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Comprehension and Knowledge

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Abstract
The ability of an agent to comprehend a sentence is tightly connected to the agent's prior experiences and background knowledge. The article suggests to interpret comprehension as a modality and proposes a complete bimodal logical system that describes an interplay between comprehension and knowledge modalities. The main technical result is a completeness theorem for the proposed system

1 Introduction

In this article, we introduce a logic-based framework for defining and reasoning about comprehension. Comprehension often requires an elimination of the ambiguity present in natural language. This usually can be done by taking into account the background knowledge. As an example, consider the following dialog that took place on January 25, 1990, near John F. Kennedy International Airport in New York:

> AIR TRAFFIC CONTROLLER Avianca 052 heavy I'm gonna bring you about fifteen miles north east and then turn you back onto the approach is that fine with you and your fuel
> FIRST OFFICER I guess so thank you very much

About 8 min after this conversation, Avianca flight 052 ran out of fuel and crashed. Out of 158 persons aboard, 73 died ((NTSB, 1991), page v). In its report, the National Transportation Safety Board lists "the lack of standardized understandable terminology" as a contributing factor to the crash ((NTSB, 1991), page v). While analyzing the crash, Helmreich points out that Colombia and the United States score very differently on cultural dimensions such as power distance, individualism-collectivism,

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and uncertainty avoidance. He argues that these cultural factors contributed to the lack of understanding between the Colombian crew and the American air traffic controller (Helmreich 1994); others agree (Orasanu, Fischer, and Davison, 1997).

In a low power distance culture, "I guess so" is an informal way to confirm that the aircraft has enough fuel while, perhaps, communicating the crew's unhappiness to make another loop in the air. In a high power distance culture, such as Colombia, it would be too disrespectful to express the same idea with "I guess so". Instead, in such cultures, "I guess so" is a mitigated expression of a concern, a respectful way to warn about an imminent danger. The United States, where this sentence could be interpreted either way[1], falls in the middle of power distance scale ((Hofstede, 2001), page 87).

Note that this ambiguity disappears if the controller has additional knowledge about the cultural background of the crew. As the example shows, knowledge might play a key role in comprehension. In this article, we propose a logic that describes the interplay between knowledge and comprehension.

2 Outline

The rest of this article is structured as follows. First, we define a model of our logical system and relate this model to the above example. Then, we define the syntax and the formal semantics of our system and review the related literature. When investigating a logical system with more than one modality, it is natural to first decide if one of these modalities can be defined through another and, thus, is redundant. In Sect. 6 and Sect. 7, we show that neither of the two modalities, knowledge and comprehension, of our logical system is definable through the other. In Sect. 8, we list the axioms of our logical system. In the three sections that follow, we prove the soundness and completeness of our system. To simplify the presentation, the proof of the completeness is divided into two parts. First, we establish completeness with respect to pseudo models and then we show how such models can be transformed into our standard semantics. A preliminary version of this work, with an outline of the proof of completeness, has appeared as (Naumov and Ros, 2021).

3 Contextual Epistemic Model

We assume a fixed countable set of propositional variables and a fixed countable set of agents $\mathcal{A}$. Let us start with a definition of a contextual epistemic model which will be used in the next section to define the semantics of our logical system.

Definition 1 A tuple $(W, \{\sim_a\}_{a\in\mathcal{A}}, M, \pi)$ is a contextual epistemic model if

[1] When American air traffic controllers were asked by the investigators what words they would respond immediately when a flight crew communicates a low fuel emergency, they replied "MAYDAY", "PAN, PAN, PAN", and "Emergency" ((NTSB, 1991) page 63). Avianca 052 communication transcripts show that the word "Emergency" was used in the communication between the pilot and the first officer, but not with the air traffic controller ((NTSB, 1991), page 10).

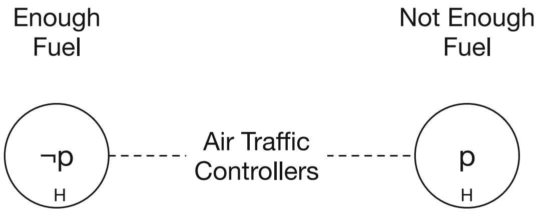


Fig. 1 Landing in Bogotá, Columbia

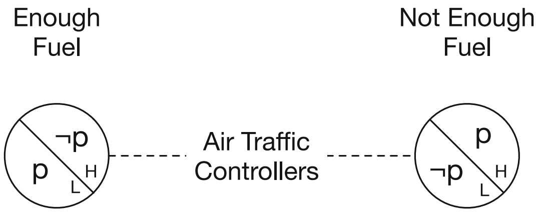


Fig. 2 Landing in New York, USA

1. W is a (possibly empty) set of *states*,
2. $\sim_a$ is an *indistinguishability* equivalence relation on set W for each agent $a \in \mathcal{A}$,
3. M is a (possibly empty) set of *contexts*,
4. π is a *valuation* function such that $\pi(p) \subseteq W \times M$ for each propositional variable p.

As we discussed in the introduction, locution "I guess so" is a real-world example of the kind of ambiguity that one should be able to reason about in order to comprehend human verbal communication. In this section, we interpret it as the statement "the words 'I guess so' give an accurate description of the current state". We denote this statement by propositional variable p.

Figure 1 depicts a contextual epistemic model capturing a hypothetical landing of Avianca 052 in Bogotá, Columbia, where the flight originated. Since the traffic controllers at Bogotá airport have the same high-power-distance cultural background as Avianca's pilots, this model has a *single* high-power-distance context H. The model has two states, "Enough Fuel" and "Not Enough Fuel", indistinguishable (before the pilots say "I guess so") to the air traffic controllers. In the high-power-distance context H, the statement p is true in the state "Not Enough Fuel" and false in the state "Enough Fuel". Once the Bogotá controllers hear "I guess so", they likely will conclude that the plane is low on fuel and issue an emergency landing order.

Figure 2 depicts a contextual epistemic model describing the actual landing of Avianca 052 at JFK International Airport in New York. It also has two states indistinguishable to the air traffic controllers. We capture the ambiguity of the locution "I guess so" to New York controllers by two distinct contexts: a low-power-distance culture context L and a high-power-distance culture context H. We visualize these contexts using lower-left and upper-right semi-circles forming each state. Statement p ("the words 'I guess so' give an accurate description of the current state") is true in state "Enough Fuel" only in context L. The same statement is true in the state "Not Enough Fuel" only in context H, see Fig. 2.

4 Syntax and Semantics

In this section, we describe the syntax and the formal semantics of our logical system. The language Φ of our system is defined by the grammar

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{K}_a\varphi \mid \mathsf{C}_a\varphi,$$

where p is a propositional variable and $a \in \mathcal{A}$ is an agent. We read $\mathsf{K}_a\varphi$ as "agent a knows that φ" and $\mathsf{C}_a\varphi$ as "agent a comprehends φ". We assume that conjunction $\wedge$, biconditional $\leftrightarrow$, and true $\top$ are defined through negation $\neg$ and implication $\to$ in the usual way. For any finite set of formulae $Y \subseteq \Phi$, by $\wedge Y$ we mean the conjunction of all formulae in set Y. By definition, $\wedge\varnothing$ is formula $\top$. Next, we define the formal semantics of our logical system.

Definition 2 For any formula $\varphi \in \Phi$, any state $w \in W$, and any context $m \in M$, the *satisfaction* relation $(w, m) \Vdash \varphi$ is defined recursively as follows:

1. $(w, m) \Vdash p$ if $(w, m) \in \pi(p)$,
2. $(w, m) \Vdash \neg\varphi$ if $(w, m) \nVdash \varphi$,
3. $(w, m) \Vdash \varphi \to \psi$ if $(w, m) \nVdash \varphi$ or $(w, m) \Vdash \psi$,
4. $(w, m) \Vdash \mathsf{K}_a\varphi$ if $(u, m') \Vdash \varphi$ for each state $u \in W$ such that $w \sim_a u$ and each context $m' \in M$,
5. $(w, m) \Vdash \mathsf{C}_a\varphi$ when for each state $u \in W$ and any contexts $m', m'' \in M$, if $w \sim_a u$ and $(u, m') \Vdash \varphi$, then $(u, m'') \Vdash \varphi$.

Item 5 of Definition 2 is the key definition of this article. It formally specifies the semantics of the comprehension modality C. As defined in item 4, the statement "an agent a knows that φ" means that φ is *true* in each context in each a-indistinguishable state. We say that agent a comprehends φ if φ is *consistent across the contexts* in each a-indistinguishable state. In other words, a comprehends φ if, for each a-indistinguishable state, φ is true in one context if and only if it is true in any other context.

In our example from Fig. 1, in high-power-distance context H, the statement

$$\mathsf{K}_{\text{Traffic Controllers}}\ p \tag{1}$$

is false in both states because p is true in the context H in the right state and is false in the context H in the left state. At the same time, the statement

$$\mathsf{C}_{\text{Traffic Controllers}}\ p \tag{2}$$

is true in both states because in both states the value of the propositional variable p is vacuously consistent across all contexts (of which there is only one). In other words, in the example from Fig. 1, the air traffic controllers do not know (before the pilots say "I guess so") if statement p is true or not, but they comprehend this statement due to the lack of multiple contexts.

In the example depicted in Fig. 2, statement (1) is still false under both contexts in both indistinguishable states because propositional variable p is false under at least one context in at least one states of the model. In addition, statement (2) is also false in both states of this example because the value of the propositional variable p is not consistent across the contexts in at least one (in our case, both) of the two indistinguishable states.

To summarize, before the pilots say "I guess so", in both examples the air traffic controllers do not know if statement p is true or not. However, the controllers in Bogotá comprehend p and the controllers in New York do not.

The next lemma holds because, by item 5 of Definition 2, the validity of $(w, m) \Vdash \mathsf{C}_a\varphi$ does not depend on the value m.

Lemma 1 $(w, m) \Vdash \mathsf{C}_a\varphi$ *iff* $(w, m') \Vdash \mathsf{C}_a\varphi$ *for any state* $w \in W$ *and any contexts* $m, m' \in M$.

5 Related Literature

In the philosophy of language, the type of semantics given in Definition 2 is often called 2D semantics. In general, 2D semantics defines the meaning of a statement based on the possible world w and some other information. That other information could be called index (Lewis, 1980), counterfactual world (Chalmers, 2004), possible world (Stalnaker, 2004), or scenario (Fritz, 2013). We use the word "context". Many completeness results for logical systems based on 2D semantics can be found in the literature. Some of them suppose that context is another state (Marx and Venema, 1997; Sano, 2010). Others consider a setting where the context is an agent (Grove and Halpern, 1991, 1993 ; Grove, 1995, Epstein and Naumov 2021, Epstein, Naumov, and Tao, 2023, Naumov and Tao, 2023, Naumov and Wu, 2024).

Halpern and Kets (2014) suggest that different agents can have different understandings of propositional variables. In our terms, this means having agent-specific contexts. They consider the cases when agents are aware and not aware of other agents interpreting propositions differently. However, the focus of their work is on probabilistic beliefs. They do not define comprehension as a modality and do not propose any logical system. Gattinger and Wang (2019) propose a logical system in which the meaning of a propositional variable is a Boolean expression. They give a sound and complete axiomatization of the expression $P \equiv Q$ that stands for "propositional formulae P and Q have the same meanings".

The most related work to ours is Li and Guo's A Logic LU for Understanding (2010), where the authors proposed a traditional semantics for modality U (understandable). The Kripke-like models that they consider have a reflexive and Euclidean reachability relation. A formula $\mathsf{U}\varphi$ is satisfied in a world w if either φ is satisfied in all worlds from the set $\{u \mid wRu\}$ or φ is not satisfied in all such worlds. In other words, they assume that a formula is understandable if its truth value is consistent among all reachable worlds. This interpretation of understanding is close to the one we use in item 5 of Definition 4. They also propose a complete logical system, but their system does not contain either knowledge modality or multiple agents. One can consider our approach

Fig. 3 Two Models

as an attempt to bring their modality into the setting of the multiagent epistemic logic.

More generally, the connection between knowledge and comprehension has long been a subject of psychology and literacy studies (Pearson, Hansen, and Gordon, 1979; Keysar, Barr, Balin, and Brauner, 2000; Hagoort, Hald, Bastiaansen, and Petersson 2004; Kennard, Anderegg, and Ewoldsen 2017) Langer (1984) states that "the knowledge and experience an individual brings to a reading task are critical factors in comprehension". Within the field of psychology, the comprehension of logical connectives is investigated in (Paris, 1973). D'Hanis (2002) suggests to use adaptive logic for capturing metaphors. Another logical system for metaphors in the Chinese language is advocated in (Zhang and Zhou, 2004). Neither of the last two papers claims a complete axiomatization.

6 Undefinability of Comprehension through Knowledge

In this section, we prove that the comprehension modality C is not definable through knowledge modality K. More precisely, we show that modality C cannot be expressed in the language $\Phi^{-\mathsf{C}}$ defined by the grammar

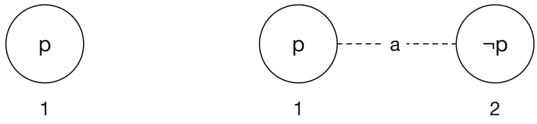
$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{K}_a\varphi.$$

We prove this by constructing two models indistinguishable in language $\Phi^{-\mathsf{C}}$, but distinguishable in the full language Φ of our logical system. Without loss of generality, we can assume that the set of agents $\mathcal{A}$ consists of a single agent a and the set of propositional variables contains a single propositional variable p. The two models that we use to prove undefinability are depicted in Fig. 3.

We refer to them as the left and the right models. Both models have two states: 1 and 2 indistinguishable by agent a. Both models also have two contexts: 1 and 2. In the diagram, the number *outside* of a circle is the name of the state, while the number *inside* of a semi-circle is the name of the context. It will be important for our proof that states and contexts have the same names. Valuation functions π_l of the left model and π_r of the right model are specified in Fig. 3. For example, $\pi_l(p) = \{(1, 1), (1, 2)\}$. In other words, in state 1 of the left model, propositional variable p is true in context 1 and in context 2. By $\Vdash_l$ and $\Vdash_r$ we denote the satisfaction relation of the left and the right model respectively. The next lemma proves that the two models are indistinguishable in language $\Phi^{-\mathsf{C}}$. Note that the order of x and y is different on the left-hand side of the two satisfaction statements in this lemma.

Lemma 2 $(x, y) \Vdash_l \varphi$ *iff* $(y, x) \Vdash_r \varphi$ *for any integers* $x, y \in \{1, 2\}$ *and any formula* $\varphi \in \Phi^{-\mathsf{C}}$.

Fig. 4 Two Models



Proof We prove the statement by induction on the structural complexity of formula φ. First, we consider the case when φ is a propositional variable p. Observe that $(x, y) \in \pi_l(p)$ iff $(y, x) \in \pi_l(p)$ for any integers $x, y \in \{1, 2\}$, see Fig. 3. Thus, $(x, y) \Vdash_l p$ iff $(y, x) \Vdash_r p$ by item 1 of Definition 2.

If formula φ is a negation or an implication, then the required follows from items 2 and 3 of Definition 2 and the induction hypothesis in the standard way.

Suppose that formula φ has the form $\mathsf{K}_a\psi$. By item 4 of Definition 2, the statement $(x, y) \Vdash_l \mathsf{K}_a\psi$ implies that $(x', y') \Vdash_l \psi$ for any integers $x', y' \in \{1, 2\}$. Hence, by the induction hypothesis, $(y', x') \Vdash_r \psi$ for any integers $x', y' \in \{1, 2\}$. Therefore, $(y, x) \Vdash_r \mathsf{K}_a\psi$ again by item 4 of Definition 2. The proof in the other direction is similar.

The next lemma shows that the left and the right models are distinguishable in the language Φ of our logical system.

Lemma 3 $(1, 1) \Vdash_l \mathsf{C}_a p$ *and* $(1, 1) \nVdash_r \mathsf{C}_a p$.

Proof Note that $(x, 1) \in \pi_l(p)$ iff $(x, 2) \in \pi_l(p)$ for any integer $x \in \{1, 2\}$, see Fig. 3. Thus, $(x, 1) \Vdash_l p$ iff $(x, 2) \Vdash_l p$ for any integer $x \in \{1, 2\}$ by item 1 of Definition 2. Therefore, $(1, 1) \Vdash_l \mathsf{C}_a p$ by item 5 of Definition 2.

Next, observe that $(1, 1) \in \pi_r(p)$ and $(1, 2) \notin \pi_r(p)$, see Fig. 3. Thus, $(1, 1) \Vdash_r p$ and $(1, 2) \nVdash_r p$ by item 1 of Definition 2. Therefore, $(1, 1) \nVdash_r \mathsf{C}_a p$ by item 5 of Definition 2.

The next theorem follows from the two lemmas above.

Theorem 1 *Comprehension modality* C *is not definable in language* $\Phi^{-\mathsf{C}}$.

7 Undefinability of Knowledge through Comprehension

In this section we prove that knowledge modality K is not definable in the language $\Phi^{-\mathsf{K}}$ specified by the grammar

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{C}_a\varphi.$$

The proof is similar to the one in the previous section. The left and the right models are depicted in Fig. 4.

The left model has a single state 1, while the right model has two states, 1 and 2, indistinguishable by agent a. Both models have only one context, which we refer to as context 1. Valuation functions π_l and π_r are defined as shown in Fig. 4. Namely, $\pi_l(p) = \pi_r(p) = \{(1, 1)\}$.

First, we show that state 1 in the left model is indistinguishable in language $\Phi^{-\mathsf{K}}$ from state 1 in the right model.

Lemma 4 $(1, 1) \Vdash_l \varphi$ *iff* $(1, 1) \Vdash_r \varphi$ *for any* $\varphi \in \Phi^{-\mathsf{K}}$.

Proof We prove the statement of the lemma by induction on structural complexity of formula φ. Note that $(1, 1) \in \pi_l(p)$ and $(1, 1) \in \pi_r(p)$, see Fig. 4. Thus, $(1, 1) \Vdash_l p$ and $(1, 1) \Vdash_r p$ by item 1 of Definition 2. Therefore, the statement of the lemma holds if formula φ is propositional variable p.

If formula φ is a negation or an implication, then the required follows from items 2 and 3 of Definition 2 and the induction hypothesis in the standard way.

Suppose that formula φ has the form $\mathsf{C}_a\psi$. Note that $(1, 1) \Vdash_l \mathsf{C}_a\psi$ by item 5 of Definition 2 because there is only one context in the left model. Similarly, $(1, 1) \Vdash_r \mathsf{C}_a\psi$ because there is only one context in the right model. Therefore, the statement of the lemma holds in the case when formula φ has the form $\mathsf{C}_a\psi$. Note that the proof of this case does not use the induction hypothesis.

The next lemma shows that the left and the right models are distinguishable in the language Φ of our logical system.

Lemma 5 $(1, 1) \Vdash_l \mathsf{K}_a p$ *and* $(1, 1) \nVdash_r \mathsf{K}_a p$.

Proof Note that $(1, 1) \in \pi_l(p)$, see Fig. 4. Thus, $(1, 1) \Vdash_l p$ by item 1 of Definition 2. Therefore, $(1, 1) \Vdash_l \mathsf{K}_a p$ by item 4 of Definition 2.

At the same time, $(1, 1) \in \pi_r(p)$ and $(2, 1) \notin \pi_r(p)$, see Fig. 4. Thus, $(1, 1) \Vdash_r p$ and $(2, 1) \nVdash_r p$ by item 1 of Definition 2. Therefore, $(1, 1) \nVdash_r \mathsf{K}_a p$ by item 4 of Definition 2 and because $1 \sim_a 2$, see Fig. 4.

The next theorem follows from the two previous lemmas.

Theorem 2 *Knowledge modality* K *is not definable in language* $\Phi^{-\mathsf{K}}$.

8 Axioms

In the rest of the article, we give a sound and complete logical system that captures the interplay between the knowledge modality K and the comprehension modality C. In addition to propositional tautologies in language Φ, our logical system contains the following axioms:

1. Truth: $\mathsf{K}_a\varphi \to \varphi$,
2. Negative Introspection: $\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\varphi$,
3. Distributivity: $\mathsf{K}_a(\varphi \to \psi) \to (\mathsf{K}_a\varphi \to \mathsf{K}_a\psi)$,
4. Comprehension of Known: $\mathsf{K}_a\varphi \to \mathsf{C}_a\varphi$,
5. Introspection of Comprehension: $\mathsf{C}_a\varphi \to \mathsf{K}_a\mathsf{C}_a\varphi$,
6. Comprehension of Negation: $\mathsf{C}_a\varphi \to \mathsf{C}_a\neg\varphi$,
7. Comprehension of Implication: $\mathsf{C}_a\varphi \to (\mathsf{C}_a\psi \to \mathsf{C}_a(\varphi \to \psi))$,
8. Substitution: $\mathsf{K}_a(\varphi \leftrightarrow \psi) \to (\mathsf{C}_a\varphi \to \mathsf{C}_a\psi)$,
9. Comprehension of Comprehension: $\mathsf{C}_a\mathsf{C}_b\varphi$,

10. Comprehension of Reflexivity: $\mathsf{C}_a(\mathsf{C}_b\varphi \to \varphi)$.

The Truth, the Negative Introspection, and the Distributivity axioms are standard axioms of epistemic logic S5. The Comprehension of Known axiom states that an agent must comprehend any statement that she knows. The Introspection of Comprehension axiom states that if an agent comprehends a statement, then she must know that she comprehends it. The Comprehension of Negation and the Comprehension of Implication axioms capture the fact that all agents are assumed to understand the meaning of Boolean connectives. Thus, if an agent comprehends φ and ψ, then she must comprehend the negation $\neg\varphi$ and the implication $\varphi \to \psi$. The Substitution axiom states that if an agent knows that two sentences are equivalent and she comprehends one of them, then she must comprehend the other. The Comprehension of Comprehension axiom states that any agent must comprehend the statement $\mathsf{C}_b\varphi$, even if she does not comprehend φ. In other words, the axiom states that the notion of comprehension is unambiguous; there is a common agreement on what it means to comprehend.

The Comprehension of Reflexivity axiom states that any agent a must comprehend the statement $\mathsf{C}_b\varphi \to \varphi$. The formal proof of soundness for this axiom is given in Lemma 13. See Lemma 16 for a related property.

We write $\vdash \varphi$, and say that formula φ is a *theorem* of our logical system, if φ is provable from the above axioms using the Modus Ponens and the Necessitation inference rules:

$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad\qquad \frac{\varphi}{\mathsf{K}_a\varphi}.$$

We write $X \vdash \varphi$ if formula φ is provable from the theorems of our logical system and the set of additional axioms X using only the Modus Ponens inference rule. Thus, the statements $\vdash \varphi$ and $\varnothing \vdash \varphi$ are equivalent.

The proof of the next standard lemma is given in the appendix.

Lemma 6 (deduction) *If $X, \varphi \vdash \psi$, then $X \vdash \varphi \to \psi$.*

9 Soundness

The Truth, the Negative Introspection, and the Distributivity axioms are standard axioms of epistemic logic S5. Below we show as a separate lemma the soundness of each of the remaining axioms in a state $w \in W$ under a context $m \in M$ of an arbitrary contextual epistemic model $(W, \{\sim_a\}_{a\in\mathcal{A}}, M, \pi)$.

Lemma 7 *If $(w, m) \Vdash \mathsf{K}_a\varphi$, then $(w, m) \Vdash \mathsf{C}_a\varphi$.*

Proof Consider any state $u \in W$ and any two contexts $m', m'' \in M$ such that $w \sim_a u$ and $(u, m') \Vdash \varphi$. By item 5 of Definition 2, it suffices to show that $(u, m'') \Vdash \varphi$.

Note that the assumption $(w, m) \Vdash \mathsf{K}_a\varphi$ of the lemma implies $(u, m'') \Vdash \varphi$ by item 4 of Definition 2 and the assumption $w \sim_a u$.

Lemma 8 *If $(w, m) \Vdash \mathsf{C}_a\varphi$, then $(w, m) \Vdash \mathsf{K}_a\mathsf{C}_a\varphi$.*

Proof Consider any state u and any context $m' \in M$ such that $w \sim_a u$. By item 4 of Definition 2, it suffices to prove that $(u, m') \Vdash \mathsf{C}_a\varphi$. Towards this proof, consider any state $v \in W$ and any two contexts $m_1, m_2 \in M$ such that $u \sim_a v$ and $(v, m_1) \Vdash \varphi$. By item 5 of Definition 2, it suffices to show that $(v, m_2) \Vdash \varphi$.

The assumptions $w \sim_a u$ and $u \sim_a v$ imply that $w \sim_a v$ because $\sim_a$ is an equivalence relation. Then, the assumption $(v, m_1) \Vdash \varphi$ implies $(v, m_2) \Vdash \varphi$ by item 5 of Definition 2 and the assumption $(w, m) \Vdash \mathsf{C}_a\varphi$ of the lemma.

Lemma 9 *If* $(w, m) \Vdash \mathsf{C}_a\varphi$*, then* $(w, m) \Vdash \mathsf{C}_a\neg\varphi$.

Proof Consider any state $u \in W$ and any two contexts $m', m'' \in M$ such that $w \sim_a u$ and

$$(u, m') \Vdash \neg\varphi. \tag{3}$$

Note that by item 5 of Definition 2, it suffices to show that $(u, m'') \Vdash \neg\varphi$.

Suppose that $(u, m'') \nVdash \neg\varphi$. Thus, $(u, m'') \Vdash \varphi$ by item 2 of Definition 2. Hence, $(u, m') \Vdash \varphi$ by item 5 of Definition 2, the assumption $(w, m) \Vdash \mathsf{C}_a\varphi$ of the lemma, and the assumption $w \sim_a u$. Therefore, $(u, m') \nVdash \neg\varphi$ by item 2 of Definition 2, which contradicts statement (3).

Lemma 10 *If* $(w, m) \Vdash \mathsf{C}_a\varphi$ *and* $(w, m) \Vdash \mathsf{C}_a\psi$*, then* $(w, m) \vdash \mathsf{C}_a(\varphi \to \psi)$.

Proof Consider any state $u \in W$ and any two contexts $m', m'' \in M$ such that $w \sim_a u$ and

$$(u, m') \Vdash \varphi \to \psi. \tag{4}$$

Note that by item 5 of Definition 2, it suffices to prove that $(u, m'') \Vdash \varphi \to \psi$. Towards this proof, suppose that $(u, m'') \Vdash \varphi$. By item 3 of Definition 2, it suffices to show that $(u, m'') \Vdash \psi$.

The assumption $(u, m'') \Vdash \varphi$ implies that $(u, m') \Vdash \varphi$ by item 5 of Definition 2, the assumption $(w, m) \Vdash \mathsf{C}_a\varphi$ of the lemma, and the assumption $w \sim_a u$. Hence, $(u, m') \Vdash \psi$ by item 3 of Definition 2 and statement (4). Thus, $(u, m'') \Vdash \psi$, by item 5 of Definition 2, the assumption $(w, m) \Vdash \mathsf{C}_a\psi$ of the lemma, and the assumption $w \sim_a u$.

Lemma 11 *If* $(w, m) \Vdash \mathsf{K}_a(\varphi \leftrightarrow \psi)$ *and* $(w, m) \Vdash \mathsf{C}_a\varphi$*, then* $(w, m) \Vdash \mathsf{C}_a\psi$.

Proof Consider any state $u \in W$ and any two contexts $m', m'' \in M$ such that $w \sim_a u$ and $(u, m') \Vdash \psi$. By item 5 of Definition 2, it suffices to show that $(u, m'') \Vdash \psi$.

By Definition 2, the assumption $(w, m) \Vdash \mathsf{K}_a(\varphi \leftrightarrow \psi)$ of the lemma implies that

$$(u, m') \Vdash \psi \to \varphi, \tag{5}$$

$$(u, m'') \Vdash \varphi \to \psi. \tag{6}$$

By item 3 of Definition 2, the assumption $(u, m') \Vdash \psi$ and statement (5) imply that $(u, m') \Vdash \varphi$. Hence, $(u, m'') \Vdash \varphi$ by item 5 of Definition 2, the assumption

$(w, m) \Vdash \mathsf{C}_a\varphi$ of the lemma, and the assumption $w \sim_a u$. Thus, $(u, m'') \Vdash \psi$ by item 3 of Definition 2 and statement (6).

Lemma 12 $(w, m) \Vdash \mathsf{C}_a\mathsf{C}_b\varphi$.

Proof Consider any state $u \in W$ and any two contexts $m', m'' \in M$ such that $w \sim_a u$ and $(u, m') \Vdash \mathsf{C}_b\varphi$. Note that by item 5 of Definition 2, it suffices to prove that $(u, m'') \Vdash \mathsf{C}_b\varphi$. The last statement is true by Lemma 1.

Lemma 13 $(w, m) \Vdash \mathsf{C}_a(\mathsf{C}_b\varphi \to \varphi)$.

Proof Consider any state $u \in W$ and any two contexts $m', m'' \in M$ such that $w \sim_a u$ and

$$(u, m') \Vdash \mathsf{C}_b\varphi \to \varphi. \tag{7}$$

Note that by item 5 of Definition 2, it suffices to prove that $(u, m'') \Vdash \mathsf{C}_b\varphi \to \varphi$. Towards this proof, suppose that $(u, m'') \Vdash \mathsf{C}_b\varphi$. By item 3 of Definition 2, it suffices to show that $(u, m'') \Vdash \varphi$. Indeed, by Lemma 1, the assumption $(u, m'') \Vdash \mathsf{C}_b\varphi$ implies that $(u, m') \Vdash \mathsf{C}_b\varphi$. It follows by item 3 of Definition 2 and statement (7) that $(u, m') \Vdash \varphi$. Thus, $(u, m'') \Vdash \varphi$ by the assumption $(u, m'') \Vdash \mathsf{C}_b\varphi$, item 5 of Definition 2 and because $u \sim_b u$.

10 Completeness for Pseudo Models

In this section, we define the class of pseudo models for our logical system and prove its completeness with respect to this class. In Sect. 11, we use this result to prove the completeness of our system with respect to the class of contextual epistemic models.

10.1 Pseudo Models

Unlike contextual epistemic models, pseudo models allow each state to have its own set of contexts. This is reflected in the definition below. In order to avoid the use of the dependent product type, the same definition also slightly changes the specification of valuation function π.

Definition 3 A tuple $(W, \{\sim_a\}_{a\in\mathcal{A}}, \{M_w\}_{w\in W}, \{\pi_w\}_{w\in W})$ is a pseudo model if

1. W is an arbitrary set of *states*,
2. $\sim_a$ is an *indistinguishability* equivalence relation on set W for each agent $a \in \mathcal{A}$,
3. M_w is a nonempty set of *contexts* for each state $w \in W$,
4. π_w is a *valuation* function from propositional variables into the powerset of M_w for each state $w \in W$.

Definition 4 For any formula $\varphi \in \Phi$, any state $w \in W$ of a pseudo model, and any context $m \in M_w$, the satisfaction relation $(w, m) \Vdash \varphi$ is defined recursively as follows:

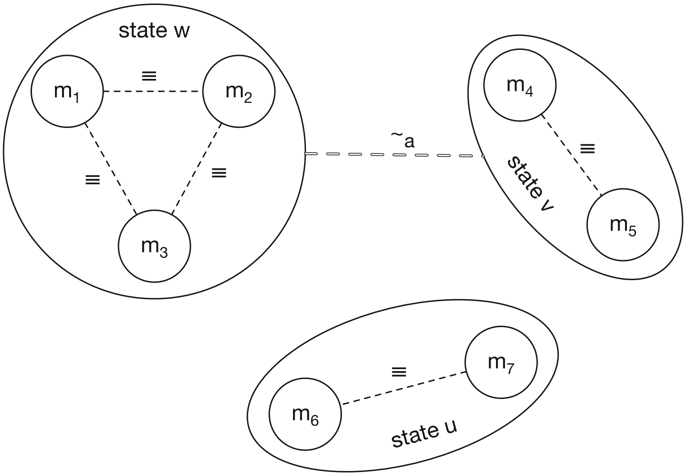


Fig. 5 Canonical pseudo model

1. $(w, m) \Vdash p$ if $m \in \pi_w(p)$,
2. $(w, m) \Vdash \neg\varphi$ if $(w, m) \nVdash \varphi$,
3. $(w, m) \Vdash \varphi \to \psi$ if $(w, m) \nVdash \varphi$ or $(w, m) \Vdash \psi$,
4. $(w, m) \Vdash \mathsf{K}_a\varphi$ if $(u, m') \Vdash \varphi$ for each state $u \in W$ such that $w \sim_a u$ and each context $m' \in M_u$,
5. $(w, m) \Vdash \mathsf{C}_a\varphi$ when for each state $u \in W$ and any contexts $m', m'' \in M_u$, if $w \sim_a u$ and $(u, m') \Vdash \varphi$, then $(u, m'') \Vdash \varphi$.

10.2 Completeness Proof Overview

In Sect. 10.3 through Sect. 10.7, we prove the completeness of our logical system with respect to pseudo models. In this subsection, we outline the main ideas behind the proof.

A completeness theorem for a modal logical system is usually proven by constructing a canonical model in which *states* are defined to be maximal consistent sets of formulae. This is different in our case because we define *contexts*, rather than states, to be maximal consistent sets of formulae. The set of all such context will be denoted by M.

Definition 3 specifies that any pseudo model should have a state-specific set of contexts M_w for each state $w \in W$. Sets of contexts M_w and M_u corresponding to distinct states w and u can but do not have to be disjoint. In our canonical pseudo model, they are disjoint. In other words, we partition the set of all contexts (maximal consistent sets of formulae) M into sets of contexts $\{M_w\}_{w\in W}$ corresponding to different states. We define this partition through an equivalence relation $\equiv$ on set M. Then, we define *states* as equivalence classes of this relation, see Fig. 5.

The exact definition of relation $\equiv$ is based on the intuition that if $\mathsf{C}_a\varphi$ is true under a context in a state, then φ must be consistent across all contexts in the given state.

To capture this, we say that $m \equiv m'$ when for each formula $\mathsf{C}_a\varphi \in m$, if $\varphi \in m$, then $\varphi \in m'$, see Definition 8.

To define indistinguishability relation $\sim_a$ between states, we first define it as a relation between contexts and then show that this relation is well-defined on states (equivalence classes of contexts with respect to relation $\equiv$). Our definition of indistinguishability of contexts by an agent a is equivalent to the standard approach in epistemic logic: $m \sim_a m'$ if contexts m and m' contain the same K-formulae.

A typical proof of completeness in modal logic includes a step where for each state w that does not contain a modal formula $\Box\varphi$ the proof constructs a "reachable" state u such that $\neg\varphi \in u$. This step is often phrased as an "existence" lemma. In our proof, such a step for modality K is very standard and it is carried out in Lemma 36. The case of modality C, however, is significantly different. Indeed, because item 5 of Definition 4 refers to two different contexts, m' and m'', the corresponding step for modality C involves construction of two maximal consistent sets corresponding to these contexts. Since m' and m'' in item 5 of Definition 4 are two contexts in the same state, we must guarantee that $m' \equiv m''$. This means that sets m' and m'' must agree on all formulae φ such that $\mathsf{C}_a\varphi$ belongs to at least one of them.

To construct sets m' and m'', we introduce a new technique that we call *perfect conforming* sets. First, we define the notion of a conforming set and consider a set Y of formulae that "must" belong to both: set m' and m''. We show that set Y is conforming. Then, we define *perfect* conforming set and show that any conforming set can be extended to a perfect conforming set. We extend set Y to a perfect conforming set Y' and show that sets $Y' \cup \{\varphi\}$ and $Y' \cup \{\neg\varphi\}$ are consistent. Finally, we use Lindenbaum's lemma to extend sets $Y' \cup \{\varphi\}$ and $Y' \cup \{\neg\varphi\}$ to maximal consistent sets of formulae m' and m'', respectively.

10.3 Derivable Formulae

In this subsection, we give several formal proofs in our logical system. The results from this section are used later in the completeness proof.

Lemma 14 *The inference rule* $\dfrac{\varphi \leftrightarrow \psi}{\mathsf{C}_a\varphi \to \mathsf{C}_a\psi}$ *is derivable in our logical system.*

Proof Suppose $\vdash \varphi \leftrightarrow \psi$. Thus, $\vdash \mathsf{K}_a(\varphi \leftrightarrow \psi)$ by the Necessitation inference rule. Therefore, $\vdash \mathsf{C}_a\varphi \to \mathsf{C}_a\psi$ by the Substitution axiom and the Modus Ponens rule.

Lemma 15 $\vdash \mathsf{C}_a\neg\mathsf{C}_b\psi$.

Proof Note that $\vdash \mathsf{C}_a\mathsf{C}_b\psi$ by the Comprehension of Comprehension axiom. Therefore, $\vdash \mathsf{C}_a\neg\mathsf{C}_b\psi$ by the Comprehension of Negation axiom and the Modus Ponens inference rule.

The next property is an interesting counterpart of the Comprehension of Reflexivity axiom.

Lemma 16 $\vdash \mathsf{C}_a(\mathsf{C}_b\varphi \to \neg\varphi)$.

Proof Note that $\neg\neg\varphi \leftrightarrow \varphi$ is a propositional tautology. Thus, $\vdash \mathsf{C}_a\neg\neg\varphi \to \mathsf{C}_a\varphi$ by Lemma 14. At the same time, $\vdash \mathsf{C}_b\neg\varphi \to \mathsf{C}_b\neg\neg\varphi$ by the Comprehension of Negation axiom. Thus, $\vdash \mathsf{C}_b\neg\varphi \to \mathsf{C}_b\varphi$ by propositional reasoning. Also, by the Comprehension of Negation axiom, $\vdash \mathsf{C}_b\varphi \to \mathsf{C}_b\neg\varphi$. Hence, $\vdash \mathsf{C}_b\neg\varphi \leftrightarrow \mathsf{C}_b\varphi$ by propositional reasoning. Then, $\vdash (\mathsf{C}_b\neg\varphi \to \neg\varphi) \leftrightarrow (\mathsf{C}_b\varphi \to \neg\varphi)$ by propositional reasoning. Hence, $\vdash \mathsf{C}_a(\mathsf{C}_b\neg\varphi \to \neg\varphi) \to \mathsf{C}_a(\mathsf{C}_b\varphi \to \neg\varphi)$ by Lemma 14. Observe that $\mathsf{C}_a(\mathsf{C}_b\neg\varphi \to \neg\varphi)$ is an instance of the Comprehension of Reflexivity axiom. Thus, $\vdash \mathsf{C}_a(\mathsf{C}_b\varphi \to \neg\varphi)$ the Modus Pones inference rule.

Lemma 17 $\vdash \mathsf{C}_a(\varphi \wedge \mathsf{C}_b\varphi)$.

Proof Note that formula $\neg(\mathsf{C}_b\varphi \to \neg\varphi) \leftrightarrow (\varphi \wedge \mathsf{C}_b\varphi)$ is a propositional tautology. Thus, by the Necessitation inference rule,

$$\vdash \mathsf{K}_a(\neg(\mathsf{C}_b\varphi \to \neg\varphi) \leftrightarrow (\varphi \wedge \mathsf{C}_b\varphi)). \tag{8}$$

At the same time, $\vdash \mathsf{C}_a(\mathsf{C}_b\varphi \to \neg\varphi)$ by Lemma 16. Thus, $\vdash \mathsf{C}_a\neg(\mathsf{C}_b\varphi \to \neg\varphi)$ by the Comprehension of Negation axiom and the Modus Ponens inference rule. Therefore, $\vdash \mathsf{C}_a(\varphi \wedge \mathsf{C}_b\varphi)$ by the Substitution axiom, statement (8), and the Modus Ponens inference rule.

The proof of the next lemma is similar to the proof of the lemma above except that Lemma 18 uses the Comprehension of Reflexivity axiom instead of Lemma 16.

Lemma 18 $\vdash \mathsf{C}_a(\neg\varphi \wedge \mathsf{C}_b\varphi)$.

Lemma 19 $\vdash \mathsf{C}_a(\varphi \to \psi) \to (\mathsf{K}_a\varphi \to \mathsf{C}_a\psi)$.

Proof Note that $\varphi \to ((\varphi \to \psi) \leftrightarrow \psi)$ is a propositional tautology. Thus, $\vdash \mathsf{K}_a(\varphi \to ((\varphi \to \psi) \leftrightarrow \psi))$ by the Necessitation inference rule. Hence,

$$\vdash \mathsf{K}_a\varphi \to \mathsf{K}_a((\varphi \to \psi) \leftrightarrow \psi) \tag{9}$$

by the Distributivity axiom and the Modus Pones inference rule. At the same time, by the Substitution axiom,

$$\vdash \mathsf{K}_a((\varphi \to \psi) \leftrightarrow \psi) \to (\mathsf{C}_a(\varphi \to \psi) \to \mathsf{C}_a\psi).$$

Thus, by propositional reasoning using statement (9),

$$\vdash \mathsf{K}_a\varphi \to (\mathsf{C}_a(\varphi \to \psi) \to \mathsf{C}_a\psi).$$

Then, $\vdash \mathsf{C}_a(\varphi \to \psi) \to (\mathsf{K}_a\varphi \to \mathsf{C}_a\psi)$ again by propositional reasoning.

Lemma 20 $\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a\psi \to \mathsf{C}_a(\varphi \wedge \psi))$.

Proof $\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a\neg\psi \to \mathsf{C}_a(\varphi \to \neg\psi))$ by the Comprehension of Implication axiom. Thus, by the Comprehension of Negation axiom and propositional reasoning, $\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a\psi \to \mathsf{C}_a(\varphi \to \neg\psi))$. Hence, again by the Comprehension of Negation axiom and propositional reasoning,

$$\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a\psi \to \mathsf{C}_a\neg(\varphi \to \neg\psi)). \tag{10}$$

Notice that $\neg(\varphi \to \neg\psi) \leftrightarrow \varphi \wedge \psi$ is a propositional tautology. Thus, by Lemma 14, statement (10), and propositional reasoning,

$$\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a\psi \to \mathsf{C}_a(\varphi \wedge \psi)).$$

Lemma 21

$$\vdash \mathsf{C}_a\gamma_1 \wedge \mathsf{C}_a\gamma_2 \wedge \mathsf{C}_a(\gamma_1 \wedge \psi \to \varphi) \wedge \mathsf{C}_a(\gamma_2 \wedge \neg\psi \to \varphi) \to \mathsf{C}_a(\gamma_1 \wedge \gamma_2 \to \varphi).$$

Proof By the Comprehension of Implication axiom,

$$\vdash \mathsf{C}_a\gamma_2 \wedge \mathsf{C}_a(\gamma_1 \to (\psi \to \varphi)) \to \mathsf{C}_a(\gamma_2 \to (\gamma_1 \to (\psi \to \varphi))).$$

Hence, by Lemma 14 and propositional reasoning,

$$\vdash \mathsf{C}_a\gamma_2 \wedge \mathsf{C}_a(\gamma_1 \to (\psi \to \varphi)) \to \mathsf{C}_a(\gamma_1 \wedge \gamma_2 \to (\psi \to \varphi)).$$

Thus, again by Lemma 14 and propositional reasoning,

$$\vdash \mathsf{C}_a\gamma_2 \wedge \mathsf{C}_a(\gamma_1 \wedge \psi \to \varphi) \to \mathsf{C}_a(\gamma_1 \wedge \gamma_2 \to (\psi \to \varphi)).$$

Similarly,

$$\vdash \mathsf{C}_a\gamma_1 \wedge \mathsf{C}_a(\gamma_2 \wedge \neg\psi \to \varphi) \to \mathsf{C}_a(\gamma_1 \wedge \gamma_2 \to (\neg\psi \to \varphi)).$$

Hence, by Lemma 20 and propositional reasoning,

$$\begin{aligned}&\vdash \mathsf{C}_a\gamma_1 \wedge \mathsf{C}_a\gamma_2 \wedge \mathsf{C}_a(\gamma_1 \wedge \psi \to \varphi) \wedge \mathsf{C}_a(\gamma_2 \wedge \neg\psi \to \varphi)\\ &\to \mathsf{C}_a((\gamma_1 \wedge \gamma_2 \to (\psi \to \varphi)) \wedge (\gamma_1 \wedge \gamma_2 \to (\neg\psi \to \varphi))).\end{aligned}$$

Finally, the following formula is a propositional tautology:

$$((\gamma_1 \wedge \gamma_2 \to (\psi \to \varphi)) \wedge (\gamma_1 \wedge \gamma_2 \to (\neg\psi \to \varphi))) \leftrightarrow (\gamma_1 \wedge \gamma_2 \to \varphi).$$

Therefore,

$$\vdash \mathsf{C}_a\gamma_1 \wedge \mathsf{C}_a\gamma_2 \wedge \mathsf{C}_a(\gamma_1 \wedge \psi \to \varphi) \wedge \mathsf{C}_a(\gamma_2 \wedge \neg\psi \to \varphi) \to \mathsf{C}_a(\gamma_1 \wedge \gamma_2 \to \varphi)$$

by Lemma 14 and propositional reasoning.

Lemma 22 $\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a(\varphi \to \neg\psi) \to \mathsf{C}_a(\varphi \to \psi))$.

Proof By Lemma 20,

$$\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a(\varphi \to \neg\psi) \to \mathsf{C}_a(\varphi \wedge (\varphi \to \neg\psi))).$$

Hence, by the Comprehension of Negation axiom and propositional reasoning,

$$\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a(\varphi \to \neg\psi) \to \mathsf{C}_a\neg(\varphi \wedge (\varphi \to \neg\psi))).$$

Note that $\neg(\varphi \wedge (\varphi \to \neg\psi)) \leftrightarrow (\varphi \to \psi)$ is a propositional tautology. Therefore,

$$\vdash \mathsf{C}_a\varphi \to (\mathsf{C}_a(\varphi \to \neg\psi) \to \mathsf{C}_a(\varphi \to \psi))$$

by Lemma 14 and propositional reasoning.

The next two lemmas state well-known properties of S5 modality. To keep the article self-contained, we give their proofs in the appendix.

Lemma 23 $\mathsf{K}_a\varphi_1, \dots, \mathsf{K}_a\varphi_n \vdash \mathsf{K}_a(\varphi_1 \wedge \dots \wedge \varphi_n)$.

Lemma 24 (Positive Introspection) $\vdash \mathsf{K}_a\varphi \to \mathsf{K}_a\mathsf{K}_a\varphi$.

10.4 Conforming Sets

In this subsection, we introduce the core notion in our construction, conforming set, and prove its basic properties. The intuition behind this notion has been discussed in Sect. 10.2.

Definition 5 For any set $X \subseteq \Phi$, any agent $a \in \mathcal{A}$, and any $\varphi \in \Phi$ such that $X \nvdash \mathsf{C}_a\varphi$, a formulae Y is (X, a, φ)-*conforming* if

1. $X \vdash \mathsf{C}_a\gamma$ for each formula $\gamma \in Y$,
2. $X \nvdash \mathsf{C}_a(\wedge Y' \to \varphi)$ for each finite set $Y' \subseteq Y$.

We use the word "conforming" to emphasize that set Y conforms to the requirements imposed by triple (X, a, φ).

Lemma 25 $X \vdash \mathsf{C}_a(\wedge Y')$ *for any finite subset* $Y' \subseteq Y$ *of any* (X, a, φ)*-conforming set* Y.

Proof If set Y' is empty, then $\wedge Y'$ is Boolean constant $\top$. Thus, $\wedge Y'$ is a tautology. Hence, $\vdash \mathsf{K}_a(\wedge Y')$ by the Necessitation inference rule. Thus, $\vdash \mathsf{C}_a(\wedge Y')$ by the Comprehension of Known axiom and the Modus Ponens inference rule. Therefore, $X \vdash \mathsf{C}_a(\wedge Y')$.

If set Y' contains a single element γ, then the required follows from item 1 of Definition 5 and the assumption of the lemma that set Y is (X, a, φ)-conforming.

Suppose that set Y' contains at $n \geq 2$ elements. Note that $X \vdash \mathsf{C}_a\gamma$ for each formula $\gamma \in Y$ by item 1 of Definition 5 and the assumption of the lemma that set Y is (X, a, φ)-conforming. Therefore, $X \vdash \mathsf{C}_a(\wedge Y')$ by propositional reasoning using $n - 1$ times Lemma 20.

Lemma 26 *If $X \nvdash \mathsf{C}_a\varphi$, then set $\{\psi \mid \mathsf{K}_a\psi \in X\}$ is (X, a, φ)-conforming.*

Proof We verify conditions 1 and 2 from Definition 5 separately:
Condition 1. Consider any $\mathsf{K}_a\psi \in X$. Then, $X \vdash \mathsf{C}_a\psi$ by the Comprehension of Known axiom and the Modus Ponens inference rule.
Condition 2. Suppose that $X \vdash \mathsf{C}_a(\psi_1 \wedge \cdots \wedge \psi_n \to \varphi)$ for some formulae $\mathsf{K}_a\psi_1, \ldots, \mathsf{K}_a\psi_n \in X$. Note that $X \vdash \mathsf{K}_a(\psi_1 \wedge \cdots \wedge \psi_n)$ by Lemma 23 and the choice of formulae $\mathsf{K}_a\psi_1, \ldots, \mathsf{K}_a\psi_n$. Therefore, $X \vdash \mathsf{C}_a\varphi$ by Lemma 19 and the Modus Ponens inference rule, which contradicts the assumption $X \nvdash \mathsf{C}_a\varphi$ of the lemma.

Lemma 27 *If set Y is (X, a, φ)-conforming, then $Y \nvdash \varphi$.*

Proof Suppose that $Y \vdash \varphi$. Thus, $Y' \vdash \varphi$ for some finite subset $Y' \subseteq Y$. Hence, $\vdash \wedge Y' \to \varphi$ by Lemma 6. Then, $\vdash \mathsf{K}_a(\wedge Y' \to \varphi)$ by the Necessitation inference rule. Thus, $\vdash \mathsf{C}_a(\wedge Y' \to \varphi)$ by the Comprehension of Known axiom and the Modus Ponens inference rule. Therefore, by item 2 of Definition 5, set Y is not (X, a, φ)-conforming.

Lemma 28 *If set Y is (X, a, φ)-conforming, then Y is $(X, a, \neg\varphi)$-conforming.*

Proof By condition 1 of Definition 5, the assumption that set Y is (X, a, φ)-conforming implies that

$$X \vdash \mathsf{C}_a\gamma \text{ for each formula } \gamma \in Y. \tag{11}$$

Suppose that set Y is not $(X, a, \neg\varphi)$-conforming. Hence, by Definition 5 and statement (11), there is a finite set $Y' \subseteq Y$ such that $X \vdash \mathsf{C}_a(\wedge Y' \to \neg\varphi)$. Note that $X \vdash \mathsf{C}_a \wedge Y'$ by Lemma 25 because $Y' \subseteq Y$. Hence, by Lemma 22 and the Modus Ponens inference rule, $X \vdash \mathsf{C}_a(\wedge Y' \to \varphi)$. Therefore, by Definition 5, set Y is not (X, a, φ)-conforming.

10.5 Perfect Sets

In this subsection, we show that any conforming set can be extended in a certain way with the result still being a conforming set. Then, we define a set to be *perfect* if it is extended in this way as much as possible. The contexts m' and m'', that have been discussed in Sect. 10.2 and are formally defined in the proof of Lemma 38, are not just conforming, but perfect conforming sets.

Lemma 29 *For any (X, a, φ)-conforming set Y and any formula $\mathsf{C}_b\psi$, at least one of the following sets is (X, a, φ)-conforming:*

1. $Y \cup \{\neg\mathsf{C}_b\psi\}$,
2. $Y \cup \{\psi \wedge \mathsf{C}_b\psi\}$,
3. $Y \cup \{\neg\psi \wedge \mathsf{C}_b\psi\}$.

Proof By condition 1 of Definition 5, the assumption that set Y is (X, a, φ)-conforming implies that

$$X \vdash \mathsf{C}_a\gamma \text{ for each formula } \gamma \in Y. \tag{12}$$

Note also that by Lemmas 15, 17, and 18, respectively,

$$\vdash \mathsf{C}_a\neg\mathsf{C}_b\psi, \quad \vdash \mathsf{C}_a(\psi \wedge \mathsf{C}_b\psi), \text{ and } \quad \vdash \mathsf{C}_a(\neg\psi \wedge \mathsf{C}_b\psi). \tag{13}$$

Suppose that none of the sets $Y \cup \{\neg\mathsf{C}_b\psi\}$, $Y \cup \{\psi \wedge \mathsf{C}_b\psi\}$, and $Y \cup \{\neg\psi \wedge \mathsf{C}_b\psi\}$ are (X, a, φ)-conforming. Thus, by Definition 5, statement (12), and statement (13), there are three finite subsets $Y_1 \subseteq Y \cup \{\neg\mathsf{C}_b\psi\}$, $Y_2 \subseteq Y \cup \{\psi \wedge \mathsf{C}_b\psi\}$, and $Y_3 \subseteq Y \cup \{\neg\psi \wedge \mathsf{C}_b\psi\}$ such that

$$X \vdash \mathsf{C}_a(\wedge Y_1 \to \varphi), \tag{14}$$
$$X \vdash \mathsf{C}_a(\wedge Y_2 \to \varphi), \tag{15}$$
$$X \vdash \mathsf{C}_a(\wedge Y_3 \to \varphi). \tag{16}$$

Note that if any of the sets Y_1, Y_2, or Y_3 is a subset of Y, then the above statements imply, by Definition 5, that set Y is *not* (X, a, φ)-conforming. The latter contradicts the assumption of the lemma. Thus, $Y_1, Y_2, Y_3 \nsubseteq Y$.

Hence, there are finite sets $Y_1', Y_2', Y_3' \subseteq Y$ such that $Y_1 = Y_1' \cup \{\neg\mathsf{C}_b\psi\}$, $Y_2 = Y_2' \cup \{\psi \wedge \mathsf{C}_b\psi\}$, and $Y_3 = Y_3' \cup \{\neg\psi \wedge \mathsf{C}_b\psi\}$. Then, by Lemma 14, statements (14), (15), and (16) imply that

$$X \vdash \mathsf{C}_a(\wedge Y_1' \wedge \neg\mathsf{C}_b\psi \to \varphi), \tag{17}$$
$$X \vdash \mathsf{C}_a(\wedge Y_2' \wedge \mathsf{C}_b\psi \wedge \psi \to \varphi), \tag{18}$$
$$X \vdash \mathsf{C}_a(\wedge Y_3' \wedge \mathsf{C}_b\psi \wedge \neg\psi \to \varphi). \tag{19}$$

Recall that set Y is (X, a, φ)-conforming by the assumption of the lemma. Thus,

$$X \vdash \mathsf{C}_a(\wedge Y_1'), \tag{20}$$
$$X \vdash \mathsf{C}_a(\wedge Y_2'), \tag{21}$$
$$X \vdash \mathsf{C}_a(\wedge Y_3'), \tag{22}$$
$$X \vdash \mathsf{C}_a(\wedge(Y_2' \cup Y_3')) \tag{23}$$

by Lemma 25 and the assumption $Y_1', Y_2', Y_3' \subseteq Y$.

Also, $\vdash \mathsf{C}_a\mathsf{C}_b\psi$ by the Comprehension of Comprehension axiom. Then, by Lemma 20 and and propositional reasoning, statements (21), (22), and (23) imply that

$$X \vdash \mathsf{C}_a(\wedge Y_2' \wedge \mathsf{C}_b\psi), \tag{24}$$
$$X \vdash \mathsf{C}_a(\wedge Y_3' \wedge \mathsf{C}_b\psi), \tag{25}$$
$$X \vdash \mathsf{C}_a(\wedge(Y_2' \cup Y_3') \wedge \mathsf{C}_b\psi). \tag{26}$$

Additionally, $\vdash \mathsf{C}_a\mathsf{C}_b\psi$ implies $\vdash \mathsf{C}_a\neg\mathsf{C}_b\psi$ by the Comprehension of Negation axiom and the Modus Ponens inference rule. Thus, again by Lemma 20 and propositional reasoning, statement (20) implies

$$X \vdash \mathsf{C}_a(\wedge Y_1' \wedge \neg \mathsf{C}_b \psi). \tag{27}$$

The following statement is an instance of Lemma 21:

$$\begin{aligned} \vdash\ & \mathsf{C}_a(\wedge Y_2' \wedge \mathsf{C}_b\psi) \wedge \mathsf{C}_a(\wedge Y_3' \wedge \mathsf{C}_b\psi) \\ & \wedge \mathsf{C}_a(\wedge Y_2' \wedge \mathsf{C}_b\psi \wedge \psi \to \varphi) \\ & \wedge \mathsf{C}_a(\wedge Y_3' \wedge \mathsf{C}_b\psi \wedge \neg\psi \to \varphi) \\ & \quad \to \mathsf{C}_a(\wedge Y_2' \wedge \mathsf{C}_b\psi \wedge (\wedge Y_3') \wedge \mathsf{C}_b\psi \to \varphi). \end{aligned}$$

Hence, by propositional reasoning using statements (24), (25), (18), and (19),

$$X \vdash \mathsf{C}_a(\wedge Y_2' \wedge \mathsf{C}_b\psi \wedge (\wedge Y_3') \wedge \mathsf{C}_b\psi \to \varphi).$$

Thus, by Lemma 14,

$$X \vdash \mathsf{C}_a(\wedge(Y_2' \cup Y_3') \wedge \mathsf{C}_b\psi \to \varphi). \tag{28}$$

Note that the following statement is also an instance of Lemma 21:

$$\begin{aligned} \vdash\ & \mathsf{C}_a(\wedge(Y_2' \cup Y_3') \wedge \mathsf{C}_b\psi) \wedge \mathsf{C}_a(\wedge Y_1' \wedge \neg\mathsf{C}_b\psi) \\ & \wedge \mathsf{C}_a(\wedge(Y_2' \cup Y_3') \wedge \mathsf{C}_b\psi \to \varphi) \\ & \wedge \mathsf{C}_a(\wedge Y_1' \wedge \neg\mathsf{C}_b\psi \to \varphi) \\ & \quad \to \mathsf{C}_a(\wedge(Y_2' \cup Y_3') \wedge (\wedge Y_1') \to \varphi). \end{aligned}$$

Hence, by propositional reasoning using statements (26), (27), (28), and (17),

$$X \vdash \mathsf{C}_a(\wedge(Y_2' \cup Y_3') \wedge (\wedge Y_1') \to \varphi).$$

Thus, by Lemma 14,

$$X \vdash \mathsf{C}_a(\wedge(Y_1' \cup Y_2' \cup Y_3') \to \varphi).$$

Therefore, set Y is not (X, a, φ)-conforming by Definition 5 and the assumption $Y_1', Y_2', Y_3' \subseteq Y$, which contradicts the assumption of the lemma.

Definition 6 A set of formulae $Y \subseteq \Phi$ is *perfect* if for any agent $b \in \mathcal{A}$ and any formula $\psi \in \Phi$ at least one of the formulae $\neg\mathsf{C}_b\psi$, $\psi \wedge \mathsf{C}_b\psi$, and $\neg\psi \wedge \mathsf{C}_b\psi$ belongs to set Y.

Note that although an (X, a, φ)-conforming set could be finite, a perfect set is always infinite because set Φ is infinite.

Lemma 30 *Any (X, a, φ)-conforming set Y could be extended to a perfect (X, a, φ)-conforming set Y'.*

Proof Consider any enumeration $\mathsf{C}_{b_1}\psi_1, \mathsf{C}_{b_2}\psi_2, \dots$ of all C-formulae in set Φ. By Lemma 29, there is a chain of (X, a, φ)-conforming sets $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ such that $Y_0 = Y$ and set Y_{k+1} is one of the following sets:

$$Y_k \cup \{\neg\mathsf{C}_{b_k}\psi_k\},\ \ Y_k \cup \{\psi_k \wedge \mathsf{C}_{b_k}\psi_k\},\ \ Y_k \cup \{\neg\psi_k \wedge \mathsf{C}_{b_k}\psi_k\}$$

for each $k \geq 0$. Let $Y' = \bigcup_k Y_k$. Set Y' is (X, a, φ)-conforming because, by Definition 5, the union of any chain of (X, a, φ)-conforming sets is (X, a, φ)-conforming.

10.6 Canonical Pseudo Model

We now define the canonical pseudo model $(W, \{\sim_a\}_{a\in\mathcal{A}}, \{M_w\}_{w\in W}, \{\pi_w\}_{w\in W})$ for our logical system. The key building blocks of this pseudo model are maximal consistent sets that we refer to as "contexts". We partition contexts into equivalence classes. As discussed in Sect. 10.2, each *state* is an equivalence class of the contexts.

Definition 7 The set of contexts M is the set of all maximal consistent sets of formulae.

Definition 8 For any contexts $m_1, m_2 \in M$, let $m_1 \equiv m_2$ when for each formula $\mathsf{C}_a\varphi \in m_1$, if $\varphi \in m_1$, then $\varphi \in m_2$.

Lemma 31 *Relation $\equiv$ is an equivalence relation on set M.*

Proof **Reflexivity:** Consider any formula $\mathsf{C}_a\varphi \in \Phi$ and any context $m \in M$. Suppose that $\varphi \in m$ and $\mathsf{C}_a\varphi \in m$. It suffices to show that $\varphi \in m$, which is our assumption.
Symmetry: Consider any formula $\mathsf{C}_a\varphi \in \Phi$ and any contexts $m_1, m_2 \in M$ such that $m_1 \equiv m_2$, $\mathsf{C}_a\varphi \in m_2$, and $\varphi \in m_2$. It suffices to show that $\varphi \in m_1$.

Claim $\mathsf{C}_a\neg\varphi \in m_1$.

Proof of Claim The assumption $\mathsf{C}_a\varphi \in m_2$ implies

$$\neg\mathsf{C}_a\varphi \notin m_2 \tag{29}$$

because set m_2 is consistent. At the same time, $\vdash \mathsf{C}_a\mathsf{C}_a\varphi$ by the Comprehension of Comprehension axiom. Thus, by Comprehension of Negation axiom and Modus Ponens inference rule, $\vdash \mathsf{C}_a\neg\mathsf{C}_a\varphi$. Hence, $\mathsf{C}_a\neg\mathsf{C}_a\varphi \in m_1$ because set m_1 is maximal. Then, $\neg\mathsf{C}_a\varphi \notin m_1$ by Definition 8 using statement (29) and the assumption $m_1 \equiv m_2$. Thus, $\mathsf{C}_a\varphi \in m_1$ because m_1 is maximal. Hence, $m_1 \vdash \mathsf{C}_a\neg\varphi$ by the Comprehension of Negation axiom and the Modus Ponens rule. Therefore, $\mathsf{C}_a\neg\varphi \in m_1$ by the maximality of m_1.

To finish the proof that relation $\equiv$ is symmetric, suppose $\varphi \notin m_1$. Thus, $\neg\varphi \in m_1$ because m_1 is a maximal consistent set. Hence, $\neg\varphi \in m_2$ by Definition 8, Claim 10.6, and the assumption $m_1 \equiv m_2$. Then, $\varphi \notin m_2$ because set m_2 is consistent, which contradicts our assumption that $\varphi \in m_2$.
Transitivity: Consider any formula $\mathsf{C}_a\varphi \in \Phi$ and any contexts $m_1, m_2, m_3 \in S$ such that $m_1 \equiv m_2$, $m_2 \equiv m_3$, $C_a\varphi \in m_1$, and $\varphi \in m_1$. It suffices to show that $\varphi \in m_3$.

Note that $\vdash \mathsf{C}_a\mathsf{C}_a\varphi$ by the Comprehension of Comprehension axiom. Thus, $\mathsf{C}_a\mathsf{C}_a\varphi \in m_1$ due to the maximality of set m_1. Hence, $\mathsf{C}_a\varphi \in m_2$ by Definition 8 and the assumptions $m_1 \equiv m_2$ and $\mathsf{C}_a\varphi \in m_1$. At the same time, $\varphi \in m_2$ by Definition 8 and the assumptions $m_1 \equiv m_2$, $\mathsf{C}_a\varphi \in m_1$, and $\varphi \in m_1$. The statements $\mathsf{C}_a\varphi \in m_2$ and $\varphi \in m_2$ imply $\varphi \in m_3$ by Definition 8 and the assumption $m_2 \equiv m_3$.

Definition 9 Set of states W is the set of equivalence classes of M with respect to relation $\equiv$.

We now are ready to define the equivalence relation $\sim_a$ on states from set W. We do this in two steps. First, we define this relation on contexts, then we show that this relation is well-defined on $\equiv$-classes of contexts, which are states.

Definition 10 For any two contexts $m_1, m_2 \in M$ and any agent $a \in \mathcal{A}$, let $m_1 \sim_a m_2$ when for each formula φ, if $\mathsf{K}_a\varphi \in m_1$, then $\varphi \in m_2$.

Alternatively, one can define $m_1 \sim_a m_2$ if sets m_1 and m_2 contain the same K-formulae. Our definition simplifies the proof of completeness, but it requires the lemma below.

Lemma 32 *Relation $\sim_a$ is an equivalence relation on set of contexts M for each agent $a \in \mathcal{A}$.*

Proof **Reflexivity:** Consider any formula $\varphi \in \Phi$. Suppose that $\mathsf{K}_a\varphi \in m$. It suffices to show that $\varphi \in m$. Indeed, the assumption $\mathsf{K}_a\varphi \in m$ implies $m \vdash \varphi$ by the Truth axiom and the Modus Ponens inference rule. Therefore, $\varphi \in m$ because set m is maximal.
Symmetry: Consider any contexts $m_1, m_2 \in M$ such that $m_1 \sim_a m_2$ and any formula $\mathsf{K}_a\varphi \in m_2$. It suffices to show that $\varphi \in m_1$. Suppose the opposite. Then, $\varphi \notin m_1$. Hence, $m_1 \nvdash \varphi$ because set m_1 is maximal. Thus, $m_1 \nvdash \mathsf{K}_a\varphi$ by the contraposition of the Truth axiom. Then, $\neg\mathsf{K}_a\varphi \in m_1$ because set m_1 is maximal. Thus, $m_1 \vdash \mathsf{K}_a\neg\mathsf{K}_a\varphi$ by the Negative Introspection axiom and the Modus Ponens inference rule. Hence, $\mathsf{K}_a\neg\mathsf{K}_a\varphi \in m_1$ because set m_1 is maximal. Then, $\neg\mathsf{K}_a\varphi \in m_2$ by the assumption $m_2 \sim_a m_1$ and Definition 10. Therefore, $\mathsf{K}_a\varphi \notin m_2$ because set m_1 is consistent, which contradicts the assumption $\mathsf{K}_a\varphi \in m_2$.
Transitivity: Consider any contexts $m_1, m_2, m_3 \in W$ such that $m_1 \sim_a m_2$ and $m_2 \sim_a m_3$ and any formula $\mathsf{K}_a\varphi \in m_1$. It suffices to show that $\varphi \in m_3$. The assumption $\mathsf{K}_a\varphi \in m_1$ implies $m_1 \vdash \mathsf{K}_a\mathsf{K}_a\varphi$ by Lemma 24 and the Modus Ponens rule. Thus, $\mathsf{K}_a\mathsf{K}_a\varphi \in m_1$ because set m_1 is maximal. Hence, $\mathsf{K}_a\varphi \in m_2$ by the assumption $m_1 \sim_a m_2$ and Definition 10. Then, $\varphi \in m_3$ by the assumption $m_2 \sim_a m_3$ and Definition 10.

Lemma 33 *If $m_1 \equiv m_2$, then $m_1 \sim_a m_2$ for each agent $a \in \mathcal{A}$.*

Proof Consider any formula $\mathsf{K}_a\varphi \in m_1$. By Definition 10, it suffices to show that $\varphi \in m_2$. Indeed, the assumption $\mathsf{K}_a\varphi \in m_1$ implies $m_1 \vdash \mathsf{K}_a\mathsf{K}_a\varphi$ by Lemma 24 and the Modus Ponens inference rule. Thus, $m_1 \vdash \mathsf{C}_a\mathsf{K}_a\varphi$ by the Comprehension of Known axiom and the Modus Ponens inference rule. Hence, $\mathsf{C}_a\mathsf{K}_a\varphi \in m_1$ because set m_1 is maximal. Then, $\mathsf{K}_a\varphi \in m_2$ by Definition 8, the assumption $m_1 \equiv m_2$, and the assumption $\mathsf{K}_a\varphi \in m_1$. Hence, $m_2 \vdash \varphi$ by the Truth axiom and the Modus Ponens inference rule. Therefore, $\varphi \in m_2$ because set m_2 is maximal.

Lemma 34 *Relation $\sim_a$ is well-defined on set W of states.*

Proof Suppose that $m_1 \sim_a m_2$, $m_1 \equiv m_1'$, and $m_2 \equiv m_2'$. It suffices to show that $m_1' \sim_a m_2'$, which follows from Lemmas 33 and 32.

The next statement follows from Lemma 32.

Lemma 35 *Relation $\sim_a$ is an equivalence relation on set W for each agent $a \in \mathcal{A}$.*

As we mentioned earlier, contexts in a state w are the contexts that belong to set w.

Definition 11 $M_w = w$ for each state $w \in W$.

Recall that each state $w \in W$ is an equivalence class of set M by Definition 9. Thus, $M_w = w$ is a nonempty set.

Definition 12 $\pi_w(p) = \{m \in w \mid p \in m\}$, for each state $w \in W$ and each propositional variable p.

This concludes the definition of the canonical pseudo model.

10.7 Completeness: Final Steps

In this subsection, we prove the "induction" or "truth" Lemma 39 for our canonical pseudo model and use it to finish the proof of strong completeness in the usual way. To keep the proof by induction of the "truth" lemma manageable, we separate three major cases of the induction into Lemmas 36, 37, and 38 below. Note that Lemma 38 is using perfect conforming sets. The four lemmas below refer to the canonical pseudo model defined in the previous subsection.

Lemma 36 *For any context m and any formula $\mathsf{K}_a\varphi \notin m$, there is a context $m' \in M$ such that $m \sim_a m'$ and $\varphi \notin m'$.*

Proof Let X be the set of formulae $\{\neg\varphi\} \cup \{\psi \mid \mathsf{K}_a\psi \in m\}$.

First, we show that set X is consistent. Assume the opposite. Thus, there are formulae $\mathsf{K}_a\psi_1, \dots, \mathsf{K}_a\psi_n \in m$ such that $\vdash \bigwedge_{i\leq n}\psi_i \to \varphi$. Hence, by the Necessitation inference rule, $\vdash \mathsf{K}_a\left(\bigwedge_{i\leq n}\psi_i \to \varphi\right)$. Then, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \mathsf{K}_a\bigwedge_{i\leq n}\psi_i \to \mathsf{K}_a\varphi$. Thus, $\mathsf{K}_a\psi_1, \dots, \mathsf{K}_a\psi_n \vdash \mathsf{K}_a\varphi$ by Lemma 23 and the Modus Ponens inference rule. Hence, $m \vdash \mathsf{K}_a\varphi$ by the choice of the formulae $\mathsf{K}_a\psi_1, \dots, \mathsf{K}_a\psi_n$. Then, $\mathsf{K}_a\varphi \in m$ because set m is maximal, which contradicts an assumption of the lemma. Therefore, set X is consistent.

Let m' be any maximal consistent extension of set X. Note that $m \sim_a m'$ by Definition 10 and the choice of sets X and m'. Also, $\neg\varphi \in X \subseteq m'$ implies that $\varphi \notin m'$ because set m' is consistent.

Lemma 37 *If $\mathsf{C}_a\varphi \in m$, $m \sim_a m'$, $m' \equiv m''$, and $\varphi \in m'$, then $\varphi \in m''$.*

Proof The assumption $\mathsf{C}_a\varphi \in m$ implies $m \vdash \mathsf{K}_a\mathsf{C}_a\varphi$ by the Introspection of Comprehension axiom and the Modus Ponens inference rule. Thus, $\mathsf{K}_a\mathsf{C}_a\varphi \in m$ because set m is maximal. Hence, $\mathsf{C}_a\varphi \in m'$ by Definition 10 and the assumption $m \sim_a m'$. Therefore, $\varphi \in m''$ by Definition 8 and the assumptions $\varphi \in m'$ and $m' \equiv m''$.

Lemma 38 *If* $\mathsf{C}_a\varphi \notin m$*, then there are contexts* $m', m'' \in M$ *such that* $m \sim_a m'$*,* $m' \equiv m''$*,* $\varphi \in m'$*, and* $\varphi \notin m''$*.*

Proof The assumption $\mathsf{C}_a\varphi \notin m$ implies that $m \nvdash \mathsf{C}_a\varphi$ because set m is maximal. Hence, set $Y = \{\psi \mid \mathsf{K}_a\psi \in m\}$ is (m, a, φ)-conforming by Lemma 26.

Let Y' be a perfect (m, a, φ)-conforming extension of set Y. Such set Y' exists by Lemma 30. Note that set Y' is also $(m, a, \neg\varphi)$-conforming by Lemma 28. Thus, the set $Y' \cup \{\varphi\}$ is consistent by Lemma 27. Let m' be any maximal consistent extension of this set. Note that $m \sim_a m'$ by Definition 10 and the choice of sets Y and Y'. Also, $\varphi \in Y' \cup \{\varphi\} \subseteq m'$ by the choice of set m'.

By Lemma 27, the set $Y' \cup \{\neg\varphi\}$ is also consistent because set Y' is (m, a, φ)-conforming. Let m'' be any maximal consistent extension of this set. Then, $\neg\varphi \in Y' \cup \{\neg\varphi\} \subseteq m''$. Thus, $\varphi \notin m''$ because set m'' is consistent.

Finally, we show that $m' \equiv m''$. Consider an arbitrary formula $\mathsf{C}_b\psi \in m'$ such that $\psi \in m'$. By Definition 8, it suffices to show that $\psi \in m''$. Indeed, the assumptions $\mathsf{C}_b\psi \in m'$ and $\psi \in m'$ imply that $\psi \wedge \mathsf{C}_b\psi \in m'$ because set m' is maximal. Thus, $\neg\mathsf{C}_b\psi \notin m'$ and $\neg\psi \wedge \mathsf{C}_b\psi \notin m'$ because set m' is consistent. Hence, $\neg\mathsf{C}_b\psi \notin Y'$ and $\neg\psi \wedge \mathsf{C}_b\psi \notin Y'$ because $Y' \subseteq Y' \cup \{\varphi\} \subseteq m'$ by the choice of set m'. Then, $\psi \wedge \mathsf{C}_b\psi \in Y'$ by Definition 6 and the assumption that set Y' is perfect. Thus, $\psi \wedge \mathsf{C}_b\psi \in m''$ because $Y' \subseteq Y' \cup \{\neg\varphi\} \subseteq m''$ by the choice of set m''. Therefore, $\psi \in m''$ because set m'' is maximal.

By $[m]$ we mean the equivalence class (state) of the context m with respect to relation $\equiv$.

Lemma 39 $([m], m) \Vdash \varphi$ *iff* $\varphi \in m$ *for any context* $m \in M$ *and any formula* $\varphi \in \Phi$*.*

Proof We prove the lemma on structural induction of formula φ. If formula φ is a propositional variable, then the required follows from item 1 of Definition 4 and Definition 12. The case when formula φ is a negation or an implication follows from items 2 and 3 of Definition 4 and the maximality and the consistency of set m in the standard way.

Assume that formula φ has the form $\mathsf{K}_a\psi$.

$(\Rightarrow)$: Suppose that $\mathsf{K}_a\psi \notin m$. Thus, by Lemma 36, there is a context $m' \in M$ such that $m \sim_a m'$ and $\psi \notin m'$. Hence, $[m] \sim_a [m']$ and, by the induction hypothesis, $([m'], m') \nVdash \psi$. Therefore, $([m], m) \nVdash \mathsf{K}_a\psi$ by item 4 of Definition 4.

$(\Leftarrow)$: Assume $\mathsf{K}_a\psi \in m$. Consider any context m' in a state $[m']$ such that $[m] \sim_a [m']$. By item 4 of Definition 4, it suffices to show that $([m'], m') \Vdash \psi$. Indeed, the assumption $[m] \sim_a [m']$ implies that $m \sim_a m'$. Hence, $\psi \in m'$ by Definition 10 and the assumption $\mathsf{K}_a\psi \in m$. Thus, $([m'], m') \Vdash \psi$ by the induction hypothesis.

Finally, suppose that formula φ has the form $\mathsf{C}_a\psi$.

$(\Rightarrow)$: Assume that $\mathsf{C}_a\psi \notin m$. Thus, by Lemma 38, there are contexts $m', m'' \in M$ such that $m \sim_a m'$, $m' \equiv m''$, $\psi \in m'$, and $\psi \notin m''$. Hence, $[m] \sim_a [m']$ and, by the induction hypothesis, $([m'], m') \Vdash \psi$ as well as $([m''], m'') \nVdash \psi$. Note that $[m'] = [m'']$ because $m' \equiv m''$. Therefore, $([m], m) \nVdash \mathsf{C}_a\psi$ by item 5 of Definition 4.

$(\Leftarrow)$: Assume that $\mathsf{C}_a\psi \in m$. Consider any state $u \in W$ and any contexts $m', m'' \in u$ such that $[m] \sim_a u$ and $(u, m') \Vdash \psi$. By item 5 of Definition 4, it suffices to prove that $(u, m'') \Vdash \psi$. Indeed, note that $[m'] = [m''] = u$ because $m', m'' \in u$. Then, by the induction hypothesis, the assumption $(u, m') \Vdash \psi$ implies that $\psi \in m'$. Also, $m \sim_a m'$ because $[m] \sim_a u$ and $m' \in u$. Additionally, $m' \equiv m''$ because $[m'] = [m'']$. Thus, $\psi \in m''$ by Lemma 37. Hence, $([m''], m'') \Vdash \psi$ by the induction hypothesis. Therefore, $(u, m'') \Vdash \psi$ because $[m''] = u$.

We are finally ready to state and prove the strong completeness theorem for our logical system with respect to pseudo models.

Theorem 3 (strong completeness for pseudo models) *If $X \nvdash \varphi$, then there is a state w of a pseudo model and a context m in state w such that $(w, m) \Vdash \chi$ for each formula $\chi \in X$ and $(w, m) \nVdash \varphi$.*

Proof The assumption $X \nvdash \varphi$ implies that the set $X \cup \{\neg\varphi\}$ is consistent. Let m be any maximal consistent extension of this set. Thus, $\chi \in m$ for each formula $\chi \in X$. Also, $\varphi \notin m$ because $\neg\varphi \in X \cup \{\neg\varphi\} \subseteq m$ and set m is consistent. Therefore, $([m], m) \Vdash \chi$ for each formula $\chi \in X$ and $([m], m) \nVdash \varphi$ by Lemma 39.

11 Completeness for Contextual Epistemic Models

In this section, we use our completeness results for pseudo models to prove the completeness with respect to the contextual epistemic models. We start the proof by defining a contextual epistemic model $(W', \{\sim'_a\}_{a\in\mathcal{A}}, M', \pi')$ for each given pseudo model $(W, \{\sim_a\}_{a\in\mathcal{A}}, \{M_w\}_{w\in W}, \{\pi_w\}_{w\in W})$.

Let $W' = W$ and $\sim'_a = \sim_a$ for each agent $a \in \mathcal{A}$. Also, let

$$M = \bigcup_{w\in W} M_w.$$

We use the above definition of the set of contexts M even if sets $\{M_w\}_{w\in W}$ are not pairwise disjoint. Recall that by Definition 3, set M_w is nonempty for each state $w \in W$. Let m_w denote a fixed arbitrary context of each state $w \in W$. In the construction below, we treat m_w as a "default" context of state w.

For each state $w \in W$, function τ_w maps set M into set M_w:

$$\tau_w(m) = \begin{cases} m, & \text{if } m \in M_w \\ m_w, & \text{otherwise.} \end{cases}$$

We are now ready to define valuation function π' for the contextual epistemic model:

$$\pi'(p) = \{(w, m) \in W \times M \mid \tau_w(m) \in \pi_w(p)\}.$$

The next lemma can be shown by induction on the structural complexity of formula φ.

Lemma 40 *$w, m \Vdash' \varphi$ iff $w, \tau_w(m) \Vdash \varphi$, where $\Vdash$ is the satisfaction relation for the pseudo model $(W, \{\sim_a\}_{a\in\mathcal{A}}, \{M_w\}_{w\in W}, \{\pi_w\}_{w\in W})$ and $\Vdash'$ is the satisfaction relation for the epistemic contextual model $(W', \{\sim'_a\}_{a\in\mathcal{A}}, M', \pi')$.*

The theorem below follows from Theorem 3 and Lemma 40.

Theorem 4 (strong completeness) *If $X \nvdash \varphi$, then there is a state w of an epistemic contextual and a context m such that $(w, m) \Vdash \chi$ for each formula $\chi \in X$ and $(w, m) \nVdash \varphi$.*

12 Alternative Languages

In this article, we considered modalities K and C defined through a combination of multiple quantifiers over different domains. In addition to them, one can potentially consider more primitive modalities: context-specific knowledge modality K^{cs}, state-specific comprehension modality C^{ss}, and "for any contexts" modality $\Box$. Their semantics is defined as follows:

$(w, m) \Vdash \mathsf{K}^{cs}\varphi$ when $(u, m) \Vdash \varphi$ for each state $u \in W$ such that $w \sim_a u$.
$(w, m) \Vdash \mathsf{C}^{ss}\varphi$ when for any contexts $m', m'' \in M$, if $(w, m') \Vdash \varphi$, then $(w, m'') \Vdash \varphi$.
$(w, m) \Vdash \Box\varphi$ if $(w, m') \Vdash \varphi$ for any context $m' \in M$.

These five modalities are not unrelated. Namely, it is easy to show that

$$\begin{aligned}
\Box\varphi &\equiv \varphi \wedge \mathsf{C}^{ss}\varphi,\\
\mathsf{C}^{ss}\varphi &\equiv \Box(\varphi \to \Box\varphi),\\
\mathsf{C}_a\varphi &\equiv \mathsf{K}_a\mathsf{C}^{ss}\varphi \equiv \mathsf{K}^{cs}_a\mathsf{C}^{ss}\varphi \equiv \mathsf{K}_a(\varphi \to \Box\varphi),\\
\mathsf{K}_a\varphi &\equiv \mathsf{K}^{cs}_a\Box\varphi \equiv \Box\mathsf{K}^{cs}_a\varphi.
\end{aligned}$$

We have chosen to study modalities K and C because we think they better reflect the intuitive notions of knowledge and comprehension. We did this at the expense of a more complicated set of axioms and a more complicated proof of the completeness.

13 Conclusion

The contribution of this article is three-fold. First, we introduced a novel modality "comprehensible" and gave its formal semantics in contextual epistemic models. Second, we have shown that this modality cannot be defined through knowledge modality and vice versa. Finally, we proposed a sound and complete logical system that describes the interplay between the knowledge and the comprehension modalities.

In modal logic, the filtration technique is often used to prove the completeness of a logical system with respect to a class of finite models (Gabbay, 1972). Such completeness normally implies the decidability of the system. For this approach to work in our case, the class of finite models would require not only the number of states to be finite, but the number of contexts to be finite as well. We have not been successful in adopting the filtration technique to achieve this. Thus, proving the decidability of the proposed logical system remains an open question.

Appendix: Auxiliary Lemmas

Lemma 6 *If* $X, \varphi \vdash \psi$, *then* $X \vdash \varphi \to \psi$.

Proof Suppose that sequence $\psi_1, \dots, \psi_n$ is a proof from set $X \cup \{\varphi\}$ and the theorems of our logical system that uses the Modus Ponens inference rule only. In other words, for each $k \leq n$, either

1. $\vdash \psi_k$, or
2. $\psi_k \in X$, or
3. ψ_k is equal to φ, or
4. there are $i, j < k$ such that formula ψ_j is equal to $\psi_i \to \psi_k$.

It suffices to show that $X \vdash \varphi \to \psi_k$ for each $k \leq n$. We prove this by induction on k through considering the four cases above separately.
Case 1: $\vdash \psi_k$. Note that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Hence, $\vdash \varphi \to \psi_k$ by the Modus Ponens inference rule. Therefore, $X \vdash \varphi \to \psi_k$.
Case 2: $\psi_k \in X$. Note again that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Therefore, by the Modus Ponens inference rule, $X \vdash \varphi \to \psi_k$.
Case 3: formula ψ_k is equal to φ. Thus, $\varphi \to \psi_k$ is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$.
Case 4: formula ψ_j is equal to $\psi_i \to \psi_k$ for some $i, j < k$. Thus, by the induction hypothesis, $X \vdash \varphi \to \psi_i$ and $X \vdash \varphi \to (\psi_i \to \psi_k)$. Note that formula $(\varphi \to \psi_i) \to ((\varphi \to (\psi_i \to \psi_k)) \to (\varphi \to \psi_k))$ is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$ by applying the Modus Ponens inference rule twice.

Lemma 23 $\mathsf{K}_a\varphi_1, \dots, \mathsf{K}_a\varphi_n \vdash \mathsf{K}_a(\varphi_1 \wedge \dots \wedge \varphi_n)$.

Proof Note that the following formula is a tautology:

$$\varphi_1 \to (\varphi_2 \to \dots (\varphi_n \to (\varphi_1 \wedge \dots \wedge \varphi_n)) \dots).$$

Thus, by the Necessitation inference rule,

$$\vdash \mathsf{K}_a(\varphi_1 \to (\varphi_2 \to \dots (\varphi_n \to (\varphi_1 \wedge \dots \wedge \varphi_n)) \dots)).$$

Hence, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{K}_a\varphi_1 \to \mathsf{K}_a(\varphi_2 \to \dots (\varphi_n \to (\varphi_1 \wedge \dots \wedge \varphi_n)) \dots)).$$

Then, again by the Modus Ponens inference rule,

$$\mathsf{K}_a\varphi_1 \vdash \mathsf{K}_a(\varphi_2 \to \ldots(\varphi_n \to (\varphi_1 \wedge \cdots \wedge \varphi_n))\ldots)).$$

Therefore, $\mathsf{K}_a\varphi_1, \ldots, \mathsf{K}_a\varphi_n \vdash \mathsf{K}_a(\varphi_1 \wedge \cdots \wedge \varphi_n)$ by repeating the previous steps $n-1$ more times.

Lemma 24 $\vdash \mathsf{K}_a\varphi \to \mathsf{K}_a\mathsf{K}_a\varphi$.

Proof Formula $\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \neg\mathsf{K}_a\varphi$ is an instance of the Truth axiom. Thus, $\vdash \mathsf{K}_a\varphi \to \neg\mathsf{K}_a\neg\mathsf{K}_a\varphi$ by contraposition. Hence, taking into account that $\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi$ is an instance of the Negative Introspection axiom, we have

$$\vdash \mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi. \tag{30}$$

At the same time, $\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\varphi$ is an instance of the Negative Introspection axiom. Thus, $\vdash \neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\varphi$ by the law of contrapositive in the propositional logic. Hence, by the Necessitation inference rule, $\vdash \mathsf{K}_a(\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\varphi)$. Thus, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \mathsf{K}_a\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\mathsf{K}_a\varphi$. The latter, together with statement (30), implies the statement of the lemma by propositional reasoning.

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