^{(1)} + B_{2,1} \delta
 \end{aligned}
 \tag{A.17}$$

where

- $\text{i.q.}(\zeta_2/\xi^2)$ is the indefinite quotient of ζ_2 by ξ^2
- $B_{2,0}$ is an arbitrary complex constant of the same quantity dimension as ζ_2
- $B_{2,1}$ is an arbitrary complex constant of the same quantity dimension as ζ_2/ξ

For $\zeta_2 = 0_{\mathbb{R}}$, Equation (A.16) and Equation (A.17) become

$$\xi^2 \chi_2 = 0_{\mathbb{R}} \tag{A.18}$$

$$\begin{aligned}
 (\chi_2)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{0_{\mathbb{R}}}{\xi^2} \right) \\
 &= \text{Pf} \left(\frac{0}{\xi^2} \right) + B_{2,0} \delta^{(1)} + B_{2,1} \delta
 \end{aligned}$$

where

- $\text{i.q.}(0_{\mathbb{R}}/\xi^2)$ is the indefinite quotient of $0_{\mathbb{R}}$ by ξ^2

Equation (A.18) can be regarded as the homogeneous equation associated with Equation (A.16).

Accordingly, the general solution $(\chi_2)_{\text{gen}}$ of Equation (A.16) can be regarded as the sum of the particular solution $\text{Pf}(\zeta_2/\xi^2)$ of Equation (A.16) and the general solution $B_{2,0} \delta^{(1)} + B_{2,1} \delta$ of Equation (A.18).

For $\zeta_2 = 1_{\mathbb{R}}$, Equation (A.16) and Equation (A.17) become

$$\xi^2 \chi_2 = 1_{\mathbb{R}}$$

$$\begin{aligned}
 (\chi_2)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{1_{\mathbb{R}}}{\xi^2} \right) \\
 &= \text{f.p.} \left(\frac{1}{\xi^2} \right) + B_{2,0} \delta^{(1)} + B_{2,1} \delta
 \end{aligned}$$

where

- $\text{i.q.}(1_{\mathbb{R}}/\xi^2)$ is the indefinite quotient of $1_{\mathbb{R}}$ by ξ^2

For $\zeta_2 = \text{sgn}$, Equation (A.16) and Equation (A.17) become

$$\xi^2 \chi_2 = \text{sgn}$$

$$\begin{aligned} (\chi_2)_{\text{gen}} &\equiv \text{i.q.}\left(\frac{\text{sgn}}{\xi^2}\right) \\ &= \text{Pf}\left(\frac{1}{\xi|\xi|}\right) + B_{2,0}\delta^{(1)} + B_{2,1}\delta \end{aligned}$$

where

- $\text{i.q.}(\text{sgn}/\xi^2)$ is the indefinite quotient of sgn by ξ^2

A.3.3.3 Unit n -tuple and n^{th} -order pseudofunctions

The *unit n -tuple* on $\mathbb{R}$, denoted by $\delta^{(n-1)}$, is the n^{th} derivative of $\mathfrak{g}$ [Schwartz, 1966].

Specifically, $\delta^{(n-1)}$ is a point-supported distribution on $\mathbb{R}$ (of order $n-1$, of L -order n , of D -order $n+1$, of C -order $n-1$, homogeneous of degree $-n$, and even or odd depending on whether $n-1$ is even or odd) with [Schwartz, 1966; Kanwal, 2004]

$$\begin{aligned} \delta^{(n-1)} &= \left(\delta^{(1)}\right)^{* (n-1)} \\ &= \mathfrak{g}^{(n)} \end{aligned}$$

with

$$\begin{aligned} \gamma_{C^{n-1}(0)} \delta^{(n-1)} &= \sum_{i=1}^n \left((-1)^{i-1} \binom{n-1}{n-i} \gamma_{C^{n-1}(0)}^{(i-1)}(0) \delta^{(n-i)} \right) \\ \psi_{\mathcal{D}'} * \delta^{(n-1)} &= \psi_{\mathcal{D}'}^{(n-1)} \end{aligned}$$

where

- $\left(\delta^{(1)}\right)^{* (n-1)}$ is the $(n-1)^{\text{th}}$ convolutive power of $\delta^{(1)}$
- $\mathfrak{g}^{(n)}$ is the n^{th} derivative of $\mathfrak{g}$
- $\gamma_{C^{n-1}(0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^{n-1} function on a neighbourhood of the origin

A.3 Mathematical framework: Compendium of notable distributions

- $\gamma_{C^{n-1}(0)}^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ is the $(i-1)^{\text{th}}$ derivative of $\gamma_{C^{n-1}(0)}$ on a neighbourhood of the origin
- $\binom{n-1}{n-i}$ is a binomial coefficient
- $\psi_{\mathcal{D}'}^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of $\psi_{\mathcal{D}'}$

We also have [Schwartz, 1966; Kanwal, 2004]

$$\begin{aligned}
 \delta_{\xi_0}^{(n-1)} &\equiv \left(\delta_{\xi_0} \right)^{(n-1)} \\
 &= \left(\delta^{(n-1)} \right)_{\xi_0} \\
 &= \delta_{\xi_0} * \delta^{(n-1)} \\
 &= \left(\mathfrak{g}_{\xi_0} \right)^{(n)} \\
 &= \left(\mathfrak{g}^{(n)} \right)_{\xi_0} \\
 &\equiv \mathfrak{g}_{\xi_0}^{(n)}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{S}_{\xi_{\emptyset}} \left(\delta^{(n-1)} \right) &= \frac{\text{sgn}(\xi_{\emptyset})}{\xi_{\emptyset}^n} \delta^{(n-1)} \\
 &= \frac{1}{\xi_{\emptyset}^{n-1} |\xi_{\emptyset}|} \delta^{(n-1)}
 \end{aligned}$$

with

$$\gamma_{C^{n-1}(\xi_0)} \delta_{\xi_0}^{(n-1)} = \sum_{i=1}^n \left((-1)^{i-1} \binom{n-1}{n-i} \gamma_{C^{n-1}(\xi_0)}^{(i-1)} (\xi_0) \delta_{\xi_0}^{(n-i)} \right)$$

$$\begin{aligned}
 \psi_{\mathcal{D}'} * \delta_{\xi_0}^{(n-1)} &\equiv \left(\psi_{\mathcal{D}'} \right)_{\xi_0}^{(n-1)} \\
 &= \left(\left(\psi_{\mathcal{D}'} \right)_{\xi_0} \right)^{(n-1)} \\
 &= \left(\psi_{\mathcal{D}'}^{(n-1)} \right)_{\xi_0}
 \end{aligned}$$

where

- $\left(\delta_{\xi_0} \right)^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of δ_{ξ_0}
- $\left(\delta^{(n-1)} \right)_{\xi_0}$ is the ξ_0 -translated $\delta^{(n-1)}$
- $\left(\mathfrak{g}_{\xi_0} \right)^{(n)}$ is the n^{th} derivative of $\mathfrak{g}_{\xi_0}$
- $\left(\mathfrak{g}^{(n)} \right)_{\xi_0}$ is the ξ_0 -translated $\mathfrak{g}^{(n)}$

- $\gamma_{C^{n-1}(\xi_0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^{n-1} function on a neighbourhood of ξ_0
- $\gamma_{C^{n-1}(\xi_0)}^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ is the $(i-1)^{\text{th}}$ derivative of $\gamma_{C^{n-1}(\xi_0)}$
- $\left((\psi_{D'})_{\xi_0}\right)^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of $(\psi_{D'})_{\xi_0}$
- $\left(\psi_{D'}^{(n-1)}\right)_{\xi_0}$ is the ξ_0 -translated $\psi_{D'}^{(n-1)}$

Let $\gamma_{C^n(-\infty, \xi_0)}$ (respectively, $\gamma_{C^n(\xi_0, +\infty)}$) be a C^n function on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) whose derivative of any order in $\mathbb{N}_{[0, n-1]}$ has limit as ξ tends to ξ_0^- (respectively, ξ_0^+) existing and finite; let $\gamma_{C^n(-\infty, \xi_0)}^{(n)}$ (respectively, $\gamma_{C^n(\xi_0, +\infty)}^{(n)}$) be the n^{th} derivative of $\gamma_{C^n(-\infty, \xi_0)}$ (respectively, $\gamma_{C^n(\xi_0, +\infty)}$). Let $\psi_{C^n(-\infty, \xi_0)}$ (respectively, $\psi_{C^n(\xi_0, +\infty)}$) be the unique regular distribution on $\mathbb{R}$ whose localisation at $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) results in the regular distribution on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) generated by $\gamma_{C^n(-\infty, \xi_0)}$ (respectively, $\gamma_{C^n(\xi_0, +\infty)}$) and whose localisation at $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$) results in the null distribution on $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$); let $\psi_{C^n(-\infty, \xi_0)}^{[n]}$ (respectively, $\psi_{C^n(\xi_0, +\infty)}^{[n]}$) be the unique regular distribution on $\mathbb{R}$ whose localisation at $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) results in the regular distribution on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) generated by $\gamma_{C^n(-\infty, \xi_0)}^{(n)}$ (respectively, $\gamma_{C^n(\xi_0, +\infty)}^{(n)}$) and whose localisation at $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$) results in the null distribution on $\mathbb{R}_{(\xi_0, +\infty)}$ (respectively, $\mathbb{R}_{(-\infty, \xi_0)}$). We have [Schwartz, 1966; Estrada and Kanwal, 2002; Kanwal, 2004]

$$\begin{aligned} \left(\psi_{C^n(-\infty, \xi_0)} + \psi_{C^n(\xi_0, +\infty)}\right)^{(n)} &= \psi_{C^n(-\infty, \xi_0)}^{[n]} + \psi_{C^n(\xi_0, +\infty)}^{[n]} \\ &\quad + \sum_{i=1}^n \left(\left(\lim_{\xi \rightarrow \xi_0^+} \gamma_{C^n(\xi_0, +\infty)}^{(i-1)} - \lim_{\xi \rightarrow \xi_0^-} \gamma_{C^n(-\infty, \xi_0)}^{(i-1)} \right) \delta_{\xi_0}^{(n-i)} \right) \end{aligned} \quad (\text{A.19})$$

where

- $\gamma_{C^n(-\infty, \xi_0)}^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ is the $(i-1)^{\text{th}}$ derivative of $\gamma_{C^n(-\infty, \xi_0)}$
- $\gamma_{C^n(\xi_0, +\infty)}^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ is the $(i-1)^{\text{th}}$ derivative of $\gamma_{C^n(\xi_0, +\infty)}$

We say that $\psi_{C^n(-\infty, \xi_0)} + \psi_{C^n(\xi_0, +\infty)}$ is a C^n distribution on $\mathbb{R}$ up to the presence of an $(n-1)^{\text{th}}$ -order jump discontinuity at ξ_0 . Equation (A.19) can be easily generalised to be applicable to C^n distributions on $\mathbb{R}$ up to the presence of countably many $(n-1)^{\text{th}}$ -order jump discontinuities.

The distributions $\text{Pf}\left(\vartheta/\xi^n\right)$ and $\text{Pf}\left(\check{\vartheta}/\xi^n\right)$ are the non-canonical pseudofunctions on $\mathbb{R}$ generated by the functions defined almost everywhere on $\mathbb{R}$ by respectively ϑ/ξ^n and $\check{\vartheta}/\xi^n$ via the Hadamard-finite-part method [Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Specifically, $\text{Pf}\left(\vartheta/\xi^n\right)$ and $\text{Pf}\left(\check{\vartheta}/\xi^n\right)$ are singular tempered distributions on $\mathbb{R}$ (of order n , of L -order n , of D -order $n+1$, of C -order $n-1$, non-homogeneous, and respectively causal and anticausal) with

$$\begin{aligned}\text{Pf}\left(\frac{\vartheta}{\xi^n}\right) &= \frac{(-1)^{n-1}}{(n-1)!} \left(\left((\ln(|\xi|)) \vartheta \right)^{(n)} + \nu_n \delta^{(n-1)} \right) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \left(\text{Pf}\left(\frac{\vartheta}{\xi}\right) + \nu_n \delta \right)^{(n-1)}\end{aligned}\tag{A.20}$$

$$\begin{aligned}\text{Pf}\left(\frac{\check{\vartheta}}{\xi^n}\right) &= \frac{(-1)^{n-1}}{(n-1)!} \left(\left((\ln(|\xi|)) \check{\vartheta} \right)^{(n)} - \nu_n \delta^{(n-1)} \right) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \left(\text{Pf}\left(\frac{\check{\vartheta}}{\xi}\right) - \nu_n \delta \right)^{(n-1)}\end{aligned}\tag{A.21}$$

with

$$\nu_n := \begin{cases} 0 & , \quad n = 1 \\ \sum_{l=1}^{n-1} \left(\frac{1}{l} \right) & , \quad n \in \mathbb{N}_{[2,+\infty)} \end{cases}$$

where

- l is a bound index taking values in $\mathbb{N}_{[1,n-1]}$
- $\left((\ln(|\xi|)) \vartheta \right)^{(n)}$ is the n^{th} derivative of $(\ln(|\xi|)) \vartheta$
- $\left(\text{Pf}\left(\vartheta/\xi\right) + \nu_n \delta \right)^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of $\text{Pf}\left(\vartheta/\xi\right) + \nu_n \delta$
- $\left((\ln(|\xi|)) \check{\vartheta} \right)^{(n)}$ is the n^{th} derivative of $(\ln(|\xi|)) \check{\vartheta}$
- $\left(\text{Pf}\left(\check{\vartheta}/\xi\right) - \nu_n \delta \right)^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of $\text{Pf}\left(\check{\vartheta}/\xi\right) - \nu_n \delta$

The distribution $\text{f.p.}(1/\xi^n)$ is the canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/\xi^n$ via the Hadamard-finite-part method [Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004].

Specifically, $\text{f.p.}\left(\frac{1}{\xi^n}\right)$ is a singular integrable distribution on $\mathbb{R}$ [Kierat and Skórnik, 1996; King, 2009a] (of order n , of H -order $(n-1)^+$ [Mišur and Palle, 2019], of L -order n , of D -order $n+1$, of C -order $n-1$, homogeneous of degree $-n$, and either odd or even depending on whether n is odd or even) with

$$\begin{aligned}\text{f.p.}\left(\frac{1}{\xi^n}\right) &= \text{Pf}\left(\frac{\mathfrak{G}}{\xi^n}\right) + \text{Pf}\left(\frac{\check{\mathfrak{G}}}{\xi^n}\right) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \left((\ln(|\xi|)) (\mathfrak{G} + \check{\mathfrak{G}}) \right)^{(n)} \\ &= \frac{(-1)^{n-1}}{(n-1)!} (\ln(|\xi|))^{(n)} \\ &= \frac{(-1)^{n-1}}{(n-1)!} \left(\text{p.v.}\left(\frac{1}{\xi}\right) \right)^{(n-1)}\end{aligned}\tag{A.22}$$

where

- $\left((\ln(|\xi|)) (\mathfrak{G} + \check{\mathfrak{G}}) \right)^{(n)}$ is the n^{th} derivative of $(\ln(|\xi|)) (\mathfrak{G} + \check{\mathfrak{G}})$
- $(\ln(|\xi|))^{(n)}$ is the n^{th} derivative of $\ln(|\xi|)$
- $(\text{p.v.}(1/\xi))^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of $\text{p.v.}(1/\xi)$

In Equation (A.20), Equation (A.21), and Equation (A.22), each of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ can be interpreted as the multiplicative product of a distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ by a function of locally bounded variation on $\mathbb{R}$; alternatively, without implying any multiplications involving distributions, each of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ can be just interpreted as the regular distribution on $\mathbb{R}$ generated by the homographic locally integrable function on $\mathbb{R}$ (in which case the multiplication sign does not denote an operator but is merely a conventional symbol). Unlike $\ln(|\xi|)$, none of $(\ln(|\xi|))\mathfrak{G}$ and $(\ln(|\xi|))\check{\mathfrak{G}}$ is generated by a function of bounded mean oscillation on $\mathbb{R}$ [Chang and Sadoski, 2006; Garnett, 2007]).

We observe that the functions defined almost everywhere on $\mathbb{R}$ by $\mathfrak{G}/\xi^n$, $\check{\mathfrak{G}}/\xi^n$, and their sum $1/\xi^n$ are not locally integrable functions on $\mathbb{R}$ (although so are the functions defined almost everywhere on $\mathbb{R}$ by $(\ln(|\xi|))\mathfrak{G}$, $(\ln(|\xi|))\check{\mathfrak{G}}$, and their sum $\ln(|\xi|)$). Furthermore, *per se*, the functions defined almost everywhere on $\mathbb{R}$ by both $\mathfrak{G}/\xi^n$ and $\check{\mathfrak{G}}/\xi^n$ generate only non-canonical

pseudofunctions on $\mathbb{R}$ such as $\text{Pf}\left(\vartheta/\xi^n\right)$ and $\text{Pf}\left(\check{\vartheta}/\xi^n\right)$; nevertheless, the function defined almost everywhere on $\mathbb{R}$ by their sum $1/\xi^n$ generates a canonical pseudofunction on $\mathbb{R}$ such as $\text{f.p.}\left(1/\xi^n\right)$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002]. From Equation (A.20) and Equation (A.21), we observe that the $(n-1)^{\text{th}}$ derivatives of $\text{Pf}\left(\vartheta/\xi\right)$ and $\text{Pf}\left(\check{\vartheta}/\xi\right)$ are not respectively $\left((n-1)!/(-1)^{n-1}\right)\text{Pf}\left(\vartheta/\xi^n\right)$ and $\left((n-1)!/(-1)^{n-1}\right)\text{Pf}\left(\check{\vartheta}/\xi^n\right)$ (as it would be were $\text{Pf}\left(\vartheta/\xi\right)$ and $\text{Pf}\left(\check{\vartheta}/\xi\right)$ canonical pseudofunctions on $\mathbb{R}$); nevertheless, from Equation (A.22), we observe that the $(n-1)^{\text{th}}$ derivative of $\text{p.v.}\left(1/\xi\right)$ is $\left((n-1)!/(-1)^{n-1}\right)\text{f.p.}\left(1/\xi^n\right)$ (as it necessarily is being $\text{p.v.}\left(1/\xi\right)$ a canonical pseudofunction on $\mathbb{R}$). The relations hold [Jones, 1982]

$$\xi \text{f.p.}\left(\frac{1}{\xi^n}\right) = \text{f.p.}\left(\frac{1}{\xi^{n-1}}\right)$$

$$\xi^n \text{f.p.}\left(\frac{1}{\xi^n}\right) = 1_{\mathbb{R}}$$

More generally, we have

$$\gamma_{C^n(0)} \text{f.p.}\left(\frac{1}{\xi^n}\right) = \text{f.p.}\left(\frac{\gamma_{C^n(0)}}{\xi^n}\right)$$

$$\gamma_{C^n(0)}^{(i-1)}(0) = 0 \quad , \quad i \in \mathbb{N}_{[1,n]} \quad \Leftrightarrow \quad \gamma_{C^n(0)} \text{f.p.}\left(\frac{1}{\xi^n}\right) = \frac{\gamma_{C^n(0)}}{\xi^n}$$

where

- $\gamma_{C^n(0)}$ is a locally integrable function on $\mathbb{R}$ that is a C^n function on a neighbourhood of the origin
- $\gamma_{C^n(0)}^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ is the $(i-1)^{\text{th}}$ derivative of $\gamma_{C^n(0)}$ on a neighbourhood of the origin

Furthermore, $\text{f.p.}\left(1/\xi^n\right)$ and $\delta^{(n-1)}$ form a linearly independent set, and the set of all their linear combinations coincides with the set of all tempered distributions homogeneous of degree $-n$ on $\mathbb{R}$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002].

The distribution $\text{Pf}\left(1/\left(\xi^{n-1}|\xi|\right)\right)$ is the non-canonical pseudofunction on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $1/\left(\xi^{n-1}|\xi|\right)$ via the Hadamard-finite-part method

[Lighthill, 1958; Gelfand and Shilov, 1964; Schwartz, 1966; Jones, 1982; Estrada and Kanwal, 1989, 2002; Kanwal, 2004]. Specifically, $\text{Pf}\left(1/(\xi^{n-1}|\xi|)\right)$ is a singular integrable distribution on $\mathbb{R}$ [Kierat and Skórnik, 1996; Ortner, 2011] (of order n , of L -order n , of D -order $n+1$, of C -order $n-1$, non-homogeneous, and either even or odd depending on whether n is odd or even) with

$$\begin{aligned} \text{Pf}\left(\frac{1}{\xi^{n-1}|\xi|}\right) &\equiv \text{Pf}\left(\frac{\text{sgn}}{\xi^n}\right) \\ &= \text{Pf}\left(\frac{\mathfrak{g}}{\xi^n}\right) - \text{Pf}\left(\frac{\check{\mathfrak{g}}}{\xi^n}\right) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \left(\left((\ln(|\xi|))(\mathfrak{g} - \check{\mathfrak{g}}) \right)^{(n)} + 2\nu_n \delta^{(n-1)} \right) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \left(\left((\ln(|\xi|))\text{sgn} \right)^{(n)} + 2\nu_n \delta^{(n-1)} \right) \\ &= \frac{(-1)^{n-1}}{(n-1)!} \left(\text{Pf}\left(\frac{1}{|\xi|}\right) + 2\nu_n \delta \right)^{(n-1)} \end{aligned} \quad (\text{A.23})$$

where

- $\left((\ln(|\xi|))(\mathfrak{g} - \check{\mathfrak{g}}) \right)^{(n)}$ is the n^{th} derivative of $(\ln(|\xi|))(\mathfrak{g} - \check{\mathfrak{g}})$
- $\left((\ln(|\xi|))\text{sgn} \right)^{(n)}$ is the n^{th} derivative of $(\ln(|\xi|))\text{sgn}$
- $\left(\text{Pf}\left(1/|\xi|\right) + 2\nu_n \delta \right)^{(n-1)}$ is the $(n-1)^{\text{th}}$ derivative of $\text{Pf}\left(1/|\xi|\right) + 2\nu_n \delta$

In Equation (A.23), each of $(\ln(|\xi|))\mathfrak{g}$, $(\ln(|\xi|))\check{\mathfrak{g}}$, and $(\ln(|\xi|))\text{sgn}$ can be interpreted as the multiplicative product of a distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ by a function of locally bounded variation on $\mathbb{R}$; alternatively, without implying any multiplications involving distributions, each of $(\ln(|\xi|))\mathfrak{g}$, $(\ln(|\xi|))\check{\mathfrak{g}}$, and $(\ln(|\xi|))\text{sgn}$ can be just interpreted as the regular distribution on $\mathbb{R}$ generated by the homographic locally integrable function on $\mathbb{R}$ (in which case the multiplication sign does not denote an operator but is merely a conventional symbol). Unlike $\ln(|\xi|)$, none of $(\ln(|\xi|))\mathfrak{g}$, $(\ln(|\xi|))\check{\mathfrak{g}}$, and $(\ln(|\xi|))\text{sgn}$ is generated by a function of bounded mean oscillation on $\mathbb{R}$ [Chang and Sadoski, 2006; Garnett, 2007]).

We observe that the functions defined almost everywhere on $\mathbb{R}$ by $\mathfrak{g}/\xi^n$, $\check{\mathfrak{g}}/\xi^n$, and their difference $1/(\xi^{n-1}|\xi|)$ are not locally integrable functions on $\mathbb{R}$ (although so are the functions

defined almost everywhere on $\mathbb{R}$ by $(\ln(|\xi|))\mathfrak{P}$, $(\ln(|\xi|))\check{\mathfrak{P}}$, and their difference $(\ln(|\xi|))\text{sgn}$.

Furthermore, *per se*, the functions defined almost everywhere on $\mathbb{R}$ by both $\mathfrak{P}/\xi^n$ and $\check{\mathfrak{P}}/\xi^n$ generate only non-canonical pseudofunctions on $\mathbb{R}$ such as $\text{Pf}(\mathfrak{P}/\xi^n)$ and $\text{Pf}(\check{\mathfrak{P}}/\xi^n)$; likewise, the function defined almost everywhere on $\mathbb{R}$ by their difference $1/(\xi^{n-1}|\xi|)$ generates only non-canonical pseudofunctions on $\mathbb{R}$ such as $\text{Pf}(1/(\xi^{n-1}|\xi|))$ [Gelfand and Shilov, 1964; Estrada and Kanwal, 2002]. From Equation (A.20) and Equation (A.21), we know that the $(n-1)^{\text{th}}$ derivatives of $\text{Pf}(\mathfrak{P}/\xi)$ and $\text{Pf}(\check{\mathfrak{P}}/\xi)$ are not respectively $((n-1)!/(-1)^{n-1})\text{Pf}(\mathfrak{P}/\xi^n)$ and $((n-1)!/(-1)^{n-1})\text{Pf}(\check{\mathfrak{P}}/\xi^n)$ (as it would be were $\text{Pf}(\mathfrak{P}/\xi)$ and $\text{Pf}(\check{\mathfrak{P}}/\xi)$ canonical pseudofunctions on $\mathbb{R}$); likewise, from Equation (A.23), we observe that the $(n-1)^{\text{th}}$ derivative of $\text{Pf}(1/|\xi|)$ is not $((n-1)!/(-1)^{n-1})\text{Pf}(1/(\xi^{n-1}|\xi|))$ (as it would be were $\text{Pf}(1/|\xi|)$ a canonical pseudofunction on $\mathbb{R}$). The relations hold [Jones, 1982]

$$\xi \text{Pf}\left(\frac{1}{\xi^{n-1}|\xi|}\right) = \text{f.p.}\left(\frac{1}{\xi^{n-2}|\xi|}\right)$$

$$\xi^n \text{Pf}\left(\frac{1}{\xi^{n-1}|\xi|}\right) = \text{sgn}$$

More generally, we have

$$\gamma_{C^n(0)} \text{Pf}\left(\frac{1}{\xi^{n-1}|\xi|}\right) = \text{Pf}\left(\frac{\gamma_{C^n(0)}}{\xi^{n-1}|\xi|}\right)$$

$$\gamma_{C^n(0)}^{(i-1)}(0) = 0 \quad , \quad i \in \mathbb{N}_{[1,n]} \quad \Leftrightarrow \quad \gamma_{C^n(0)} \text{Pf}\left(\frac{1}{\xi^{n-1}|\xi|}\right) = \frac{\gamma_{C^n(0)}}{\xi^{n-1}|\xi|}$$

Let ζ_n be a known distribution on $\mathbb{R}$. The n^{th} -degree algebraic equation

$$\xi^n \chi_n = \zeta_n \tag{A.24}$$

of constant term $-\zeta_n$ in the unknown distribution χ_n is satisfied by infinitely many distributions on $\mathbb{R}$ that are quotients of ζ_n by ξ^n [Schwartz, 1966; Jones, 1982]. In particular, if ζ_n is a regular

distribution on $\mathbb{R}$, the general solution $(\chi_n)_{\text{gen}}$ of Equation (A.24) is the singular distribution on $\mathbb{R}$ given by [Misra and Lavoine, 1986]

$$\begin{aligned} (\chi_n)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{\zeta_n}{\xi^n} \right) \\ &= \text{Pf} \left(\frac{\zeta_n}{\xi^n} \right) + \sum_{i=1}^n (B_{n,i-1} \delta^{(n-i)}) \end{aligned} \quad (\text{A.25})$$

where

- $\text{i.q.}(\zeta_n / \xi^n)$ is the indefinite quotient of ζ_n by ξ^n
- $B_{n,i-1}$ with i in $\mathbb{N}_{[1,n]}$ is an arbitrary complex constant of the same quantity dimension as ζ_n / ξ^{i-1}

For $\zeta_n = 0_{\mathbb{R}}$, Equation (A.24) and Equation (A.25) become

$$\xi^n \chi_n = 0_{\mathbb{R}} \quad (\text{A.26})$$

$$\begin{aligned} (\chi_n)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{0_{\mathbb{R}}}{\xi^n} \right) \\ &= \text{Pf} \left(\frac{0}{\xi^n} \right) + \sum_{i=1}^n (B_{n,i-1} \delta^{(n-i)}) \end{aligned}$$

where

- $\text{i.q.}(0_{\mathbb{R}} / \xi^n)$ is the indefinite quotient of $0_{\mathbb{R}}$ by ξ^n

Equation (A.26) can be regarded as the homogeneous equation associated with Equation (A.24).

Accordingly, the general solution $(\chi_n)_{\text{gen}}$ of Equation (A.24) can be regarded as the sum of the

particular solution $\text{Pf}(\zeta_n / \xi^n)$ of Equation (A.24) and the general solution $\sum_{i=1}^n (B_{n,i-1} \delta^{(n-i)})$ of

Equation (A.26).

For $\zeta_n = 1_{\mathbb{R}}$, Equation (A.24) and Equation (A.25) become

$$\xi^n \chi_n = 1_{\mathbb{R}}$$

$$\begin{aligned} (\chi_n)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{1_{\mathbb{R}}}{\xi^n} \right) \\ &= \text{f.p.} \left(\frac{1}{\xi^n} \right) + \sum_{i=1}^n (B_{n,i-1} \delta^{(n-i)}) \end{aligned}$$

where

- $\text{i.q.}(1_{\mathbb{R}}/\xi^n)$ is the indefinite quotient of $1_{\mathbb{R}}$ by ξ^n

For $\xi_n = \text{sgn}$, Equation (A.24) and Equation (A.25) become

$$\xi^n \chi_n = \text{sgn}$$

$$\begin{aligned} (\chi_n)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{\text{sgn}}{\xi^n} \right) \\ &= \text{Pf} \left(\frac{1}{\xi^{n-1} |\xi|} \right) + \sum_{i=1}^n (B_{n,i-1} \delta^{(n-i)}) \end{aligned}$$

where

- $\text{i.q.}(\text{sgn}/\xi^n)$ is the indefinite quotient of sgn by ξ^n

A.4 Signals: Integral transformations

A.4.1 Introduction

A distribution on $\mathbb{R}^2$ is a continuous linear form on a test function space of suitable smooth functions on $\mathbb{R}^2$ [Schwartz, 1966]. Many of the concepts and results pertaining to distributions on $\mathbb{R}$ can be immediately transferred to distributions on $\mathbb{R}^2$ [Schwartz, 1966]; hereinafter, we tacitly make such extension whenever appropriate. By abuse of language, we say that the free real variables of the elements of the test function space on which a distribution on $\mathbb{R}^2$ is defined are also the free real variables of the distribution on $\mathbb{R}^2$. The localisation of a distribution on $\mathbb{R}^2$ at a point of $\mathbb{R}^2$, whenever well-defined, results in a complex constant [Łojasiewicz, 1958]; besides, the localisation of a distribution on $\mathbb{R}^2$ at a straight line of $\mathbb{R}^2$ that is parallel to either coordinate axis, whenever well-defined, results in a distribution on $\mathbb{R}$ [Łojasiewicz, 1958].

An *integral transformation* is a certain type of continuous linear map between two Hausdorff locally convex topological linear spaces over $\mathbb{F}$ [Schwartz, 1952, 1954d, 1954e, 1955, 1957, 1958, 1966; Grothendieck, 1955; Hirata and Shiraishi, 1960; Gelfand and Vilenkin, 1964; Treves, 1967; Hörmander, 1990; Friedlander and Joshi, 1998; Duistermaat and Kolk, 2010]. Any integral transformation is associated uniquely with a fixed distribution on $\mathbb{R}^2$ that is said to be its *kernel distribution* [Schwartz, 1952, 1954d, 1954e, 1955, 1957, 1958, 1966; Grothendieck, 1955; Hirata and Shiraishi, 1960; Gelfand and Vilenkin, 1964; Treves, 1967; Hörmander, 1990; Friedlander and Joshi, 1998; Duistermaat and Kolk, 2010]. The kernel distribution of an integral transformation has two free real variables when considered in itself, whereas it has one free real variable and one bound real variable when considered in the integral transformation; hereinafter, unless otherwise stated, the kernel distribution of an integral transformation is considered in the integral transformation. We qualify the kernel distribution of an integral transformation as either *symmetric* or *asymmetric* depending on whether its free real variable and bound real variable can or cannot be exchanged [Schwartz, 1952, 1957, 1958]. Whenever well-defined, the multiplication and the convolution by a fixed distribution on $\mathbb{R}$ are examples of integral transformations.

We say that an element of the domain and the corresponding element of the codomain of an integral transformation are respectively an *integral transformand* and its *integral transform*. Furthermore, we say that the ordered pair formed by an integral transformand and its integral transform is an *integral-transform pair*. It is convenient to introduce the concept of *partial integral*

on $\mathbb{R}$ with respect to the bound real variable of the kernel distribution of an integral transformation [Schwartz, 1952, 1957, 1958; Hirata and Shiraishi, 1960]. Let there be given an integral transformation (and hence its kernel distribution). Remarkably, the integral transform of an integral transformand coincides with the partial integral on $\mathbb{R}$ of the multiplicative product of the integral transformand by the kernel distribution with respect to the bound real variable of the kernel distribution [Schwartz, 1952, 1957, 1958; Hirata and Shiraishi, 1960].

By defining *information* as “intelligence or knowledge capable of being represented in forms suitable for communication, storage or processing” [International Electrotechnical Commission, 1988], a *signal* can be defined as a “physical phenomenon whose presence, absence, or variation is considered as representing information” [International Electrotechnical Commission, 2019]; besides, a *system* can be defined as a “set of interrelated elements considered in a defined context as a whole and separated from their environment” [International Organization for Standardization, 2018]. In the *theory of signals and systems* [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], the mathematical model of a signal or a system is often provided by maps that can be interpreted as characterisations in *time*, in either *frequency* or *angular-frequency*, or in either *complex frequency* or *complex angular-frequency*; by abuse of language, we say that such characterisations ‘live’ respectively in the *time domain*, in the *frequency domain* (also known as *Fourier domain*), or in the *complex frequency domain* (also known as *Laplace domain*). Hereinafter, signals are modelled by means of distributions (or, occasionally, functions), whereas systems are modelled by means of maps between distribution spaces (or, occasionally, function spaces). With reference to time-domain characterisations, each signal and each system that we consider are respectively a *scalar-valued continuous-time deterministic signal* (hereinafter *signal*) and a *single-input–single-output continuous-time deterministic system* (hereinafter *system*).

In the following, some information about Fourier transformation, Laplace transformation, and Hilbert transformation is summarised.

A.4.2 Fourier transformation

Let f_s be a smooth function of rapid decay on $\mathbb{R}$ of the free real variable t . The *direct Fourier transform* (hereinafter *Fourier transform*) F_s of the *direct Fourier transformand* (hereinafter *Fourier transformand*) f_s is the smooth function of rapid decay on $\mathbb{R}$ of the free real variable ω defined by [Lavoine, 1963; Schwartz, 1966; Lewis, 2022a; Weisstein, 2024]

$$F_s \equiv \mathcal{F}(f_s) \quad (A.27)$$

$$:= \left(\frac{|\nu|}{(2\pi)^{1-\lambda}} \right)^{\frac{1}{2} + \infty} \int_{-\infty}^{+\infty} (f_s e^{j\nu\omega t'}) dt'$$

where

- $\mathcal{F}$ is the *direct Fourier transformation* (hereinafter *Fourier transformation*)
- λ is a real constant of quantity dimension one whose value depends on the adopted definition of Fourier transformation
- ν is a non-null real constant of quantity dimension one whose value depends on the adopted definition of Fourier transformation
- t' is a bound real variable of the same quantity dimension as t

We say that the ordered pair formed by a Fourier transformand and its Fourier transform is a *direct Fourier-transform pair* (hereinafter *Fourier-transform pair*).

Let $F_{\bullet s}$ be a smooth function of rapid decay on $\mathbb{R}$ of the free real variable ω . The *inverse Fourier transform* $f_{\bullet s}$ of the *inverse Fourier transformand* $F_{\bullet s}$ is the smooth function of rapid decay on $\mathbb{R}$ of the free real variable t defined by [Lavoine, 1963; Schwartz, 1966; Lewis, 2022a; Weisstein, 2024]

$$f_{\bullet s} = \mathcal{F}_{\text{inv}}(F_{\bullet s}) \quad (A.28)$$

$$:= \left(\frac{|\nu|}{(2\pi)^{1+\lambda}} \right)^{\frac{1}{2} + \infty} \int_{-\infty}^{+\infty} (F_{\bullet s} e^{-j\nu t \omega'}) d\omega'$$

where

- $\mathcal{F}_{\text{inv}}$ is the *inverse Fourier transformation*
- ω' is a bound real variable of the same quantity dimension as ω

We say that the ordered pair formed by an inverse Fourier transform and its inverse Fourier transform is an *inverse Fourier-transform pair*.

Remarkably, $\dot{f}_s$ coincides with f_s if and only if $\dot{F}_s$ coincides with F_s [Schwartz, 1966]. In particular, the explicit representation of $\dot{f}_s$ as the inverse Fourier transform of F_s is said to be the *Fourier integral* representation of $\dot{f}_s$ [Ghizzetti and Ossicini, 1971; Folland, 1999; Lewis, 2022a].

Hereinafter, as is customary in *pure mathematics* and *systems engineering* [Lewis, 2022a; Weisstein, 2024], we adopt

$$\lambda = 1$$

$$\nu = -1$$

By letting $\dot{f}_s \equiv f_s$ and $\dot{F}_s \equiv F_s$, Equation (A.27) and Equation (A.28) become

$$\begin{aligned} F_s &\equiv \mathcal{F}(f_s) \\ &:= \int_{-\infty}^{+\infty} (f_s e^{-j\omega t'}) dt' \\ f_s &= \mathcal{F}_{\text{inv}}(F_s) \\ &:= \frac{1}{2\pi} \int_{-\infty}^{+\infty} (F_s e^{jt\omega'}) d\omega' \end{aligned} \tag{A.29}$$

The domain of the Fourier transformation is enlarged from the function space of the smooth functions of rapid decay on $\mathbb{R}$ to the function space of the $p'_{[1,2]}$ -integrable functions on $\mathbb{R}$ with $p'_{[1,2]}$ in $\mathbb{R}_{[1,2]}$ (in which case it is embodied in the *Fourier–Plancherel transformation*), to the Radon-measure space of the integrable Radon measures on $\mathbb{R}$ (in which case it is embodied in the *Fourier–Stieltjes transformation*), and to the distribution space of the tempered distributions on $\mathbb{R}$ (in which case it is embodied in the *Fourier–Schwartz transformation*) [Widder, 1941; Schwartz, 1966; Folland, 1999; Katznelson, 2004]. Further enlargements require resorting to mathematical entities that are more general than distributions [Gelfand and Shilov, 1964, 1968; Zemanian, 1965a; Beltrami and Wohlers, 1966; Carmichael and Mitrović, 1989]. Hereinafter, we refer to the Fourier

transformation (respectively, inverse Fourier transformation) without specifying its particular embodiment for a certain Fourier transformand (respectively, inverse Fourier transformand).

A.4.2.1 Generalities

The (symmetric) kernel distribution of the Fourier transformation and of the inverse Fourier transformation are respectively $e^{-j\omega t'}$ and $(1/(2\pi))e^{-jt\omega'}$ [Schwartz, 1952, 1957, 1958; Hirata and Shiraishi, 1960; Duistermaat and Kolk, 2010]. When t is interpreted as *time*, ω is interpreted as *angular frequency*.

The *spectrum* of a Fourier transformand is the support of its Fourier transform [Schwartz, 1966; Bass, 1984]; in the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], the spectrum of an element of $\mathcal{S}'(\mathbb{R})$ is its *band*. Two Fourier transformands have the same Fourier transform if and only if they coincide [Schwartz, 1966]; accordingly, a Fourier transformand vanishes on $\mathbb{R}$ if and only if its Fourier transform vanishes on $\mathbb{R}$ (namely, if and only if it has empty spectrum) [Schwartz, 1966].

As seen below, the Fourier transformation and the inverse Fourier transformation establish as many automorphisms both from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$ and from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$, an automorphism established by one transformation having as its inverse an automorphism established by the other transformation [Schwartz, 1966; Poletti, 2015]. Within $\mathcal{S}'(\mathbb{R})$, the Fourier transformation inherits many properties valid within $\mathcal{S}(\mathbb{R})$.

The following properties hold [Schwartz, 1966; Kwakernaak and Sivan, 1991]:

- The Fourier transform (respectively, inverse Fourier transform) of a real-valued tempered distribution on $\mathbb{R}$ is a conjugate-symmetric tempered distribution on $\mathbb{R}$; inversely, the Fourier transform (respectively, inverse Fourier transform) of a conjugate-symmetric tempered distribution on $\mathbb{R}$ is a real-valued tempered distribution on $\mathbb{R}$.
- The Fourier transform (respectively, inverse Fourier transform) of an imaginary-valued tempered distribution on $\mathbb{R}$ is a conjugate-antisymmetric tempered distribution on $\mathbb{R}$; inversely, the Fourier transform (respectively, inverse Fourier transform) of a conjugate-antisymmetric tempered distribution on $\mathbb{R}$ is an imaginary-valued tempered distribution on $\mathbb{R}$.
- The Fourier transform (respectively, inverse Fourier transform) of an even tempered distribution on $\mathbb{R}$ is an even tempered distribution on $\mathbb{R}$.

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- The Fourier transform (respectively, inverse Fourier transform) of an odd tempered distribution on $\mathbb{R}$ is an odd tempered distribution on $\mathbb{R}$.
- The Fourier transform (respectively, inverse Fourier transform) of an even real-valued (or, equivalently, conjugate-symmetric real-valued or, equivalently, conjugate-symmetric even) tempered distribution on $\mathbb{R}$ is an even real-valued (or, equivalently, conjugate-symmetric real-valued or, equivalently, conjugate-symmetric even) tempered distribution on $\mathbb{R}$.
- The Fourier transform (respectively, inverse Fourier transform) of an even imaginary-valued (or, equivalently, conjugate-antisymmetric imaginary-valued or, equivalently, conjugate-antisymmetric even) tempered distribution on $\mathbb{R}$ is an even imaginary-valued (or, equivalently, conjugate-antisymmetric imaginary-valued or, equivalently, conjugate-antisymmetric even) tempered distribution on $\mathbb{R}$.
- The Fourier transform (respectively, inverse Fourier transform) of an odd real-valued (or, equivalently, conjugate-antisymmetric real-valued or, equivalently, conjugate-antisymmetric odd) tempered distribution on $\mathbb{R}$ is an odd imaginary-valued (or, equivalently, conjugate-symmetric imaginary-valued or, equivalently, conjugate-symmetric odd) tempered distribution on $\mathbb{R}$; inversely, the Fourier transform (respectively, inverse Fourier transform) of an odd imaginary-valued (or, equivalently, conjugate-symmetric imaginary-valued or, equivalently, conjugate-symmetric odd) tempered distribution on $\mathbb{R}$ is an odd real-valued (or, equivalently, conjugate-antisymmetric real-valued or, equivalently, conjugate-antisymmetric odd) tempered distribution on $\mathbb{R}$.

We have [Schwartz, 1966]

$$\begin{aligned}
 \mathcal{F}_{\text{inv}} \circ \mathcal{F} &= \mathcal{F} \circ \mathcal{F}_{\text{inv}} \\
 &= \mathcal{C} \circ \mathcal{C} \\
 &= \mathcal{R} \circ \mathcal{R} \\
 &= \mathcal{P} \circ \mathcal{P} \\
 &= \mathcal{I}
 \end{aligned} \tag{A.30}$$

$$\left\{ \begin{aligned}
 \mathcal{C} \circ \mathcal{F} &= \mathcal{F} \circ \mathcal{P} \\
 &= 2\pi \mathcal{F}_{\text{inv}} \circ \mathcal{C} \\
 \mathcal{R} \circ \mathcal{F} &= \mathcal{F} \circ \mathcal{R} \\
 &= 2\pi \mathcal{F}_{\text{inv}} \\
 \mathcal{P} \circ \mathcal{F} &= \mathcal{F} \circ \mathcal{C} \\
 &= 2\pi \mathcal{F}_{\text{inv}} \circ \mathcal{P}
 \end{aligned} \right. \tag{A.31}$$

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$$\left\{ \begin{array}{l} \mathcal{C} \circ \mathcal{F}_{\text{inv}} = \mathcal{F}_{\text{inv}} \circ \mathcal{P} \\ \quad = \frac{1}{2\pi} \mathcal{F} \circ \mathcal{C} \\ \mathcal{R} \circ \mathcal{F}_{\text{inv}} = \mathcal{F}_{\text{inv}} \circ \mathcal{R} \\ \quad = \frac{1}{2\pi} \mathcal{F} \\ \mathcal{P} \circ \mathcal{F}_{\text{inv}} = \mathcal{F}_{\text{inv}} \circ \mathcal{C} \\ \quad = \frac{1}{2\pi} \mathcal{F} \circ \mathcal{P} \end{array} \right. \quad (\text{A.32})$$

The second of Equation (A.31) and the second of Equation (A.32) can be generalised. We have [Schwartz, 1966]

$$\mathcal{S}_{\xi_{\theta}} \circ \mathcal{F} = \frac{1}{|\xi_{\theta}|} \mathcal{F} \circ \mathcal{S}_{\frac{1}{\xi_{\theta}}}$$

$$\mathcal{S}_{\xi_{\theta}} \circ \mathcal{F}_{\text{inv}} = \frac{1}{|\xi_{\theta}|} \mathcal{F}_{\text{inv}} \circ \mathcal{S}_{\frac{1}{\xi_{\theta}}}$$

From Equation (A.30), Equation (A.31), and Equation (A.32), it can be easily verified that we have

$$\left\{ \begin{array}{l} \mathcal{F}^0 \equiv \mathcal{I} \\ \mathcal{F}^1 \equiv \mathcal{F} \\ \mathcal{F}^2 \equiv \mathcal{F} \circ \mathcal{F} \\ \quad = 2\pi \mathcal{R} \\ \mathcal{F}^3 \equiv \mathcal{F} \circ \mathcal{F} \circ \mathcal{F} \\ \quad = (2\pi)^2 \mathcal{F}_{\text{inv}} \\ \mathcal{F}^4 \equiv \mathcal{F} \circ \mathcal{F} \circ \mathcal{F} \circ \mathcal{F} \\ \quad = (2\pi)^2 \mathcal{I} \end{array} \right. \quad (\text{A.33})$$

$$\left\{ \begin{array}{l} \mathcal{F}_{\text{inv}}^0 \equiv \mathcal{I} \\ \mathcal{F}_{\text{inv}}^{-1} \equiv \mathcal{F}_{\text{inv}} \\ \mathcal{F}_{\text{inv}}^{-2} \equiv \mathcal{F}_{\text{inv}} \circ \mathcal{F}_{\text{inv}} \\ \quad = \frac{1}{2\pi} \mathcal{R} \\ \mathcal{F}_{\text{inv}}^{-3} \equiv \mathcal{F}_{\text{inv}} \circ \mathcal{F}_{\text{inv}} \circ \mathcal{F} \\ \quad = \left(\frac{1}{2\pi} \right)^2 \mathcal{F} \\ \mathcal{F}_{\text{inv}}^{-4} \equiv \mathcal{F}_{\text{inv}} \circ \mathcal{F}_{\text{inv}} \circ \mathcal{F}_{\text{inv}} \circ \mathcal{F}_{\text{inv}} \\ \quad = \left(\frac{1}{2\pi} \right)^2 \mathcal{I} \end{array} \right. \quad (\text{A.34})$$

where

- $\mathcal{F}^{n_0}$ is the n_0^{th} iterate of the Fourier transformation
- $\mathcal{F}^{-n_0}$ is the n_0^{th} iterate of the inverse Fourier transformation

Within both $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, the Fourier transformation and the inverse Fourier transformation prove to be 4-periodic up to multiplication by 2π to the power of an integer constant exponent. Specifically, for any z_1 in $\mathbb{Z}$ and z_2 in $\mathbb{Z}$, we have [McBride and Kerr, 1987; Kerr, 1988a, 1988b; Benedetto, 1997; Ozaktas *et al.*, 1999]

$$\mathcal{F}^{z_1+4z_2} = (2\pi)^{2z_2} \mathcal{F}^{z_1} \quad (\text{A.35})$$

As an important implication of Equation (A.33), Equation (A.34), and Equation (A.35), if the Fourier transformation (respectively, inverse Fourier transformation) establishes an isomorphism from a first subspace of $\mathcal{S}(\mathbb{R})$ or $\mathcal{S}'(\mathbb{R})$ that is invariant under reflection into a second subspace of respectively $\mathcal{S}(\mathbb{R})$ or $\mathcal{S}'(\mathbb{R})$, then the second subspace is invariant under reflection, the inverse Fourier transformation (respectively, Fourier transformation) establishes an isomorphism from the first subspace into the second subspace, and the Fourier transformation and the inverse Fourier transformation establish as many isomorphisms from the second subspace into the first subspace [Gelfand and Shilov, 1968].

Finally, we mention that $\mathcal{F}^z$ can be generalised to give meaning to the iterate of any order in $\mathbb{R}$ of the Fourier transformation, which leads to defining the *fractional Fourier transformation* [McBride, 1979; McBride and Kerr, 1987; Kerr, 1988a, 1988b; Ozaktas *et al.*, 1999; King, 2009b].

A.4.2.2 Functions

Let $p_{[2,+\infty]}$ in $\overline{\mathbb{R}}_{[2,+\infty]}$ and $p'_{[1,2]}$ in $\mathbb{R}_{[1,2]}$ form a pair of Hölder-conjugate exponents. The Fourier transform (respectively, inverse Fourier transform) of a $p'_{[1,2]}$ -integrable function on $\mathbb{R}$ with $p'_{[1,2]}$ in $\mathbb{R}_{[1,2]}$ is a $p_{[2,+\infty]}$ -integrable function on $\mathbb{R}$ with $p_{[2,+\infty]}$ in $\overline{\mathbb{R}}_{[2,+\infty]}$ [Schwartz, 1966; Katznelson, 2004]. If a suitable *summability method* is adopted, the Fourier integral representation of a $p'_{[1,2]}$ -integrable function on $\mathbb{R}$ has both *almost-everywhere convergence* on $\mathbb{R}$ and *convergence in $L^{p'_{[1,2]}}$ norm* (also known as *mean-square convergence* in the special case in which both $p'_{[1,2]}$ and $p_{[2,+\infty]}$ are 2) on $\mathbb{R}$ to the $p'_{[1,2]}$ -integrable function on $\mathbb{R}$ [Folland, 1992; Katznelson, 2004]. In particular, the Fourier

transform (respectively, inverse Fourier transform) of a square integrable function on $\mathbb{R}$ is a square integrable function on $\mathbb{R}$. In fact, the Fourier transformation and the inverse Fourier transformation establish as many automorphisms from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$, the automorphism established by one transformation having as its inverse the automorphism established by the other transformation [Schwartz, 1966; Poletti, 2015]. With a suitable choice of λ and ν in Equation (A.27) and Equation (A.28) (for example, $\lambda = 0$ and $\nu = -2\pi$ as is customary in *signal processing*, or $\lambda = 0$ and $\nu = 1$ as is customary in *modern physics* [Lewis, 2022a; Weisstein, 2024]), the automorphisms established by the Fourier transformation and by the inverse Fourier transformation from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ can be made unitary maps from a Hilbert space over $\mathbb{F}$ into itself [Schwartz, 1966; Kwakernaak and Sivan, 1991]. Furthermore, the Fourier transform (respectively, inverse Fourier transform) of the convolutive product of two square integrable functions on $\mathbb{R}$ is an integrable function on $\mathbb{R}$ [Poletti, 2015]; inversely, the Fourier transform (respectively, inverse Fourier transform) of an integrable function on $\mathbb{R}$ is the convolutive product of two square integrable functions on $\mathbb{R}$ [Poletti, 2015]. In fact, denoting by $L^2(\mathbb{R}) * L^2(\mathbb{R})$ the function space of the convolutive products of two square integrable functions on $\mathbb{R}$, which is a subspace of $L^\infty(\mathbb{R})$ (in fact, of $\dot{\mathcal{B}}^0(\mathbb{R})$) coinciding with $\mathcal{A}(\mathbb{R})$ as introduced below, the Fourier transformation and the inverse Fourier transformation establish as many isomorphisms both from $L^2(\mathbb{R}) * L^2(\mathbb{R})$ into $L^1(\mathbb{R})$ and from $L^1(\mathbb{R})$ into $L^2(\mathbb{R}) * L^2(\mathbb{R})$, an isomorphism established by one transformation having as its inverse an isomorphism established by the other transformation [Benedetto, 1975; Poletti, 2015]. The Fourier transform (respectively, inverse Fourier transform) of an integrable function on $\mathbb{R}$ is a C^0 function vanishing at infinity on $\mathbb{R}$ [Schwartz, 1966; Katznelson, 2004]; in particular, the Fourier transform (respectively, inverse Fourier transform) of a smooth integrable function on $\mathbb{R}$ is a C^0 function of rapid decay on $\mathbb{R}$ [Schwartz, 1966; Barros-Neto, 1973]. The Fourier integral representation of a representative of an integrable function on $\mathbb{R}$ (calculated by means of Equation (A.29) via the Cauchy-principal-value method) has *pointwise convergence* on $\mathbb{R}$ to the *precise representative* of the integrable function on $\mathbb{R}$ if this is a function of locally bounded variation on $\mathbb{R}$ (the precise representative being the representative whose value at any point of $\mathbb{R}$ is half the sum of the one-sided limits thereat [Ambrosio *et al.*, 2000; Evans and Gariepy, 2015]) [Ghizzetti and Ossicini, 1971; Folland, 1992; Lewis, 2022a]; furthermore, the Fourier integral representation of a representative of an integrable function on $\mathbb{R}$ whose Fourier transform is an integrable function on $\mathbb{R}$ (namely, of a representative of an element of $L^1(\mathbb{R}) \cap \mathcal{A}(\mathbb{R})$ or, equivalently, of $L^1(\mathbb{R}) \cap \mathcal{B}(\mathbb{R})$ as introduced below, which is a subspace of $L^1(\mathbb{R})$ that is invariant

under Fourier transformation) has pointwise convergence on $\mathbb{R}$ to the continuous representative of the integrable function on $\mathbb{R}$ (both the integrable function on $\mathbb{R}$ and its Fourier transform being necessarily elements of $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap \mathcal{E}^0(\mathbb{R})$ and hence of $L^p(\mathbb{R}) \cap \mathcal{E}^0(\mathbb{R})$) [Ghizzetti and Ossicini, 1971; Folland, 1992, 1999; Benedetto, 1997; Lewis, 2022a]. It is worth mentioning that, despite having pointwise convergence on $\mathbb{R}$, the first of the aforementioned Fourier integral representations need not have *uniform convergence* on $\mathbb{R}$, the lack of which results in the undesired *Gibbs phenomenon* (namely, any ‘partial sum’ calculated by means of Equation (A.29) as an integral on an interval of $\mathbb{R}$ symmetric with respect to the origin presents intrinsic *overshoot* and *undershoot* on any neighbourhood of any point of jump discontinuity of the Fourier transformand) [Benedetto, 1997].

The Fourier transform (respectively, inverse Fourier transform) of a smooth function of rapid decay on $\mathbb{R}$ is a smooth function of rapid decay on $\mathbb{R}$ [Schwartz, 1966; Poletti, 2015]. In fact, the Fourier transformation and the inverse Fourier transformation establish as many automorphisms from $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$, the automorphism established by one transformation having as its inverse the automorphism established by the other transformation [Schwartz, 1966; Poletti, 2015]. The Fourier integral representation of a smooth function of rapid decay on $\mathbb{R}$ has pointwise convergence on $\mathbb{R}$ to the smooth function of rapid decay on $\mathbb{R}$ itself [Folland, 1992, 1999].

The Fourier transform (respectively, inverse Fourier transform) of a smooth function of compact support on $\mathbb{R}$ is a smooth function of rapid decay of exponential type on $\mathbb{R}$ [Gelfand and Shilov, 1964, 1968; Zemanian, 1965a; Schwartz, 1966; Treves, 1967; Barros-Neto, 1973; Misra and Lavoine, 1986; Hoskins and Sousa Pinto, 2005]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a smooth function of rapid decay of exponential type on $\mathbb{R}$ is a smooth function of compact support on $\mathbb{R}$ [Schwartz, 1966; Beltrami and Wohlers, 1966; Misra and Lavoine, 1986; Carmichael and Mitrović, 1989; Hoskins and Sousa Pinto, 2005]. Accordingly, the Fourier transform (respectively, inverse Fourier transform) of a smooth function of compact support on $\mathbb{R}$ can be extended to a holomorphic function of exponential type on $\mathbb{C}$; by choosing suitably the free complex variable, such holomorphic function is the Laplace transform (respectively, the Fourier–Laplace transform), as introduced below, of the smooth function of compact support on $\mathbb{R}$ [Zemanian, 1965a; Beltrami and Wohlers, 1966; Misra and Lavoine, 1986; Carmichael and Mitrović, 1989]. In fact, as an implication of the *Paley–Wiener–Schwartz theorem*, the Fourier transformation and the inverse Fourier transformation establish as many isomorphisms both from $\mathcal{D}(\mathbb{R})$ into $\exp\mathcal{S}(\mathbb{R})$ and from $\exp\mathcal{S}(\mathbb{R})$ into $\mathcal{D}(\mathbb{R})$, an isomorphism established by one transformation having as its inverse an isomorphism established by the other transformation [Gelfand and Shilov,

1964, 1968; Schwartz, 1966; Zemanian, 1965a; Beltrami and Wohlers, 1966; Misra and Lavoine, 1986; Carmichael and Mitrović, 1989; Hoskins and Sousa Pinto, 2005].

Let $f_{\exp \mathcal{S}}$ be a smooth function of rapid decay of exponential type on $\mathbb{R}$, and let $\mathbb{R}_{[\omega_1, \omega_2]}$ with ω_1 in $\mathbb{R}$, ω_2 in $\mathbb{R}$, and ω_2 greater than ω_1 be the smallest closed interval of $\mathbb{R}$ containing the spectrum of $f_{\exp \mathcal{S}}$. Let $\mathbb{R}_{(-\Omega/2, +\Omega/2)}$ with Ω in $\mathbb{R}_{(\Omega_N, +\infty)}$ and Ω_N the maximum of $2|\omega_1|$ and $2|\omega_2|$ be an open interval of $\mathbb{R}$ symmetric with respect to the origin and containing the spectrum of $f_{\exp \mathcal{S}}$; in particular, the spectrum of $f_{\exp \mathcal{S}}$ is itself symmetric with respect to the origin if $f_{\exp \mathcal{S}}$ is either real-valued or imaginary-valued. For any Ω_S in $\mathbb{R}_{(\Omega_N, +\infty)}$ (or, equivalently, for any Ω_S in $\mathbb{R}_{[\Omega, +\infty)}$), the *sampling theorem* (also known as *Whittaker–Shannon–Kotelnikov sampling theorem* or *WSK sampling theorem*) holds [Zayed, 1993; Duistermaat and Kolk, 2010]

$$f_{\exp \mathcal{S}} = T_S \lim_{R \rightarrow +\infty} \left(\sum_{r=-R}^{+R} \left((f_{\exp \mathcal{S}}(t_r))(f_S)_{t_r} \right) \right) \quad (\text{A.36})$$

with

$$T_S = \frac{2\pi}{\Omega_S}$$

$$t_r = rT_S, \quad r \in \mathbb{Z}$$

$$f_S(t) = \frac{1}{T_S} \operatorname{sinc}\left(\frac{\Omega_S}{2}t\right) = \begin{cases} \frac{\sin\left(\frac{\Omega_S}{2}t\right)}{\pi t}, & t \in \mathbb{R}_{(-\infty, 0) \cup (0, +\infty)} \\ \frac{1}{T_S}, & t = 0 \end{cases}$$

where

- R is a bound index taking values in $\mathbb{N}$
- r is a bound index taking values in $\mathbb{Z}_{[-R, +R]}$
- Ω_N is the *Nyquist rate*
- Ω_S is the *sampling rate*
- T_S is the *sampling interval*

- t_r with r in $\mathbb{Z}$ is the r^{th} sampling point
- $\{t_r, r \in \mathbb{Z}\}$ is the sampling sequence
- $\{f_{\exp S}(t_r), r \in \mathbb{Z}\}$ is the sampled sequence
- f_s is the sampling function
- $(f_s)_{t_r}$ with r in $\mathbb{Z}$ is the t_r -translated sampling function

The series in Equation (A.36) has pointwise convergence on $\mathbb{R}$ [Zayed, 1993; Duistermaat and Kolk, 2010]. Remarkably, the series in Equation (A.36) has pointwise convergence on $\mathbb{R}$ (as well as convergence in L^p norm on $\mathbb{R}$) even in the case, more general, in which $f_{\exp S}$ is replaced by a p -integrable function of compact spectrum on $\mathbb{R}$ and, when p in $\bar{\mathbb{R}}_{[1,+\infty)}$, even for any Ω_s in $\mathbb{R}_{[\Omega_N,+\infty)}$ [Butzer *et al.*, 1988; Zayed, 1993; Benedetto, 1997; García *et al.*, 1998; Hoskins and Sousa Pinto, 2005]; as seen below, when p is in $\bar{\mathbb{R}}_{(2,+\infty]}$, the spectrum of such p -integrable function on $\mathbb{R}$ need be regarded as the spectrum of the p -integrable distribution on $\mathbb{R}$ generated by it. If Ω_s is not in $\mathbb{R}_{[\Omega_N,+\infty)}$ or if $f_{\exp S}$ is replaced by a function on $\mathbb{R}$ that is not a function of compact spectrum on $\mathbb{R}$ (in particular, if $f_{\exp S}$ is replaced by a function of compact support on $\mathbb{R}$), the application of Equation (A.36) results in the undesired phenomenon of *aliasing* [Zayed, 1993; Benedetto, 1997].

A.4.2.3 Radon measures

The Fourier transform (respectively, inverse Fourier transform) of an integrable Radon measure on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ that is generable by a C^0 (in fact, uniformly continuous) bounded function on $\mathbb{R}$ [Schwartz, 1966; Katznelson, 2004; Moody and Strungaru, 2017]. In particular, the Fourier transform (respectively, inverse Fourier transform) of an integrable Radon measure on $\mathbb{R}$ that is generable by an integrable function on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ that is generable by a C^0 function vanishing at infinity on $\mathbb{R}$ [Schwartz, 1966; Katznelson, 2004]. An integrable Radon measure on $\mathbb{R}$ is non-negative-definite if and only if its Fourier transform (respectively, inverse Fourier transform) is a Radon measure on $\mathbb{R}$ that is generable by a non-negative real-valued C^0 (in fact, uniformly continuous) bounded function on $\mathbb{R}$ [Schwartz, 1966; Argabright and Gil de Lamadrid, 1974; Berg and Forst, 1975; Katznelson, 2004].

According to the *Bochner theorem*, the Fourier transform (respectively, inverse Fourier transform) of a non-negative real-valued integrable Radon measure on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ that is generable by a non-negative-definite C^0 function on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Schwartz,

1966; Berg and Forst, 1975; Bass, 1984; Benedetto, 1997; Vladimirov, 2002; Katznelson, 2004; Benedetto and Czaja, 2009; Schilling *et al.*, 2012; Moody and Strungaru, 2017]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a Radon measure on $\mathbb{R}$ that is generable by a non-negative-definite C^0 function on $\mathbb{R}$ is a non-negative real-valued integrable Radon measure on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Schwartz, 1966; Berg and Forst, 1975; Bass, 1984; Benedetto, 1997; Vladimirov, 2002; Katznelson, 2004; Benedetto and Czaja, 2009; Schilling *et al.*, 2012; Moody and Strungaru, 2017]. In particular, the Fourier transform (respectively, inverse Fourier transform) of a non-negative real-valued integrable Radon measure on $\mathbb{R}$ that is generable by a non-negative real-valued integrable function on $\mathbb{R}$ is a Radon measure on $\mathbb{R}$ that is generable by a non-negative-definite C^0 function vanishing at infinity on $\mathbb{R}$ [Schwarz, 1966; Berg and Forst, 1975; Figà-Talamanca, 1977]; in contrast, the Fourier transform (respectively, inverse Fourier transform) of a Radon measure on $\mathbb{R}$ that is generable by a non-negative-definite C^0 function vanishing at infinity on $\mathbb{R}$ need not be a non-negative real-valued integrable Radon measure on $\mathbb{R}$ that is generable by a non-negative real-valued integrable function on $\mathbb{R}$ [Schwarz, 1966; Berg and Forst, 1975; Figà-Talamanca, 1977].

The Fourier transform (respectively, inverse Fourier transform) of a non-negative-definite non-negative real-valued Radon measure on $\mathbb{R}$ is a non-negative-definite non-negative real-valued Radon measure on $\mathbb{R}$ [Berg and Forst, 1975; Moody and Strungaru, 2017].

A.4.2.4 Distributions

The Fourier transform (respectively, inverse Fourier transform) of a tempered distribution on $\mathbb{R}$ is a tempered distribution on $\mathbb{R}$ [Schwarz, 1966; Poletti, 2015]. In fact, the Fourier transformation and the inverse Fourier transformation establish as many automorphisms from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}'(\mathbb{R})$, the automorphism established by one transformation having as its inverse the automorphism established by the other transformation [Schwarz, 1966; Poletti, 2015]. We say that a tempered distribution on $\mathbb{R}$ is a *Fourier-transformable distribution* on $\mathbb{R}$. In particular, the Fourier transform (respectively, inverse Fourier transform) of a tempered distribution on $\mathbb{R}$ that is homogeneous of degree c_{hom} in $\mathbb{C}$ is a tempered distribution on $\mathbb{R}$ that is homogeneous of degree $-(c_{\text{hom}} + 1)$ in $\mathbb{C}$ [Hörmander, 1990]. Furthermore, the Fourier transform (respectively, inverse Fourier transform) of a tempered distribution of order 0 (in particular, of an integrable distribution of $\mathcal{S}$ -order 0) on $\mathbb{R}$ is a uniformly almost periodic distribution on $\mathbb{R}$ (in particular, of a distribution that is generable by a uniformly almost periodic function on $\mathbb{R}$) if and only if the tempered distribution of order 0 (in particular, the integrable distribution of $\mathcal{S}$ -order 0) on $\mathbb{R}$ is generable by an atomic Radon

measure on $\mathbb{R}$ [Katznelson, 2004; Strungaru and Terauds, 2016; Moody and Strungaru, 2017; Strungaru, 2020].

We denote by $\mathcal{R}'_{\text{RR}}(\mathbb{R})$ (respectively, $\mathcal{R}'_{\text{LR}}(\mathbb{R})$) the distribution space of the Fourier transforms of right-sided (respectively, left-sided) tempered distributions on $\mathbb{R}$. We refer to the elements of $\mathcal{R}'_{\text{RR}}(\mathbb{R})$, $\mathcal{R}'_{\text{LR}}(\mathbb{R})$, and $\mathcal{R}'_{\text{RR}}(\mathbb{R}) \cup \mathcal{R}'_{\text{LR}}(\mathbb{R})$ as respectively *right-retarded distributions* on $\mathbb{R}$, *left-retarded distributions* on $\mathbb{R}$, and *retarded distributions* on $\mathbb{R}$. The Fourier transform (respectively, inverse Fourier transform) of a one-sided tempered distribution on $\mathbb{R}$ is a retarded distribution on $\mathbb{R}$ [Shambayati and Zieleźny, 1983; Oberguggenberger, 1992; Carlsson and Wittsten, 2017]; inversely, a retarded distribution on $\mathbb{R}$ is the Fourier transform (respectively, inverse Fourier transform) of a one-sided tempered distribution on $\mathbb{R}$ [Shambayati and Zieleźny, 1983; Oberguggenberger, 1992; Carlsson and Wittsten, 2017]. Specifically, we have

$$\begin{aligned} \mathcal{S}'_{\text{R}}(\mathbb{R}) &\xleftrightarrow[\mathcal{F}_{\text{inv}}]{\mathcal{F}} \mathcal{R}'_{\text{RR}}(\mathbb{R}) \xleftrightarrow[\frac{1}{2\pi}\mathcal{F}_{\text{inv}}]{\frac{1}{2\pi}\mathcal{F}} \mathcal{S}'_{\text{L}}(\mathbb{R}) \xleftrightarrow[\mathcal{F}_{\text{inv}}]{\mathcal{F}} \mathcal{R}'_{\text{LR}}(\mathbb{R}) \xleftrightarrow[\frac{1}{2\pi}\mathcal{F}_{\text{inv}}]{\frac{1}{2\pi}\mathcal{F}} \mathcal{S}'_{\text{R}}(\mathbb{R}) \\ \mathcal{S}'_{\text{R}}(\mathbb{R}) &\xleftrightarrow[\mathcal{R}]{\mathcal{R}} \mathcal{S}'_{\text{L}}(\mathbb{R}) \\ \mathcal{R}'_{\text{RR}}(\mathbb{R}) &\xleftrightarrow[\mathcal{R}]{\mathcal{R}} \mathcal{R}'_{\text{LR}}(\mathbb{R}) \end{aligned}$$

A distribution is a right-retarded distribution on $\mathbb{R}$ if and only if it is a tempered distribution on $\mathbb{R}$ that can be extended to a holomorphic function on $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (respectively, $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$) whose modulus satisfies a certain majorisation condition [Shambayati and Zieleźny, 1983; Carlsson and Wittsten, 2017]; by choosing suitably the free complex variable, such holomorphic function is the Laplace transform (respectively, the Fourier–Laplace transform), as introduced below, of the right-sided distribution on $\mathbb{R}$ of which the right-retarded distribution on $\mathbb{R}$ is the Fourier transform [Zemanian, 1965a, 1968b]. Besides, a distribution is a left-retarded distribution on $\mathbb{R}$ if and only if it is a tempered distribution on $\mathbb{R}$ that can be extended to a holomorphic function on $\mathbb{C}_{\text{Re} \in (-\infty, 0)}$ (respectively, $\mathbb{C}_{\text{Im} \in (0, +\infty)}$) whose modulus satisfies a certain majorisation condition [Shambayati and Zieleźny, 1983; Carlsson and Wittsten, 2017]; by choosing suitably the free complex variable, such holomorphic function is the Laplace transform (respectively, the Fourier–Laplace transform), as introduced below, of the left-sided distribution on $\mathbb{R}$ of which the left-retarded distribution on $\mathbb{R}$ is the Fourier transform [Zemanian, 1965a, 1968b].

The Fourier transform (respectively, inverse Fourier transform) of a $(-n)$ -weighted integrable distribution on $\mathbb{R}$ is the $(n-1)^{\text{th}}$ derivative of a tempered distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ [Schwartz, 1962; Nussenzveig, 1972]; equivalently, the Fourier transform (respectively, inverse Fourier transform) of a $(-n)$ -weighted integrable distribution on $\mathbb{R}$ is a tempered distribution of C -order in $\mathbb{Z}_{[-\infty, n-2]}$. In particular, the Fourier transform (respectively, inverse Fourier transform) of a $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$ is a bounded distribution on $\mathbb{R}$ [Schwartz, 1966]. Furthermore, the Fourier transform (respectively, inverse Fourier transform) of a regular $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$ is a distribution vanishing at infinity on $\mathbb{R}$ [Schwartz, 1966]. It is worth noting that the countable union for all n in $\mathbb{N}$ of the distribution spaces of the regular $(-n)$ -weighted integrable distributions of S -order 0 on $\mathbb{R}$ coincides with the distribution space of the distributions generated by the elements of $\mathcal{P}_L(\mathbb{R})$ [Schwartz, 1966; Poletti, 2015].

According to the *Bochner–Schwartz theorem*, the Fourier transform (respectively, inverse Fourier transform) of a non-negative real-valued $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$ is a non-negative-definite distribution on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Zemanian, 1965b; Schwartz, 1966; Bertrandias *et al.*, 1987; Vladimirov, 2002]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a non-negative-definite distribution on $\mathbb{R}$ is a non-negative real-valued $(-n)$ -weighted integrable distribution of S -order 0 on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Zemanian, 1965b; Schwartz, 1966; Bertrandias *et al.*, 1987; Vladimirov, 2002]. Furthermore, the Fourier transform (respectively, inverse Fourier transform) of a weakly non-negative tempered distribution on $\mathbb{R}$ (namely, of a tempered distribution on $\mathbb{R}$ whose real part is non-negative) is a weakly non-negative-definite tempered distribution on $\mathbb{R}$ (namely, a tempered distribution on $\mathbb{R}$ whose conjugate-symmetric part is non-negative-definite) [Staffans, 1976; Gripenberg *et al.*, 1990]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a weakly non-negative-definite tempered distribution on $\mathbb{R}$ (namely, of a tempered distribution on $\mathbb{R}$ whose conjugate-symmetric part is non-negative-definite) is a weakly non-negative tempered distribution on $\mathbb{R}$ (namely, a tempered distribution on $\mathbb{R}$ whose real part is non-negative) [Staffans, 1976; Gripenberg *et al.*, 1990].

We denote by $B^\wedge(\mathbb{R}) \equiv \mathcal{F}(\mathcal{D}_L^{\prime 0}(\mathbb{R}))$ the distribution space of the Fourier transforms of the elements of $\mathcal{D}_L^{\prime 0}(\mathbb{R})$ and by $\mathcal{F}(B^\wedge(\mathbb{R}))$ the distribution space of the Fourier transforms of the

elements of $B^\wedge(\mathbb{R})$; besides, we denote by $B(\mathbb{R})$ the function space of the functions generating the elements of $B^\wedge(\mathbb{R})$. Furthermore, we denote by $\dot{B}^{n_0\wedge}(\mathbb{R})$ the distribution space of the distributions generated by the elements of $\dot{B}^{n_0}(\mathbb{R})$ and by $\mathcal{F}(\dot{B}^{n_0\wedge}(\mathbb{R}))$ the distribution space of the Fourier transforms of the elements of $\dot{B}^{n_0\wedge}(\mathbb{R})$; in particular, we denote by $PF(\mathbb{R}) \equiv \mathcal{F}(\dot{B}^{0\wedge}(\mathbb{R}))$ the distribution space of the Fourier transforms of the elements of $\dot{B}^{0\wedge}(\mathbb{R})$ and by $\mathcal{F}(PF(\mathbb{R}))$ the distribution space of the Fourier transforms of the elements of $PF(\mathbb{R})$. Somehow unfortunately, an element of $PF(\mathbb{R})$ is said to be a *pseudofunction* on $\mathbb{R}$ [Benedetto, 1975; Graham and McGehee, 1979; Katznelson, 2004]; that said, to avoid confusion with a different concept, we do not use the term *pseudofunction* in this sense hereinafter. From Equation (A.33), we have

$$\mathcal{F}(B^\wedge(\mathbb{R})) \equiv \mathcal{D}_L^{\prime 0}(\mathbb{R})$$

$$\mathcal{F}(PF(\mathbb{R})) \equiv \dot{B}^{0\wedge}(\mathbb{R})$$

Denoting by $(PF(\mathbb{R}))'$ the topological dual space of $PF(\mathbb{R})$, we have [Benedetto, 1975; Graham and McGehee, 1979]

$$(PF(\mathbb{R}))' \equiv B(\mathbb{R})$$

We denote by $A(\mathbb{R}) \equiv \mathcal{F}(L^1(\mathbb{R}))$ the function space of the Fourier transforms of the elements of $L^1(\mathbb{R})$, which coincides with $L^2(\mathbb{R}) * L^2(\mathbb{R})$, and by $\mathcal{F}(A(\mathbb{R}))$ the function space of the Fourier transforms of the elements of $A(\mathbb{R})$. Furthermore, we denote by $L^{p'\wedge}(\mathbb{R})$ the distribution space of the distributions generated by the elements of $L^{p'}(\mathbb{R})$ and by $\mathcal{F}(L^{p'\wedge}(\mathbb{R}))$ the distribution space of the Fourier transforms of the elements of $L^{p'\wedge}(\mathbb{R})$; in particular, we denote by $PM(\mathbb{R}) \equiv \mathcal{F}(L^{\infty\wedge}(\mathbb{R}))$ the distribution space of the Fourier transforms of the elements of $L^{\infty\wedge}(\mathbb{R})$ and by $\mathcal{F}(PM(\mathbb{R}))$ the distribution space of the Fourier transforms of the elements of $PM(\mathbb{R})$. An element of $PM(\mathbb{R})$ is said to be a *pseudomeasure* on $\mathbb{R}$ [Brainerd and Edwards, 1966; Larsen, 1971; Benedetto, 1975, 1997; Dieudonné, 1978; Graham and McGehee, 1979; Katznelson, 2004]. From Equation (A.33), we have

A.4 Signals: Integral transformations

$$\mathcal{F}(A(\mathbb{R})) \equiv L^1(\mathbb{R})$$

$$\mathcal{F}(PM(\mathbb{R})) \equiv L^{\infty\wedge}(\mathbb{R})$$

Denoting by $(A(\mathbb{R}))'$ and $(\mathcal{F}(L^{p'_{[1,+\infty)}^\wedge}(\mathbb{R})))'$ the topological dual spaces of respectively $A(\mathbb{R})$ and $\mathcal{F}(L^{p'_{[1,+\infty)}^\wedge}(\mathbb{R}))$, we have [Brainerd and Edwards, 1966; Larsen, 1971; Benedetto, 1975, 1997; Katznelson, 2004]

$$(A(\mathbb{R}))' \equiv PM(\mathbb{R})$$

$$(\mathcal{F}(L^{p'_{[1,+\infty)}^\wedge}(\mathbb{R})))' \equiv \mathcal{F}(L^{p_{[1,+\infty)}^\wedge}(\mathbb{R}))$$

$$(L^1(\mathbb{R}) \cap B(\mathbb{R})) \equiv (L^1(\mathbb{R}) \cap A(\mathbb{R}))$$

The chains of inclusion relations hold [Benedetto, 1975, 1997; Figà-Talamanca, 1977; Dieudonné, 1978; Graham and McGehee, 1979; Katznelson, 2004]

$$\begin{aligned} (L^\infty(\mathbb{R}) \cap UC(\mathbb{R})) &\supset B(\mathbb{R}) \\ \cup &\cup \\ \dot{B}^0(\mathbb{R}) &\supset (B(\mathbb{R}) \cap \dot{B}^0(\mathbb{R})) \supset A(\mathbb{R}) \supset \mathcal{D}^2(\mathbb{R}) \\ \\ (S'(\mathbb{R}) \cap \mathcal{D}'^1(\mathbb{R})) &\supset PM(\mathbb{R}) \supset \mathcal{D}'^0_L(\mathbb{R}) \\ \cup &\cup \\ PF(\mathbb{R}) &\supset (\mathcal{D}'^0_L(\mathbb{R}) \cap PF(\mathbb{R})) \supset L^{1\wedge}(\mathbb{R}) \end{aligned}$$

Interestingly, $PM(\mathbb{R})$ contains some elements of $\mathcal{E}'^1(\mathbb{R}) \setminus \mathcal{E}'^0(\mathbb{R})$ [Brainerd and Edwards, 1966; Katznelson, 2004]. To the best of our knowledge, the problem of satisfactorily characterising $B(\mathbb{R})$ within $L^\infty(\mathbb{R})$ and $A(\mathbb{R})$ within $\dot{B}^0(\mathbb{R})$ is still unsolved [Benedetto, 1975, 1997; Graham and McGehee, 1979; Katznelson, 2004; Benedetto and Czaja, 2009]; in fact, this difficulty is one of the reasons motivating the introduction of $PM(\mathbb{R})$ as the superspace of $\mathcal{D}'^0_L(\mathbb{R})$ whose elements have as their Fourier transforms precisely the elements of the superspace $L^\infty(\mathbb{R})$ of $B(\mathbb{R})$ as well as the introduction of $PF(\mathbb{R})$ as the superspace of $L^{1\wedge}(\mathbb{R})$ whose elements have as their Fourier transforms precisely the elements of the superspace $\dot{B}^0(\mathbb{R})$ of $A(\mathbb{R})$. Examples of elements of

$L^\infty(\mathbb{R}) \setminus (B(\mathbb{R}) \cup \dot{B}^0(\mathbb{R}))$ are sgn and the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$, whereas an example of element of $(L^\infty(\mathbb{R}) \cap UC(\mathbb{R})) \setminus (B(\mathbb{R}) \cup \dot{B}^0(\mathbb{R}))$ is the function defined almost everywhere on $\mathbb{R}$ by $e^{j|\xi|}$. An example of element of $B(\mathbb{R}) \setminus A(\mathbb{R})$ is $1_{\mathbb{R}}$. Examples of elements of $\dot{B}^0(\mathbb{R}) \setminus A(\mathbb{R})$ are the functions defined almost everywhere on $\mathbb{R}$ by $\xi \left((1 + |\xi|) (\ln(e + |\xi|)) \right)^{-1}$ and $\xi^{1/2} e^{-(\xi + j/\xi)} \mathfrak{g}(\xi)$ [Lewis, 2022a; Ortner and Wagner, 2023]; other examples can be constructed [Chandrasekharan, 1989]. An example of element of $(B(\mathbb{R}) \cap \dot{B}^0(\mathbb{R})) \setminus A(\mathbb{R})$ can be constructed [Figà-Talamanca, 1977]. Interestingly, $A(\mathbb{R})$ (and, *a fortiori*, $B(\mathbb{R})$) is not a superspace of $\mathcal{D}^0(\mathbb{R})$ [Herrero, 1985]. An example of element of $A(\mathbb{R}) \setminus \mathcal{D}^2(\mathbb{R})$ is sinc . An example of element of $(\mathcal{S}'(\mathbb{R}) \cap \mathcal{D}'^1(\mathbb{R})) \setminus PM(\mathbb{R})$ is $\delta^{(1)}$. Examples of elements of $PM(\mathbb{R}) \setminus (\mathcal{D}'^0(\mathbb{R}) \cup PF(\mathbb{R}))$ (each element of $\mathcal{D}'^0(\mathbb{R}) \cap PF(\mathbb{R})$ being generable by an element of $L^1_{\text{loc}}(\mathbb{R})$ [Benedetto, 1975]) are $\text{p.v.}(1/\xi)$ and the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$. An example of element of $\mathcal{D}'^0(\mathbb{R}) \setminus L^1(\mathbb{R})$ is δ . An example of element of $PF(\mathbb{R}) \setminus L^1(\mathbb{R})$ is the inverse Fourier transform of any element of $\dot{B}^0(\mathbb{R}) \setminus A(\mathbb{R})$ [Chandrasekharan, 1989; Lewis, 2022a]. An example of element of $(\mathcal{D}'^0(\mathbb{R}) \cap PF(\mathbb{R})) \setminus L^1(\mathbb{R})$ is the inverse Fourier transform of any element of $(B(\mathbb{R}) \cap \dot{B}^0(\mathbb{R})) \setminus A(\mathbb{R})$ [Figà-Talamanca, 1977].

The Fourier transform (respectively, inverse Fourier transform) of a distribution on $\mathbb{R}$ generated by a $p'_{[2,+\infty]}$ -integrable function on $\mathbb{R}$ with $p'_{[2,+\infty]}$ in $\overline{\mathbb{R}}_{[2,+\infty]}$ is the sum of a distribution on $\mathbb{R}$ generated by a square integrable function on $\mathbb{R}$ and a pseudomeasure on $\mathbb{R}$, and it is a *quasimeasure* on $\mathbb{R}$ (namely, it belongs to a certain superspace $QM(\mathbb{R})$ of $PM(\mathbb{R}) \cap \mathcal{D}'^0(\mathbb{R})$ whose each element can be expressed, the expression proving non-unique, as a possibly infinite but convergent and locally finite linear combination of pseudomeasures of compact support on $\mathbb{R}$) [Gaudry, 1966; Larsen, 1971; Dieudonné, 1978; Katznelson, 2004]; in particular, if the Fourier transform (respectively, inverse Fourier transform) of a distribution on $\mathbb{R}$ generated by a $p'_{[2,+\infty]}$ -integrable function on $\mathbb{R}$ with $p'_{[2,+\infty]}$ in $\overline{\mathbb{R}}_{[2,+\infty]}$ is a distribution of compact support on $\mathbb{R}$, then it is a pseudomeasure on $\mathbb{R}$ (and hence the $p'_{[2,+\infty]}$ -integrable function on $\mathbb{R}$ is a bounded function on $\mathbb{R}$) [Katznelson, 2004]. More generally, as an immediate application of the differentiation property of the Fourier transformation (as introduced below), we obtain that the

Fourier transform (respectively, inverse Fourier transform) of a $p'_{[2,+\infty]}$ -integrable distribution on $\mathbb{R}$ with $p'_{[2,+\infty]}$ in $\overline{\mathbb{R}}_{[2,+\infty]}$ is the multiplicative product of a polynomial function on $\mathbb{R}$ by the sum of a distribution on $\mathbb{R}$ generated by a square integrable function on $\mathbb{R}$ and a pseudomeasure on $\mathbb{R}$. The Fourier transform (respectively, inverse Fourier transform) of a $p'_{[1,2]}$ -integrable distribution on $\mathbb{R}$ with $p'_{[1,2]}$ in $\mathbb{R}_{[1,2]}$ is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by a $p_{[2,+\infty]}$ -integrable function on $\mathbb{R}$ with $p_{[2,+\infty]}$ in $\overline{\mathbb{R}}_{[2,+\infty]}$ [Schwartz, 1966]; besides, the Fourier transform (respectively, inverse Fourier transform) of a distribution on $\mathbb{R}$ generated by the multiplicative product of a polynomial function on $\mathbb{R}$ by a $p'_{[1,2]}$ -integrable function on $\mathbb{R}$ with $p'_{[1,2]}$ in $\mathbb{R}_{[1,2]}$ is a $p_{[2,+\infty]}$ -integrable distribution on $\mathbb{R}$ with $p_{[2,+\infty]}$ in $\overline{\mathbb{R}}_{[2,+\infty]}$ [Schwartz, 1966]. In particular, the Fourier transform (respectively, inverse Fourier transform) of a square integrable distribution on $\mathbb{R}$ is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by a square integrable function on $\mathbb{R}$ [Schwartz, 1966; Poletti, 2015]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a distribution on $\mathbb{R}$ generated by the multiplicative product of a polynomial function on $\mathbb{R}$ by a square integrable function on $\mathbb{R}$ is a square integrable distribution on $\mathbb{R}$ [Schwartz, 1966; Poletti, 2015]. In fact, the Fourier transformation and the inverse Fourier transformation establish as many isomorphisms both from $\mathcal{D}'_L(\mathbb{R})$ into $\mathcal{P}_L(\mathbb{R})$ and from $\mathcal{P}_L(\mathbb{R})$ into $\mathcal{D}'_L(\mathbb{R})$, an isomorphism established by one transformation having as its inverse an isomorphism established by the other transformation [Schwartz, 1966; Poletti, 2015]. Furthermore, the Fourier transform (respectively, inverse Fourier transform) of the convolutive product of two square integrable distributions on $\mathbb{R}$ is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by an integrable function on $\mathbb{R}$ [Schwartz, 1966; Poletti, 2015]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a distribution on $\mathbb{R}$ generated by the multiplicative product of a polynomial function on $\mathbb{R}$ by an integrable function on $\mathbb{R}$ is the convolutive product of two square integrable distributions on $\mathbb{R}$ [Schwartz, 1966; Poletti, 2015]. In fact, denoting by $\mathcal{D}'_L(\mathbb{R}) * \mathcal{D}'_L(\mathbb{R})$ the distribution space of the convolutive products of two square integrable distributions on $\mathbb{R}$, which is a subspace of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (in fact, of $\mathcal{B}'(\mathbb{R})$), the Fourier transformation and the inverse Fourier transformation establish as many isomorphisms both from $\mathcal{D}'_L(\mathbb{R}) * \mathcal{D}'_L(\mathbb{R})$ into $\mathcal{P}_L(\mathbb{R})$ and from $\mathcal{P}_L(\mathbb{R})$ into $\mathcal{D}'_L(\mathbb{R}) * \mathcal{D}'_L(\mathbb{R})$, an isomorphism established by one transformation having as its inverse an isomorphism established by the other transformation [Schwartz, 1966; Poletti, 2015]. The Fourier transform (respectively, inverse Fourier transform) of an integrable distribution on $\mathbb{R}$ is generable by a C^0 function of slow growth on $\mathbb{R}$, the value at the

origin of such C^0 function of slow growth on $\mathbb{R}$ equalling the 0th moment in the ordinary sense (and hence in the Cesàro sense) of the integrable distribution on $\mathbb{R}$ or, equivalently, the integral on $\mathbb{R}$ (and hence the improper integral on $\mathbb{R}$) of the integrable distribution on $\mathbb{R}$ [Schwartz, 1966; Barros-Neto, 1973; Estrada and Kanwal, 2002; Katznelson, 2004]. Remarkably, the Fourier transform (respectively, inverse Fourier transform) of an integrable distribution on $\mathbb{R}$ can be obtained by evaluating the integrable distribution on $\mathbb{R}$ against the test function $e^{-j\omega t}$ (respectively, $e^{j\omega t}/(2\pi)$), the evaluation reducing to an integration on $\mathbb{R}$ if the integrable distribution on $\mathbb{R}$ is generable by an integrable function on $\mathbb{R}$ [Schwartz, 1966]. In particular, the Fourier transform (respectively, inverse Fourier transform) of a distribution that is generable by an integrable Radon measure on $\mathbb{R}$ is generable by a C^0 (in fact, uniformly continuous) bounded function on $\mathbb{R}$, and the Fourier transform (respectively, inverse Fourier transform) of a distribution that is generable by an integrable Radon measure on $\mathbb{R}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ is generable by a C^0 function vanishing at infinity on $\mathbb{R}$ [Schwartz, 1966; Katznelson, 2004]. It is worth emphasising that a tempered distribution on $\mathbb{R}$ having Fourier transform (respectively, inverse Fourier transform) for which the localisation at the origin is possible (in particular, for which the localisation at a neighbourhood of the origin results in a distribution generated by a C^0 function thereon) need not be an integrable distribution on $\mathbb{R}$ [Wawak, 1987, 1988]; nevertheless, although the result of the localisation at the origin of the Fourier transform (respectively, inverse Fourier transform) of such a tempered distribution on $\mathbb{R}$ cannot be interpreted as its integral on $\mathbb{R}$, it can be interpreted as its improper integral on $\mathbb{R}$ [Sikorski, 1961; Wawak, 1987, 1988; Richards and Youn, 1990; Estrada and Kanwal, 2002].

The Fourier transform (respectively, inverse Fourier transform) of a smooth function of polynomial growth on $\mathbb{R}$ is a distribution of very rapid decay on $\mathbb{R}$ that can be expressed as the sum of a finite linear combination of derivatives of order in $\mathbb{N}$ of δ and a regular distribution of very rapid decay on $\mathbb{R}$ whose localisation at $\mathbb{R}_{(-\infty,0)\cup(0,+\infty)}$ results in a smooth function of rapid decay thereon [Durán *et al.*, 1998]. The Fourier transform (respectively, inverse Fourier transform) of an asymptotically small distribution on $\mathbb{R}$ is a tempered distribution on $\mathbb{R}$ that is smooth at the origin [Estrada and Kanwal, 2002]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a tempered distribution on $\mathbb{R}$ that is smooth at the origin is an asymptotically small distribution on $\mathbb{R}$ [Durán *et al.*, 1998; Estrada and Kanwal, 2002]. Remarkably, the Fourier transform (respectively, inverse Fourier transform) of an asymptotically small distribution on $\mathbb{R}$ can be obtained by evaluating in the Cesàro sense the asymptotically small distribution on $\mathbb{R}$ against the test function $e^{-j\omega t}$ (respectively, $e^{j\omega t}/(2\pi)$) [Estrada and Kanwal, 2002].

The Fourier transform (respectively, inverse Fourier transform) of a distribution of rapid decay on $\mathbb{R}$ is generable by a smooth function of slow growth on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Larcher, 2012; Poletti, 2015; Kisyński, 2017]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a distribution that is generable by a smooth function of slow growth on $\mathbb{R}$ is a distribution of rapid decay on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966; Treves, 1967; Larcher, 2012; Poletti, 2015; Kisyński, 2017]. Furthermore, the Fourier transform (respectively, inverse Fourier transform) of a distribution of very rapid decay on $\mathbb{R}$ is generable by a smooth function of very slow growth on $\mathbb{R}$ [Larcher, 2012]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a distribution that is generable by a smooth function of very slow growth on $\mathbb{R}$ is a distribution of very rapid decay on $\mathbb{R}$ [Larcher, 2012]. In fact, the Fourier transformation and the inverse Fourier transformation establish as many isomorphisms both from $\mathcal{O}'_c(\mathbb{R})$ into $\mathcal{O}_m(\mathbb{R})$ and from $\mathcal{O}_m(\mathbb{R})$ into $\mathcal{O}'_c(\mathbb{R})$ (respectively, both from $\mathcal{O}'_m(\mathbb{R})$ into $\mathcal{O}_c(\mathbb{R})$ and from $\mathcal{O}_c(\mathbb{R})$ into $\mathcal{O}'_m(\mathbb{R})$), an isomorphism established by one transformation having as its inverse an isomorphism established by the other transformation [Schwartz, 1966; Horváth, 1966; Treves, 1967; Larcher, 2012; Kisyński, 2017]. The n_0 th derivative of the Fourier transform (respectively, inverse Fourier transform) of a distribution of rapid decay on $\mathbb{R}$ is generable by a smooth function of slow growth on $\mathbb{R}$, the value at the origin multiplied by j^{n_0} of such smooth function of slow growth on $\mathbb{R}$ equalling the n_0 th moment in the ordinary sense (and hence in the Cesàro sense) of the distribution of rapid decay on $\mathbb{R}$ [Schwartz, 1966; Estrada and Kanwal, 2002]. Remarkably, the n_0 th derivative of the Fourier transform (respectively, inverse Fourier transform) of a distribution of rapid decay on $\mathbb{R}$ can be obtained by evaluating the distribution of rapid decay on $\mathbb{R}$ against the test function $t^{n_0} e^{-j\omega t} / j^{n_0}$ (respectively, $t^{n_0} e^{j\omega t} / (-j^{n_0} 2\pi)$), the evaluation reducing to an integration on $\mathbb{R}$ if the multiplicative product of t^{n_0} by the integrable distribution on $\mathbb{R}$ is generable by an integrable function on $\mathbb{R}$ [Schwartz, 1966; Estrada and Kanwal, 2002]. As is evident, the Fourier transformation and the inverse Fourier transformation establish as many automorphisms from $\mathcal{O}_m^\wedge(\mathbb{R}) \cap \mathcal{O}'_c(\mathbb{R})$ into $\mathcal{O}_m^\wedge(\mathbb{R}) \cap \mathcal{O}'_c(\mathbb{R})$, the automorphism established by one transformation having as its inverse the automorphism established by the other transformation. It is worth noting that $\mathcal{O}_m^\wedge(\mathbb{R}) \cap \mathcal{O}'_c(\mathbb{R})$ is a superspace of $\mathcal{S}^\wedge(\mathbb{R})$ [Schwartz, 1966]; an example of element of $(\mathcal{O}_m^\wedge(\mathbb{R}) \cap \mathcal{O}'_c(\mathbb{R})) \setminus \mathcal{S}^\wedge(\mathbb{R})$ is the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$.

The Fourier transform (respectively, inverse Fourier transform) of a distribution of compact support on $\mathbb{R}$ is generable by a smooth function of very slow growth of exponential type on $\mathbb{R}$ [Schwartz,

1966; Treves, 1967; Barros-Neto, 1973; García *et al.*, 1998; Hoskins and Sousa Pinto, 2005]; inversely, the Fourier transform (respectively, inverse Fourier transform) of a distribution on $\mathbb{R}$ generated by a smooth function of very slow growth of exponential type on $\mathbb{R}$ is a distribution of compact support on $\mathbb{R}$ [Schwartz, 1966; García *et al.*, 1998; Hoskins and Sousa Pinto, 2005]. Accordingly, the Fourier transform (respectively, inverse Fourier transform) of a distribution of compact support on $\mathbb{R}$ can be extended to a holomorphic function of exponential type on $\mathbb{C}$; by choosing suitably the free complex variable, such holomorphic function is the Laplace transform (respectively, the Fourier–Laplace transform), as introduced below, of the distribution of compact support on $\mathbb{R}$ [Zemanian, 1965a; Beltrami and Wohlers, 1966; Misra and Lavoine, 1986]. In fact, as an implication of the Paley–Wiener–Schwartz theorem, the Fourier transformation and the inverse Fourier transformation establish as many isomorphisms both from $\mathcal{E}'(\mathbb{R})$ into $\exp\mathcal{O}_\mathbb{C}(\mathbb{R})$ and from $\exp\mathcal{O}_\mathbb{C}(\mathbb{R})$ into $\mathcal{E}'(\mathbb{R})$, an isomorphism established by one transformation having as its inverse an isomorphism established by the other transformation [Schwartz, 1966; García *et al.*, 1998; Hoskins and Sousa Pinto, 2005]. As a consequence of the fact that a holomorphic function on $\mathbb{C}$ either vanishes only at isolated points of $\mathbb{C}$ or vanishes at each point of $\mathbb{C}$, no tempered distribution on $\mathbb{R}$ other than $0_\mathbb{R}$ can have both compact support and compact spectrum. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], a distribution of compact support on $\mathbb{R}$ is a *transient signal*, whereas a distribution of compact spectrum on $\mathbb{R}$ is a *band-limited signal*.

Let $f_{\exp\mathcal{O}_\mathbb{C}}$ be a distribution on $\mathbb{R}$ generated by a smooth function of very slow growth of exponential type on $\mathbb{R}$, and let $\mathbb{R}_{[\omega_1, \omega_2]}$ with ω_1 in $\mathbb{R}$, ω_2 in $\mathbb{R}$, and ω_2 greater than ω_1 be the smallest closed interval of $\mathbb{R}$ containing the spectrum of $f_{\exp\mathcal{O}_\mathbb{C}}$. Let $\mathbb{R}_{(-\Omega/2, +\Omega/2)}$ with Ω in $\mathbb{R}_{(\Omega_N, +\infty)}$ and Ω_N the maximum of $2|\omega_1|$ and $2|\omega_2|$ be an open interval of $\mathbb{R}$ symmetric with respect to the origin and containing the spectrum of $f_{\exp\mathcal{O}_\mathbb{C}}$; in particular, the spectrum of $f_{\exp\mathcal{O}_\mathbb{C}}$ is itself symmetric with respect to the origin if $f_{\exp\mathcal{O}_\mathbb{C}}$ is either real-valued or imaginary-valued. For any Ω_s in $\mathbb{R}_{(\Omega_N, +\infty)}$ (or, equivalently, for any Ω_s in $\mathbb{R}_{[\Omega, +\infty)}$), the *generalised sampling theorem* (also known as *generalised Whittaker–Shannon–Kotelnikov sampling theorem* or *generalised WSK sampling theorem*) holds [Pfaffelhuber, 1971a]

A.4 Signals: Integral transformations

$$\begin{aligned}
f_{\exp \mathcal{O}_C} &= T_S \lim_{R \rightarrow +\infty} \left(\sum_{r=-R}^{+R} \left((f_{\exp \mathcal{O}_C}(t_r))(f_S)_{t_r} \right) \right) \\
&= T_S f_{\exp \mathcal{O}_C} \text{III}_{T_S} * f_S \\
&= T_S \lim_{R \rightarrow +\infty} \left(\sum_{r=-R}^{+R} \left((f_{\exp \mathcal{O}_C}(t_r))\delta_{t_r} \right) \right) \\
&= T_S f_{\exp \mathcal{O}_C} \text{III}_{T_S}
\end{aligned} \tag{A.37}$$

with

$$\text{III}_{T_S} := \lim_{R \rightarrow +\infty} \left(\sum_{r=-R}^{+R} (\delta_{t_r}) \right)$$

where

- δ_{t_r} with r in $\mathbb{Z}$ is the t_r -translated δ
- III_{T_S} is the T_S -periodic *unit-impulse train* (also known as *Dirac comb* or *shah distribution*) on $\mathbb{R}$
- $\{f_{\exp \mathcal{O}_C}(t_r), r \in \mathbb{Z}\}$ is the *sampled sequence*

The series in Equation (A.37) converges in the topological dual space of the function space of the elements of $\exp \mathcal{S}(\mathbb{R})$ whose spectra do not exceed $\mathbb{R}_{[-\Omega/2, +\Omega/2]}$ [Pfaffelhuber, 1971a]. Remarkably, the series in Equation (A.37) has pointwise convergence on $\mathbb{R}$ if f_S is replaced by an element of $\exp \mathcal{S}(\mathbb{R})$ (in particular, if f_S is replaced by its convolutive product by an element of $\exp \mathcal{S}(\mathbb{R})$) whose spectrum is contained in $\mathbb{R}_{(-\Omega_S/2, +\Omega_S/2)}$ and whose Fourier transform coincides with $1_{\mathbb{R}}$ on an open subset of $\mathbb{R}$ containing $\mathbb{R}_{[\omega_1, \omega_2]}$ [García *et al.*, 1998; Hoskins and Sousa Pinto, 2005]. If Ω_S is not in $\mathbb{R}_{(\Omega_N, +\infty)}$ or if $f_{\exp \mathcal{O}_C}$ is replaced by a distribution on $\mathbb{R}$ that is not a distribution of compact spectrum on $\mathbb{R}$ (in particular, if $f_{\exp \mathcal{O}_C}$ is replaced by a distribution of compact support on $\mathbb{R}$), the application of Equation (A.37) results in the undesired phenomenon of aliasing [Zayed, 1993; Benedetto, 1997].

A.4.2.5 Fourier exchange theorem

We now introduce the *Fourier exchange theorem* [Schwartz, 1966; Horváth, 1966; Treves, 1967; Kisiński, 2017], which is key for our purposes. In loose terms, the Fourier exchange theorem states that the Fourier transformation and the inverse Fourier transformation convert a multiplicative

product into a convolutive product (and *vice versa*) between the time domain and the frequency domain. Remarkably, the operation of multiplication is conditioned by local singularity but not by growth at infinity, whereas the operation of convolution is conditioned by growth at infinity but not by local singularity [Schwartz, 1966]. Indeed, the Fourier transformation and the inverse Fourier transformation exchange local singularity with growth at infinity (and *vice versa*) between the time domain and the frequency domain; specifically, the greater the local singularity in the time domain (respectively, frequency domain), the faster the growth at infinity in the frequency domain (respectively, time domain) [Schwartz, 1966].

Given the distribution ς in $\mathcal{S}'(\mathbb{R})$, the function μ in $\mathcal{O}_M(\mathbb{R})$, and the distribution κ in $\mathcal{O}'_C(\mathbb{R})$, the equivalent relations hold [Schwartz, 1966; Horváth, 1966; Treves, 1967; Kisyński, 2017]

$$\mathcal{F}(\mu \cdot \varsigma) = \frac{1}{2\pi} \left(\mathcal{F}(\mu^\wedge) * \mathcal{F}(\varsigma) \right) \quad (\text{A.38})$$

$$\mathcal{F}(\kappa * \varsigma) = \mathcal{F}(\kappa)^\vee \cdot \mathcal{F}(\varsigma) \quad (\text{A.39})$$

where

- $\mathcal{F}(\mu \cdot \varsigma)$ in $\mathcal{S}'(\mathbb{R})$ is the Fourier transform of $\mu \cdot \varsigma$ in $\mathcal{S}'(\mathbb{R})$
- $\mu^\wedge$ in $\mathcal{O}_M^\wedge(\mathbb{R})$ is the distribution on $\mathbb{R}$ generated by μ in $\mathcal{O}_M(\mathbb{R})$
- $\mathcal{F}(\mu^\wedge)$ in $\mathcal{O}'_C(\mathbb{R})$ is the Fourier transform of $\mu^\wedge$ in $\mathcal{O}_M^\wedge(\mathbb{R})$
- $\mathcal{F}(\varsigma)$ in $\mathcal{S}'(\mathbb{R})$ is the Fourier transform of ς in $\mathcal{S}'(\mathbb{R})$
- $\mathcal{F}(\kappa * \varsigma)$ in $\mathcal{S}'(\mathbb{R})$ is the Fourier transform of $\kappa * \varsigma$ in $\mathcal{S}'(\mathbb{R})$
- $\mathcal{F}(\kappa)$ in $\mathcal{O}_M^\wedge(\mathbb{R})$ is the Fourier transform of κ in $\mathcal{O}'_C(\mathbb{R})$
- $\mathcal{F}(\kappa)^\vee$ in $\mathcal{O}_M(\mathbb{R})$ is the function generating $\mathcal{F}(\kappa)$ in $\mathcal{O}_M^\wedge(\mathbb{R})$

We refer to the pair formed by Equation (A.38) and Equation (A.39) as the Fourier exchange theorem; in particular, we refer to Equation (A.38) and Equation (A.39) as respectively the *multiplication Fourier exchange theorem* and the *convolution Fourier exchange theorem*.

Remarkably, Equation (A.38) and Equation (A.39) remain valid if $\mathcal{F}$ is replaced by $2\pi\mathcal{F}_{\text{inv}}$ [Newcomb and Oliveira, 1963]. Hereinafter, for simplicity, we refer only to Fourier transformation (as opposed to the inverse Fourier transformation), and we omit the superscripts $^\wedge$ and $^\vee$.

As immediate applications of Equation (A.38) and Equation (A.39), we obtain respectively the *differentiation property* and the *translation property* of the Fourier transformation. Let $f_{S'}$ be a tempered distribution on $\mathbb{R}$, and let $F_{S'}$ be its Fourier transform. We have [Schwartz, 1966]

$$\begin{aligned}\mathcal{F}\left(f_{S'}^{(n)}\right) &= \mathcal{F}\left(\delta^{(n)} * f_{S'}\right) \\ &= \mathcal{F}\left(\delta^{(n)}\right) \cdot \mathcal{F}\left(f_{S'}\right) \\ &= (\mathrm{j}\omega)^n F_{S'}\end{aligned}$$

$$\begin{aligned}\mathcal{F}\left((-j t)^n \cdot f_{S'}\right) &= \frac{1}{2 \pi}\left(\mathcal{F}\left((-j t)^n\right) * \mathcal{F}\left(f_{S'}\right)\right) \\ &= \frac{1}{2 \pi}\left(2 \pi \delta^{(n)} * F_{S'}\right) \\ &= F_{S'}^{(n)}\end{aligned}$$

$$\begin{aligned}\mathcal{F}\left(\left(f_{S'}\right)_{t_0}\right) &= \mathcal{F}\left(\delta_{t_0} * f_{S'}\right) \\ &= \mathcal{F}\left(\delta_{t_0}\right) \cdot \mathcal{F}\left(f_{S'}\right) \\ &= \mathrm{e}^{-j t_0 \omega} F_{S'}\end{aligned}$$

$$\begin{aligned}\mathcal{F}\left(\mathrm{e}^{j \omega_0 t} \cdot f_{S'}\right) &= \frac{1}{2 \pi}\left(\mathcal{F}\left(\mathrm{e}^{j \omega_0 t}\right) * \mathcal{F}\left(f_{S'}\right)\right) \\ &= \frac{1}{2 \pi}\left(2 \pi \delta_{\omega_0} * F_{S'}\right) \\ &= \left(F_{S'}\right)_{\omega_0}\end{aligned}$$

where

- $f_{S'}^{(n)}$ is the n^{th} derivative of $f_{S'}$
- $F_{S'}^{(n)}$ is the n^{th} derivative of $F_{S'}$
- t_0 is a real constant of the same quantity dimension as t
- $\left(f_{S'}\right)_{t_0}$ is the t_0 -translated $f_{S'}$
- δ_{t_0} is the t_0 -translated δ
- ω_0 is a real constant of the same quantity dimension as ω
- δ_{ω_0} is the ω_0 -translated δ
- $\left(F_{S'}\right)_{\omega_0}$ is the ω_0 -translated $F_{S'}$

We say that, applied in the manner just described, Equation (A.38) and Equation (A.39) express the standard form of the Fourier exchange theorem. Remarkably, Equation (A.38) can be enhanced to express the Fourier transform of the multiplicative product of a tempered distribution on $\mathbb{R}$ by any finite set of smooth functions of slow growth on $\mathbb{R}$; similarly, Equation (A.39) can be enhanced to express the Fourier transform of the convolutive product of a tempered distribution on $\mathbb{R}$ by any finite set of distributions of rapid decay on $\mathbb{R}$.

In fact, by defining special operations of multiplication and convolution, Equation (A.38) and Equation (A.39) can be generalised significantly [Hirata and Ogata, 1958; Shiraishi, 1959; Hirata and Shiraishi, 1960; Schwartz, 1954c, 1957, 1958, 1962, 1966; Reed and Simon, 1975; Dierolf and Voigt, 1978; Kamiński, 1982; Dierolf, 1984; Oberguggenberger, 1986, 1992; Vladimirov, 2002; Ortner, 2004, 2011; Ortner and Wagner, 2023].

The multiplication of two tempered distributions on $\mathbb{R}$ may result in a distribution on $\mathbb{R}$ that is not a tempered distribution on $\mathbb{R}$ [Reed and Simon, 1975]. Therefore, we introduce a special operation of multiplication, to which we refer as *tempered multiplication* (also known as *S' -multiplication*), having tempered distributions on $\mathbb{R}$ as operands and resulting in a tempered distribution on $\mathbb{R}$ whenever well-defined [Reed and Simon, 1975; Oberguggenberger, 1986, 1992; Ortner and Wagner, 2023]. Whenever the multiplication of two tempered distributions on $\mathbb{R}$ is well-defined and results in a tempered distribution on $\mathbb{R}$, the same is true for their tempered multiplication, and the two operations result in the same tempered distribution on $\mathbb{R}$ [Oberguggenberger, 1992; Ortner and Wagner, 2023]; inversely, whenever the tempered multiplication of two tempered distributions on $\mathbb{R}$ is well-defined and hence results in a tempered distribution on $\mathbb{R}$, the same is true for their multiplication, and the two operations result in the same tempered distribution on $\mathbb{R}$ [Ortner and Wagner, 2023]. In particular, the *ad hoc* operation of multiplication having a distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ and a function of locally bounded variation on $\mathbb{R}$ as operands can be regarded as a tempered multiplication whenever the distribution of C -order in $\mathbb{Z}_{[-\infty, -1]}$ on $\mathbb{R}$ and the distribution on $\mathbb{R}$ generated by the function of locally bounded variation on $\mathbb{R}$ are tempered distributions on $\mathbb{R}$ and the *ad hoc* operation of multiplication results in a tempered distribution on $\mathbb{R}$ [Schwartz, 1962; Nussenzveig, 1972; Oberguggenberger, 1992; Ortner and Wagner, 2023]. Like multiplication, tempered multiplication is commutative but not associative [Oberguggenberger, 1992]; however, if the tempered multiplication of two tempered distributions on $\mathbb{R}$ is well-defined, the tempered multiplication of them by any finite set of smooth functions of slow growth on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a tempered distribution on $\mathbb{R}$. It is worth

noting that the tempered multiplication is well-defined for some pair of distributions having non-disjoint singular supports [Reed and Simon, 1975; Oberguggenberger, 1992].

The convolution of two tempered distributions on $\mathbb{R}$ may result in a distribution on $\mathbb{R}$ that is not a tempered distribution on $\mathbb{R}$ [Dierolf and Voigt, 1978; Dierolf, 1984]. Therefore, we introduce a special operation of convolution, to which we refer as *tempered convolution* (also known as *$\mathcal{S}'$ -convolution*), having tempered distributions on $\mathbb{R}$ as operands and resulting in a tempered distribution on $\mathbb{R}$ whenever well-defined [Hirata and Ogata, 1958; Shiraishi, 1959; Hirata and Shiraishi, 1960; Schwartz, 1962; Dierolf and Voigt, 1978; Kamiński, 1982; Dierolf, 1984; Oberguggenberger, 1986, 1992; Ortner, 2011; Ortner and Wagner, 2023]. Whenever the convolution of two tempered distributions on $\mathbb{R}$ is well-defined and results in a tempered distribution on $\mathbb{R}$, the same is true for their tempered convolution, and the two operations result in the same tempered distribution on $\mathbb{R}$ [Shiraishi, 1959]; inversely, whenever the tempered convolution of two tempered distributions on $\mathbb{R}$ is well-defined and hence results in a tempered distribution on $\mathbb{R}$, the same is true for their convolution, and the two operations result in the same tempered distribution on $\mathbb{R}$ [Hirata and Ogata, 1958]. Like convolution, tempered convolution is commutative but not associative [Shiraishi, 1959; Oberguggenberger, 1986, 1992]; however, if the tempered convolution of two tempered distributions on $\mathbb{R}$ is well-defined, the tempered convolution of them by any finite set of distributions of rapid decay on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a tempered distribution on $\mathbb{R}$ [Hirata and Ogata, 1958; Oberguggenberger, 1986, 1992; Čížek and Jelínek, 1996]. Given a tempered distribution on $\mathbb{R}$, there exists a distribution space of $(-z)$ -weighted integrable distributions on $\mathbb{R}$ such that the tempered convolution of the tempered distribution on $\mathbb{R}$ by each element of the distribution space of $(-z)$ -weighted integrable distributions on $\mathbb{R}$ is well-defined [Álvarez and Moyo, 2007].

Equation (A.38) and Equation (A.39) are still valid if the multiplications and convolutions therein are interpreted as respectively tempered multiplications and tempered convolutions [Hirata and Ogata, 1958; Schwartz, 1962; Reed and Simon, 1975; Dierolf and Voigt, 1978; Kamiński, 1982; Dierolf, 1984; Oberguggenberger, 1986, 1992; Vladimirov, 2002; Ortner, 2011; Ortner and Wagner, 2023]. We say that, applied in the manner just described, Equation (A.38) and Equation (A.39) express the generalised form of the Fourier exchange theorem. Remarkably, Equation (A.38) can be enhanced to express the Fourier transform of the tempered multiplicative product of two tempered distributions on $\mathbb{R}$ whose tempered multiplicative product is well-defined by any finite set of smooth functions of slow growth on $\mathbb{R}$; similarly, Equation (A.39) can be enhanced to express the Fourier transform of the tempered convolutive product of two tempered distributions on $\mathbb{R}$ whose tempered

convolutive product is well-defined by any finite set of distributions of rapid decay on $\mathbb{R}$.

Hereinafter, the multiplication and the convolution of two tempered distributions on $\mathbb{R}$ are to be intended as respectively a tempered multiplication and a tempered convolution; for simplicity, we refer to them as respectively multiplication and convolution.

As a first particular case of application of the generalised form of the Fourier exchange theorem, the right-hand side of Equation (A.38) is well-defined if both $\mathcal{F}(\mu^\wedge)$ and $\mathcal{F}(\varsigma)$ are one-sided tempered distributions on $\mathbb{R}$, and the left-hand side of Equation (A.39) is well-defined if both κ and ς are one-sided tempered distributions on $\mathbb{R}$. Accordingly, the multiplicative product of two right-retarded (respectively, left-retarded) distributions on $\mathbb{R}$ can be defined as the inverse Fourier transform of the right-hand side of Equation (A.38) [Reed and Simon, 1975; Oberguggenberger, 1986, 1992]. Equivalently, the multiplicative product of two right-retarded distributions on $\mathbb{R}$ can be defined as the limit of the multiplicative product of the holomorphic functions on $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (respectively, on $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$) to which they can be extended as the real part σ (respectively, as the imaginary part $\underline{\sigma}$) of the free complex variable $s \equiv \sigma + j\omega$ (respectively, $\underline{s} \equiv \omega + j\underline{\sigma}$) tends to 0 from the right (respectively, from below), the limit being within $\mathcal{S}'(\mathbb{R})$, which implies that the multiplicative product of the holomorphic functions on $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (respectively, on $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$) generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\mathbb{R}_{(0, +\infty)}$ (respectively, for each fixed $\underline{\sigma}$ in $\mathbb{R}_{(-\infty, 0)}$) [Shambayati and Zieleźny, 1983; Oberguggenberger, 1992; Carlsson and Wittsten, 2017]. Besides, the multiplicative product of two left-retarded distributions on $\mathbb{R}$ can be defined as the limit of the multiplicative product of the holomorphic functions on $\mathbb{C}_{\text{Re} \in (-\infty, 0)}$ (respectively, on $\mathbb{C}_{\text{Im} \in (0, +\infty)}$) to which they can be extended as the real part σ (respectively, as the imaginary part $\underline{\sigma}$) of the free complex variable $s \equiv \sigma + j\omega$ (respectively, $\underline{s} \equiv \omega + j\underline{\sigma}$) tends to 0 from the left (respectively, from above), the limit being within $\mathcal{S}'(\mathbb{R})$, which implies that the multiplicative product of the holomorphic functions on $\mathbb{C}_{\text{Re} \in (-\infty, 0)}$ (respectively, on $\mathbb{C}_{\text{Im} \in (0, +\infty)}$) generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\mathbb{R}_{(-\infty, 0)}$ (respectively, for each fixed $\underline{\sigma}$ in $\mathbb{R}_{(0, +\infty)}$) [Shambayati and Zieleźny, 1983; Oberguggenberger, 1992; Carlsson and Wittsten, 2017]. Just like the convolution of any finite set of right-sided (respectively, left-sided) distributions on $\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a right-sided (respectively, left-sided) distribution on $\mathbb{R}$ that coincides with $0_{\mathbb{R}}$ if and only if at least one of the convolved distributions is $0_{\mathbb{R}}$, the multiplication of any finite set of right-retarded (respectively, left-retarded) distributions on

$\mathbb{R}$ is always well-defined, commutative, and associative, and it results in a right-retarded (respectively, left-retarded) distribution on $\mathbb{R}$ that coincides with $0_{\mathbb{R}}$ if and only if at least one of the multiplied distributions is $0_{\mathbb{R}}$ [Oberguggenberger, 1992]; in other words, just like the distribution space of the right-sided (respectively, left-sided) distributions on $\mathbb{R}$ is a convolution algebra (namely, a complex algebra in which the operation of multiplication of the algebraic structure is the operation of convolution), and in fact a complex commutative weak topological domain algebra, the distribution space of the right-retarded (respectively, left-retarded) distributions on $\mathbb{R}$ is a *multiplication algebra* (namely, a complex algebra in which the operation of multiplication of the algebraic structure is the operation of multiplication), and in fact a complex commutative weak topological domain algebra [Oberguggenberger, 1992]. Accordingly, Equation (A.38) can be enhanced to express the Fourier transform of the multiplicative product of any finite set of right-retarded (respectively, left-retarded) distributions on $\mathbb{R}$; similarly, Equation (A.39) can be enhanced to express the Fourier transform of the convolutive product of any finite set of one-sided tempered distribution on $\mathbb{R}$.

As a second particular case of application of the generalised form of the Fourier exchange theorem, Equation (A.39) is valid if κ and ς are distributions that are elements of respectively $\mathcal{D}'_{L^{p_1}}(\mathbb{R})$ and $\mathcal{D}'_{L^{p_2}}(\mathbb{R})$ with p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$ and p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2 - 1$ is in $\mathbb{R}_{[0,1]}$ (in which case $\kappa * \varsigma$ is a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_{12}}}(\mathbb{R})$ with p_{12} in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{12}$ equals $1/p_1 + 1/p_2 - 1$) [Ortner, 2011]. Furthermore, we know that the convolution of three distributions on $\mathbb{R}$ that are elements of respectively $\mathcal{D}'_{L^{p_1}}(\mathbb{R})$, $\mathcal{D}'_{L^{p_2}}(\mathbb{R})$, and $\mathcal{D}'_{L^{p_3}}(\mathbb{R})$ with p_1 in $\bar{\mathbb{R}}_{[1,+\infty]}$, p_2 in $\bar{\mathbb{R}}_{[1,+\infty]}$, and p_3 in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_1 + 1/p_2 + 1/p_3 - 2$ is in $\mathbb{R}_{[0,1]}$ as well as one the equivalent additional conditions that $1/p_1 + 1/p_2 - 1$ is in $\mathbb{R}_{[0,1]}$, $1/p_2 + 1/p_3 - 1$ is in $\mathbb{R}_{[0,1]}$, or $1/p_1 + 1/p_3 - 1$ is in $\mathbb{R}_{[0,1]}$ is well-defined, commutative, and associative, and it results in a distribution on $\mathbb{R}$ that is an element of $\mathcal{D}'_{L^{p_{123}}}(\mathbb{R})$ with p_{123} in $\bar{\mathbb{R}}_{[1,+\infty]}$ satisfying the condition that $1/p_{123}$ equals $1/p_1 + 1/p_2 + 1/p_3 - 2$ [Čížek and Jelínek, 1996; Ortner, 2004; Ciampa *et al.*, 2008]. Accordingly, Equation (A.39) can be enhanced to express the Fourier transform of the convolutive product of three such p -integrable distributions on $\mathbb{R}$.

A.4.3 Laplace transformation

Let there be given a distribution on $\mathbb{R}$ of the free real variable t whose multiplicative product by $e^{-\sigma t}$ is a tempered distribution on $\mathbb{R}$ for all and only the values of σ in a nonempty subset of $\overline{\mathbb{R}}$. Such a distribution on $\mathbb{R}$ is a distribution of finite order on $\mathbb{R}$, and the nonempty subset of $\overline{\mathbb{R}}$ is a nonempty interval of $\overline{\mathbb{R}}$ [Newcomb and Oliveira, 1963; Zemanian, 1965a, 1966a; Beltrami and Wohlers, 1966; Schwartz, 1966]. If and only if such nonempty interval of $\overline{\mathbb{R}}$ is a non-singleton interval of $\overline{\mathbb{R}}$ (or, equivalently, if and only if the interior of such nonempty interval of $\overline{\mathbb{R}}$ is a nonempty interval of $\overline{\mathbb{R}}$), we say that the distribution of finite order on $\mathbb{R}$ is a *Laplace-transformable distribution* on $\mathbb{R}$ and that it is associated with the non-singleton interval of $\overline{\mathbb{R}}$. Remarkably, for each value of σ in the interior of the non-singleton interval of $\overline{\mathbb{R}}$ associated with a Laplace-transformable distribution on $\mathbb{R}$, the multiplicative product of the Laplace-transformable distribution on $\mathbb{R}$ by $e^{-\sigma t}$ is not just a tempered distribution on $\mathbb{R}$ but in fact a distribution of rapid decay on $\mathbb{R}$ (and hence its Fourier transform is generable by a smooth function of slow growth on $\mathbb{R}$) [Newcomb and Oliveira, 1963; Schwartz, 1966]. The distribution space of the Laplace-transformable distributions on $\mathbb{R}$ associated with the same open non-singleton interval of $\overline{\mathbb{R}}$ is a convolution algebra (in fact, a complex commutative topological domain algebra) [Schwartz, 1966]. It is worth noting that, whilst there are many Laplace-transformable distributions on $\mathbb{R}$ that are not Fourier-transformable distributions on $\mathbb{R}$ (for example, the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{\sigma_0 t} \mathfrak{g}(t)$, σ_0 being a positive real constant of the same quantity dimension as ω), there are also Fourier-transformable distributions on $\mathbb{R}$ that are not Laplace-transformable distributions on $\mathbb{R}$ (for example, the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $\left(1 + (t/t_0)^2\right)^{-1}$, t_0 being a non-null real constant of the same quantity dimension as t) [Zemanian, 1965a]. As is evident, a Laplace-transformable distribution on $\mathbb{R}$ is a Fourier-transformable distribution on $\mathbb{R}$ if and only if its multiplicative product by $e^{-\sigma t}$ is a tempered distribution on $\mathbb{R}$ for $\sigma = 0$; besides, a Fourier-transformable distribution on $\mathbb{R}$ is a Laplace-transformable distribution on $\mathbb{R}$ if and only if its multiplicative product by $e^{-\sigma t}$ is a tempered distribution on $\mathbb{R}$ not only for $\sigma = 0$. Many Laplace-transformable distributions on $\mathbb{R}$ that are not Fourier-transformable distributions on $\mathbb{R}$ and are relevant for applications are one-sided distributions on $\mathbb{R}$; an example of Laplace-transformable distribution on $\mathbb{R}$ that is neither a Fourier-transformable distribution on $\mathbb{R}$ nor a one-sided distribution on $\mathbb{R}$ is the distribution on $\mathbb{R}$ generated by the function defined almost

everywhere on $\mathbb{R}$ by $e^{\sigma_L t} \mathfrak{G}(t) + e^{\sigma_U t} \tilde{\mathfrak{G}}(t)$ with either σ_U greater than σ_L greater than 0 or 0 greater than σ_U greater than σ_L (σ_L and σ_U being real constants of the same quantity dimension as ω), which is associated with the non-singleton interval $\mathbb{R}_{[\sigma_L, \sigma_U]}$.

Let Γ_{f_L} be a non-singleton interval of $\overline{\mathbb{R}}$, and let $\Gamma_{f_L}^0 \equiv (\sigma_1, \sigma_2)$ and $\partial\Gamma_{f_L} \equiv \{\sigma_1, \sigma_2\}$ with σ_1 in $\overline{\mathbb{R}}$, σ_2 in $\overline{\mathbb{R}}$, and σ_2 greater than σ_1 be respectively its (necessarily nonempty) interior and its (necessarily doubleton) frontier; it is worth noting that Γ_{f_L} may include none, one, or both of σ_1 and σ_2 , whereas $\Gamma_{f_L}^0$ and $\partial\Gamma_{f_L}$ include respectively none and both of σ_1 and σ_2 . Let f_L be a Laplace-transformable distribution on $\mathbb{R}$ of the free real variable t associated with Γ_{f_L} . The *direct two-sided Laplace transform* (also known as *direct bilateral Laplace transform*, hereinafter *Laplace transform*) F_L of the *direct two-sided Laplace transformand* (hereinafter *Laplace transformand*) f_L is the holomorphic function on $\mathbb{C}_{\text{Re} \in \Gamma_{f_L}^0}$ of the free complex variable $s \equiv \sigma + j\omega$, having modulus majorised by some polynomial function on each subset of $\mathbb{C}_{\text{Re} \in \Gamma_{f_L}^0}$ bounded from the left and from the right by, and including, a pair of straight lines parallel to $\mathbb{C}_{\text{Re}=0}$, defined by [Newcomb and Oliveira, 1963; Zemanian, 1965a, 1966a, 1968b; Beltrami and Wohlers, 1966; Schwartz, 1966]

$$F_L \equiv \mathcal{L}(f_L) := \left(\mathcal{F}(f_L e^{-\sigma t}) \right)_{\sigma + j\omega = s}, \quad \begin{cases} \sigma \in \Gamma_{f_L}^0 \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_L}^0} \end{cases} \quad (\text{A.40})$$

where

- $\mathcal{L}$ is the *direct two-sided Laplace transformation* (also known as *direct bilateral Laplace transformation*, hereinafter *Laplace transformation*)

We say that the ordered pair formed by a Laplace transformand and its Laplace transform is a *direct Laplace-transform pair* (hereinafter *Laplace-transform pair*).

Furthermore, $\mathbb{C}_{\text{Re} \in \Gamma_{f_L}^0}$ is said to be the *region of definition* (also known as *strip of definition*, *region of existence*, *strip of existence*, or, by abuse of language, *region of convergence* or *strip of convergence*) of F_L [Newcomb and Oliveira, 1963; Zemanian, 1965a, 1966a, 1966b, 1968b]; besides, σ_1 and σ_2 are said to be respectively the *left abscissa of definition* (also known as *left abscissa of existence* or, by abuse of language, *left abscissa of convergence*) of F_L and the

right abscissa of definition (also known as *right abscissa of existence* or, by abuse of language, *right abscissa of convergence*) of $F_{\mathcal{L}}$ [Zemanian, 1965a, 1968b]. It is worth emphasising that the region of definition of $F_{\mathcal{L}}$ is by definition $\mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0}$ even when $f_{\mathcal{L}} e^{-\sigma t}$ is a tempered distribution on $\mathbb{R}$ for some values of σ in $\partial \Gamma_{f_{\mathcal{L}}}$; by enforcing this restriction (which is standard but not universal [Newcomb and Oliveira, 1963]), we have the advantage that $F_{\mathcal{L}}$ is a holomorphic function on $\mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0}$. If, instead, $\mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}}$ is taken as the region of definition of $F_{\mathcal{L}}$, then $F_{\mathcal{L}}$ is a distribution-valued meromorphic function on $\mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}}$ [Carmichael and Mitrović, 1989; Ortner and Wagner, 2013, 2015].

Both the Laplace transformation and the Laplace transform are qualified as *right-sided* (respectively, *left-sided*) whenever the Laplace transformand is right-sided (respectively, left-sided), [Lavoine, 1959; Zemanian, 1965a; Misra and Lavoine, 1986; Poletti, 2005]. In particular, a Laplace transformand is right-sided (respectively, left-sided) only if σ_2 is $+\infty$ (respectively, σ_1 is $-\infty$) [Zemanian, 1965a, 1968b]; in fact, a Laplace transformand is right-sided (respectively, left-sided) if and only if σ_2 is $+\infty$ (respectively, σ_1 is $-\infty$) and its Laplace transform has modulus satisfying a certain majorisation condition [Zemanian, 1965a, 1968b]. Furthermore, a Laplace transformand has compact support if and only if its Laplace transform is a holomorphic function of exponential type on $\mathbb{C}$ (and hence σ_1 is $-\infty$ and σ_2 is $+\infty$) [Zemanian, 1965a; Beltrami and Wohlers, 1966; Misra and Lavoine, 1986].

As expressed by Equation (A.40), $F_{\mathcal{L}}$ does not depend on the value of σ in $\Gamma_{f_{\mathcal{L}}}^0$ [Newcomb and Oliveira, 1963; Zemanian, 1965a]. Remarkably, $F_{\mathcal{L}}$ can be obtained by evaluating $f_{\mathcal{L}}$ against the test function e^{-st} for any fixed s in $\mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0}$ (which implies defining suitable test function spaces and distribution spaces though) [Newcomb and Oliveira, 1963; Beltrami and Wohlers, 1966; Zemanian, 1966a, 1968b]. It is worth emphasising that the Laplace transform of a Laplace-transformable distribution on $\mathbb{R}$ is determined uniquely only when its region of definition is specified [Zemanian, 1968b]; for example, $\mathfrak{g}$ (which has Fourier transform $\text{p.v.}(1/(j\omega)) + \pi\delta$) has Laplace transform $1/s$ with region of definition $\mathbb{C}_{\text{Re} \in (0, +\infty)}$, whereas $-\check{\mathfrak{g}}$ (which has Fourier transform $\text{p.v.}(1/(j\omega)) - \pi\delta$) has Laplace transform $1/s$ with region of definition $\mathbb{C}_{\text{Re} \in (-\infty, 0)}$ [Newcomb and Oliveira, 1963; Zemanian, 1968b].

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If 0 is in $\Gamma_{f_{\mathcal{L}}}^0$, Equation (A.40) allows expressing the Fourier transform of $f_{\mathcal{L}}$ as the Laplace transform of $f_{\mathcal{L}}$ localised at $\mathbb{C}_{\text{Re}=0}$ and the Laplace transform of $f_{\mathcal{L}}$ as the Fourier transform of $f_{\mathcal{L}}$ extended to a holomorphic function on $\Gamma_{f_{\mathcal{L}}}^0$.

If σ_1 is in $\Gamma_{f_{\mathcal{L}}} \cap \mathbb{R}$, we have [Beltrami and Wohlers, 1966]

$$\begin{aligned} \lim_{\sigma \rightarrow \sigma_1^+} \left((F_{\mathcal{L}})_{s=\sigma+j\omega} \right) &\equiv \lim_{\sigma \rightarrow \sigma_1^+} \left((\mathcal{L}(f_{\mathcal{L}}))_{s=\sigma+j\omega} \right) \\ &= \mathcal{F} \left(f_{\mathcal{L}} e^{-\sigma_1 t} \right) \end{aligned} \quad , \quad \begin{cases} \sigma \in \Gamma_{f_{\mathcal{L}}}^0 \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0} \end{cases} \quad (\text{A.41})$$

each limit in Equation (A.41) being within $\mathcal{S}'(\mathbb{R})$, which implies that $(F_{\mathcal{L}})_{s=\sigma+j\omega}$ generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\Gamma_{f_{\mathcal{L}}}^0$ as well as for σ equalling σ_1 . We refer to the quantity expressed by Equation (A.41) as the *left boundary value* of $F_{\mathcal{L}}$. In particular, if σ_1 is 0, Equation (A.41) allows expressing the Fourier transform of $f_{\mathcal{L}}$ as the left boundary value of the Laplace transform of $f_{\mathcal{L}}$. We mention that, if σ_1 is 0, it is possible but not trivial to express the Laplace transform of $f_{\mathcal{L}}$ in terms of the Fourier transform of $f_{\mathcal{L}}$ [Lavoine, 1963].

If σ_2 is in $\Gamma_{f_{\mathcal{L}}} \cap \mathbb{R}$, we have [Beltrami and Wohlers, 1966]

$$\begin{aligned} \lim_{\sigma \rightarrow \sigma_2^-} \left((F_{\mathcal{L}})_{s=\sigma+j\omega} \right) &\equiv \lim_{\sigma \rightarrow \sigma_2^-} \left((\mathcal{L}(f_{\mathcal{L}}))_{s=\sigma+j\omega} \right) \\ &= \mathcal{F} \left(f_{\mathcal{L}} e^{-\sigma_2 t} \right) \end{aligned} \quad , \quad \begin{cases} \sigma \in \Gamma_{f_{\mathcal{L}}}^0 \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0} \end{cases} \quad (\text{A.42})$$

each limit in Equation (A.42) being within $\mathcal{S}'(\mathbb{R})$, which implies that $(F_{\mathcal{L}})_{s=\sigma+j\omega}$ generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\Gamma_{f_{\mathcal{L}}}^0$ as well as for σ equalling σ_2 . We refer to the quantity expressed by Equation (A.42) as the *right boundary value* of $F_{\mathcal{L}}$. In particular, if σ_2 is 0, Equation (A.42) allows expressing the Fourier transform of $f_{\mathcal{L}}$ as the right boundary value of the Laplace transform of $f_{\mathcal{L}}$. We mention that, if σ_2 is 0, it is possible but not trivial to express the Laplace transform of $f_{\mathcal{L}}$ in terms of the Fourier transform of $f_{\mathcal{L}}$ [Lavoine, 1963].

Let $\Gamma_{f_{\mathcal{L}}}$ be a non-singleton interval of $\overline{\mathbb{R}}$, and let $\Gamma_{f_{\mathcal{L}}}^0 \equiv (\sigma_1, \sigma_2)$ and $\partial\Gamma_{f_{\mathcal{L}}} \equiv \{\sigma_1, \sigma_2\}$ with σ_1 in $\overline{\mathbb{R}}$, σ_2 in $\overline{\mathbb{R}}$, and σ_2 greater than σ_1 be respectively its (necessarily nonempty) interior and its

(necessarily doubleton) frontier; it is worth noting that $\Gamma_{f_{\mathcal{L}}}^0$ may include none, one, or both of σ_1 and σ_2 , whereas $\Gamma_{f_{\mathcal{L}}}^0$ and $\partial\Gamma_{f_{\mathcal{L}}}^0$ include respectively none and both of σ_1 and σ_2 . Let $F_{\mathcal{L}}$ be a holomorphic function on $\mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0}$ of the free complex variable $s \equiv \sigma + j\omega$ having modulus majorised by some polynomial function on each subset of $\mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0}$ bounded from the left and from the right by, and including, a pair of straight lines parallel to $\mathbb{C}_{\text{Re}=0}$. The *inverse two-sided Laplace transform* (also known as *inverse bilateral Laplace transform*, hereinafter *inverse Laplace transform*) $f_{\mathcal{L}}$ of the *inverse two-sided Laplace transformand* (hereinafter *inverse Laplace transformand*) $F_{\mathcal{L}}$ is the Laplace-transformable distribution on $\mathbb{R}$ of the free real variable t associated with $\Gamma_{f_{\mathcal{L}}}^0$ defined by [Newcomb and Oliveira, 1963; Zemanian, 1965a, 1966a, 1966b, 1968b]

$$\begin{aligned} f_{\mathcal{L}} &= \mathcal{L}_{\text{inv}} \left(F_{\mathcal{L}} \right) \\ &:= \left(\mathcal{F}_{\text{inv}} \left(\left(F_{\mathcal{L}} \right)_{s=\sigma+j\omega} \right) \right) e^{\sigma t} \quad , \quad \begin{cases} \sigma \in \Gamma_{f_{\mathcal{L}}}^0 \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0} \end{cases} \end{aligned} \quad (\text{A.43})$$

where

- $\mathcal{L}_{\text{inv}}$ is the *inverse two-sided Laplace transformation* (also known as *inverse bilateral Laplace transformation*, hereinafter *inverse Laplace transformation*)

We say that the ordered pair formed by an inverse Laplace transformand and its inverse Laplace transform is an *inverse Laplace-transform pair*.

As expressed by Equation (A.43), $f_{\mathcal{L}}$ does not depend on the value of σ in $\Gamma_{f_{\mathcal{L}}}^0$ [Newcomb and

Oliveira, 1963]. Remarkably, $f_{\mathcal{L}}$ can be obtained by evaluating $\left(F_{\mathcal{L}} \right)_{s=\sigma+j\omega}$ against the test function

$\left(e^{st} / (2\pi) \right)_{s=\sigma+j\omega}$ for any fixed σ in $\Gamma_{f_{\mathcal{L}}}^0$ (which implies defining suitable test function spaces and distribution spaces though) [Newcomb and Oliveira, 1963].

Furthermore, $f_{\mathcal{L}}$ coincides with $f_{\mathcal{L}}$ if and only if $\Gamma_{f_{\mathcal{L}}}^0$ coincides with $\Gamma_{f_{\mathcal{L}}}^0$ (or, equivalently, $\Gamma_{f_{\mathcal{L}}}^0$ coincides with $\Gamma_{f_{\mathcal{L}}}^0$) and $F_{\mathcal{L}}$ coincides with $F_{\mathcal{L}}$ (in fact, if and only if $\Gamma_{f_{\mathcal{L}}}^0$ coincides with $\Gamma_{f_{\mathcal{L}}}^0$ and $F_{\mathcal{L}}$

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localised at a straight line parallel to $\mathbb{C}_{\text{Re}=0}$ in $\Gamma_{f_{\mathcal{L}}}^0$ coincides with $F_{\mathcal{L}}$ localised at the same straight line parallel to $\mathbb{C}_{\text{Re}=0}$ in $\Gamma_{f_{\mathcal{L}}}^0$ or, equivalently, if and only if $\Gamma_{f_{\mathcal{L}}}^0 \cap \Gamma_{f_{\mathcal{L}}}^0$ is nonempty and $F_{\mathcal{L}}$ localised at a straight line parallel to $\mathbb{C}_{\text{Re}=0}$ in $\Gamma_{f_{\mathcal{L}}}^0 \cap \Gamma_{f_{\mathcal{L}}}^0$ coincides with $F_{\mathcal{L}}$ localised at the same straight line parallel to $\mathbb{C}_{\text{Re}=0}$ in $\Gamma_{f_{\mathcal{L}}}^0 \cap \Gamma_{f_{\mathcal{L}}}^0$ [Zemanian, 1965a, 1966a, 1968b; Beltrami and Wohlers, 1966].

Accordingly, a Laplace transformand vanishes on $\mathbb{R}$ if and only if its Laplace transform vanishes on $\mathbb{C}$ (in fact, if and only if its Laplace transform vanishes on the region of definition or, equivalently, if and only if its Laplace transform vanishes at a straight line parallel to $\mathbb{C}_{\text{Re}=0}$ in the region of definition) [Zemanian, 1965a].

By letting $f_{\mathcal{L}} \equiv f_{\mathcal{L}}$, $F_{\mathcal{L}} \equiv F_{\mathcal{L}}$, and $\Gamma_{f_{\mathcal{L}}}^0 \equiv \Gamma_{f_{\mathcal{L}}}^0$, Equation (A.40) and Equation (A.43) become

$$\begin{aligned} F_{\mathcal{L}} &\equiv \mathcal{L}(f_{\mathcal{L}}) \\ &:= \left(\mathcal{F}(f_{\mathcal{L}} e^{-\sigma t}) \right)_{\sigma + j\omega = s}, \quad \begin{cases} \sigma \in \Gamma_{f_{\mathcal{L}}}^0 \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0} \end{cases} \end{aligned} \quad (\text{A.44})$$

$$\begin{aligned} f_{\mathcal{L}} &= \mathcal{L}_{\text{inv}}(F_{\mathcal{L}}) \\ &:= \left(\mathcal{F}_{\text{inv}}((F_{\mathcal{L}})_{s=\sigma+j\omega}) \right) e^{\sigma t}, \quad \begin{cases} \sigma \in \Gamma_{f_{\mathcal{L}}}^0 \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0} \end{cases} \end{aligned} \quad (\text{A.45})$$

The (symmetric) kernel distribution of the Laplace transformation and of the inverse Laplace transformation are respectively $e^{-st'}$ and $(1/(2\pi j))e^{ts'}$ (s' being a bound complex variable of the same quantity dimension as s) [Widder, 1941, 1971]. When t is interpreted as time, s is interpreted as *complex angular frequency*.

In the literature, several results involving the Laplace transformation and the inverse Laplace transformation are expressed in an equivalent formulation, be it done explicitly [Treves, 1967, Barros-Neto, 1973; Carmichael and Mitrović, 1989; Friedlander and Joshi, 1998; Carlsson and Wittsten, 2017] or implicitly [Schwartz, 1962; Nussenzveig, 1972; Shambayati and Zieleźny, 1983; Gripenberg *et al.*, 1990; King, 2009b]. Such equivalent formulation can be obtained from the usual formulation by letting

$$\begin{cases} \sigma \equiv -\underline{\sigma} \\ s \equiv j\underline{s} \end{cases} \quad (\text{A.46})$$

which implies introducing the free real variable $\underline{\sigma}$ and the free complex variable $\underline{s} \equiv \omega + j\underline{\sigma}$. By letting

$$\begin{cases} \sigma_1 \equiv -\underline{\sigma}_2 \\ \sigma_2 \equiv -\underline{\sigma}_1 \end{cases} \quad (\text{A.47})$$

with $\underline{\sigma}_2$ greater than $\underline{\sigma}_1$, we then have the correspondences

$$\begin{aligned} \sigma \in \Gamma_{f_{\mathcal{L}}} &\leftrightarrow \underline{\sigma} \in \underline{\Gamma}_{f_{\mathcal{L}}} \\ \sigma \in \Gamma_{f_{\mathcal{L}}}^0 \equiv (\sigma_1, \sigma_2) &\leftrightarrow \underline{\sigma} \in \underline{\Gamma}_{f_{\mathcal{L}}}^0 \equiv (\underline{\sigma}_1, \underline{\sigma}_2) \\ \sigma \in \partial\Gamma_{f_{\mathcal{L}}} \equiv \{\sigma_1, \sigma_2\} &\leftrightarrow \underline{\sigma} \in \partial\underline{\Gamma}_{f_{\mathcal{L}}} \equiv \{\underline{\sigma}_1, \underline{\sigma}_2\} \\ \sigma = \sigma_1 &\leftrightarrow \underline{\sigma} = \underline{\sigma}_2 \\ \sigma = \sigma_2 &\leftrightarrow \underline{\sigma} = \underline{\sigma}_1 \\ \sigma \rightarrow \sigma_1^+ &\leftrightarrow \underline{\sigma} \rightarrow \underline{\sigma}_2^- \\ \sigma \rightarrow \sigma_2^- &\leftrightarrow \underline{\sigma} \rightarrow \underline{\sigma}_1^+ \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}} &\leftrightarrow \underline{s} \in \mathbb{C}_{\text{Im} \in \underline{\Gamma}_{f_{\mathcal{L}}}} \\ s \in \mathbb{C}_{\text{Re} \in \Gamma_{f_{\mathcal{L}}}^0} &\leftrightarrow \underline{s} \in \mathbb{C}_{\text{Im} \in \underline{\Gamma}_{f_{\mathcal{L}}}^0} \\ s \in \mathbb{C}_{\text{Re}=0} &\leftrightarrow \underline{s} \in \mathbb{C}_{\text{Im}=0} \\ \mathcal{L} &\leftrightarrow \mathcal{FL} \\ \mathcal{L}_{\text{inv}} &\leftrightarrow \mathcal{FL}_{\text{inv}} \\ F_{\mathcal{L}} &\leftrightarrow \underline{F}_{\mathcal{L}} \end{aligned} \quad (\text{A.48})$$

where

- $\mathcal{FL}$ is the *direct Fourier–Laplace transformation* (also known as *direct Fourier–Borel transformation*, hereinafter *Fourier–Laplace transformation*)
- $\mathcal{FL}_{\text{inv}}$ is the *inverse Fourier–Laplace transformation* (also known as *inverse Fourier–Borel transformation*)

For the rest of the terminology, the adaptations are straightforward.

From Equation (A.44) and Equation (A.45), by using Equation (A.46), Equation (A.47), and Equation (A.48), we have

$$\begin{aligned} \underline{F}_{\mathcal{L}} &\equiv \mathcal{FL}(f_{\mathcal{L}}) \\ &:= \left(\mathcal{F}(f_{\mathcal{L}} e^{\sigma t}) \right)_{\omega + j\underline{\sigma} = \underline{s}}, \quad \begin{cases} \underline{\sigma} \in \underline{\Gamma}_{f_{\mathcal{L}}}^0 \\ \underline{s} \in \mathbb{C}_{\text{Im} \in \underline{\Gamma}_{f_{\mathcal{L}}}^0} \end{cases} \end{aligned}$$

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$$f_{\mathcal{L}} = \mathcal{FL}_{\text{inv}}(E_{\mathcal{L}})$$

$$:= \left(\mathcal{F}_{\text{inv}} \left(\left(\underline{E}_{\mathcal{L}} \right)_{\underline{s} = \omega + j\underline{\sigma}} \right) \right) e^{-\underline{\sigma}t} \quad , \quad \begin{cases} \underline{\sigma} \in \Gamma_{f_{\mathcal{L}}}^0 \\ \underline{s} \in \mathbb{C}_{\text{Im} \in \Gamma_{f_{\mathcal{L}}}^0} \end{cases}$$

The (symmetric) kernel distribution of the Fourier–Laplace transformation and of the inverse Fourier–Laplace transformation are respectively $e^{-j\underline{s}t'}$ and $(1/(2\pi j))e^{j\underline{s}t'}$ ($\underline{s}'$ being a bound complex variable of the same quantity dimension as $\underline{s}$) [Treves, 1967, Barros-Neto, 1973; Carmichael and Mitrović, 1989; Friedlander and Joshi, 1998; Carlsson and Wittsten, 2017]. When t is interpreted as time, $\underline{s}$ is interpreted as *complementary complex angular frequency*.

A.4.4 Hilbert transformation

Let $f_{\mathcal{H}}$ be a (-1) -weighted integrable distribution on $\mathbb{R}$ of the free real variable ξ . The *direct Hilbert transform* (hereinafter *Hilbert transform*) $F_{\mathcal{H}}$ of the *direct Hilbert transformand* (hereinafter *Hilbert transformand*) $f_{\mathcal{H}}$ is the tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$) of the free real variable η defined by [Schwartz, 1962, 1966; Nussenzveig, 1972; Carton-Lebrun, 1995, 2005; Pandey, 1996; Álvarez and Carton-Lebrun, 1999; Álvarez and Guzmán-Partida, 2002; Ortner, 2004, 2011; Álvarez *et al.*, 2007; Álvarez and Moyo, 2007; King, 2009a; Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013; Wagner, 2014; Garrett, 2020; Kleiner and Hilfer, 2022]

$$\begin{aligned} F_{\mathcal{H}} &\equiv \mathcal{H}(f_{\mathcal{H}}) \\ &:= -\frac{1}{\pi} \text{p.v.} \left(\frac{1}{\eta} \right) * f_{\mathcal{H}} \end{aligned} \quad (\text{A.49})$$

where

- $\mathcal{H}$ is the *direct Hilbert transformation* (hereinafter *Hilbert transformation*)

We say that the ordered pair formed by a Hilbert transformand and its Hilbert transform is a *direct Hilbert-transform pair* (hereinafter *Hilbert-transform pair*).

Let $g_{\mathcal{H}}$ be the inverse Fourier transform of $f_{\mathcal{H}}$. As an application of the generalised form of the Fourier exchange theorem, we have [Schwartz, 1962; Nussenzveig, 1972; Pandey, 1996; Ortner, 2004; King, 2009a; Garrett, 2020]

$$\mathcal{F}(\mathcal{H}(f_{\mathcal{H}})) = j \text{sgn} \cdot \mathcal{F}(f_{\mathcal{H}})$$

$$\mathcal{H}(\mathcal{F}(g_{\mathcal{H}})) = -j \mathcal{F}(\text{sgn} \cdot g_{\mathcal{H}})$$

Let $F_{\mathcal{H}}$ be a (-1) -weighted integrable distribution on $\mathbb{R}$ of the free real variable η . The *inverse Hilbert transform* $f_{\mathcal{H}}$ of the *inverse Hilbert transformand* $F_{\mathcal{H}}$ is the tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$) of the free real variable ξ defined by [Schwartz, 1962, 1966; Nussenzveig, 1972; Carton-Lebrun, 1995, 2005; Pandey, 1996; Álvarez and Carton-Lebrun, 1999; Álvarez and Guzmán-Partida, 2002; Ortner, 2004, 2011; Álvarez *et al.*, 2007; Álvarez and Moyo, 2007; King,

2009a; Guzmán-Partida *et al.*, 2011; Ortner and Wagner, 2013; Wagner, 2014; Garrett, 2020; Kleiner and Hilfer, 2022]

$$\begin{aligned} f_{\mathcal{H}} &\equiv \mathcal{H}_{\text{inv}} \left(F_{\mathcal{H}} \right) \\ &:= \frac{1}{\pi} \text{p.v.} \left(\frac{1}{\xi} \right) * F_{\mathcal{H}} \end{aligned} \quad (\text{A.50})$$

where

– $\mathcal{H}_{\text{inv}}$ is the *inverse Hilbert transformation*

We say that the ordered pair formed by an inverse Hilbert transformand and its inverse Hilbert transform is an *inverse Hilbert-transform pair*.

Let $g_{\mathcal{H}}$ be the inverse Fourier transform of $f_{\mathcal{H}}$. As an application of the generalised form of the convolution Fourier exchange theorem, we have [Schwartz, 1962; Nussenzveig, 1972; Pandey, 1996; Ortner, 2004; King, 2009a; Garrett, 2020]

$$\begin{aligned} \mathcal{F} \left(\mathcal{H}_{\text{inv}} \left(f_{\mathcal{H}} \right) \right) &= -j \text{sgn} \cdot \mathcal{F} \left(f_{\mathcal{H}} \right) \\ \mathcal{H}_{\text{inv}} \left(\mathcal{F} \left(g_{\mathcal{H}} \right) \right) &= j \mathcal{F} \left(\text{sgn} \cdot g_{\mathcal{H}} \right) \end{aligned}$$

The following properties hold [King, 2009a]:

- The Hilbert transform (respectively, inverse Hilbert transform) of a real-valued (-1) -weighted integrable distribution on $\mathbb{R}$ is a real-valued tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$).
- The Hilbert transform (respectively, inverse Hilbert transform) of an imaginary-valued (-1) -weighted integrable distribution on $\mathbb{R}$ is an imaginary-valued tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$).

- The Hilbert transform (respectively, inverse Hilbert transform) of an even (-1) -weighted integrable distribution on $\mathbb{R}$ is an odd tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$).
- The Hilbert transform (respectively, inverse Hilbert transform) of an odd (-1) -weighted integrable distribution on $\mathbb{R}$ is an even tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$).
- The Hilbert transform (respectively, inverse Hilbert transform) of a conjugate-symmetric (-1) -weighted integrable distribution on $\mathbb{R}$ is a conjugate-antisymmetric tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$).
- The Hilbert transform (respectively, inverse Hilbert transform) of a conjugate-antisymmetric (-1) -weighted integrable distribution on $\mathbb{R}$ is a conjugate-symmetric tempered distribution on $\mathbb{R}$ (in fact, (-1) -weighted bounded distribution on $\mathbb{R}$ as well as $(-1, -1)$ -weighted integrable distribution on $\mathbb{R}$).

For both the Hilbert transformation and the inverse Hilbert transformation, the largest possible domain is $\mathcal{D}'_{L^1, -1}(\mathbb{R})$ [Schwartz, 1962; Nussenzveig, 1972; Carton-Lebrun, 1995, 2005; Álvarez and Carton-Lebrun, 1999; Álvarez and Guzmán-Partida, 2002; Guzmán-Partida *et al.*, 2011; Kleiner and Hilfer, 2022]. Accordingly, we say that a (-1) -weighted integrable distribution on $\mathbb{R}$ is a *Hilbert-transformable distribution* on $\mathbb{R}$. We mention that the Hilbert transformation and the inverse Hilbert transformation can be generalised to have a larger domain than $\mathcal{D}'_{L^1, -1}(\mathbb{R})$, but such generalised transformations cannot be interpreted in terms of convolution and may determine non-unique results [Fefferman and Stein, 1972; Carton-Lebrun, 1995, 2005; Estrada and Kanwal, 2000] or require resorting to mathematical entities that are more general than distributions [Gelfand and Shilov, 1968; Pandey, 1982, 1996; King, 2009a]. We have [Ortner and Wagner, 2013]

$$\left(\mathcal{D}'_{L^{\infty}, -1}(\mathbb{R}) \cap \mathcal{D}'_{L^1, -1, -1}(\mathbb{R})\right) \supset \mathcal{D}'_{L^1, -1}(\mathbb{R})$$

Therefore, it is evident that the Hilbert transformation and the inverse Hilbert transformation cannot be iterated unrestrictedly. The largest possible domain such that the Hilbert transformation and the inverse Hilbert transformation can be iterated unrestrictedly is $\mathcal{D}'_{L^1, -1, +1}(\mathbb{R})$ [Yang, 2008;

A.4 Signals: Integral transformations

Kleiner and Hilfer, 2022]; for $p_{[1,+\infty)}$ in $\mathbb{R}_{[1,+\infty)}$, we have [Ortner and Wagner, 2013; Kleiner and Hilfer, 2022]

$$\mathcal{D}'_{L^1, -1}(\mathbb{R}) \supset \mathcal{D}'_{L^1, -1, +1}(\mathbb{R}) \supset \mathcal{D}'_{L^p[1,+\infty)}(\mathbb{R})$$

In particular, an element of $\mathcal{D}'_{L^1, -1}(\mathbb{R})$ is an element of $\mathcal{D}'_{L^1, -1, +1}(\mathbb{R})$ if the localisation of its Fourier transform at a neighbourhood of the origin results in a regular distribution thereon [Garrett, 2020].

Within $\mathcal{D}'_{L^1, -1, +1}(\mathbb{R})$, $f_{\mathcal{H}}$ coincides with $f_{\mathcal{H}}$ if and only if $F_{\mathcal{H}}$ coincides with $F_{\mathcal{H}}$ [Carton-Lebrun, 1995, 2005; Pandey, 1996; Ortner, 2004, 2011; King, 2009a; Garrett, 2020].

Within $\mathcal{D}'_{L^1, -1, +1}(\mathbb{R})$, by letting $f_{\mathcal{H}} \equiv f_{\mathcal{H}}$ and $F_{\mathcal{H}} \equiv F_{\mathcal{H}}$, Equation (A.49) and Equation (A.50) become

$$\begin{aligned} F_{\mathcal{H}} &\equiv \mathcal{H}(f_{\mathcal{H}}) \\ &:= -\frac{1}{\pi} \text{p.v.} \left(\frac{1}{\eta} \right) * f_{\mathcal{H}} \end{aligned}$$

$$\begin{aligned} f_{\mathcal{H}} &\equiv \mathcal{H}_{\text{inv}}(F_{\mathcal{H}}) \\ &:= \frac{1}{\pi} \text{p.v.} \left(\frac{1}{\xi} \right) * F_{\mathcal{H}} \end{aligned}$$

The (asymmetric) kernel distribution of the Hilbert transformation and of the inverse Hilbert transformation are respectively $-(1/\pi) \text{p.v.}(1/(\eta - \xi'))$ and $(1/\pi) \text{p.v.}(1/(\xi - \eta'))$ (η' being a bound real variable of the same quantity dimension as η) [Schwartz, 1952; King, 2009a, 2009b]. When ξ is interpreted as time (respectively, angular frequency), so is η ; accordingly, in applications, it is customary to denote both ξ and η by t (respectively, ω).

Within $\mathcal{D}'_{L^1, -1, +1}(\mathbb{R})$, we have [Carton-Lebrun, 1995, 2005; Pandey, 1996; Ortner, 2004, 2011; King, 2009a; Garrett, 2020]

$$\begin{aligned} \mathcal{H}_{\text{inv}} \circ \mathcal{H} &= \mathcal{H} \circ \mathcal{H}_{\text{inv}} \\ &= \mathcal{C} \circ \mathcal{C} \\ &= \mathcal{R} \circ \mathcal{R} \\ &= \mathcal{P} \circ \mathcal{P} \\ &= \mathcal{I} \end{aligned} \tag{A.51}$$

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$$\left\{ \begin{array}{l} \mathcal{C} \circ \mathcal{H} = \mathcal{H} \circ \mathcal{C} \\ \quad \quad \quad = -\mathcal{H}_{\text{inv}} \circ \mathcal{C} \\ \mathcal{R} \circ \mathcal{H} = -\mathcal{H} \circ \mathcal{R} \\ \quad \quad \quad = \mathcal{H}_{\text{inv}} \circ \mathcal{R} \\ \mathcal{P} \circ \mathcal{H} = -\mathcal{H} \circ \mathcal{P} \\ \quad \quad \quad = \mathcal{H}_{\text{inv}} \circ \mathcal{P} \end{array} \right.$$

$$\left\{ \begin{array}{l} \mathcal{C} \circ \mathcal{H}_{\text{inv}} = \mathcal{H}_{\text{inv}} \circ \mathcal{C} \\ \quad \quad \quad = -\mathcal{H} \circ \mathcal{C} \\ \mathcal{R} \circ \mathcal{H}_{\text{inv}} = -\mathcal{H}_{\text{inv}} \circ \mathcal{R} \\ \quad \quad \quad = \mathcal{H} \circ \mathcal{R} \\ \mathcal{P} \circ \mathcal{H}_{\text{inv}} = -\mathcal{H}_{\text{inv}} \circ \mathcal{P} \\ \quad \quad \quad = \mathcal{H} \circ \mathcal{P} \end{array} \right.$$

Within $\mathcal{D}'_{L^1, -1, +1}(\mathbb{R})$, from Equation (A.51), it can be easily verified that we have

$$\left\{ \begin{array}{l} \mathcal{H}^0 \equiv \mathcal{I} \\ \mathcal{H}^1 \equiv \mathcal{H} \\ \quad \quad \quad = -\mathcal{H}_{\text{inv}} \\ \mathcal{H}^2 \equiv \mathcal{H} \circ \mathcal{H} \\ \quad \quad \quad = -\mathcal{I} \\ \mathcal{H}^3 \equiv \mathcal{H} \circ \mathcal{H} \circ \mathcal{H} \\ \quad \quad \quad = -\mathcal{H} \\ \quad \quad \quad = \mathcal{H}_{\text{inv}} \\ \mathcal{H}^4 \equiv \mathcal{H} \circ \mathcal{H} \circ \mathcal{H} \circ \mathcal{H} \\ \quad \quad \quad = \mathcal{I} \end{array} \right.$$

$$\left\{ \begin{array}{l} \mathcal{H}_{\text{inv}}^0 \equiv \mathcal{I} \\ \mathcal{H}^{-1} \equiv \mathcal{H}_{\text{inv}} \\ \quad \quad \quad = -\mathcal{H} \\ \mathcal{H}^{-2} \equiv \mathcal{H}_{\text{inv}} \circ \mathcal{H}_{\text{inv}} \\ \quad \quad \quad = -\mathcal{I} \\ \mathcal{H}^{-3} \equiv \mathcal{H}_{\text{inv}} \circ \mathcal{H}_{\text{inv}} \circ \mathcal{H}_{\text{inv}} \\ \quad \quad \quad = -\mathcal{H}_{\text{inv}} \\ \quad \quad \quad = \mathcal{H} \\ \mathcal{H}^{-4} \equiv \mathcal{H}_{\text{inv}} \circ \mathcal{H}_{\text{inv}} \circ \mathcal{H}_{\text{inv}} \circ \mathcal{H}_{\text{inv}} \\ \quad \quad \quad = \mathcal{I} \end{array} \right.$$

where

- $\mathcal{H}^{n_0}$ is the n_0^{th} iterate of the Hilbert transformation

- $\mathcal{H}^{-n_0}$ is the n_0^{th} iterate of the inverse Hilbert transformation

Within $\mathcal{D}'_{L^1, -1, 1}(\mathbb{R})$, the Hilbert transformation and the inverse Hilbert transformation prove to be 4-periodic. Specifically, for any $z_1 \in \mathbb{Z}$ and $z_2 \in \mathbb{Z}$, we have

$$\mathcal{H}^{z_1+4z_2} = \mathcal{H}^{z_1}$$

The Hilbert transform (respectively, inverse Hilbert transform) of a q -integrable distribution on $\mathbb{R}$ is a q -integrable distribution on $\mathbb{R}$ [Schwartz, 1966; Pandey, 1996; Ortner, 2004, 2011; King, 2009a; Guzmán-Partida *et al.*, 2011]; in particular, the Hilbert transform (respectively, inverse Hilbert transform) of a distribution on $\mathbb{R}$ generated by a q -integrable function on $\mathbb{R}$ is a distribution on $\mathbb{R}$ generated by a q -integrable function on $\mathbb{R}$ [Pandey, 1996; Ortner, 2004; King, 2009a; Guzmán-Partida *et al.*, 2011]. In fact, the Hilbert transformation and the inverse Hilbert transformation establish as many automorphisms from $\mathcal{D}'_{L^q}(\mathbb{R})$ into $\mathcal{D}'_{L^q}(\mathbb{R})$ (in particular, from $L^{q^*}(\mathbb{R})$ into $L^{q^*}(\mathbb{R})$), the automorphism established by one transformation having as its inverse the automorphism established by the other transformation [Schwartz, 1966; Pandey, 1996; Ortner, 2004, 2011; King, 2009a; Guzmán-Partida *et al.*, 2011]. Furthermore, the Hilbert transform (respectively, inverse Hilbert transform) of an integrable distribution on $\mathbb{R}$ (in particular, a distribution on $\mathbb{R}$ generated by an integrable function on $\mathbb{R}$) is a p -integrable distribution on $\mathbb{R}$ [Schwartz, 1966]. Finally, the Hilbert transform (respectively, inverse Hilbert transform) of a real-variable Hardy distribution of exponent 1 on $\mathbb{R}$ is a real-variable Hardy distribution of exponent 1 on $\mathbb{R}$ [Ortner, 2004]; in particular, the Hilbert transform (respectively, inverse Hilbert transform) of a distribution on $\mathbb{R}$ generated by a real-variable Hardy function of exponent 1 on $\mathbb{R}$ is a distribution on $\mathbb{R}$ generated by a real-variable Hardy function of exponent 1 on $\mathbb{R}$ [Fefferman and Stein, 1972; Ortner, 2004; Chang and Sadosky, 2006; Garnett, 2007].

A (-1) -weighted integrable distribution on $\mathbb{R}$ is said to satisfy the *Kramers–Kronig relations* (also known as *Sokhotski–Plemelj formulae* or *dispersion relations*) if and only if it is j times its own Hilbert transform or, equivalently, if and only if its real part and imaginary part form a Hilbert-transform pair (namely, if and only if its imaginary part is the Hilbert transform of its real part) or, equivalently, if and only if its imaginary part and real part form an inverse Hilbert-transform pair (namely, if and only if its real part is the inverse Hilbert transform of its imaginary part) [Schwartz, 1962; Beltrami and Wohlers, 1966; Nussenzveig, 1972; Carmichael and Mitrović, 1989; King, 2009b]. A (-1) -weighted integrable distribution on $\mathbb{R}$ has anticausal Fourier transform (respectively, causal

inverse Fourier transform) if and only if it satisfies the Kramers–Kronig relations [Schwartz, 1962; Beltrami and Wohlers, 1966; Nussenzveig, 1972; King, 2009b]; besides, a (-1) -weighted integrable distribution on $\mathbb{R}$ has causal Fourier transform (respectively, anticausal inverse Fourier transform) if and only if its reflected satisfies the Kramers–Kronig relations [Schwartz, 1962; Beltrami and Wohlers, 1966; Nussenzveig, 1972; King, 2009b]. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], an element of $\mathcal{S}'(\mathbb{R})$ whose Fourier transform is causal is an *analytic signal*. Finally, we mention that there exist *generalised Kramers–Kronig relations* (which are said to involve *subtractions*) that are well-defined for any $(-n)$ -weighted square integrable distribution on $\mathbb{R}$ [Beltrami and Wohlers, 1966; Nussenzveig, 1972].

A.5 Signals: Energy and power characterisations

A.5.1 Introduction

In the following, some information about signal energy and signal power is summarised, and then some examples are provided.

A.5.2 Signal energy

We separately discuss functions and distributions.

A.5.2.1 Functions

The cross-correlation function of an ordered pair of elements of $L^1(\mathbb{R})$ is an element of $L^1(\mathbb{R})$, and hence its Fourier transform (respectively, inverse Fourier transform) is an element of $L^2(\mathbb{R}) * L^2(\mathbb{R})$. In particular, whenever the auto-correlation function of an element of $L^1(\mathbb{R})$ is a C^0 function on $\mathbb{R}$, it is a non-negative-definite element of $L^1(\mathbb{R}) \cap \mathcal{E}^0(\mathbb{R})$ whose Fourier transform (respectively, inverse Fourier transform) is a non-negative real-valued element of $L^2(\mathbb{R}) * L^2(\mathbb{R})$ [Bass, 1984]. The cross-correlation function of an ordered pair of elements one of $L^1(\mathbb{R})$ and the other of $L^2(\mathbb{R})$ is an element of $L^2(\mathbb{R})$, and hence its Fourier transform (respectively, inverse Fourier transform) is an element of $L^2(\mathbb{R})$. The cross-correlation function of an ordered pair of elements one of $L^1(\mathbb{R})$ and the other of $L^\infty(\mathbb{R})$ is an element of $\dot{\mathcal{B}}^0(\mathbb{R})$, and hence its Fourier transform (respectively, inverse Fourier transform) is an element of $PF(\mathbb{R})$.

The cross-correlation function of an ordered pair of elements of $L^2(\mathbb{R})$ is an element of $L^\infty(\mathbb{R})$ (in fact, of $\dot{\mathcal{B}}^0(\mathbb{R})$) whose Fourier transform (respectively, inverse Fourier transform) is an element of $L^1(\mathbb{R})$ [Schwartz, 1966; Katznelson, 2004; Poletti, 2015]. In particular, the auto-correlation function of an element of $L^2(\mathbb{R})$ is a non-negative-definite element of $L^\infty(\mathbb{R})$ (in fact, of $\dot{\mathcal{B}}^0(\mathbb{R})$) whose Fourier transform (respectively, inverse Fourier transform) is a non-negative real-valued element of $L^1(\mathbb{R})$ [Schwartz, 1966; Katznelson, 2004; Poletti, 2015]; in fact, a non-negative-definite C^0 function on $\mathbb{R}$ is the auto-correlation function of some square integrable function on $\mathbb{R}$, not uniquely determined, if and only if the Fourier transform (respectively, inverse Fourier transform) of the non-negative-definite C^0 function on $\mathbb{R}$ is a non-negative real-valued element of $L^1(\mathbb{R})$ [Schwartz, 1966; Katznelson, 2004; Poletti, 2015].

In agreement with the fact that correlation is not commutative, the cross-correlation function of an ordered pair of elements one of $L^1(\mathbb{R})$ and the other of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, or $L^\infty(\mathbb{R})$, or of an ordered pair of elements of $L^2(\mathbb{R})$, coincides with the reflected complex conjugate (or, equivalently, the complex conjugate reflected) of the cross-correlation function of the reverse ordered pair of

elements. Given an ordered pair of elements one of $L^1(\mathbb{R})$ and the other of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, or $L^\infty(\mathbb{R})$ each of which translated by some translation vector, or an ordered pair of elements of $L^2(\mathbb{R})$ each of which translated by some translation vector, its cross-correlation function coincides with the translated by the difference of the first translation vector and the second translation vector of the cross-correlation function of the ordered pair of elements. In particular, given an element of $L^1(\mathbb{R})$ translated by some translation vector or an element of $L^2(\mathbb{R})$ translated by some translation vector, its auto-correlation function coincides with the auto-correlation function of the element.

Let f_{L^2} and h_{L^2} be two elements of $L^2(\mathbb{R})$, and let $h_{L^2}^*$ be the complex conjugate of h_{L^2} . The quantity expressed by

$$\begin{aligned} R_{f_{L^2}h_{L^2}} &= f_{L^2} \star h_{L^2} \\ &= f_{L^2} * \mathcal{P}(h_{L^2}) \\ &= \mathcal{P}(R_{h_{L^2}f_{L^2}}) \end{aligned} \quad (\text{A.52})$$

with

$$R_{f_{L^2}h_{L^2}}(0) = \int_{-\infty}^{+\infty} (f_{L^2} h_{L^2}^*) dt' \quad (\text{A.53})$$

where

- $\star$ is the *correlation operator*
- $R_{f_{L^2}h_{L^2}}$ is the cross-correlation function of the ordered pair $\{f_{L^2}, h_{L^2}\}$
- $R_{h_{L^2}f_{L^2}}$ is the cross-correlation function of the ordered pair $\{h_{L^2}, f_{L^2}\}$
- $\int_{-\infty}^{+\infty} (f_{L^2} h_{L^2}^*) dt'$ is the L^2 inner product of the ordered pair $\{f_{L^2}, h_{L^2}\}$

is an element of $L^2(\mathbb{R}) * L^2(\mathbb{R})$. Let F_{L^2} and H_{L^2} be the Fourier transforms of respectively f_{L^2} and h_{L^2} , and let $H_{L^2}^*$ be the complex conjugate of H_{L^2} . The quantity expressed by

$$\begin{aligned}
 S_{f_{L^2}h_{L^2}} &= \frac{1}{2\pi} \mathcal{F} \left(R_{f_{L^2}h_{L^2}} \right) \\
 &= \frac{1}{2\pi} \left(F_{L^2} \cdot H_{L^2}^* \right) \\
 &= \mathcal{C} \left(S_{h_{L^2}f_{L^2}} \right)
 \end{aligned} \tag{A.54}$$

where

- $S_{f_{L^2}h_{L^2}}$ is $1/(2\pi)$ times the Fourier transform of $R_{f_{L^2}h_{L^2}}$
- $S_{h_{L^2}f_{L^2}}$ is $1/(2\pi)$ times the Fourier transform of $R_{h_{L^2}f_{L^2}}$

is an element of $L^1(\mathbb{R})$. We have

$$R_{f_{L^2}h_{L^2}}(0) = \int_{-\infty}^{+\infty} \left(S_{f_{L^2}h_{L^2}} \right) d\omega' \tag{A.55}$$

Remarkably, Equation (A.55) can be re-expressed as the *Plancherel identity* [Wiener, 1930, 1988; Treves, 1967; Kwakernaak and Sivan, 1991; Benedetto, 1997; Katznelson, 2004]

$$\int_{-\infty}^{+\infty} \left(f_{L^2} h_{L^2}^* \right) dt' = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(F_{L^2} H_{L^2}^* \right) d\omega' \tag{A.56}$$

which is equivalent to the identity

$$\int_{-\infty}^{+\infty} \left(f_{L^2} h_{L^2} \right) dt' = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(F_{L^2} \tilde{H}_{L^2} \right) d\omega'$$

where

- $\int_{-\infty}^{+\infty} \left(F_{L^2} H_{L^2}^* \right) d\omega'$ is the L^2 inner product of the ordered pair $\{F_{L^2}, H_{L^2}\}$
- $\tilde{H}_{L^2}$ is the reflected H_{L^2}

When, with a suitable choice of λ and ν , the automorphisms established by the Fourier transformation and by the inverse Fourier transformation from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ are made unitary maps from a Hilbert space over $\mathbb{F}$ into itself, Equation (A.56) loses the multiplying factor $1/(2\pi)$ and expresses the preservation of L^2 inner product [Schwartz, 1966; Kwakernaak and Sivan, 1991]. If $R_{f_{L^2}h_{L^2}}$ is an element of $L^1(\mathbb{R})$, the homologue of Equation (A.55) holds

A.5 Signals: Energy and power characterisations

$$S_{f_{L^2}h_{L^2}}(0) = \int_{-\infty}^{+\infty} \left(R_{f_{L^2}h_{L^2}} \right) dt'$$

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $R_{f_{L^2}h_{L^2}}(0)$ is the *cross-energy* of the ordered pair $\{f_{L^2}, h_{L^2}\}$, whereas $f_{L^2}h_{L^2}^*$ and $S_{f_{L^2}h_{L^2}}$ are respectively the *instantaneous cross-power* (which may be interpreted as *cross-energy temporal density* when f_{L^2} and h_{L^2} ‘live’ in the time domain) of the ordered pair $\{f_{L^2}, h_{L^2}\}$ and the *cross-energy spectral density* of the ordered pair $\{f_{L^2}, h_{L^2}\}$.

In the special case $f_{L^2} = h_{L^2}$, in place of Equation (A.52) and Equation (A.53) we have

$$\begin{aligned} R_{f_{L^2}f_{L^2}} &= f_{L^2} \star f_{L^2} \\ &= f_{L^2} * \mathcal{P}(f_{L^2}) \end{aligned}$$

with

$$R_{f_{L^2}f_{L^2}}(0) = \int_{-\infty}^{+\infty} \left(|f_{L^2}|^2 \right) dt'$$

where

- $R_{f_{L^2}f_{L^2}}$ is the auto-correlation function of f_{L^2}
- $\int_{-\infty}^{+\infty} \left(|f_{L^2}|^2 \right) dt'$ is the square of the L^2 magnitude of f_{L^2}

in place of Equation (A.54) we have

$$\begin{aligned} S_{f_{L^2}f_{L^2}} &= \frac{1}{2\pi} \mathcal{F} \left(R_{f_{L^2}f_{L^2}} \right) \\ &= \frac{1}{2\pi} \left(|F_{L^2}|^2 \right) \end{aligned}$$

where

- $S_{f_{L^2}f_{L^2}}$ is $1/(2\pi)$ times the Fourier transform of $R_{f_{L^2}f_{L^2}}$

and in place of Equation (A.55) we have

$$R_{f_{L^2}f_{L^2}}(0) = \int_{-\infty}^{+\infty} \left(S_{f_{L^2}f_{L^2}} \right) d\omega' \quad (\text{A.57})$$

Remarkably, Equation (A.57) can be re-expressed as the *Parseval identity* [Wiener, 1930, 1988; Schwartz, 1966; Treves, 1967; Kwakernaak and Sivan, 1991; Benedetto, 1997; Katznelson, 2004]

$$\int_{-\infty}^{+\infty} \left(|f_{L^2}|^2 \right) dt' = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(|F_{L^2}|^2 \right) d\omega' \quad (\text{A.58})$$

where

$$- \int_{-\infty}^{+\infty} \left(|F_{L^2}|^2 \right) d\omega' \text{ is the square of the } L^2 \text{ magnitude of } F_{L^2}$$

When, with a suitable choice of λ and ν , the automorphisms established by the Fourier transformation and by the inverse Fourier transformation from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ are made unitary maps from a Hilbert space over $\mathbb{F}$ into itself, Equation (A.58) loses the multiplying factor $1/(2\pi)$ and expresses the preservation of L^2 magnitude [Schwartz, 1966; Kwakernaak and Sivan, 1991]. If $R_{f_{L^2}f_{L^2}}$ is an element of $L^1(\mathbb{R})$, the homologue of Equation (A.57) holds

$$S_{f_{L^2}f_{L^2}}(0) = \int_{-\infty}^{+\infty} \left(R_{f_{L^2}f_{L^2}} \right) dt'$$

By definition, the *quadratic integral* on $\mathbb{R}$ of a square integrable function on $\mathbb{R}$ is the integral on $\mathbb{R}$ of the square of its modulus, which equals the integral on $\mathbb{R}$ of $1/(2\pi)$ times the non-negative real-valued element of $L^1(\mathbb{R})$ coinciding with the Fourier transform of its auto-correlation function.

The spectrum of the cross-correlation function of an ordered pair of elements one of $L^1(\mathbb{R})$ and the other of $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, or $L^\infty(\mathbb{R})$ and the spectrum of the cross-correlation function of an ordered pair of elements of $L^2(\mathbb{R})$ is no larger than the intersection of the spectra of the two elements; accordingly, if the elements of such an ordered pair have disjoint spectra, their cross-correlation function is $0_{\mathbb{R}}$. In particular, the spectrum of the auto-correlation function of an element of $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ is no larger than the spectrum of the element.

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $R_{f_{L^2}f_{L^2}}(0)$ is the *auto-energy* of f_{L^2} , whereas $|f_{L^2}|^2$

and $S_{f_{L^2}f_{L^2}}$ are respectively the *instantaneous auto-power* (which may be interpreted as *auto-energy temporal density* when f_{L^2} ‘lives’ in the time domain) of f_{L^2} and the *auto-energy spectral density* of f_{L^2} ; accordingly, an element of $L^2(\mathbb{R})$ is a *finite-auto-energy signal* (hereinafter *finite-energy signal*).

Whenever well-defined, we introduce the quantities [Wiener, 1930; Masani, 1966; Gardner, 1992; Bendat and Piersol, 2010]

$$\begin{aligned}\rho_{f_{L^2}h_{L^2}} &= \frac{|R_{f_{L^2}h_{L^2}}|^2}{R_{f_{L^2}f_{L^2}}(0)R_{h_{L^2}h_{L^2}}(0)} \\ &= \frac{\mathcal{R}\left(|R_{h_{L^2}f_{L^2}}|^2\right)}{R_{h_{L^2}h_{L^2}}(0)R_{f_{L^2}f_{L^2}}(0)} \\ &= \mathcal{R}\left(\rho_{h_{L^2}f_{L^2}}\right)\end{aligned}\tag{A.59}$$

$$\begin{aligned}\gamma_{f_{L^2}h_{L^2}} &= \frac{|S_{f_{L^2}h_{L^2}}|^2}{S_{f_{L^2}f_{L^2}}S_{h_{L^2}h_{L^2}}} \\ &= \frac{|S_{h_{L^2}f_{L^2}}|^2}{S_{h_{L^2}h_{L^2}}S_{f_{L^2}f_{L^2}}} \\ &= \gamma_{h_{L^2}f_{L^2}}\end{aligned}\tag{A.60}$$

where

- $R_{h_{L^2}h_{L^2}}$ is the auto-correlation function of h_{L^2}
- $S_{h_{L^2}h_{L^2}}$ is $1/(2\pi)$ times the Fourier transform of $R_{h_{L^2}h_{L^2}}$

Specifically, Equation (A.59) and Equation (A.60) are well-defined if and only if their denominators do not vanish. We refer to $\rho_{f_{L^2}h_{L^2}}$ (respectively, $\rho_{h_{L^2}f_{L^2}}$) as the *correlation coefficient function* of the ordered pair $\{f_{L^2}, h_{L^2}\}$ (respectively, $\{h_{L^2}, f_{L^2}\}$) and to $\gamma_{f_{L^2}h_{L^2}}$ (respectively, $\gamma_{h_{L^2}f_{L^2}}$) as the *coherence function* of the ordered pair $\{f_{L^2}, h_{L^2}\}$ (respectively, $\{h_{L^2}, f_{L^2}\}$). As a consequence of the Cauchy–Schwarz inequality, both $\rho_{f_{L^2}h_{L^2}}$ (respectively, $\rho_{h_{L^2}f_{L^2}}$) and $\gamma_{f_{L^2}h_{L^2}}$ (respectively, $\gamma_{h_{L^2}f_{L^2}}$) take value in $\mathbb{R}_{[0,1]}$ [Bass, 1963, 1974; Masani, 1966]. On the one hand, we say that the ordered pair

$\{f_{L^2}, h_{L^2}\}$ (or, equivalently, $\{h_{L^2}, f_{L^2}\}$) is *completely uncorrelated* if and only if $\rho_{f_{L^2}h_{L^2}}$ (or, equivalently, $\rho_{h_{L^2}f_{L^2}}$) coincides with $0_{\mathbb{R}}$ and *completely correlated* if and only if $\rho_{f_{L^2}h_{L^2}}$ (or, equivalently, $\rho_{h_{L^2}f_{L^2}}$) coincides with $1_{\mathbb{R}}$; on the other hand, we say that the ordered pair $\{f_{L^2}, h_{L^2}\}$ (or, equivalently, $\{h_{L^2}, f_{L^2}\}$) is *completely incoherent* if and only if $\gamma_{f_{L^2}h_{L^2}}$ (or, equivalently, $\gamma_{h_{L^2}f_{L^2}}$) coincides with $0_{\mathbb{R}}$ and *completely coherent* if and only if $\gamma_{f_{L^2}h_{L^2}}$ (or, equivalently, $\gamma_{h_{L^2}f_{L^2}}$) coincides with $1_{\mathbb{R}}$. As is evident, the ordered pair $\{f_{L^2}, h_{L^2}\}$ (or, equivalently, $\{h_{L^2}, f_{L^2}\}$) is completely uncorrelated if and only if it is completely incoherent. Any ordered pair of elements of $L^2(\mathbb{R})$ having disjoint spectra is completely uncorrelated or, equivalently, completely incoherent, because its cross-correlation function is $0_{\mathbb{R}}$. No ordered pair of elements of $L^2(\mathbb{R})$ is completely correlated, because its cross-correlation function is an element of $L^\infty(\mathbb{R})$ (in fact, of $\mathcal{B}^0(\mathbb{R})$). As seen below, any ordered pair of elements of $L^2(\mathbb{R})$ formed by the input variable and the output variable of a convolution system is completely coherent.

A.5.2.2 Distributions

The cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_{L^1}(\mathbb{R})$ is an element of $\mathcal{D}'_{L^1}(\mathbb{R})$, and hence its Fourier transform (respectively, inverse Fourier transform) is generable by a C^0 function of slow growth on $\mathbb{R}$. In particular, the auto-correlation distribution of an element of $\mathcal{D}'_{L^1}(\mathbb{R})$ is a non-negative-definite element of $\mathcal{D}'_{L^1}(\mathbb{R})$ whose Fourier transform (respectively, inverse Fourier transform) is generable by a non-negative real-valued C^0 function of slow growth on $\mathbb{R}$. The cross-correlation distribution of an ordered pair of elements one of $\mathcal{D}'_{L^1}(\mathbb{R})$ and the other of $\mathcal{D}'_{L^2}(\mathbb{R})$ is an element of $\mathcal{D}'_{L^2}(\mathbb{R})$, and hence its Fourier transform (respectively, inverse Fourier transform) is generable by an element of $\mathcal{P}_{L^2}(\mathbb{R})$. The cross-correlation distribution of an ordered pair of elements one of $\mathcal{D}'_{L^1}(\mathbb{R})$ and the other of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ is an element of $\mathcal{D}'_{L^\infty}(\mathbb{R})$, and hence its Fourier transform (respectively, inverse Fourier transform) is the sum of a distribution on $\mathbb{R}$ generated by an element of $\mathcal{P}_{L^2}(\mathbb{R})$ and the multiplicative product of a polynomial function on $\mathbb{R}$ by an element of $PM(\mathbb{R})$.

The cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ is an element of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (in fact, of $\dot{\mathcal{B}}'(\mathbb{R})$) whose Fourier transform (respectively, inverse Fourier transform) is generable by an element of $\mathcal{P}_{L^1}(\mathbb{R})$ [Schwartz, 1966; Poletti, 2015]. In particular, the auto-correlation distribution of an element of $\mathcal{D}'_{L^2}(\mathbb{R})$ is a non-negative-definite element of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (in fact, of $\dot{\mathcal{B}}'(\mathbb{R})$) whose Fourier transform (respectively, inverse Fourier transform) is generable by a non-negative real-valued element of $\mathcal{P}_{L^1}(\mathbb{R})$ [Schwarz, 1966]; in fact, a non-negative-definite distribution on $\mathbb{R}$ is the auto-correlation distribution of some square integrable distribution on $\mathbb{R}$, not uniquely determined, if and only if the Fourier transform (respectively, inverse Fourier transform) of the non-negative-definite distribution on $\mathbb{R}$ is generable by a non-negative real-valued element of $\mathcal{P}_{L^1}(\mathbb{R})$ [Schwartz, 1966].

In agreement with the fact that correlation is not commutative, the cross-correlation distribution of an ordered pair of elements one of $\mathcal{D}'_{L^1}(\mathbb{R})$ and the other of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (or, *a fortiori*, $\mathcal{D}'_{L^1}(\mathbb{R})$ or $\mathcal{D}'_{L^2}(\mathbb{R})$), or of an ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$, coincides with the reflected complex conjugate (or, equivalently, the complex conjugate reflected) of the cross-correlation distribution of the reverse ordered pair of elements. Given an ordered pair of elements one of $\mathcal{D}'_{L^1}(\mathbb{R})$ and the other of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (or, *a fortiori*, $\mathcal{D}'_{L^1}(\mathbb{R})$ or $\mathcal{D}'_{L^2}(\mathbb{R})$) each of which translated by some translation vector, or an ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ each of which translated by some translation vector, its cross-correlation distribution coincides with the translated by the difference of the first translation vector and the second translation vector of the cross-correlation distribution of the ordered pair of elements. In particular, given an element of $\mathcal{D}'_{L^2}(\mathbb{R})$ (or, *a fortiori*, $\mathcal{D}'_{L^1}(\mathbb{R})$) translated by some translation vector, its auto-correlation distribution coincides with the auto-correlation distribution of the element.

By definition, the *quadratic integral* on $\mathbb{R}$ of a square integrable distribution on $\mathbb{R}$ is the integral on $\mathbb{R}$ of $1/(2\pi)$ times the non-negative real-valued element of $\mathcal{P}_{L^1}(\mathbb{R})$ generating the Fourier transform of its auto-correlation distribution. Accordingly, a distribution on $\mathbb{R}$ has finite quadratic integral on $\mathbb{R}$ if and only if it is generable by a square integrable function on $\mathbb{R}$ [Deny, 1950; Schwartz, 1966]. By convention, a square integrable distribution on $\mathbb{R}$ that is not generable by a square integrable function on $\mathbb{R}$ has quadratic integral on $\mathbb{R}$ equalling $+\infty$.

The spectrum of the cross-correlation distribution of an ordered pair of elements one of $\mathcal{D}'_{L^1}(\mathbb{R})$ and the other of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ (or, *a fortiori*, $\mathcal{D}'_{L^1}(\mathbb{R})$ or $\mathcal{D}'_{L^2}(\mathbb{R})$) and the spectrum of the cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ is no larger than the intersection of the spectra of the two elements; accordingly, if the elements of such an ordered pair have disjoint spectra, their cross-correlation distribution is $0_{\mathbb{R}}$. In particular, the spectrum of the auto-correlation distribution of an element of $\mathcal{D}'_{L^2}(\mathbb{R})$ (or, *a fortiori*, $\mathcal{D}'_{L^1}(\mathbb{R})$) is no larger than the spectrum of the element.

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $1/(2\pi)$ times the Fourier transform of the cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ is the *cross-energy spectral density* of the ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$, whereas $1/(2\pi)$ times the Fourier transform of the auto-correlation distribution of an element of $\mathcal{D}'_{L^2}(\mathbb{R})$ is its *auto-energy spectral density* of the element of $\mathcal{D}'_{L^2}(\mathbb{R})$.

Finally, we immediately generalise the concepts of correlation coefficient function and coherence function to be applicable to any ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ for which they are well-defined.

A.5.3 Signal power

We separately discuss functions and distributions.

A.5.3.1 Functions

Correlation need not be well-defined for the ordered pairs of elements of a function space other than $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$. However, an operation of the same nature can be introduced in its place for the ordered pairs of elements of some suitable function space other than $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ [Wiener, 1930, 1988; Bass, 1962, 1963, 1974, 1984, 1989; Masani, 1966, 1979; Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Benedetto, 1997; Mäkilä *et al.*, 1998; Partington, 2004; Benedetto and Czaja, 2009]. We refer to such operation as *mean correlation*, and we denote such a function space by $P(\mathbb{R})$. The mean correlation of a first element of $P(\mathbb{R})$ by a second element of $P(\mathbb{R})$ results in a function on $\mathbb{R}$ to which we refer as the *mean cross-correlation function* of the ordered pair of elements of $P(\mathbb{R})$; in particular, the mean correlation of an element of $P(\mathbb{R})$ by itself results in a function on $\mathbb{R}$ to which we refer as the *mean auto-correlation function* of the element of $P(\mathbb{R})$.

More specifically, $P(\mathbb{R})$ is any subspace of $L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$ that is invariant under translation and is such that the mean correlation of any ordered pair of elements is well-defined and results in a C^0 function on $\mathbb{R}$ [Wiener, 1930, 1988; Bass, 1962, 1963, 1974, 1984, 1989; Masani, 1966, 1979; Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Mäkilä *et al.*, 1998; Partington, 2004]; remarkably, such a function space can be canonically made a Hilbert space over $\mathbb{F}$ [Bass, 1963, 1984; Masani, 1979]. For convenience, we also assume that $P(\mathbb{R})$ is invariant under reflection. To the best of our knowledge, no satisfactory characterisation of the maximal of all the possible function spaces satisfying the aforementioned conditions is available; important examples (non-maximal, intersecting, and incomparable with respect to inclusion) are the function space of the square integrable functions on $\mathbb{R}$, the function space of the bounded functions on $\mathbb{R}$ having limit at infinity, and the function space of the uniformly almost periodic functions on $\mathbb{R}$ [Mari, 1996; Zhang, 2004]. Remarkably, the mean cross-correlation function (in particular, mean auto-correlation function) of an ordered pair of uniformly almost periodic functions on $\mathbb{R}$ (in particular, of a uniformly almost periodic function on $\mathbb{R}$) is a uniformly almost periodic function on $\mathbb{R}$, and the mean cross-correlation function (in particular, mean auto-correlation function) of an ordered pair of C^0 periodic functions on $\mathbb{R}$ (in particular, of a C^0 periodic function on $\mathbb{R}$) is a C^0 periodic function on $\mathbb{R}$ [Wiener, 1930, 1988; Bass, 1962, 1963, 1974, 1984; Pfaffelhuber, 1975; Partington,

2004]. For the ordered pairs of elements of $L^1(\mathbb{R})$ and for the ordered pairs of elements of $L^2(\mathbb{R})$, mean correlation is well-defined but of limited interest; indeed, the mean cross-correlation function (in particular, mean auto-correlation function) of an ordered pair of elements (in particular, of an element) of $L^1(\mathbb{R})$ or of an ordered pair of elements (in particular, of an element) of $L^2(\mathbb{R})$ is $0_{\mathbb{R}}$ [Wiener, 1930, 1988; Pfaffelhuber, 1975; Partington, 2004]. It is worth noting that $L^1(\mathbb{R})$ is not a subspace of $L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$ and hence is not an example of $P(\mathbb{R})$. An example of element of $L^1(\mathbb{R}) \setminus (L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R}))$ is the function defined almost everywhere on $\mathbb{R}$ by $(|\xi|)^{-1/2} \text{rect}(\xi)$. Examples of elements of $(L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})) \setminus L^1(\mathbb{R})$ are the function defined almost everywhere on $\mathbb{R}$ by $(|\xi|)^{1/4}$ and the function defined almost everywhere on $\mathbb{R}$ by $(|\ln(|\xi|)|)^{1/2}$. Examples of elements of $L^1(\mathbb{R}) \cap L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$ are all the elements of $\mathcal{D}^0(\mathbb{R})$.

In agreement with the fact that, like correlation, mean correlation is not commutative, the mean cross-correlation function of an ordered pair of elements of $P(\mathbb{R})$ coincides with the reflected complex conjugate (or, equivalently, the complex conjugate reflected) of the mean cross-correlation function of the reverse ordered pair of elements of $P(\mathbb{R})$ [Wiener, 1930; Bass, 1974; Masani, 1979]. Given an ordered pair of elements of $P(\mathbb{R})$ each of which translated by some translation vector, its mean cross-correlation function coincides with the mean cross-correlation function, translated by the difference of the first translation vector and the second translation vector, of the ordered pair of elements of $P(\mathbb{R})$ [Masani, 1979; Bertrandias *et al.*, 1987].

Let f_p and h_p be two elements of $P(\mathbb{R})$, and let h_p^* be the complex conjugate of h_p . The quantity expressed by

$$\begin{aligned} \bar{R}_{f_p h_p} &= f_p \bar{\star} h_p \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(f_p \star \left(h_p \Pi_{-\frac{T}{2}, \frac{T}{2}} \right) \right) \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(f_p * \mathcal{P} \left(h_p \Pi_{-\frac{T}{2}, \frac{T}{2}} \right) \right) \right) \\ &= \mathcal{P}(\bar{R}_{h_p f_p}) \end{aligned} \tag{A.61}$$

with

$$\begin{aligned}\bar{R}_{f_P h_P}(0) &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} (f_P h_P^*) dt' \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{t_0 - T/2}^{t_0 + T/2} (f_P h_P^*) dt' \right)\end{aligned}\tag{A.62}$$

where

- $\star$ is the *mean-correlation operator*
- $\bar{R}_{f_P h_P}$ is the mean cross-correlation function of the ordered pair $\{f_P, h_P\}$
- $\bar{R}_{h_P f_P}$ is the mean cross-correlation function of the ordered pair $\{h_P, f_P\}$
- T is a bound positive real variable of the same quantity dimension as t
- $\Pi_{-T/2, +T/2}$ is the boxcar function with support $\mathbb{R}_{[-T/2, +T/2]}$

is a C^0 function on $\mathbb{R}$ [Wiener, 1930, 1988; Bass, 1962, 1963, 1974, 1984, 1989; Masani, 1966, 1979; Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Benedetto, 1997; Mäkilä *et al.*, 1998; Partington, 2004; Benedetto and Czaja, 2009]. It is worth noting that we have

$$\begin{aligned}\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(f_P \star \left(h_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) \right) \right) &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(\left(f_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) \star h_P \right) \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(\left(f_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) \star \left(h_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) \right) \right)\end{aligned}\tag{A.63}$$

$$\begin{aligned}\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(f_P * \mathcal{P} \left(h_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) \right) \right) &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(\left(f_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) * \mathcal{P}(h_P) \right) \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(\left(f_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) * \mathcal{P} \left(h_P \Pi_{-\frac{T}{2}, +\frac{T}{2}} \right) \right) \right)\end{aligned}\tag{A.64}$$

but none of the equalities in Equation (A.63) and Equation (A.64) need hold before the limits are taken (and hence with T free as opposed to bound).

If $1_{\mathbb{R}}$ is an element of $P(\mathbb{R})$ (as is the case when $P(\mathbb{R})$ coincides with the function space of the bounded functions on $\mathbb{R}$ having limit at infinity or with the function space of the uniformly almost periodic functions on $\mathbb{R}$), the quantity expressed by

$$\begin{aligned}\bar{R}_{f_P 1_{\mathbb{R}}}(0) &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} (f_P) dt' \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} (f_P) dt' \right)\end{aligned}$$

where

- $\bar{R}_{f_P 1_{\mathbb{R}}}$ is the mean cross-correlation function of the ordered pair $\{f_P, 1_{\mathbb{R}}\}$

exists and is finite [Bass, 1962, 1963, 1974, 1984, 1989; Masani, 1979; Bertrandias *et al.*, 1987; Partington, 2004]. By definition, $\bar{R}_{f_P 1_{\mathbb{R}}}(0)$ is the *mean value* on $\mathbb{R}$ of f_P . We immediately generalise the concept of mean value on $\mathbb{R}$ to be applicable to any function on $\mathbb{R}$ for which such value at the origin exists and is finite; in particular, an element of $L^1(\mathbb{R})$ has null mean value on $\mathbb{R}$ [Bass, 1984]. The function space of the elements of $L^1_{\text{loc}}(\mathbb{R})$ whose mean value on $\mathbb{R}$ exists and is finite can be equipped with the algebraic structure of linear space over $\mathbb{F}$ [Bass, 1974, 1984].

Unlike the Fourier transform (respectively, inverse Fourier transform) of an element of $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ and the Fourier transform (respectively, inverse Fourier transform) of the cross-correlation function of an ordered pair of elements of $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, the Fourier transform (respectively, inverse Fourier transform) of an element of $P(\mathbb{R})$ and the Fourier transform (respectively, inverse Fourier transform) of the mean cross-correlation function of an ordered pair of elements of $P(\mathbb{R})$ need not be well-defined as functions on $\mathbb{R}$ [Wiener, 1930, 1988; Bass, 1962, 1963, 1974, 1984, 1989; Masani, 1966, 1979; Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Benedetto, 1997; Mäkilä *et al.*, 1998; Partington, 2004; Benedetto and Czaja, 2009]. Nevertheless, if the quantity expressed by

$$\begin{aligned}\bar{S}_{f_P h_P} &= \frac{1}{2\pi} \mathcal{F}(\bar{R}_{f_P h_P}) \\ &= \mathcal{C}(\bar{S}_{h_P f_P})\end{aligned}$$

where

- $\bar{S}_{f_P h_P}$ is $1/(2\pi)$ times the Fourier transform of $\bar{R}_{f_P h_P}$
- $\bar{S}_{h_P f_P}$ is $1/(2\pi)$ times the Fourier transform of $\bar{R}_{h_P f_P}$

is well-defined as a function on $\mathbb{R}$ (which implies that it is an element of $L^1(\mathbb{R})$) [Bass, 1963, 1974, 1984]), we have

$$\bar{R}_{f_p h_p}(0) = \int_{-\infty}^{+\infty} (\bar{S}_{f_p h_p}) d\omega' \quad (\text{A.65})$$

If $\bar{R}_{f_p h_p}$ is an element of $L^1(\mathbb{R})$, the homologue of Equation (A.65) holds

$$\bar{S}_{f_p h_p}(0) = \int_{-\infty}^{+\infty} (\bar{R}_{f_p h_p}) dt'$$

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $\bar{R}_{f_p h_p}(0)$ is the *mean cross-power* (hereinafter *cross-power*) of the ordered pair $\{f_p, h_p\}$, whereas $f_p \cdot h_p^*$ and $\bar{S}_{f_p h_p}$ are respectively the *instantaneous cross-power* (which may be interpreted as *cross-energy temporal density* when f_p and h_p ‘live’ in the time domain) of the ordered pair $\{f_p, h_p\}$ and the *mean-cross-power spectral density* (hereinafter *cross-power spectral density*) of the ordered pair $\{f_p, h_p\}$.

In the special case $f_p = h_p$, in place of Equation (A.61) and Equation (A.62) we have

$$\begin{aligned} \bar{R}_{f_p f_p} &= f_p \star f_p \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(f_p \star \left(f_p \Pi_{\frac{T}{2}, \frac{T}{2}} \right) \right) \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \left(f_p * \mathcal{P} \left(f_p \Pi_{\frac{T}{2}, \frac{T}{2}} \right) \right) \right) \end{aligned}$$

with

$$\begin{aligned} \bar{R}_{f_p f_p}(0) &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} (|f_p|^2) dt' \right) \\ &= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{t_0 - T/2}^{t_0 + T/2} (|f_p|^2) dt' \right) \end{aligned}$$

where

- $\bar{R}_{f_p f_p}$ is the mean auto-correlation function of f_p

By definition, $\left(\bar{R}_{f_P f_P}(0)\right)^{1/2}$ is the *root-mean-square value* on $\mathbb{R}$ of f_P . We immediately generalise the concept of root-mean-square value on $\mathbb{R}$ to be applicable to any function on $\mathbb{R}$ for which such value at the origin exists and is finite; in particular, an element of $L^2(\mathbb{R})$ has null root-mean-square value on $\mathbb{R}$ [Bass, 1984]. The function space of the elements of $L_{-1}^2(\mathbb{R}) \cap L_{\text{loc}}^2(\mathbb{R})$ whose root-mean-square value on $\mathbb{R}$ exists and is finite and its subspace formed by the elements of $L_{-1}^2(\mathbb{R}) \cap L_{\text{loc}}^2(\mathbb{R})$ whose mean auto-correlation functions are well-defined as functions on $\mathbb{R}$ (in particular, C^0 functions on $\mathbb{R}$) are not invariant under addition [Bass, 1962, 1963, 1974, 1984; Masani, 1966, 1979; Bertrandias *et al.*, 1987; Mari, 1996; Partington, 2004; Zhang, 2004]; hence, unlike their common subspace $P(\mathbb{R})$, they have the serious pathology that they cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$ (least of all can they be canonically made Hilbert spaces over $\mathbb{F}$) [Bass, 1962, 1963, 1974, 1984; Masani, 1966, 1979; Bertrandias *et al.*, 1987; Mari, 1996; Partington, 2004; Zhang, 2004]. However, the function space of the elements of $L_{-1}^2(\mathbb{R}) \cap L_{\text{loc}}^2(\mathbb{R})$ with null root-mean-square value on $\mathbb{R}$ can be equipped with the algebraic structure of linear space over $\mathbb{F}$ [Bass, 1984]; in fact, such function space (which is a superspace of $L^2(\mathbb{R})$) is as an example of $P(\mathbb{R})$ [Bass, 1984]. Ultimately, $P(\mathbb{R})$ plays in essence the role of a well-behaved (albeit difficult to characterise in full generality) function space of functions on $\mathbb{R}$ whose root-mean-square value on $\mathbb{R}$ exists and is finite. Examples of elements of $L_{-1}^2(\mathbb{R}) \cap L_{\text{loc}}^2(\mathbb{R})$ whose root-mean-square values on $\mathbb{R}$ do not exist are the function defined almost everywhere on $\mathbb{R}$ by $(|\xi|)^{1/4}$ and the function defined almost everywhere on $\mathbb{R}$ by $(|\ln(|\xi|)|)^{1/2}$ [Bass, 1984]. An example of element of $L_{-1}^2(\mathbb{R}) \cap L_{\text{loc}}^2(\mathbb{R})$ whose root-mean-square value on $\mathbb{R}$ exists and is finite but whose mean auto-correlation function is not well defined as a function on $\mathbb{R}$ is the function defined almost everywhere on $\mathbb{R}$ by $\xi^{j\xi} \mathfrak{G}(\xi)$ [Bass, 1963, 1974, 1984]. The standard example of element of $L_{-1}^2(\mathbb{R}) \cap L_{\text{loc}}^2(\mathbb{R})$ whose mean auto-correlation function, although not well-defined as a function on $\mathbb{R}$ (let alone a C^0 function on $\mathbb{R}$), is well-defined as a pointwise function on $\mathbb{R}$ (equalling 1 at the origin and 0 on $\mathbb{R}_{(-\infty, 0) \cup (0, +\infty)}$) is the function defined almost everywhere on $\mathbb{R}$ by $e^{j\xi^2}$ [Wiener, 1930, 1988; Bass, 1963, 1974, 1984; Masani, 1966, 1979; Benedetto, 1983, 1997].

Unlike the Fourier transform (respectively, inverse Fourier transform) of an element of $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ and the Fourier transform (respectively, inverse Fourier transform) of the auto-correlation function of an element of $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, the Fourier transform (respectively, inverse Fourier

transform) of an element of $P(\mathbb{R})$ and the Fourier transform (respectively, inverse Fourier transform) of the mean auto-correlation function of an element of $P(\mathbb{R})$ need not be well-defined as functions on $\mathbb{R}$ [Wiener, 1930, 1988; Pfaffelhuber, 1975]. Nevertheless, if the quantity expressed by

$$\bar{S}_{f_P f_P} = \frac{1}{2\pi} \mathcal{F}(\bar{R}_{f_P f_P})$$

where

$$- \quad \bar{S}_{f_P f_P} \text{ is } 1/(2\pi) \text{ times the Fourier transform of } \bar{R}_{f_P f_P}$$

is well-defined as a function on $\mathbb{R}$ (which implies that it is an element of $L^1(\mathbb{R})$) [Bass, 1963, 1974, 1984]), in place of Equation (A.65) we have

$$\bar{R}_{f_P f_P}(0) = \int_{-\infty}^{+\infty} (\bar{S}_{f_P f_P}) d\omega' \quad (\text{A.66})$$

If $\bar{R}_{f_P f_P}$ is an element of $L^1(\mathbb{R})$, the homologue of Equation (A.66) holds

$$\bar{S}_{f_P f_P}(0) = \int_{-\infty}^{+\infty} (\bar{R}_{f_P f_P}) dt'$$

Whenever the involved Fourier transforms are well-defined as functions on $\mathbb{R}$, the spectrum of the mean cross-correlation function of an ordered pair of elements of $P(\mathbb{R})$ (in particular of the mean auto-correlation function of an element of $P(\mathbb{R})$) is no larger than the intersection of the spectra of the two elements of $P(\mathbb{R})$ (in particular, than the spectrum of the element of $P(\mathbb{R})$) [Pfaffelhuber, 1975]; accordingly, the mean cross-correlation function of an ordered pair of elements of $P(\mathbb{R})$ having disjoint spectra is $0_{\mathbb{R}}$.

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $\bar{R}_{f_P f_P}(0)$ is the *mean auto-power* (hereinafter *auto-power*) of f_P , whereas $|f_P|^2$ and $\bar{S}_{f_P f_P}$ are respectively the *instantaneous auto-power* (which may be interpreted as *auto-energy temporal density* when f_P 'lives' in the time domain) of f_P and the *mean-auto-power spectral density* (hereinafter *auto-power spectral density*) of f_P ; accordingly,

an element of $L^1_{\text{loc}}(\mathbb{R})$ (in particular, of $L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$) whose root-mean-square value on $\mathbb{R}$ exists and is finite is a *finite-mean-auto-power signal* (hereinafter *finite-power signal*).

Whenever well-defined, we introduce the quantities [Wiener, 1930; Masani, 1966; Gardner, 1992; Bendat and Piersol, 2010]

$$\begin{aligned}\bar{\rho}_{f_P h_P} &= \frac{|\bar{R}_{f_P h_P}|^2}{\bar{R}_{f_P f_P}(0) \bar{R}_{h_P h_P}(0)} \\ &= \frac{\mathcal{R}\left(|\bar{R}_{h_P f_P}|^2\right)}{\bar{R}_{h_P h_P}(0) \bar{R}_{f_P f_P}(0)} \\ &= \mathcal{R}\left(\bar{\rho}_{h_P f_P}\right)\end{aligned}\tag{A.67}$$

$$\begin{aligned}\bar{\gamma}_{f_P h_P} &= \frac{|\bar{S}_{f_P h_P}|^2}{\bar{S}_{f_P f_P} \bar{S}_{h_P h_P}} \\ &= \frac{|\bar{S}_{h_P f_P}|^2}{\bar{S}_{h_P h_P} \bar{S}_{f_P f_P}} \\ &= \bar{\gamma}_{h_P f_P}\end{aligned}\tag{A.68}$$

where

- $\bar{R}_{h_P h_P}$ is the mean auto-correlation function of h_P
- $\bar{S}_{h_P h_P}$ is $1/(2\pi)$ times the Fourier transform of $\bar{R}_{h_P h_P}$

Specifically, Equation (A.67) and Equation (A.68) are well-defined only if their denominators do not vanish. We refer to $\bar{\rho}_{f_P h_P}$ (respectively, $\bar{\rho}_{h_P f_P}$) as the *mean correlation coefficient function* of the ordered pair $\{f_P, h_P\}$ (respectively, $\{h_P, f_P\}$) and to $\bar{\gamma}_{f_P h_P}$ (respectively, $\bar{\gamma}_{h_P f_P}$) as the *mean coherence function* of the ordered pair $\{f_P, h_P\}$ (respectively, $\{h_P, f_P\}$). As a consequence of the Cauchy–Schwarz inequality, both $\bar{\rho}_{f_P h_P}$ (respectively, $\bar{\rho}_{h_P f_P}$) and $\bar{\gamma}_{f_P h_P}$ (respectively, $\bar{\gamma}_{h_P f_P}$) take value in $\mathbb{R}_{[0,1]}$ [Bass, 1963, 1974; Masani, 1966]. On the one hand, we say that the ordered pair $\{f_P, h_P\}$ (or, equivalently, $\{h_P, f_P\}$) is *completely mean uncorrelated* if and only if $\bar{\rho}_{f_P h_P}$ (or, equivalently, $\bar{\rho}_{h_P f_P}$) coincides with $0_{\mathbb{R}}$ and *completely mean correlated* if and only if $\bar{\rho}_{f_P h_P}$ (or, equivalently, $\bar{\rho}_{h_P f_P}$) coincides with $1_{\mathbb{R}}$; on the other hand, we say that the ordered pair $\{f_P, h_P\}$ (or, equivalently, $\{h_P, f_P\}$) is *completely mean incoherent* if and only if $\bar{\gamma}_{f_P h_P}$ (or, equivalently, $\bar{\gamma}_{h_P f_P}$) coincides with

$0_{\mathbb{R}}$ and *completely mean coherent* if and only if $\overline{\gamma}_{f_P h_P}$ (or, equivalently, $\overline{\gamma}_{h_P f_P}$) coincides with $1_{\mathbb{R}}$. As is evident, the ordered pair $\{f_P, h_P\}$ (or, equivalently, $\{h_P, f_P\}$) is completely mean uncorrelated if and only if it is completely mean incoherent. Any ordered pair of elements of $P(\mathbb{R})$ having disjoint spectra is completely mean uncorrelated or, equivalently, completely mean incoherent, because its mean cross-correlation function is $0_{\mathbb{R}}$. Any ordered pair of elements of $P(\mathbb{R})$ formed by constant functions on $\mathbb{R}$ is completely mean correlated. As seen below, any ordered pair of elements of $P(\mathbb{R})$ formed by the input variable and the output variable of a convolution system is completely mean coherent.

We say that an element of $L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$ is a *stationary function* on $\mathbb{R}$ if and only if its mean auto-correlation function is well-defined as a C^0 function on $\mathbb{R}$ [Bass, 1974, 1984; Bertrandias *et al.*, 1987]; an element of $L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$ is a stationary function on $\mathbb{R}$ if and only if its convolution by any element of $\mathcal{D}(\mathbb{R})$ (in fact, by any element of $\mathcal{S}(\mathbb{R})$) results in a stationary function on $\mathbb{R}$ [Bertrandias *et al.*, 1987]. In particular, we say that a stationary function on $\mathbb{R}$ is an *amenable stationary function* on $\mathbb{R}$ if and only if its mean value on $\mathbb{R}$ exists and is finite [Bass, 1974, 1984; Bertrandias *et al.*, 1987]. We qualify an amenable stationary function on $\mathbb{R}$ as either *centred* or *non-centred* depending on whether it has null or non-null mean value on $\mathbb{R}$. The mean value on $\mathbb{R}$ of the mean auto-correlation function of an amenable stationary function on $\mathbb{R}$ exists and is finite, and it is greater than or equal to the square of the mean value on $\mathbb{R}$ of the amenable stationary function on $\mathbb{R}$ [Bass, 1974, 1984]; accordingly, an amenable stationary function on $\mathbb{R}$ has null mean value on $\mathbb{R}$ if its mean auto-correlation function has null mean value on $\mathbb{R}$ [Bass, 1984]. The function space of the (possibly amenable) stationary functions on $\mathbb{R}$ is not invariant under addition [Bass, 1974]; hence, it has the serious pathology that it cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$ (least of all can it be canonically made a Hilbert space over $\mathbb{F}$) [Bass, 1974].

We say that a (necessarily centred) amenable stationary function on $\mathbb{R}$ is a *centred pseudorandom function* on $\mathbb{R}$ if and only if its mean auto-correlation function has Fourier transform (respectively, inverse Fourier transform) well-defined as a function on $\mathbb{R}$ (and hence its mean auto-correlation function is an element of $\dot{\mathcal{B}}^0(\mathbb{R})$ and has not only null mean value on $\mathbb{R}$ but also null root-mean-square value on $\mathbb{R}$) [Bass, 1962, 1963, 1974, 1984, 1989; Masani, 1979; Bertrandias *et al.*, 1987]. We say that a non-centred amenable stationary function on $\mathbb{R}$ is a *non-centred pseudorandom function* on $\mathbb{R}$ if and only if the difference of it and its mean value on $\mathbb{R}$ is a centred

pseudorandom function on $\mathbb{R}$ [Bass, 1974; Masani, 1979]. We say that an amenable stationary function on $\mathbb{R}$ is a *pseudorandom function* on $\mathbb{R}$ if and only if it is either a centred pseudorandom function on $\mathbb{R}$ or a non-centred pseudorandom function on $\mathbb{R}$ (or, equivalently, if and only if it is the sum of a centred pseudorandom function on $\mathbb{R}$ and a constant function on $\mathbb{R}$) [Bass, 1974; Masani, 1979]. A right-sided (respectively, left-sided) real-valued centred pseudorandom function on $\mathbb{R}$ whose mean auto-correlation function is not $0_{\mathbb{R}}$ has no limit at negative infinity (respectively, positive infinity) and, on each neighbourhood of negative infinity (respectively, positive infinity), either is unbounded or changes sign infinitely many times [Bass, 1962, 1974, 1984; Masani, 1979]. It is worth noting that what justifies qualifying a centred pseudorandom function on $\mathbb{R}$ as pseudorandom is precisely the fact that it has mean auto-correlation function vanishing at infinity (which, in loose terms, means that such a function and a translated of it by some translation vector are little ‘statistically correlated’ whenever the translation vector has sufficiently large magnitude) [Bass, 1984]; in fact, the stronger condition that the mean auto-correlation function of a centred pseudorandom function on $\mathbb{R}$ has Fourier transform (respectively, inverse Fourier transform) well-defined as a function on $\mathbb{R}$ is required only for convenience [Masani, 1979]. The function space of the pseudorandom functions on $\mathbb{R}$ is not invariant under addition [Bass, 1963, 1974]; hence, it has the serious pathology that it cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$ (least of all can it be canonically made a Hilbert space over $\mathbb{F}$) [Bass, 1963, 1974].

The mean auto-correlation function of a stationary function on $\mathbb{R}$ (in particular, of an element of $P(\mathbb{R})$) is a non-negative-definite C^0 function on $\mathbb{R}$ [Wiener, 1930, 1988; Bass, 1963, 1974, 1984, 1989; Masani, 1966, 1979; Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Partington, 2004]; accordingly, whenever well-defined as a function on $\mathbb{R}$, its Fourier transform is a non-negative real-valued integrable function on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Schwartz, 1966; Vladimirov, 2002; Katznelson, 2004; Schilling *et al.*, 2012]. Inversely, according to the *Wiener–Wintner theorem*, a non-negative-definite C^0 function on $\mathbb{R}$ is the mean auto-correlation function of a stationary function on $\mathbb{R}$ (in fact, of a locally bounded function on $\mathbb{R}$ defined up to translation by any translation vector and up to modification on a bounded subset of $\mathbb{R}$ so as to coincide thereon with an element of $L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})$) [Bass, 1984, 1989; Benedetto, 1997].

A.5.3.2 Distributions

Correlation need not be well-defined for the ordered pairs of elements of a distribution space other than $\mathcal{D}'_1(\mathbb{R})$ or $\mathcal{D}'_2(\mathbb{R})$. However, an operation of the same nature can be introduced in its place

for the ordered pairs of elements of some suitable distribution space other than $\mathcal{D}'_{L^1}(\mathbb{R})$ or $\mathcal{D}'_{L^2}(\mathbb{R})$ [Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Bass, 1989]. We refer to such operation as *mean correlation*, and we denote such a distribution space by $\mathcal{D}'_p(\mathbb{R})$. The mean correlation of a first element of $\mathcal{D}'_p(\mathbb{R})$ by a second element of $\mathcal{D}'_p(\mathbb{R})$ results in a distribution on $\mathbb{R}$ to which we refer as the *mean cross-correlation distribution* of the ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$; in particular, the mean correlation of an element of $\mathcal{D}'_p(\mathbb{R})$ by itself results in a distribution on $\mathbb{R}$ to which we refer as the *mean auto-correlation distribution* of the element of $\mathcal{D}'_p(\mathbb{R})$.

More specifically, $\mathcal{D}'_p(\mathbb{R})$ is any subspace of $\mathcal{D}'_{L^2,-1}(\mathbb{R})$ that is invariant under translation and is such that the mean correlation of any ordered pair of elements is well-defined and results in a tempered distribution on $\mathbb{R}$ [Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Bass, 1989]. For convenience, we also assume that $\mathcal{D}'_p(\mathbb{R})$ is invariant under reflection. The elements of any $P(\mathbb{R})$ generate elements of some $\mathcal{D}'_p(\mathbb{R})$ [Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Bass, 1989]. We denote by $P^\wedge(\mathbb{R})$ the distribution space of the distributions on $\mathbb{R}$ generated by the elements of $P(\mathbb{R})$. To the best of our knowledge, no satisfactory characterisation of the maximal of all the distribution spaces satisfying the aforementioned conditions is available; important examples (non-maximal, intersecting, and incomparable with respect to inclusion) are the distribution space of the square integrable distributions on $\mathbb{R}$ and the distribution space of the uniformly almost periodic distributions on $\mathbb{R}$ [Pfaffelhuber, 1975; Bertrandias *et al.*, 1987]. Remarkably, the mean cross-correlation distribution (in particular, mean auto-correlation distribution) of an ordered pair of uniformly almost periodic distributions on $\mathbb{R}$ (in particular, of a uniformly almost periodic distribution on $\mathbb{R}$) is a uniformly almost periodic distribution on $\mathbb{R}$, and the mean cross-correlation distribution (in particular, mean auto-correlation distribution) of an ordered pair of periodic distributions on $\mathbb{R}$ (in particular, of a periodic distribution on $\mathbb{R}$) is a periodic distribution on $\mathbb{R}$ [Pfaffelhuber, 1975; Bertrandias *et al.*, 1987]. For the ordered pairs of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ (or, *a fortiori*, $\mathcal{D}'_{L^1}(\mathbb{R})$), mean correlation is well-defined but of limited interest; indeed, the mean cross-correlation distribution (in particular, mean auto-correlation distribution) of an ordered pair of elements (in particular, of an element) of $\mathcal{D}'_{L^2}(\mathbb{R})$ (or, *a fortiori*, $\mathcal{D}'_{L^1}(\mathbb{R})$) is $0_{\mathbb{R}}$ [Pfaffelhuber, 1975].

In agreement with the fact that, like correlation, mean correlation is not commutative, the mean cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ coincides with the reflected complex conjugate (or, equivalently, the complex conjugate reflected) of the mean cross-correlation

distribution of the reverse ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$. Given an ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ each of which translated by some translation vector, its mean cross-correlation distribution coincides with the mean cross-correlation distribution, translated by the difference of the first translation vector and the second translation vector, of the ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$.

If $1_{\mathbb{R}}$ is an element of $\mathcal{D}'_p(\mathbb{R})$ (as is the case when $\mathcal{D}'_p(\mathbb{R})$ coincides with the distribution space of the uniformly almost periodic distributions on $\mathbb{R}$), it makes sense to consider the value at the origin of the mean cross-correlation distribution of the ordered pair formed by the element of $\mathcal{D}'_p(\mathbb{R})$ and $1_{\mathbb{R}}$. By definition, whenever it exists, such value at the origin is the *mean value* on $\mathbb{R}$ of the element of $\mathcal{D}'_p(\mathbb{R})$. We immediately generalise the concept of mean value on $\mathbb{R}$ to be applicable to any distribution on $\mathbb{R}$ for which such value at the origin exists and is finite; in particular, an element of $\mathcal{D}'_L(\mathbb{R})$ has null mean value on $\mathbb{R}$.

It makes sense to consider the value at the origin of the mean auto-correlation distribution of an element of $\mathcal{D}'_p(\mathbb{R})$. By definition, whenever it exists, the square root of such value at the origin is the *root-mean-square value* on $\mathbb{R}$ of the element of $\mathcal{D}'_p(\mathbb{R})$. We immediately generalise the concept of root-mean-square value on $\mathbb{R}$ to be applicable to any distribution on $\mathbb{R}$ for which such value at the origin exists and is finite; in particular, an element of $\mathcal{D}'_L(\mathbb{R})$ has null root-mean-square value on $\mathbb{R}$.

The spectrum of the mean cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ (in particular of the mean auto-correlation distribution of an element of $\mathcal{D}'_p(\mathbb{R})$) is no larger than the intersection of the spectra of the two elements of $\mathcal{D}'_p(\mathbb{R})$ (in particular, than the spectrum of the element of $\mathcal{D}'_p(\mathbb{R})$) [Pfaffelhuber, 1975]; accordingly, the mean cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ having disjoint spectra is $0_{\mathbb{R}}$.

We immediately generalise the concepts of mean correlation coefficient function and mean coherence function to be applicable to any ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ for which they are well-defined.

We say that an element of $\mathcal{D}'_{L^2, -1}(\mathbb{R})$ is a *stationary distribution* on $\mathbb{R}$ if and only if its mean auto-correlation distribution is well-defined as a tempered distribution on $\mathbb{R}$ [Bertrandias *et al.*,

1987]; an element of $\mathcal{D}'_{L^2, -1}(\mathbb{R})$ is a stationary distribution on $\mathbb{R}$ if and only if its convolution by any element of $\mathcal{D}^\wedge(\mathbb{R})$ (in fact, by any element of $\mathcal{S}^\wedge(\mathbb{R})$) results in a distribution on $\mathbb{R}$ generated by a stationary function on $\mathbb{R}$ [Bertrandias *et al.*, 1987]. Each stationary function on $\mathbb{R}$ generates a stationary distribution on $\mathbb{R}$ [Bertrandias *et al.*, 1987]. In particular, we say that a stationary distribution on $\mathbb{R}$ is an *amenable stationary distribution* on $\mathbb{R}$ if and only if its mean value on $\mathbb{R}$ exists and is finite. We qualify an amenable stationary distribution on $\mathbb{R}$ as either *centred* or *non-centred* depending on whether it has null or non-null mean value on $\mathbb{R}$. Each (possibly centred or non-centred) amenable stationary function on $\mathbb{R}$ generates a (respectively centred or non-centred) amenable stationary distribution on $\mathbb{R}$. The distribution space of the (possibly amenable) stationary distributions on $\mathbb{R}$ is not invariant under addition [Pfaffelhuber, 1975]; hence, it has the serious pathology that it cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$ (least of all can it be canonically made a Hilbert space over $\mathbb{F}$).

We say that a (necessarily centred) amenable stationary distribution on $\mathbb{R}$ is a *centred pseudorandom distribution* on $\mathbb{R}$ if and only if its convolution by any element of $\mathcal{D}^\wedge(\mathbb{R})$ (in fact, by any element of $\mathcal{S}^\wedge(\mathbb{R})$) results in a distribution on $\mathbb{R}$ generated by a centred pseudorandom function on $\mathbb{R}$ [Bass, 1989]. We say that a non-centred amenable stationary distribution on $\mathbb{R}$ is a *non-centred pseudorandom distribution* on $\mathbb{R}$ if and only if the difference of it and its mean value on $\mathbb{R}$ is a centred pseudorandom distribution on $\mathbb{R}$. We say that amenable stationary distribution on $\mathbb{R}$ is a *pseudorandom distribution* on $\mathbb{R}$ if and only if it is either a centred pseudorandom distribution on $\mathbb{R}$ or a non-centred pseudorandom distribution on $\mathbb{R}$ (or, equivalently, if and only if it is the sum of a centred pseudorandom distribution on $\mathbb{R}$ and a constant distribution on $\mathbb{R}$). Each (possibly centred or non-centred) pseudorandom function on $\mathbb{R}$ generates a (respectively centred or non-centred) pseudorandom distribution on $\mathbb{R}$. The distribution space of the pseudorandom distributions on $\mathbb{R}$ is not invariant under addition; hence, it has the serious pathology that it cannot be equipped with the algebraic structure of linear space over $\mathbb{F}$ (least of all can it be canonically made a Hilbert space over $\mathbb{F}$).

The mean auto-correlation distribution of a stationary distribution on $\mathbb{R}$ (in particular, of an element of $\mathcal{D}'_p(\mathbb{R})$) is a non-negative-definite distribution on $\mathbb{R}$ [Pfaffelhuber, 1975; Bertrandias *et al.*, 1987]; accordingly, its Fourier transform is a non-negative real-valued $(-n)$ -weighted integrable distribution of $\mathcal{S}$ -order 0 on $\mathbb{R}$ [Gelfand and Vilenkin, 1964; Zemanian, 1965b; Schwartz, 1966; Vladimirov, 2002; Katznelson, 2004; Schilling *et al.*, 2012]. Inversely, to the best of our knowledge, the problem of establishing whether the Wiener–Wintner theorem can be generalised

to state that a non-negative-definite distribution on $\mathbb{R}$ is the mean auto-correlation distribution of a stationary distribution on $\mathbb{R}$ is still unsolved [Bertrandias *et al.*, 1987].

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $1/(2\pi)$ times the Fourier transform of the mean cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ is the *mean-cross-power spectral density* (hereinafter *cross-power spectral density*) of the ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$, whereas $1/(2\pi)$ times the Fourier transform of the mean auto-correlation distribution of an element of $\mathcal{D}'_p(\mathbb{R})$ is the *mean-auto-power spectral density* (hereinafter *auto-power spectral density*) of the element of $\mathcal{D}'_p(\mathbb{R})$.

A.5.4 Examples

The distribution δ is an element of $\mathcal{D}'_L(\mathbb{R})$ whose auto-correlation distribution is δ . Accordingly, $1/(2\pi)$ times the Fourier transform of such auto-correlation distribution is well-defined as a regular tempered distribution on $\mathbb{R}$ (namely, $(1/(2\pi))1_{\mathbb{R}}$). Because δ is an element of $\mathcal{D}'_L(\mathbb{R})$, its mean auto-correlation distribution is $0_{\mathbb{R}}$. Accordingly, $1/(2\pi)$ times the Fourier transform of such mean auto-correlation distribution is $0_{\mathbb{R}}$. More generally, we refer to a square integrable distribution on $\mathbb{R}$ whose auto-correlation distribution is a positive real constant times δ (in particular, is δ) as an *energy white noise* on $\mathbb{R}$ (in particular, a *unit energy white noise* on $\mathbb{R}$).

Let ω_g be a non-null real constant of the same quantity dimension as ω . The function defined almost everywhere on $\mathbb{R}$ by $e^{j(\omega_g t)^2}$ is an element of $(L^2_{-1}(\mathbb{R}) \cap L^2_{\text{loc}}(\mathbb{R})) \setminus L^2(\mathbb{R})$ whose auto-correlation function is not well-defined as a function on $\mathbb{R}$ (let alone a C^0 function on $\mathbb{R}$) but whose mean auto-correlation function, although not well-defined as a function on $\mathbb{R}$ (let alone a C^0 function on $\mathbb{R}$), is well-defined as a pointwise function on $\mathbb{R}$ (equalling 1 at the origin and 0 on $\mathbb{R}_{(-\infty,0) \cup (0,+\infty)}$) [Wiener, 1930, 1988; Bass, 1963, 1974, 1984; Masani, 1966, 1979; Benedetto, 1983, 1997]. Accordingly, neither $1/(2\pi)$ times the Fourier transform of such auto-correlation function nor $1/(2\pi)$ times the Fourier transform of such mean auto-correlation function is well-defined as a function on $\mathbb{R}$. However, an interesting heuristic interpretation can be provided. Indeed, the mean auto-correlation function of the function defined almost everywhere on $\mathbb{R}$ by $e^{j(\omega_g t)^2}$ can be regarded as the limit of *sinc* dilated by a dilation coefficient of the same quantity dimension as ω in $\mathbb{R}_{(0,+\infty)}$ as the dilation coefficient tends to $+\infty$. Accordingly, $1/(2\pi)$ times the Fourier transform of such mean auto-correlation function can be regarded as the limit of the multiplicative inverse of twice the dilation coefficient times *rect* dilated by the multiplicative inverse of twice the dilation coefficient as the dilation coefficient tends to $+\infty$. To the latter limit, which is denoted by ε , reference is made as *epsilon functional* on $\mathbb{R}$ (or, somehow unfortunately, *epsilon distribution* on $\mathbb{R}$) [Vallée, 1994; Beffa, 2024]. Although ε can be regarded as the continuous linear form under which the image of a test function is its mean value on $\mathbb{R}$, it does not qualify as a distribution on $\mathbb{R}$ (because its continuity relies on equipping its domain with a non-canonical topological structure) [Vallée, 1994; Beffa, 2024]. Nevertheless, in accordance with the fact that the image of any test function of compact support on $\mathbb{R}$ under ε is null (as it is under $0_{\mathbb{R}}$), a support in $\overline{\mathbb{R}} \setminus \mathbb{R} \equiv \{-\infty, +\infty\}$

can be attributed to ε [Masani, 1966; Benedetto, 1983, 1997]. It is worth noting that ε is to some extent dual to δ , which we know to be the continuous linear form under which the image of a test function is its value at the origin and which has support in the origin. Indeed, δ can be obtained as the limit within $\mathcal{S}'(\mathbb{R})$ of the distribution on $\mathbb{R}$ generated by the multiplicative inverse of a dilation coefficient of the same quantity dimension as t in $\mathbb{R}_{(0,+\infty)}$ times rect dilated by the multiplicative inverse of the dilation coefficient as the dilation coefficient tends to 0^+ . Accordingly, the Fourier transform of δ (namely, $1_{\mathbb{R}}$) can be obtained as the limit within $\mathcal{S}'(\mathbb{R})$ of the tempered distribution on $\mathbb{R}$ generated by sinc dilated by half the dilation coefficient as the dilation coefficient tends to 0^+ .

The distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{j(\omega_{\theta}t)^2}$ is a regular element of $\mathcal{D}'_2(\mathbb{R}) \setminus L^{\infty}(\mathbb{R})$ whose auto-correlation distribution is well-defined as a singular tempered distribution on $\mathbb{R}$ (namely, $(\pi/\omega_{\theta}^2)\delta$). Accordingly, $1/(2\pi)$ times the Fourier transform of such auto-correlation distribution is well-defined as a regular tempered distribution on $\mathbb{R}$ (namely, $(\pi/\omega_{\theta}^2)1_{\mathbb{R}}$). Because the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{j(\omega_{\theta}t)^2}$ is an element of $\mathcal{D}'_2(\mathbb{R})$, its mean auto-correlation distribution is $0_{\mathbb{R}}$. Accordingly, $1/(2\pi)$ times the Fourier transform of such mean auto-correlation distribution is $0_{\mathbb{R}}$.

The distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{j|\omega_0 t|}$ (which coincides with $1_{\mathbb{R}}$ when ω_0 equals 0) is an element of $L^{\infty}(\mathbb{R})$ whose mean auto-correlation distribution is the distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $\cos(\omega_0 t)$ (which coincides with $1_{\mathbb{R}}$ when ω_0 equals 0) [Benedetto, 1997]. Accordingly, $1/(2\pi)$ times the Fourier transform of such mean auto-correlation distribution is $(\delta_{-\omega_0} + \delta_{+\omega_0})/2$ (which coincides with δ when ω_0 equals 0), $\delta_{-\omega_0}$ and $\delta_{+\omega_0}$ being respectively the $-\omega_0$ -translated δ and the $+\omega_0$ -translated δ .

The distribution on $\mathbb{R}$ generated by the function defined almost everywhere on $\mathbb{R}$ by $e^{j(|\omega_0 t|)^{1/2}}$ (which coincides with $1_{\mathbb{R}}$ when ω_0 equals 0) is an element of $L^{\infty}(\mathbb{R})$ whose mean auto-correlation distribution is $1_{\mathbb{R}}$ [Wiener, 1930, 1988; Benedetto, 1997]. Accordingly, $1/(2\pi)$ times the Fourier transform of such mean auto-correlation distribution is δ .

The distribution expressed by

A.5 Signals: Energy and power characterisations

$$T_S e^{j(\omega_\theta t)^2} \text{III}_{T_S} = T_S \lim_{R \rightarrow +\infty} \left(\sum_{r=-R}^{+R} \left(e^{j(\omega_\theta t_r)^2} \delta_{t_r} \right) \right)$$

with

$$t_r = rT_S \quad , \quad r \in \mathbb{Z}$$

$$\frac{(\omega_\theta T_S)^2}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$$

is an element of $\mathcal{D}'_{L^2}(\mathbb{R})$ whose mean auto-correlation distribution is δ [Pfaffelhuber, 1975].

Accordingly, $1/(2\pi)$ times the Fourier transform of such mean auto-correlation distribution is

$(1/(2\pi))1_{\mathbb{R}}$. More generally, we refer to a stationary distribution on $\mathbb{R}$ whose mean

auto-correlation distribution is a positive real constant times δ (in particular, is δ) as a *power white noise* on $\mathbb{R}$ (in particular, a *unit power white noise* on $\mathbb{R}$) [Masani, 1966; Pfaffelhuber, 1975; Benedetto, 1983; Partington, 2004].

A.6 Systems: Generalities

A.6.1 Introduction

Hereinafter, we adopt the *immittance formalism*, which is appropriate whenever it is possible to distinguish between *cause* (namely, *input variable*, or just *input*) and *effect* (namely, *output variable*, or just *output*) of a system [Newcomb, 1963a, 1964a; Zemanian, 1965a, 1968a, 1972; Beltrami and Wohlers, 1966; Wohlers, 1969; Anderson and Newcomb, 1976; Vladimirov, 2002]; we mention that the alternative *scattering formalism* can be adopted in situations that are more general [Newcomb, 1963a, 1964a; Zemanian, 1968a, 1972; Beltrami and Wohlers, 1966; Wohlers, 1969; Anderson and Newcomb, 1976; Vladimirov, 2002]. The *input space* and the *output space* of a system are respectively the domain and the codomain of the map characterising the system [Poletti, 2015; Ciampa and Poletti, 2021a]; we refer to such map as the *input–output representation* of the system (and, whenever appropriate, we qualify it as *time-domain*, *frequency-domain*, or *complex-frequency-domain*). Therefore, for a system, the input space is a distribution space containing each admissible input variable and no other elements, whereas the output space is a distribution space containing each corresponding output variable and possibly some other elements; in the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], the input variable and the output variable of a system are respectively its *excitation* and its *response*. We say that two systems are *equivalent* if and only if they have the same input space and the same output space as well as the same output variable whenever they have the same input variable. We say that a system is a *trivial system* if and only if its output variable coincides with $0_{\mathbb{R}}$ for any input variable. A system is said to have the property of *realness* (also known as *reality*) if and only if its output variable is real-valued whenever its input variable is real-valued [Zemanian, 1965a; Vladimirov, 2002; Beffa, 2024].

In the following, some information about classification, interconnection, causality, passivity, stability, and invertibility is summarised.

A.6.2 Classification

A *kernel system* is a system whose output variable is the integral transform of the input variable according to an integral transformation associated with a fixed distribution on $\mathbb{R}^2$ to which we refer as *kernel distribution* [Schwartz, 1952, 1954d, 1954e, 1955, 1957, 1958, 1966; Grothendieck, 1955; Hirata and Shiraishi, 1960; Gelfand and Vilenkin, 1964; Newcomb, 1963a, 1964a; Treves, 1967; Zemanian, 1968a, 1972; Meidan, 1972a, 1972b; Anderson and Newcomb, 1976; Hörmander, 1990; Friedlander and Joshi, 1998; Duistermaat and Kolk, 2010; Ciampa and Poletti, 2021b; Beffa, 2024]. As is evident, a kernel system is completely characterised by its input space, output space, and kernel distribution; in particular, a kernel system is a trivial system if (but, in general, not only if) its kernel distribution is the null distribution on $\mathbb{R}^2$. A kernel system has the property of realness if and only if its kernel distribution is real-valued [Meidan, 1972b; Kwakernaak and Sivan, 1991]. When certain conditions on input space and output space are satisfied (as happens apart from some pathological cases), a system is a kernel system if and only if it has the properties of single-valuedness, continuity, and linearity [Schwartz, 1952, 1957, 1958; Treves, 1967; Zemanian, 1968a; Ciampa and Poletti, 2021b]; in particular, the aforementioned conditions are satisfied when the input space is any of $\mathcal{D}'(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, $\mathcal{D}'_{L^p}(\mathbb{R})$ (including $\mathcal{D}'_{L^\infty}(\mathbb{R})$, provided that it is equipped with a certain topological structure), $\dot{\mathcal{B}}'(\mathbb{R})$, $\mathcal{O}'_C(\mathbb{R})$, $\mathcal{O}'_M(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$ and the output space is $\mathcal{D}'(\mathbb{R})$ or when the input space is any of $\mathcal{D}'_R(\mathbb{R})$, $\mathcal{D}'_L(\mathbb{R})$, $\mathcal{S}'_R(\mathbb{R})$, or $\mathcal{S}'_L(\mathbb{R})$ and the output space is respectively $\mathcal{D}'_R(\mathbb{R})$, $\mathcal{D}'_L(\mathbb{R})$, $\mathcal{S}'_R(\mathbb{R})$, or $\mathcal{S}'_L(\mathbb{R})$ [Schwartz, 1952, 1957, 1958, 1966]. Hereinafter, unless otherwise stated, a kernel system has non-pathological input space and output space (which, amongst other things, implies that both its input space and its output space contain $\mathcal{D}^\wedge(\mathbb{R})$); accordingly, it has the properties of single-valuedness, continuity, and linearity, and its input–output representation is a morphism from the input space into the output space. Examples of distribution spaces not containing $\mathcal{D}^\wedge(\mathbb{R})$ are the distribution space of the uniformly almost periodic distributions on $\mathbb{R}$ and the distribution space of the distributions on $\mathbb{R}$ generated by smooth functions of very slow growth of exponential type on $\mathbb{R}$ (namely, the distribution space of the tempered distributions on $\mathbb{R}$ having compact spectra) [Ciampa and Poletti, 2021a]. For a kernel system admitting δ_{ξ_0} as an input variable for each ξ_0 in $\mathbb{R}$, the result of the localisation of the kernel distribution at the straight line of $\mathbb{R}^2$ at which the bound real variable of the integral transformation equals ξ_0 can be regarded as the *translated unit-impulse response* (namely, as the output variable corresponding to an input variable coinciding with δ_{ξ_0}), δ_{ξ_0} and the translated unit-impulse response being regarded

as respectively a distribution-valued smooth function on $\mathbb{R}$ of the free real variable ξ_0 and a distribution-valued C^0 function on $\mathbb{R}$ of the free real variable ξ_0 [Schwartz, 1952, 1955, 1957, 1958; Łojasiewicz, 1958; Hirata and Shiraishi, 1960; Newcomb, 1963b; Zemanian, 1965a; Meidan, 1972a, 1972b; Kwakernaak and Sivan, 1991; Ortner and Wagner, 2015; Ciampa and Poletti, 2021b]. As a consequence of the properties of continuity and linearity, the input–output representation of a kernel system is a locally bounded map; in the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], a kernel system whose input space and output space are normed linear spaces over $\mathbb{F}$ is a *finite-gain system*, whereas the magnitude of the locally bounded map constituting the input–output representation of such kernel system is its *maximum gain* [Meidan, 1977; Khalil, 2002]. As a consequence of the properties of single-valuedness and linearity, the output variable of a kernel system coincides with $0_{\mathbb{R}}$ whenever the input variable coincides with $0_{\mathbb{R}}$; accordingly, in the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], the output variable need be interpreted as *forced* (or *zero-state*) as opposed to *free* (or *zero-input*).

A *convolution system* is a kernel system whose output variable is the convolutive product of the input variable by a fixed distribution on $\mathbb{R}$ to which we refer as *convolution kernel distribution* [Schwartz, 1962, 1966; Newcomb, 1963a, 1964a; Zemanian, 1965a, 1968a, 1972; Anderson and Newcomb, 1976; Vladimirov, 2002; Corduneanu, 2009; Poletti, 2015; Ciampa and Poletti, 2021b; Beffa, 2024]. The convolution kernel distribution of a convolution system (having free real variable t_C and endowed with a fictitious bound real variable t'_C) and the kernel distribution of the same convolution system regarded as a kernel system (having free real variable t_K and bound real variable t'_K) can be obtained from each other with the real-valued affine change of free real variable and bound real variable [Albertoni and Cugiani, 1951, 1953; Schwartz, 1952, 1957, 1958]

$$\begin{cases} t_K = t_C + t'_C \\ t'_K = t'_C \end{cases} \Leftrightarrow \begin{cases} t_C = t_K - t'_K \\ t'_C = t'_K \end{cases}$$

As is evident, a convolution system is completely characterised by its input space, output space, and convolution kernel distribution; in particular, a convolution system is a trivial system if (but, in general, not only if) its convolution kernel distribution is $0_{\mathbb{R}}$. A convolution system has the property of realness if and only if its convolution kernel distribution is real-valued [Zemanian, 1965a; Beffa, 2024]. When certain conditions on input space and output space are satisfied (as happens apart from some pathological cases), a system is a convolution system if and only if it has the properties of

single-valuedness, continuity, linearity, and translation-invariance (also known as shift-invariance), which, amongst other things, implies that both the input space and the output space are invariant under translation [Schwartz, 1962; Zemanian, 1965a, 1968a; Ciampa *et al.*, 2006, 2008; Poletti, 2015; Ciampa and Poletti, 2021a]; in particular, the aforementioned conditions are satisfied when the input space is any of $\mathcal{D}'(\mathbb{R})$, $\mathcal{S}'(\mathbb{R})$, $\mathcal{D}'_{L^p}(\mathbb{R})$ (including $\mathcal{D}'_{L^\infty}(\mathbb{R})$, provided that it is equipped with a certain topological structure), $\mathcal{B}'(\mathbb{R})$, $\mathcal{O}'_C(\mathbb{R})$, $\mathcal{O}'_M(\mathbb{R})$, or $\mathcal{E}'(\mathbb{R})$ and the output space is $\mathcal{D}'(\mathbb{R})$ or when the input space is any of $\mathcal{D}'_R(\mathbb{R})$, $\mathcal{D}'_L(\mathbb{R})$, $\mathcal{S}'_R(\mathbb{R})$, or $\mathcal{S}'_L(\mathbb{R})$ and the output space is respectively $\mathcal{D}'_R(\mathbb{R})$, $\mathcal{D}'_L(\mathbb{R})$, $\mathcal{S}'_R(\mathbb{R})$, or $\mathcal{S}'_L(\mathbb{R})$ [Schwartz, 1966; Poletti, 2005; Ciampa *et al.*, 2006, 2008]. Hereinafter, unless otherwise stated, a convolution system has non-pathological input space and output space (which, amongst other things, implies that both its input space and its output space contain $\mathcal{D}^\wedge(\mathbb{R})$); accordingly, it has the properties of single-valuedness, continuity, linearity, and translation-invariance, and its input–output representation is a morphism from the input space into the output space. For a convolution system admitting δ as an input variable, the convolution kernel distribution can be regarded as the *unit-impulse response* (namely, as the output variable corresponding to an input variable coinciding with δ) [Schwartz, 1962; Newcomb, 1963b; Zemanian, 1965a, 1968a; Ciampa *et al.*, 2006, 2008; Poletti, 2015; Ciampa and Poletti, 2021b]. Whenever they exist, we refer to the Fourier transform and to the Laplace transform (respectively, Fourier–Laplace transform) of the convolution kernel distribution of a convolution system as respectively the *frequency-response distribution* and the *transfer function* (respectively, *complementary transfer function*) of the convolution system. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], the spectrum of the convolution kernel distribution of a convolution system is the *band* of the convolution system. In particular, whenever the convolution kernel distribution of a convolution system has both a Fourier transform and a Laplace transform (respectively, Fourier–Laplace transform), they are respectively a tempered distribution on $\mathbb{R}$ and a holomorphic function on a subset of $\mathbb{C}$ including or bounded by $\mathbb{C}_{\text{Re}=0}$ (respectively, $\mathbb{C}_{\text{Im}=0}$), the interior of such subset of $\mathbb{C}$ being the region of definition of the Laplace transform, and the holomorphic function on such subset of $\mathbb{C}$ satisfying a certain majorisation condition [Zemanian, 1965a; Beltrami and Wohlers, 1966; Schwartz, 1966; Wohlers, 1969]. Remarkably, whenever the input space of a convolution system includes the bounded distributions on $\mathbb{R}$ generated by the functions defined almost everywhere on $\mathbb{R}$ by $e^{j\omega_0 t}$ for each ω_0 in $\mathbb{R}$, the frequency-response distribution is a smooth function on $\mathbb{R}$ such that the multiplicative product of its value at ω_0 by $e^{j\omega_0 t}$ coincides with the output variable corresponding to the input variable $e^{j\omega_0 t}$ [Poletti, 2015].

A *differential system* is a system whose output variable is related to the input variable by a *constant-coefficient linear non-homogeneous ordinary differential equation* (hereinafter *differential equation*) [Ciampa *et al.*, 2019; Ciampa, 2021]; specifically, the differential equation equates a nontrivial finite linear combination of derivatives of order in $\mathbb{N}$ of the output variable to a nontrivial finite linear combination of derivatives of order in $\mathbb{N}$ of the input variable [Ciampa *et al.*, 2019; Ciampa, 2021]. By abuse of language, we say that the input space and the output space of a differential system are also the input space and the output space of the differential equation associated with the differential system. There is no one-to-one correspondence between the set of the differential equations and the set of the differential systems; indeed, although a differential system is by definition associated with one and only one differential equation, a differential equation can be associated with no, one, or more than one differential system [Ciampa *et al.*, 2019; Ciampa, 2021]. Nevertheless, when equipped with a suitable existence and uniqueness condition, a differential equation (possibly satisfying some requirement on the multiplicity of the imaginary-valued solutions of its associated homogeneous equation) defines a unique differential system (such differential system being a convolution system with input space and output space determined by the existence and uniqueness condition) [Ciampa *et al.*, 2019; Ciampa, 2021]. We mention some remarkable existence and uniqueness conditions. The *condition of initial rest* (respectively, the *condition of final rest*), which applies to any differential equation with input space $\mathcal{D}'_R(\mathbb{R})$ (respectively, $\mathcal{D}'_L(\mathbb{R})$), prescribes that the output space coincides with $\mathcal{D}'_R(\mathbb{R})$ (respectively, $\mathcal{D}'_L(\mathbb{R})$) [Zemanian, 1965a; Ciampa *et al.*, 2019; Ciampa, 2021]. The *condition of boundedness*, which applies to any differential equation with input space $\mathcal{D}'_{L^\infty}(\mathbb{R})$ whose associated homogeneous equation has no imaginary-valued solution, prescribes that the output space coincides with $\mathcal{D}'_{L^\infty}(\mathbb{R})$ [Ciampa *et al.*, 2019; Ciampa, 2021]. The *condition of vanishing at infinity*, which applies to any differential equation with input space $\dot{\mathcal{B}}'(\mathbb{R})$ whose associated homogeneous equation has no imaginary-valued solution, prescribes that the output space coincides with $\dot{\mathcal{B}}'(\mathbb{R})$ [Ciampa *et al.*, 2019; Ciampa, 2021]. The *condition of boundedness and vanishing at negative infinity* (respectively, the *condition of boundedness and vanishing at positive infinity*), which applies to any differential equation with input space $\mathcal{D}'_{L^\infty}(\mathbb{R})$ whose associated homogeneous equation has no imaginary-valued solution of multiplicity greater than 1, prescribes that the output space coincides with $\mathcal{D}'_{L^\infty}(\mathbb{R}) \cap \dot{\mathcal{B}}'_{-\infty}(\mathbb{R})$ (respectively, $\mathcal{D}'_{L^\infty}(\mathbb{R}) \cap \dot{\mathcal{B}}'_{+\infty}(\mathbb{R})$) [Ciampa *et al.*, 2019; Ciampa, 2021]. More generally, the *condition of $(-n_0)$ -weighted boundedness and vanishing at negative infinity* (respectively, the *condition of $(-n_0)$ -weighted boundedness and vanishing at positive infinity*), which

applies to any differential equation with input space $\mathcal{D}'_{L',n_0}(\mathbb{R})$ whose associated homogeneous equation has no imaginary-valued solution of multiplicity greater than $n_0 + 1$, prescribes that the output space coincides with $\mathcal{D}'_{L',-n_0}(\mathbb{R}) \cap \dot{\mathcal{B}}'_{-\infty}(\mathbb{R})$ (respectively, $\mathcal{D}'_{L',-n_0}(\mathbb{R}) \cap \dot{\mathcal{B}}'_{+\infty}(\mathbb{R})$) [Ortner and Wagner, 2013]. An *initially-at-rest differential system* (respectively, a *finally-at-rest differential system*) is a differential system whose associated differential equation is equipped with the condition of initial rest (respectively, final rest), such differential system being a convolution system with input space and output space coinciding with $\mathcal{D}'_R(\mathbb{R})$ (respectively, $\mathcal{D}'_L(\mathbb{R})$) [Zemanian, 1965a; Schwartz, 1966; Kwakernaak and Sivan, 1991; Poletti, 2005; Ciampa *et al.*, 2019; Ciampa, 2021]. The convolution kernel distribution of an initially-at-rest (respectively, finally-at-rest) differential system is a nontrivial finite linear combination of derivatives of order in $\mathbb{N}$ of the right-sided (respectively, left-sided) convolutive inverse of a nontrivial finite linear combination of derivatives of order in $\mathbb{N}$ of δ [Schwartz, 1966; Poletti, 2005; Ciampa *et al.*, 2019; Ciampa, 2021].

A system whose input space contains $\mathcal{D}^\wedge(\mathbb{R})$ has the property of *reciprocity* if, for each ordered pair of input variables in $\mathcal{D}^\wedge(\mathbb{R})$ (to which corresponds an ordered pair of output variables in $\mathcal{D}'(\mathbb{R})$), the convolutive product of the first input variable and the second output variable coincides with the convolutive product of the second input variable and the first output variable [Newcomb, 1963a, 1964a; Anderson and Newcomb, 1965, 1967; Wohlers, 1969]. A system whose input space contains $\mathcal{D}^\wedge(\mathbb{R})$ has the property of reciprocity if it has the properties of single-valuedness and linearity (in particular, if it is a kernel system) [Newcomb, 1963a]. The *norator* (namely, the system having arbitrary and independent input variable and output variable) is an example of system having the properties of both linearity and translation-invariance but having the properties of neither single-valuedness nor reciprocity [Newcomb, 1963a; Carlin, 1964].

Systems not having the property of single-valuedness can be said to have the property of *multi-valuedness* [Zemanian, 1965a; 1968a]. A system having the property of multi-valuedness is not a kernel system, let alone a convolution system; however, under appropriate conditions, an *augmented system* having the property of single-valuedness can be associated with it, such augmented system being possibly a kernel system or even a convolution system [Newcomb, 1963a, 1964a; Zemanian, 1968a; Anderson and Newcomb, 1976].

A.6.3 Interconnection

Let there be given two systems such that the output space of one (to which we refer as *upstream system*) is included in the input space of the other (to which we refer as *downstream system*). The two systems are said to be in *series connection* if and only if there exists a system (to which we refer as *series-equivalent system*) whose input space and output space are respectively the input space of the upstream system and the output space of the downstream system and whose input–output representation is the composite of the input–output representations of the two systems [Kwakernaak and Sivan, 1991]. Let there be given two systems for which the output space of any one coincides with the input space of the other. Because either of the two systems can act both as upstream system and as downstream system, there may exist two different series-equivalent systems [Oppenheim *et al.*, 1997; Sandberg, 2000; King, 2009b]; if and only if the two series-equivalent systems are equivalent, we say that the series connection of the two systems is *commutative*. In particular, it can be easily verified that the series connection of two initially-at-rest (respectively, finally-at-rest) differential systems having right-sided (respectively, left-sided) convolution kernel distributions is commutative. Under certain conditions, the series-equivalent system of two kernel systems in series connection is a kernel system whose kernel distribution coincides with the Volterra composition product of the kernel distributions of the two kernel systems [Schwartz, 1952, 1957, 1958; Newcomb, 1963a, 1964a; Anderson and Newcomb, 1976; Duistermaat and Kolk, 2010]. Similarly, under certain conditions, the series-equivalent system of two convolution systems in series connection is a convolution system whose convolution kernel distribution coincides with the convolution product of the convolution kernel distributions of the two convolution systems [Schwartz, 1966]; furthermore, if the frequency-response distributions of the two convolution systems exist, the frequency-response distribution of the series-equivalent system exists too and coincides with the multiplicative product of the frequency-response distributions of the two convolution systems [Schwartz, 1966]. In particular, it can be easily verified that the aforementioned conditions are satisfied whenever the convolutive product of the convolution kernel distributions of the two convolution systems and any input variable is well-defined, commutative, and associative, and the convolution Fourier exchange theorem applies to it.

Let there be given two systems having the same input space and the same output space. The two systems are said to be in *parallel connection* if and only if there exists a system (to which we refer as *parallel-equivalent system*) whose input space and output space are respectively the input space and the output space of the two systems and whose input–output representation is the sum of the

input–output representations of the two systems [Kwakernaak and Sivan, 1991]. The parallel-equivalent system of two kernel systems in parallel connection is a kernel system whose kernel distribution coincides with the sum of the kernel distributions of the two kernel systems [Kwakernaak and Sivan, 1991]. Similarly, the parallel-equivalent system of two convolution systems in parallel connection is a convolution system whose convolution kernel distribution coincides with the sum of the convolution kernel distributions of the two convolution systems [Kwakernaak and Sivan, 1991]; furthermore, if the frequency-response distributions of the two convolution systems exist, the frequency-response distribution of the parallel-equivalent system exists too and coincides with the sum of the frequency-response distributions of the two convolution systems [Kwakernaak and Sivan, 1991].

A.6.4 Causality

A system has the property of *causality*, also known as *non-anticipativeness* or *physical realisability*, (respectively, *anticausality*) if and only if, for each ξ_0 in $\mathbb{R}$, the condition that two arbitrary input variables coincide on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) implies that also the corresponding output variables coincide on $\mathbb{R}_{(-\infty, \xi_0)}$ (respectively, $\mathbb{R}_{(\xi_0, +\infty)}$) [Zemanian, 1965a, 1968a, 1972; Coxson, 1983]; in particular, we say that a system not having the property of causality has the property of *acausality*. Under certain conditions, a system having the properties of single-valuedness, linearity, translation-invariance, and causality ‘automatically’ has the property of continuity [Neumann, 1980]. Each initially-at-rest (respectively, finally-at-rest) differential system has the property of causality (respectively, anticausality) [Kwakernaak and Sivan, 1991]. A kernel system has the property of causality (respectively, anticausality) if and only if its kernel distribution has support contained in the subset of $\mathbb{R}^2$ at which the free real variable of the integral transformation is greater than or equal to (respectively, less than or equal to) the bound real variable of the integral transformation [Zemanian, 1968a, 1972]. A convolution system has the property of causality (respectively, anticausality) if and only if its convolution kernel distribution is a causal distribution on $\mathbb{R}$ (respectively, an anticausal distribution on $\mathbb{R}$) [Beltrami and Wohlers, 1966; Zemanian, 1968a, 1972; Wohlers, 1969]. The following results provide necessary and sufficient conditions on either the transfer function or the frequency-response distribution of a convolution system for it to have the property of causality; they can be easily adapted to provide the corresponding results for the property of anticausality. A convolution system has the property of causality if and only if its transfer function (respectively, complementary transfer function), if it exists, has region of definition in a subset of $\mathbb{C}$ unbounded from the right (respectively, from below) and has modulus majorised by some polynomial function on some subset of the region of definition bounded from the left (respectively, from above) by, and including, a straight line parallel to $\mathbb{C}_{\text{Re}=0}$ (respectively, to $\mathbb{C}_{\text{Im}=0}$) [Zemanian, 1965a, 1972]. Equivalently, a convolution system has the property of causality if and only if its frequency-response distribution, if it exists, is the limit of a holomorphic function on $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (respectively, on $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$) as the real part σ (respectively, as the imaginary part $\underline{\sigma}$) of the free complex variable $s \equiv \sigma + j\omega$ (respectively, $\underline{s} \equiv \omega + j\underline{\sigma}$) tends to 0 from the right (respectively, from below), the holomorphic function on $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (respectively, on $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$) having modulus majorised by some polynomial function on each subset of $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (respectively, of $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$) bounded from the left (respectively, from above) by, and including, a straight line parallel to $\mathbb{C}_{\text{Re}=0}$

(respectively, to $\mathbb{C}_{\text{Im}=0}$), and the limit being within $\mathcal{S}'(\mathbb{R})$, which implies that the holomorphic function on $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ (respectively, on $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$) generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\mathbb{R}_{(0, +\infty)}$ (respectively, for each fixed $\underline{\sigma}$ in $\mathbb{R}_{(-\infty, 0)}$) [Schwartz, 1962; Nussenzveig, 1972; King, 2009b]. Equivalently, a convolution system has the property of causality if and only if its frequency-response distribution, if it exists and is a $(-n)$ -weighted square integrable distribution on $\mathbb{R}$ (in particular, a (-1) -weighted integrable distribution on $\mathbb{R}$), satisfies the generalised Kramers–Kronig relations (in particular, the Kramers–Kronig relations) [Schwartz, 1962; Beltrami and Wohlers, 1966; Wohlers, 1969; Nussenzveig, 1972; King, 2009b]. Equivalently, according to the *Paley–Wiener logarithmic integral theorem*, a convolution system has the property of causality if and only if its frequency-response distribution, if it exists and is a $(-n)$ -weighted square integrable distribution on $\mathbb{R}$, and hence can be expressed as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by $(-n)$ -weighted square integrable functions on $\mathbb{R}$, satisfies the condition that the natural logarithm of the modulus of each of such $(-n)$ -weighted square integrable functions on $\mathbb{R}$ is a (-2) -weighted integrable function on $\mathbb{R}$ [Masani, 1966; Pfaffelhuber, 1971b, 1975; Benedetto, 1997]; in particular, a convolution system has the property of causality if and only if its frequency-response distribution, if it exists and is generable by the multiplicative product of a polynomial function on $\mathbb{R}$ by a square integrable function on $\mathbb{R}$ (which happens if and only if the convolution kernel distribution of the convolution system is a square integrable distribution on $\mathbb{R}$), satisfies the condition that the natural logarithm of the modulus of such multiplicative product is a (-2) -weighted integrable function on $\mathbb{R}$. The frequency-response distribution of a causal convolution system vanishes on an open interval of $\mathbb{R}$ if and only if it vanishes on $\mathbb{R}$ (namely, if and only if it coincides with $0_{\mathbb{R}}$) [Pfaffelhuber, 1971b]. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], a convolution system having the property of realness whose band consists of a pair (respectively, of the complement of a pair) of disjoint bounded intervals of $\mathbb{R}$ symmetric with respect to the origin is a *band-pass filter* (respectively, a *band-stop filter*), such a convolution system becoming a *low-pass filter* (respectively, a *high-pass filter*) in the limiting case in which the pair of disjoint bounded intervals of $\mathbb{R}$ symmetric with respect to the origin degenerates into a bounded interval of $\mathbb{R}$ symmetric with respect to the origin.

Restricting ourselves to the case of kernel systems (in particular, convolution systems), we now introduce the property of *memorylessness*, which only systems having the properties of both

causality and anticausality can have [Coxson, 1983]. A kernel system has the property of memorylessness if and only if its output variable is the multiplicative product of the input variable by a fixed distribution on $\mathbb{R}$ that is such as to make the multiplication well-defined (which happens, in particular, if the fixed distribution on $\mathbb{R}$ is generable by a C^0 function on $\mathbb{R}$) [Schwartz, 1952, 1957, 1958; Kwakernaak and Sivan, 1991]; equivalently, a kernel system has the property of memorylessness if and only if its (necessarily symmetric) kernel distribution is the multiplicative product of a certain singular distribution of order 0 on $\mathbb{R}^2$ having as support the bisector of the first and third quadrant (namely, the continuous linear form under which the image of a test function is the integral on $\mathbb{R}$ of the result of its localisation at the bisector of the first and third quadrant) by a fixed distribution on $\mathbb{R}$ that is such as to make the multiplication well-defined (which happens, in particular, if the fixed distribution on $\mathbb{R}$ is generable by a C^0 function on $\mathbb{R}$) [Schwartz, 1952, 1957, 1958; Albertoni and Cugiani, 1953]. A convolution system has the property of memorylessness if and only if its output variable is the multiplicative product of the input variable by a fixed constant function on $\mathbb{R}$ [Schwartz, 1952, 1957, 1958; Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997]; equivalently, a convolution system has the property of memorylessness if and only if its convolution kernel distribution is the multiplicative product of δ by a fixed constant function on $\mathbb{R}$ [Schwarz, 1952, 1957, 1958; Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997]. As is evident, a convolution system having the property of memorylessness is a differential system (but a differential system need not have the property of memorylessness). In particular, a convolution system having the property of memorylessness is said to be the *identity system* if and only if its output variable coincides with the input variable or, equivalently, if and only if its convolution kernel distribution coincides with δ [Schwarz, 1952, 1957, 1958; Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997]. In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], a convolution system having the property of realness and having frequency-response distribution that is generable by a locally integrable function on $\mathbb{R}$ with modulus coinciding with $1_{\mathbb{R}}$ is an *all-pass filter* (an example of it being the identity system).

A.6.5 Passivity

For a given system having the properties of realness and single-valuedness, we consider an input variable and the corresponding output variable. Then, we define the *input–output product* as the integral on $\mathbb{R}$ of half the sum of two terms the first being the multiplicative product of the input variable by the complex conjugate of the output variable and the second being the multiplicative product of the output variable by the complex conjugate of the input variable (which is the same as the integral on $\mathbb{R}$ of the real part of the multiplicative product of the input variable by the complex conjugate of the output variable and the same as the integral on $\mathbb{R}$ of the real part of the multiplicative product of the output variable by the complex conjugate of the input variable) [Newcomb, 1963a, 1964a; Zemanian, 1965a, 1968a, 1972; Beltrami and Wohlers, 1966; Wohlers, 1969; Anderson and Newcomb, 1976]. By localising both the input variable and the output variable at an open subset of $\mathbb{R}$ having left endpoint $-\infty$ and right endpoint a variable point of $\overline{\mathbb{R}}$, the input–output product can be regarded as a pointwise function of the variable point of $\overline{\mathbb{R}}$ taking values in $\overline{\mathbb{R}}$; we refer to such pointwise function as *input–output product function*. By definition, the value of the input–output product function at $-\infty$ is null (in accordance with the property of single-valuedness); however, the value of the input–output product function at any other point of $\overline{\mathbb{R}}$ may or may not be well-defined, and, if it is well-defined, it may or may not be finite. For a convolution system, the value of the input–output product function at each point of $\overline{\mathbb{R}}$ is surely well-defined and finite for each input variable in $\mathcal{D}^{\wedge}(\mathbb{R})$ [Zemanian, 1965a, 1968a; Beltrami and Wohlers, 1966; Wohlers, 1969].

A system having the properties of realness and single-valuedness has the property of *passivity* if and only if the value of the input–output product function at each point of $\overline{\mathbb{R}}$ is well-defined, finite, and non-negative for each input variable in $\mathcal{D}^{\wedge}(\mathbb{R})$ [Newcomb, 1963a, 1964a; Zemanian, 1965a, 1968a, 1972; Beltrami and Wohlers, 1966; Wohlers, 1969; Anderson and Newcomb, 1976; Wyatt *et al.*, 1981]; in particular, a convolution system having the property of realness has the property of passivity if and only if the value of the input–output product function at each point of $\overline{\mathbb{R}}$ is well-defined, finite, and non-negative not only for each input variable in $\mathcal{D}^{\wedge}(\mathbb{R})$ but also for each input variable in $\mathcal{S}^{\wedge}(\mathbb{R})$ [Zemanian, 1965a; Beltrami and Wohlers, 1966; Wohlers, 1969]. Under certain conditions, a system having the properties of single-valuedness, linearity, and passivity ‘automatically’ has the property of continuity [Neumann, 1980]. A system having the properties of realness and single-valuedness and not having the property of passivity has the property of *activity* [Anderson and Newcomb, 1976; Wyatt *et al.*, 1981, 1982].

A system having the properties of realness and single-valuedness as well as the property of linearity has the property of *losslessness* (also known as *conservativeness*) if and only if the value of the input–output product function at $+\infty$ is well-defined and null for each input variable in $L^2(\mathbb{R})$ with corresponding output variable in $L^2(\mathbb{R})$ [Wyatt *et al.*, 1982]. In particular, if the input space and the output space of a system having the properties of realness and single-valuedness as well as the property of linearity are subspaces of $L^2(\mathbb{R})$ (respectively, are $L^2(\mathbb{R})$), the system has the property of losslessness if and only if its input–output representation is a map from the input space into the output space such that any input variable and the corresponding output variable are orthogonal [Wyatt *et al.*, 1982]. Unlike the property of passivity (respectively, activity), the property of losslessness is introduced here only for systems having the property of linearity; indeed, when adopting the immittance formalism, it is problematic to introduce it situations that are more general [Newcomb, 1963a, 1964a; Beltrami and Wohlers, 1966; Wohlers, 1969; Zemanian, 1972; Anderson and Newcomb, 1976; Wyatt *et al.*, 1982]. The properties of passivity (respectively, activity) and losslessness are independent [Wyatt *et al.*, 1982]; we say that a system having the property of passivity (respectively, activity) but not having the property of losslessness has the property of *dissipativity* (respectively, *generativeness*).

If the input variable and the output variable of a kernel system ‘live’ in the time domain, if they are *power-conjugate quantities* (namely, if their multiplicative product has the quantity dimension of power and hence the corresponding input–output product function has the quantity dimension of energy), and if the *motor convention* (also known as *passive sign convention*) [International Electrotechnical Commission, 2018] is adopted as a group of reference conventions, which implies that a non-negative (respectively, non-positive) value of the input–output product function corresponds to energy absorbed (respectively, released) by the system, then the value of the input–output product function at any time can be interpreted as energy absorbed by the system from time $-\infty$ to that time, the energy stored in the system at time $-\infty$ being null by definition (in accordance with the property of single-valuedness) [Newcomb, 1963a, 1964a; Zemanian, 1965a, 1968a, 1972; Anderson and Newcomb, 1976]. We refer to a system whose input variable and output variable are power-conjugate quantities as a *power-conjugate system*. Hereinafter, we consider the properties of passivity (respectively, activity) and losslessness only for power-conjugate systems. We warn that several non-equivalent definitions are available in the literature for the properties of passivity (respectively, activity) and losslessness. In particular, we observe that the adopted definitions rely only on the concepts of input variable and output variable; in fact, by introducing the

concept of *state variable*, such properties can be investigated more in depth [Willems, 1972a, 1972b; Wyatt *et al.*, 1981, 1982].

A kernel system has the property of passivity only if it has the property of causality [Zemanian, 1968a, 1972]. A convolution system has the property of passivity if and only if its convolution kernel distribution is a real-valued causal bounded distribution on $\mathbb{R}$ that can be expressed as the sum of a non-negative real constant times $\delta^{(1)}$ and a weakly non-negative-definite real-valued causal distribution of C -order in $\mathbb{Z}_{[-\infty, 0]}$ on $\mathbb{R}$ (namely, that can be expressed as the sum of a non-negative real constant times $\delta^{(1)}$ and a real-valued causal distribution of C -order in $\mathbb{Z}_{[-\infty, 0]}$ on $\mathbb{R}$ whose conjugate-symmetric part or, equivalently, whose even part is non-negative-definite) [König and Zemanian, 1965; Zemanian, 1965a, 1965b, 1968a; Beltrami and Wohlers, 1966; Wohlers, 1969]; in particular, this implies that the frequency-response distribution and the transfer function (respectively, complementary transfer function) of a convolution system having the property of passivity always exist. Equivalently, a convolution system has the property of passivity if and only if its transfer function is a *positive-real function* (namely, if and only if its transfer function is a holomorphic function on $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ whose real part is non-negative and whose localisation at $\mathbb{C}_{\text{Re} \in (0, +\infty)} \cap \mathbb{C}_{\text{Im} = 0}$ results in a real-valued function thereon) [König and Zemanian, 1965; Zemanian, 1965b, 1968a; Beltrami and Wohlers, 1966; Wohlers, 1969; Vladimirov, 2002]. Equivalently, a convolution system has the property of passivity if and only if its frequency-response distribution is the limit of a positive-real function as the real part σ of the free complex variable $s \equiv \sigma + j\omega$ tends to 0 from the right, the limit being within $\mathcal{S}'(\mathbb{R})$, which implies that the positive-real function generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\mathbb{R}_{(0, +\infty)}$ [Beltrami and Wohlers, 1966; Zemanian, 1968a; Wohlers, 1969].

A.6.6 Stability

A system has the property of *input–output stability* if and only if its output space coincides with its input space [Ciampa, 2021]; we say that a system not having the property of input–output stability has the property of *input–output instability*. Therefore, a system has the property of input–output stability if and only if its input–output representation, whenever it qualifies as a morphism, is an endomorphism from the input space into the output space. As is evident, a convolution system has the property of input–output stability if and only if its convolution kernel distribution coincides with the distribution space of the convulators of the input space. In particular, a convolution system having as input space any of $\mathcal{D}'(\mathbb{R})$, $\mathcal{D}^\wedge(\mathbb{R})$, $\mathcal{E}'(\mathbb{R})$, and $\mathcal{E}^\wedge(\mathbb{R})$, any of $\mathcal{O}'_M(\mathbb{R})$ and $\mathcal{O}^\wedge_M(\mathbb{R})$, any of $\mathcal{S}'(\mathbb{R})$, $\mathcal{S}^\wedge(\mathbb{R})$, $\mathcal{O}'_C(\mathbb{R})$, and $\mathcal{O}^\wedge_C(\mathbb{R})$, any of $\mathcal{D}'_R(\mathbb{R})$ and $\mathcal{D}^\wedge_R(\mathbb{R})$, and any of $\mathcal{D}'_L(\mathbb{R})$ and $\mathcal{D}^\wedge_L(\mathbb{R})$ has the property of input–output stability if and only if its convolution kernel distribution is an element of respectively $\mathcal{E}'(\mathbb{R})$, $\mathcal{O}'_M(\mathbb{R})$, $\mathcal{O}'_C(\mathbb{R})$, $\mathcal{D}'_R(\mathbb{R})$, and $\mathcal{D}'_L(\mathbb{R})$ [Schwartz, 1966; Larcher, 2013]. Somehow unfortunately, the input–output representation of a convolution system (in particular, of a convolution system having the property of input–output stability) is also said to be a *multiplier* from the input space into the output space (in particular, a *multiplier* of the input space) [Larsen, 1971]; that said, to avoid confusion with a different (albeit related) concept, we do not use the term *multiplier* in this sense hereinafter. We now consider some important special cases.

On the one hand, a system has the property of $\mathcal{D}'_{L^p}$ *input–output stability* (in particular, *generalised bounded-input–bounded-output stability*, hereinafter *generalised BIBO stability*) if its input space and output space coincide with $\mathcal{D}'_{L^p}(\mathbb{R})$ (in particular, $\mathcal{D}'_{L^\infty}(\mathbb{R})$) [Beltrami and Wohlers, 1965; Beffa, 2024]. On the other hand, a system has the property of L^p *input–output stability* (in particular, *bounded-input–bounded-output stability*, hereinafter *BIBO stability*) if its input space and output space coincide with $L^{p^\wedge}(\mathbb{R})$ (in particular, $L^{\infty^\wedge}(\mathbb{R})$) [Newcomb, 1964b; Meidan, 1977; Unser, 2020; Ciampa, 2021].

We denote by $A^q(\mathbb{R})$ the function space of the possibly infinite but convergent (in norm) linear combinations of convolutive products of pairs formed by an element of $L^q(\mathbb{R})$ and an element of $L^{q'}(\mathbb{R})$ such that the sum on all the pairs of the linear combination of the multiplicative product of the L^q magnitude of the element of $L^q(\mathbb{R})$ by the $L^{q'}$ magnitude of the element of $L^{q'}(\mathbb{R})$ is finite. We have [Figà-Talamanca, 1965; Benedetto, 1975]

A.6 Systems: Generalities

$$A^q(\mathbb{R}) \equiv A^{q'}(\mathbb{R})$$

$$A^2(\mathbb{R}) \equiv A(\mathbb{R})$$

By definition, we have

$$A^1(\mathbb{R}) \equiv A^\infty(\mathbb{R}) \equiv \dot{\mathcal{B}}^0(\mathbb{R})$$

Let $q_{(2,+\infty)}$ in $\mathbb{R}_{(2,+\infty)}$ and $q'_{(1,2)}$ in $\mathbb{R}_{(1,2)}$ form a pair of Hölder-conjugate exponents. Denoting by $(A^p(\mathbb{R}))'$ the topological dual space of $A^p(\mathbb{R})$, we have [Figà-Talamanca, 1965; Brainerd and Edwards, 1966; Larsen, 1971; Benedetto, 1975]

$$PM(\mathbb{R}) \equiv (A^2(\mathbb{R}))' \supset (A^{q_{(2,+\infty)}}(\mathbb{R}))' \equiv (A^{q'_{(1,2)}}(\mathbb{R}))' \supset (A^1(\mathbb{R}))' \equiv (A^\infty(\mathbb{R}))' \equiv \mathcal{D}'_{L^1}(\mathbb{R})$$

A convolution system having as input space either $\mathcal{D}'_{L^p}(\mathbb{R})$ (in particular, $L^{p\wedge}(\mathbb{R})$) or $\mathcal{D}^{\wedge}_{L^p}(\mathbb{R})$ has the property of input–output stability if and only if its convolution kernel distribution is an element of $\mathcal{D}'_{L^1}(\mathbb{R})$ (in particular, $(A^p(\mathbb{R}))'$), whereas a convolution system having as input space either $\dot{\mathcal{B}}'(\mathbb{R})$ (in particular, $\dot{\mathcal{B}}^{0\wedge}(\mathbb{R})$) or $\dot{\mathcal{B}}^{\wedge}(\mathbb{R})$ has the property of input–output stability if and only if its convolution kernel distribution is an element of $\mathcal{D}'_{L^1}(\mathbb{R})$ (in particular, $\mathcal{D}'^{0}_{L^1}(\mathbb{R})$) [Newcomb, 1964b; Beltrami and Wohlers, 1965; Figà-Talamanca, 1965; Brainerd and Edwards, 1966; Larsen, 1971; Benedetto, 1975; Larcher, 2013; Unser, 2020; Beffa, 2024].

A.6.6.1 Energy convolution systems

A convolution system has the property of $\mathcal{D}'_{L^2}$ input–output stability (respectively, L^2 input–output stability) if and only if its convolution kernel distribution is an element of $\mathcal{D}'_{L^1}(\mathbb{R})$ (respectively, $PM(\mathbb{R})$) [Figà-Talamanca, 1965; Brainerd and Edwards, 1966; Larsen, 1971; Benedetto, 1975; Larcher, 2013; Unser, 2020]. We refer to a convolution system having the property of $\mathcal{D}'_{L^2}$ input–output stability (respectively, L^2 input–output stability) as an *energy convolution system*.

Let g_2 , u_2 , and w_2 be respectively the convolution kernel distribution, the input variable, and the output variable of an energy convolution system. Let G_2 , U_2 , and W_2 be the Fourier transforms of respectively g_2 , u_2 , and w_2 . Let $R_{g_2 g_2} \equiv \mathcal{P}(R_{g_2 g_2})$, $R_{u_2 u_2} \equiv \mathcal{P}(R_{u_2 u_2})$, and $R_{w_2 w_2} \equiv \mathcal{P}(R_{w_2 w_2})$ be the

auto-correlation distributions of respectively g_2 , u_2 , and w_2 , and let $S_{g_2 g_2} \equiv \mathcal{C}(S_{g_2 g_2})$, $S_{u_2 u_2} \equiv \mathcal{C}(S_{u_2 u_2})$, and $S_{w_2 w_2} \equiv \mathcal{C}(S_{w_2 w_2})$ be $1/(2\pi)$ times the Fourier transforms of respectively $R_{g_2 g_2}$, $R_{u_2 u_2}$, and $R_{w_2 w_2}$. Let $R_{u_2 w_2} \equiv \mathcal{P}(R_{u_2 w_2})$ (respectively, $R_{w_2 u_2} \equiv \mathcal{P}(R_{w_2 u_2})$) be the cross-correlation distribution of the ordered pair $\{u_2, w_2\}$ (respectively, $\{w_2, u_2\}$), and let $S_{u_2 w_2} \equiv \mathcal{C}(S_{u_2 w_2})$ (respectively, $S_{w_2 u_2} \equiv \mathcal{C}(S_{w_2 u_2})$) be $1/(2\pi)$ times the Fourier transform of $R_{u_2 w_2}$ (respectively, $R_{w_2 u_2}$). The time-domain relations hold

$$w_2 = g_2 * u_2 \Rightarrow \begin{cases} R_{w_2 w_2} = g_2 * R_{u_2 w_2} \\ R_{w_2 u_2} = g_2 * R_{u_2 u_2} \end{cases} \Rightarrow R_{w_2 w_2} = R_{g_2 g_2} * R_{u_2 u_2} \quad (\text{A.69})$$

Equivalently, the frequency-domain relations hold

$$W_2 = G_2 \cdot U_2 \Rightarrow \begin{cases} S_{w_2 w_2} = G_2 \cdot S_{u_2 w_2} \\ S_{w_2 u_2} = G_2 \cdot S_{u_2 u_2} \end{cases} \Rightarrow S_{w_2 w_2} = 2\pi(S_{g_2 g_2} \cdot S_{u_2 u_2}) \quad (\text{A.70})$$

Furthermore, the frequency-domain relation holds

$$S_{u_2 w_2} S_{w_2 u_2} = S_{u_2 u_2} S_{w_2 w_2} \quad (\text{A.71})$$

Equation (A.71) expresses the remarkable fact that the ordered pair formed by an input variable and the corresponding output variable of an energy convolution system is completely coherent. It is worth noting that neither the standard form nor the generalised form of the multiplication Fourier exchange theorem need apply to Equation (A.71).

Although on the one hand the set $\{g_2, u_2, w_2\}$ (respectively, $\{G_2, U_2, W_2\}$) does determine uniquely the set $\{R_{g_2 g_2}, R_{u_2 u_2}, R_{w_2 w_2}, R_{u_2 w_2}, R_{w_2 u_2}\}$ (respectively, $\{S_{g_2 g_2}, S_{u_2 u_2}, S_{w_2 w_2}, S_{u_2 w_2}, S_{w_2 u_2}\}$), on the other hand the set $\{R_{g_2 g_2}, R_{u_2 u_2}, R_{w_2 w_2}, R_{u_2 w_2}, R_{w_2 u_2}\}$ (respectively, $\{S_{g_2 g_2}, S_{u_2 u_2}, S_{w_2 w_2}, S_{u_2 w_2}, S_{w_2 u_2}\}$) does not determine uniquely the set $\{g_2, u_2, w_2\}$ (respectively, $\{G_2, U_2, W_2\}$); for example, the cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ (respectively, L^2) is the same as the cross-correlation distribution of the ordered pair of elements of $\mathcal{D}'_{L^2}(\mathbb{R})$ (respectively, L^2) translated by any translation vector, and the auto-correlation distribution of an element of $\mathcal{D}'_{L^2}(\mathbb{R})$ (respectively, L^2) is the same as the auto-correlation distribution of the element of $\mathcal{D}'_{L^2}(\mathbb{R})$ (respectively, L^2) translated by any translation vector. Accordingly, in Equation (A.69) (respectively,

Equation (A.70)), none of the implications can be reversed, and hence the relation on the right can be regarded as a weak version of the pair of relations in the centre, which in turn can be regarded as a weak version of the relation on the left. In fact, the importance of the relations in the centre and on the right of Equation (A.69) (respectively, Equation (A.70)) lies in the fact that, unlike the relation on the left, they can be applied not only in deterministic settings but also (with some adjustments) in stochastic settings [Itô, 1954; Gelfand and Vilenkin, 1964; Benedetto, 1997; Bendat and Piersol, 2010].

Remarkably, despite allowing only a weak characterisation of an energy convolution system, the knowledge of $R_{g_2 g_2}$ (or, equivalently, $S_{g_2 g_2}$) is sufficient to determine whether such convolution system has the property of causality (which can be done by applying the Paley–Wiener logarithmic integral theorem in light of the fact that $(2\pi S_{g_2 g_2})^{1/2}$ coincides with $|G_2|$ [Masani, 1966; Pfaffelhuber, 1971b, 1975; Benedetto, 1997]). In particular, for an energy convolution system having the property of $\mathcal{D}'_{L^2}$ input–output stability and hence admitting δ as an input variable, $R_{g_2 g_2}$ can be regarded as the *unit-energy-white-noise response* (namely, as the auto-correlation distribution of the output variable corresponding to an input variable having auto-correlation distribution coinciding with δ).

A.6.6.2 Power convolution systems

On the one hand, a system has the property of $\mathcal{D}'_P$ input–output stability if and only if its input space and output space coincide with $\mathcal{D}'_P(\mathbb{R})$. On the other hand, a system has the property of P input–output stability if and only if its input space and output space coincide with $P^\wedge(\mathbb{R})$.

We are not aware of any sufficient condition of general validity ensuring that a convolution system has the property of $\mathcal{D}'_P$ input–output stability (respectively, P input–output stability). However, a convolution system that has a stationary distribution on $\mathbb{R}$ (in particular, a distribution on $\mathbb{R}$ generated by a stationary function on $\mathbb{R}$) as input variable has a stationary distribution on $\mathbb{R}$ (in particular, a distribution on $\mathbb{R}$ generated by a stationary function on $\mathbb{R}$) as output variable if its convolution kernel distribution is an element of $\mathcal{D}'_{L^1, +1}(\mathbb{R})$ that can be expressed as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by elements of $L^1_{+1}(\mathbb{R}) \cap L^2_{+1}(\mathbb{R})$ (in particular, is a distribution on $\mathbb{R}$ generated by an element of $L^1_{+1}(\mathbb{R}) \cap L^2_{+1}(\mathbb{R})$) [Masani, 1966; Pfaffelhuber, 1975; Wiener, 1988]. In fact, a convolution system that has a distribution on $\mathbb{R}$ generated by a stationary function on $\mathbb{R}$ (in particular, by respectively a centred or a non-centred pseudorandom function on $\mathbb{R}$) as input variable has a distribution on $\mathbb{R}$ generated

by a stationary function on $\mathbb{R}$ (in particular, by respectively a centred or a non-centred pseudorandom function on $\mathbb{R}$) as output variable if its convolution kernel distribution is an element of $\mathcal{D}'^0_L(\mathbb{R})$ [Bass, 1962, 1974, 1984, 1989; Masani, 1979; Bertrandias *et al.*, 1987]. Furthermore, a convolution system that has a uniformly almost periodic distribution on $\mathbb{R}$ (in particular, a distribution on $\mathbb{R}$ generated by a uniformly almost periodic function on $\mathbb{R}$) as input variable has a uniformly almost periodic distribution on $\mathbb{R}$ (in particular, a distribution on $\mathbb{R}$ generated by a uniformly almost periodic function on $\mathbb{R}$) as output variable if its convolution kernel distribution is an element of $\mathcal{D}'_L(\mathbb{R})$ (in particular, is a distribution on $\mathbb{R}$ generated by an element of $L^1(\mathbb{R})$) [Schwartz, 1966; Amerio and Prouse, 1971; Bass, 1974, 1984; Partington, 2004]. Hereinafter, we assume that the convolution kernel distribution of a convolution system having the property of $\mathcal{D}'_P$ input–output stability (respectively, P input–output stability) is an element of $\mathcal{D}'_{L^{l+1}}(\mathbb{R})$ that can be expressed as a finite linear combination of derivatives of order in $\mathbb{N}$ of distributions on $\mathbb{R}$ generated by elements of $L^1_{+1}(\mathbb{R}) \cap L^2_{+1}(\mathbb{R})$ (in particular, is a distribution on $\mathbb{R}$ generated by an element of $L^1_{+1}(\mathbb{R}) \cap L^2_{+1}(\mathbb{R})$). We refer to a convolution system having the property of $\mathcal{D}'_P$ input–output stability (respectively, P input–output stability) as a *power convolution system*.

Let g_P , u_P , and w_P be respectively the convolution kernel distribution, the input variable, and the output variable of a power convolution system. Let G_P , U_P , and W_P be the Fourier transforms of respectively g_P , u_P , and w_P . Let $R_{g_P g_P} \equiv \mathcal{P}(R_{g_P g_P})$ be the auto-correlation distribution of g_P , let $\bar{R}_{u_P u_P} \equiv \mathcal{P}(\bar{R}_{u_P u_P})$ and $\bar{R}_{w_P w_P} \equiv \mathcal{P}(\bar{R}_{w_P w_P})$ be the mean auto-correlation distributions of respectively u_P and w_P , and let $S_{g_P g_P} \equiv \mathcal{C}(S_{g_P g_P})$, $\bar{S}_{u_P u_P} \equiv \mathcal{C}(\bar{S}_{u_P u_P})$, and $\bar{S}_{w_P w_P} \equiv \mathcal{C}(\bar{S}_{w_P w_P})$ be $1/(2\pi)$ times the Fourier transforms of respectively $R_{g_P g_P}$, $\bar{R}_{u_P u_P}$, and $\bar{R}_{w_P w_P}$. Let $\bar{R}_{u_P w_P} \equiv \mathcal{P}(\bar{R}_{u_P w_P})$ (respectively, $\bar{R}_{w_P u_P} \equiv \mathcal{P}(\bar{R}_{w_P u_P})$) be the mean cross-correlation distribution of the ordered pair $\{u_P, w_P\}$ (respectively, $\{w_P, u_P\}$), and let $\bar{S}_{u_P w_P} \equiv \mathcal{C}(\bar{S}_{u_P w_P})$ (respectively, $\bar{S}_{w_P u_P} \equiv \mathcal{C}(\bar{S}_{w_P u_P})$) be $1/(2\pi)$ times the Fourier transform of $\bar{R}_{u_P w_P}$ (respectively, $\bar{R}_{w_P u_P}$). The time-domain relations hold [Masani, 1966; Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Wiener, 1988]

$$w_P = g_P * u_P \Rightarrow \begin{cases} \bar{R}_{w_P w_P} = g_P * \bar{R}_{u_P u_P} \\ \bar{R}_{w_P u_P} = g_P * \bar{R}_{u_P u_P} \end{cases} \Rightarrow \bar{R}_{w_P w_P} = R_{g_P g_P} * \bar{R}_{u_P u_P} \quad (\text{A.72})$$

Equivalently, the frequency-domain relations hold [Masani, 1966; Pfaffelhuber, 1975; Bertrandias *et al.*, 1987; Wiener, 1988]

$$W_p = G_p \cdot U_p \Rightarrow \begin{cases} \bar{S}_{w_p w_p} = G_p \cdot \bar{S}_{u_p w_p} \\ \bar{S}_{w_p u_p} = G_p \cdot \bar{S}_{u_p u_p} \end{cases} \Rightarrow \bar{S}_{w_p w_p} = 2\pi \left(S_{g_p g_p} \cdot \bar{S}_{u_p u_p} \right) \quad (\text{A.73})$$

Furthermore, whenever well-defined, the frequency-domain relation holds

$$\bar{S}_{u_p w_p} \bar{S}_{w_p u_p} = \bar{S}_{u_p u_p} \bar{S}_{w_p w_p} \quad (\text{A.74})$$

Whenever well-defined, Equation (A.74) expresses the remarkable fact that the ordered pair formed by an input variable and the corresponding output variable of a power convolution system is completely mean coherent. It is worth noting that neither the standard form nor the generalised form of the multiplication Fourier exchange theorem need apply to Equation (A.74).

Although on the one hand the set $\{g_p, u_p, w_p\}$ (respectively, $\{G_p, U_p, W_p\}$) does determine uniquely the set $\{R_{g_p g_p}, \bar{R}_{u_p u_p}, \bar{R}_{w_p w_p}, \bar{R}_{u_p w_p}, \bar{R}_{w_p u_p}\}$ (respectively, $\{S_{g_p g_p}, \bar{S}_{u_p u_p}, \bar{S}_{w_p w_p}, \bar{S}_{u_p w_p}, \bar{S}_{w_p u_p}\}$), on the other hand the set $\{R_{g_p g_p}, \bar{R}_{u_p u_p}, \bar{R}_{w_p w_p}, \bar{R}_{u_p w_p}, \bar{R}_{w_p u_p}\}$ (respectively, $\{S_{g_p g_p}, \bar{S}_{u_p u_p}, \bar{S}_{w_p w_p}, \bar{S}_{u_p w_p}, \bar{S}_{w_p u_p}\}$) does not determine uniquely the set $\{g_p, u_p, w_p\}$ (respectively, $\{G_p, U_p, W_p\}$); for example, the mean cross-correlation distribution of an ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ (respectively, $P(\mathbb{R})$) is the same as the mean cross-correlation distribution of the ordered pair of elements of $\mathcal{D}'_p(\mathbb{R})$ (respectively, $P(\mathbb{R})$) each of which translated by any translation vector, and the mean auto-correlation distribution of an element of $\mathcal{D}'_p(\mathbb{R})$ (respectively, $P(\mathbb{R})$) is the same as the mean auto-correlation distribution of the element of $\mathcal{D}'_p(\mathbb{R})$ (respectively, $P(\mathbb{R})$) translated by any translation vector. Accordingly, in Equation (A.72) (respectively, Equation (A.73)), none of the implications can be reversed, and hence the relation on the right can be regarded as a weak version of the pair of relations in the centre, which in turn can be regarded as a weak version of the relation on the left. In fact, the importance of the relations in the centre and on the right of Equation (A.72) (respectively, Equation (A.73)) lies in the fact that, unlike the relation on the left, they can be applied not only in deterministic settings but also (with some adjustments) in stochastic settings [Itô, 1954; Gelfand and Vilenkin, 1964; Benedetto, 1983, 1997; Bendat and Piersol, 2010].

Remarkably, despite allowing only a weak characterisation of a power convolution system, the knowledge of $R_{g_p g_p}$ (or, equivalently, $S_{g_p g_p}$) is sufficient to determine whether such convolution

system has the property of causality (which can be done by applying the Paley–Wiener logarithmic integral theorem in light of the fact that $\left(2\pi S_{g_P g_P}\right)^{1/2}$ coincides with $|G_P|$ [Masani, 1966; Pfaffelhuber, 1971b, 1975; Benedetto, 1997]). In particular, for a power convolution system having the property of $\mathcal{D}'_P$ input–output stability and admitting δ as an input variable, $R_{g_P g_P}$ can be regarded as the *unit-power-white-noise response* (namely, as the mean auto-correlation distribution of the output variable corresponding to an input variable having mean auto-correlation distribution coinciding with δ).

A.6.7 Invertibility

A system has the property of *left-invertibility* (respectively, *right-invertibility*) if and only if there exists a *left-inverse system* (respectively, *right-inverse system*) of it such that the two systems in series connection are equivalent to the identity system when the system acts as upstream system (respectively, downstream system) [Bonilla Estrada *et al.*, 2007; Ciampa *et al.*, 2019; Ciampa, 2021]; we say that a system not having the property of left-invertibility (respectively, right-invertibility) has the property of *non-left-invertibility* (respectively, *non-right-invertibility*). Accordingly, two systems in series connection are equivalent to the identity system if and only if the upstream system has the property of left-invertibility (and is the right-inverse system of the downstream system), and the downstream system has the property of right-invertibility (and is the left-inverse system of the upstream system). As is evident, a system has the property of left-invertibility (respectively, right-invertibility) if and only if its input–output representation, whenever it qualifies as a morphism, is a section (respectively, a retraction) from the input space into the output space; furthermore, the output space of a system having the property of left-invertibility is included in the input space of its left-inverse system, and the input space of a system having the property of right-invertibility includes the output space of its right-inverse system.

A system has the property of *invertibility* if and only if there exists an *inverse system* of it such that the two systems in series connection are equivalent to the identity system regardless of whether the system acts as upstream system or downstream system (or, equivalently, if and only if it has the properties of both left-invertibility and right-invertibility) [Bonilla Estrada *et al.*, 2007; Ciampa *et al.*, 2019; Ciampa, 2021]; we say that a system not having the property of invertibility has the property of *non-invertibility*. Accordingly, two systems in commutative series connection are equivalent to the identity system regardless of which of the two acts as upstream system or downstream system if and only if they have the property of invertibility (and are the inverse system of each other); we say that two systems being the inverse of each other are *mutually inverse*. As is evident, a system has the property of invertibility if and only if its input–output representation, whenever it qualifies as a morphism, is an isomorphism from the input space into the output space; furthermore, the output space of a system having the property of invertibility coincides with the input space of its inverse system.

The properties of stability and invertibility are independent. However, we say that a system has the property of *bistability* if and only if it has the properties of both stability and invertibility and its inverse system has the property of stability (and necessarily the property of invertibility). As is evident, a system has the property of bistability if and only if its input–output representation,

whenever it qualifies as a morphism, is an automorphism from the input space into the output space; furthermore, the coinciding input space and output space of a system having the property of bistability coincide with the coinciding input space and output space of its inverse system. In particular, it is possible to give necessary and sufficient conditions for a convolution system to have the property of bistability when input space and output space coincide with the distribution space of the tempered distributions on $\mathbb{R}$ [Abdullah, 2002] and when input space and output space coincide with the distribution space of the square integrable distributions on $\mathbb{R}$ [Abdullah, 2000]; furthermore, it is possible to give necessary and sufficient conditions for a convolution system having the property of causality and a causal distribution on $\mathbb{R}$ (in particular, a causal tempered distribution on $\mathbb{R}$) as convolution kernel distribution to have the property of bistability when input space and output space coincide with the distribution space of the causal distributions on $\mathbb{R}$ (in particular, the distribution space of the causal tempered distributions on $\mathbb{R}$) [Shambayati and Zieleźny, 1985].

A.7 Systems: Antidifferentiators, differentiators, and generalised differentiators

A.7.1 Introduction

For the systems considered hereinafter, we denote by g , u , and w respectively the convolution kernel distribution, the input variable, and the output variable; furthermore, we denote by G , U , and W the Fourier transforms of respectively g , u , and w . As seen below, all such systems are convolution systems admitting δ as an input variable; accordingly, the convolution kernel distribution of any such system can be interpreted as unit-impulse response.

In the following, antidifferentiators, differentiators, and generalised differentiators are discussed.

A.7.2 Antidifferentiators

Each function on $\mathbb{R}$ or distribution on $\mathbb{R}$ to be antidifferentiated is assumed *right-sided* (although, with minor modifications, it can be assumed left-sided). Such assumption is convenient for our applications and is important, because it makes the pertinent input space a convolution algebra. Nevertheless, we emphasise that it is possible to waive it; if this is done, to determine uniquely an antiderivative, the property of right-sidedness need be replaced by the property of vanishing at negative infinity.

We now introduce systems for antidifferentiation.

A.7.2.1 Time domain

Hereinafter, each function on $\mathbb{R}$ or distribution on $\mathbb{R}$ in the time domain is assumed *real-valued*. Such assumption is convenient for our applications but is not crucial.

Let u be a right-sided locally integrable function on $\mathbb{R}$ of t . We denote by u_1 the *integral on $\mathbb{R}$* of u defined as

$$u_1 := \int_{-\infty}^{+\infty} (u) dt' \quad (\text{A.75})$$

Furthermore, we denote by u_Q the *quadratic integral on $\mathbb{R}$* of u defined as

$$u_Q := \int_{-\infty}^{+\infty} (u^2) dt' \quad (\text{A.76})$$

In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], u_Q is the *auto-energy* of u , and, if u_Q is finite, u is a *finite-energy signal*. Each (possibly right-sided) square integrable function on $\mathbb{R}$ has finite quadratic integral on $\mathbb{R}$, and each (possibly right-sided) locally integrable function on $\mathbb{R}$ having finite quadratic integral on $\mathbb{R}$ is a square integrable function on $\mathbb{R}$.

We denote by $\bar{u}$ the *right-sided mean value on $\mathbb{R}$* of u defined as

$$\begin{aligned}
\bar{u} &:= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{t_u}^{t_u+T} (u) dt' \right) \\
&= 2 \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} (u) dt' \right) \right) \\
&= \lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (u) dt' \right)
\end{aligned} \tag{A.77}$$

where

- t_u is the left endpoint of the support of u

As is evident, $\bar{u}$ is 2 times the *mean value on $\mathbb{R}$* of u . If $\bar{u}$ is finite, we have

$$u = \bar{u} \mathfrak{G}_{t_u} + \tilde{u}$$

with

$$\begin{cases} \bar{\mathfrak{G}}_{t_u} = 1 \\ \bar{\tilde{u}} = 0 \end{cases}$$

where

- $\mathfrak{G}_{t_u}$ is the t_u -translated $\mathfrak{G}$
- $\bar{\mathfrak{G}}_{t_u}$ is the right-sided mean value on $\mathbb{R}$ of $\mathfrak{G}_{t_u}$
- $\tilde{u}$ is the right-sided centred u
- $\bar{\tilde{u}}$ is the right-sided mean value on $\mathbb{R}$ of $\tilde{u}$

Furthermore, we denote by $\bar{u}_q$ the *right-sided root-mean-square value on $\mathbb{R}$* of u defined as

$$\begin{aligned}
\bar{u}_q &:= \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{t_u}^{t_u+T} (u^2) dt' \right) \right)^{\frac{1}{2}} \\
&= 2^{\frac{1}{2}} \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} (u^2) dt' \right) \right)^{\frac{1}{2}} \\
&= \left(\lim_{T \rightarrow +\infty} \left(\frac{1}{T} \int_{-T}^{+T} (u^2) dt' \right) \right)^{\frac{1}{2}}
\end{aligned} \tag{A.78}$$

As is evident, $\bar{u}_q$ is $2^{1/2}$ times the *root-mean-square value* on $\mathbb{R}$ of u . In the language of the theory of signals and systems [Kwakernaak and Sivan, 1991; Oppenheim *et al.*, 1997; Poletti, 2005, 2015; Lewis, 2022a, 2022b], $(\bar{u}_q)^2$ is 2 times the *auto-power* of u , and, if $(\bar{u}_q)^2$ is finite, u is a *finite-power signal*. A (possibly right-sided) square integrable function on $\mathbb{R}$ has null (possibly right-sided) root-mean-square value on $\mathbb{R}$, but a (possibly right-sided) locally integrable function on $\mathbb{R}$ having null (possibly right-sided) root-mean-square value on $\mathbb{R}$ need not be a (possibly right-sided) square integrable function on $\mathbb{R}$; for example, the function defined almost everywhere on $\mathbb{R}$ by $\left(1 + (t/t_0)^2\right)^{-1/4} \mathfrak{G}(t)$ has null right-sided root-mean-square value on $\mathbb{R}$ but is not a square integrable function on $\mathbb{R}$. If both $\bar{u}$ and $\bar{u}_q$ are finite, we have

$$\begin{aligned} (\bar{u}_q)^2 &= (\bar{u})^2 + 2\bar{u}\bar{u}_q + (\bar{u}_q)^2 \\ &= (\bar{u})^2 + (\bar{u}_q)^2 \end{aligned}$$

where

- $\bar{u}_q$ is the right-sided root-mean-square value on $\mathbb{R}$ of $\tilde{u}$

Hereinafter, for convenience, we allow the quantities just defined to take values in $\overline{\mathbb{R}}$. We observe that the existence of u_1 implies the existence of $\bar{u}$. Indeed, whenever u_1 exists, we have

$$\begin{aligned} u_1 = -\infty &\Rightarrow \bar{u} \in \overline{\mathbb{R}}_{[-\infty, 0]} \\ u_1 \in \mathbb{R} &\Rightarrow \bar{u} = 0 \\ u_1 = +\infty &\Rightarrow \bar{u} \in \overline{\mathbb{R}}_{[0, +\infty]} \end{aligned} \tag{A.79}$$

However, the existence of $\bar{u}$ does not imply the existence of u_1 ; for example, the function defined almost everywhere on $\mathbb{R}$ by $\sin(\omega_0 t) \mathfrak{G}(t)$ has null right-sided mean value on $\mathbb{R}$ but does not have integral on $\mathbb{R}$. Whenever both u_1 and $\bar{u}$ exist, we have

$$\begin{aligned} \bar{u} \in \overline{\mathbb{R}}_{[-\infty, 0)} &\Rightarrow u_1 = -\infty \\ \bar{u} = 0 &\Rightarrow u_1 \in \overline{\mathbb{R}} \\ \bar{u} \in \overline{\mathbb{R}}_{(0, +\infty]} &\Rightarrow u_1 = +\infty \end{aligned} \tag{A.80}$$

From Equation (A.79) and Equation (A.80), we observe in particular that a null u_1 implies a null $\bar{u}$ but a null $\bar{u}$ does not imply a null u_1 .

We know that, in general, distributions on $\mathbb{R}$ cannot be evaluated against test functions that are not smooth functions on $\mathbb{R}$ and cannot be squared [Schwartz, 1966]. Accordingly, the concepts of integral on $\mathbb{R}$, quadratic integral on $\mathbb{R}$, (possibly right-sided) mean value on $\mathbb{R}$, and (possibly right-sided) root-mean-square value on $\mathbb{R}$ cannot be immediately applied to them. Nevertheless, such concepts can be generalised to be applicable to some distributions on $\mathbb{R}$ and then, by convention, to some others.

A distribution on $\mathbb{R}$ has finite integral on $\mathbb{R}$ if and only if it is an integrable distribution on $\mathbb{R}$ (but a distribution on $\mathbb{R}$ that is not an integrable distribution on $\mathbb{R}$ may have finite improper integral on $\mathbb{R}$). By convention, a non-negative-definite distribution on $\mathbb{R}$ that is not an integrable distribution on $\mathbb{R}$ has integral on $\mathbb{R}$ equalling $+\infty$.

A distribution on $\mathbb{R}$ has finite quadratic integral on $\mathbb{R}$ if and only if it is generable by a square integrable function on $\mathbb{R}$. By convention, a square integrable distribution on $\mathbb{R}$ that is not generable by a square integrable function on $\mathbb{R}$ has quadratic integral on $\mathbb{R}$ equalling $+\infty$.

The mean value on $\mathbb{R}$ (respectively, right-sided mean value on $\mathbb{R}$) of a distribution on $\mathbb{R}$ (respectively, right-sided distribution on $\mathbb{R}$), whenever it exists and is finite, is the value at the origin (respectively, 2 times the value at the origin) of the mean cross-correlation distribution of the ordered pair formed by it and the unit distribution on $\mathbb{R}$.

The root-mean-square value on $\mathbb{R}$ (respectively, right-sided root-mean-square value on $\mathbb{R}$) of a distribution on $\mathbb{R}$ (respectively, right-sided distribution on $\mathbb{R}$), whenever it exists and is finite, is the value at the origin (respectively, $2^{1/2}$ times the value at the origin) of its mean auto-correlation distribution.

Let $w \equiv u^{(-n)}$ be the right-sided n^{th} antiderivative of u . Provisionally, we confine ourselves to the case that u , and thus w , is a right-sided locally integrable function on $\mathbb{R}$. Almost everywhere on $\mathbb{R}$, we have

$$\begin{aligned} w(t) &\equiv u^{(-n)}(t) \\ &:= \int_{-\infty}^t \int_{-\infty}^{t'_1} \cdots \int_{-\infty}^{t'_{n-1}} (u) dt'_n \cdots dt'_2 dt'_1 \end{aligned} \tag{A.81}$$

where

- t'_i with i in $\mathbb{N}_{[1,n]}$ is a bound real variable of the same quantity dimension as t

Because u is right-sided, we observe that the lower limit of integration in Equation (A.81) is $-\infty$ only formally. Furthermore, because u is a locally integrable function on $\mathbb{R}$, by the Lebesgue's form of the fundamental theorem of calculus [Gordon, 1994] we observe that w exists and is a locally absolutely continuous function on $\mathbb{R}$ with

$$w^{(n)} = u \quad (\text{A.82})$$

where

- $w^{(n)}$ is the n^{th} derivative of w

For any representatives of w and u , Equation (A.82) holds almost everywhere on $\mathbb{R}$ (and even everywhere on $\mathbb{R}$ if u is a C^0 function on $\mathbb{R}$) [Gordon, 1994].

Almost everywhere on $\mathbb{R}$, by applying the Cauchy formula for higher-order antidifferentiation [McBride, 1979] to Equation (A.81), we have

$$\begin{aligned} w(t) &= \frac{1}{(n-1)!} \int_{-\infty}^t (t-t')^{n-1} u \, dt' \\ &= \frac{1}{(n-1)!} \int_{-\infty}^{+\infty} (t-t')^{n-1} (\check{\mathfrak{G}})_t u \, dt' \\ &= \frac{1}{(n-1)!} \int_{-\infty}^{+\infty} ((\check{\rho}_{n-1})_t u) \, dt' \\ &= \frac{1}{(n-1)!} \int_{-\infty}^{+\infty} (\rho_{n-1}(\check{u})_t) \, dt' \\ &= \int_{-\infty}^{+\infty} (\mathfrak{G}^{*n}(\check{u})_t) \, dt' \end{aligned}$$

and hence we have

$$\boxed{w = \mathfrak{G}^{*n} * u} \quad (\text{A.83})$$

where

- $(\check{\mathfrak{G}})_t$ is the t -translated reflected $\mathfrak{G}$
- $(\check{\rho}_{n-1})_t$ is the t -translated reflected ρ_{n-1}
- $(\check{u})_t$ is the t -translated reflected u

We observe that both u and $\mathfrak{G}^{*n}$ are right-sided locally integrable functions on $\mathbb{R}$. Hence, the convolution in Equation (A.83) is well-defined and commutative (as well as associative if $\mathfrak{G}^{*n}$ is expressed as the n -fold convolutive product of $\mathfrak{G}$ by itself), and w is a right-sided locally integrable function on $\mathbb{R}$. Furthermore, we observe that Equation (A.83) is still valid when $\mathfrak{G}^{*n}$ is regarded as a right-sided distribution on $\mathbb{R}$ and u , and thus w , is a right-sided distribution on $\mathbb{R}$ [Gelfand and Shilov, 1964; Zemanian, 1965a; Schwartz, 1966; Misra and Lavoine, 1986]; also in such case, the convolutions in it are well-defined, commutative, and associative [Schwartz, 1966].

We now re-express Equation (A.82) as

$$\delta^{(n)} * w = u \quad (\text{A.84})$$

By comparing Equation (A.83) with Equation (A.84), it is evident that $\mathfrak{G}^{*n}$ is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are convolutive inverses of $\delta^{(n)}$.

From Equation (A.81) and Equation (A.83), we introduce the *right-sided n^{th} -order antidifferentiator* (hereinafter *n^{th} -order antidifferentiator*) as the convolution system of input variable u and output variable w characterised for n in $\mathbb{N}_{[1,+\infty)}$ by the input–output representation

$$\begin{aligned} w &:= \mathfrak{G}^{*n} * u \\ &= g * u \end{aligned} \quad (\text{A.85})$$

where

- g is the convolution kernel distribution of the n^{th} -order antidifferentiator defined by

$$\begin{aligned} g &:= \mathfrak{G}^{*n} * \delta \\ &= \mathfrak{G}^{*n} \end{aligned} \quad (\text{A.86})$$

From Equation (A.82), by replacing w with the unknown distribution $\left(u^{(-n)}\right)_{\text{gen}}$, we obtain an n^{th} -order differential equation of constant term $-u$. We have

$$\left(\left(u^{(-n)}\right)_{\text{gen}}\right)^{(n)} = u \quad (\text{A.87})$$

where

- $\left(\left(a^{(-n)}\right)_{\text{gen}}\right)^{(n)}$ is the n^{th} derivative of $\left(a^{(-n)}\right)_{\text{gen}}$

We recognise that w , as expressed by Equation (A.85), is a particular solution of Equation (A.87). Indeed, w is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are n^{th} antiderivatives of u , which are provided by the general solution of Equation (A.87) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. Let $u^{(-i)}$ with i in $\mathbb{N}_{[1,n]}$ be the right-sided i^{th} antiderivative of u , and let $u^{(0)}$ be the 0^{th} antiderivative of u as well as the 0^{th} derivative of u . We have

$$\begin{aligned} \left(u^{(-n)}\right)_{\text{gen}} &\equiv \int \int \cdots \int (u) dt'_n \cdots dt'_2 dt'_1 \\ &= g * u + \sum_{i=1}^n \hat{u}^{(-i)} t^{n-i} \\ &= w + \sum_{i=1}^n \hat{u}^{(-i)} t^{n-i} \end{aligned} \tag{A.88}$$

where

- $\int \int \cdots \int (u) dt'_n \cdots dt'_2 dt'_1$ is the n^{th} indefinite integral of u
- $\hat{u}^{(-i)}$ with i in $\mathbb{N}_{[1,n]}$ is an arbitrary complex constant of the same quantity dimensions as $u^{(-i)}$

In agreement with the fact that we assume u real-valued, we assume $\hat{u}^{(-i)}$ with i in $\mathbb{N}_{[1,n]}$ real.

From Equation (A.86), it can be easily verified that the n^{th} -order antidifferentiator has the property of causality; furthermore, the n^{th} -order antidifferentiator has the property of input–output stability if its input space and output space coincide with the distribution space of the distributions of rapid decay on $\mathbb{R}$ with null moments of any order in $\mathbb{N}_{[0,n-1]}$.

We observe that Equation (A.83) is valid even if u is an $(n-1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case w (which is a $(-n+1)$ -weighted bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are n^{th} antiderivatives of u . Furthermore, we observe that the convolutive product of u by any finite set of convolutive powers of $\mathfrak{g}$ having convolutive exponents summing to n is well-defined, commutative, and associative if u is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if u is an $(n-1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither).

A.7.2.2 Frequency domain

For the following developments, we now assume that u is a right-sided distribution of rapid decay on $\mathbb{R}$. We recall that, for a distribution of rapid decay on $\mathbb{R}$, the moment of any order in $\mathbb{N}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense). In particular, each distribution of rapid decay on $\mathbb{R}$ is an integrable distribution on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966]; accordingly, from Equation (A.79) and Equation (A.80) we observe that u_1 is finite and $\bar{u}$ is null.

Because u is a right-sided distribution of rapid decay on $\mathbb{R}$, its Fourier transform is generable by a smooth function of slow growth on $\mathbb{R}$ [Schwartz, 1966]. Let U be the Fourier transform of u .

Let w be expressed, from Equation (A.85), as

$$w = g * u \quad (\text{A.89})$$

Under the assumptions made, we know that the convolution in Equation (A.89) is well-defined and commutative. In fact, g is a right-sided $(-n+1)$ -weighted bounded distribution on $\mathbb{R}$, and u is a right-sided distribution of rapid decay on $\mathbb{R}$; hence, w , as the convolutive product of g and u , is a right-sided $(-n+1)$ -weighted bounded distribution on $\mathbb{R}$, and its Fourier transform is a tempered distribution on $\mathbb{R}$. Accordingly, the convolution Fourier exchange theorem applies in its standard form, and we have

$$\mathcal{F}(g * u) = \mathcal{F}(g) \cdot \mathcal{F}(u) \quad (\text{A.90})$$

Let $W \equiv U_{(-n)}$ be the Fourier transform of $w \equiv u^{(-n)}$. From Equation (A.89) and Equation (A.90), we have

$$\begin{aligned} W &= \mathcal{F}(w) \\ &= \mathcal{F}(g * u) \\ &= \mathcal{F}(g) \cdot \mathcal{F}(u) \\ &= G \cdot U \end{aligned}$$

and hence

$$\begin{aligned}
 W &= \left(\text{f.p.} \left(\frac{1}{(j\omega)^n} \right) + j^{n-1} \frac{\pi}{(n-1)!} \delta^{(n-1)} \right) U \\
 &= \text{f.p.} \left(\frac{U}{(j\omega)^n} \right) + \sum_{i=1}^n \left((-1)^{i-1} j^{n-1} \frac{\pi}{(n-i)!(i-1)!} U^{(i-1)}(0) \delta^{(n-i)} \right)
 \end{aligned} \tag{A.91}$$

with [Estrada and Kanwal, 2002]

$$\begin{aligned}
 j^{i-1} U^{(i-1)}(0) &= \int_{-\infty}^{+\infty} \left((t')^{i-1} u \right) dt' \\
 &= \lim_{t \rightarrow +\infty} \left((t^{i-1} u)^{(-1)} \right)
 \end{aligned} , \quad i \in \mathbb{N}_{[1,n]} \tag{A.92}$$

where

- G is the frequency-response distribution of the n^{th} -order antidifferentiator defined by

$$\begin{aligned}
 G &:= \mathcal{F}(g) \\
 &= \mathcal{F}(\mathfrak{g}^{*n}) \\
 &= \Theta_{*n} \\
 &= \frac{1}{(n-1)!} \mathbf{P}_{n-1} \\
 &= \text{f.p.} \left(\frac{1}{(j\omega)^n} \right) + j^{n-1} \frac{\pi}{(n-1)!} \delta^{(n-1)}
 \end{aligned}$$

- Θ_{*n} is the Fourier transform of $\mathfrak{g}^{*n}$
- $\mathbf{P}_{n-1}$ is the Fourier transform of ρ_{n-1}
- $U^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ is the $(i-1)^{\text{th}}$ derivative of U
- $\int_{-\infty}^{+\infty} \left((t')^{i-1} u \right) dt'$ with i in $\mathbb{N}_{[1,n]}$ is the $(i-1)^{\text{th}}$ moment of u
- $(t^{i-1} u)^{(-1)}$ with i in $\mathbb{N}_{[1,n]}$ is the right-sided 1st antiderivative of $t^{i-1} u$

The expression of $\mathbf{P}_{n-1}$ can be obtained by recalling well-known Fourier-transform pairs [Kanwal, 2004] and by applying the multiplication Fourier exchange theorem in its standard form. We have

$$\begin{aligned}
P_{n-1} &= \mathcal{F}(\rho_{n-1}) \\
&= \mathcal{F}(t^{n-1} \mathfrak{g}) \\
&= \mathcal{F}\left(\frac{t^{n-1} \operatorname{sgn} + t^{n-1}}{2}\right) \\
&= \mathcal{F}\left(t^{n-1} \cdot \frac{\operatorname{sgn}}{2}\right) + \mathcal{F}\left(\frac{t^{n-1}}{2}\right) \\
&= \frac{1}{2\pi} \left(\mathcal{F}(t^{n-1}) * \mathcal{F}\left(\frac{\operatorname{sgn}}{2}\right) \right) + \mathcal{F}\left(\frac{t^{n-1}}{2}\right) \\
&= \frac{1}{2\pi} \left(j^{n-1} 2\pi \delta^{(n-1)} * \text{p.v.}\left(\frac{1}{j\omega}\right) \right) + j^{n-1} \pi \delta^{(n-1)} \\
&= \text{f.p.}\left(\frac{(n-1)!}{(j\omega)^n}\right) + j^{n-1} \pi \delta^{(n-1)}
\end{aligned}$$

Because u is a right-sided distribution of rapid decay on $\mathbb{R}$, U and all the $U^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ are generable by smooth functions of slow growth on $\mathbb{R}$. Accordingly, in Equation (A.91), the multiplications of U by both $\text{f.p.}\left(1/(j\omega)^n\right)$ and $\delta^{(n-1)}$ are well-defined; furthermore, in Equation (A.92), all the $U^{(i-1)}(0)$ with i in $\mathbb{N}_{[1,n]}$ are finite, and so are all the integrals on $\mathbb{R}$. Specifically, $U(0)$ equals the integral on $\mathbb{R}$ of u as well as the limit (in general, in the Cesàro sense) of $u^{(-1)}$ as t tends to $+\infty$. It is worth noting that $j^{i-1}U^{(i-1)}(0)$ with i in $\mathbb{N}_{[1,n]}$ equals the $(i-1)^{\text{th}}$ moment of u . In fact, because u is a right-sided distribution of rapid decay on $\mathbb{R}$, its moment of any order in $\mathbb{N}$ is well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense).

By applying to Equation (A.84) the convolution Fourier exchange theorem in its standard form, we have

$$(j\omega)^n \cdot W = U \quad (\text{A.93})$$

By comparing Equation (A.91) with Equation (A.93), it is evident that

$\text{f.p.}\left(1/(j\omega)^n\right) + j^{n-1}\left(\pi/((n-1)!)\right)\delta^{(n-1)}$ is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are multiplicative inverses of $(j\omega)^n$. Besides, it is worth noting that we have

$$\begin{aligned}
 \text{f.p.} \left(\frac{1}{(j\omega)^n} \right) + j^{n-1} \frac{\pi}{(n-1)!} \delta^{(n-1)} &= \lim_{\sigma \rightarrow 0^-} \left(\left(\frac{1}{(j\underline{s})^n} \right)_{\underline{s} = \omega + j\underline{\sigma}} \right), \quad \begin{cases} \underline{\sigma} \in \mathbb{R}_{(-\infty, 0)} \\ \underline{s} \in \mathbb{C}_{\text{Im} \in (-\infty, 0)} \end{cases} \\
 &= \lim_{\sigma \rightarrow 0^+} \left(\left(\frac{1}{s^n} \right)_{s = \sigma + j\omega} \right), \quad \begin{cases} \sigma \in \mathbb{R}_{(0, +\infty)} \\ s \in \mathbb{C}_{\text{Re} \in (0, +\infty)} \end{cases}
 \end{aligned} \tag{A.94}$$

$1/(j\underline{s})^n$ and $\mathbb{C}_{\text{Im} \in (-\infty, 0)}$ being respectively the Fourier–Laplace transform of ρ_{n-1} and its region of definition, $1/s^n$ and $\mathbb{C}_{\text{Re} \in (0, +\infty)}$ being respectively the Laplace transform of ρ_{n-1} and its region of definition, and the limits in Equation (A.94) being within the distribution space of the tempered distributions on $\mathbb{R}$, which implies that $\left(1/(j\underline{s})^n\right)_{\underline{s} = \omega + j\underline{\sigma}}$ generates a tempered distribution on $\mathbb{R}$ for each fixed $\underline{\sigma}$ in $\mathbb{R}_{(-\infty, 0)}$ and $\left(1/s^n\right)_{s = \sigma + j\omega}$ generates a tempered distribution on $\mathbb{R}$ for each fixed σ in $\mathbb{R}_{(0, +\infty)}$.

From Equation (A.93), by replacing W with the unknown distribution $\left(U_{(-n)}\right)_{\text{gen}}$, we obtain an n^{th} -degree algebraic equation of constant term $-U$. We have

$$(j\omega)^n \left(U_{(-n)}\right)_{\text{gen}} = U \tag{A.95}$$

We recognise that W , as expressed by Equation (A.91), is a particular solution of Equation (A.95). Indeed, W is the only one that is right-sided amongst the infinitely many distributions on $\mathbb{R}$ that are quotients of U by $(j\omega)^n$, which are provided by the general solution of Equation (A.95) [Schwartz, 1966; Jones, 1982; Misra and Lavoine, 1986]. We have

$$\begin{aligned}
 \left(U_{(-n)}\right)_{\text{gen}} &\equiv \text{i.q.} \left(\frac{U}{(j\omega)^n} \right) \\
 &= \text{f.p.} \left(\frac{U}{(j\omega)^n} \right) + \sum_{i=1}^n \left((-1)^{i-1} j^{n-1} \frac{\pi}{(n-i)!(i-1)!} \hat{U}^{(i-1)} \delta^{(n-i)} \right) \\
 &= W + \sum_{i=1}^n \left(j^{n-i} 2\pi \hat{u}^{(-i)} \delta^{(n-i)} \right)
 \end{aligned} \tag{A.96}$$

where

– $\text{i.q.} \left(U/(j\omega)^n \right)$ is the indefinite quotient of U by $(j\omega)^n$

A.7 Systems: Antidifferentiators, differentiators, and generalised differentiators

- $\hat{U}^{(i-1)}$ with i in $\mathbb{N}_{[1,n]}$ is an arbitrary complex constant of the same quantity dimension as $U^{(i-1)}$ with

$$\hat{U}^{(i-1)} = U^{(i-1)}(0) + j^{i-1} 2(n-i)!(i-1)! \hat{u}^{(-i)} \quad , \quad i \in \mathbb{N}_{[1,n]}$$

If we assume $\hat{u}^{(-i)}$ real (in agreement with the fact that we assume u real-valued), then $\hat{U}^{(i-1)}$ is either real (when $i-1$ is even) or imaginary (when $i-1$ is odd). As is evident, Equation (A.95) and Equation (A.96) are the Fourier transforms of respectively Equation (A.87) and Equation (A.88).

We observe that the localisation of f.p. $\left(1/(j\omega)^n\right)$ at any open subset of $\mathbb{R}$ not containing the origin results in a regular tempered distribution thereon; furthermore, its multiplication by a smooth function of slow growth on $\mathbb{R}$ whose derivative of any order in $\mathbb{N}_{[0,n-1]}$ vanishes at the origin results in a regular tempered distribution on $\mathbb{R}$. Therefore, from Equation (A.91), we have

$$\boxed{\forall i \in \mathbb{N}_{[1,n]} \quad U^{(i-1)}(0) = 0 \quad \Leftrightarrow \quad W = \frac{U}{(j\omega)^n}} \quad (\text{A.97})$$

As an implication of Equation (A.97), a necessary and sufficient condition for $U^{(i-1)}(0)$ (or, equivalently, the $(i-1)^{\text{th}}$ moment of u) to be null for each i in $\mathbb{N}_{[1,n]}$ is that W , like U , is generable by a smooth function of slow growth on $\mathbb{R}$ (and hence w , like u , is a right-sided distribution of rapid decay on $\mathbb{R}$).

Let $U_{(-i)}$ with i in $\mathbb{N}_{[1,n]}$ be the Fourier transform of $u^{(-i)}$, and let $U_{(0)}$ be the same as $U^{(0)}$. Let $\underline{n} \equiv \underline{n}_1$ be a natural constant in $\mathbb{N}_{[1,n]}$, and let $\underline{i} \equiv \underline{i}_1$ be the generic element of the index set consisting of the natural constants $\mathbb{N}_{[1,\underline{n}]}$. When n is in $\mathbb{N}_{[2,+\infty)}$, let $\underline{n}_2$ be a natural constant in $\mathbb{N}_{[2,n]}$, and let $\underline{i}_2$ be the generic element of the index set consisting of the natural constants in $\mathbb{N}_{[1,\underline{n}_2-1]}$.

We now show that the fact that all $U^{(\underline{i}-1)}(0)$ with $\underline{i}$ in $\mathbb{N}_{[1,\underline{n}]}$ are null is equivalent to the fact that all $U_{(-\underline{i}+1)}(0)$ with $\underline{i}$ in $\mathbb{N}_{[1,\underline{n}]}$ are null. The case in which n equals 1 is trivial. If n is in $\mathbb{N}_{[2,+\infty)}$, by recalling the differentiation property of the Fourier transformation, we have

$$\begin{aligned} \forall \underline{i}_2 \in \mathbb{N}_{[1, n_2-1]} \quad U^{(\underline{i}_2)} &= \left((j\omega)^{\underline{i}_2} U_{(-\underline{i}_2)} \right)^{(\underline{i}_2)} \\ &= \sum_{y=0}^{\underline{i}_2} \binom{\underline{i}_2}{y} \left((j\omega)^{\underline{i}_2} \right)^{(\underline{i}_2-y)} \left(U_{(-\underline{i}_2)} \right)^{(y)} \end{aligned} \quad (\text{A.98})$$

where

- y is a bound index taking values in $\mathbb{N}_{[0, \underline{i}_2]}$
- $\left((j\omega)^{\underline{i}_2} U_{(-\underline{i}_2)} \right)^{(\underline{i}_2)}$ with $\underline{i}_2$ in $\mathbb{N}_{[1, n_2-1]}$ is the $\underline{i}_2^{\text{th}}$ derivative of $(j\omega)^{\underline{i}_2} U_{(-\underline{i}_2)}$
- $\left((j\omega)^{\underline{i}_2} \right)^{(\underline{i}_2-y)}$ with $\underline{i}_2$ in $\mathbb{N}_{[1, n_2-1]}$ and y in $\mathbb{N}_{[0, \underline{i}_2]}$ is the $(\underline{i}_2 - y)^{\text{th}}$ derivative of $(j\omega)^{\underline{i}_2}$
- $\left(U_{(-\underline{i}_2)} \right)^{(y)}$ with $\underline{i}_2$ in $\mathbb{N}_{[1, n_2-1]}$ and y in $\mathbb{N}_{[0, \underline{i}_2]}$ is the y^{th} derivative of $U_{(-\underline{i}_2)}$
- $\binom{\underline{i}_2}{y}$ is a binomial coefficient

Equation (A.98) can be localised at the origin only in special circumstances; in particular, this is the case when the derivative of any order in $\mathbb{N}_{[0, n_2-1]}$ of each $U_{(-\underline{i}_2)}$ with $\underline{i}_2$ in $\mathbb{N}_{[1, n_2-1]}$ is a regular distribution on $\mathbb{R}$. We observe that a necessary and sufficient condition for each $U_{(-\underline{i}_2)}$ with $\underline{i}_2$ in $\mathbb{N}_{[1, n_2-1]}$ (as well as its derivative of any order in $\mathbb{N}$) to be not only regular but also generable by a smooth function of slow growth on $\mathbb{R}$ is that each $U^{(\underline{i}_2-1)}(0)$ with $\underline{i}_2$ in $\mathbb{N}_{[1, n_2-1]}$ is null. Therefore, from Equation (A.98), we have

$$\forall \underline{i}_2 \in \mathbb{N}_{[1, n_2-1]} \quad U^{(\underline{i}_2-1)}(0) = 0 \quad \Leftrightarrow \quad \forall \underline{i}_2 \in \mathbb{N}_{[1, n_2-1]} \quad U^{(\underline{i}_2)}(0) = j^{\underline{i}_2} \underline{i}_2! U_{(-\underline{i}_2)}(0)$$

and hence, with no restrictions on n , we conclude that we have

$$\forall \underline{i} \in \mathbb{N}_{[1, n]} \quad U^{(\underline{i}-1)}(0) = 0 \quad \Leftrightarrow \quad \forall \underline{i} \in \mathbb{N}_{[1, n]} \quad U_{(-\underline{i}+1)}(0) = 0$$

Ultimately, we can generalise Equation (A.97) to obtain

$$\boxed{\begin{aligned} \forall \underline{i} \in \mathbb{N}_{[1, n]} \quad U^{(\underline{i}-1)}(0) = 0 &\quad \Leftrightarrow \quad \forall \underline{i} \in \mathbb{N}_{[1, n]} \quad U_{(-\underline{i}+1)}(0) = 0 \\ &\quad \Leftrightarrow \quad \forall \underline{i} \in \mathbb{N}_{[1, n]} \quad U_{(-\underline{i})} = \frac{U_{(-\underline{i}+1)}}{(j\omega)^{\underline{i}-1}} \end{aligned}} \quad (\text{A.99})$$

As an implication of Equation (A.99), a necessary and sufficient condition for $U^{(i-1)}(0)$ (or, equivalently, the $(i-1)^{\text{th}}$ moment of u) to be null for each i in $\mathbb{N}_{[1, \underline{n}]}$ is that $U_{(-i+1)}(0)$ (or, equivalently, the 0^{th} moment of $u^{(-i+1)}$) is null for each i in $\mathbb{N}_{[1, \underline{n}]}$ or, equivalently, that $U_{(-i)}$, like U , is generable by a smooth function of slow growth on $\mathbb{R}$ for each i in $\mathbb{N}_{[1, \underline{n}]}$ (and hence $u^{(-i)}$, like u , is a right-sided distribution of rapid decay on $\mathbb{R}$ for each i in $\mathbb{N}_{[1, \underline{n}]}$).

We know that Equation (A.83) is valid even if u is an $(n-1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither), in which case w (which is a $(-n+1)$ -weighted bounded distribution on $\mathbb{R}$) is the only one that is vanishing at negative infinity amongst the infinitely many distributions on $\mathbb{R}$ that are n^{th} antiderivatives of u ; accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, Equation (A.91) is still valid too. Furthermore, we know that the convolutive product of u by any finite set of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to n is well-defined, commutative, and associative if u is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if u is an $(n-1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither); accordingly, by virtue of the generalised form of the convolution Fourier exchange theorem, the multiplicative product of the Fourier transform of u by any finite set of Fourier transforms of convolutive powers of $\mathfrak{G}$ having convolutive exponents summing to n is well-defined, commutative, and associative, and it coincides with the Fourier transform of the convolutive product of u by the corresponding finite set of convolutive powers of $\mathfrak{G}$, if u is a distribution of rapid decay on $\mathbb{R}$ (be it right-sided, one-sided, or neither) but not, in general, if u is an $(n-1)$ -weighted integrable distribution on $\mathbb{R}$ (be it right-sided, one-sided, or neither).

Finally, we re-express Equation (A.91) and Equation (A.93) as

$$\boxed{\mathcal{F}(\mathcal{D}^{-n}(u)) = \left(\text{f.p.} \left(\frac{1}{(j\omega)^n} \right) + j^{n-1} \frac{\pi}{(n-1)!} \delta^{(n-1)} \right) \mathcal{F}(u)} \quad (\text{A.100})$$

$$\boxed{\mathcal{F}(\mathcal{D}^n(w)) = (j\omega)^n \mathcal{F}(w)} \quad (\text{A.101})$$

where

- $\mathcal{D}^{-n}$ is the *right-sided n^{th} -order antidifferentiation operator*
- $\mathcal{D}^n$ is the *n^{th} -order differentiation operator*

It can be easily verified that we have

$$\boxed{\mathcal{D}^{-n}(\mathcal{F}(u)) = \mathcal{F}\left(\left(\text{f.p.}\left(\frac{1}{(-jt)^n}\right) + (-j)^{n-1} \frac{\pi}{(n-1)!} \delta^{(n-1)}\right)u\right)} \quad (\text{A.102})$$

$$\boxed{\mathcal{D}^n(\mathcal{F}(w)) = \mathcal{F}\left((-jt)^n w\right)} \quad (\text{A.103})$$

As is evident, Equation (A.102) and Equation (A.103) are the duals to Equation (A.100) and Equation (A.101).

A.7.2.3 Example

As an example, we consider the case in which we have

$$u = u_0 t_\delta^{N+1} \delta^{(N)} \quad (\text{A.104})$$

where

- N is a natural constant of quantity dimension one
- u_0 is a complex constant of the same quantity dimension as u

We observe that u is a point-supported distribution on $\mathbb{R}$.

In the time domain, it can be easily verified that we have

$$\begin{aligned} w &\equiv u^{(-n)} \\ &= \begin{cases} u_0 t_\delta^{N+1} \delta^{(N-n)} & , \quad n \in \mathbb{N}_{[1, N]} \\ u_0 t_\delta^{N+1} \mathfrak{G}^{*(n-N)} & , \quad n \in \mathbb{N}_{[N, +\infty)} \end{cases} \end{aligned} \quad (\text{A.105})$$

In the frequency domain, it can be easily verified that we have

$$\begin{aligned} U &= u_0 t_\delta^{N+1} (j\omega)^N \\ W &\equiv U_{(-n)} \\ &= \begin{cases} u_0 t_\delta^{N+1} (j\omega)^{N-n} & , \quad n \in \mathbb{N}_{[1, N]} \\ u_0 t_\delta^{N+1} \Theta_{*(n-N)} & , \quad n \in \mathbb{N}_{[N, +\infty)} \end{cases} \end{aligned}$$

A.7.3 Differentiators

We now introduce systems for differentiation.

A.7.3.1 Time domain

Let u be a distribution on $\mathbb{R}$.

Let $w \equiv u^{(n)}$ be the n^{th} derivative of u . We have

$$\boxed{w = \delta^{(n)} * u} \quad (\text{A.106})$$

We observe that $\delta^{(n)}$ is a point-supported distribution on $\mathbb{R}$. Hence, the convolution in Equation (A.106) is well-defined and commutative, and w is a distribution on $\mathbb{R}$ [Schwartz, 1966].

From Equation (A.106), we introduce the n^{th} -order differentiator as the convolution system of input variable u and output variable w characterised for n in $\mathbb{N}_{[1,+\infty)}$ by the input–output representation

$$\begin{aligned} w &:= \delta^{(n)} * u \\ &= g * u \end{aligned} \quad (\text{A.107})$$

where

- g is the convolution kernel distribution of the n^{th} -order differentiator defined by

$$\begin{aligned} g &:= \delta^{(n)} * \delta \\ &= \delta^{(n)} \end{aligned} \quad (\text{A.108})$$

From Equation (A.108), it can be easily verified that the n^{th} -order differentiator has the property of causality; furthermore, the n^{th} -order differentiator has the property of input–output stability if its input space and output space coincide with any of many distribution spaces (including the distribution space of the distributions on $\mathbb{R}$, the distribution space of the tempered distributions on $\mathbb{R}$, and the distribution space of the distributions of rapid decay on $\mathbb{R}$).

We observe that the convolutive product of u by any finite set of convolutive powers of $\delta^{(1)}$ having convolutive exponents summing to n is well-defined, commutative, and associative.

A.7.3.2 Frequency domain

For the following developments, we now assume that u is a tempered distribution on $\mathbb{R}$.

Because u is a tempered distribution on $\mathbb{R}$, its Fourier transform is a tempered distribution on $\mathbb{R}$ [Schwartz, 1966]. Let U be the Fourier transform of u .

Let w be expressed, from Equation (A.107), as

$$w = g * u \quad (\text{A.109})$$

Under the assumptions made, we know that the convolution in Equation (A.109) is well-defined and commutative. In fact, g is a point-supported distribution on $\mathbb{R}$, and u is a tempered distribution on $\mathbb{R}$; hence, w , as the convolutive product of g and u , is a tempered distribution on $\mathbb{R}$, and its Fourier transform is a tempered distribution on $\mathbb{R}$. Accordingly, the convolution Fourier exchange theorem applies in its standard form, and we have

$$\mathcal{F}(g * u) = \mathcal{F}(g) \cdot \mathcal{F}(u) \quad (\text{A.110})$$

Let $W \equiv U_{(n)}$ be the Fourier transform of $w \equiv u^{(n)}$. From Equation (A.109) and Equation (A.110), we have

$$\begin{aligned} W &= \mathcal{F}(w) \\ &= \mathcal{F}(g * u) \\ &= \mathcal{F}(g) \cdot \mathcal{F}(u) \\ &= G \cdot U \end{aligned}$$

and hence

$$\boxed{W = (j\omega)^n U}$$

where

- G is the frequency-response distribution of the n^{th} -order differentiator defined by

$$\begin{aligned} G &:= \mathcal{F}(g) \\ &= \mathcal{F}(\delta^{(n)}) \\ &= (j\omega)^n \end{aligned}$$

We know that the convolutive product of u by any finite set of convolutive powers of $\delta^{(1)}$ having convolutive exponents summing to n is well-defined, commutative, and associative; accordingly, by virtue of the standard form of the convolution Fourier exchange theorem, the multiplicative product of the Fourier transform of u by any finite set of Fourier transforms of convolutive powers of $\delta^{(1)}$ having convolutive exponents summing to n is well-defined, commutative, and associative, and it coincides with the Fourier transform of the convolutive product of u by the corresponding finite set of convolutive powers of $\delta^{(1)}$.

A.7.3.3 Example

As an example, we consider the case in which we have

$$u = \frac{u_0}{t_{\vartheta}^{N-1}} \mathfrak{G}^{*N}$$

We observe that u is a right-sided tempered distribution on $\mathbb{R}$.

In the time domain, it can be easily verified that we have

$$\begin{aligned} w &\equiv u^{(n)} \\ &= \begin{cases} \frac{u_0}{t_{\vartheta}^{N-1}} \mathfrak{G}^{*(N-n)} & , \quad n \in \mathbb{N}_{[1,N]} \\ \frac{u_0}{t_{\vartheta}^{N-1}} \delta^{(n-N)} & , \quad n \in \mathbb{N}_{[N,+\infty)} \end{cases} \end{aligned}$$

In the frequency domain, it can be easily verified that we have

$$U = \frac{u_0}{t_{\vartheta}^{N-1}} \Theta_{*N}$$

$$\begin{aligned} W &\equiv U_{(n)} \\ &= \begin{cases} \frac{u_0}{t_{\vartheta}^{N-1}} \Theta_{*(N-n)} & , \quad n \in \mathbb{N}_{[1,N]} \\ \frac{u_0}{t_{\vartheta}^{N-1}} (j\omega)^{n-N} & , \quad n \in \mathbb{N}_{[N,+\infty)} \end{cases} \end{aligned}$$

A.7.4 Generalised differentiators

We now introduce systems for generalised differentiation to unify the treatment of systems for antidifferentiation and systems for differentiation.

A.7.4.1 Time domain

Let u be a right-sided locally integrable function on $\mathbb{R}$ of t . We denote by u_1 , u_Q , $\bar{u}$, and $\bar{u}_q$ respectively its integral on $\mathbb{R}$, its quadratic integral on $\mathbb{R}$, its right-sided mean value on $\mathbb{R}$, and its right-sided root-mean-square value on $\mathbb{R}$ defined in accordance with respectively Equation (A.75), Equation (A.76), Equation (A.77), and Equation (A.78).

Let $w \equiv u^{(z)}$ be the *right-sided z^{th} generalised derivative* of u . By recalling Equation (A.81), Equation (A.83), Equation (A.104) and Equation (A.105) with $N = 0$ and $u_0 t_0^{N+1} = 1$, and Equation (A.106), we have

$$\boxed{w = \delta^{(z)} * u} \quad (\text{A.111})$$

where

– $\delta^{(z)}$ is the right-sided z^{th} generalised derivative of δ expressed as

$$\delta^{(z)} = \begin{cases} \mathfrak{g}^{*|z|} & , \quad z \in \mathbb{Z}_{(-\infty, 0)} \\ \delta^{(z)} & , \quad z \in \mathbb{Z}_{[0, +\infty)} \end{cases}$$

We observe that $\delta^{(z)}$ is either a right-sided tempered distribution on $\mathbb{R}$ (when z is in $\mathbb{Z}_{(-\infty, 0)}$) or a point-supported distribution on $\mathbb{R}$ (when z is in $\mathbb{Z}_{[0, +\infty)}$). Hence, the convolution in Equation (A.111) is well-defined and commutative, and w is a right-sided distribution on $\mathbb{R}$ [Schwartz, 1966].

From Equation (A.111), we introduce the *right-sided z^{th} -order generalised differentiator* (hereinafter *z^{th} -order generalised differentiator*) as the convolution system of input variable u and output variable w characterised for z in $\mathbb{Z}$ by the input–output representation

$$\begin{aligned} w &:= \delta^{(z)} * u \\ &= g * u \end{aligned} \quad (\text{A.112})$$

where

- g is the convolution kernel distribution of the z^{th} -order generalised differentiator defined by

$$\begin{aligned} g &:= \delta^{(z)} * \delta \\ &= \delta^{(z)} \end{aligned} \quad (\text{A.113})$$

From Equation (A.113), it can be easily verified that the z^{th} -order generalised differentiator has the property of causality; furthermore, the z^{th} -order generalised differentiator of order in $\mathbb{Z}_{(-\infty, 0)}$ has the property of input–output stability if its input space and output space coincide with the distribution space of the distributions of rapid decay on $\mathbb{R}$ with null moments of any order in $\mathbb{N}_{|z|-1}$, whereas the z^{th} -order generalised differentiator of order in $\mathbb{Z}_{[0, +\infty)}$ has the property of input–output stability if its input space and output space coincide with any of many distribution spaces (including the distribution space of the distributions on $\mathbb{R}$, the distribution space of the tempered distributions on $\mathbb{R}$, and the distribution space of the distributions of rapid decay on $\mathbb{R}$).

It can be easily verified that, for any z_1 in $\mathbb{Z}$ and z_2 in $\mathbb{Z}$, the series-equivalent system of the z_1^{th} -order generalised differentiator and the z_2^{th} -order generalised differentiator admits δ as an input variable, the series connection being commutative; furthermore, the convolution kernel distribution g_{eq} of the series-equivalent system coincides with the convolutive product of the convolution kernel distributions g_1 and g_2 of the two convolution systems, and we have

$$\begin{aligned} g_{\text{eq}} &= \delta^{(z_1)} * \delta^{(z_2)} \\ &= \delta^{(z_1 + z_2)} \end{aligned} \quad (\text{A.114})$$

In particular, the z^{th} -order generalised differentiator and the $-z^{\text{th}}$ -order generalised differentiator are mutually inverse if the input spaces of both their series-equivalent systems coincide with the distribution space of the distributions of rapid decay on $\mathbb{R}$.

Equation (A.114) can be immediately generalised to be applicable to the case of an arbitrary number of z^{th} -order generalised differentiators in series connection.

A.7.4.2 Frequency domain

For the following developments, we now assume that u is a right-sided distribution of rapid decay on $\mathbb{R}$. We recall that, for a distribution of rapid decay on $\mathbb{R}$, the moment of any order in $\mathbb{N}$ is

well-defined in the ordinary sense (and hence in the Cesàro sense) and is finite (the moment of any order in $\mathbb{N}$ in the ordinary sense being the same as the one in the Cesàro sense). In particular, each distribution of rapid decay on $\mathbb{R}$ is an integrable distribution on $\mathbb{R}$ [Schwartz, 1966; Horváth, 1966]; accordingly, from Equation (A.79) and Equation (A.80) we observe that u_1 is finite and $\bar{u}$ is null.

Because u is a right-sided distribution of rapid decay on $\mathbb{R}$, its Fourier transform is generable by a smooth function of slow growth on $\mathbb{R}$ [Schwartz, 1966]. Let U be the Fourier transform of u .

Let w be expressed, from Equation (A.112), as

$$w = g * u \quad (\text{A.115})$$

Under the assumptions made, we know that the convolution in Equation (A.115) is well-defined and commutative. Furthermore, the convolution Fourier exchange theorem applies in its standard form, and we have

$$\mathcal{F}(g * u) = \mathcal{F}(g) \cdot \mathcal{F}(u) \quad (\text{A.116})$$

Let $W \equiv U_{(z)}$ be the Fourier transform of $w \equiv u^{(z)}$. From Equation (A.115) and Equation (A.116), we have

$$\begin{aligned} W &= \mathcal{F}(w) \\ &= \mathcal{F}(g * u) \\ &= \mathcal{F}(g) \cdot \mathcal{F}(u) \\ &= G \cdot U \end{aligned}$$

and hence

$$\boxed{W = \Delta_{(z)} U}$$

where

- G is the frequency-response distribution of the z^{th} -order generalised differentiator defined by

$$\begin{aligned} G &:= \mathcal{F}(g) \\ &= \mathcal{F}(\delta^{(z)}) \\ &= \Delta_{(z)} \end{aligned}$$

- $\Delta_{(z)}$ is the Fourier transform of $\delta^{(z)}$ expressed as

$$\Delta_{(z)} = \begin{cases} \text{f.p.} \left(\frac{1}{(j\omega)^{|z|}} \right) + j^{|z|-1} \frac{\pi}{(|z|-1)!} \delta^{(|z|-1)} & , \quad z \in \mathbb{Z}_{(-\infty, 0)} \\ (j\omega)^z & , \quad z \in \mathbb{Z}_{[0, +\infty)} \end{cases}$$

We know that, for any z_1 in $\mathbb{Z}$ and z_2 in $\mathbb{Z}$, the series-equivalent system of the z_1^{th} -order generalised differentiator and the z_2^{th} -order generalised differentiator admits δ as an input variable, the series connection being commutative. In its standard form, the convolution Fourier exchange theorem can be applied to the input–output representation of the series-equivalent system if and only if at least one of z_1 and z_2 is in $\mathbb{Z}_{[0, +\infty)}$. However, in its generalised form, it can be applied for any z_1 in $\mathbb{Z}$ and z_2 in $\mathbb{Z}$; furthermore, the frequency-response distribution G_{eq} of the series-equivalent system coincides with the multiplicative product of the frequency-response distributions g_1 and g_2 of the two convolution systems, and we have

$$\begin{aligned} G_{\text{eq}} &= \Delta_{(z_1)} \Delta_{(z_2)} \\ &= \Delta_{(z_1 + z_2)} \end{aligned} \tag{A.117}$$

Equation (A.117) can be immediately generalised to be applicable to the case of an arbitrary number of z^{th} -order generalised differentiators in series connection all of which except at most one are order in $\mathbb{Z}_{[0, +\infty)}$.

Finally, we generalise Equation (A.100) and Equation (A.101) as

$$\boxed{\mathcal{F}(\mathcal{D}^z(u)) = \Delta_{(z)} \mathcal{F}(u)}$$

and their duals Equation (A.102) and Equation (A.103) as

$$\boxed{\mathcal{D}^z(\mathcal{F}(w)) = \mathcal{F}(\tilde{\Delta}_{(z)} w)}$$

where

- $\mathcal{D}^z$ is the *right-sided* z^{th} -order generalised differentiation operator
- $\tilde{\Delta}_{(z)}$ is the time-domain reflected $\Delta_{(z)}$ expressed as

$$\begin{aligned}\tilde{\Delta}_{(z)} &= \left(\Delta_{(z)} \right)_{\theta=-t} \\ &= \begin{cases} \text{f.p.} \left(\frac{1}{(-jt)^{|z|}} \right) + j^{|z|-1} \frac{\pi}{(|z|-1)!} \left((-1)^{|z|-1} \delta^{(|z|-1)} \right) & , \quad z \in \mathbb{Z}_{(-\infty, 0)} \\ (-jt)^z & , \quad z \in \mathbb{Z}_{[0, +\infty)} \end{cases}\end{aligned}$$

A.7.4.3 Example

As an example, we consider the case in which we have

$$u = u_0 t_{\theta}^{Z+1} \delta^{(Z)}$$

where

- Z is an integer constant of quantity dimension one

We observe that u is either a right-sided tempered distribution on $\mathbb{R}$ (when z is in $\mathbb{Z}_{(-\infty, 0)}$) or a point-supported distribution on $\mathbb{R}$ (when z is in $\mathbb{Z}_{[0, +\infty)}$).

In the time domain, it can be easily verified that we have

$$\begin{aligned}w &\equiv u^{(z)} \\ &= u_0 t_{\theta}^{Z+1} \delta^{(Z+z)}\end{aligned}$$

In the frequency domain, it can be easily verified that we have

$$U = u_0 t_{\theta}^{Z+1} \Delta_{(z)}$$

$$\begin{aligned}W &\equiv U_{(z)} \\ &= u_0 t_{\theta}^{Z+1} \Delta_{(Z+z)}\end{aligned}$$

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