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Classical Implication for Three-Valued Logic

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Abstract
The article proposes a new implication for three-valued logical systems. The tautologies of this implication are exactly the same as for the classical implication in the two-valued Boolean logic. In the setting of this new implication, the conjunction and disjunction can be viewed as binary modalities. The article studies the definability and complete axiomatisation of the three versions of these modalities capturing Kleene's, weak Kleene's, and short-circuit versions of the connectives.

Keywords Three-valued logic · Axiomatisation · Completeness

1 Three Implications

In this article, we propose a new implication for three-valued logics. Unlike a traditional ("Boolean") logic, a three-valued logic assumes that each statement has one of three values: false (F), uncertain (U), and true (T). The two most commonly studied implications for three-valued logics are Kleene (1938) and Łukasiewicz's (1932, p.213) implications. In this article, we denote these implications by $\to_{\mathrm{K}}$ and $\to_{\mathrm{Ł}}$, respectively.

The truth table for Kleene's implication is shown at the left of Figure 1. For example, symbol T at the intersection of F-row and U-column means that $\mathsf{F} \to_{\mathrm{K}} \mathsf{U}$ is equal to T. The truth table for Łukasiewicz's implication is shown in the centre of Figure 1. Note that the only difference between Kleene's and Łukasiewicz's implications is at the centre cell: $\mathsf{U} \to_{\mathrm{K}} \mathsf{U}$ has value U and $\mathsf{U} \to_{\mathrm{Ł}} \mathsf{U}$ has value T.

The truth table for the implication that we propose is shown at the right of Figure 1. Since this implication is the focus of this article, we denote it simply by $\to$. We call our implication "classical". Although this name sounds pretentious, we will justify its use later in this section.

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$\to_K$	F	U	T
F	T	T	T
U	U	U	T
T	F	U	T

$\to_Ł$	F	U	T
F	T	T	T
U	U	T	T
T	F	U	T

$\to$	F	U	T
F	T	T	T
U	T	T	T
T	F	U	T

Fig. 1 Kleene's, Łukasiewicz's, and classical implications

φ	$\varphi \to_K$ F	$\varphi \to_Ł$ F	$\varphi \to$ F
F	T	T	T
U	U	U	T
T	F	F	F

Fig. 2 Three negations

The negation in a Boolean logic is often defined as an implication to false. In other words, the formula $\neg\phi$ is often viewed in a Boolean logic as an abbreviation for $\phi \to$ F. We can use this intuition to define three negations in three-valued logic as $\phi \to_K$ F, $\phi \to_Ł$ F, and $\phi \to$ F. We denote the expression $\phi \to_K$ F by $\neg_K\phi$ and call it "Kleene's negation". Figure 2 shows the truth tables for the expressions $\phi \to_K$ F, $\phi \to_Ł$ F, and $\phi \to$ F. As one can see from this figure, the expressions $\phi \to_K$ F and $\phi \to_Ł$ F are equivalent. Because of this, we will not consider a "Łukasiewicz's negation". At the same time, as the same figure shows, $\phi \to$ F defines a different negation, which we call "classical" negation and denote by $\neg\phi$. Informally, the classical negation $\neg\phi$ has the meaning "ϕ does not have value T". Although the classical implication is new to the current article, the classical negation has been considered in the literature before (Hernández-Tello et al., 2017, 2021).

To justify the use of the terms "classical implication" and "classical negation" we need first to introduce the notions of a valuation and a three-valued tautology. Throughout the rest of the article, we assume a fixed set of propositional variables.

Definition 1 A valuation is a function from the set of propositional variables into the set {F, U, T}.

Definition 2 A valuation is Boolean if its range is the set {F, T}.

Recall that we interpreted the negation as an implication to false in order to justify the meaning of Kleene's and classical negations. In the rest of this article, it will be slightly more convenient to assume that constant false is not present in our language and to consider both of these negations as primitive connectives.

In the context of this section, by a formula, we mean any propositional formula that uses only connectives $\to_K$, $\to_Ł$, $\to$, $\neg_K$, and $\neg$. For any formula ϕ and any valuation $*$ we define the truth value ϕ^* recursively, using Figure 1 (for $\to_K$, $\to_Ł$, and $\to$) and Figure 2 (for $\neg_K$ and $\neg$).

Definition 3 A formula ϕ is a three-valued tautology if $\phi^* =$ T for each valuation $*$.

Definition 4 A formula ϕ is a Boolean tautology if $\phi^* =$ T for each Boolean valuation $*$.

Note that Boolean tautologies are exactly the tautologies in the classical Boolean two-valued logic.

Theorem 1 *Each three-valued tautology is a Boolean tautology.*

Proof Consider any three-valued tautology ϕ and any Boolean valuation $*$. By Definition 4, it suffices to show that $\phi^* = \mathsf{T}$. By Definition 2, function $*$ is a valuation. Therefore, $\phi^* = \mathsf{T}$ by Definition 3 and the assumption of the theorem that formula ϕ is a three-valued tautology. □

Note that the converse of Theorem 1 is not true. For example, formula $p \rightarrow_{\mathsf{K}} p$ is a Boolean tautology but it is not a three-valued tautology. Indeed, if $*$ is any valuation that maps propositional variable p into U, then $(p \rightarrow_{\mathsf{K}} p)^* = \mathsf{U}$, see Figure 1. What is *surprising* is that, as we show in the next theorem, the converse of Theorem 1 is true if one considers formulae that use only classical implication $\rightarrow$ and classical negation $\neg$. In other words, in the language containing only those two connectives the set of three-valued tautologies is the same as the set of Boolean tautologies! This means that we can take any tautology in the classical Boolean propositional logic that uses only implication and negation, such as, for instance, $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$, and this formula will have to be a three-valued tautology where $\rightarrow$ and $\neg$ are interpreted as classical implication and classical negation in three-valued logic. To phrase this in yet another way, the set of three-valued tautologies in the language containing only $\rightarrow$ and $\neg$ is *exactly* the set of tautologies in the classical Boolean propositional logic. This unexpected observation justifies the use of the terms "classical implication" and "classical negation" when referring to connectives $\rightarrow$ and $\neg$ in a three-valued logic.

Theorem 2 *Each Boolean tautology containing only classical implication and classical negation is a three-valued tautology.*

Proof Let us consider the set of all tautologies in the classical Boolean two-valued logic that uses only implication and negation. It is well known (Mendelson, 2009) that the set of such tautologies can be axiomatised by the logical system consisting of the Modus Ponens inference rule and the following three axioms:

$$\phi \rightarrow (\psi \rightarrow \phi),$$
$$(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi)),$$
$$(\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi).$$

Thus, to finish the proof of the theorem, it suffices to show that each of these three axioms is a three-valued tautology and that the set of three-valued tautologies is closed with respect to the Modus Ponens inference rule. We prove this in the four claims below.

Claim 1 $(\phi \rightarrow (\psi \rightarrow \phi))^* = \mathsf{T}$ *for each valuation* $*$.

PROOF OF CLAIM Suppose that $(\phi \rightarrow (\psi \rightarrow \phi))^* \neq \mathsf{T}$ for some valuation $*$. Thus,

$$\phi^* = \mathsf{T} \tag{1}$$

and $(\psi \to \phi)^* \neq \mathsf{T}$, see the right table in Figure 1. The last statement implies that $\psi^* = \mathsf{T}$ and $\phi^* \neq \mathsf{T}$, see the same table, which contracts statement (1).

Claim 2 $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))^* = \mathsf{T}$ *for each valuation* $*$.

PROOF OF CLAIM Suppose $((\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi)))^* \neq \mathsf{T}$ for some valuation $*$. Thus, see the right table in Figure 1,

$$(\phi \to (\psi \to \chi))^* = \mathsf{T} \tag{2}$$

and $((\phi \to \psi) \to (\phi \to \chi))^* \neq \mathsf{T}$. The last statement, see the same table, implies

$$(\phi \to \psi)^* = \mathsf{T} \tag{3}$$

and $(\phi \to \chi)^* \neq \mathsf{T}$. Similarly, the last statement implies that

$$\chi^* \neq \mathsf{T} \tag{4}$$

and $\phi^* = \mathsf{T}$. The last statement, using the same truth table and statements (2) and (3) imply that $(\psi \to \chi)^* = \mathsf{T}$ and $\psi^* = \mathsf{T}$. Therefore, by the same truth table, $\chi^* = \mathsf{T}$, which contradicts statement (4).

Claim 3 $((\neg\phi \to \neg\psi) \to (\psi \to \phi))^* = \mathsf{T}$ *for each valuation* $*$.

PROOF OF CLAIM Suppose that $((\neg\phi \to \neg\psi) \to (\psi \to \phi))^* \neq \mathsf{T}$ for some valuation $*$. Thus, see the right table in Figure 1,

$$(\neg\phi \to \neg\psi)^* = \mathsf{T} \tag{5}$$

and $(\psi \to \phi)^* \neq \mathsf{T}$. The last statement, by the same table, implies that $\psi^* = \mathsf{T}$ and $\phi^* \neq \mathsf{T}$. Hence, $(\neg\psi)^* \neq \mathsf{T}$ and $(\neg\phi)^* = \mathsf{T}$, see the last column in Figure 2. Therefore, $(\neg\phi \to \neg\psi)^* \neq \mathsf{T}$ by the right table in Figure 1, which contradicts statement (5).

Claim 4 *If formulae* ϕ *and* $\phi \to \psi$ *are three-valued tautologies, then formula* ψ *is also a three-valued tautology.*

PROOF OF CLAIM Suppose that $\psi^* \neq \mathsf{T}$ for some valuation $*$. Note that $\phi^* = \mathsf{T}$ by the assumption of the claim. Therefore, $(\phi \to \psi)^* \neq \mathsf{T}$ by the right table in Figure 1, which contradicts the assumption of the lemma that formula $\phi \to \psi$ is a three-valued tautology.

This concludes the proof of the theorem. ◻

Together, the two theorems above imply that the set of three-valued tautologies in the language containing only the classical implication and the classical negation is exactly the set of tautologies in the classical two-valued logics. The two observations below show that a similar result does not hold for Kleene's and Łukasiewicz's implications.

Fact 1 *If the formulae* ϕ *and* ψ *have value* U*, then the value of the formulae* $\phi \to_K (\psi \to_K \phi)$ *and* $(\neg\phi \to_K \neg\psi) \to_K (\psi \to_K \phi)$ *is* U.

	F	U	T
F	F	U	F
U	U	U	U
T	F	U	T

	F	U	T
F	F	F	F
U	U	U	U
T	F	U	T

	F	U	T
F	F	F	F
U	F	U	U
T	F	U	T

Fig. 3 Weak Kleene's (left), short-circuit (centre), and Kleene's (right) conjunctions

	F	U	T
F	F	U	T
U	U	U	U
T	T	U	T

	F	U	T
F	F	U	T
U	U	U	U
T	T	T	T

	F	U	T
F	F	U	T
U	U	U	T
T	T	T	T

Fig. 4 Weak Kleene's (left), short-circuit (centre), and Kleene's (right) disjunctions

Fact 2 *If the values of formulae* ϕ, ψ, *and* χ *are* U, U, *and* F, *respectively, then the formula* $(\phi \to_{K} (\psi \to_{K} \chi)) \to_{K} ((\phi \to_{K} \psi) \to_{K} (\phi \to_{K} \chi))$ *and the formula* $(\phi \to_{Ł} (\psi \to_{Ł} \chi)) \to_{Ł} ((\phi \to_{Ł} \psi) \to_{Ł} (\phi \to_{Ł} \chi))$ *have value* U.

As we observed above, the logical system containing just classical implication and classical negation is exactly the classical Boolean propositional logic. The language of this system can be potentially extended with other connectives. For example, one can consider an extension of this logical system by the so-called weak Kleene conjunction $\wedge$. As we will see in the next section, in three-valued logic, weak Kleene conjunction is not definable through classical implication and classical negation. Thus, intuitively, weak Kleene conjunction acts as a *binary modality* in the classical propositional logic. Because of this, later in this article, we will be able to claim that Lindenbaum's lemma and other standard properties of maximal consistent sets from classical logic are valid for 3-valued logical systems based on classical implication and classical negation.

In this article, we study three possible extensions of the above logical system by different sets of additional connectives. We introduce these three extensions in the next section.

2 Three Logical Systems for Conjunction and Disjunction

There are at least three different ways to define conjunction and disjunction in three-valued logics that have been proposed in the literature. Surprisingly, all three of them are used in a significant way in programming languages. We show the three definitions of conjunction and of disjunction in Figures 3 and 4, respectively.

The left-most truth tables in those two figures define so-called weak Kleene's (1952, p.334) conjunction and disjunction, respectively. These operations have been originally introduced by Bochvar (1937). Many programming languages have a type that they call "Boolean" but it is actually a three-valued type with the third value being *undefined*. For instance, the Boolean expression $(0/3 < 2) \wedge (3/0 < 2)$ has the undefined value because the result of the division of number 3 by number 0 is undefined. Different programming languages name and handle undefined values slightly differently. In Java, for example, the undefined value is called "an exception". The left-most truth tables in Figures 3 and 4 define the property of connectives `&` (conjunction) and `|` (disjunction) in Java if U is interpreted as throwing an exception.

In this article, by *Weak Kleene's Logic* we mean the logical system that describes the three-valued tautologies in the language containing classical implication, classical negation, weak Kleene's conjunction, and weak Kleene's disjunction. One of the contributions of this article is the proof that the connectives of Weak Kleene's Logic are independent. That is, none of them is definable through any combination of the three others. The other important contribution is a sound and complete axiomatisation of Weak Kleene's Logic. We will also observe that Kleene's negation is definable through classical negation, weak Kleene's disjunction, and weak Kleene's conjunction.

The truth tables in the middle of Figures 3 and 4 define so-called short-circuit conjunction and disjunction. To understand the intuition behind these operations, consider the Boolean expression $(0/3 > 2) \wedge (3/0 > 2)$. A short-circuit evaluation of this expression observes that the statement $0/3 > 2$ is false and, *without evaluation* of the statement $3/0 > 2$, returns false as the value of the entire expression $(0/3 > 2) \wedge (3/0 > 2)$. Similarly, a short-circuit evaluation of the expression $(0/3 > -2) \vee (3/0 > 2)$ returns the value true even though the value of the right disjunct is undefined. Short-circuit connectives can be found in many programming languages. In Java, they are called "conditional-and" (Gosling et al. (2023), Section 15.23) and "conditional-or" (Gosling et al. (2023), Section 15.24) and denoted by `&&` and `||`, respectively.

In this article, by *Short-Circuit Logic*, we mean the logical system that describes the set of all three-valued tautologies in the language containing classical implication, classical negation, short-circuit conjunction, and short-circuit disjunction. We will analyse dependencies between these connectives and give a complete axiomatisation of Short-Circuit Logic. We also will show that Kleene's negation is definable through classical negation, short-circuit conjunction, and short-circuit disjunction.

The truth tables at the right of Figures 3 and 4 define so-called Kleene's conjunction and disjunction (Kleene, 1938). They are also sometimes called "strong" Kleene's conjunction and disjunction. These conjunctions are not used in Java, but they are used in modern SQL database language to handle unknown Boolean values (Winand, 2023).

The situation with Kleene's connectives is a bit different from weak Kleene connectives and short-circuit connectives. First, Kleene's negation is *not* definable through any combination of classical implication, classical negation, Kleene's conjunction, and Kleene's disjunction. Second, our proof technique, in the case of Kleene's connectives, requires the presence of Kleene's negation in the language. Thus, we include Kleene's negation as one of the primitive connectives in the logical system describing the properties of Kleene's conjunction and disjunction.

Furthermore, classical negation is definable through classical implication and Kleene's negation and Kleene's conjunction are definable through Kleene's disjunction and Kleene's negation. As a result, for our third logical system, we have chosen the language containing classical implication, Kleene's negation, and Kleene's disjunction. By *Kleene's Logic* we mean the set of all three-valued tautologies in this language. In this article, we will show that primitive connectives of Kleene's Logic are independent and give a sound and complete axiomatisation of this logic.

3 Weak Kleene's Logic

A complete tableau system for the original Weak Kleene's logic can be found, for example, in the online supplement of Beall and Logan (2017). Bonzio et al. (2017) gives axiomatisation of *paraconsistent* original Weak Kleene's Logic (the set of all formulae whose truth value is never false). These works include neither classical implication nor classical negation. The goal of this section is to give a complete axiomatisation of weak Kleene's connectives in the language containing classical implication $\rightarrow$ and classical negation $\neg$.

Within the context of this section, by $\wedge$ and $\vee$ we mean *weak* Kleene's conjunction and *weak* Kleene's disjunction, respectively. In this section, by a formula, we mean any formula that uses only connectives $\rightarrow$, $\neg$, $\wedge$, and $\vee$. For any valuation $*$, we assume that the value $\phi^* \in \{\mathsf{F}, \mathsf{U}, \mathsf{T}\}$ is defined recursively using the right truth table in Figure 1 for implication $\rightarrow$, the right-most column in Figure 2 for negation $\neg$, the left truth table in Figure 3 for weak Kleene's conjunction $\wedge$, and the left truth table in Figure 4 for weak Kleene's disjunction $\vee$.

3.1 Undefinability results

Before giving a complete axiomatisation of the logical system containing connectives $\rightarrow$, $\neg$, $\wedge$, and $\vee$, we show that these four connectives are *independent*. In other words, we show that neither of them can be expressed through a combination of the three others. More precisely, we show that using each of these connectives one can construct a formula that is not *semantically equivalent* to a formula that uses only the remaining three connectives, where semantical equivalence is defined as follows:

Definition 5 Formulae ϕ and ψ are semantically equivalent if ϕ^* and ψ^* have the same value for each possible valuation $*$.

We obtain the undefinability results using the "truth set algebra" technique proposed in Knight et al. (2022) and also used in Deuser et al. (2024). To illustrate the technique on an easier example, we first show the following result:

Theorem 3 *Formulae $p \vee q$, $p \wedge q$, and $\neg p$ are not semantically equivalent to any formula that uses only connectives $\rightarrow$.*

Proof Let us first show that the formula $p \vee q$ is not semantically equivalent to any formula that uses only classical implication $\rightarrow$. Without loss of generality, we can assume that our language only contains propositional variables p and q. Then, the truth table for each formula can be given as a 3 by 3 table whose rows represent three possible values of propositional variable p and whose columns represent three possible values of propositional variable q. For example, the upper-left diagram in Figure 5 shows the truth table for formula $p \rightarrow q$.

It is relatively easy to verify that the set of seven truth tables shown in Figure 5 is *closed with respect to the implication*. For example, the second diagram in the second row of this table shows the truth table for the formula $(q \rightarrow p) \rightarrow p$. The third diagram in the same row shows the truth table for the formula q. The truth table for the formula

$$((q \to p) \to p) \to q$$

could be computed by computing the classical implication $\to$ at each of the nine cells separately. For example, the low-left cell of the truth table for the formula $(q \to p) \to p$ contains T and the low-left cell of the truth table for the formula q contains F. Note that the value of T $\to$ F is F. Thus, the low-left cell of the truth table for the formula $((q \to p) \to p) \to q$ must contain F. By repeating this computation eight more times, one can see that the truth table for the formula $((q \to p) \to p) \to q$ is captured by the first diagram of the first row in Figure 5. As a side note, observe that because the first diagram of the first row depicts the truth table for the formula $p \to q$, we have just shown that formulae $((q \to p) \to p) \to q$ and $p \to q$ are semantically equivalent.

One can similarly consider all 49 possible pairs of truth tables shown in Figure 5 and see that the cell-wise implication of any two of them is one of the seven truth tables depicted in Figure 5. Therefore, the truth table of any formula that contains only propositional variables p and q and classical implication $\to$ is one of the seven truth tables depicted in Figure 5.

Finally, observe that the truth table for weak Kleene's disjunction $\vee$, see centre truth table in Figure 4, is not among the seven truth tables shown in Figure 5. Therefore, the formula $p \vee q$ is not semantically equivalent to any formula that uses only classical implication $\to$.

A similar argument can be made to show that the formulae $p \wedge q$ and $\neg p$ are also not semantically equivalent to any formula that uses only the classical implication $\to$. □

The proofs of the next four theorems are very similar to the proof of Theorem 3. However, instead of the set of seven truth tables depicted in Figure 5, these proofs use the sets of 658, 90, 656, and 576 truth tables, respectively. We used a computer program to verify that those sets are closed with respect to the required operations.

Theorem 4 *Formula $p \wedge q$ is not semantically equivalent to any formula that uses only connectives $\to$, $\vee$, and $\neg$.*

Theorem 5 *Formula $p \vee q$ is not semantically equivalent to any formula that uses only connectives $\to$, $\wedge$, and $\neg$.*

Theorem 6 *Formula $p \to q$ is not semantically equivalent to any formula that uses only connectives $\vee$, $\wedge$, and $\neg$.*

Theorem 7 *Formula $\neg p$ is not semantically equivalent to any formula that uses only connectives $\to$, $\vee$, and $\wedge$.*

3.2 Axioms

In this section, by $\mathtt{B}\phi$ we denote the excluded middle law formula $\phi \vee \neg\phi$. Note that, in three-valued logic, the formula $\phi \vee \neg\phi$ has value T if and only if the value of ϕ

	F	U	T
F	T	T	T
U	T	T	T
T	F	U	T

	F	U	T
F	F	U	T
U	F	U	T
T	T	T	T

	F	U	T
F	T	T	F
U	T	T	U
T	T	T	T

	F	U	T
F	T	T	T
U	T	T	T
T	T	T	T

	F	U	T
F	F	F	T
U	U	U	T
T	T	T	T

	F	U	T
F	F	U	T
U	F	U	T
T	F	U	T

	F	U	T
F	F	F	F
U	U	U	U
T	T	T	T

Fig. 5 The first row shows the truth tables for the formulae $p \to q$, $(p \to q) \to q$, and $q \to p$. The second row shows the truth tables for the formulae $p \to p$, $(q \to p) \to p$, and q. The third row shows the truth table for the formula p

is one of the two Boolean values: T and F. We read $\mathtt{B}\phi$ as "formula ϕ has a Boolean value". By the axioms of Weak Kleene's Logic, we mean axioms A1 through A19 listed in Table 1. In the context of this section, we write $\vdash \phi$ and say that formula ϕ is a *theorem* of Weak Kleene's Logic if ϕ can be obtained from the axioms of the Weak Kleene's Logic using the Modus Ponens inference rule:

$$\frac{\phi, \qquad \phi \to \psi}{\psi}.$$

In the context of this section, we write $X \vdash \phi$ if a formula ϕ is provable from axioms A1 through A19 and an additional set of formulae X using the Modus Ponens inference rule. Note that statements $\varnothing \vdash \phi$ and $\vdash \phi$ are equivalent. We say that the set of formulae X is inconsistent if there is a formula ϕ such that $X \vdash \phi$ and $X \vdash \neg\phi$. Note that because axioms A1, A2, and A3 of the classical Boolean logic and the Modus Ponens inference rule are present in this system, the Weak Kleene's Logic is an extension of the classical Boolean logic by two new connectives ($\wedge$ and $\vee$) in the same sense as, for example, S4 modal logic is an extension of the classical Boolean logic. This means that the maximal consistent sets can be defined in the standard way and they have the same properties as in any other extension of the classical logic. For example, as we state in Lemma 3, Lindenbaum's lemma is true for such sets. Also, for instance, if a formula is derivable from a maximal consistent set, then this formula must belong to such a set.

Table 1 also contains axioms A20 through A31. These are not axioms of Weak Kleene's Logic. Some of them are provable in our logical system and some are not three-valued tautologies. The column labelled with "WK" specifies the status of each of the formulae A1 through A31 in Weak Kleene's Logic. The columns labelled with

Table 1 Axioms

	Formula	WK	SC	K
A1	$\phi \to (\psi \to \phi)$	Axiom	Axiom	Axiom
A2	$(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$	Axiom	Axiom	Axiom
A3	$(\neg\phi \to \neg\psi) \to (\psi \to \phi)$	Axiom	Axiom	Axiom
A4	$\mathtt{B}\neg\phi$	Axiom	Axiom	Provable
A5	$\phi \to \mathtt{B}\phi$	Axiom	Provable	Provable
A6	$\mathtt{B}\phi \to \mathtt{B}(\psi \to \phi)$	Axiom	Axiom	Axiom
A7	$\mathtt{B}(\phi \wedge \psi) \to \mathtt{B}\phi$	Axiom	Axiom	False
A8	$\mathtt{B}(\phi \wedge \psi) \to \mathtt{B}\psi$	Axiom	Axiom	False
A9	$\mathtt{B}(\phi \vee \psi) \to \mathtt{B}\phi$	Axiom	Axiom	False
A10	$\mathtt{B}(\phi \vee \psi) \to \mathtt{B}\psi$	Axiom	False	False
A11	$\mathtt{B}\phi \to (\mathtt{B}\psi \to \mathtt{B}(\phi \wedge \psi))$	Axiom	Axiom	Provable
A12	$\mathtt{B}\phi \to (\mathtt{B}\psi \to \mathtt{B}(\phi \vee \psi))$	Axiom	Axiom	Axiom
A13	$\phi \wedge \psi \to \phi$	Axiom	Axiom	Provable
A14	$\phi \wedge \psi \to \psi$	Axiom	Axiom	Provable
A15	$\phi \to (\psi \to (\phi \wedge \psi))$	Axiom	Axiom	Provable
A16	$\phi \to (\mathtt{B}\psi \to \phi \vee \psi)$	Axiom	Provable	Provable
A17	$\mathtt{B}\phi \to (\psi \to \phi \vee \psi)$	Axiom	Axiom	Provable
A18	$\mathtt{B}(\phi \to \psi) \to (\phi \to \mathtt{B}\psi)$	Axiom	Axiom	Axiom
A19	$\phi \vee \psi \to (\neg\phi \to \psi)$	Axiom	Axiom	Axiom
A20	$\phi \to \phi \vee \psi$	False	Axiom	Axiom
A21	$\mathtt{B}(\phi \wedge \psi) \to (\phi \to \mathtt{B}\psi)$	Provable	Axiom	Provable
A22	$\neg\phi \to (\mathtt{B}\phi \to \mathtt{B}(\phi \wedge \psi))$	False	Axiom	Provable
A23	$\mathtt{B}(\phi \vee \psi) \to (\neg\phi \to \mathtt{B}\psi)$	Provable	Axiom	Provable
A24	$\neg_{\mathsf{K}}\phi \to \neg\phi$	Provable	Provable	Axiom
A25	$\neg_{\mathsf{K}}\phi \to \mathtt{B}\phi$	Provable	Provable	Axiom
A26	$\neg\phi \to (\mathtt{B}\phi \to \neg_{\mathsf{K}}\phi)$	Provable	Provable	Axiom
A27	$\mathtt{B}\neg_{\mathsf{K}}\phi \to \mathtt{B}\phi$	Provable	Provable	Axiom
A28	$\mathtt{B}\phi \to \mathtt{B}\neg_{\mathsf{K}}\phi$	Provable	Provable	Axiom
A29	$\phi \to \psi \vee \phi$	False	False	Axiom
A30	$\neg\phi \to (\neg\psi \to (\mathtt{B}(\phi \vee \psi) \to \mathtt{B}\phi))$	Provable	Provable	Axiom
A31	$\neg\phi \to (\neg\psi \to (\mathtt{B}(\phi \vee \psi) \to \mathtt{B}\psi))$	Provable	Provable	Axiom

"SC" and "K" show the status of the same formulae in Short-Circuit Logic and Kleene's Logic. We further discuss these columns later in the article.

Theorem 8 *(soundness) If $\vdash \phi$, then ϕ is a three-valued tautology.*

Proof We verified the soundness of axioms A1, A2, and A3 as well as of the Modus Ponens inference rules in Claim 1, Claim 2, Claim 3, and Claim 4, respectively. The proofs of soundness of the remaining axioms are similar. □

3.3 Completeness

In this subsection, until the proof of Theorem 9, we assume a fixed maximal consistent set of formulae X. We define valuation $*$ as follows:

Definition 6 For any propositional variable p, let

$$*(p) = \begin{cases} \mathsf{T}, & \text{if } p \in X, \\ \mathsf{U}, & \text{if } \mathtt{B}p \notin X, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

Lemma 1 *Function $*$ is well-defined.*

Proof It suffices to show that if $p \in X$, then $\mathtt{B}p \in X$. Indeed, suppose that $p \in X$. Hence, $X \vdash \mathtt{B}p$ by axiom A5 and the Modus Ponens inference rule. Therefore, $\mathtt{B}p \in X$ because X is a maximal consistent set[1]. □

Lemma 2 *(truth lemma) For any formula ϕ,*

1. $\phi \in X$ iff $\phi^* = \mathsf{T}$,
2. $\mathtt{B}\phi \in X$ iff $\phi^* \neq \mathsf{U}$.

Proof We prove the statement of the lemma by structural induction on formula ϕ. In the case when formula ϕ is a propositional variable p, we have $\phi \in X$ iff $p \in X$ iff $*(p) = \mathsf{T}$ iff $p^* = \mathsf{T}$ iff $\phi^* = \mathsf{T}$ by Definition 6. Similarly, $\mathtt{B}\phi \in X$ iff $\mathtt{B}p \in X$ iff $*(p) \neq \mathsf{U}$ iff $p^* \neq \mathsf{U}$ iff $\phi^* \neq \mathsf{U}$.

Suppose that formula ϕ has the form $\psi \to \chi$. We prove the two parts of the statement of the lemma separately.

Part 1. $(\Rightarrow)$: Assume that $\psi \to \chi \in X$. Then, by the Modus Ponens inference rule, if $\psi \in X$, then $X \vdash \chi$. Thus, because X is a maximal consistent set, if $\psi \in X$, then $\chi \in X$. Hence, by the induction hypothesis, if $\psi^* = \mathsf{T}$, then $\chi^* = \mathsf{T}$. Therefore, $(\psi \to \chi)^* = \mathsf{T}$ by the right-most truth table in Figure 1.

$(\Leftarrow)$: Assume that $(\psi \to \chi)^* = \mathsf{T}$. Thus, by the right-most truth table in Figure 1, if $\psi^* = \mathsf{T}$, then $\chi^* = \mathsf{T}$. Hence, by the induction hypothesis, if $\psi \in X$, then $\chi \in X$. Then, one of the following cases takes place:

Case A: $\psi \notin X$. Thus, $\neg\psi \in X$ because X is a maximal consistent set. Note that the formula $\neg\psi \to (\psi \to \chi)$ is a Boolean tautology. Thus, due to the completeness theorem for Boolean logic (Mendelson, 2009), this formula is provable from axioms A1-A3. Hence, $X \vdash \psi \to \chi$ by the Modus Ponens inference rule. Therefore, $\psi \to \chi \in X$ because X is a maximal consistent set.

Case B: $\chi \in X$. Then, $X \vdash \psi \to \chi$ by axiom A1 and the Modus Ponens inference rule. Therefore, $\psi \to \chi \in X$ because X is a maximal consistent set.

Part 2. $(\Rightarrow)$: Suppose that $(\psi \to \chi)^* = \mathsf{U}$. Thus, $\psi^* = \mathsf{T}$ and $\chi^* = \mathsf{U}$ by the right-most truth table in Figure 1. Hence, $\psi \in X$ and $\mathtt{B}\chi \notin X$ by the induction hypothesis. Then, $\psi \in X$ and $X \nvdash \mathtt{B}\chi$ because X is a maximal consistent set. Thus, $X \nvdash \psi \to \mathtt{B}\chi$

[1] As we observed in Section 3.2, if a formula is derivable from a maximal consistent set, then it must belong to this set.

by the Modus Ponens rule applied contrapositively. Hence, $X \nvdash \mathtt{B}(\psi \to \chi)$ by axiom A18 and the Modus Ponens rule applied contrapositively. Therefore, $\mathtt{B}(\psi \to \chi) \notin X$.
$(\Leftarrow)$. Suppose that $\mathtt{B}(\psi \to \chi) \notin X$. Then, $X \nvdash \mathtt{B}(\psi \to \chi)$ because X is a maximal consistent set. Thus, by the Modus Ponens rule, applied contrapositively,

$$X \nvdash \psi \to \chi \qquad \text{and} \qquad X \nvdash \mathtt{B}\chi, \tag{6}$$

using axioms A5 and A6, respectively.

At the same time, note that the formula $\neg\psi \to (\psi \to \chi)$ is a Boolean tautology. Thus, due to the completeness theorem for Boolean logic (Mendelson, 2009), this formula is provable from axioms A1-A3. Hence, $X \nvdash \neg\psi$ by the part $X \nvdash \psi \to \chi$ of statement (6) and the Modus Ponens inference rule applied contrapositively. Then, $\neg\psi \notin X$. Thus, $\psi \in X$ because X is a maximal consistent set. Also, $\mathtt{B}\chi \notin X$ by the part $X \nvdash \mathtt{B}\chi$ of statement (6). Hence, $\psi^* = \mathsf{T}$ and $\chi^* = \mathsf{U}$ by the induction hypothesis. Therefore, $(\psi \to \chi)^* = \mathsf{U}$ by the right-most truth table in Figure 1.

Suppose that formula ϕ has the form $\neg\psi$. We again prove the two parts of the lemma separately.
Part 1. $(\Rightarrow)$: Suppose that $\neg\psi \in X$. Then, $\psi \notin X$ because set X is consistent. Thus, $\psi^* \neq \mathsf{T}$ by the induction hypothesis. Therefore, $(\neg\psi)^* = \mathsf{T}$ the right-most column in Figure 2.
$(\Leftarrow)$: Suppose that $(\neg\psi)^* = \mathsf{T}$. Thus, $\psi^* \neq \mathsf{T}$ by the right-most column in Figure 2. Then, $\psi \notin X$ by the induction hypothesis. Therefore, $\neg\psi \in X$ because X is a maximal consistent set.
Part 2. $(\Rightarrow)$: It suffices to prove that $(\neg\psi)^* \neq \mathsf{U}$. The last statement is true for *any* formula ψ by the right-most column in Figure 2.
$(\Leftarrow)$: It suffices to show that $\mathtt{B}\neg\psi \in X$. The last statement is true by axiom A4 because X is a maximal consistent set of formulae.

Suppose that formula ϕ has the form $\psi \wedge \chi$. We again prove the two parts of the lemma separately.
Part 1. $(\Rightarrow)$: Assume that $\psi \wedge \chi \in X$. Then, $X \vdash \psi$ and $X \vdash \chi$ by the Modus Ponens inference rule using, respectively, axiom A13 and axiom A14. Hence, $\psi \in X$ and $\chi \in X$ because X is a maximal consistent set of formulae. Thus, $\psi^* = \mathsf{T}$ and $\chi^* = \mathsf{T}$ by the induction hypothesis. Therefore, $(\psi \wedge \chi)^* = \mathsf{T}$ by the truth table at the left of Figure 3.
$(\Leftarrow)$: Assume that $(\psi \wedge \chi)^* = \mathsf{T}$. Thus, $\psi^* = \mathsf{T}$ and $\chi^* = \mathsf{T}$ by the truth table at the left of Figure 3. Then, $\psi \in X$ and $\chi \in X$ by the induction hypothesis. Hence, $X \vdash \psi \wedge \chi$ by axiom A15 and the Modus Ponens inference rule applied twice. Therefore, $\psi \wedge \chi \in X$ because X is a maximal consistent set of formulae.
Part 2. $(\Rightarrow)$: Assume that $\mathtt{B}(\psi \wedge \chi) \in X$. Then, $X \vdash \mathtt{B}\psi$ and $X \vdash \mathtt{B}\chi$ by the Modus Ponens inference rule and, respectively, axioms A7 and A8. Hence, $\mathtt{B}\psi \in X$ and $\mathtt{B}\chi \in X$ because X is a maximal consistent set of formulae. Thus, $\psi^* \neq \mathsf{U}$ and $\chi^* \neq \mathsf{U}$ by the induction hypothesis. Therefore, $(\psi \wedge \chi)^* \neq \mathsf{U}$ by the truth table at the left of Figure 3.
$(\Leftarrow)$: Assume that $\mathtt{B}(\psi \wedge \chi) \notin X$. Thus, $X \nvdash \mathtt{B}(\psi \wedge \chi)$ because X is a maximal consistent set of formulae. Hence, either $\mathtt{B}\psi \notin X$ or $\mathtt{B}\chi \notin X$ by axiom A11 and the Modus Ponens inference rule applied contrapositively twice. Then, either $\psi^* = \mathsf{U}$ or

$\chi^* = \mathsf{U}$ by the induction hypothesis. Therefore, $(\psi \wedge \chi)^* = \mathsf{U}$ by the truth table at the left of Figure 3.

Suppose that formula ϕ has the form $\psi \vee \chi$. We once again prove the two parts of the lemma separately.

Part 1. $(\Rightarrow)$: Assume that $\psi \vee \chi \in X$. Thus, $X \vdash \mathtt{B}(\psi \vee \chi)$ by axiom A5 and the Modus Ponens inference rule. Hence, $X \vdash \mathtt{B}\psi$ and $X \vdash \mathtt{B}\chi$ by, respectively, axioms A9 and A10 and the Modus Ponens inference rule. Then, $\mathtt{B}\psi \in X$ and $\mathtt{B}\chi \in X$ because X is a maximal consistent set. Thus, by the induction hypothesis,

$$\psi^* \neq \mathsf{U} \quad \text{and} \quad \chi^* \neq \mathsf{U}. \tag{7}$$

At the same time, the assumption $\psi \vee \chi \in X$ also implies $X \vdash \neg\psi \to \chi$ by axiom A19 and the Modus Ponens inference rule. Thus, again by the Modus Ponens inference rule, if $X \vdash \neg\psi$, then $X \vdash \chi$. Hence, because X is a maximal consistent set, if $\psi \notin X$, then $\chi \in X$. Then, by the induction hypothesis, if $\psi^* \neq \mathsf{T}$, then $\chi^* = \mathsf{T}$. Therefore, $(\psi \vee \chi)^* = \mathsf{T}$ by the truth table at the left of Figure 3 and statements (7).
$(\Leftarrow)$: Suppose that $(\psi \vee \chi)^* = \mathsf{T}$. Thus, by the truth table at the left of Figure 3,

$$\psi^* \neq \mathsf{U} \quad \text{and} \quad \chi^* \neq \mathsf{U}$$

and, in addition, at least one of the following statements is true:

$$\psi^* = \mathsf{T} \quad \text{or} \quad \chi^* = \mathsf{T}.$$

Hence, by the induction hypothesis,

$$\mathtt{B}\psi \in X \quad \text{and} \quad \mathtt{B}\chi \in X$$

and either

$$\psi \in X \quad \text{or} \quad \chi \in X.$$

Thus, $X \vdash \psi \vee \chi$ by the Modus Ponens inference rule and either axiom A16 or axiom A17. Therefore, $\psi \vee \chi \in X$ because X is a maximal consistent set.

Part 2. $(\Rightarrow)$: Assume that $\mathtt{B}(\psi \vee \chi) \in X$. Then, $X \vdash \mathtt{B}\psi$ and $X \vdash \mathtt{B}\chi$ by, respectively, axiom A9 and axiom A10 and the Modus Ponens inference rule. Hence, $\mathtt{B}\psi \in X$ and $\mathtt{B}\chi \in X$ because X is a maximal consistent set. Then, $\psi^* \neq \mathsf{U}$ and $\psi^* \neq \mathsf{U}$ by the induction hypothesis. Therefore, $(\psi \vee \chi)^* \neq \mathsf{U}$ by the truth table at the left of Figure 3.
$(\Leftarrow)$: Suppose that $(\psi \vee \chi)^* \neq \mathsf{U}$. Then, $\psi^* \neq \mathsf{U}$ and $\psi^* \neq \mathsf{U}$ by the truth table at the left of Figure 3. Hence, $\mathtt{B}\psi \in X$ and $\mathtt{B}\chi \in X$ by the induction hypothesis. Thus, $X \vdash \mathtt{B}(\psi \vee \chi)$ by axiom A12 and the Modus Ponens inference rule. Therefore, $\mathtt{B}(\psi \vee \chi) \in X$ because X is a maximal consistent set. □

Lemma 3 *[Lindenbaum] Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

Proof Recall that our logical system is an extension of the classical propositional logic by two binary modalities: $\wedge$ and $\vee$. Thus, the standard proof of Lindenbaum's lemma (Mendelson (2009), Proposition 2.14) applies here. □

Theorem 9 *(strong completeness) For any set of formulae* Γ *and any formula* ϕ*, if* $\Gamma \nvdash \phi$*, then there is a valuation* $*$ *such that* $\gamma^* = \mathsf{T}$ *for each formula* $\gamma \in \Gamma$ *and* $\phi^* \neq \mathsf{T}$.

Proof Suppose that $\Gamma \nvdash \phi$. Then, the set $\Gamma \cup \{\neg\phi\}$ is consistent. By Lemma 3, it can be extended to a maximal consistent set X. Note that $\phi \notin X$ because set X is consistent. Therefore, $\gamma^* = \mathsf{T}$ for each formula $\gamma \in \Gamma$ and $\phi^* \neq \mathsf{T}$ by Lemma 2. □

The logical system introduced in this section includes classical negation $\neg$ but does not include Kleene's negation $\neg_{\kappa}$. Note however that Kleene's negation $\neg_{\kappa}\phi$ is definable in our language:

Lemma 4 *The formulae* $\neg_{\kappa} p$ *and* $(p \vee \neg p) \wedge \neg p$ *are semantically equivalent.*

4 Short-Circuit Logic

In the context of this section, $\wedge$ and $\vee$ mean short-circuit conjunction and disjunction as defined in the centre of Figure 3 and Figure 4, respectively.

4.1 Undefinability results

Using the technique discussed in Subsection 3.1, one can prove the following results. The proofs of these results use sets of 642, 72, and 576 truth tables, respectively.

Theorem 10 *Formula* $p \wedge q$ *is not semantically equivalent to any formula that uses only connectives* $\rightarrow$, $\vee$, *and* $\neg$.

Theorem 11 *Formula* $p \vee q$ *is not semantically equivalent to any formula that uses only connectives* $\rightarrow$, $\wedge$, *and* $\neg$.

Theorem 12 *Formula* $\neg p$ *is not semantically equivalent to any formula that uses only connectives* $\rightarrow$, $\wedge$, *and* $\vee$.

The three results above, of course, are very similar to the corresponding undefinability results in Subsection 3.1. So, it is perhaps surprising that, unlike in the case of Weak Kleene's Logic, classical implication is definable through short-circuit disjunction and classical negation:

Lemma 5 *The formulae* $p \rightarrow q$ *and* $\neg p \vee q$ *are semantically equivalent.*

As a result, in this section, we technically do not have to consider the classical implication at all. In spite of this, in this section, we still decided to treat $\rightarrow$ as a primitive connective. We do it not only to be faithful to the title of the article but also because we still want to be able to claim that any Boolean tautology using only $\rightarrow$ and $\neg$ is provable from axioms A1-A3. If $\rightarrow$ is treated as an abbreviation, this result is still true but is harder to argue for. Thus, in this section, by a formula, we still mean any formula that uses connectives $\rightarrow$, $\neg$, $\wedge$, and $\vee$.

4.2 Complete Axiomatisation

Short-Circuit Logic has been proposed in Bergstra and Ponse (2012). Work (Bergstra et al., 2021) gives a complete set of equivalences for this logic in the language containing short-circuit conjunction, short-circuit disjunction, and Kleene's negation (earlier denoted by $\neg_\kappa$). It is easy to see that classical implication and classical negation are not definable in this language. Indeed, if all propositional variables are assigned value U, then all expressions build out of short-circuit conjunction, short-circuit disjunction, and Kleene's negation also have value U. At the same time, $\mathsf{U} \to \mathsf{U}$ and $\neg\mathsf{U}$ both have value T. In this subsection, we give an axiomatisation of Short Circuit Logic in the language with connectives $\to$, $\neg$, $\wedge$, and $\vee$.

As in the previous section, by $\mathtt{B}\phi$ we mean the excluded middle law formula $\phi\vee\neg\phi$. Of course, here, unlike the previous section, $\vee$ is the short-circuit disjunction. It is easy to see, however, that just like in the case of Weak Kleene's Logic, the formula $\phi \vee \neg\phi$ has value T if and only if formula ϕ has one of two Boolean values: T or F.

The axioms of Short-Circuit Logic are those of formula A1-A31 in Table 1 that are marked as axioms in the "SC" column of the table. In the context of this section, we write $X \vdash \phi$ if a formula ϕ is provable from these axioms and the additional set of formulae X using the Modus Ponens inference rule. Just like before statements $\varnothing \vdash \phi$ and $\vdash \phi$ are equivalent. We say that a set of formulae X is inconsistent if there is a formula ϕ such that $X \vdash \phi$ and $X \vdash \neg\phi$.

Among the remaining formulae in Table 1, some are theorems and some are not three-valued tautologies. In the table, we marked them as "provable" and "false", respectively.

Theorem 13 *[soundness] If $\vdash \phi$, then ϕ is a three-valued tautology.*

Note that Lindenbaum's lemma holds in Short-Circuit Logic for the same reason as in Weak Kleene's Logic.

Lemma 6 *[Lindenbaum] Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

The completeness theorem for Short-Circuit Logic is stated as Theorem 14 at the end of this subsection. Its proof follows the same pattern as the completeness proof for Weak Kleene's Logic. Here, we also assume that X is fixed, until the proof of Theorem 14, maximal consistent set of formulae.

Let valuation $*$ be defined as follows:

Definition 7 For any propositional variable p,

$$*(p) = \begin{cases} \mathsf{T}, & \text{if } p \in X, \\ \mathsf{U}, & \text{if } \mathtt{B}p \notin X, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

Lemma 7 *Function $*$ is well-defined.*

Proof It suffices to show that if $p \in X$, then $\mathtt{B}p \in X$. Indeed, suppose that $p \in X$. Hence, $X \vdash p \vee \neg p$ by axiom A20 and the Modus Ponens inference rule. Then, $X \vdash \mathtt{B}p$ by the definition of abbreviation $\mathtt{B}$. Therefore, $\mathtt{B}p \in X$ because X is a maximal consistent set. □

Lemma 8 *(truth lemma) For any formula ϕ,*

1. $\phi \in X$ *iff* $\phi^* = \mathsf{T}$,
2. $\mathtt{B}\phi \in X$ *iff* $\phi^* \neq \mathsf{U}$.

Proof We prove the statement of the lemma by induction on the structural complexity of the formula ϕ. If ϕ is a propositional variable, then the argument is the same as in Lemma 2.

Suppose that formula ϕ is an implication. In this case, the proof of lemma is the same as the corresponding proof for Lemma 2. Note that the corresponding proof of for Lemma 2 is using axiom A5, which is technically not an axiom of Short-Circuit Logic. However, this axiom is a special case of axiom A20, which is an axiom of Short-Circuit Logic.

If formula ϕ has the form $\neg\psi$, then the proof is also the same as the corresponding proof for Lemma 2.

Suppose that formula ϕ has the form $\psi \wedge \chi$. In this case, the proof of Part 1 of the lemma is the same as the corresponding proof in Lemma 2. Of course, instead of the truth table at the left of Figure 3 one should refer to the truth table at the centre of the same figure. We now prove Part 2 of the lemma.
$(\Rightarrow)$: Assume that $\mathtt{B}(\psi \wedge \chi) \in X$. Thus, $X \vdash \mathtt{B}\psi$ by axiom A7 and the Modus Ponens inference rule. Hence, $\mathtt{B}\psi \in X$ because X is a maximal consistent set. Then, by the induction hypothesis,

$$\psi^* \neq \mathsf{U}. \tag{8}$$

At the same time, the assumption $\mathtt{B}(\psi \wedge \chi) \in X$ also implies $X \vdash \psi \rightarrow \mathtt{B}\chi$ by axiom A21 and the Modus Ponens inference rule. Hence, again by the Modus Ponens inference rule, if $X \vdash \psi$, then $X \vdash \mathtt{B}\chi$. Thus, if $\psi \in X$, then $\mathtt{B}\chi \in X$ because X is a maximal consistent set. Then, if $\psi^* = \mathsf{T}$, then $\chi^* \neq \mathsf{U}$ by the induction hypothesis. Therefore, $(\psi \wedge \chi)^* \neq \mathsf{U}$ by statement (8) and the truth table at the centre of Figure 3.
$(\Leftarrow)$: Assume that $(\psi \wedge \chi)^* \neq \mathsf{U}$. Thus, $\psi^* \neq \mathsf{U}$ by the truth table at the centre of Figure 3. Hence, by the induction hypothesis,

$$\mathtt{B}\psi \in X. \tag{9}$$

We consider the following two cases separately:
Case A: $\psi \in X$. The assumption $(\psi \wedge \chi)^* \neq \mathsf{U}$, by the truth table at the centre of Figure 3, implies that if $\psi^* = \mathsf{T}$, then $\chi^* \neq \mathsf{U}$. Then, by the induction hypothesis, if $\psi \in X$ then $\mathtt{B}\chi \in X$. Hence, $\mathtt{B}\chi \in X$ by the assumption of the case. Thus, $X \vdash \mathtt{B}(\phi \wedge \psi)$ by statement 9, axiom A11 and the Modus Ponens inference rule applied twice. Therefore, $\mathtt{B}(\phi \wedge \psi) \in X$ because X is a maximal consistent set.

Case B: $\psi \notin X$. Then, $\neg\psi \in X$ because X is a maximal consistent set. Hence, $X \vdash \mathtt{B}(\phi \wedge \psi)$ by statement (9), axiom A22 and the Modus Ponens inference rule applied twice. Therefore, $\mathtt{B}(\phi \wedge \psi) \in X$ because X is a maximal consistent set.

Suppose that formula ϕ has the form $\psi \vee \chi$.

Part 1. $(\Rightarrow)$: Assume $\psi \vee \chi \in X$. We consider the following two cases separately.
Case A: $\psi \in X$. Thus, $\psi^* = \mathsf{T}$ by the induction hypothesis. Then $(\psi \vee \chi)^* = \mathsf{T}$ by the truth table at the centre of Figure 4.
Case B: $\psi \notin X$. The assumption $\psi \vee \chi \in X$, by axiom A20 and the Modus Ponens inference rule, implies that $X \vdash (\psi \vee \chi) \vee \neg(\psi \vee \chi)$. In other words, $X \vdash \mathtt{B}(\phi \vee \psi)$. Then, $X \vdash \mathtt{B}\psi$ by axiom A9 and the Modus Ponens inference rule. Thus, $\mathtt{B}\psi \in X$ because X is a maximal consistent set. Hence, by the induction hypothesis,

$$\psi^* \neq \mathsf{U}. \tag{10}$$

At the same time, the assumption $\psi \vee \chi \in X$ implies $X \vdash \neg\psi \to \chi$ by axiom A19 and the Modus Ponens inference rule. Note that the assumption $\psi \notin X$ of the case implies that $\neg\psi \in X$ because X is a maximal consistent set. Hence, $X \vdash \chi$. Thus, $\chi \in X$ because X is a maximal consistent set. Then, $\chi^* = \mathsf{T}$ by the induction hypothesis. Therefore, $(\psi \vee \chi)^* = \mathsf{T}$ by statement (10) and the truth table at the centre of Figure 4.

$(\Leftarrow)$: Suppose $(\psi \vee \chi)^* = \mathsf{T}$. Then, see the truth table at the centre of Figure 4, one of the following two cases takes place:
Case A: $\psi^* = \mathsf{T}$. Thus, $\psi \in X$ by the induction hypothesis. Hence, $X \vdash \psi \vee \chi$ by axiom A20 and the Modus Ponens inference rule. Therefore, $\psi \vee \chi \in X$ because X is a maximal consistent set.
Case B: $\psi^* \neq \mathsf{U}$ and $\chi^* = \mathsf{T}$. Thus, $\mathtt{B}\psi \in X$ and $\chi \in X$ by the induction hypothesis. Hence, $X \vdash \psi \vee \chi$ by axiom A17 and the Modus Ponens inference rule applied twice. Therefore, $\psi \vee \chi \in X$ because X is a maximal consistent set.
Part 2. $(\Rightarrow)$: Suppose that $(\psi \vee \chi)^* = \mathsf{U}$. Then, see the truth table at the centre of Figure 4, one of the following two cases takes place:
Case A: $\psi^* = \mathsf{U}$. Hence, $\mathtt{B}\psi \notin X$ by the induction hypothesis. Thus, $X \nvdash \mathtt{B}\psi$ because X is a maximal consistent set of formulae. Hence, $X \nvdash \mathtt{B}(\psi \vee \chi)$ by axiom A9 and the Modus Ponens inference rule applied contrapositively. Therefore, $\mathtt{B}(\psi \vee \chi) \notin X$.
Case B: $\psi^* = \mathsf{F}$ and $\chi^* = \mathsf{U}$. Then, $\psi^* \neq \mathsf{T}$ and $\chi^* = \mathsf{U}$. Hence, $\psi \notin X$ and $\mathtt{B}\chi \notin X$ by the induction hypothesis. Thus, $\neg\psi \in X$ and $X \nvdash \mathtt{B}\chi$ because X is a maximal consistent set of formulae. Then, $X \nvdash \neg\psi \to \mathtt{B}\chi$ by the Modus Ponens inference rule applied contrapositively. Hence, $X \nvdash \mathtt{B}(\psi \vee \chi)$ by axiom A23 and the Modus Ponens inference rule applied contrapositively. Therefore, $\mathtt{B}(\psi \vee \chi) \notin X$.
$(\Leftarrow)$ Suppose that $(\psi \vee \chi)^* \neq \mathsf{U}$. Then, see the truth table at the centre of Figure 4, one of the following two cases takes place:
Case A: $\psi^* = \mathsf{T}$. Then, $\psi \in X$ by the induction hypothesis. Thus, $X \vdash \psi \vee \chi$ by axiom A20 and the Modus Ponens inference rule. Hence, $X \vdash (\psi \vee \chi) \vee \neg(\psi \vee \chi)$ again by axiom A20 and the Modus Ponens inference rule. Then, $X \vdash \mathtt{B}(\psi \vee \chi)$ by the definition of notation $\mathtt{B}$. Therefore, $\mathtt{B}(\psi \vee \chi) \in X$ because X is a maximal consistent set of formulae.

Case B: $\psi^* = \mathsf{F}$ and $\chi^* \neq \mathsf{U}$. Then, $\psi^* \neq \mathsf{U}$ and $\chi^* \neq \mathsf{U}$. Hence, $\mathtt{B}\psi \in X$ and $\mathtt{B}\chi \in X$ by the induction hypothesis. Thus, $X \vdash \mathtt{B}(\psi \vee \chi)$ by axiom A12 and the Modus Ponens inference rule applied twice. Therefore, $\mathtt{B}(\psi \vee \chi) \in X$ because X is a maximal consistent set of formulae. □

The proof of the next theorem is similar to the proof of Theorem 9, but it uses Lemma 8 instead of Lemma 2.

Theorem 14 *[strong completeness] For any set of formulae* Γ *and any formula* ϕ*, if* $\Gamma \nvdash \phi$*, then there is a valuation* $*$ *such that* $\gamma^* = \mathsf{T}$ *for each formula* $\gamma \in \Gamma$ *and* $\phi^* \neq \mathsf{T}$.

The next lemma shows that Kleene's negation is definable in the Short-Circuit Logic. Note that, unlike Lemma 4, here, $\vee$ and $\wedge$ denote the short-circuit disjunction and conjunction.

Lemma 9 *The formulae* $\neg_K p$ *and* $(p \vee \neg p) \wedge \neg p$ *are semantically equivalent.*

5 Kleene's Logic

In this section, we study the definability of connectives and axiomatisation of Kleene's Logic. Recall that, see Lemma 4 and Lemma 9, Kleene's negation $\neg_K$ is definable through $\neg$, $\wedge$, and $\vee$ in Weak Kleene's and Short Circuit logics. As a result, in the previous two sections, we were not considering $\neg_K$ as a primitive connective.

5.1 Undefinability results

As the next theorem shows, the situation is different in Kleene's Logic. In this theorem, and the rest of the section, by $\wedge$ and $\vee$ we mean Kleene's conjunction and Kleene's disjunction.

Theorem 15 *Formula* $\neg_K p$ *is not semantically equivalent to any formula that uses only connectives* $\rightarrow$, $\neg$, $\wedge$, *and* $\vee$.

The proof of the above theorem uses the same technique as our other undefinability results. It is based on 108 truth tables.

Due to the above theorem, let us start by first considering all five connectives: $\rightarrow$, $\neg$, $\neg_K$, $\wedge$, and $\vee$. Next, observe that

Theorem 16 *The following pairs of formulae are semantically equivalent:*

1. $p \wedge q$ *and* $\neg_K(\neg_K p \vee \neg_K q)$,
2. $p \vee q$ *and* $\neg_K(\neg_K p \wedge \neg_K q)$.

Due to the above theorem, either of the connectives $\wedge$ and $\vee$ can be omitted from our language. We have chosen to omit $\wedge$ and to keep $\vee$. Finally, observe that

Theorem 17 *The following pairs of formulae are semantically equivalent:*

1. *$\neg p$ and $p \rightarrow \neg_{K} p$,*
2. *$p \rightarrow q$ and $\neg p \vee q$.*

This means either of the connectives $\neg$ and $\rightarrow$ can be omitted from our language. We have chosen to omit $\neg$ and to keep $\rightarrow$. This leaves us with connectives $\rightarrow$, $\neg_{K}$, and $\vee$. The next two theorems, in combination with Theorem 15, show that neither of these three connectives can be defined through the other two. The proofs of these theorems use 2688 and 82 truth tables respectively.

Theorem 18 *Formula $p \vee q$ is not semantically equivalent to any formula that uses only connectives $\rightarrow$ and $\neg_{K}$.*

Theorem 19 *Formula $p \rightarrow q$ is not semantically equivalent to any formula that uses only connectives $\vee$, and $\neg_{K}$.*

5.2 Complete Axiomatisation

Out of the three logical systems considered in this paper, Kleene's Logic is the most studied in the literature. Weak completeness for Hilbert-style and Genzen-style axiomatisations have been proven in Kearns (1974) and Cleave (1974), respectively. Strong completeness is shown in Kearns (1979). Neither of those works deals with the classical implication or classical negation.

In this subsection, we give a complete axiomatisation of Kleene's Logic in the language containing the connectives $\rightarrow$, $\neg_{K}$, and $\vee$. We define $\mathtt{B}\phi$ and $\neg\phi$ as abbreviations for the formulae $\phi \vee \neg_{K}\phi$ and $p \rightarrow \neg_{K} p$, respectively. Note that in this section the definition of $\mathtt{B}$ is using Kleene's negation $\neg_{K}$. In the previous two sections, to define $\mathtt{B}$, we used the classical negation $\neg$. This change is necessary because the formula $\phi \vee \neg\phi$ is a three-valued tautology in Kleene's Logic.

The axioms of Kleene's Logic are those formulae in Table 1, that are labelled with the word "axiom" in column "K" of the table. The only inference rule of the logic is Modus Ponens.

Theorem 20 *[soundness] If $\vdash \phi$, then ϕ is a three-valued tautology.*

The proof of the completeness theorem follows the same pattern as the completeness results in the two previous sections. We fix a maximal consistent set of formulae X and define valuation $*$ as follows:

Definition 8 For any propositional variable p,

$$*(p) = \begin{cases} \mathsf{T}, & \text{if } p \in X, \\ \mathsf{U}, & \text{if } \mathtt{B}p \notin X, \\ \mathsf{F}, & \text{otherwise.} \end{cases}$$

Lemma 10 *Function $*$ is well-defined.*

Proof It suffices to show that if $p \in X$, then $\mathtt{B}p \in X$. Indeed, suppose that $p \in X$. Hence, $X \vdash p \vee \neg_{K} p$ by axiom A20 and the Modus Ponens inference rule. Therefore, $\mathtt{B}p \in X$ because X is a maximal consistent set. □

Lemma 11 *For any formula ϕ,*

1. $\phi \in X$ *iff* $\phi^* = \mathsf{T}$,
2. $\mathtt{B}\phi \in X$ *iff* $\phi^* \neq \mathsf{U}$.

Proof We prove the statement of the lemma by induction on the structural complexity of the formula ϕ. If ϕ is a propositional variable or an implication, then the argument is the same as in Lemma 8.

Suppose that formula ϕ has the form $\neg\psi$. We prove the two parts of the lemma separately.

Part 1. $(\Rightarrow)$: Suppose that $\neg_{\mathsf{K}}\psi \in X$. Then, $X \vdash \neg\psi$ and $X \vdash \mathtt{B}\psi$ by axioms A24 and A25, respectively, and the Modus Ponens inference rule. Hence, $\psi \notin X$ and $\mathtt{B}\psi \in X$ because X is a maximal consistent set of formulae. Then, $\psi^* \neq \mathsf{T}$ and $\psi^* \neq \mathsf{U}$ by the induction hypothesis. Thus, $\psi^* = \mathsf{F}$. Therefore, $(\neg_{\mathsf{K}}\psi)^* = \mathsf{T}$ by the truth table for $\neg_{\mathsf{K}}$, see Figure 2.
$(\Leftarrow)$: Suppose that $(\neg_{\mathsf{K}}\psi)^* = \mathsf{T}$. Then, $\psi^* = \mathsf{F}$ by the truth table for $\neg_{\mathsf{K}}$, see Figure 2. Then, $\psi^* \neq \mathsf{T}$ and $\psi^* \neq \mathsf{U}$. Hence, $\psi \notin X$ and $\mathtt{B}\psi \in X$ by the induction hypothesis. Thus, $\neg\psi \in X$ and $\mathtt{B}\psi \in X$ because X is a maximal consistent set of formulae. Then, $X \vdash \neg_{\mathsf{K}}\psi$ by axiom A26 and Modus Ponens inference rule applied twice. Therefore, $\neg_{\mathsf{K}}\psi \in X$ because X is a maximal consistent set of formulae.
Part 2. The statement $\mathtt{B}\neg_{\mathsf{K}}\psi \in X$ is equivalent to the statement $\mathtt{B}\psi \in X$ by axioms A27 and A28, the Modus Ponens inference rule, and because X is a maximal consistent set of formulae. The statement $\mathtt{B}\psi \in X$ is equivalent to the statement $\psi^* \neq \mathsf{U}$ by the induction hypothesis. The statement $\psi^* \neq \mathsf{U}$ is equivalent to the statement $(\neg_{\mathsf{K}}\psi)^* \neq \mathsf{U}$ by the truth table for $\neg_{\mathsf{K}}$, see Figure 2.

Finally, suppose that formula ϕ has the form $\psi \vee \chi$.
Part 1. $(\Rightarrow)$: Assume that $(\psi \vee \chi) \in X$. Then, $X \vdash \neg\psi \to \chi$ by axiom A19 and the Modus Ponens inference rule. Thus, again by the Modus Ponens inference rule, if $\neg\psi \in X$, then $X \vdash \chi$. Hence, because X is a maximal consistent set of formulae, if $\psi \notin X$, then $\chi \in X$. Thus, by the induction hypothesis, if $\psi^* \neq \mathsf{T}$, then $\chi^* = \mathsf{T}$. In other words, $\psi^* = \mathsf{T}$ or $\chi^* = \mathsf{T}$. Therefore, $(\psi \vee \chi)^* = \mathsf{T}$ by the right-most truth table in Figure 4.
$(\Leftarrow)$: Suppose that $(\psi \vee \chi)^* = \mathsf{T}$ Then, either $\psi^* = \mathsf{T}$ or $\chi^* = \mathsf{T}$ by the right-most truth table in Figure 4. Hence, by the induction hypothesis, either $\psi \in X$ or $\chi \in X$. Thus, $X \vdash \psi \vee \chi$ by either axiom A20 or axiom A29 and the Modus Ponens inference rule. Therefore, $\psi \vee \chi \in X$ because X is a maximal consistent set of formulae.
Part 2. $(\Rightarrow)$: Assume that $(\psi \vee \chi)^* = \mathsf{U}$. Thus, by the right-most truth table in Figure 4, one of the following cases takes place:
Case A: $\psi^* = \mathsf{U}$, $\psi^* \neq \mathsf{T}$, and $\chi^* \neq \mathsf{T}$. Then, by the induction hypothesis, $\mathtt{B}\psi \notin X$, $\neg\psi \in X$, and $\neg\chi \in X$. Hence, $X \nvdash \mathtt{B}\psi \notin X$ and $X \vdash \mathtt{B}(\psi \vee \chi) \to \mathtt{B}\psi$ by axiom A30 and the Modus Ponens rule applied twice. Hence, $X \nvdash \mathtt{B}(\psi \vee \chi)$ by the Modus Ponens inference rule applied contrapositively. Therefore, $\mathtt{B}(\psi \vee \chi) \in X$ because X is a maximal consistent set of formulae.
Case B: $\psi^* \neq \mathsf{T}$, $\psi^* \neq \mathsf{T}$, and $\chi^* \neq \mathsf{T}$. This case is similar to Case A except that it uses axiom A31 instead of axiom A30.
$(\Leftarrow)$: Suppose that $(\psi \vee \chi)^* \neq \mathsf{U}$. Thus, by the right-most truth table in Figure 4, one of the following three cases takes place:

Case A: $\psi^* = \mathsf{T}$. Then, $\psi \in X$ by the induction hypothesis. Thus, $X \vdash \psi \vee \chi$ by axiom A20 and the Modus Ponens rule. Hence, $X \vdash (\psi \vee \chi) \vee \neg(\psi \vee \chi) \in X$ again by axiom A20 and the Modus Ponens inference rule. In other words, $X \vdash \mathtt{B}(\psi \vee \chi)$. Therefore, $\mathtt{B}(\psi \vee \chi) \in X$ because X is a maximal consistent set of formulae.
Case B: $\chi^* = \mathsf{T}$. The proof, in this case, is similar to the proof in Case A except that instead of using axiom A20 twice, it first uses axiom A29 and then axiom A20.
Case C: $\psi^* = \mathsf{F}$ and $\chi^* = \mathsf{F}$. Then, $\psi^* \neq \mathsf{U}$ and $\chi^* \neq \mathsf{U}$. Hence, $\mathtt{B}\psi \in X$ and $\mathtt{B}\chi \in X$ by the induction hypothesis. Then, $X \vdash \mathtt{B}(\psi \vee \chi)$ by axiom A12. Thus, $\mathtt{B}(\psi \vee \chi) \in X$ because X is a maximal consistent set of formulae. □

The proof of the next theorem is similar to the proof of Theorem 9, but it uses Lemma 11 instead of Lemma 2.

Theorem 21 *[strong completeness] For any set of formulae Γ and any formula ϕ, if $\Gamma \nvdash \phi$, then there is a valuation $*$ such that $\gamma^* = \mathsf{T}$ for each formula $\gamma \in \Gamma$ and $\phi^* \neq \mathsf{T}$.*

6 Conclusion

In this article, we proposed a new implication for three-valued logic that, together with the matching negation, has exactly the same properties as the implication and the negation in the two-valued Boolean logic. In the setting with this implication and this negation, other connectives of three-valued logic can be viewed as modal operators. We considered three sets of such operators corresponding to Weak Kleene Logic, Short-Circuit Logic, and Kleene's Logic. In all three cases, we gave a sound and strongly complete axiomatisation of the corresponding logical system.

In the future, we would like to explore the applicability of our approach to multi-valued logical systems with more than three values.

Declarations

Conflicts of Interest The authors wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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