UNIVERSITY OF SOUTHAMPTON

Faculty of Social Sciences
School of Mathematical Sciences

AdS/CFT at Loop Order

by

Ernesto Bianchi
ORCiD: 0000-0002-8758-1105

A thesis for the degree of
Doctor of Philosophy

June 2025

University of Southampton

Abstract

Faculty of Social Sciences
School of Mathematical Sciences

Doctor of Philosophy

AdS/CFT at Loop Order

by Ernesto Bianchi

Since the early days of AdS/CFT, rigorous obtention of tree-level holographic correlators from a given bulk theory is known through the process of holographic renormalization, however the picture where quantum corrections are also taken into account is currently lacking. It is the purpose of this thesis to fill this gap in the literature and propose a method of holographic renormalization valid to all orders in the bulk loop expansion, where it is found that the same prescription from classical order is valid provided one replaces the classical action by the effective action. This gives a first principle derivation of Witten-Feynman rules for diagrams in AdS analogous to those in flat space.

In addition to the usual IR divergences present in holography, at loop order there are also UV divergences coming from the short-distance singularities of the bulk propagators, and this led us to construct a novel AdS invariant regularization scheme which we denote as geodesic point-splitting. Its derivation from a regularized action in AdS and its connection with an IR regulator for the dual CFT is also discussed.

The quantum corrections to the correlators take the form of loop Witten diagrams in AdS, and we show they obey the conformal Ward identities to all loop orders by explicitly writing a general loop diagram in the expected CFT form. Direct computation of loop Witten diagrams is challenging, and we also make progress in this direction by providing new and exact results for the most basic yet essential loop vertices in AdS appearing in almost every theory. How these are constrained by AdS symmetry is also discussed.

As an example of our methods, we work out in detail the case of a scalar Φ^4 theory in the bulk, obtaining the renormalized 2-point function of the dual operator to 2 loops and the 4-point function to 1 loop, for operators of arbitrary dimension $\Delta > d/2$ and bulk spacetime dimensions up to $d + 1 = 7$.

Contents

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

    - [15] Máximo Bañados, Ernesto Bianchi, Iván Muñoz, and Kostas Skenderis. Bulk renormalization and the AdS/CFT correspondence. *Phys. Rev. D*, 107(2):L021901, 2023
    - [16] Maximo Bañados, Ernesto Bianchi, Ivan Muñoz, and Kostas Skenderis. AdS amplitudes as CFT correlators. *JHEP*, 01:101, 2025

    and parts to be published in:

    - [23] Ernesto Bianchi, Ivan Muñoz, and Kostas Skenderis. Bulk renormalization and AdS/CFT at loop order. (to appear), 2025

Signed:... Date:..................

To Bárbara, who embarked on this journey with me
To Luca, who joined us along the way

Acknowledgements

These nearly 4 years in Southampton culminate with the writing of this thesis which, although carries only my name as author, there are many people who have brought me to this point and therefore receive my utmost gratitude.

Starting with my family: papá, mamá, Isi, Agus, Mami, and all the rest, thank you for your unconditional support and for rejoicing in my successes as if they were yours. I can only imagine how proud el Papo would be and the conversations we would have. Bárbara, thank you for your company during these years, for everything you do for our own family, and for always reminding me to enjoy the present.

I must thank my supervisor and mentor, Kostas Skenderis, for his patience, availability, advices, and constant guidance, but also for sharing with me his way of seeing and doing physics, which has deeply influenced and shaped my own. Thank you Kostas.

I also thank Máx Bañados for his mentoring during my early steps and, together with Iván Muñoz, for the countless discussions we have had, which have led to a large portion of this work.

I end by thanking the string theory group at Southampton, in particular to Arvind, Callum, Chandra, David, Davide, Filip, Orestis, Radu, and Zezhuang, for all the discussions, chatting and good moments. A special thank to Marika Taylor for her undoubted willingness to help when I needed it.

Thank you all.

"If one is working from the point of view of getting beauty in one's equations, ... one is on a sure line of progress."

Paul Dirac

Chapter 1

Introduction

One of the main challenges of current theoretical physics is the construction of a quantum theory of gravity, motivated by the study of black holes and perhaps more profoundly by the origin of our universe. Standard techniques of quantization manage to elucidate the first quantum effects of gravity at large distances, however as one forces these methods to smaller distances incurable divergences appear, losing all predictive power. In modern language of effective field theory, gravity has an irrelevant coupling resulting in the theory being non-renormalizable. This is exciting as it suggests gravity in the UV is not simply the Einstein-Hilbert action with a few tweaked parameters but something radically new, possibly redefining our notions of space and time. This is also however uncertain, as then how to proceed in this direction is not clear, difficulted by the lack of access to experimental data and with mathematical consistency the only guidance.

Current efforts attempt to make progress in the formulation of quantum gravity from many different angles, but what has proven to be useful is to look at its known non-perturbative properties, standing out among them its holographic nature [102, 100]: the information of a gravitational region of spacetime can be stored in a non-gravitational surface that surrounds it. This is remarkable, as it allows one to map questions about gravity to a different, usually much more familiar, scenario. This motivates the study of holography, and in particular the concrete examples where it has been explicitly realized.

The most celebrated example of holography is the AdS/CFT correspondence [80], where a gravitational theory in Anti-de Sitter spacetime may be equivalently described by a non-gravitational conformal field theory living at its boundary. In this example, the usual UV divergences of quantum field theories present in the boundary theory are mapped to IR divergences of the bulk theory due to the infinite volume of AdS, resulting in a UV/IR or strong/weak duality [101]. Foundational work on this matter has focused on the construction of a renormalized dictionary, known as holographic renormalization [44, 98], between these two theories at leading order in the duality, taking classical gravity on the bulk side and thus describing a strongly-coupled CFT at the boundary. In this approximation, structural evidence is found that supports the correspondence, with the divergences renormalized by a finite number of local covariant counterterms, and with the resulting holographic boundary correlators obtained from an AdS computation obeying the conformal Ward identities.

Holography may ultimately lead to quantum gravity, and this requires a better understanding of the holographic dictionary beyond the classical approximation, where quantum effects of gravity on the bulk side are taken into account. A rigorous renormalization scheme of AdS/CFT at loop order is currently lacking in the literature, and this would provide stronger evidence for the conjectured holographic nature of gravity.

It is the objective of this thesis to fill this gap in the literature and propose a method of holographic renormalization valid to all orders in the bulk loop expansion. This material has been organized as follows:

In Chapter 2, we review the standard method of holographic renormalization in AdS/CFT valid in the classical approximation, introducing the usual IR regulator in the bulk and boundary counterterms, the renormalized on-shell action, and the exact holographic 1-point functions once divergences have been renormalized. As an example, the case of an interacting scalar field with Dirichlet boundary condition is discussed in detail, obtaining novel and exact formulas for the boundary counterterms and holographic 1-point functions, for arbitrary interaction terms in the Lagrangian and values of $\nu \equiv \Delta - d/2 > 0$, including the special cases $\nu \in \mathbb{N}$.

In Chapter 3, we perform the first analysis of AdS/CFT at loop order by studying the conformal structure of a general loop Witten diagram. The close connection between AdS and CFT suggests one can write AdS isometries in the language of conformal transformations. Such language is indeed constructed with AdS isometries seen as constrained conformal transformations, making the conformal properties of AdS objects manifest. We use this to show that a general loop Witten diagram obey the conformal Ward identities by explicitly writing it in the form of a CFT n-point function. How conformal invariance at the boundary follows from bulk diffeomorphism in this new language is also discussed.

In Chapter 4, we present the general method of holographic renormalization in AdS/CFT valid to all orders in the bulk loop expansion. Subleading corrections in the correspondence also involve UV divergences in the bulk and we construct a novel AdS invariant regularization scheme, denoted geodesic point-splitting. Its derivation from a regularized action in AdS and its connection with an IR regulator for the dual CFT is also discussed. We introduce the counterterms needed to renormalize these new divergences, the obtention of the renormalized 1PI on-shell effective action in the bulk, and the modified holographic dictionary at loop order, with the CFT data renormalized due to quantum corrections in AdS.

In Chapter 5, we show new and exact computations of many bulk vertices appearing in loop Witten diagrams. These include the convergent integrals $\int G \int G \int \cdots$, the IR divergent integrals $\int KK$ and $\int GK$ for arbitrary and integer values of ν, and the UV divergent integrals $\int G^N$, $\int G^N K$ and $\int G^N KK$, including in the latter the terminating and logarithmic cases. The power of AdS isometries in bulk vertices has not been fully appreciated, and we show how these constrain the form of these integrals.

In Chapter 6, as an illustration of our methods we consider the example of a scalar Φ^4 theory. Holographic renormalization at loop order is performed for this theory, obtaining explicit formulas for the counterterms, for the renormalized bulk parameters, and for the renormalized holographic correlators, with the CFT data corrected order by order in the bulk coupling. This is done up to two loops in the 2-point function and up to 1-loop in the 4-point function, for dual operators of arbitrary dimension $\Delta > d/2$ and bulk spacetime dimensions up to $d+1=7$.

In Chapter 7, we conclude with a discussion of the main points of the thesis, and possible future directions.

The conventions throughout the thesis are $c = \hbar = \ell_{\text{AdS}} = 1$, unless otherwise stated.

Chapter 2

AdS/CFT at tree-level

The AdS/CFT correspondence dictionary [70, 103] relates quantities that are formally divergent. At leading, tree-level order in the bulk there are IR divergences due to the infinite volume of the spacetime, which are mapped to the usual UV divergences of QFT at the boundary. To construct a sensible dictionary between both sides these divergences must be renormalized. The relevant object in the bulk and at tree-level is the renormalized on-shell action $S^{\text{Ren}}_{\text{AdS}}[\varphi^I_{(0)}]$ as a function the fields $\varphi^I_{(0)}$ parametrizing the boundary conditions for the bulk fields Φ^I. This then acts as a generating function of connected correlators of primary operators $\mathcal{O}_{\Delta_I}$. The on-shell equations are obtained by minimizing the bulk gravitational action, $\delta S^{\text{Ren}}_{\text{AdS}}/\delta\Phi^I = 0$, while keeping fixed the fields that parametrize the boundary conditions. The construction of a holographic dictionary that is valid at the renormalized level is known as holographic renormalization [44, 98], and it is the purpose of this introductory chapter to review the general methodology valid at classical order. The case for scalar fields is discussed in more detail, as they will be the relevant object of study in the rest of the thesis.

2.1 Holographic renormalization at classical order

2.1.1 Regularization

Infrared divergences come from the infinite volume of AdS as one approaches its conformal boundary situated at $z = 0$ in Poincaré coordinates, with z the bulk radial direction. We will adopt the usual scheme and regularize the AdS volume by adding a hard cut-off $\varepsilon \ll 1$ to the bulk radial direction away from the conformal boundary [73]: $z \geq \varepsilon > 0$.

2.1.2 Counterterms

In this regularization scheme, divergent terms are regulated to the surface $z = \varepsilon$. These may then be subtracted with the addition of a boundary counterterm $B[\Phi^I; \varepsilon]$ located at this region, where AdS covariance implies it can be written in terms of the bulk fields Φ^I and the induced metric at the regulated surface.

2.1.3 Renormalized on-shell action

At classical order, the theory on AdS is approximated by its saddle-point

$$Z_{\text{AdS}}[\varphi^I_{(0)}] = e^{-S_{\text{AdS}}[\Phi^I]}\,, \tag{2.1}$$

which is a formal expression as it is ill-defined due to the IR divergences. Regularizing it for instance with the IR regulator ε, leads to the regularized action $S^{\text{Reg}}_{\text{AdS}}[\varphi^I_{(0)}; \varepsilon]$ whose divergences can be absorbed with the boundary counterterm $B[\Phi^I; \varepsilon]$, leading to the subtracted action

$$S^{\text{Sub}}_{\text{AdS}}[\varphi^I_{(0)}; \varepsilon] = S^{\text{Reg}}_{\text{AdS}}[\varphi^I_{(0)}; \varepsilon] + B[\Phi^I; \varepsilon]\,. \tag{2.2}$$

Once IR divergences have been renormalized, the renormalized on-shell action is then obtained in the limit of vanishing regulator

$$S^{\text{Ren}}_{\text{AdS}}[\varphi^I_{(0)}] = \lim_{\varepsilon\to 0} S^{\text{Sub}}_{\text{AdS}}[\varphi^I_{(0)}; \varepsilon]\,, \tag{2.3}$$

as a function of the boundary conditions $\varphi^I_{(0)}$.

2.1.4 Exact 1-point functions

Once $S^{\text{Ren}}_{\text{AdS}}$ has been constructed, the holographic dictionary reads

$$Z_{\text{CFT}}[\varphi^I_{(0)}] = Z_{\text{AdS}}[\varphi^I_{(0)}] \implies W_{\text{CFT}}[\varphi^I_{(0)}] = S^{\text{Ren}}_{\text{AdS}}[\varphi^I_{(0)}]\,, \tag{2.4}$$

where the $\varphi^I_{(0)}$ are identified as the sources for some conformal operators $\mathcal{O}_{\Delta_I}$ in the dual theory. Correlation functions for $\mathcal{O}_{\Delta_I}$ may then be computed from the theory on AdS by functionally differentiating $S^{\text{Ren}}_{\text{AdS}}$ with respect to $\varphi^I_{(0)}$, leading to the exact holographic 1-point function in the presence of sources

$$\langle \mathcal{O}_{\Delta_I}(\vec{x})\rangle_{\varphi^I_{(0)}} = \frac{-1}{\sqrt{g_{(0)}}}\frac{\delta S^{\text{Ren}}_{\text{AdS}}[\varphi^I_{(0)}]}{\delta \varphi^I_{(0)}(\vec{x})}\,, \tag{2.5}$$

with $g_{(0)ij}$ the metric at the boundary theory.

This summarizes the standard holographic renormalization procedure in very general terms. In the next section, we will review in more detail the case for scalar fields under Dirichlet boundary conditions.

2.2 Example: scalar field

2.2.1 Holographic renormalization

Consider the example of an interacting scalar theory in the bulk

$$S_{\mathrm{AdS}}[\Phi] = \int \mathrm{d}^{d+1}x\sqrt{g}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{m^2}{2}\Phi^2 + V(\Phi)\right]. \tag{2.6}$$

Adding the IR regulator ε and boundary counterterm B, the renormalized on-shell action reads

$$S_{\mathrm{AdS}}^{\mathrm{Ren}}[\Phi] = \lim_{\varepsilon\to 0}\int_{z\geq\varepsilon} \mathrm{d}^{d+1}x\sqrt{g}\left[\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{m^2}{2}\Phi^2 + V(\Phi)\right] + B[\Phi;\varepsilon]\,, \tag{2.7}$$

with B properly chosen as to renormalize all the divergences from the action. The variation of $S_{\mathrm{AdS}}^{\mathrm{Ren}}$ consists only in a boundary term

$$\delta S_{\mathrm{AdS}}^{\mathrm{Ren}}[\varphi_{(0)}] = \lim_{\varepsilon\to 0}\,\delta B[\Phi;\varepsilon] - \int_{z=\varepsilon}\mathrm{d}^d x\,\sqrt{g}\,\partial^z\Phi\delta\Phi\,, \tag{2.8}$$

as the resulting bulk term in the variation vanishes given the classical equation of motion for Φ

$$(-\Box + m^2)\Phi = -V'(\Phi)\,. \tag{2.9}$$

There are 2 independent Green's functions associated to the differential operator $(-\Box + m^2)$: G_Δ and $G_{\bar{\Delta}}$, with Δ and $\bar{\Delta}$ the greater/lower solutions to $m^2 = \Delta(\Delta - d)$. Imposing Dirichlet boundary conditions for the bulk field Φ picks the former, allowing us to write the equation above as the integral equation

$$\Phi(x) = \Phi_0(x) - \int \mathrm{d}^{d+1}x'\sqrt{g'}\,G_\Delta(x,x')V'\left(\Phi(x')\right)\,, \tag{2.10}$$

where Φ_0 solves the homogeneous case: $(-\Box + m^2)\Phi_0 = 0$. The full solution for Φ may then be obtained in powers of Φ_0 by recursively replacing the expression for Φ on $V'(\Phi)$

$$\Phi(x) = \Phi_0(x) - \int \mathrm{d}^{d+1}x'\sqrt{g'}\,G_\Delta(x,x')V'\left(\Phi_0(x')\right) + \cdots\,. \tag{2.11}$$

In Poincaré coordinates $g_{\mu\nu} = \delta_{\mu\nu}/z^2$, the near-boundary expansion of Φ_0 is worked out in powers of the radial coordinate z. For arbitrary values of $\nu \equiv \Delta - d/2 > 0$, it takes the form

$$\Phi_0(x) = z^{d-\Delta} \sum_{n=0}^{\lfloor \nu \rfloor} \frac{\Gamma(\nu - n)}{\Gamma(\nu)\, n!} \left(\frac{z^2 \partial^2}{4} \right)^n \varphi_{(0)}(\vec{x}) + z^{\Delta} \varphi_{(2\nu)}(\vec{x}) + \mathcal{O}(z^{\Delta <})\,, \tag{2.12}$$

where $\lfloor \nu \rfloor$ is the integer part of ν, and ∂^2 is the flat Laplacian. Here $\varphi_{(0)}$ and $\varphi_{(2\nu)}$ are the 2 linearly independent solutions of the homogeneous equation, and by Dirichlet boundary conditions one is fixing the value of $\varphi_{(0)}$ at the boundary $z = 0$. When $\nu \in \mathbb{N}$, the 2 series become degenerate and one must add a logarithmic term at order z^{Δ} to have a solution

$$\begin{aligned} \Phi_0(x) = {} & z^{d-\Delta} \sum_{n=0}^{\nu-1} \frac{\Gamma(\nu - n)}{\Gamma(\nu)\, n!} \left(\frac{z^2 \partial^2}{4} \right)^n \varphi_{(0)}(\vec{x}) \\ & + z^{\Delta} \left[\varphi_{(2\nu)}(\vec{x}) - \frac{2}{\nu \Gamma(\nu)^2} \left(\frac{\partial^2}{4} \right)^{\nu} \varphi_{(0)}(\vec{x}) \ln(\mu z) \right] + \mathcal{O}(z^{\Delta <})\,, \end{aligned} \tag{2.13}$$

where μ is an arbitrary scale.

The explicit form of the Green's function $G_\Delta(x, x')$, also known as bulk-to-bulk propagator, is given by

$$G_\Delta(x, x') = \frac{c_\Delta}{2^{\Delta+1} \nu} \xi^{\Delta} {}_2F_1 \left(\begin{matrix} \frac{\Delta}{2}, \frac{\Delta+1}{2} \\ \Delta - \frac{d}{2} + 1 \end{matrix} ; \xi^2 \right), \qquad \xi = \frac{2 z z'}{z^2 + z'^2 + (\vec{x} - \vec{x}\,')^2}\,, \tag{2.14}$$

where ${}_2F_1$ is Gauss' hypergeometric function, and $c_\Delta \equiv \Gamma(\Delta)/[\pi^{\frac{d}{2}} \Gamma(\nu)]$. It is a function of the bi-scalar ξ and as such, it is invariant under simultaneous isometry transformations of the points x and x'. It has a near-boundary expansion of the form

$$G_\Delta(x, x') = \frac{z^{\Delta}}{2\nu} K_\Delta(x', \vec{x}) + \mathcal{O}(z^{\Delta <})\,, \tag{2.15}$$

where the function $K_\Delta(x', \vec{x})$ is known as bulk-to-boundary propagator

$$K_\Delta(x', \vec{x}) = c_\Delta \left[\frac{z'}{z'^2 + (\vec{x}\,' - \vec{x})^2} \right]^{\Delta}. \tag{2.16}$$

One may readily see the near-boundary expansion of the full solution Φ takes the same form as the expansion for the free case Φ_0, with the normalizable mode $\varphi_{(2\nu)}$ replaced by

$$\varphi^V_{(2\nu)}(\vec{x}) = \varphi_{(2\nu)}(\vec{x}) - \frac{1}{2\nu} \int \mathrm{d}^{d+1} x' \sqrt{g'}\, K_\Delta(x', \vec{x}) V'\left(\Phi_0(x')\right) + \cdots\,, \tag{2.17}$$

receiving contributions from the interactions. For scalar fields with this asymptotic, plugging it in the variation (2.8) leads to a number of IR divergences which are renormalized by a finite number of local covariant boundary counterterms of the form

$$B[\Phi;\varepsilon] = \int_{z=\varepsilon} \mathrm{d}^d x \sqrt{\gamma} \left[\frac{(d-\Delta)}{2} \Phi^2(x) + \frac{1}{2} \sum_{n=1}^{\lfloor \nu \rfloor} c_n(\nu) \Phi(x) \Box_\gamma^n \Phi(x) \right] , \tag{2.18}$$

with $\gamma_{ij} = \delta_{ij}/\varepsilon^2$ the induced metric on the regulated surface. A direct computation of $c_n(\nu)$ yields

$$c_n(\nu) = \frac{1}{4^n \Gamma(\nu)^2} \sum_{i=1}^{n} \frac{\Gamma(2\nu - i)\Gamma(\nu - i)^2}{\Gamma(2\nu - 2i)(i-1)!} \sum_{j=0}^{n-i} b_j b_{n-i-j} , \tag{2.19}$$

in terms of the coefficients b_n which are determined recursively starting from b_0

$$b_0 = 1 , \quad b_{n>0} = - \sum_{i=0}^{n-1} \frac{\Gamma(\nu - n + i)}{\Gamma(\nu)(n-i)!} b_i . \tag{2.20}$$

For instance, the first few numbers are

$$c_1(\nu) = \frac{1}{2(\nu - 1)} , \quad c_2(\nu) = \frac{1}{8(\nu-1)^2(\nu-2)} , \quad \cdots . \tag{2.21}$$

It would be interesting to solve the recursion formula and find a closed-form expression for b_n, as that would lead in turn to a closed-form for the numbers $c_n(\nu)$. Nevertheless, using the formulas above they may be determined up to the desired value.

The numbers $c_n(\nu)$ have poles at $\nu = n$, and for $\nu \in \mathbb{N}$ the last counterterm of the series becomes logarithmic

$$\frac{1}{2} c_{\lfloor \nu \rfloor}(\nu) \Phi(x) \Box_\gamma^{\lfloor \nu \rfloor} \Phi(x) \to - \frac{2^{1-2\nu}}{\Gamma(\nu)^2} \Phi(x) \Box_\gamma^\nu \Phi(x) \ln(\mu\varepsilon) . \tag{2.22}$$

In this case, finite additions to $B[\Phi;\varepsilon]$ are possible and these are captured by changes in the arbitrary scale μ, representing the scheme-dependence associated with the logarithmic subtractions. For special Δ's and bulk interaction terms, additional contributions to the logarithmic terms are present and these are related to conformal anomalies due to higher-point functions [26]. Once this renormalization process has been carried out, the resulting boundary term is now finite and the limit $\varepsilon \to 0$ may be safely taken, leading to the renormalized variation

$$\delta S^{\text{Ren}}_{\text{AdS}}[\varphi_{(0)}] = - \int \mathrm{d}^d x \, 2\nu \varphi^V_{(2\nu)}(\vec{x}) \delta\varphi_{(0)}(\vec{x}) , \tag{2.23}$$

up to local terms in $\varphi_{(0)}$ when $\nu \in \mathbb{N}$, that may be absorbed in the scale μ. Functionally differentiating with respect to $\varphi_{(0)}$ then leads to the exact holographic 1-point function in presence of sources

$$\langle \mathcal{O}_\Delta(\vec{x}) \rangle_{\varphi_{(0)}} = 2\nu \varphi^V_{(2\nu)}(\vec{x}) . \tag{2.24}$$

Explicit computation of the holographic correlators requires exact (as opposed to asymptotic) solutions to the bulk field equations, and these will be determined next.

2.2.2 Renormalized correlators

To construct an exact solution for the field Φ, boundary conditions must be supplemented. We will impose Dirichlet boundary conditions at the conformal boundary and regularity in the interior of AdS

$$\text{Dirichlet:} \quad \Phi(z \to 0, \vec{x}) \to z^{d-\Delta} \varphi_{(0)}(\vec{x}) \,, \tag{2.25}$$

$$\text{Regularity:} \quad \Phi(z \to \infty, \vec{x}) \to 0 \,. \tag{2.26}$$

These conditions completely fix the free part of the field, Φ_0, to be of the form

$$\Phi_0(x) = \int \mathrm{d}^d y \, K_\Delta(x, \vec{y}) \varphi_{(0)}(\vec{y}) \,, \tag{2.27}$$

with K_Δ the bulk-to-boundary propagator. The exact solution for Φ is then read from (2.11)

$$\begin{aligned} \Phi(x) = & \int \mathrm{d}^d y \, K_\Delta(x, \vec{y}) \varphi_{(0)}(\vec{y}) \\ & - \int \mathrm{d}^{d+1} x' \sqrt{g'} \, G_\Delta(x, x') V' \left[\int \mathrm{d}^d y \, K_\Delta(x', \vec{y}) \varphi_{(0)}(\vec{y}) \right] + \cdots \,, \end{aligned} \tag{2.28}$$

expressed in increasing powers of $\varphi_{(0)}$. This is a convenient representation as terms of a given order in $\varphi_{(0)}$ will contribute to a specific holographic correlator: contributions to the $(n+1)$-point function come from the terms of order $\varphi_{(0)}^n$ in Φ. If one is interested in lower-point functions, then keeping the first few terms in (2.28) is sufficient.

To compute correlators we need to expand Φ and identify the mode $\varphi_{(2\nu)}^V$, now as a functional of $\varphi_{(0)}$. The expansion of G_Δ was given in (2.15), while for K_Δ it is more easily derived from its representation in momentum

$$K_\Delta(x, \vec{y}) = \frac{z^{\frac{d}{2}}}{2^{\nu-1}\Gamma(\nu)} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \, p^\nu K_\nu(pz) e^{-i\vec{p}(\vec{x}-\vec{y})} \,, \tag{2.29}$$

where $K_\nu(pz)$ is the modified Bessel function of second kind. The expression for $\varphi_{(2\nu)}^V$ may then be obtained using the series representation (B.3) of the Bessel function and computing the resulting momentum integrals using the results of appendix C, leading to the identification

$$\varphi^V_{(2\nu)}(\vec{x}) = \int \mathrm{d}^d y \, \frac{c_\Delta}{|\vec{x}-\vec{y}|^{2\Delta}} \varphi_{(0)}(\vec{y}) \\ - \frac{1}{2\nu} \int \mathrm{d}^{d+1}x' \sqrt{g'} \, K_\Delta(x',\vec{x}) V' \left[\int \mathrm{d}^d y \, K_\Delta(x',\vec{y}) \varphi_{(0)}(\vec{y}) \right] + \cdots . \tag{2.30}$$

Naively, this expression seems to be valid for all values of $\nu > 0$, however for $\nu \in \mathbb{N}$ a more careful treatment shows the first term in $\varphi^V_{(2\nu)}$ is ill-defined as a distribution, i.e., its Fourier transform diverges. This can be seen, for instance, from properly expanding this term around ν integer: $\nu = n + \epsilon$, $n \in \mathbb{N}$, obtaining an expansion in ϵ of the form

$$\frac{1}{|\vec{x}-\vec{y}|^{d+2n+2\epsilon}} \sim \frac{1}{\epsilon} \Box^n \delta(\vec{x}-\vec{y}) + \mathcal{O}(\epsilon^0) \, . \tag{2.31}$$

As $\epsilon \to 0$, the leading term in the expansion diverges and the LHS is ill-defined as a distribution at the coincident point $\vec{x} = \vec{y}$. For a more detailed analysis of this issue, see the discussion in appendix C.

In the case $\nu \in \mathbb{N}$, one must use instead the series representation (B.4) of the Bessel function, which leads to

$$\varphi^V_{(2\nu)}(\vec{x}) = \int \mathrm{d}^d y \, \mathcal{R}_M \left[\frac{c_\Delta}{|\vec{x}-\vec{y}|^{2\Delta}} \right] \varphi_{(0)}(\vec{y}) \\ - \frac{1}{2\nu} \int \mathrm{d}^{d+1}x' \sqrt{g'} \, K_\Delta(x',\vec{x}) V' \left[\int \mathrm{d}^d y \, K_\Delta(x',\vec{y}) \varphi_{(0)}(\vec{y}) \right] + \cdots . \tag{2.32}$$

Here $\mathcal{R}_M$ denotes the renormalized version of the function, defined in (C.24) for a value of $M^2 = 4\mu^2 e^{\psi(1)+\psi(\nu+1)}$, and where μ is the arbitrary scale introduced before. It has the property that $\mathcal{R}_M[f(|\vec{x}-\vec{y}|)] = f(|\vec{x}-\vec{y}|)$ for $\vec{x} \neq \vec{y}$, however unlike the bare function, it is well-behaved as a distribution including the singular point $\vec{x} = \vec{y}$ and as such it has a Fourier transform, which is given by (C.23).

From the exact 1-point function of the dual operator $\mathcal{O}_\Delta$ in (2.24), functionally differentiating with respect to $\varphi_{(0)}$ and setting the sources to 0, leads to the holographic 2-point function

$$\langle \mathcal{O}_\Delta(\vec{y}_1) \mathcal{O}_\Delta(\vec{y}_2) \rangle = \begin{cases} \dfrac{2\nu c_\Delta}{|\vec{y}_1 - \vec{y}_2|^{2\Delta}} \, , & \nu \notin \mathbb{N} \\ \mathcal{R}_M \left[\dfrac{2\nu c_\Delta}{|\vec{y}_1 - \vec{y}_2|^{2\Delta}} \right] \, , & \nu \in \mathbb{N} \end{cases} \tag{2.33}$$

valid for $\Delta > d/2$. These are precisely the CFT 2-point functions for a scalar operator of conformal dimension Δ.

For higher-point functions, we need the specific form of the potential $V(\Phi)$ in the Lagrangian. As an example, consider the case of a quartic interaction

$$V(\Phi) = \frac{\lambda}{4!}\Phi^4 . \tag{2.34}$$

In this case, the mode $\varphi^V_{(2\nu)}(\vec{x})$ in (2.30) (or (2.32) when $\nu \in \mathbb{N}$) evaluates to

$$\begin{aligned}\varphi^V_{(2\nu)}(\vec{x}) &= \int \mathrm{d}^d y \, \frac{c_\Delta}{|\vec{x}-\vec{y}|^{2\Delta}} \varphi_{(0)}(\vec{y}) \\ &\quad - \frac{\lambda}{12\nu} \int \mathrm{d}^{d+1}x' \sqrt{g'}\, K_\Delta(x',\vec{x}) \left[\int \mathrm{d}^d y \, K_\Delta(x',\vec{y}) \varphi_{(0)}(\vec{y}) \right]^3 + \cdots . \end{aligned} \tag{2.35}$$

The leading interaction term results in a contribution to the holographic 4-point function for the dual operator

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\mathcal{O}_\Delta(\vec{y}_3)\mathcal{O}_\Delta(\vec{y}_4)\rangle = -\lambda \int \mathrm{d}^{d+1}x \sqrt{g}\, K_\Delta(x,\vec{y}_1) K_\Delta(x,\vec{y}_2) K_\Delta(x,\vec{y}_3) K_\Delta(x,\vec{y}_4) . \tag{2.36}$$

Diagrammatically, these contributions may be represented in terms of Witten diagrams (see fig. 2.1) analogous to those in flat space, consisting in lines starting from the insertion points at the boundary and with the interactions taking place in the bulk.

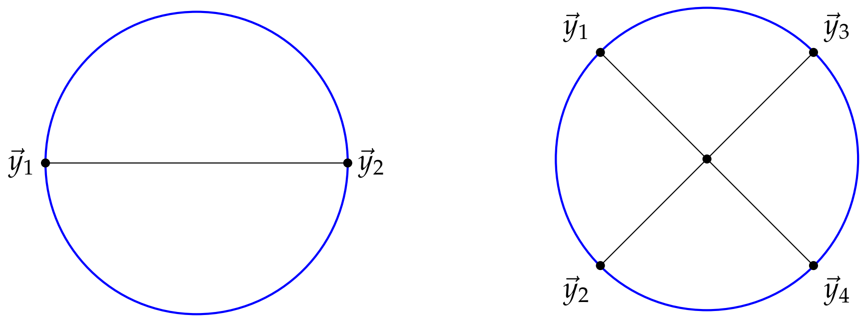

FIGURE 2.1: Witten diagrams contributing to the connected holographic 2- and 4-point functions dual to Φ^4 theory in AdS.

Contact Witten diagrams constructed from bulk-to-boundary propagators are known as D-functions, defined by the integral

$$D_{\Delta_1,\ldots,\Delta_n} \equiv \int \mathrm{d}^{d+1}x \sqrt{g}\, K^{\Delta_1}(x,\vec{y}_1) \cdots K^{\Delta_n}(x,\vec{y}_n) , \tag{2.37}$$

where $K(x,\vec{y}_i) = z/[z^2 + (\vec{x}-\vec{y}_i)^2]$. Since D-functions appear as contributions to conformal correlators, they must have the expected form of a CFT n-point function. This is indeed the case, as we will prove in the next chapter using AdS symmetry arguments.

Of special interest is the case $n = 4$. A direct treatment of the integral computes this D-function as a sum of Appell F_4 hypergeometric functions, which may be written more compactly in terms of the function $H(\alpha, \beta, \gamma, \delta; u, v)$ introduced in [53] (see also [52])

$$D_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} = \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\left(\frac{\Delta_T - d}{2}\right)}{\prod_n \Gamma(\Delta_n)} \frac{u^{\frac{1}{2}(\Delta_1+\Delta_2-\frac{1}{3}\Delta_T)} v^{\frac{1}{2}(\Delta_2+\Delta_3-\frac{1}{3}\Delta_T)}}{\prod_{i<j} (y_{ij}^2)^{\frac{1}{2}(\Delta_i+\Delta_j-\frac{1}{3}\Delta_T)}}$$
$$\times H\left(\Delta_2, \frac{\Delta_T}{2} - \Delta_4, \Delta_1 + \Delta_2 - \frac{\Delta_T}{2} + 1, \Delta_1 + \Delta_2; u, v\right), \tag{2.38}$$

where $\Delta_T = \sum_n \Delta_n$ and u, v are the conformal invariants (cross-ratios)

$$u = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \qquad v = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}. \tag{2.39}$$

This representation for the D-function allows us, for instance, to express the 4-point function (2.36) as

$$\langle \mathcal{O}_\Delta(\vec{y}_1)\mathcal{O}_\Delta(\vec{y}_2)\mathcal{O}_\Delta(\vec{y}_3)\mathcal{O}_\Delta(\vec{y}_4)\rangle = -\lambda c_\Delta^4 \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\left(2\Delta - \frac{d}{2}\right)}{\Gamma(\Delta)^4} \frac{(uv)^{\frac{\Delta}{3}}}{\prod_{i<j} (y_{ij}^2)^{\frac{\Delta}{3}}} H(\Delta, \Delta, 1, 2\Delta; u, v). \tag{2.40}$$

recovering its CFT form.

Chapter 3

AdS amplitudes as CFT correlators

In this thesis we are interested in extending the picture of Chapter 2 to also include the corrections to the holographic correlators coming from the bulk loops. This is precisely the content of Chapter 4, with a fully worked out example in Chapter 6. But before doing this, as a first study in this direction, in this chapter we will look at the conformal structure of a general loop Witten diagram. At loop order, the AdS/CFT correspondence could have been invalidated simply due to the breaking of conformal symmetry in the dual theory. As we will prove here, conformal invariance is preserved to all orders in the bulk loop expansion thanks to the AdS covariance of the bulk propagators.

This chapter has been previously published as a paper in [16].

3.1 Introduction

In a theory of quantum gravity there are no bulk local invariants (because diffeomorpshims act on spacetime points). In spacetimes with asymptotia, we need to impose boundary conditions at infinity, and one may define local operators at the (conformal) boundary via the boundary conditions. For example, one may require that a bulk scalar field takes a prescribed value at the boundary. The gravitational path-integral computed with such boundary conditions would then compute observables that depends on boundary points. Such observables may be organized according to their transformation properties under the asymptotic symmetry group, the group of transformations that preserves the boundary conditions.

In the case of asymptotically (locally) AdS gravity, the boundary carries a conformal structure, so this construction naturally produces n-point functions, which we will call AdS amplitudes, that transform as CFT correlators. The AdS/CFT conjectures [80, 70, 103] asserts that AdS gravity is equivalent to a local CFT in one dimension less and in particular AdS amplitudes are equal to CFT correlators[1]. The relation between AdS amplitudes computed via Witten diagrams and CFT correlators has been tested with tree-level examples already in the foundational papers [70, 103] and numerous explicit evaluations of AdS amplitudes appeared in the early AdS/CFT literature; see [61, 59, 78, 14, 54] for a sample of early papers and [49] for a review. In more recent times explicit loop-level diagrams have also been computed, see, for example, [87, 56, 1, 6, 13, 67, 105, 21, 22, 64, 90, 33, 81, 5, 43, 34, 71, 15], and they are all in agreement with CFT expectations. To a large extent, the community takes for granted that AdS amplitudes are CFT correlators. It is the purpose of this chapter to provide an explicit proof that this is the case to all orders in bulk perturbation theory. We will discuss in detail the case the external operators are scalars, but all steps have a straightforward generalisation to spinning operators. It would be interesting to spell out all technical details but we leave this for future work.

In the next section, we summarize the constraints imposed by conformal invariance on CFT correlations functions. Then in Section 3.3 we show that AdS amplitudes satisfy these contraints. In particular, this derivation shows that the constants and functions of cross-ratios that appear in CFT correlators are determined in terms of bulk data. In Section 3.4 we show how the constraints of conformal invariance emerge from bulk diffeomorpshisms and illustrate how our results for scalar correlators extend to spinning ones by considering the case of conserved currents. We finish with a discussion of our results in Section 3.5.

3.2 CFT correlators

We review in this section the constraints of conformal invariance on CFT correlation functions of primary operators. This is a topic with long history, see [89, 83, 42] for some of the original literature and [66, 50, 97, 84] for reviews.

Conformal transformations are diffeomorphisms that results in a Weyl transformation: under $\vec{x} \to \vec{x}\,'$,

$$\mathrm{d}s^2 \to \mathrm{d}s'^2 = \Omega^2(\vec{x})\mathrm{d}s^2\,. \tag{3.1}$$

[1]This perspective on the duality has been emphasised early on in [65].

We work with Euclidean signature and coordinates x^α, $\alpha = 1, \ldots, d$. We will also use a vector notation, $\vec{x} = \{x^\alpha\}$. In flat space, where $\mathrm{d}s^2 = \mathrm{d}\vec{x}^2$, these are given by

$$\text{Poincare:} \quad x'^\alpha = a^\alpha{}_\beta x^\beta + a^\alpha\,,, \qquad \Omega(\vec{x}) = 1\,, \tag{3.2}$$

$$\text{Dilation:} \quad x'^\alpha = \lambda x^\alpha\,, \qquad \Omega(\vec{x}) = \lambda\,, \tag{3.3}$$

$$\text{Inversion:} \quad x'^\alpha = \frac{x^\alpha}{\vec{x}^2}\,, \qquad \Omega(\vec{x}) = \frac{1}{\vec{x}^2}\,, \tag{3.4}$$

$$\text{Special conformal:} \quad x'^\alpha = \frac{x^\alpha + b^\alpha \vec{x}^2}{1 + 2\vec{b}\cdot\vec{x} + \vec{b}^2\vec{x}^2}\,, \qquad \Omega(\vec{x}) = \frac{1}{1 + 2\vec{b}\cdot\vec{x} + \vec{b}^2\vec{x}^2}\,. \tag{3.5}$$

The factors $\Omega(\vec{x})$ are related to the Jacobian $|\partial\vec{x}'/\partial\vec{x}|$ via $\Omega(\vec{x}) = |\partial\vec{x}'/\partial\vec{x}|^{1/d}$ and $\partial x'^\alpha/\partial x^\beta = \Omega(\vec{x}) R^\alpha{}_\beta(\vec{x})$, where $R^\alpha{}_\beta \in O(d)$ is the orthogonal matrix

$$R^\alpha{}_\beta(\vec{x}) = \left\{ a^\alpha{}_\beta,\ \delta^\alpha_\beta,\ I^\alpha_\beta(\vec{x}),\ I^\alpha_\gamma\left(\frac{\vec{x}}{\vec{x}^2} + \vec{b}\right) I^\gamma_\beta(\vec{x}) \right\}\,, \tag{3.6}$$

for Poincare, dilations, inversions and special conformal transformation, correspondingly, and $\det R = \pm 1$ with -1 for inversions, $+1$ for the rest, where

$$I^\alpha_\beta(\vec{x}) = \delta^\alpha_\beta - 2\frac{x^\alpha x_\beta}{\vec{x}^2}\,. \tag{3.7}$$

One may check conformal transformations satisfy

$$(\vec{x}_1' - \vec{x}_2')^2 = \Omega(\vec{x}_1)\Omega(\vec{x}_2)(\vec{x}_1 - \vec{x}_2)^2\,. \tag{3.8}$$

and

$$I_{\alpha\beta}(\vec{x}_1' - \vec{x}_2') = R_\alpha{}^\gamma(\vec{x}_1) R_\beta{}^\delta(\vec{x}_2) I_{\gamma\delta}(\vec{x}_1 - \vec{x}_2)\,. \tag{3.9}$$

For concreteness, and to keep the technicalities to the minimum we will primarily focus on scalar operators, and we will briefly discuss spinning operator at the end of this section.

Scalar primary operators $\mathcal{O}$ of dimension Δ transform as

$$\mathcal{O}'(\vec{x}') = \Omega(\vec{x})^{-\Delta}\mathcal{O}(\vec{x}), \tag{3.10}$$

and n-point functions should therefore satisfy

$$\langle \mathcal{O}_1(\vec{x}_1') \cdots \mathcal{O}_n(\vec{x}_n') \rangle = \Omega(\vec{x}_1)^{-\Delta_1} \cdots \Omega(\vec{x}_n)^{-\Delta_n} \langle \mathcal{O}_1(\vec{x}_1) \cdots \mathcal{O}_n(\vec{x}_n) \rangle\,. \tag{3.11}$$

Following the presentation in [30], the solution of (3.11) is given by

$$\langle \mathcal{O}_1(\vec{x}_1) \ldots \mathcal{O}_n(\vec{x}_n) \rangle = \frac{C_n(u_{ijkl})}{\prod_{1\le i<j\le n} (x_{ij}^2)^{\Delta_{ij}^{(n)}}}, \tag{3.12}$$

where $x_{ij}^2 \equiv |\vec{x}_{ij}|^2 \equiv (\vec{x}_i - \vec{x}_j)^2$ and the parameters $\Delta_{ij}^{(n)}$ are related to the scaling dimensions by the relations,

$$\Delta_i = \sum_{j=1}^{n} \Delta_{ij}^{(n)}, \qquad i = 1, 2, \ldots, n\,, \tag{3.13}$$

where we have assumed without loss of generality that $\Delta_{ji}^{(n)} = \Delta_{ij}^{(n)}, \Delta_{ii}^{(n)} = 0$.

The functions $C_n(u_{ijkl})$ are arbitrary functions of the conformal cross ratios,

$$u_{ijkl} = \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}, \qquad i \neq j \neq k \neq l, \,. \tag{3.14}$$

These functions encode theory-specific information. Cross-ratios exist from 4-point function on, so C_2 and C_3 are constants. Not all cross ratios are independent. For instance:

$$u_{ijkl} = u_{jilk} = u_{klij} = u_{lkji} = \frac{1}{u_{ikjl}} = \frac{1}{u_{jlik}} = \frac{1}{u_{kilj}} = \frac{1}{u_{ljki}}\,, \tag{3.15}$$

and there are more relations involving product of cross ratios. A simple counting suggests there are $n(n-3)/2$ independent cross-ratios (this is an over-counting when $n > d+2$, see for example [84] – this is not going to play a role here). One may choose the following combinations as independent cross-ratios,

$$u_i = u_{123i} = \frac{x_{12}^2 x_{3i}^2}{x_{13}^2 x_{2i}^2}\,, \quad v_i = u_{321i} = \frac{x_{1i}^2 x_{23}^2}{x_{13}^2 x_{2i}^2}\,, \quad w_{ij} = u_{23ij} = \frac{x_{23}^2 x_{ij}^2}{x_{2i}^2 x_{3j}^2}\,, \tag{3.16}$$

where $i, j = 4, \ldots, n$ and $i < j$.

Equation (3.13) is a set of n linear equations that may be used to determine $\Delta_{ij}^{(n)}$ given Δ_i. When $n = 2$, we find

$$\Delta_1 = \Delta_{12}^{(2)} = \Delta_2, \tag{3.17}$$

encoding the fact that only operators with same dimension have non-vanishing 2-point functions. When $n = 3$, the unique solution is

$$\Delta_{ij}^{(3)} = \Delta_i + \Delta_j - \frac{\Delta_T}{2}, \tag{3.18}$$

where Δ_T denotes the sum over all dimensions, $\Delta_T = \sum \Delta_i$.

For $n > 3$ there are more unknowns than equations: $\Delta_{ij}^{(n)}$ is a symmetric hollow matrix (*i.e.* symmetric with zero in the diagonals) so it has $n(n-1)/2$ independent matrix elements and we have n equations to satisfy. It follows that the solution of (3.13) is determined up to $n(n-3)/2$ constants, which is precisely the number of cross-ratios.

The general solution is given by

$$\Delta_{ij}^{(n)} = \hat{\Delta}_{ij}^{(n)} + \delta_{ij}^{(n)} \tag{3.19}$$

where

$$\hat{\Delta}_{ij}^{(n)} = \frac{1}{n-2}\left(\Delta_i + \Delta_j - \frac{\Delta_T}{n-1}\right), \qquad i < j\,. \tag{3.20}$$

is a particular solution and $\delta_{ij}^{(n)}$ is a symmetric hollow matrix satisfying the homogeneous linear equations:

$$\sum_{j=1}^{n} \delta_{ij}^{(n)} = 0, \qquad i = 1, 2, \ldots, n. \tag{3.21}$$

These equations may be solved by linearly expressing any n of the $n(n-1)/2$ parameters $\delta_{ij}^{(n)}$ in terms of the remaining $n(n-1)/2 - n = n(n-3)/2$ ones. For example, when $n = 4$ we may solve $\delta_{12}^{(4)}, \delta_{23}^{(4)}, \delta_{24}^{(4)}, \delta_{34}^{(4)}$ in terms of $\delta_{13}^{(4)}$ and $\delta_{14}^{(4)}$:

$$\delta_{12}^{(4)} = \delta_{34}^{(4)} = -\delta_{13}^{(4)} - \delta_{14}^{(4)}, \quad \delta_{23}^{(4)} = \delta_{14}^{(4)}, \quad \delta_{24}^{(4)} = \delta_{13}^{(4)}\,. \tag{3.22}$$

Then

$$\langle \mathcal{O}_1(\vec{y}_1) \ldots \mathcal{O}_4(\vec{y}_4) \rangle = \frac{C_4(u_4, v_4)}{\prod\limits_{1 \le i < j \le 4} (x_{ij}^2)^{\Delta_{ij}^{(4)}}} = \frac{\hat{C}_4(u_4, v_4)}{\prod\limits_{1 \le i < j \le 4} (x_{ij}^2)^{\hat{\Delta}_{ij}^{(4)}}}, \tag{3.23}$$

where $\hat{C}_4(u_4, v_4) = C_4(u_4, v_4) u_4^{\delta_{13}^{(4)} + \delta_{14}^{(4)}} v_4^{-\delta_{14}^{(4)}}$. Thus the freedom in the solution of (3.13) just amounts to redefining the arbitrary function of cross-ratios. The same is true for any n. To have an unambiguous definition of the function of cross-ratios one needs to choose a solution of (3.13).

The formulas for spinning operators are similar but more involved. Here we will quote the results for the case of vector primaries as we will needed it later. Vector primaries $\mathcal{J}_\alpha$ of dimension Δ transform

$$\mathcal{J}'_\alpha(\vec{x}\,') = \Omega^{-\Delta}(\vec{x}) R_\alpha{}^\beta(\vec{x}) \mathcal{J}_\beta(\vec{x})\,, \tag{3.24}$$

and this implies that n-point function should satisfy,

$$\begin{aligned} &\langle \mathcal{J}^1_{\alpha_1}(\vec{x}_1') \cdots \mathcal{J}^n_{\alpha_n}(\vec{x}_n') \rangle = \\ &\quad \Omega(\vec{x}_1)^{-\Delta_1} \cdots \Omega(\vec{x}_n)^{-\Delta_n} R_{\alpha_1}{}^{\beta_1}(\vec{x}_1) \cdots R_{\alpha_n}{}^{\beta_n}(\vec{x}_n) \langle \mathcal{J}^1_{\beta_1}(\vec{x}_1) \cdots \mathcal{J}^n_{\beta_n}(\vec{x}_n) \rangle\,, \end{aligned} \tag{3.25}$$

where $R_\alpha{}^\beta(\vec{x})$ is given in (3.6).

3.3 AdS amplitudes

The objects of interest are AdS amplitudes, which may be computed via Witten diagrams. The basic structure is well known [103]: a Witten diagram for an n-point function is constructed by n bulk-to-boundary propagators which are linked to a number of bulk-to-bulk propagators connected via bulk vertices, which are integrated over all of AdS.

As just reviewed, conformal invariance fixes the form of 2-point and 3-point functions, up to a number of constants, and the form of higher-point functions up to a functions of cross-ratios. The constants and the function of cross-ratios depend on the specific CFT but the form of the correlators is independent of it. In AdS/CFT correspondence the AdS isometries play the role of conformal transformations, so one should be able to establish the same results using AdS isometries only. We will show that this is indeed the case, and along the way we will also show the relation of the arbitrary constants and functions of cross-ratios with bulk quantities. We will establish this result to all orders in bulk perturbation theory and for scalar correlators. We will discuss the generalisation to general spinning operators afterwards.

3.3.1 AdS isometries as constrained conformal transformations

We work in Euclidean signature and use coordinates where AdS metric is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{\ell^2}{z^2}(\mathrm{d}z^2 + \mathrm{d}\vec{x}^2) = \ell^2 \frac{\delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu}{z^2}, \tag{3.26}$$

where ℓ is the AdS radius (set to 1). The conformal boundary is at $z = 0$ and this is the place where we need to impose boundary conditions. We will denote bulk point by $x^\mu = (z, x^\alpha) = (z, \vec{x})$, where $\mu = 0, 1, \ldots, d$ is a bulk index, $\alpha = 1, 2, \ldots, d$ is a boundary index and $x^0 = z$ is the radial coordinate.

It is well known that the AdS metric is invariant under the following transformations (AdS isometries),

$$z' = z\,, \qquad x'^\alpha = a^\alpha{}_\beta x^\beta + a^\alpha \qquad \to \text{Poincaré} \tag{3.27}$$

$$z' = \lambda z\,, \qquad x'^\alpha = \lambda x^\alpha \qquad \to \text{Dilation} \tag{3.28}$$

$$z' = \frac{z}{(z^2 + \vec{x}^2)}\,, \qquad x'^\alpha = \frac{x^\alpha}{(z^2 + \vec{x}^2)} \qquad \to \text{Inversion} \tag{3.29}$$

$$z' = \frac{z}{1 + 2\vec{b}\cdot\vec{x} + \vec{b}^2(z^2 + \vec{x}^2)}\,, \qquad x'^\alpha = \frac{x^\alpha + b^\alpha(z^2 + \vec{x}^2)}{1 + 2\vec{b}\cdot\vec{x} + \vec{b}^2(z^2 + \vec{x}^2)} \qquad \to \text{SCT} \tag{3.30}$$

where we have also indicated the conformal transformation they limit to at the conformal boundary as $z \to 0$.

It is less known that these transformations can be thought of as constrained flat-space $(d+1)$-dimensional conformal transformations. We will denote these transformations as in (3.2)-(3.5) but with the parameters carrying a tilde (and the indices being $(d+1)$-dimensional indices): $a^{\alpha}{}_{\beta} \to \tilde{a}^{\mu}{}_{\nu}, a^{\alpha} \to \tilde{a}^{\mu}, \lambda \to \tilde{\lambda}, b^{\alpha} \to \tilde{b}^{\mu}$. Under such transformations

$$\delta_{\mu\nu} \mathrm{d}x'^{\mu} \mathrm{d}x'^{\nu} = \tilde{\Omega}(x)^2 \delta_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}, \tag{3.31}$$

where $\tilde{\Omega}(x)$ is $(d+1)$ version of $\Omega(\vec{x})$ in (3.2)-(3.5). For these conformal transformations to be AdS isometries the transformation of z must cancel the factor of $\tilde{\Omega}$:

$$z' = \tilde{\Omega}(x) z\,. \tag{3.32}$$

This is indeed satisfied if we impose:

$$\tilde{a}^{z}{}_{\nu} = \delta^{z}_{\nu}, \qquad \tilde{a}^{\nu}{}_{z} = \delta^{\nu}_{z}, \qquad \tilde{a}^{z} = 0, \qquad \tilde{b}^{z} = 0\,. \tag{3.33}$$

Thus, altogether and after dropping the tildes we obtain

$$x'^{\mu} = a^{\mu}{}_{\nu} x^{\nu} + a^{\mu}\,, \qquad \text{with } a^{z}{}_{\nu} = \delta^{z}_{\nu}, \quad a^{\mu}{}_{z} = \delta^{\mu}_{z}, \quad a^{z} = 0\,, \tag{3.34}$$

$$x'^{\mu} = \lambda x^{\mu}\,, \tag{3.35}$$

$$x'^{\mu} = \frac{x^{\mu}}{x^2}\,, \tag{3.36}$$

$$x'^{\mu} = \frac{x^{\mu} + b^{\mu} x^2}{1 + 2b \cdot x + b^2 x^2}\,, \qquad \text{with } b^{z} = 0\,, \tag{3.37}$$

where $x^2 = \delta_{\mu\nu} x^{\mu} x^{\nu} = z^2 + \vec{x}^2$, $b \cdot x = \delta_{\mu\nu} b^{\mu} x^{\nu} = \vec{b} \cdot \vec{x}$. One may readily check that (3.34)-(3.37) agree with (3.27)-(3.30).

The advantage of viewing AdS isometries as constrained conformal transformations is that we can immediately inherit all CFT properties that are independent of specific rotations $a^{\mu}{}_{\nu}$ or translations a^{μ}, b^{μ}. For instance, the Jacobian of AdS isometries can be immediately obtained

$$\frac{\partial x'^{\mu}}{\partial x^{\nu}} = \tilde{\Omega}(x) \tilde{R}^{\mu}{}_{\nu}(x)\,, \tag{3.38}$$

where $\tilde{\Omega}$ (the one from $z' = \tilde{\Omega} z$) and $\tilde{R}^{\mu}{}_{\nu} \in O(d+1)$ are those of a CFT for constrained rotations and translations

$$\text{C. Poincare:} \quad \tilde{\Omega}(x) = 1\,, \qquad \tilde{R}^{\mu}{}_{\nu}(x) = a^{\mu}{}_{\nu}\,,\ a^{z}{}_{\nu} = \delta^{z}_{\nu}\,,\ a^{\mu}{}_{z} = \delta^{\mu}_{z}\,, \tag{3.39}$$

$$\text{Dilation:} \quad \tilde{\Omega}(x) = \lambda\,, \qquad \tilde{R}^{\mu}{}_{\nu}(x) = \delta^{\mu}_{\nu}\,, \tag{3.40}$$

$$\text{Inversion:} \quad \tilde{\Omega}(x) = \frac{1}{x^2}\,, \qquad \tilde{R}^{\mu}{}_{\nu}(x) = I^{\mu}_{\nu}(x) = \delta^{\mu}_{\nu} - 2\frac{x^{\mu} x_{\nu}}{x^2}\,, \tag{3.41}$$

$$\text{C. SCT:} \quad \tilde{\Omega}(x) = \frac{1}{1 + 2\vec{b} \cdot \vec{x} + \vec{b}^2 x^2}\,, \qquad \tilde{R}^{\mu}{}_{\nu}(x) = I^{\mu}_{\rho}\left(\frac{x}{x^2} + \vec{b}\right) I^{\rho}_{\nu}(x)\,. \tag{3.42}$$

This implies for example that the inversion property of AdS is inherited from that of flat space

$$(x_1' - x_2')^2 = \tilde{\Omega}(x_1)\tilde{\Omega}(x_2)(x_1 - x_2)^2 \,, \tag{3.43}$$

and

$$I_{\mu\nu}(x_1' - x_2') = \tilde{R}_\mu^{\ \rho}(x_1)\tilde{R}_\nu^{\ \sigma}(x_2)I_{\rho\sigma}(x_1 - x_2) \,. \tag{3.44}$$

We can further obtain useful formulas by taking the limit of bulk points to the boundary. In this limit, $\tilde{\Omega}$ and the boundary components of $\tilde{R}^\mu{}_\nu$ reduce to those of an unconstrained CFT in d dimensions

$$\lim_{z\to 0} \tilde{\Omega}(x) = \Omega(\vec{x}) \,, \quad \lim_{z\to 0} \tilde{R}^\alpha{}_\beta(x) = R^\alpha{}_\beta(\vec{x}) \,, \tag{3.45}$$

and thus one also recovers the Jacobian

$$\lim_{z\to 0} \frac{\partial x'^\alpha}{\partial x^\beta} = \lim_{z\to 0} \tilde{\Omega}(x)\tilde{R}^\alpha{}_\beta(x) = \Omega(\vec{x})R^\alpha{}_\beta(\vec{x}) \,. \tag{3.46}$$

When one of the points in (3.43) and (3.44) are taken to the boundary, one obtains the useful relations

$$(x' - \vec{y}\,')^2 = \tilde{\Omega}(x)\Omega(\vec{y})(x - \vec{y})^2 \,, \tag{3.47}$$

and

$$I_{\mu\alpha}(x' - \vec{y}\,') = \tilde{R}_\mu^{\ \nu}(x)R_\alpha^{\ \beta}(\vec{y})I_{\nu\beta}(x - \vec{y}) \,. \tag{3.48}$$

3.3.2 AdS propagators

The bulk-to-boundary propagator for a bulk field dual to an operator of dimension Δ is the regular solution of the bulk equation

$$(-\Box + m^2)K_\Delta(x_1, \vec{x}_2) = 0 \tag{3.49}$$

where $m^2 = \Delta(\Delta - d)$ and it is given by

$$K_\Delta(x_1, \vec{x}_2) = c_\Delta \left(\frac{z_1}{(x_1 - \vec{x}_2)^2}\right)^\Delta \,, \qquad c_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma\left(\Delta - \frac{d}{2}\right)} . \tag{3.50}$$

where $(x_1 - \vec{x}_2)^2 = z_1^2 + (\vec{x}_1 - \vec{x}_2)^2$. This is normalized such that as we approach the AdS boundary the propagator tends to a delta function

$$\lim_{z_1\to 0} K_\Delta(x_1, \vec{x}_2) \to z_1^{d-\Delta}\delta(\vec{x}_1 - \vec{x}_2) \,. \tag{3.51}$$

The bulk-to-boundary propagator $K_\Delta(x_1, \vec{x}_2)$ transforms as a CFT primary field of dimension Δ at $\vec{x}_2$ under the AdS isometries (3.27)-(3.30) acting simultaneously on x_1

and $\vec{x}_2$

$$K_\Delta(x_1', \vec{x}_2') = \Omega(\vec{x}_2)^{-\Delta} K_\Delta(x_1, \vec{x}_2)\,, \tag{3.52}$$

where the factors of Ω are those given in (3.2)-(3.5). This is most easily shown using the perspective of the AdS isometries as constrained conformal transformations. Indeed, using (3.32) and (3.47) we obtain

$$K_\Delta(x_1', \vec{x}_2') = c_\Delta \left(\frac{z_1'}{(x_1' - \vec{x}_2')^2} \right)^\Delta = c_\Delta \left(\frac{\tilde{\Omega}(x_1) z_1}{\tilde{\Omega}(x_1)\Omega(\vec{x}_2)(x_1 - \vec{x}_2)^2} \right)^\Delta = \Omega(\vec{x}_2)^{-\Delta} K_\Delta(x_1, \vec{x}_2) \tag{3.53}$$

We will also explain in Section 3.4 that this transformation rule follows from bulk diffeomorphism invariance.

The bulk-to-bulk propagator for the same field is the regular solution of the equation

$$(-\Box + m^2) G_\Delta(x_1, x_2) = \frac{1}{\sqrt{g}} \delta(x_1 - x_2)\,, \tag{3.54}$$

with normalizable behavior at infinity, $G_\Delta \sim z_1^\Delta$ as x_1 approaches the conformal boundary (and $\sim z_2^\Delta$ when x_2 approaches the conformal boundary). AdS invariance implies that the propagator is a function of an AdS invariant distance, which we may take to be the chordal distance

$$\xi = \frac{2 z_1 z_2}{z_1^2 + z_2^2 + (\vec{x}_1 - \vec{x}_2)^2}\,. \tag{3.55}$$

The invariance of the chordal distance under transformations (3.27)-(3.30) (or equivalently (3.34)-(3.37)) that act simultaneously on both x_1 and x_2 follows by inspection upon use of (3.32) and (3.43). By explicit computation

$$G_\Delta(x_1, x_2) = \frac{2^{-\Delta} c_\Delta}{2\Delta - d} \xi^\Delta\, {}_2F_1\left(\frac{\Delta}{2}, \frac{\Delta + 1}{2}, \Delta - \frac{d}{2} + 1; \xi^2 \right)\,. \tag{3.56}$$

It follows that

$$G_\Delta(x_1', x_2') = G_\Delta(x_1, x_2)\,. \tag{3.57}$$

One may similarly obtain the transformation properties for propagators of spinning fields. We report here the result for the bulk-to-boundary propagator of an (Abelian) gauge field, as this is a case we discuss later. The bulk-to-boundary propagator has been obtained in the early AdS/CFT literature [62]. Up to gauge transformations it is given by

$$K_{\mu\alpha}(x_1, \vec{x}_2) = C \frac{z_1^{d-2}}{[(x_1 - \vec{x}_2)^2]^{d-1}} I_{\mu\alpha}(x_1 - \vec{x}_2)\,, \tag{3.58}$$

where C is a constant and $I_{\mu\alpha}$ the inversion tensor, with μ and α bulk and boundary indices, respectively. Using (3.32), (3.47) and (3.48) one may work out how this bulk-to-boundary propagator transforms under bulk isometries

$$
\begin{aligned}
K_{\mu\alpha}(x_1', \vec{x}_2') &= C \frac{z_1'^{d-2}}{\left[(x_1' - \vec{x}_2')^2\right]^{d-1}} I_{\mu\alpha}(x_1' - \vec{x}_2') \\
&= C \frac{\tilde{\Omega}^{d-2}(x_1) z_1^{d-2}}{\left[\tilde{\Omega}(x_1)\Omega(\vec{x}_2)(x_1 - \vec{x}_2)^2\right]^{d-1}} \tilde{R}_\mu^{\ \nu}(x_1) R_\alpha^{\ \beta}(\vec{x}_2) I_{\nu\beta}(x_1 - \vec{x}_2)\,, \\
&= \Omega^{-(d-1)}(\vec{x}_2)\tilde{\Omega}^{-1}(x_1)\tilde{R}_\mu^{\ \nu}(x_1) C \frac{z_1^{d-2}}{\left[(x_1 - \vec{x}_2)^2\right]^{d-1}} I_{\nu\beta}(x_1 - \vec{x}_2) R_\alpha^{\ \beta}(\vec{x}_2)\,, \\
&= \Omega^{-(d-1)}(\vec{x}_2) \frac{\partial x_1^\nu}{\partial x_1'^\mu} K_{\nu\beta}(x_1, \vec{x}_2) R_\alpha^{\ \beta}(\vec{x}_2)\,.
\end{aligned}
\tag{3.59}
$$

It follows that the vector bulk-to-boundary propagators transforms as a vector in the bulk index μ and a CFT conserved current in the boundary index α (compare with (3.24) with $\Delta = d - 1$). We will rederive this transformation property from bulk diffeomorphism invariance in Section 3.4.

3.3.3 AdS amplitudes are CFT correlators

We now discuss the computation of AdS amplitudes, i.e. bulk n-point functions with all legs in AdS boundary. This can be computed via Witten-Feynman diagrams, involving n bulk-to-boundary propagators connecting the n boundary points to an "amputated" bulk n-point function $G_n(x_1, \ldots, x_n)$ and integrating over $x_1, \ldots, x_n$. The amputated bulk n-point function is constructed from bulk-to-bulk propagator connected via vertices that come from the bulk action, and integrating over the position of each vertex. As long as the bulk action is invariant under AdS isometries, the invariance of the bulk-to-bulk propagator guarantees that $G_n(x_1, \ldots, x_n)$ is also invariant under (3.27)-(3.30) that act simultaneously on all $x_1, \ldots, x_n$

$$
G_n(x_1', \ldots, x_n') = G_n(x_1, \ldots, x_n)\,. \tag{3.60}
$$

This could have been invalidated by short-distance singularities, but as we discuss in Subsection 3.3.4 we can regulate the short-distance singularities while respecting the AdS isometries. More generally, (3.60) is guaranteed by diffeomorphism invariance (in a theory with no diffeomorphism anomalies), and we will discuss in Section 3.4 the extension to tensorial correlators. When the bulk points x_i tend to the boundary, IR divergences appear. These correspond via the AdS/CFT correspondence to UV divergences in the dual CFT and lead to conformal anomalies and anomalous dimensions. This will be discussed in Subsection 3.3.4, but for ease in presentation we suppress the IR issues in this subsection. We will now show that the dependence of the correlators on the external positions $\vec{y}_i$ is the same with that of a CFT, without computing any integral.

3.3.3.1 2-point function

Let us start with the 2-point function, whch is illustrated in Fig. 3.1,

$$I_2(\vec{y}_1,\vec{y}_2) = \int_{x_1}\int_{x_2} K_{\Delta_1}(x_1,\vec{y}_1)G_2(x_1,x_2)K_{\Delta_2}(x_2,\vec{y}_2)\,, \tag{3.61}$$

where we use the shorthand notation, $\int_x = \int d^{d+1}x\sqrt{\det g}$. We can extract all the

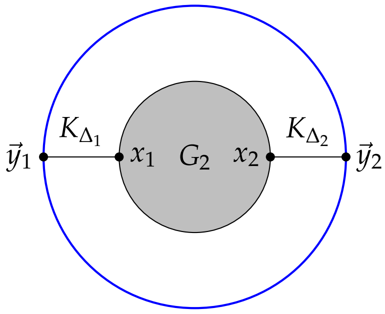

FIGURE 3.1: General 2-point function. The blue (outer) circle represents the boundary of AdS, and the shaded region represents general (loop) interactions that connect the bulk points x_1, x_2.

dependence of the external points $\vec{y}_1$ and $\vec{y}_2$ from the integral by performing change variables in integration variables x_1 and x_2 that account to AdS isometries. First, note that by using the inversion property of the bulk-to-boundary propagators we find,

$$I_2(\vec{y}_1',\vec{y}_2') = (\vec{y}_1^{\,2})^{\Delta_1}(\vec{y}_2^{\,2})^{\Delta_2} I_2(\vec{y}_1,\vec{y}_2)\,. \tag{3.62}$$

We now shift the integration variables $\vec{x}_1, \vec{x}_2$ by $\vec{y}_1$ to obtain

$$I_2 = \int_{x_1}\int_{x_2} K_{\Delta_1}(x_1,\vec{0})G_2(x_1,x_2)K_{\Delta_2}(x_2,\vec{y}_{21})\,, \tag{3.63}$$

where $\vec{y}_{21} = (\vec{y}_2 - \vec{y}_1)$. We can now change variables by rescaling x_1, x_2 by $|\vec{y}_{12}|$ and use (3.52) to find

$$I_2 = \frac{1}{|\vec{y}_{12}|^{\Delta_1+\Delta_2}}\int_{x_1}\int_{x_2} K_{\Delta_1}(x_1,\vec{0})G_2(x_1,x_2)K_{\Delta_2}(x_2,\hat{y}_{21})\,. \tag{3.64}$$

Thus

$$I_2(\vec{y}_1,\vec{y}_2) = \frac{C_2}{|\vec{y}_{12}|^{\Delta_1+\Delta_2}}, \tag{3.65}$$

with C_2 equal to

$$C_2 = \int_{x_1}\int_{x_2} K_{\Delta_1}(x_1,\vec{0})G_2(x_1,x_2)K_{\Delta_2}(x_2,\hat{y}_{21})\,. \tag{3.66}$$

Finally, rotational invariance implies that the integral does not depend on the direction specified by $\hat{y}_{21}$ and thus it is a constant. Equation (3.65) should be consistent with the transformation in (3.62) and this implies $\Delta_1 = \Delta_2 = \Delta$, thus reproducing the expected CFT answer

$$I_2(\vec{y}_1, \vec{y}_2) = \frac{C_2}{|\vec{y}_{12}|^{2\Delta}}. \tag{3.67}$$

3.3.3.2 3-point function

The general 3-point function, see Fig. 3.2, is given by

$$I_3(\vec{y}_1, \vec{y}_2, \vec{y}_3) = \int_{x_1}\int_{x_2}\int_{x_3} K_{\Delta_1}(x_1, \vec{y}_1) K_{\Delta_2}(x_2, \vec{y}_2) K_{\Delta_3}(x_3, \vec{y}_3) G_3(x_1, x_2, x_3), \tag{3.68}$$

where $G_3(x_1, x_2, x_3)$ is the amputated bulk 3-point function. We first shift the

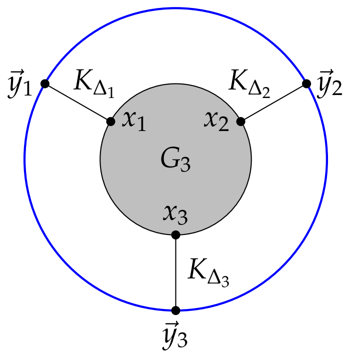

FIGURE 3.2: General 3-point function. The blue (outer) circle represents the boundary of AdS, and the shaded region represents general (loop) interactions that connect the bulk points x_1, x_2, x_3.

integration variables $\vec{x}_2, \vec{x}_2, \vec{x}_3$ by $\vec{y}_1$ to obtain

$$I_3 = \int_{x_1}\int_{x_2}\int_{x_3} K_{\Delta_1}(x_1, \vec{0}) K_{\Delta_2}(x_2, \vec{y}_{21}) K_{\Delta_3}(x_3, \vec{y}_{31}) G_3(x_1, x_2, x_3)\,. \tag{3.69}$$

Then we make a change of variable that amounts to an inversion on all integration variables and use the transformation of the bulk-to-boundary propagator (3.52) to obtain

$$I_3 = |\vec{y}\,'_{12}|^{2\Delta_2} |\vec{y}\,'_{13}|^{2\Delta_3} \int_{x_1}\int_{x_2}\int_{x_3} c_{\Delta_1} z_1^{\Delta_1} K_{\Delta_2}(x_2, \vec{y}\,'_{21}) K_{\Delta_3}(x_3, \vec{y}\,'_{31}) G_3(x_1, x_2, x_3), \tag{3.70}$$

where here and in the remainder of this section prime indicates a (boundary) inversion

$$\vec{y}\,' = \frac{\vec{y}}{\vec{y}^2}\,. \tag{3.71}$$

After this step, only two bulk-to-boundary propagators depend on the external positions, so we can proceed analogously to the case of 2-point function to obtain

$$I_3 = \frac{|\vec{y}\,'_{12}|^{2\Delta_2} |\vec{y}\,'_{13}|^{2\Delta_3}}{|\vec{y}\,'_{31} - \vec{y}\,'_{21}|^{\Delta_2+\Delta_3-\Delta_1}} \int_{x_1}\int_{x_2}\int_{x_3} c_{\Delta_1} z_1^{\Delta_1} K_{\Delta_2}(x_2, \vec{0}) K_{\Delta_3}(x_3, \hat{y}'_{31,21}) G_3(x_1, x_2, x_3) \tag{3.72}$$

where $\hat{y}'_{31,21}$ is the unit vector of $\vec{y}'_{31} - \vec{y}'_{21}$, *i.e.*

$$\hat{y}'_{31,21} = \frac{\vec{y}'_{31} - \vec{y}'_{21}}{|\vec{y}'_{31} - \vec{y}'_{21}|} . \tag{3.73}$$

Let

$$C_3 = \int_{x_1}\int_{x_2}\int_{x_3} c_{\Delta_1} z_1^{\Delta_1} K_{\Delta_2}(x_2, \vec{0}) K_{\Delta_3}(x_3, \hat{y}'_{31,21}) G_3(x_1, x_2, x_3) \tag{3.74}$$

Rotation invariance implies that C_3 is independent of $\hat{y}'_{31,21}$ that thus it is a constant. Using (3.71) to re-express the answer in terms of the original insertion points we finally get

$$I_3(\vec{y}_1, \vec{y}_2, \vec{y}_3) = \frac{C_3}{|\vec{y}_{12}|^{\Delta_1+\Delta_2-\Delta_3} |\vec{y}_{13}|^{\Delta_1+\Delta_3-\Delta_2} |\vec{y}_{23}|^{\Delta_2+\Delta_3-\Delta_1}} , \tag{3.75}$$

which is precisely the expected form for a CFT 3-point function.

3.3.3.3 4-point functions

The general 4-point function, see Fig. 3.3, is given by

$$I_4(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4) = \int_{x_1}\int_{x_2}\int_{x_3}\int_{x_4} K_{\Delta_1}(x_1, \vec{y}_1) K_{\Delta_2}(x_2, \vec{y}_2) K_{\Delta_3}(x_3, \vec{y}_3) K_{\Delta_4}(x_4, \vec{y}_4) G_4(x_1, x_2, x_3, x_4), \tag{3.76}$$

where $G_4(x_1, x_2, x_3, x_4)$ is the amputated bulk-to-bulk 4-point function. Following the

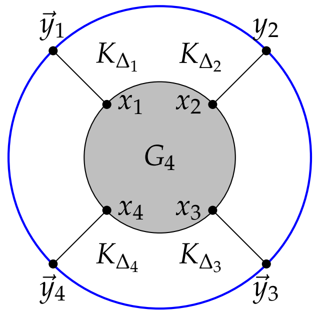

FIGURE 3.3: General 4-point function. The blue (outer) circle represents the boundary of AdS, and the shaded region represents general (loop) interactions that connect the bulk points x_1, x_2, x_3, x_4.

same steps[2] as in the case of 3-point functions we arrive at

$$I_4 = \frac{|\vec{y}_{12}|^{\Delta_3+\Delta_4-\Delta_1-\Delta_2} |\vec{y}_{13}|^{\Delta_2+\Delta_4-\Delta_1-\Delta_3}}{|\vec{y}_{14}|^{2\Delta_4} |\vec{y}_{23}|^{\Delta_2+\Delta_3+\Delta_4-\Delta_1}} \tag{3.77}$$

$$\times \int_{x_1}\int_{x_2}\int_{x_3}\int_{x_4} c_{\Delta_1} z_1^{\Delta_1} K_{\Delta_2}(x_2, \vec{0}) K_{\Delta_3}(x_3, \hat{y}'_{31,21}) K_{\Delta_4}\left(x_4, \frac{|\vec{y}_{13}||\vec{y}_{24}|}{|\vec{y}_{14}||\vec{y}_{23}|}\hat{y}'_{41,21}\right) G_4(x_1, x_2, x_3, x_4)$$

[2] In more detail: we translate the internal coordinates by $\vec{y}_1$, invert them together with the external coordinates, translate them again by $\vec{y}'_{21}$, rescale them by $|\vec{y}'_{31} - \vec{y}'_{21}|$, and finally, write the inverted external points in terms of the original positions.

where $\hat{y}'_{31,21}$ is the unit vector of $\vec{y}'_{31} - \vec{y}'_{21}$, $\hat{y}'_{41,21}$ the unit vector of $\vec{y}'_{41} - \vec{y}'_{21}$, and $\vec{y}'_{ij} = \vec{y}_{ij}/\vec{y}^2_{ij}$, is the inversion of $\vec{y}_{ij}$ (and $\vec{y}_{ij} = (\vec{y}_i - \vec{y}_j)$). Using the conformal cross-ratios from (3.16) (and relabeling $u_4 \to u$ and $v_4 \to v$) we find that the 4-point function takes the expected form for a CFT 4-point function,

$$I_4(\vec{y}_1, \vec{y}_2, \vec{y}_3, \vec{y}_4) = \frac{C_4(u,v)}{\prod_{1\le i<j\le 4} (y^2_{ij})^{\Delta^{(4)}_{ij}}} , \tag{3.78}$$

where the dimensions $\Delta^{(4)}_{ij}$ satisfy the conformal constraints (3.13), and

$$\begin{aligned} C_4(u,v) &= u^{\Delta^{(4)}_{34}} v^{\Delta^{(4)}_{14} - \Delta_4} \\ &\times \int_{x_1}\int_{x_2}\int_{x_3}\int_{x_4} c_{\Delta_1} z_1^{\Delta_1} K_{\Delta_2}(x_2, \vec{0}) K_{\Delta_3}(x_3, \hat{y}'_{31,21}) K_{\Delta_4}\left(x_4, \frac{\hat{y}'_{41,21}}{\sqrt{v}}\right) G_4(x_1, x_2, x_3, x_4) \end{aligned} \tag{3.79}$$

where in asserting that C_4 depends only on u, v we used the fact that rotational invariance implies that the integral may depend on $\hat{y}'_{31,21}$ and $\hat{y}'_{41,21}$ only via their inner product and as we now explain this inner product is a function of u and v. Indeed, since $\hat{y}'_{31,21}$ and $\hat{y}'_{41,21}$ are unit vectors their inner product depends only on the angle between them and conformal transformation preserves angles. $\hat{y}'_{31,21} \cdot \hat{y}'_{41,21}$ being conformal invariant that depends on four positions is necessarily is a function u and v. We can compute this function explicit as follows. Reverting to the original variables we find,

$$\hat{y}'_{31,21} \cdot \hat{y}'_{41,21} = \frac{y^2_{12} y_{13} y_{14}}{y_{23} y_{24}} \left(\frac{\vec{y}_{13}}{y^2_{13}} - \frac{\vec{y}_{12}}{y^2_{12}}\right) \cdot \left(\frac{\vec{y}_{14}}{y^2_{14}} - \frac{\vec{y}_{12}}{y^2_{12}}\right) . \tag{3.80}$$

Expanding the product and using the formula

$$\vec{y}_{1i} \cdot \vec{y}_{1j} = \frac{1}{2}\left(y^2_{1i} + y^2_{1j} - y^2_{ij}\right) , \tag{3.81}$$

leads to the result

$$\hat{y}'_{31,21} \cdot \hat{y}'_{41,21} = \frac{1 + v - u}{2\sqrt{v}} , \tag{3.82}$$

which is a function of u and v, as claimed.

Note that the integral in (3.79) depends on u only through the inner product in (3.82). This appears to be special to holographic CFT and it will be interesting to investigate its implications.

3.3.3.4 n-point function

We now discuss the general case. Starting from

$$I_n(\vec{y}_1, \cdots, \vec{y}_n) = \int_{x_1} \cdots \int_{x_n} \prod_{i=1}^{n} K_{\Delta_i}(x_i, \vec{y}_i)\, G_n(x_1, \cdots, x_n)\,, \tag{3.83}$$

which is represented by Fig. 3.4, and repeating the same steps one finds:

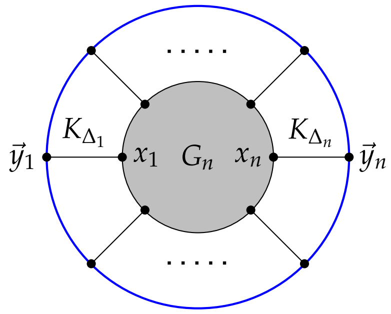

FIGURE 3.4: General n-point function. The blue (outer) circle represents the boundary of AdS, and the shaded region represents general (loop) interactions that connect the bulk points $x_1, \ldots, x_n$.

$$\begin{aligned} I_n =& \frac{y_{12}^{\Delta_T-2\Delta_1-2\Delta_2} y_{13}^{\Delta_T-2\Delta_1-2\Delta_3}}{y_{23}^{\Delta_T-2\Delta_1} \prod_{i=4}^{n} y_{1i}^{2\Delta_i}} \\ &\times \int_{x_1} \cdots \int_{x_n} c_{\Delta_1} z_1^{\Delta_1} K_{\Delta_2}(x_2, \vec{0}) K_{\Delta_3}(x_3, \hat{y}'_{31,21}) \prod_{i=4}^{n} K_{\Delta_i}\left(x_i, \frac{y_{13}y_{2i}}{y_{1i}y_{23}} \hat{y}'_{i1,21}\right)\, G_n(x_1, \cdots, x_n)\,, \end{aligned} \tag{3.84}$$

which may be processed to

$$I_n(\vec{y}_1, \ldots, \vec{y}_n) = \frac{C_n(u_i, v_i, w_{ij})}{\prod\limits_{1 \le i < j \le n} (y_{ij}^2)^{\Delta_{ij}^{(n)}}}\,, \tag{3.85}$$

where the dimensions $\Delta_{ij}^{(n)}$ satisfy the conformal constraints (3.13), and

$$\begin{aligned} C_n =& \prod_{i=4}^{n} u_i^{\Delta_i - \Delta_{1i}^{(n)} - \Delta_{2i}^{(n)} - \sum\limits_{i<k\le n} \Delta_{ik}^{(n)}} v_i^{\Delta_{1i}^{(n)} - \Delta_i} \prod_{4 \le j < l \le n} w_{jl}^{\Delta_{jl}^{(n)}} \\ &\times \int_{x_1} \cdots \int_{x_n} c_{\Delta_1} z_1^{\Delta_1} K_{\Delta_2}(x_2, \vec{0}) K_{\Delta_3}(x_3, \hat{y}'_{31,21}) \prod_{i=4}^{n} K_{\Delta_i}\left(x_i, \frac{\hat{y}'_{i1,21}}{\sqrt{v_i}}\right)\, G_n(x_1, \ldots, x_n)\,. \end{aligned} \tag{3.86}$$

The remaining integral is a function of cross-ratios. Indeed, by rotational invariance it is a function of the inner product between the unit vectors $\hat{y}'_{i1,21}$. A similar computation as above leads to:

$$\hat{y}'_{i1,21} \cdot \hat{y}'_{j1,21} = \frac{1}{2}\left(\sqrt{u_{1ij2}} + \sqrt{u_{1ji2}} - \sqrt{u_{12ij}}\sqrt{u_{12ji}}\right)\,. \tag{3.87}$$

which is a function of cross-ratios, as claimed. For $i = j$, the product is just 1, as expected. For $j = 3$ and $i > 3$, it can be expressed in terms of u_i and v_i:

$$\hat{y}'_{i1,21} \cdot \hat{y}'_{31,21} = \frac{1 + v_i - u_i}{2\sqrt{v_i}}, \qquad i > 3, \tag{3.88}$$

while for $i, j > 3$ also in terms of w_{ij}:

$$\hat{y}'_{i1,21} \cdot \hat{y}'_{j1,21} = \frac{v_i + v_j - u_j w_{ij}}{2\sqrt{v_i v_j}}, \qquad i, j > 3. \tag{3.89}$$

As in the case of 4-point functions, the integral in (3.86) appears to have special dependence on some of the cross-ratios (u_i and w_{ij}) and it will be interesting to investigate the implications.

3.3.4 Regularization and renormalization

We have just shown that AdS amplitudes can be brought to a form that manifestly satisfied the CFT Ward identities by a sequence of steps that involved changing integration variables amounting to AdS isometries. Such manipulations are well-posed if the integrals are finite. However, the integrals may diverge both in the UV and the IR. The UV divergences come from bulk loops, while the IR divergences are due to the infinite volume of AdS. The bulk IR divergences correspond to CFT UV divergences via the AdS/CFT correspondence.

One can regulate the UV divergences in a way that preserves the AdS invariance. This is expected as bulk UV divergences are mapped by the AdS/CFT correspondence to IR divergences in the CFT. Such divergences should cancel on their own and they should not lead to breaking of the conformal symmetry. The regulator amounts to separating (bulk) coincident points along a geodesic by affine distance τ [15], which thus acts an UV regulator. This results to modifying the argument of the bulk-to-bulk propagator by changing $\xi \to \xi / \cosh \tau$ in (3.56), recovering the prescription in [22, 21]. We will discuss in detail this regulator in Chapter 4, where we will also show that it can be derived from a regulated action. Since the regulated theory is invariant under AdS isometries, all steps outlined above are valid in the regulated theory. The discussion in this chapter is about bulk scalar propagators, but we expect the results to extend to general tensorial fields (the metric, gauge fields, antisymmetric tensor fields, etc.).

The issues with the IR divergences is more subtle. These correspond to UV divergences in the dual CFT and such divergences give rise to anomalous dimensions and conformal anomalies, and thus the breaking of conformal symmetry is inevitable. One can regulate the IR divergence by imposing an explicit IR cut-off, $z \geq \varepsilon$, as in the original works on holographic renormalization [72, 44, 98]. In this case the explicit cut-off results into additional terms when one follows the manipulations described above. This has already been discussed in [15] and we will discuss in detail how to compute such integrals in Chapter 5. The results is that the AdS amplitudes still take the form of the CFT correlators but now the dimensions may renormalize (there are anomalous dimensions) and conformal anomalies appear.

An alternative approach is to use dimensional regularization, where the spacetime dimension d and dimensions Δ_i of operators are shifted as in [25]. For generic values of Δ_i the correlators may be defined by analytic continuation. For such cases the analysis above holds unchanged. However, there are also cases where genuine singularities appear and boundary counterterms and renormalization is needed [26, 27, 28, 29]. In the case where both UV and IR issues are present one would need to renormalize the parameters in the bulk action (masses and coupling constants), the fields that specify the boundary conditions (sources of the dual operators) and add appropriate boundary counterterms, as discussed in [15] and in the next chapter.

3.4 From bulk diffeomorphism to conformal invariance

In this section we present an alternative derivation of (3.52) that makes clear how conformal symmetry emerges from bulk diffeomorphism. This derivation also easily extends to general fields and we discuss the case of gauge fields.

Recall that the boundary field $\varphi_{(0)}(\vec{x})$ that parametrizes the boundary condition of a bulk scalar field $\phi(x)$ is given by

$$\varphi_{(0)}(\vec{x}) = \lim_{z\to 0} z^{\Delta-d}\phi(x)\,. \tag{3.90}$$

As we discussed in Subsection 3.3.1 the radial coordinate under the isometry transformations in (3.27)-(3.30) transforms as,

$$z' = \tilde{\Omega}(x) z \tag{3.91}$$

where $\tilde{\Omega}(x)$ has the property

$$\lim_{z\to 0}\tilde{\Omega}(x) = \Omega(\vec{x}), \tag{3.92}$$

with $\Omega(\vec{x})$ the Jacobian factor of the conformal transformations listed in (3.2)-(3.5). Then, adapting an argument from [24, 98], we find that the source transforms as follows

$$\varphi'_{(0)}(\vec{x}\,') = \lim_{z'\to 0} z'^{\Delta-d}\phi'(x') = \lim_{z\to 0}\tilde{\Omega}^{\Delta-d}(x) z^{\Delta-d}\phi(x) = \Omega^{\Delta-d}(\vec{x})\varphi_{(0)}(\vec{x})\,, \tag{3.93}$$

where in the second equality we used the fact that ϕ is a scalar under bulk diffeos and equation (3.91), and in the last equality we used (3.92). This is the expected transformation rule for a source that couples to a scalar operator of dimension Δ. Now, at linearized order in the sources the bulk field is given by

$$\phi(x) = \int \mathrm{d}^d y\, K_\Delta(x,\vec{y})\varphi_{(0)}(\vec{y}) \tag{3.94}$$

Thus,

$$\begin{aligned} \phi'(x') &= \int \mathrm{d}^d y'\, K_\Delta(x',\vec{y}\,')\varphi_0'(\vec{y}\,')\,, \\ &= \int \mathrm{d}^d y\, \Omega^d(\vec{y})K_\Delta(x',\vec{y}\,')\Omega^{\Delta-d}(\vec{y})\varphi_{(0)}(\vec{y})\,, \\ &= \int \mathrm{d}^d y\, \Omega^\Delta(\vec{y})K_\Delta(x',\vec{y}\,')\varphi_{(0)}(\vec{y})\,, \end{aligned} \tag{3.95}$$

Since this a scalar field, $\phi'(x') = \phi(x)$, and comparing (3.94) with (3.95) we conclude that the bulk-to-boundary propagator transforms as a scalar primary field:

$$K_\Delta(x',\vec{y}\,') = \Omega^{-\Delta}(\vec{y})K_\Delta(x,\vec{y})\,, \tag{3.96}$$

Another way to see this is to note that (3.94) has the same form as the coupling of the source to the operator: $\int \mathrm{d}^d y\, \mathcal{O}(\vec{y})\phi_{(0)}(\vec{y})$.

3.4.1 Generalization to spinning operator

This discussion readily generalises to spinning fields. The higher the spin the more complex the formulas and to keep the technicalities to the minimum we will present the details for a gauge field. All the steps, however, are the same in all cases. As in the case of a scalar, the first step is to establish that the sources indeed transforms as a source of a spinning primary operator. For a gauge field the source is given by

$$a_{(0)\alpha}(\vec{x}) = \lim_{z\to 0} A_\alpha(x)\,. \tag{3.97}$$

We now follow the same steps as in (3.93)

$$\begin{aligned} a'_{(0)\alpha}(\vec{x}\,') &= \lim_{z'\to 0} A'_\alpha(x') = \lim_{z\to 0} \frac{\partial x^\mu}{\partial x'^\alpha} A_\mu(x) = \lim_{z\to 0} \tilde{\Omega}^{-1}(x)\tilde{R}_\alpha^{\ \mu}(x)A_\mu(x) \\ &= \Omega^{-1}(\vec{x})R_\alpha^{\ \beta}(\vec{x})a_{(0)\beta}(\vec{x})\,, \end{aligned} \tag{3.98}$$

where we used (3.38), (3.46) and the fact that the radial component of the field, A_z, is subleading in z and thus vanishes as $z \to 0$. This is indeed the correct transformation for a source that couples to a conserved current of dimension $\Delta = d-1$.

The bulk gauge field to linear order in the sources is given by

$$A_\mu(x) = \int \mathrm{d}^d y\, K_\mu^{\ \alpha}(x,\vec{y})a_{(0)\alpha}(\vec{y})\,. \tag{3.99}$$

Following the same steps as in (3.95) we find

$$A'_\mu(x') = \int \mathrm{d}^d y' \; K_\mu^{\ \alpha}(x', \vec{y}\,') a'_{(0)\alpha}(\vec{y}\,') \tag{3.100}$$

$$= \int \mathrm{d}^d y \; \Omega^d(\vec{y}) K_\mu^{\ \alpha}(x', \vec{y}\,') \Omega^{-1}(\vec{y}) R_\alpha^{\ \beta}(\vec{y}) a_{(0)\beta}(\vec{y})$$

$$= \int \mathrm{d}^d y \; \Omega^{d-1}(\vec{y}) K_\mu^{\ \alpha}(x', \vec{y}\,') R_\alpha^{\ \beta}(\vec{y}) a_{(0)\beta}(\vec{y})\,. \tag{3.101}$$

By diffeomorphism invariance

$$A'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu(x) \quad \Rightarrow \quad K_{\mu\alpha}(x', \vec{y}\,') = \Omega^{-(d-1)}(\vec{y}) \frac{\partial x^\nu}{\partial x'^\mu} K_{\nu\beta}(x, \vec{y}) R_\alpha^{\ \beta}(\vec{y})\,, \tag{3.102}$$

and we reproduce (3.59).

A general AdS amplitude of n conserved currents is given by

$$I_{\alpha_1 \ldots \alpha_n}(\vec{y}_1, \ldots, \vec{y}_n) = \int_{x_1} \cdots \int_{x_n} K_{\mu_1\alpha_1}(x_1, \vec{y}_1) \cdots K_{\mu_n\alpha_n}(x_n, \vec{y}_n) G^{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n)\,, \tag{3.103}$$

where $G^{\mu_1 \cdots \mu_n}(x_1, \ldots, x_n)$ is the amputated bulk n-point function of the gauge field A_μ (to any loop order). Provided it transforms under diffeomorphisms as indicated by its indices

$$G^{\mu_1 \cdots \mu_n}(x'_1, \ldots, x'_n) = \frac{\partial x_1'^{\mu_1}}{\partial x_1^{\nu_1}} \cdots \frac{\partial x_n'^{\mu_n}}{\partial x_n^{\nu_n}} G^{\nu_1 \cdots \nu_n}(x_1, \ldots, x_n)\,, \tag{3.104}$$

a straightforward computation shows that the amplitudes transform as

$$I_{\alpha_1 \ldots \alpha_n}(\vec{y}\,'_1, \ldots, \vec{y}\,'_n) = \Omega^{-(d-1)}(\vec{y}_1) \cdots \Omega^{-(d-1)}(\vec{y}_n) R_{\alpha_1}^{\ \beta_1}(\vec{y}_1) \cdots R_{\alpha_n}^{\ \beta_n}(\vec{y}_n) I_{\beta_1 \ldots \beta_n}(\vec{y}_1, \ldots, \vec{y}_n)\,, \tag{3.105}$$

which is indeed the transformation property of a CFT n-point function of conserved currents, see (3.25).

3.5 Conclusions

We have shown that AdS amplitudes satisfy the conformal Ward identities and we obtained explicit formulas that compute the constants and functions of cross-ratios that appear in the CFT correlators in terms of bulk quantities. These are given in (3.66), (3.74), (3.79), (3.86) for scalar n-point functions. The same analysis can be carried out for spinning operators and we worked out explicitly the case of conserved currents. The constraints of conformal invariance originate from diffeomorphsim invariance in the bulk.

Altogether these results imply that the AdS gravity is a CFT, but they do not yet imply that it is a local CFT. Local CFTs have local UV divergences, and thus the corresponding bulk IR divergences should also be local. This has been established at tree-level in [72, 44, 85, 26] and for scalar fields in AdS up to two loops in [15] and in the next chapter. In addition, the conformal anomalies should be that of a local CFT, and they are [72, 44]. One should contrast these results with the case of de Sitter, where the bulk isometries also match that of (Euclidean) CFT. The Ward identities due to de Sitter isometries also take the form of conformal Ward identities, but the IR divergences of de Sitter in-in correlators and corresponding anomalies only partially match that of a local CFT [31]. Local CFTs are further constrained by OPEs. These may be used to express 4- and higher-point functions in terms of CFT data: conformal dimensions (encoded in 2-point functions) and OPE coefficients (encoded in 3-point functions, and these should satisfy bootstrap equations. We note that the functions of cross-ratios that appear in our analysis have special dependence on some of the cross-ratios (see comments below (3.82) and (3.89)) and it would be interesting to understand the implication of this in the context of the bootstrap program.

The connection between CFT correlators and AdS amplitudes depends on the amputated bulk correlators transforming properly under bulk diffeomorphism. Such tranformation properties could be invalidated by UV and/or IR divergences. We used an AdS invariant regulator to ensure that UV issues do not cause any problems, but full details have only been worked out for scalar fields. It would be interesting to work out the regularised bulk-to-bulk propagators for general spinning field. At loop order and for gauge field the analysis would likely require to properly take into account the contribution of ghost fields. IR divergences do break (part of) the AdS isometries, but this breaking is linked to conformal anomalies and anomalous dimensions and it is a feature, not a problem. We also note any AdS covariant n-point function, irrespectively of how it is obtained would automatically yield a solution of the CFT Ward identities – here we assumed they are computed by bulk perturbation theory, but *a priori* there could be other non-perturbative constructions.

Chapter 4

Holographic renormalization at loop order

In Chapter 2 we reviewed holographic renormalization at leading, tree-level order in the AdS/CFT correspondence. In this case, one has the usual IR boundary divergences present in holography and the relevant object to construct in the bulk is the renormalized on-shell action $S_{\mathrm{AdS}}^{\mathrm{Ren}}$. At subleading, loop-level order in the correspondence one also has UV divergences in the bulk coming from quantum corrections, which given the UV/IR relation [101] are mapped to IR divergences in the boundary theory. In this case, to have a sensible dictionary at loop-level the relevant object to construct is the renormalized on-shell 1PI effective action in the bulk $\Gamma_{\mathrm{AdS}}^{\mathrm{Ren}}$, where now the boundary operators are dual not to the classical fields in the bulk but to the true minima $\delta\Gamma_{\mathrm{AdS}}^{\mathrm{Ren}}/\delta\Phi^I = 0$ of the full quantum theory on AdS. At tree-level Γ_{AdS} reduces to S_{AdS} and one recovers the standard AdS/CFT prescription. It is the purpose of this chapter then to present the construction of the holographic dictionary valid to all orders in the bulk loop expansion. We will follow the same structure as the discussion at tree-level: in Section 4.1 we present our regularization schemes to regulate the corresponding divergences of the bulk theory, while in Section 4.2 we present the counterterms needed to renormalize them. Once regulators and counterterms have been introduced, in Section 4.3 we discuss the construction of the renormalized bulk 1PI effective action, leading to the renormalized dictionary with the boundary theory in Section 4.4.

Chapter to be published as a paper in [23].

4.1 Regularization

The physics of IR and UV divergences is different and one needs to be able to distinguish between them, needing separate regulators. Dimensional regularization is often a convenient scheme, however it regulates both divergences at the same time and one would need to devise a way to separate the two before it can be meaningfully used at loop order in AdS. Of course, if only one type of divergence is present at a given diagram then dimensional regularization can be used unambiguously. Here we aim for a setup that is always valid. We shall use the same IR regulator ε introduced in the analysis at tree-level in 2.1.1, while the UV regulator is an AdS invariant point-splitting method, which we call geodesic point-splitting. This UV regularization prescription was first introduced in [21, 22], and we showed in [15] that it can be understood precisely as an AdS invariant point-splitting.

4.1.1 UV regulator

Ultraviolet divergences arise from short-distance singularities due to quantum corrections in AdS and thus they become relevant in the AdS/CFT correspondence at subleading order in the $1/N^2$ expansion. As the short-distance behavior should be independent of long-distance properties, one expects these divergences to be similar to the corresponding ones in flat space.

The UV-IR connection implies that the bulk UV divergences are mapped to IR divergences in the dual CFT[1]. Now, IR divergences in CFTs do not spoil the conformal invariance of the theory, and conformal invariance in AdS/CFT follows from AdS covariance in the bulk. This suggests one should be able to regularize the UV divergences in the bulk in a way that preserves the AdS symmetry. Here we will show this is indeed possible by explicitly constructing one such regularization scheme: geodesic point-splitting. The bulk-to-bulk propagator $G_\Delta(x_1, x_2)$ is a function of the invariant (chordal) distance $u(x_1, x_2)$ between the 2 points, and UV divergences come from the coincident point $x_1 = x_2$, where $u = 0$. The idea of geodesic point-splitting consists in replacing one of the 2 points in u by a geodesic parameterized by its (Euclidean) proper time τ that passes through it at $\tau = 0$: $x_2 \to x_2(\tau),\ x_2(0) = x_2$. As long as $\tau \neq 0$, then $u(x_2, x_2(\tau)) \neq 0$ and short-distance singularities are effectively regularized, with τ acting as the regulator. As we will see, from all possible geodesics that pass through the point x_2, there is a subset that precisely leaves $u(x_1, x_2(\tau))$ invariant under simultaneous isometry transformations of both points x_1 and x_2,

[1]Originally, the UV-IR connection [101] related UV divergences of the boundary theory with near-boundary IR divergences in the bulk. However, given the relation between the AdS radial coordinate and the RG scale of the dual QFT and the monotonicity properties of RG flows, a UV-IR connection must also hold in the opposite direction with UV in the bulk and IR in the boundary.

preserving the AdS covariance in the bulk and hence the conformal invariance at the boundary.

To discuss the UV regulator in more detail we need AdS geodesics. These can be derived from the Lagrangian

$$L = \frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{1}{z}\sqrt{\left(\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau}\right)^2}\,, \tag{4.1}$$

with $z, \vec{x}$ the Poincaré coordinates, and τ the proper time as the affine parameter along the geodesics. The resulting equation for $\vec{x}$ can be recast in the form

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} = \frac{\vec{A}z}{\sqrt{1-A^2z^2}}\left|\frac{\mathrm{d}z}{\mathrm{d}\tau}\right|\,, \tag{4.2}$$

where $\vec{A}$ is an integration constant. Integrating it directly between $\tau = 0$ and some $\tau > 0$

$$\vec{x} - \vec{x}_0 = \pm\frac{\vec{A}}{A^2}\left(\sqrt{1-A^2z_0^2} - \sqrt{1-A^2z^2}\right)\,, \tag{4.3}$$

where $(z_0, \vec{x}_0)$ is the position at $\tau = 0$, and the upper and lower signs correspond to the cases $z > z_0$ and $z < z_0$, respectively. The expression for z as a function of τ is more easily derived from the line element, leading to the equation

$$\frac{1}{z\sqrt{1-A^2z^2}}\left|\frac{\mathrm{d}z}{\mathrm{d}\tau}\right| = 1\,, \tag{4.4}$$

which integrates to

$$z = \frac{z_0}{\cosh\tau \mp \sinh\tau\sqrt{1-A^2z_0^2}}\,. \tag{4.5}$$

Replacing this result back in the expression for $\vec{x}$ we obtain,

$$\vec{x} = \vec{x}_0 + \frac{z_0^2\sinh\tau\vec{A}}{\cosh\tau \mp \sinh\tau\sqrt{1-A^2z_0^2}}\,. \tag{4.6}$$

Equations (4.5) and (4.6) constitute an infinite family of geodesics $x = (z, \vec{x})$ in AdS parameterized by τ, one for each value and direction of $\vec{A}$, that pass through the point $x_0 = (z_0, \vec{x}_0)$ at a proper time $\tau = 0$. Geodesic point-splitting consists in replacing one of the points in $u(x_1, x_2) = (x_1 - x_2)^2/(2z_1z_2)$, say x_2, by such geodesics resulting in the expression

$$\begin{aligned} u\left(x_1, x_2(\tau)\right) = & -1 + \cosh\tau\left[1 + u(x_1,x_2)\right] \\ & - \frac{\sinh\tau}{2z_1z_2}\left(2z_2^2\vec{A}\cdot(\vec{x}_1 - \vec{x}_2) \pm \sqrt{1-A^2z_2^2}\left[z_1^2 - z_2^2 + (\vec{x}_1-\vec{x}_2)^2\right]\right)\,. \end{aligned} \tag{4.7}$$

The first two terms are invariant under AdS isometries, however in general the last term breaks AdS invariance for arbitrary geodesics connecting the points x_2 and $x_2(\tau)$, encoded in this term by $\vec{A}$. A sufficient case where the invariance of the chordal distance u is preserved is for the subset of geodesics for which this last term vanishes. This is achieved for the special value

$$\vec{A} = \mp \frac{\text{sign}\left[\hat{n}\cdot(\vec{x}_1-\vec{x}_2)\right]\left[z_1^2 - z_2^2 + (\vec{x}_1-\vec{x}_2)^2\right]}{\sqrt{\left[z_1^2 - z_2^2 + (\vec{x}_1-\vec{x}_2)^2\right]^2 + 4z_2^2\left[\hat{n}\cdot(\vec{x}_1-\vec{x}_2)\right]^2}}\,\frac{\hat{n}}{z_2}\,, \tag{4.8}$$

where $\hat{n}$ is a unit vector pointing in the direction of $\vec{x}_2(\tau) - \vec{x}_2$. Note that this value of $\vec{A}$ still represents infinitely many possible geodesics, namely one for each direction $\hat{n}$ which can be arbitrarily chosen. For $d > 1$, a simple choice is where the regularized position $x_2(\tau)$ is placed perpendicular to the line of $\vec{x}_1$ and $\vec{x}_2$: $\hat{n}\cdot(\vec{x}_1-\vec{x}_2) = 0$, resulting in the geodesic

$$\vec{A} = \frac{\hat{n}}{z_2}\,, \qquad x_2(\tau) = \left(\frac{z_2}{\cosh\tau},\, \vec{x}_2 + z_2\tanh\tau\,\hat{n}\right)\,. \tag{4.9}$$

The class of geodesics described by (4.8) allows us then to define a regularized version of the chordal distance

$$u_\tau(x_1,x_2) \equiv u\left(x_1, x_2(\tau)\right) = -1 + \cosh\tau\left[1 + u(x_1,x_2)\right]\,, \tag{4.10}$$

which manifestly preserves its invariance under AdS isometries. At coincident points: $u_\tau(x_2,x_2) = -1+\cosh\tau > 0$, and the regularized distance is strictly greater than 0, as expected. One might be worried that u_τ may vanish for non-trivial values of x_1, x_2, τ producing a singularity elsewhere. However, solving for $u_\tau = 0$ one finds the condition $1 + u = 1/\cosh\tau$. Since $u \geq 0$ and $\cosh\tau > 1$ for $\tau > 0$, the LHS is equal or greater than 1 while the RHS is always less than 1, implying the condition is never met. The only possibility to have $u_\tau = 0$ is when $\tau = 0$, corresponding simply to the original singularity at $u = 0$.

Geodesic point-splitting further allows us to define a regularized bulk-to-bulk propagator. The propagator being a function of $u(x_1,x_2)$ can be regularized by replacing this quantity by its regularized version $u_\tau(x_1,x_2)$, a procedure that is valid at the level of perturbation theory and Witten diagrams. In terms of $\xi = 1/(1+u)$, the regularized propagator becomes a function of $\xi_\tau \equiv 1/(1+u_\tau) = \xi/\cosh\tau$. We define then the regularized version of $G_\Delta(\xi)$ by

$$G_{\tau,\Delta}(\xi) \equiv G_\Delta(\xi_\tau)\,, \qquad \xi_\tau = \frac{\xi}{\cosh\tau}\,. \tag{4.11}$$

Divergences come from the coincident point $\xi = 1$. For $\tau > 0$, ξ_τ is strictly less than 1 and short-distance singularities coming from the bulk propagator are regularized in an AdS invariant way. For small τ the regulated ξ_τ becomes $\xi_\tau = \xi/(1+\epsilon)$ with

$\epsilon = \tau^2/2$, and we recover the scheme used in [21, 22]. This discussion suffices for perturbative bulk computations: one can check (and we will do so explicitly later on) that using regularized propagators is sufficient to regulate loop diagrams.

The derivation we presented here makes manifest that this is a bulk UV regulator. We will now present a different derivation that also makes manifest its connection to a boundary IR regulator for holographic CFTs. Note that formulas (4.5) and (4.6) for geodesics can be also be seen to define a coordinate system in AdS, with the bulk point $(z, \vec{x})$ expressed in the variables $(\tau, \vec{A})$ around the reference point $(z_0, \vec{x}_0)$. In other words, we label the spacetime points by the geodesic that connects them to $(z_0, \vec{x}_0)$ (labeled by $\vec{A}$) and the affine parameter it takes to go to this point along this geodesic. We will refer to these coordinates as geodesic coordinates. Introducing ξ and ψ via $\cosh\tau = 1/\xi$ and $Az_0 = \sin\psi$ with $\xi \in [0,1]$ and $\psi \in [0,\pi]$, we can relate the original Poincaré coordinates $(z, \vec{x})$ to the new coordinates $(\xi, \psi, \hat{n})$ via

$$z = \frac{z_0 \xi}{1 \mp |\cos\psi| \sqrt{1-\xi^2}}\,, \tag{4.12}$$

$$\vec{x} = \vec{x}_0 + \frac{z_0 \sin\psi \sqrt{1-\xi^2}}{1 \mp |\cos\psi| \sqrt{1-\xi^2}} \hat{n}\,, \tag{4.13}$$

with the upper and lower signs for $z > z_0$ and $z < z_0$, respectively. One can invert these relations to express (ξ, ψ) in terms of $(z, |\vec{x}|)$

$$\xi = \frac{2 z z_0}{z^2 + z_0^2 + (\vec{x} - \vec{x}_0)^2}\,, \tag{4.14}$$

$$|\tan\psi| = \pm \frac{2 z_0 |\vec{x} - \vec{x}_0|}{z^2 - z_0^2 + (\vec{x} - \vec{x}_0)^2}\,. \tag{4.15}$$

It follows from (4.14) that ξ is the chordal distance between the reference point x_0 and the point x.

In geodesic coordinates and by writing $\hat{n}$ in spherical coordinates, the AdS line element takes the simple form

$$\mathrm{d}s^2 = \frac{\mathrm{d}\xi^2}{\xi^2(1-\xi^2)} + \frac{(1-\xi^2)}{\xi^2} \left(\mathrm{d}\psi^2 + \sin^2\psi \, \mathrm{d}\Omega_{d-1}^2 \right)\,, \tag{4.16}$$

$$= \frac{\mathrm{d}\xi^2}{\xi^2(1-\xi^2)} + \frac{(1-\xi^2)}{\xi^2} \mathrm{d}\Omega_d^2\,, \tag{4.17}$$

independent of the reference point $x_0 = (z_0, \vec{x}_0)$, as expected for a maximally symmetric space. This metric can be seen to parametrize AdS space as a foliation of d-spheres, where ξ acts as the bulk radial direction with the conformal boundary at $\xi = 0$ and the center of AdS at $\xi = 1$. From (4.14), one can see that these two regions correspond to taking the point x_0 as our center, with the conformal boundary reached once one is infinitely far from it. Thus, the independence of the metric on x_0 is equivalent to the statement that on AdS taking any point as its center is a valid choice. Note that the metric can be brought into a more familiar form by reparametrizing the radial coordinate by $\xi = 4\rho/(4+\rho^2)$, resulting in the line element

$$\mathrm{d}s^2 = \frac{1}{\rho^2}\left[\mathrm{d}\rho^2 + \left(1-\frac{\rho^2}{4}\right)^2 \mathrm{d}\Omega_d^2\right] . \tag{4.18}$$

This is a standard representation of AdS in a Fefferman-Graham coordinate system when the boundary conformal structure is represented by the standard metric on a unit d-sphere.

The Laplacian in geodesic coordinates takes the form

$$\Box = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) = \Box_\xi + \frac{\xi^2}{1-\xi^2}\Box_{S^d} , \tag{4.19}$$

with $\Box_{S^d}$ the angular part constructed from the metric of a unit d-sphere, and

$$\Box_\xi = \xi^2(1-\xi^2)\partial_\xi^2 + (1-d-2\xi^2)\xi\partial_\xi . \tag{4.20}$$

We are interested in the bulk-to-bulk propagator. It corresponds to the Green's function of the wave operator $-\Box + m^2$ in AdS, which in geodesic coordinates can be formulated as

$$\left(-\Box_\xi - \frac{\xi^2}{1-\xi^2}\Box_{S^d} + m^2\right) G_\Delta(\xi,\vec{\Omega};\xi',\vec{\Omega}') = \frac{1}{\sqrt{g}}\delta(\xi-\xi')\delta(\vec{\Omega}-\vec{\Omega}') , \tag{4.21}$$

where $\vec{\Omega}$ are the angular coordinates on the unit d-sphere. In principle, one can solve this equation by expanding G_Δ in eigenfunctions of $\Box_{S^d}$ (spherical harmonics) and dealing with the resulting radial equation for ξ, however one can simplify the calculation by exploiting the fact that the propagator is an AdS bi-scalar in $(\xi,\vec{\Omega})$ and $(\xi',\vec{\Omega}')$, and as explained above ξ itself is the bi-scalar $\xi(x,x_0)$ (chordal distance) when expressed in Poincaré coordinates. Since there is only one independent bi-scalar function of two points, this implies one can always write the propagator as $G_\Delta(\xi(x,x_0))$, independent of $\vec{\Omega}(x,x_0)$ (which is not a bi-scalar, otherwise it would be expressible in terms of ξ). Thus, the analysis of the Green's equation is simplified by changing variables to x, x_0, allowing us to completely drop the angular part of the Laplacian and focusing only in its spherically symmetric piece

$$(-\Box_\xi + m^2)G_\Delta(\xi(x,x_0)) = \frac{1}{\sqrt{g}}\delta(x-x_0) . \tag{4.22}$$

A complete derivation of the propagator from this equation is given in appendix A.1. This discussion parallels that of the derivation of the Green's function in flat space and spherical coordinates, since the radial coordinate r also happens to be the Euclidean bi-scalar $r(\vec{x},\vec{x}_0) = |\vec{x}-\vec{x}_0|$ of flat space, and thus one can express the propagator solely in terms of this variable, with the equation for $G(r(\vec{x},\vec{x}_0))$ corresponding only to the radial part of the full Laplacian.

The regularized bulk-to-bulk propagator is a function of $\xi_\tau = \xi/\cosh\tau$. Having spelled the equation for the bare propagator in terms of ξ we can now present a differential equation whose solution is the regularized propagator. Indeed, one can proceed by replacing ξ by ξ_τ in (4.22)

$$(-\Box_\xi + m^2) \;\to\; c(\xi,\tau)(-\Box_{\xi_\tau} + m^2)\,, \tag{4.23}$$

with $\Box_{\xi_\tau}$ corresponding to (4.20) with $\xi \to \xi_\tau$, and $c(\xi,\tau)$ a factor (to be determined shortly) which should be regular in $0 < \xi < 1$. Demanding Hermicity of the regularized operator completely fixes the ξ-dependence of this factor up to a constant

$$c(\xi,\tau) = c(\tau)\sqrt{\frac{g_{\xi_\tau}}{g_\xi}}\,, \qquad \sqrt{g_\xi} = \frac{(1-\xi^2)^{\frac{d-1}{2}}}{\xi^{d+1}}\,, \tag{4.24}$$

where $\sqrt{g_\xi}$ is the radial part of the determinant $\sqrt{g}$ in geodesic coordinates. Further imposing $c(0) = 1$ for the remaining constant ensures the deformed Green's equation reduces to the original one for vanishing regulator τ, with its solutions continuously connected to $G_\Delta(\xi)$. The regularized equation one is interested then is

$$c(\tau)\sqrt{\frac{g_{\xi_\tau}}{g_\xi}}(-\Box_{\xi_\tau} + m^2)G_{\tau,\Delta}(\xi(x,x_0)) = \frac{1}{\sqrt{g}}\delta(x-x_0)\,, \qquad c(0) = 1\,. \tag{4.25}$$

The solution to this equation has been worked out in complete detail in A.2. By construction, it is precisely proportional to $G_\Delta(\xi_\tau)$ with its normalization constant dependent on the value of $c(\tau)$, with the case $G_{\tau,\Delta}(\xi) = G_\Delta(\xi_\tau)$ corresponding to the special value given in (A.38).

The defining equation (4.25) for the regularized bulk-to-bulk propagator can now be understood as coming from the following kinetic term in Poincaré coordinates

$$\int \mathrm{d}^{d+1}x\sqrt{g}\;\Phi\, c(\tau)\sqrt{\frac{g_{\xi_\tau}}{g_\xi}}(-\Box_{\xi_\tau} + m^2)\Phi\,. \tag{4.26}$$

Note that if one writes this term in geodesic coordinates and after a rescaling $\xi \to \xi\cosh\tau$ (and ignoring unimportant factors of τ) one obtains

$$\int \mathrm{d}\Omega_d \int_0^{1/\cosh\tau} \mathrm{d}\xi\sqrt{g_\xi}\;\Phi(-\Box_\xi + m^2)\Phi\,, \tag{4.27}$$

recovering the standard kinetic term for the scalar field, with all dependence on the regulator moved to the upper limit of the radial integral. Thus, similar to the IR regulator ε, the UV regulator τ can also be thought of as a cut-off but now cutting off the deep-interior of AdS. In holography, the deep interior of AdS corresponds to the IR of the CFT, so τ acts as an IR regulator for the dual theory at the boundary.

4.2 Counterterms

The bulk theory is a function of the bare sources $\varphi^I_{B(0)}(\vec{x})$ that parametrize the boundary conditions for the fields $\Phi^I(x)$, and a number of bare couplings p^i_B that describe their masses and interactions. As discussed in the previous section, this theory will in general be divergent and we will regulate the infrared divergences by adding a hard cut-off to the bulk radial direction while for ultraviolet divergences we will adopt a geodesic point-splitting scheme, introducing the IR and UV regulators ε and τ respectively. At classical level, a boundary counterterm $B[\Phi^I; p^i_B; \varepsilon]$ located at the regulated surface $z = \varepsilon$ suffices to absorb all divergences of the bulk theory, however at loop order where one also has divergences in the deep interior of AdS this is no longer the case. As usual for QFTs, we will absorb these by also introducing Z-factors for the bare bulk parameters

$$\varphi^I_{B(0)}(\vec{x}) = Z^I{}_J(\varepsilon, \tau)\varphi^J_{(0)}(\vec{x}) , \qquad p^i_B = Z^i(\varepsilon, \tau)p^i , \tag{4.28}$$

which, in principle, may depend on both regulators. Note that there is no summation over the i index, but there is summation over the J index allowing for operator mixing. In fact, already at tree-level one needs to consider such source mixing to renormalize a certain class of 3- and higher-point functions [26]. Source counterterms are also expected for bulk fields receiving mass renormalization: since bulk masses are directly related to conformal dimensions at the boundary, sources of operators must renormalize to account for the corrected dimensions, implying (as we will see) the proper ε-dependence for the factors $Z^I{}_J$.

4.3 Renormalized 1PI effective action

Having introduced regulators and counterterms, the next step is to obtain the renormalized on-shell 1PI effective action in the bulk, and consequently the exact holographic 1-point functions. The theory on AdS is often given in terms of a bare action $S_{\text{AdS}}[\Phi^I; p^i_B]$ depending on the off-shell fields Φ^I and their bare couplings p^i_B, which in turn defines the (Euclidean) gravitational path integral

$$Z_{\text{AdS}}[\varphi^I_{B(0)}; p^i_B] = \int_{\Phi^I \sim \varphi^I_{B(0)}} \mathrm{D}\Phi^I \, e^{-S_{\text{AdS}}[\Phi^I; p^i_B]} , \tag{4.29}$$

as a functional on the boundary conditions $\varphi^I_{B(0)}$ for the bulk fields. When bulk sources J_I for the fields Φ^I are present, one defines the generating functional of connected graphs $W_{\text{AdS}}[J_I] = -\ln Z_{\text{AdS}}[J_I]$, and the effective action is constructed from this object through a Legendre transform

$$\Gamma_{\text{AdS}}[\Phi^I] = W_{\text{AdS}}[J_I] - \int \mathrm{d}^{d+1}x\sqrt{g}\, J_I(x)\Phi^I(x)\,, \tag{4.30}$$

trading the variable J_I for Φ^I. Since $W_{\text{AdS}}[J_I]$ is independent of Φ^I and $\Gamma_{\text{AdS}}[\Phi^I]$ independent of J_I, one has the variations

$$\frac{\delta W_{\text{AdS}}[J_I]}{\delta J_I} = \Phi^I\,, \qquad \frac{\delta \Gamma_{\text{AdS}}[\Phi^I]}{\delta \Phi^I} = -J_I\,. \tag{4.31}$$

In the weak coupling regime of the bulk theory where W_{AdS} may be computed perturbatively, the first equation above results in the implicit relation $\Phi^I = \Phi^I[J_I]$ which inverted to have instead $J_I = J_I[\Phi^I]$ and replaced in (4.30), allows for the perturbative construction of Γ_{AdS} as a function only on Φ^I in terms of loop corrections around the classical action S_{AdS}. By having set $\hbar = 1$, these corrections are then controlled by the bulk interacting couplings which in the weak regime are taken to be small. This leads to their identification with the rank N of the gauge group of the boundary theory, with the bulk loop corrections corresponding to $1/N^2$ corrections in the dual CFT.

In AdS/CFT, the correspondence involves the partition function in the presence of non-trivial boundary condition but no sources turned on, as in (4.29). We can still proceed to compute the effective action by adding sources to Z_{AdS} and follow the steps above, but at the end we should set the sources to zero. One ends up with an object $\Gamma_{\text{AdS}}[\Phi^I]$ in terms of now on-shell fields Φ^I that satisfy the equations of motion

$$\frac{\delta \Gamma_{\text{AdS}}}{\delta \Phi^I} = 0\,, \tag{4.32}$$

where this equation comes from setting $J_I = 0$ in (4.31). This is reminiscent of the tree-level holographic prescription, with Γ_{AdS} replaced by S_{AdS} and the fields satisfying instead the classical equations of motion. Of course, at tree-level $\Gamma_{\text{AdS}} = S_{\text{AdS}}$, and one recovers the standard AdS/CFT prescription. We now find however that the same prescription is valid to all orders in the bulk loop expansion provided we replace the classical action by the effective action[2].

Equivalently, one can construct the effective action directly from (4.29) without the addition of bulk sources using the method of background fields. This method consists in expanding the bulk fields Φ^I around on-shell solutions ϕ^I and their quantum fluctuations h^I: $\Phi^I = \phi^I + h^I$. In principle, any such expansion is valid (the usual

[2]There is a corresponding discussion in flat space: the effective action contains the flat-space S-matrix in its on-shell expansion [63, 75].

choice being taking the ϕ^I as the minima of the classical theory: $\delta S_{\text{AdS}}/\delta\phi^I = 0$) with each choice involving a different resummation but leading at the end to the same results, however this resummation is simplest when the bulk fields are expanded not around the classical minima but the true minima of the full theory: $\delta\Gamma_{\text{AdS}}/\delta\phi^I = 0$. This choice also has the advantage of allowing the direct identification of the background fields ϕ^I with the on-shell fields (4.32) obtained from the Legendre transform method. In this decomposition, the ϕ^I only appear as external lines with the internal loops run by the fluctuations h^I. Therefore, in this expansion the boundary conditions $\varphi^I_{B(0)}$ for the bulk fields Φ^I are carried by the background fields ϕ^I. All this can then be summarized as the change of variables

$$\Phi^I = \phi^I[\varphi^I_{B(0)}] + h^I\,, \qquad \frac{\delta\Gamma_{\text{AdS}}}{\delta\phi^I} = 0\,. \tag{4.33}$$

Performing this change of variables to Z_{AdS} in (4.29), one integrates out h^I perturbatively in the small bulk couplings and the effective action is identified from the resulting normalized path integral restricted to 1PI terms in ϕ^I [3]

$$e^{-\Gamma_{\text{AdS}}[\varphi^I_{B(0)};p^i_B]} = \frac{Z_{\text{AdS}}[\varphi^I_{B(0)};p^i_B]}{Z_{\text{AdS}}[0;p^i_B]}\Bigg|_{\text{1PI}}\,. \tag{4.34}$$

The bare 1PI effective action, derived either from the Legendre transform or the background field method, is a formal quantity, as it suffers from both infrared and ultraviolet divergences. Regularizing (for example, as discussed in Section 4.1) results in the regularized effective action $\Gamma^{\text{Reg}}_{\text{AdS}}[\varphi^I_{B(0)};p^i_B;\varepsilon,\tau]$ whose divergences are canceled by introducing boundary counterterms and Z-factors for the bare bulk parameters $\varphi^I_{B(0)}$ and p^i_B, as discussed in 4.2, leading to the subtracted effective action,

$$\Gamma^{\text{Sub}}_{\text{AdS}}[\varphi^I_{(0)};p^i+\Pi^i;F;\varepsilon,\tau] = \Gamma^{\text{Reg}}_{\text{AdS}}[Z^I{}_J\varphi^J_{(0)};Z^ip^i;\varepsilon,\tau] + B[\Phi^I;Z^ip^i;\varepsilon,\tau]\,, \tag{4.35}$$

where the boundary counterterms B should be expressed covariantly in terms of the on-shell fields Φ^I and the induced metric at the regulated surface $z=\varepsilon$. Here Π^i on the LHS denotes the finite contributions that the bulk couplings p^i receive from the loop corrections, after the subtraction of all divergent terms has been made. This subtraction comes with a set of arbitrary constants F that parametrize its scheme-dependence, capturing the fact that subtractions in different schemes may differ by finite pieces. To fix the scheme-dependent constants F we need to supply renormalization conditions. These could be provided either by comparing the same observable with a string theory computation or via the AdS/CFT by comparing with

[3] The background field method often involves terms linear in the quantum fluctuations h^I, which after integrated lead to non-vanishing contributions to the 1-point functions (tadpoles), and for higher-point functions in non-1PI terms. Since in the derivation of the effective action one is instructed to restrict the path integral only to 1PI contributions, these terms linear in h^I can be simply discarded or made to vanish exactly by defining a modified action with the tadpoles subtracted: $S_{\text{AdS}}[\Phi^I] \to S_{\text{AdS}}[\Phi^I] - h^I\delta S_{\text{AdS}}[\phi^I]/\delta\phi^I$.

the dual CFT. In the absence of reliable renormalization conditions, one may consider combinations of observables that are scheme independent. After this process has been carried out, the renormalized on-shell 1PI effective action is finally obtained by taking the limit of vanishing regulators

$$\Gamma^{\text{Ren}}_{\text{AdS}}[\varphi^I_{\mathcal{R}(0)}; p^i_{\mathcal{R}}] = \lim_{\varepsilon\to 0}\lim_{\tau\to 0}\Gamma^{\text{Sub}}_{\text{AdS}}[\varphi^I_{(0)}; p^i + \Pi^i; F; \varepsilon, \tau]\,, \tag{4.36}$$

in terms of renormalized sources $\varphi^I_{\mathcal{R}(0)}$ and renormalized bulk couplings $p^i_{\mathcal{R}}$. Note the order in which the limits are taken, first taking $\tau \to 0$ with ε fixed, and at the end taking $\varepsilon \to 0$, implying that one should be able to resolve the UV divergences in the presence of the regulator ε, leaving the IR divergences for last. In general, dealing first with the UV divergences through the bulk Z-factors allows for the construction of an (effective) action in terms of (renormalized) sources and bulk couplings, analogous to the action in the classical case. With the $\tau \to 0$ limit taken, this leads to the correct identification of the asymptotic expansion of the bulk fields, with the remaining IR divergences being the usual boundary divergences present in holography dealt by the boundary counterterm B, after which the last limit $\varepsilon \to 0$ may be evaluated.

4.4 Exact 1-point functions

The standard prescription $W_{\text{CFT}} = S^{\text{Ren}}_{\text{AdS}}$ is modified at loop order to

$$W_{\text{CFT}}[\varphi^I_{\mathcal{R}(0)}; \Delta^I_{\mathcal{R}}, C^{IJK}_{\mathcal{R}}] = \Gamma^{\text{Ren}}_{\text{AdS}}[\varphi^I_{\mathcal{R}(0)}; p^i_{\mathcal{R}}]\,, \tag{4.37}$$

with the CFT data that defines the boundary theory (renormalized dimensions $\Delta^I_{\mathcal{R}}$ and renormalized OPE coefficients $C^{IJK}_{\mathcal{R}}$) expressed in terms of the renormalized bulk data $p^i_{\mathcal{R}}$. Correlation functions of boundary operators $\mathcal{O}_{\Delta^I_{\mathcal{R}}}$ dual to the bulk fields Φ^I, are computed by functionally differentiating $\Gamma^{\text{Ren}}_{\text{AdS}}$ with respect to the renormalized sources $\varphi^I_{\mathcal{R}(0)}$. The variation of $\Gamma^{\text{Ren}}_{\text{AdS}}$ consists of two terms, a bulk term that vanishes given the equations of motion for the fields Φ^I, and a boundary term expressible in terms of the variations $\delta\varphi^I_{\mathcal{R}(0)}$. Correlators come from the latter, giving rise to the exact holographic 1-point functions in the presence of sources

$$\langle \mathcal{O}_{\Delta^I_{\mathcal{R}}}(\vec{x})\rangle_{\varphi^I_{\mathcal{R}(0)}} = \frac{-1}{\sqrt{g_{(0)}}}\frac{\delta\Gamma^{\text{Ren}}_{\text{AdS}}[\varphi^I_{\mathcal{R}(0)}; p^i_{\mathcal{R}}]}{\delta\varphi^I_{\mathcal{R}(0)}(\vec{x})}\,, \tag{4.38}$$

with $g_{(0)ij}$ the boundary metric.

To illustrate the method of holographic renormalization at loop order, in Chapter 6 we work out in full detail the renormalization of an interacting scalar field with Dirichlet boundary condition on a fixed AdS background. In this case, once the bulk theory has been properly renormalized, the scalar field has an asymptotic expansion of the form

$$\Phi(x) = z^{d-\Delta_{\mathcal{R}}}\varphi_{\mathcal{R}(0)}(\vec{x}) + \cdots + z^{\Delta_{\mathcal{R}}}\varphi_{(2\nu_{\mathcal{R}})}(\vec{x}) + \cdots \,, \tag{4.39}$$

where $\varphi_{(2\nu_{\mathcal{R}})}$ is the VEV term containing the non-trivial information to loop order about the boundary correlators. The tree-level mass and coupling m^2, λ of the field have been renormalized to the values $m^2_{\mathcal{R}}, \lambda_{\mathcal{R}}$, with the conformal dimension $\Delta_{\mathcal{R}}$ written in terms of the renormalized mass: $\Delta_{\mathcal{R}}(\Delta_{\mathcal{R}} - d) = m^2_{\mathcal{R}}$. The transformation properties of $\varphi_{\mathcal{R}(0)}$ and $\varphi_{(2\nu_{\mathcal{R}})}$ under conformal transformations are precisely those of a source and VEV for an operator of dimension $\Delta_{\mathcal{R}}$. In this example, the variation of the renormalized effective action is found to be

$$\delta\Gamma^{\text{Ren}}_{\text{AdS}}[\varphi_{\mathcal{R}(0)}; m^2_{\mathcal{R}}, \lambda_{\mathcal{R}}] = -\int \mathrm{d}^d x \; 2\nu_{\mathcal{R}}\varphi_{(2\nu_{\mathcal{R}})}(\vec{x})\delta\varphi_{\mathcal{R}(0)}(\vec{x})\,, \tag{4.40}$$

leading to the exact 1-point function

$$\langle \mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{x})\rangle_{\varphi_{\mathcal{R}(0)}} = 2\nu_{\mathcal{R}}\varphi_{(2\nu_{\mathcal{R}})}(\vec{x})\,, \tag{4.41}$$

up to local terms in $\varphi_{\mathcal{R}(0)}$ when $\nu_{\mathcal{R}} = \Delta_{\mathcal{R}} - d/2 \in \mathbb{N}$. The explicit construction of the exact 1-point functions requires working out a number of loop integrals on AdS. These will be analyzed next in Chapter 5, before moving to the concrete example of a scalar Φ^4 theory in Chapter 6.

Chapter 5

Integrals

Since the early days the computation of amplitudes in AdS, known as Witten diagrams, have been limited mostly to tree-level graphs, originally by brute force calculations in position space [60, 62, 48, 46, 45, 47] and with more recent techniques including momentum space, Mellin space, spectral/split representations, embedding and ambient formalisms, and the conformal bootstrap, with each approach exhibiting different but important features of the amplitudes. For a non-exhaustive list of the more recent treatments of tree-level Witten diagrams, we refer the reader to [3, 4, 2, 91, 92, 26, 30, 29, 87, 57, 95, 96, 86, 82, 94, 37, 40, 41, 99, 19, 20, 7, 68, 36, 74, 77, 10]. In the last decade or so, there have been more attempts to understand Witten diagrams at loop order using these different methods (see e.g. [3, 39, 38, 22, 21, 15, 16, 33, 34, 35, 104, 105, 32, 1, 6, 13, 12, 9, 8, 11, 67]), however the progress in this direction has been gradual and one of the main reasons is simply technical: the expressions for the AdS propagators are complicated, having to deal with hard integrals. In this chapter we go back to those earlier approaches and make progress in the computation of loop integrals in AdS directly in position space. We begin by discussing their convergence in Section 5.1 determining when to expect IR and UV divergences, while in Section 5.2 before moving to their computation we briefly discuss a class of convergent integrals relevant for the resummation of the mass-shift diagrams. Then, in Section 5.3 we study the IR divergent integrals responsible for the anomalous dimensions in the boundary theory, to then in Section 5.4 move to the computation of the UV divergent bulk loop integrals directly in position space. We end this section by illustrating our methodology, working out a number of concrete examples in Section 5.5.

Chapter to be published as a paper in [23].

5.1 Convergence

Many bulk loop diagrams are constructed from vertices of the schematic form

$$\int \mathrm{d}^{d+1}x_2 \sqrt{g_2}\, G_\Delta^N(x_1,x_2) K_{\Delta_1}(x_2,\vec{y}_1)\cdots K_{\Delta_n}(x_2,\vec{y}_n)\,, \tag{5.1}$$

consisting of N internal legs attached to some other point x_1 in the interior of AdS,

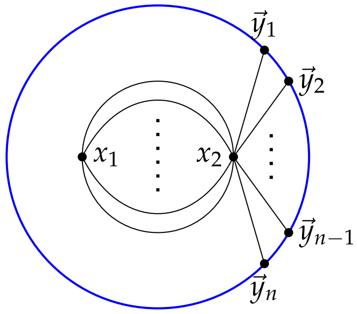

FIGURE 5.1: General vertex integral of N internal legs in the bulk, and n external legs extending to the boundary of AdS.

and n external legs extending to the boundary points $\vec{y}_i$ (see Fig. 5.1). For such general vertex, we would like to know when to expect infinities. Infrared (IR) divergences can arise as the integral approaches the conformal boundary $z_2 = 0$, while ultraviolet (UV) divergences are expected at the coincident point $x_1 = x_2$. Both type of divergences can be studied with simple power-counting in the appropriate variable. In the case of IR, this is of course the radial coordinate z_2, where each one of the elements in (5.1) has an expansion near $z_2 = 0$ of the form

$$\sqrt{g_2} = z_2^{-d-1}\,, \qquad G_\Delta(x_1,x_2) \sim z_2^{\Delta}\,, \qquad K_{\Delta_i}(x_2,\vec{y}_i) \sim z_2^{d-\Delta_i}(\text{local}) + z_2^{\Delta_i}(\text{non-local})\,. \tag{5.2}$$

For $n = 0$, that is no external legs, the integrand goes as

$$z_2 \to 0: \qquad \sqrt{g_2}\, G_\Delta^N(x_1,x_2) \sim z_2^{N\Delta-d-1}\,, \tag{5.3}$$

and thus the vertex of N bulk-to-bulk propagators between the points x_1 and x_2 is IR convergent as long as $N\Delta - d > 0$. At loop level where one has $N = 2, 3, \ldots$, this condition is always satisfied by bulk propagators with Dirichlet boundary conditions ($2\Delta - d > 0$), hence in this case the vertex is IR finite for any (positive) values of N, Δ and d. However, in the case of propagators with Neumann boundary conditions ($2\Delta - d < 0$), this condition is not always met and thus additional infinities might be expected. For instance, when $N = 2$ the vertex is always IR divergent, while for $N = 3$ it will only converge in the IR if $\Delta > d - 2\Delta$.

For $n \neq 0$, the integrand of (5.1) goes instead as

$$z_2 \to 0: \quad \sqrt{g_2}\, G_\Delta^N(x_1,x_2) K_{\Delta_1}(x_2,\vec{y}_1)\cdots K_{\Delta_n}(x_2,\vec{y}_n) \sim z_2^{N\Delta-\Delta_T+d(n-1)-1}(\text{local}) + \sum_i z_2^{N\Delta+\Delta_T-2\Delta_i-1}(\text{non-local})\,, \tag{5.4}$$

where $\Delta_T \equiv \sum \Delta_i$. Local terms are in general IR divergent, but dealt with boundary counterterms through holographic renormalization. We will be more interested in the non-local terms which survive this process and contribute to the holographic correlators. Given the exponent of z_2, we expect these to be IR convergent as long as $N\Delta + \Delta_T - 2\Delta_i > 0$, for all external legs dimensions Δ_i. A relevant case is when the dimensions of all internal and external legs composing the vertex are equal: $\Delta_i = \Delta$. When this is the case, one can factor out the Δ in the convergence condition reducing to $N + n - 2 > 0$, $n \neq 0$. The 2 divergent cases of interest are $N = 0$, $n = 2$, and $N = n = 1$, corresponding to the $\int K_\Delta K_\Delta$ and $\int G_\Delta K_\Delta$ integrals. These will be analyzed in detail in 5.3.

Moving now to the UV, the appropriate variable to perform power-counting is in the chordal distance $\xi(x_1, x_2)$. It is convenient then to recast the general vertex (5.1) in geodesic coordinates (defined in (4.12)), where precisely ξ acts as one of the coordinates identified with the bulk radial direction, the other d boundary coordinates being the sphere S^d. In these coordinates the vertex reads

$$\int \mathrm{d}\Omega_d \int \mathrm{d}\xi \sqrt{g_\xi}\, G_\Delta^N(\xi) K_{\Delta_1}\left(x_2(x_1,\xi,\vec{\Omega}),\vec{y}_1\right)\cdots K_{\Delta_n}\left(x_2(x_1,\xi,\vec{\Omega}),\vec{y}_n\right)\,. \tag{5.5}$$

Near the coincident point $\xi = 1$, the relevant terms composing the vertex have an expansion of the form

$$\sqrt{g_\xi} \sim (1-\xi^2)^{\frac{d-1}{2}}\,, \quad G_\Delta(\xi) \sim (1-\xi^2)^{-\frac{d-1}{2}}\,, \quad x_2(x_1,\xi,\vec{\Omega}) \sim x_1\,, \tag{5.6}$$

and thus the integrand goes as

$$\xi \to 1: \quad \sqrt{g_\xi}\, G_\Delta^N(\xi) K_{\Delta_1}\left(x_2(x_1,\xi,\vec{\Omega}),\vec{y}_1\right)\cdots K_{\Delta_n}\left(x_2(x_1,\xi,\vec{\Omega}),\vec{y}_n\right) \sim (1-\xi^2)^{-\frac{1}{2}(d-1)(N-1)}\,. \tag{5.7}$$

Convergence in the UV requires then $-\frac{1}{2}(d-1)(N-1)+1 > 0$, or equivalently

$$d < \frac{N+1}{N-1}\,. \tag{5.8}$$

For a given d, the UV convergence of the general vertex (5.1) depends only on the power N of the bulk-to-bulk propagator. For instance, when $N = 1$ the vertex is UV finite for all d, when $N = 2$ it converges for $d < 3$, when $N = 3$ it converges for $d < 2$, and so on. Since the UV is independent of the curvature of spacetime at large scales, one expects the same convergence criteria to apply for the equivalent diagram in flat space: at large momentum k, each of the N propagators running the loop contributes with a factor of $1/k^2$, integrated over the $N-1$ unconstrained momenta $\mathrm{d}^{d+1}k$. The superficial degree of divergence D, defined as the number of momenta in the numerator minus in the denominator, is then

$$\int \frac{(\mathrm{d}^{d+1}k)^{N-1}}{(k^2)^N} \implies D = (d+1)(N-1) - 2N\,. \tag{5.9}$$

Convergence requires $D < 0$, or equivalently $d < (N+1)/(N-1)$, agreeing with the condition (5.8) found in AdS.

5.2 Finite Integrals

Before diving into the computation of IR and UV divergent integrals, an interesting class of integrals we will discuss in this section are those composed by a chain of bulk-to-bulk propagators (see Fig. 5.2), appearing in the so-called self-energy or mass-shift diagrams for the bulk propagators

$$I_n(x_1, x_2) = \tag{5.10}$$
$$\int \mathrm{d}^{d+1}w_1\sqrt{g_1}\cdots\int \mathrm{d}^{d+1}w_n\sqrt{g_n}\; G_\Delta(x_1,w_1)G_\Delta(w_1,w_2)\cdots G_\Delta(w_{n-1},w_n)G_\Delta(w_n,x_2)\,,$$

where a simple power counting suggests it is IR convergent for $\Delta > d/2$, and UV

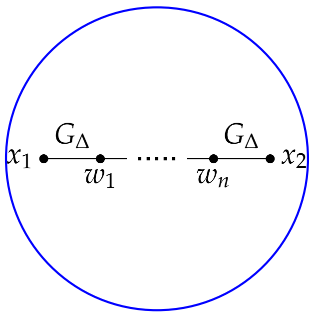

FIGURE 5.2: Integral composed by a chain of bulk-to-bulk propagators G_Δ.

convergent as long as the chain is not closed into a loop, i.e., $x_1 \neq x_2$. One could try solving for I_n directly using the known expression for the propagator, however there is a more clever way that comes from noticing that this tower of integrals arises as a perturbative manipulation of Green's equation in the parameter Δ. The calculation goes as follows: consider the Green's equation for G_Δ under the shift $\Delta \to \Delta + \gamma$

$$(-\Box + m^2 + 2\nu\gamma + \gamma^2)G_{\Delta+\gamma}(x_1,x_2) = \frac{1}{\sqrt{g}}\delta(x_1 - x_2)\,, \tag{5.11}$$

where $m^2 = \Delta(\Delta - d)$ and $\nu = \Delta - d/2$. The RHS is left unchanged as it is independent of Δ. By treating γ as a small parameter, we can expand the Green's function

$$G_{\Delta+\gamma}(x_1, x_2) = F_{\{0\}}(x_1, x_2) + \gamma F_{\{1\}}(x_1, x_2) + \gamma^2 F_{\{2\}}(x_1, x_2) + \mathcal{O}(\gamma^3)\,, \tag{5.12}$$

and solve the Green's equation perturbatively at each order in γ, leading to the set of equations up to order γ^2

$$\mathcal{O}(\gamma^0): \quad (-\Box + m^2)F_{\{0\}}(x_1, x_2) = \frac{1}{\sqrt{g}}\delta(x_1 - x_2)\,, \tag{5.13}$$

$$\mathcal{O}(\gamma^1): \quad (-\Box + m^2)F_{\{1\}}(x_1, x_2) = -2\nu F_{\{0\}}(x_1, x_2)\,, \tag{5.14}$$

$$\mathcal{O}(\gamma^2): \quad (-\Box + m^2)F_{\{2\}}(x_1, x_2) = -F_{\{0\}}(x_1, x_2) - 2\nu F_{\{1\}}(x_1, x_2)\,. \tag{5.15}$$

The leading solution is of course the unperturbed propagator G_Δ, while subleading terms correspond to the chain of propagators (5.10)

$$G_{\Delta+\gamma}(x_1, x_2) = G_\Delta(x_1, x_2) - 2\nu\gamma I_1(x_1, x_2) + \gamma^2\left[4\nu^2 I_2(x_1, x_2) - I_1(x_1, x_2)\right] + \mathcal{O}(\gamma^3)\,. \tag{5.16}$$

Since the exact form of G_Δ is known, we can contrast this expansion with the actual expansion of the propagator in Δ, obtaining the values for I_1 and I_2

$$I_1(x_1, x_2) = -\frac{1}{2\nu}\frac{\mathrm{d}}{\mathrm{d}\Delta}G_\Delta(x_1, x_2)\,, \qquad I_2(x_1, x_2) = \frac{1}{8\nu^2}\frac{\mathrm{d}^2}{\mathrm{d}\Delta^2}G_\Delta(x_1, x_2) - \frac{1}{8\nu^3}\frac{\mathrm{d}}{\mathrm{d}\Delta}G_\Delta(x_1, x_2)\,. \tag{5.17}$$

In general, the expressions for $I_1, \ldots, I_n$ are obtained by solving the Green's equation to order γ^n in the perturbative parameter. As a consistency check of these results, note that the chain I_n must satisfy the equation

$$(-\Box + m^2)I_n(x_1, x_2) = I_{n-1}(x_1, x_2)\,, \qquad I_0(x_1, x_2) \equiv G_\Delta(x_1, x_2)\,. \tag{5.18}$$

This can be checked with the formula

$$\Box\frac{\mathrm{d}^k}{\mathrm{d}\Delta^k}G_\Delta(x_1, x_2) = \frac{\mathrm{d}^k}{\mathrm{d}\Delta^k}\left[\Delta(\Delta - d)G_\Delta(x_1, x_2)\right]\,, \qquad k \in \mathbb{N}\,, \tag{5.19}$$

obtained by acting with $\mathrm{d}^k/\mathrm{d}\Delta^k$ on the Green's equation, and commuting it with $\Box$. When $k = 1$ and $k = 2$, one obtains the relations

$$(-\Box + m^2)\frac{\mathrm{d}}{\mathrm{d}\Delta}G_\Delta = -2\nu G_\Delta\,, \qquad (-\Box + m^2)\frac{\mathrm{d}^2}{\mathrm{d}\Delta^2}G_\Delta = -2G_\Delta - 4\nu\frac{\mathrm{d}}{\mathrm{d}\Delta}G_\Delta\,, \tag{5.20}$$

leading for the cases of I_1 and I_2

$$(-\Box + m^2)I_1 = G_\Delta = I_0\,, \qquad (-\Box + m^2)I_2 = -\frac{1}{2\nu}\frac{\mathrm{d}}{\mathrm{d}\Delta}G_\Delta = I_1\,, \tag{5.21}$$

as expected from (5.18).

5.3 IR Integrals

In this section we show the computation of the most basic, yet essential IR divergent vertices present in every bulk theory at loop order: the $\int K_\Delta K_\Delta$ and $\int G_\Delta K_\Delta$ integrals, relevant for the computation of the anomalous dimension of the boundary theory to subleading order in the $1/N^2$ expansion. Keeping track of the IR regulator $0 < \varepsilon \leq z$ needed at tree-level for the holographic renormalization of the boundary divergences, naturally regularizes the contributions at loop-level by adding a hard cut-off to the integrals in the bulk radial direction. These integrals will then be computed under such regulator, for arbitrary values of d and Δ, and for the special case $\nu = \Delta - d/2 \in \mathbb{N}$. How to proceed in the case $\nu = 0$ will be commented at the end.

5.3.1 $\int K_\Delta K_\Delta$ Integral

The regularized integral we want to compute is

$$I(\vec{y}_1, \vec{y}_2; \varepsilon) = \int_{z \geq \varepsilon} \mathrm{d}^{d+1}x \sqrt{g}\, K_\Delta(x, \vec{y}_1) K_\Delta(x, \vec{y}_2)\,, \tag{5.22}$$

(for an alternative computation of this integral with a different regulator, see e.g.

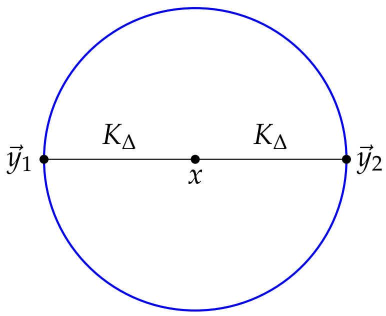

FIGURE 5.3: $\int K_\Delta K_\Delta$ integral

[67]). Before computing its value, we can study the dependence on ε by the usual trick of differentiating with respect to the regulator, taking the small regulator limit, and then integrating the resulting expression back. In general, one obtains the regulator-expansion of the integral in terms of integrals in the non-regularized coordinates. However, in our case since the bulk-to-boundary propagator is local in these coordinates at leading order in the radial expansion, the remaining integrals are trivially evaluated, and this process delivers the exact ε-expansion of the integral almost completely (up to an integration constant). To do this, it is convenient to

remove the $|\vec{y}_{12}|$ dependence of the integrand through isometry transformations, by shifting the vertex point x by $x \mapsto x + \vec{y}_2$, and then rescaling it by $x \mapsto |\vec{y}_{12}|x$

$$I = \frac{1}{|\vec{y}_{12}|^{2\Delta}} \int_{z \geq \sigma} \mathrm{d}^{d+1}x\sqrt{g}\, K_\Delta(x, \hat{y}_{12}) K_\Delta(x, \vec{0})\,. \tag{5.23}$$

After removing the conformal factor $|\vec{y}_{12}|^{2\Delta}$, the remaining integral is actually a function on the ratio $\sigma \equiv \varepsilon / |\vec{y}_{12}|$. Differentiating it then with respect to σ

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(|\vec{y}_{12}|^{2\Delta} I\right) = -\int \mathrm{d}^d x\sqrt{g}\, K_\Delta(x, \hat{y}_{12}) K_\Delta(x, \vec{0})\Big|_{z=\sigma}\,. \tag{5.24}$$

Evaluating at $z = \sigma$, as $\varepsilon \to 0$, $\sigma \to 0$ and we can use the known expansion of K_Δ, where the Dirac deltas trivially compute the boundary integrals

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(|\vec{y}_{12}|^{2\Delta} I\right) = -\sigma^{-2\nu-1}\delta(\hat{y}_{12}) + \cdots - \frac{2c_\Delta}{\sigma} + \mathcal{O}(\sigma^{-1<})\,. \tag{5.25}$$

Integrating back in σ one obtains

$$|\vec{y}_{12}|^{2\Delta} I = \frac{\sigma^{-2\nu}}{2\nu}\delta(\hat{y}_{12}) + \cdots - 2c_\Delta \ln\sigma - c_\Delta C + \mathcal{O}(\sigma^{0<})\,, \tag{5.26}$$

with C an integration constant. And finally, reverting back to ε one finds

$$I(\vec{y}_1, \vec{y}_2; \varepsilon) = \frac{\varepsilon^{-2\nu}}{2\nu}\delta(\vec{y}_{12}) + \cdots + \frac{c_\Delta}{|\vec{y}_{12}|^{2\Delta}} \ln\left(\frac{|\vec{y}_{12}|^2}{\varepsilon^2 e^C}\right) + \mathcal{O}(\varepsilon^{0<})\,. \tag{5.27}$$

After very little effort, one obtains the correct expansion of (5.22), where as we will see next, $C = \psi(\Delta) - \psi(\nu)$. Notice the leading terms are divergent and local, consistent with the analysis in (5.4). The non-local term, logarithmically divergent, is the clear signature of anomalous dimension.

Let us now move to the direct computation of the integral. The distributional behavior of the result above suggests we should solve it in momentum space. We shall use then the momentum representation of the bulk-to-boundary propagator for $\nu > 0$

$$K_\Delta(x, \vec{y}) = \frac{z^{\frac{d}{2}}}{2^{\nu-1}\Gamma(\nu)} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^\nu K_\nu(pz) e^{-i\vec{p}(\vec{x}-\vec{y})}\,, \tag{5.28}$$

where $p = |\vec{p}|$ and $K_\nu(pz)$ is the modified Bessel function of the second kind (Macdonald function). Replacing it in (5.22), performing the $\vec{x}$ integral in terms of a Dirac delta, and using it to trivially evaluate one of the momentum integrals, one obtains

$$I = \frac{1}{4^{\nu-1}\Gamma(\nu)^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^{2\nu} e^{-i\vec{p}(\vec{y}_1-\vec{y}_2)} \int_\varepsilon^\infty \frac{\mathrm{d}z}{z}\, K_\nu^2(pz)\,. \tag{5.29}$$

Using (D.13), the regularized z integral can be computed in terms of a Meijer G-function

$$\int_{\varepsilon}^{\infty} \frac{\mathrm{d}z}{z}\, K_\nu^2(pz) = \frac{\sqrt{\pi}}{4} G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix} \middle| \, p^2\varepsilon^2 \right) , \tag{5.30}$$

thus obtaining

$$I = \frac{\sqrt{\pi}}{4^\nu \Gamma(\nu)^2} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^{2\nu} G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix} \middle| \, p^2\varepsilon^2 \right) e^{-i\vec{p}(\vec{y}_1 - \vec{y}_2)} . \tag{5.31}$$

So far this exact in ε. When ν is not an integer, the asymptotic expansion of the Meijer G-function for small argument has been derived in (D.22), leading to the type of Fourier transforms studied in appendix C (namely formulas (C.10) and (C.18)). Using these results then, one finally obtains

$$I = \frac{\varepsilon^{-2\nu}}{\Gamma(\nu)^2} \sum_{k=0}^{\lfloor \nu \rfloor} \frac{\Gamma(2\nu - k)\Gamma(\nu - k)^2}{\Gamma(2\nu + 1 - 2k) k!} \left(\frac{\varepsilon^2 \Box}{4} \right)^k \delta(\vec{y}_{12}) + \frac{c_\Delta}{|\vec{y}_{12}|^{2\Delta}} \ln\left[\frac{|\vec{y}_{12}|^2}{\varepsilon^2 e^{\psi(\Delta) - \psi(\nu)}} \right] + \mathcal{O}(\varepsilon^{0<}) , \tag{5.32}$$

where $\lfloor \nu \rfloor$ is the integer part of ν. This confirms (5.27) and determines the integration constant $C = \psi(\Delta) - \psi(\nu)$, as claimed.

When ν is an integer, the asymptotic expansion of the Meijer is given instead by (D.26), leading to the Fourier transforms (C.18) and (C.35), the latter for the mass scale $M^2 \equiv 4e^{\psi(\nu)+\psi(\nu+1)}/\varepsilon^2$. In this case, one obtains the result

$$\begin{aligned} I = {} & \frac{\varepsilon^{-2\nu}}{\Gamma(\nu)^2} \sum_{k=0}^{\nu-1} \frac{\Gamma(2\nu - k)\Gamma(\nu - k)^2}{\Gamma(2\nu + 1 - 2k) k!} \left(\frac{\varepsilon^2 \Box}{4} \right)^k \delta(\vec{y}_{12}) + \frac{1}{4^\nu \Gamma(\nu)^2 2\nu^3} \Box^\nu \delta(\vec{y}_{12}) \\ & + \mathcal{R}_M \left(\frac{c_\Delta}{|\vec{y}_{12}|^{2\Delta}} \ln\left[\frac{|\vec{y}_{12}|^2}{\varepsilon^2 e^{\psi(\Delta) - \psi(\nu)}} \right] \right) + \mathcal{O}(\varepsilon^{0<}) , \end{aligned} \tag{5.33}$$

where $\mathcal{R}_M$ denotes the renormalized version of the function, defined in (C.33). It has the property that $\mathcal{R}_M[f(\vec{y}_{12})] = f(\vec{y}_{12})$ for $\vec{y}_{12} > 0$, however unlike the bare function, it is well-behaved as a distribution including the singular point $\vec{y}_{12} = 0$ and as such it has a Fourier transform, given by (C.35). Note that different definitions for $\mathcal{R}_M$ are possible, differing only by local terms at $\vec{y}_{12} = 0$. In this sense, the finite term in (5.33) proportional to $\Box^\nu \delta(\vec{y}_{12})$ is scheme-dependent and absorbable in the definition of $\mathcal{R}_M$.

For the case of $\nu = \Delta - \frac{d}{2} = 0$, the value of the integral $\int K_{\frac{d}{2}} K_{\frac{d}{2}}$ may be worked out using the appropriate representation for the propagator $K_{\frac{d}{2}}$, corresponding to (5.28) evaluated at $\nu = 0$ and with the factor $2^{\nu-1}\Gamma(\nu)$ replaced by 1. This leads to the same expression for (5.31) up to some numeric factor, with the asymptotic expansion of the Meijer G-function given by (D.29) and thus for the computation of the integral in terms of the Fourier transforms (C.18) and (C.38), with the result given by the renormalized version of the function $\ln^2(|\vec{y}_{12}|^2)/|\vec{y}_{12}|^d$.

5.3.2 $\int G_\Delta K_\Delta$ Integral

The other important IR divergent integral is

$$I(x_1,\vec{y}_2;\varepsilon) = \int_{z_2\geq\varepsilon} \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G_\Delta(x_1,x_2)K_\Delta(x_2,\vec{y}_2)\,. \tag{5.34}$$

Similarly to the $\int KK$ integral, we can study the dependence of the integral on the

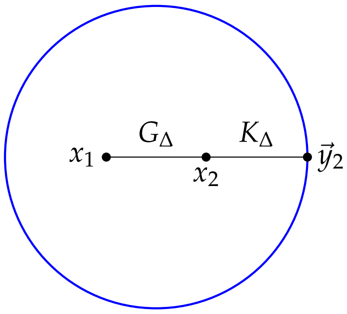

FIGURE 5.4: $\int G_\Delta K_\Delta$ integral

regulator ε with the differentiation trick, and exploiting the fact that to leading order in the ε-expansion the bulk-to-boundary propagator is local in the boundary coordinates, thus obtaining the exact value of the integral to the relevant order in ε almost completely, up to an integration constant. Before doing this, it is convenient to remove all dependence of the external points x_1 and $\vec{y}_2$ from the integrand through isometry transformations on the vertex x_2. This is achieved by the following sequence: translating by $x_2 \mapsto x_2 + \vec{y}_2$, inverting $x_2 \mapsto x_2/x_2^2$, translating again by $x_2 \mapsto x_2 + \vec{x}_1''$, rescaling $x_2 \mapsto z_1''x_2$, and finally inverting back

$$I = z_1''^{\Delta} \int_{z_2\geq\sigma} \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G_\Delta\left((1,\vec{0}),x_2\right) K_\Delta(x_2,\vec{0})\,, \tag{5.35}$$

where $x_1'' = x_1'/x_1'^2$ and $x_1' = x_1 - \vec{y}_2$. The remaining integral is actually a function on $\sigma \equiv \varepsilon z_1''$. Differentiating it then with respect to σ

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{I}{z_1''^{\Delta}}\right) = -\int \mathrm{d}^d x_2\sqrt{g_2}\, G_\Delta\left((1,\vec{0}),x_2\right) K_\Delta(x_2,\vec{0})\Big|_{z_2=\sigma}\,. \tag{5.36}$$

Evaluating at $z=\sigma$, as $\varepsilon\to 0$, $\sigma\to 0$ and we can use the known expansions of K_Δ and G_Δ, where the Dirac deltas trivially compute the boundary integrals

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{I}{z_1''^{\Delta}}\right) = -\frac{c_\Delta}{2\nu\sigma} + \mathcal{O}(\sigma^{-1<})\,. \tag{5.37}$$

Integrating back in σ

$$\frac{I}{z_1''^{\Delta}} = -\frac{c_\Delta}{2\nu}\ln\sigma - \frac{c_\Delta}{2\nu}C + \mathcal{O}(\sigma^{0<})\,, \tag{5.38}$$

where C is an integration constant. Finally, reverting back to ε and to the original coordinates one obtains the result

$$I(x_1,\vec{y}_2;\varepsilon) = -\frac{1}{2\nu}K_\Delta(x_1,\vec{y}_2)\ln\left[\varepsilon K(x_1,\vec{y}_2)e^C\right] + \mathcal{O}(\varepsilon^{0<}), \tag{5.39}$$

where $K(x_1,\vec{y}_2) = z_1/[z_1^2 + (\vec{x}_1 - \vec{y}_2)^2]$. This turns out to be the correct expansion of (5.34), with a value of $C = \psi(\Delta) - \psi(\nu) - 1/(2\nu)$.

Moving now to the explicit computation of the integral, as for $\int KK$, we will do it in momentum space. In addition to (5.28), we also need the momentum representation of the bulk-to-bulk propagator, given by

$$G_\Delta(x,x') = (zz')^{\frac{d}{2}}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,e^{-i\vec{p}(\vec{x}-\vec{x}')}\begin{cases} I_\nu(pz)K_\nu(pz'),\ z<z' \\ I_\nu(pz')K_\nu(pz),\ z>z' \end{cases}, \tag{5.40}$$

with $I_\nu(pz)$ the modified Bessel function of the first kind. After performing the $\vec{x}$ integral in terms of a Dirac delta and using it to evaluate one of the momentum integrals

$$I = \frac{z_1^{\frac{d}{2}}}{2^{\nu-1}\Gamma(\nu)}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,p^\nu e^{-i\vec{p}(\vec{x}_1-\vec{y}_2)}\int_\varepsilon^\infty\frac{\mathrm{d}z_2}{z_2}\,K_\nu(pz_2)\begin{cases} I_\nu(pz_1)K_\nu(pz_2),\ z_1<z_2 \\ I_\nu(pz_2)K_\nu(pz_1),\ z_1>z_2 \end{cases}. \tag{5.41}$$

Using (D.13) and (D.14), the regularized z_2 integral can be computed in terms of Meijer G-functions

$$\int_\varepsilon^{z_1}\frac{\mathrm{d}z_2}{z_2}\,K_\nu(pz_2)I_\nu(pz_2) = \frac{1}{4\sqrt{\pi}}G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu\end{matrix}\,\middle|\,p^2\varepsilon^2\right) - \frac{1}{4\sqrt{\pi}}G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu\end{matrix}\,\middle|\,p^2 z_1^2\right), \tag{5.42}$$

$$\int_{z_1}^\infty\frac{\mathrm{d}z_2}{z_2}\,K_\nu^2(pz_2) = \frac{\sqrt{\pi}}{4}G^{4,0}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu\end{matrix}\,\middle|\,p^2 z_1^2\right). \tag{5.43}$$

The 2 Meijers dependent on z_1 combine together to form $\partial_\nu K_\nu(pz_1)$ through the identity (D.51), thus obtaining

$$I = \frac{z_1^{\frac{d}{2}}}{2^{\nu-1}\Gamma(\nu)}\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,p^\nu\left[\frac{K_\nu(pz_1)}{4\sqrt{\pi}}G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu\end{matrix}\,\middle|\,p^2\varepsilon^2\right) - \frac{1}{2\nu}\partial_\nu K_\nu(pz_1)\right]e^{-i\vec{p}(\vec{x}_1-\vec{y}_2)}. \tag{5.44}$$

This expression is exact in ε. To leading order, the asymptotic expansion of the Meijer G-function for small argument is the same for both $\nu > 0$ integer and non-integer, as it can be seen from (D.34) and (D.37), leading to the momentum representation of K_Δ

(5.28) and its derivative

$$I = -\frac{1}{2\nu}\left(-\frac{1}{2\nu} + \ln\varepsilon + \partial_\nu\right) K_\Delta(x_1, \vec{y}_2) + \mathcal{O}(\varepsilon^{0<})\,. \tag{5.45}$$

By acting with ∂_ν on the position space representation of the propagator, one finally obtains

$$I = -\frac{1}{2\nu} K_\Delta(x_1, \vec{y}_2) \ln\left[\varepsilon K(x_1, \vec{y}_2) e^{\psi(\Delta)-\psi(\nu)-\frac{1}{2\nu}}\right] + \mathcal{O}(\varepsilon^{0<})\,, \tag{5.46}$$

confirming the claim in (5.39).

For $\nu = \Delta - d/2 = 0$, the value of the integral $\int G_{\frac{d}{2}} K_{\frac{d}{2}}$ may be worked out in a similar manner, using the appropriate representation for the propagator $K_{\frac{d}{2}}$, the identity for the Meijer G-functions (D.52), and the corresponding asymptotic expansion (D.40).

5.4 UV Integrals

We discussed the role of IR divergences of loops in AdS, leading to corrections to the conformal dimensions in the dual CFT. By how much the dimensions are corrected however is dictated by the UV divergences of the loop integrals, while for higher-point functions they also dictate the amount of correction to the OPE coefficients. Formally, the CFT data is corrected from the UV divergences in AdS by an infinite amount, and a subtraction scheme (supplemented by renormalization conditions) must be adopted in the bulk to extract the finite, physical values for these corrections. We shall adopt the UV regularization scheme of geodesic point-splitting introduced in 4.1.1, and show in this section the computation of bulk loop vertices under such regulator. We will do so for the general loop vertex (5.1) of the schematic form $\int G^N K^n$, for arbitrary N and for the cases of $n = 0, 1, 2$ external legs, where for simplicity no additional IR divergences are present in the vertices. The strategy will be to solve for these 3 vertices directly in position space, writing the bulk propagator $G_\Delta(\xi)$ in its series representation in ξ and compute them in terms of the more fundamental vertices between ξ and K defined in (E.1), whose master formulas has been derived in appendix E. When possible, these series may then be resummed back to find their closed-form expressions.

5.4.1 Series for G_Δ^N

Many bulk loop integrals involve copies of the bulk-to-bulk propagator between the same 2 points. We want to write down then a convenient expression for G_Δ^N for some positive integer number N. Consider the position space representation of G_Δ in the variable ξ

$$G_\Delta(\xi) = \frac{c_\Delta}{2^{\Delta+1}\nu}\xi^\Delta {}_2F_1\left(\begin{matrix}\frac{\Delta}{2}, \frac{\Delta+1}{2} \\ \Delta - \frac{d}{2} + 1\end{matrix}; \xi^2\right)\,. \tag{5.47}$$

By taking its N-th power, we can write it as the following series

$$G_\Delta^N(\xi) = \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \sum_{k=0}^{\infty} g_{k,N}\xi^{N\Delta+2k}\,, \tag{5.48}$$

where we introduced the coefficient $g_{k,N}$ defined by

$${}_2F_1\left(\begin{matrix}\frac{\Delta}{2}, \frac{\Delta+1}{2} \\ \Delta-\frac{d}{2}+1\end{matrix};\xi^2\right)^N \equiv \sum_{k=0}^{\infty} g_{k,N}\xi^{2k}\,. \tag{5.49}$$

One can directly check that

$$g_{0,N} = 1\,, \qquad g_{k,1} = \frac{\left(\frac{\Delta}{2}\right)_k\left(\frac{\Delta+1}{2}\right)_k}{\left(\Delta-\frac{d}{2}+1\right)_k\, k!}\,. \tag{5.50}$$

The simplicity of G_Δ^N boils down to the simplicity of $g_{k,N}$. In general, one can work out the value of this coefficient when a closed-form expression for the hypergeometric defining it is known. When this is the case, not only G_Δ^N can be expressed in terms of known functions but also, in general, the loop integrals that involve this quantity, at least for the class of integrals we are interested in. More on this at the end of the section where we study concrete examples.

Divergences are expected at the coincident point $\xi = 1$. Checking for the convergence of $G_\Delta^N(\xi)$, from (5.49) one sees that for large k

$$\frac{g_{k+1,N}}{g_{k,N}}\frac{\xi^{2k+2}}{\xi^{2k}} = \left[1 + \left(\frac{N(d-1)}{2}-1\right)\frac{1}{k} + \mathcal{O}\left(\frac{1}{k^2}\right)\right]\xi^2\,, \tag{5.51}$$

and thus performing the ratio test

$$\lim_{k\to\infty}\frac{g_{k+1,N}}{g_{k,N}}\frac{\xi^{2k+2}}{\xi^{2k}} = \xi^2\,, \tag{5.52}$$

the series converges for $0 \leq \xi^2 < 1$. As $\xi \to 1$, the ratio test is inconclusive and one must look at the subleading term in the expansion of $g_{k+1,N}/g_{k,N}$ through Raabe's test

$$\lim_{k\to\infty} k\left(\frac{g_{k+1,N}}{g_{k,N}} - 1\right) = \frac{N(d-1)}{2} - 1\,, \tag{5.53}$$

where convergence requires the limit to be < -1, implying in our case $d < 1$. Since we are interested in $d \geq 1$, the bulk propagator at coincident points is always divergent. Despite this, bulk loop integrals involving $G_\Delta^N(\xi)$ are expected to have a softer divergence near the region $\xi = 1$, given by the convergence analysis in (5.8).

The UV regularization scheme amounts to replace the argument of $G_\Delta(\xi)$ by $\xi \to \xi_\tau \equiv \xi/\cosh\tau$ with $0 < \tau \ll 1$, rendering ξ_τ strictly less than 1, and consequently the series for the regularized propagator $G_{\tau,\Delta}(\xi) \equiv G_\Delta(\xi_\tau)$ always convergent, as it can be seen from the analysis above. The regularized series representation for G_Δ^N we will use to compute the UV integrals is then

$$G_{\tau,\Delta}^N(\xi) = \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \sum_{k=0}^{\infty} g_{k,N} \left(\frac{\xi}{\cosh\tau}\right)^{N\Delta+2k}, \tag{5.54}$$

with the bare integrals simply recovered for vanishing regulator $\tau = 0$.

5.4.2 $\int G_\Delta^N$ Integral

The first UV divergent integrals we still study are of the type $\int G_\Delta^N$, that is, the bulk-to-bulk propagator to some positive integer power N. Writing it in terms of the regularized propagator $G_{\tau,\Delta}$, the regularized integral to study is

$$I_\Delta^N(x_1;\tau) = \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G_{\tau,\Delta}^N(x_1,x_2)\,, \tag{5.55}$$

where IR convergence requires $N\Delta - d > 0$. As we will prove now, the integral turns

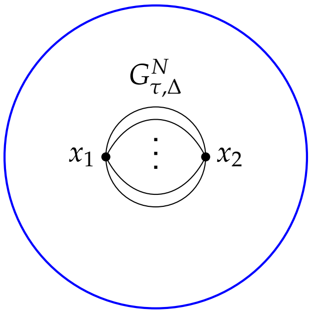

FIGURE 5.5: $\int G_\Delta^N$ integral

out to be constant, independent of the point x_1

$$I_\Delta^N(x_1;\tau) \equiv \mu_\Delta^{d,N}(\tau)\,. \tag{5.56}$$

This can be shown by removing the x_1 dependence from the integral through AdS isometry transformations on the vertex x_2. Indeed, by translating $x_2 \mapsto x_2 + \vec{x}_1$, and then rescaling $x_2 \mapsto z_1 x_2$

$$I_\Delta^N(x_1;\tau) = \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G_{\tau,\Delta}^N(x_1,x_2)\,, \tag{5.57}$$

$$= \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G_{\tau,\Delta}^N\left((z_1,\vec{0}),x_2\right)\,, \tag{5.58}$$

$$= \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G_{\tau,\Delta}^N\left((1,\vec{0}),x_2\right)\,, \tag{5.59}$$

$$= I_\Delta^N\left((1,\vec{0});\tau\right)\,, \tag{5.60}$$

$$\equiv \mu_\Delta^{d,N}(\tau)\,. \tag{5.61}$$

To determine the value of the coefficient $\mu_\Delta^{d,N}(\tau)$, we will compute (5.55) using the series representation of the regularized propagator (5.54). This leads to the computation of $I_\Delta^N(x_1;\tau)$ in terms of the fundamental vertices (E.1) defined in appendix E

$$I_\Delta^N(x_1;\tau) = \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \sum_{k=0}^{\infty} g_{k,N} \left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k} V_{N\Delta+2k}(x_1)\,, \tag{5.62}$$

where $V_{N\Delta+2k}(x_1)$ corresponds to the fundamental vertex with no external legs to the boundary, and whose general solution has been derived in (E.11). Using this result, it leads to the coefficient

$$\mu_\Delta^{d,N}(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \sum_{k=0}^{\infty} g_{k,N} \frac{\Gamma\left(\frac{N\Delta-d}{2}+k\right)}{\Gamma\left(\frac{N\Delta+1}{2}+k\right)} \left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k}\,. \tag{5.63}$$

It is represented in terms of an infinite series, however when a simple expression for $g_{k,N}$ defined by (5.49) is known, the series can be computed in closed form, in general in terms of a generalized hypergeometric function. Note that as $\tau \to 0$, performing a similar study as in (5.53), convergence of $\mu_\Delta^{d,N}(\tau)$ requires $d < (N+1)/(N-1)$, consistent with the analysis in (5.8).

5.4.3 $\int G_\Delta^N K_{\Delta_2}$ Integral

Another type of UV divergent integrals are of the form $\int G_\Delta^N K_{\Delta_2}$, with the bulk-to-bulk propagator to some positive integer power N attached to a bulk-to-boundary propagator. The regularized object to analyze is

$$I_{\Delta,\Delta_2}^N(x_1,\vec{y}_2;\tau) = \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\; G_{\tau,\Delta}^N(x_1,x_2)K_{\Delta_2}(x_2,\vec{y}_2)\,, \tag{5.64}$$

where IR convergence requires $N\Delta > \Delta_2$. In this case, isometry transformations at the

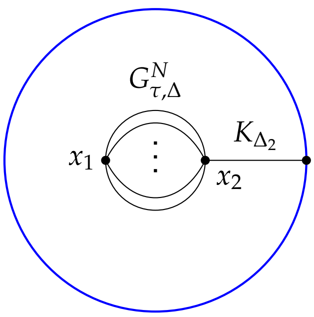

FIGURE 5.6: $\int G_\Delta^N K_{\Delta_2}$ integral

point x_2 show that the integral is proportional to a bulk-to-boundary propagator

$$I_{\Delta,\Delta_2}^N(x_1,\vec{y}_2;\tau) \equiv \eta_{\Delta,\Delta_2}^{d,N}(\tau)K_{\Delta_2}(x_1,\vec{y}_2)\,. \tag{5.65}$$

Indeed, by translating $x_2 \mapsto x_2 + \vec{y}_2$, inverting $x_2 \mapsto x_2/x_2^2$, translating again $x_2 \mapsto x_2 + \vec{x}_1''$, rescaling $x_2 \mapsto z_1'' x_2$, and finally inverting back, with $x_1'' = x_1'/x_1'^2$ and $x_1' = x_1 - \vec{y}_2$

$$I^N_{\Delta,\Delta_2}(x_1,\vec{y}_2;\tau) = \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1,x_2)K_{\Delta_2}(x_2,\vec{y}_2)\,, \tag{5.66}$$

$$= \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1',x_2)K_{\Delta_2}(x_2,\vec{0})\,, \tag{5.67}$$

$$= \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1'',x_2)c_{\Delta_2}z_2^{\Delta_2}\,, \tag{5.68}$$

$$= \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}\left((z_1'',\vec{0}),x_2\right)c_{\Delta_2}z_2^{\Delta_2}\,, \tag{5.69}$$

$$= z_1''^{\Delta_2}\int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}\left((1,\vec{0}),x_2\right)c_{\Delta_2}z_2^{\Delta_2}\,, \tag{5.70}$$

$$= K^{\Delta_2}(x_1,\vec{y}_2)\int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}\left((1,\vec{0}),x_2\right)K_{\Delta_2}(x_2,\vec{0})\,, \tag{5.71}$$

$$= K^{\Delta_2}(x_1,\vec{y}_2)I^N_{\Delta,\Delta_2}\left((1,\vec{0}),\vec{0};\tau\right)\,, \tag{5.72}$$

$$\equiv \eta^{d,N}_{\Delta,\Delta_2}(\tau)K_{\Delta_2}(x_1,\vec{y}_2)\,. \tag{5.73}$$

As we did for the coefficient $\mu^{d,N}_{\Delta}(\tau)$, we will determine $\eta^{d,N}_{\Delta,\Delta_2}(\tau)$ by computing (5.64) using the series representation of the regularized propagator (5.54), leading to the computation of $I^N_{\Delta,\Delta_2}(x_1,\vec{y}_2;\tau)$ in terms of the fundamental vertices (E.1)

$$I^N_{\Delta,\Delta_2} = c_{\Delta_2}\left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N\sum_{k=0}^{\infty} g_{k,N}\left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k} V_{N\Delta+2k,\Delta_2}(x_1,\vec{y}_2)\,, \tag{5.74}$$

where $V_{N\Delta+2k,\Delta_2}(x_1,\vec{y}_2)$ corresponds to the fundamental vertex with 1 external leg extended to the boundary, computed in (E.13). Using this result, one obtains the coefficient

$$\eta^{d,N}_{\Delta,\Delta_2}(\tau) = \pi^{\frac{d+1}{2}}\left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N\sum_{k=0}^{\infty} g_{k,N}\frac{\Gamma\left(\frac{N\Delta+\Delta_2-d}{2}+k\right)\Gamma\left(\frac{N\Delta-\Delta_2}{2}+k\right)}{\Gamma\left(\frac{N\Delta}{2}+k\right)\Gamma\left(\frac{N\Delta+1}{2}+k\right)}\left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k}\,. \tag{5.75}$$

As before, the coefficient is represented as an infinite series and it can be computed in closed form for simple $g_{k,N}$. In the limit $\tau\to 0$, Raabe's test shows that $\eta^{d,N}_{\Delta,\Delta_2}(\tau)$ converges for $d < (N+1)/(N-1)$, as expected.

5.4.4 $\int G^N_\Delta K_{\Delta_3}K_{\Delta_4}$ Integral

The last type of UV divergent integrals we will study are of the form $\int G^N_\Delta K_{\Delta_3}K_{\Delta_4}$, with the bulk-to-bulk propagator to some power N now attached to 2 bulk-to-boundary propagators. The regularized object to analyze in this case is

$$I^N_{\Delta,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4;\tau) = \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1,x_2)K_{\Delta_3}(x_2,\vec{y}_3)K_{\Delta_4}(x_2,\vec{y}_4)\,, \tag{5.76}$$

where IR convergence requires $N\Delta > |\Delta_3 - \Delta_4|$. In this case, isometry

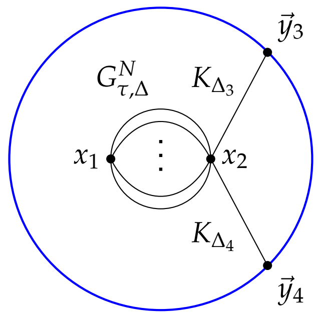

FIGURE 5.7: $\int G^N_\Delta K_{\Delta_3} K_{\Delta_4}$ integral

transformations show that the integral is of the form

$$I^N_{\Delta,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4;\tau) \equiv K_{\Delta_3}(x_1,\vec{y}_3)K_{\Delta_4}(x_1,\vec{y}_4)\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}(X;\tau)\,, \tag{5.77}$$

with the coefficient $\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}$ as a function of the combination $X \equiv K(x_1,\vec{y}_3)K(x_1,\vec{y}_4)|\vec{y}_{34}|^2$. Indeed, by translating $x_2 \mapsto x_2 + \vec{y}_4$, inverting $x_2 \mapsto x_2/x_2^2$, and translating again $x_2 \mapsto x_2 + \vec{y}\,'_{34}$

$$\begin{aligned}
I^N_{\Delta,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4;\tau) &= \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1,x_2)K_{\Delta_3}(x_2,\vec{y}_3)K_{\Delta_4}(x_2,\vec{y}_4)\,, && (5.78)\\
&= \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1',x_2)K_{\Delta_3}(x_2,\vec{y}_{34})K_{\Delta_4}(x_2,\vec{0})\,, && (5.79)\\
&= |\vec{y}\,'_{34}|^{2\Delta_3}\int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1'',x_2)K_{\Delta_3}(x_2,\vec{y}\,'_{34})c_{\Delta_4}z_2^{\Delta_4}\,, && (5.80)\\
&= |\vec{y}\,'_{34}|^{2\Delta_3}\int \mathrm{d}^{d+1}x_2\sqrt{g_2}\, G^N_{\tau,\Delta}(x_1''',x_2)K_{\Delta_3}(x_2,\vec{0})c_{\Delta_4}z_2^{\Delta_4}\,, && (5.81)\\
&\equiv |\vec{y}\,'_{34}|^{2\Delta_3} f(x_1''';\tau)\,, && (5.82)
\end{aligned}$$

where we called $x_1''' = x_1'' - \vec{y}\,'_{34}$, $x_1'' = x_1'/x_1'^2$, $x_1' = x_1 - \vec{y}_4$ and $\vec{y}\,'_{34} = \vec{y}_{34}/|\vec{y}_{34}|^2$. The remaining integral is just a function of x_1''', which we named $f(x_1''';\tau)$. Notice that under rescaling of the bulk point x_1''' or rotations of the boundary coordinates $\vec{x}_1'''$

$$f(\lambda x_1''';\tau) = \lambda^{\Delta_4-\Delta_3} f(x_1''';\tau)\,, \qquad f(z_1''',R\,\vec{x}_1''';\tau) = f(z_1''',\vec{x}_1''';\tau)\,. \tag{5.83}$$

Then, $f(x_1''';\tau)$ must be of the form

$$\begin{aligned}
f(x_1''';\tau) &= (z_1''')^{\Delta_4-\Delta_3} g\left(\frac{|\vec{x}_1'''|}{z_1'''};\tau\right)\,, && (5.84)\\
&= \left(\frac{z_1'''}{z_1'''^2 + |\vec{x}_1'''|^2}\right)^{\Delta_3} z_1'''^{\Delta_4}\, h\left(\frac{z_1'''^2}{z_1'''^2 + |\vec{x}_1'''|^2};\tau\right)\,. && (5.85)
\end{aligned}$$

Putting everything together, and writing back in terms of the original coordinates

$$I^N_{\Delta,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4;\tau) = |\vec{y}\,'_{34}|^{2\Delta_3}\left(\frac{z_1'''}{z_1'''^2+|\vec{x}_1'''|^2}\right)^{\Delta_3} z_1'''^{\Delta_4}\, h\left(\frac{z_1'''^2}{z_1'''^2+|\vec{x}_1'''|^2};\tau\right)\,, \tag{5.86}$$

$$= K^{\Delta_3}(x_1,\vec{y}_3)K^{\Delta_4}(x_1,\vec{y}_4)h\left(K(x_1,\vec{y}_3)K(x_1,\vec{y}_4)|\vec{y}_{34}|^2;\tau\right)\,, \tag{5.87}$$

$$\equiv K_{\Delta_3}(x_1,\vec{y}_3)K_{\Delta_4}(x_1,\vec{y}_4)\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}(X;\tau)\,, \tag{5.88}$$

as claimed. The explicit form of χ may be determined from a direct computation of the integral, in terms of the representation of the regularized propagator (5.54) and the fundamental vertices (E.1)

$$I^N_{\Delta,\Delta_3,\Delta_4} = c_{\Delta_3}c_{\Delta_4}\left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N\sum_{k=0}^{\infty} g_{k,N}\left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k} V_{N\Delta+2k,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4)\,, \tag{5.89}$$

using the result for the fundamental vertex $V_{N\Delta+2k,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4)$ with 2 external legs extended to the boundary (E.16), the expression for $\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}$ is identified with

$$\chi^{d,N}_{\Delta,\Delta_3,\Delta_4} = \pi^{\frac{d+1}{2}}\left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N\sum_{k=0}^{\infty} g_{k,N}\frac{\Gamma\left(\frac{N\Delta+\Delta_3+\Delta_4-d}{2}+k\right)\Gamma\left(\frac{N\Delta+\Delta_3-\Delta_4}{2}+k\right)\Gamma\left(\frac{N\Delta+\Delta_4-\Delta_3}{2}+k\right)}{\Gamma\left(\frac{N\Delta}{2}+k\right)\Gamma\left(\frac{N\Delta+1}{2}+k\right)\Gamma\left(\frac{N\Delta+\Delta_3+\Delta_4}{2}+k\right)}$$
$$\times\left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k} {}_2F_1\left(\begin{matrix}\Delta_3,\ \Delta_4\\ \frac{N\Delta+\Delta_3+\Delta_4}{2}+k\end{matrix};1-K(x_1,\vec{y}_3)K(x_1,\vec{y}_4)|\vec{y}_{34}|^2\right)\,. \tag{5.90}$$

In general, it is convenient to express it as a series in $X = K(x_1,\vec{y}_3)K(x_1,\vec{y}_4)|\vec{y}_{34}|^2$, rather than $1-X$. In the case where $N\Delta \neq \Delta_3+\Delta_4+2\mathbb{Z}$, this can be achieved using the linear transformation of the hypergeometric function (B.13), obtaining a series of the form

$$\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}(X;\tau) = \sum_{i=0}^{\infty}\left[a_i(\tau)+b_i(\tau)X^{\frac{N\Delta-\Delta_3-\Delta_4}{2}}\right]X^i\,, \tag{5.91}$$

with the coefficients $a_i(\tau)$ and $b_i(\tau)$ given by

$$a_i(\tau) = \pi^{\frac{d+1}{2}}\left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N\frac{(\Delta_3)_i(\Delta_4)_i\Gamma\left(\frac{N\Delta-\Delta_3-\Delta_4}{2}\right)}{\left(1+\frac{\Delta_3+\Delta_4-N\Delta}{2}\right)_i i!}$$
$$\times\sum_{k=0}^{\infty} g_{k,N}\frac{\Gamma\left(\frac{N\Delta+\Delta_3+\Delta_4-d}{2}+k\right)\left(\frac{N\Delta-\Delta_3-\Delta_4}{2}-i\right)_k}{\Gamma\left(\frac{N\Delta}{2}+k\right)\Gamma\left(\frac{N\Delta+1}{2}+k\right)}\left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k}\,, \tag{5.92}$$

$$b_i(\tau) = \pi^{\frac{d+1}{2}}\left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N\frac{\Gamma\left(\frac{\Delta_3+\Delta_4-N\Delta}{2}\right)\Gamma\left(\frac{N\Delta+\Delta_3-\Delta_4}{2}+i\right)\Gamma\left(\frac{N\Delta-\Delta_3+\Delta_4}{2}+i\right)}{\Gamma(\Delta_3)\Gamma(\Delta_4)\left(1+\frac{N\Delta-\Delta_3-\Delta_4}{2}\right)_i i!}$$
$$\times\sum_{k=0}^{i} g_{k,N}\frac{\Gamma\left(\frac{N\Delta+\Delta_3+\Delta_4-d}{2}+k\right)(-i)_k}{\Gamma\left(\frac{N\Delta}{2}+k\right)\Gamma\left(\frac{N\Delta+1}{2}+k\right)}\left(\frac{1}{\cosh\tau}\right)^{N\Delta+2k}\,. \tag{5.93}$$

Note that the coefficient $b_i(\tau)$ consists in a terminating sum of finite terms, thus it is always convergent in the limit $\tau \to 0$. All divergences in $\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}$ are contained then in the coefficient $a_i(\tau)$. Indeed, from a Raabe's test of its k-series, convergence of this coefficient requires

$$d < \frac{N+1+2i}{N-1}\,. \tag{5.94}$$

That is, $a_0(\tau)$ converges for $d < (N+1)/(N-1)$, $a_1(\tau)$ for $d < (N+3)/(N-1)$, and so on. As a consistency check on the values obtained for $a_i(\tau)$ and $b_i(\tau)$, note that for $N = 1$ and $\tau = 0$ using the expression for $g_{k,1}$ in (5.50), they reduce to

$$a_i = \frac{\Gamma\left(\frac{\Delta+\Delta_3+\Delta_4-d}{2}\right)\Gamma\left(\frac{\Delta-\Delta_3-\Delta_4}{2}\right)(\Delta_3)_i(\Delta_4)_i}{4\,\Gamma\left(1+\frac{\Delta-\Delta_3-\Delta_4}{2}\right)\Gamma\left(1+\frac{\Delta+\Delta_3+\Delta_4-d}{2}+i\right)\left(1+\frac{\Delta_3+\Delta_4-\Delta}{2}\right)_i}\,, \tag{5.95}$$

$$b_i = \frac{\Gamma\left(\frac{\Delta+\Delta_3+\Delta_4-d}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-\Delta}{2}\right)\Gamma\left(\frac{\Delta+\Delta_3-\Delta_4}{2}+i\right)\Gamma\left(\frac{\Delta-\Delta_3+\Delta_4}{2}+i\right)}{4\,\Gamma(\Delta_3)\Gamma(\Delta_4)\Gamma\left(\Delta-\frac{d}{2}+1+i\right)i!}\,, \tag{5.96}$$

recovering the known tree-level results [52].

The expressions for $a_i(\tau)$ and $b_i(\tau)$ are obtained with the assumption $N\Delta \neq \Delta_3 + \Delta_4 + 2\mathbb{Z}$, however under the analytic continuation $N\Delta \to \Delta_3 + \Delta_4 - 2\mathbb{N}$, the series in $a_i(\tau)$ terminates and one can make the identification $a_i(\tau) = -b_{i+\frac{\Delta_3+\Delta_4-N\Delta}{2}}(\tau)$. This leads to a cancellation in pairs of all $X^{i\geq 0}$ terms in (5.91), leaving only the sum of a finite number of convergent terms (for the case $N = 1$, see e.g. [46])

$$\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}(X;\tau) = \sum_{i=0}^{\frac{\Delta_3+\Delta_4-N\Delta}{2}-1} b_i(\tau)X^{\frac{N\Delta-\Delta_3-\Delta_4}{2}+i}\,. \tag{5.97}$$

When $N\Delta = \Delta_3 + \Delta_4 + 2\mathbb{N}_0$, the coefficients $a_i(\tau)$ and $b_i(\tau)$ become ill-defined and this can be traced back to the hypergeometric linear transformation performed to (5.90). In this case one must use instead (B.14), which introduces a logarithmic term

$$\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}(X;\tau) = \sum_{i=0}^{\infty}\left[c_i(\tau) + d_i(\tau)X^{\frac{N\Delta-\Delta_3-\Delta_4}{2}}\ln X\right]X^i\,, \tag{5.98}$$

for some coefficients $c_i(\tau)$ and $d_i(\tau)$. An important case that falls into this category is the quartic vertex with the dimension of all 4 legs equal: $N = 2$, $\Delta_3 = \Delta_4 = \Delta$. From (5.90) and (B.14), one can indeed see $\chi^{d,2}_{\Delta,\Delta,\Delta}$ takes the form of (5.98) with

$N\Delta - \Delta_3 - \Delta_4 = 0$, and with the coefficients $c_i(\tau)$ and $d_i(\tau)$ given by

$$c_i(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 \frac{(\Delta)_i^2}{i!^2} \sum_{k=0}^{i} g_{k,2} \frac{\Gamma\left(2\Delta - \frac{d}{2} + k\right)(-i)_k}{\Gamma(\Delta+k)\Gamma\left(\Delta+\frac{1}{2}+k\right)} \times \left[\psi(1+i-k) + \psi(1+i) - 2\psi(\Delta+i)\right] \left(\frac{1}{\cosh\tau}\right)^{2\Delta+2k} + \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 (-1)^i \frac{(\Delta)_i^2}{i!} \sum_{k=i+1}^{\infty} g_{k,2} \frac{\Gamma\left(2\Delta - \frac{d}{2} + k\right)\Gamma(k-i)}{\Gamma(\Delta+k)\Gamma\left(\Delta+\frac{1}{2}+k\right)} \left(\frac{1}{\cosh\tau}\right)^{2\Delta+2k}, \tag{5.99}$$

$$d_i(\tau) = -\pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 \frac{(\Delta)_i^2}{i!^2} \sum_{k=0}^{i} g_{k,2} \frac{\Gamma\left(2\Delta - \frac{d}{2} + k\right)(-i)_k}{\Gamma(\Delta+k)\Gamma\left(\Delta+\frac{1}{2}+k\right)} \left(\frac{1}{\cosh\tau}\right)^{2\Delta+2k}. \tag{5.100}$$

Similar to the previous case, the series in $d_i(\tau)$ terminates and thus the coefficient converges in the limit $\tau \to 0$, all the divergences then of $\chi^{d,2}_{\Delta,\Delta,\Delta}$ being in $c_i(\tau)$, in particular from the second series in (5.99) that starts at $k = i+1$. From a Raabe's test of this series, one concludes the coefficient $c_i(\tau)$ converges for $d < 3 + 2i$, same convergence as $a_i(\tau)$ derived in (5.94) for the particular case $N = 2$.

From the computational side, $c_i(\tau)$ is in general harder to compute in closed form compared to the other coefficients as it also contains series involving digamma functions $\psi(x)$, resulting in more complicated expressions for $c_i(\tau)$ than generalized hypergeometric functions such as Appell functions, or more generally Kampé de Fériet functions. For the case of (5.99), a more manageable expression can be obtained using the digamma property

$$\psi(1+i-k) = \psi(1+i) + \sum_{l=0}^{k-1} \frac{1}{l-i}, \tag{5.101}$$

being able to express the coefficient in terms of series involving only gamma functions

$$c_i(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 \frac{(\Delta)_i^2}{i!^2} 2\left[\psi(1+i) - \psi(\Delta+i)\right] \times \sum_{k=0}^{i} g_{k,2} \frac{\Gamma\left(2\Delta - \frac{d}{2} + k\right)(-i)_k}{\Gamma(\Delta+k)\Gamma\left(\Delta+\frac{1}{2}+k\right)} \left(\frac{1}{\cosh\tau}\right)^{2\Delta+2k} + \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 \frac{(\Delta)_i^2}{i!^2} \sum_{l=0}^{i}(-i)_l \sum_{k=l+1}^{\infty} g_{k,2} \frac{\Gamma\left(2\Delta - \frac{d}{2} + k\right)(l-i+1)_{k-l-1}}{\Gamma(\Delta+k)\Gamma\left(\Delta+\frac{1}{2}+k\right)} \left(\frac{1}{\cosh\tau}\right)^{2\Delta+2k}, \tag{5.102}$$

with all divergences contained in the term $l = i$ of the double series.

5.5 Example: relevant operator $\mathcal{O}_\Delta$

To illustrate the computation of the UV divergent integrals discussed in Section 5.4, as an example consider the case where the internal lines representing the bulk-to-bulk propagators in figs. 5.5, 5.6 and 5.7, correspond to a relevant operator $\mathcal{O}_\Delta$ of dimension $\Delta < d$. For simplicity, to avoid additional IR divergences $N\Delta$ will be taken to satisfy the 3 inequalities $N\Delta > d,\ \Delta_2,\ |\Delta_3 - \Delta_4|$ discussed in the previous section, that ensure the IR convergence of the 3 coefficients $\mu_\Delta^{d,N}$, $\eta_{\Delta,\Delta_2}^{d,N}$ and $\chi_{\Delta,\Delta_3,\Delta_4}^{d,N}$, respectively. For integer conformal dimensions $\Delta \in \mathbb{N}$ and up to $d = 6$ in the boundary, there are a total of 6 cases of relevant operators that comply with these constrains

$$d = 3\,, \quad \Delta = 2\,, \tag{5.103}$$
$$d = 4\,, \quad \Delta = 3\,, \tag{5.104}$$
$$d = 5\,, \quad \Delta = 3,\ 4\,, \tag{5.105}$$
$$d = 6\,, \quad \Delta = 4,\ 5\,. \tag{5.106}$$

Since either $\Delta = d - 1$ or $\Delta = d - 2$, from the definition of $g_{k,N}$ in (5.49) one sees that for all these cases

$$ {}_2F_1\left(\begin{matrix}\frac{\Delta}{2},\ \frac{\Delta+1}{2} \\ \Delta - \frac{d}{2} + 1\end{matrix};\, \xi^2\right)^N = {}_1F_0\left(\frac{d-1}{2};\, \xi^2\right)^N \tag{5.107}$$
$$= {}_1F_0\left(\frac{N(d-1)}{2};\, \xi^2\right)\,, \tag{5.108}$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{N(d-1)}{2}\right)_k \xi^{2k}\,, \tag{5.109}$$

from where we can read the coefficient

$$g_{k,N} = \frac{1}{k!}\left(\frac{N(d-1)}{2}\right)_k\,. \tag{5.110}$$

Then for the 6 cases (5.103)-(5.106), the bulk loop integrals $\int G_\Delta^N$, $\int G_\Delta^N K_{\Delta_2}$ and $\int G_\Delta^N K_{\Delta_3} K_{\Delta_4}$ encoded in the coefficients $\mu_\Delta^{d,N}$, $\eta_{\Delta,\Delta_2}^{d,N}$ and $\chi_{\Delta,\Delta_3,\Delta_4}^{d,N}$ in (5.63), (5.75) and (5.91), the latter through the coefficients a_i and b_i in (5.92) and (5.93), can be computed in closed form in terms of hypergeometric functions

$$\mu_{\Delta}^{d,N}(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \frac{\Gamma\left(\frac{N\Delta-d}{2}\right)}{\Gamma\left(\frac{N\Delta+1}{2}\right)} \left(\frac{1}{\cosh\tau}\right)^{N\Delta} {}_2F_1\left(\begin{matrix}\frac{N(d-1)}{2},\ \frac{N\Delta-d}{2} \\ \frac{N\Delta+1}{2}\end{matrix};\ \frac{1}{\cosh^2\tau}\right), \tag{5.111}$$

$$\eta_{\Delta,\Delta_2}^{d,N}(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \frac{\Gamma\left(\frac{N\Delta+\Delta_2-d}{2}\right)\Gamma\left(\frac{N\Delta-\Delta_2}{2}\right)}{\Gamma\left(\frac{N\Delta}{2}\right)\Gamma\left(\frac{N\Delta+1}{2}\right)} \left(\frac{1}{\cosh\tau}\right)^{N\Delta}$$
$$\times\, {}_3F_2\left(\begin{matrix}\frac{N(d-1)}{2},\ \frac{N\Delta+\Delta_2-d}{2},\ \frac{N\Delta-\Delta_2}{2} \\ \frac{N\Delta}{2},\ \frac{N\Delta+1}{2}\end{matrix};\ \frac{1}{\cosh^2\tau}\right), \tag{5.112}$$

$$a_i(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \frac{\Gamma\left(\frac{N\Delta+\Delta_3+\Delta_4-d}{2}\right)\Gamma\left(\frac{N\Delta-\Delta_3-\Delta_4}{2}\right)(\Delta_3)_i(\Delta_4)_i}{\Gamma\left(\frac{N\Delta}{2}\right)\Gamma\left(\frac{N\Delta+1}{2}\right)\left(1+\frac{\Delta_3+\Delta_4-N\Delta}{2}\right)_i i!} \left(\frac{1}{\cosh\tau}\right)^{N\Delta}$$
$$\times\, {}_3F_2\left(\begin{matrix}\frac{N(d-1)}{2},\ \frac{N\Delta+\Delta_3+\Delta_4-d}{2},\ \frac{N\Delta-\Delta_3-\Delta_4}{2}-i \\ \frac{N\Delta}{2},\ \frac{N\Delta+1}{2}\end{matrix};\ \frac{1}{\cosh^2\tau}\right), \tag{5.113}$$

$$b_i(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^N \frac{\Gamma\left(\frac{N\Delta+\Delta_3-\Delta_4}{2}+i\right)\Gamma\left(\frac{N\Delta-\Delta_3+\Delta_4}{2}+i\right)\Gamma\left(\frac{N\Delta+\Delta_3+\Delta_4-d}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_4-N\Delta}{2}\right)}{\Gamma(\Delta_3)\Gamma(\Delta_4)\left(1+\frac{N\Delta-\Delta_3-\Delta_4}{2}\right)_i\Gamma\left(\frac{N\Delta}{2}\right)\Gamma\left(\frac{N\Delta+1}{2}\right) i!}$$
$$\times \left(\frac{1}{\cosh\tau}\right)^{N\Delta} {}_3F_2\left(\begin{matrix}\frac{N(d-1)}{2},\ \frac{N\Delta+\Delta_3+\Delta_4-d}{2},\ -i \\ \frac{N\Delta}{2},\ \frac{N\Delta+1}{2}\end{matrix};\ \frac{1}{\cosh^2\tau}\right). \tag{5.114}$$

In the last 2, when $N\Delta = \Delta_3 + \Delta_4 - 2\mathbb{N}$, $\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}$ only involves the coefficient b_i as in (5.97), while for $N\Delta = \Delta_3 + \Delta_4 + 2\mathbb{N}_0$ both coefficients a_i and b_i become ill-defined and $\chi^{d,N}_{\Delta,\Delta_3,\Delta_4}$ takes instead the form of (5.98), expressed in terms of some other coefficients c_i and d_i well-defined. For the particular case $N = 2$ and $\Delta_3 = \Delta_4 = \Delta$, these were derived in (5.99) (or (5.102)) and (5.100), which for the current examples can also be computed in closed form

$$c_i(\tau) = \pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 \frac{2\,\Gamma\left(2\Delta-\frac{d}{2}\right)(\Delta)_i^2\left[\psi(1+i)-\psi(\Delta+i)\right]}{\Gamma(\Delta)\Gamma\left(\Delta+\frac{1}{2}\right) i!^2}\left(\frac{1}{\cosh\tau}\right)^{2\Delta}$$
$$\times\, {}_3F_2\left(\begin{matrix} d-1,\, 2\Delta-\frac{d}{2},\, -i \\ \Delta,\, \Delta+\frac{1}{2}\end{matrix};\frac{1}{\cosh^2\tau}\right)$$
$$+\pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 \frac{(d-1)\Gamma\left(2\Delta-\frac{d}{2}+1\right)(\Delta)_i^2}{\Gamma(\Delta+1)\Gamma\left(\Delta+\frac{3}{2}\right) i!^2}\left(\frac{1}{\cosh\tau}\right)^{2(\Delta+1)}$$
$$\times\, F^{3,2,1}_{3,1,0}\left(\begin{matrix} d,\, 2\Delta-\frac{d}{2}+1,\, 1-i \\ \Delta+1,\, \Delta+\frac{3}{2},\, 2\end{matrix};\begin{matrix}1,\, -i\\ 1-i\end{matrix};\begin{matrix}1\\ -\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right), \tag{5.115}$$

$$d_i(\tau) = -\,\pi^{\frac{d+1}{2}} \left(\frac{c_\Delta}{2^{\Delta+1}\nu}\right)^2 \frac{\Gamma\left(2\Delta-\frac{d}{2}\right)(\Delta)_i^2}{\Gamma(\Delta)\Gamma\left(\Delta+\frac{1}{2}\right) i!^2}\left(\frac{1}{\cosh\tau}\right)^{2\Delta}$$
$$\times\, {}_3F_2\left(\begin{matrix} d-1,\, 2\Delta-\frac{d}{2},\, -i \\ \Delta,\, \Delta+\frac{1}{2}\end{matrix};\frac{1}{\cosh^2\tau}\right). \tag{5.116}$$

where the function $F^{3,2,1}_{3,1,0}$ is known as Kampé de Fériet function, whose general form is defined in (B.22), and is represented by the double series

$$F^{3,2,1}_{3,1,0}\left(\begin{matrix} d,\, 2\Delta-\frac{d}{2}+1,\, 1-i \\ \Delta+1,\, \Delta+\frac{3}{2},\, 2\end{matrix};\begin{matrix}1,\, -i\\ 1-i\end{matrix};\begin{matrix}1\\ -\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right)$$
$$=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(d)_{l+k}\left(2\Delta-\frac{d}{2}+1\right)_{l+k}(1-i)_{l+k}(1)_l(-i)_l(1)_k}{(\Delta+1)_{l+k}\left(\Delta+\frac{3}{2}\right)_{l+k}(2)_{l+k}(1-i)_l\, l!\, k!}\left(\frac{1}{\cosh\tau}\right)^{2(l+k)}. \tag{5.117}$$

Noting that the l-series terminates at $l=i$, by computing the k-series one can express $F^{3,2,1}_{3,1,0}$ as the sum of a finite number of generalized hypergeometrics

$$F^{3,2,1}_{3,1,0}\left(\begin{matrix} d,\, 2\Delta-\frac{d}{2}+1,\, 1-i \\ \Delta+1,\, \Delta+\frac{3}{2},\, 2\end{matrix};\begin{matrix}1,\, -i\\ 1-i\end{matrix};\begin{matrix}1\\ -\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right)$$
$$=\sum_{l=0}^{i}\frac{(d)_l\left(2\Delta-\frac{d}{2}+1\right)_l(-i)_l}{(\Delta+1)_l\left(\Delta+\frac{3}{2}\right)_l(2)_l}\left(\frac{1}{\cosh\tau}\right)^{2l}$$
$$\times\, {}_4F_3\left(\begin{matrix} d+l,\, 2\Delta-\frac{d}{2}+1+l,\, l-i+1,\, 1 \\ \Delta+1+l,\, \Delta+\frac{3}{2}+l,\, 2+l\end{matrix};\frac{1}{\cosh^2\tau}\right). \tag{5.118}$$

These results constitute the closed form values for the regularized bulk loop integrals discussed in Section 5.4, for the particular examples (5.103)-(5.106). When these integrals are divergent, one requires the explicit expansion in the UV regulator τ to perform renormalization of the given theory on AdS. Take for instance the case of a single scalar field Φ in the bulk with a $\lambda\Phi^4$ interaction. Up to order λ^2 in the self-interacting coupling constant the coefficients appearing in the loop expansion are $\mu^{d,2}_{\Delta}$, $\eta^{d,3}_{\Delta,\Delta}$ and $\chi^{d,2}_{\Delta,\Delta,\Delta}$ (logarithmic case), the first 2 from the eight and sunset diagrams in the 2-point holographic correlator, and the latter from the double exchange diagram in the 4-point holographic correlator. For the current examples, these are given in closed form by (5.111), (5.112), and the latter through (5.115) and (5.116). To exemplify the regulator expansion of the divergent integrals appearing in this theory, we will work them out explicitly for the cases (5.103) and (5.104). For the rest of the cases, they can be obtained in a similar manner.

5.5.1 Case $d = 3,\ \Delta = 2$

In this case, bulk renormalization of a $\lambda\Phi^4$ theory requires the τ-expansion of the coefficients $\mu_2^{3,2}(\tau)$, $\eta_{2,2}^{3,3}(\tau)$ and $\chi_{2,2,2}^{3,2}(X;\tau)$. From the general expressions obtained above, these are evaluated to

$$\mu_2^{3,2}(\tau) = \frac{1}{12\pi^2}\left(\frac{1}{\cosh\tau}\right)^4 {}_2F_1\left(2,\frac{1}{2};\frac{5}{2};\frac{1}{\cosh^2\tau}\right), \tag{5.119}$$

$$\eta_{2,2}^{3,3}(\tau) = \frac{1}{320\pi^4}\left(\frac{1}{\cosh\tau}\right)^6 {}_2F_1\left(\frac{5}{2},2;\frac{7}{2};\frac{1}{\cosh^2\tau}\right), \tag{5.120}$$

$$\begin{aligned} c_i(\tau) = &-\frac{1}{8\pi^2}(1+i)\left(\frac{1}{\cosh\tau}\right)^4 {}_1F_0\left(-i;\frac{1}{\cosh^2\tau}\right) \\ &+\frac{1}{16\pi^2}(1+i)^2\left(\frac{1}{\cosh\tau}\right)^6 F_{1,1,0}^{1,2,1}\left(\begin{matrix}1-i\\2\end{matrix};\begin{matrix}1,\ -i\\1-i\end{matrix};\begin{matrix}1\\-\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right), \end{aligned} \tag{5.121}$$

$$d_i(\tau) = -\frac{1}{16\pi^2}(1+i)^2\left(\frac{1}{\cosh\tau}\right)^4 {}_1F_0\left(-i;\frac{1}{\cosh^2\tau}\right). \tag{5.122}$$

The τ-expansion of the first 2 can be worked out directly

$$\mu_2^{3,2}(\tau) = -\frac{\ln\tau}{8\pi^2} + \frac{2\ln 2 - 1}{16\pi^2} + \mathcal{O}(\tau), \tag{5.123}$$

$$\eta_{2,2}^{3,3}(\tau) = \frac{1}{128\pi^4\tau^2} + \frac{3\ln\tau}{128\pi^4} + \frac{5-9\ln 2}{384\pi^4} + \mathcal{O}(\tau). \tag{5.124}$$

For the last 2, since the hypergeometric ${}_1F_0$ is convergent in the limit $\tau\to 0$, it can be seen to have an expansion of the form

$${}_1F_0\left(-i;\frac{1}{\cosh^2\tau}\right) = {}_1F_0(-i;1) + \mathcal{O}(\tau) = \delta_{i,0} + \mathcal{O}(\tau), \tag{5.125}$$

while for the Kampé de Fériet function it corresponds to the case (B.28), being able to express it as the product of 2 hypergeometrics

$$\begin{aligned} F_{1,1,0}^{1,2,1}\left(\begin{matrix}1-i\\2\end{matrix};\begin{matrix}1,\ -i\\1-i\end{matrix};\begin{matrix}1\\-\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right) &= {}_1F_0\left(-i;\frac{1}{\cosh^2\tau}\right){}_2F_1\left(1,1;2;\frac{1}{\cosh^2\tau}\right), \\ &= -2\ln\tau\,\delta_{i,0} + \mathcal{O}(\tau). \end{aligned} \tag{5.126}$$

When the expression for the Kampé de Fériet function in terms of simpler functions is not known, its regulator expansion can be obtained directly from its series representation as a sum of hypergeometrics. For the current example this takes the form

$$\begin{aligned} &F^{1,2,1}_{1,1,0}\left(\begin{matrix}1-i\\2\end{matrix};\begin{matrix}1,\ -i\\1-i\end{matrix};\begin{matrix}1\\-\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right)\\ &\quad=\sum_{l=0}^{i}\frac{(-i)_l}{\Gamma(2+l)}\left(\frac{1}{\cosh\tau}\right)^{2l}{}_2F_1\left(\begin{matrix}l-i+1,\ 1\\2+l\end{matrix};\frac{1}{\cosh^2\tau}\right). \end{aligned} \tag{5.127}$$

From a convergence analysis, as $\tau\to 0$ the series is expected to diverge for $i=0$ and converge for $i>0$. This is of course consistent with the convergence region $d<3+2i$ of the coefficient $c_i(\tau)$ derived previously, for the particular case $d=3$. Consider then separating in these 2 cases: first for $i=0$

$$F^{1,2,1}_{1,1,0}\left(\begin{matrix}1-i\\2\end{matrix};\begin{matrix}1,\ -i\\1-i\end{matrix};\begin{matrix}1\\-\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right)\Bigg|_{i=0}={}_2F_1\left(\begin{matrix}1,\ 1\\2\end{matrix};\frac{1}{\cosh^2\tau}\right), \tag{5.128}$$

$$=-2\ln\tau+\mathcal{O}(\tau)\,, \tag{5.129}$$

and then for the case $i>0$, safely expanding around $\tau=0$

$$F^{1,2,1}_{1,1,0}\left(\begin{matrix}1-i\\2\end{matrix};\begin{matrix}1,\ -i\\1-i\end{matrix};\begin{matrix}1\\-\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right)\Bigg|_{i>0}$$

$$=\sum_{l=0}^{i}\frac{(-i)_l}{\Gamma(2+l)}{}_2F_1\left(\begin{matrix}l-i+1,\ 1\\2+l\end{matrix};1\right)+\mathcal{O}(\tau)\,, \tag{5.130}$$

$$=\frac{1}{i}{}_1F_0(-i;1)+\mathcal{O}(\tau)\,, \tag{5.131}$$

$$=\mathcal{O}(\tau)\,, \tag{5.132}$$

recovering the same expansion as before. Putting everything together, this results in the τ-expansion of the coefficients c_i and d_i

$$c_i(\tau)=-\frac{1}{8\pi^2}(\ln\tau+1)\delta_{i,0}+\mathcal{O}(\tau)\,,\qquad d_i(\tau)=-\frac{1}{16\pi^2}\delta_{i,0}+\mathcal{O}(\tau)\,, \tag{5.133}$$

and from (5.98), in the expression for $\chi^{3,2}_{2,2,2}$

$$\chi^{3,2}_{2,2,2}(X;\tau)=-\frac{\ln\tau}{8\pi^2}-\frac{\ln X+2}{16\pi^2}+\mathcal{O}(\tau), \tag{5.134}$$

in the variable $X\equiv K(x_1,\vec{y}_3)K(x_1,\vec{y}_4)|\vec{y}_{34}|^2$.

5.5.2 Case $d = 4,\ \Delta = 3$

In this case, bulk renormalization of a $\lambda\Phi^4$ theory requires the τ-expansion of the coefficients $\mu_3^{4,2}(\tau)$, $\eta_{3,3}^{4,3}(\tau)$ and $\chi_{3,3,3}^{4,2}(X;\tau)$. From the general expressions obtained previously

$$\mu_3^{4,2}(\tau) = \frac{1}{120\pi^2}\left(\frac{1}{\cosh\tau}\right)^6 {}_2F_1\left(3,1;\frac{7}{2};\frac{1}{\cosh^2\tau}\right), \tag{5.135}$$

$$\eta_{3,3}^{4,3}(\tau) = \frac{1}{6720\pi^4}\left(\frac{1}{\cosh\tau}\right)^9 {}_2F_1\left(4,3;5;\frac{1}{\cosh^2\tau}\right), \tag{5.136}$$

$$\begin{aligned} c_i(\tau) = &-\frac{\Gamma(3+i)(3+2i)}{80\pi^2\, i!}\left(\frac{1}{\cosh\tau}\right)^6 {}_2F_1\left(4,-i;\frac{7}{2};\frac{1}{\cosh^2\tau}\right) \\ &+\frac{\Gamma(3+i)^2}{140\pi^2\, i!^2}\left(\frac{1}{\cosh\tau}\right)^8 F^{2,2,1}_{2,1,0}\left(\begin{matrix}5,\ 1-i\\ \frac{9}{2},\ 2\end{matrix};\begin{matrix}1,\ -i\\ 1-i\end{matrix};\begin{matrix}1\\ -\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right), \end{aligned} \tag{5.137}$$

$$d_i(\tau) = -\frac{\Gamma(3+i)^2}{160\pi^2\, i!^2}\left(\frac{1}{\cosh\tau}\right)^6 {}_2F_1\left(4,-i;\frac{7}{2};\frac{1}{\cosh^2\tau}\right). \tag{5.138}$$

The first 2 have a τ-expansion of the form

$$\mu_3^{4,2}(\tau) = \frac{1}{128\pi\tau} - \frac{1}{24\pi^2} + \mathcal{O}(\tau), \tag{5.139}$$

$$\eta_{3,3}^{4,3}(\tau) = \frac{1}{3360\pi^4\tau^4} - \frac{31}{20160\pi^4\tau^2} - \frac{\ln\tau}{280\pi^4} + \frac{109}{172800\pi^4} + \mathcal{O}(\tau). \tag{5.140}$$

For the last 2, since the hypergeometric ${}_2F_1$ is convergent in the limit $\tau \to 0$

$${}_2F_1\left(4,-i;\frac{7}{2};\frac{1}{\cosh^2\tau}\right) = {}_2F_1\left(4,-i;\frac{7}{2};1\right) + \mathcal{O}(\tau) = -\frac{15\,\Gamma\left(-\frac{1}{2}+i\right)}{16\,\Gamma\left(\frac{7}{2}+i\right)} + \mathcal{O}(\tau). \tag{5.141}$$

No simpler expression is known for the Kampé de Fériet function, however as mentioned in the previous example its regulator expansion can be obtained directly from its representation as a sum of hypergeometrics

$$\begin{aligned} &F^{2,2,1}_{2,1,0}\left(\begin{matrix}5,\ 1-i\\ \frac{9}{2},\ 2\end{matrix};\begin{matrix}1,\ -i\\ 1-i\end{matrix};\begin{matrix}1\\ -\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right) \\ &\qquad = \sum_{l=0}^{i}\frac{(5)_l(-i)_l}{\left(\frac{9}{2}\right)_l(2)_l}\left(\frac{1}{\cosh\tau}\right)^{2l} {}_3F_2\left(\begin{matrix}5+l,\ l-i+1,\ 1\\ \frac{9}{2}+l,\ 2+l\end{matrix};\frac{1}{\cosh^2\tau}\right). \end{aligned} \tag{5.142}$$

A convergence analysis suggests the series diverges for $i = 0$ and converges for $i > 0$ as $\tau \to 0$. Separate then in these 2 cases: first for $i = 0$

$$F^{2,2,1}_{2,1,0}\left(\begin{matrix}5,\ 1-i\\ \frac{9}{2},\ 2\end{matrix};\begin{matrix}1,\ -i\\ 1-i\end{matrix};\begin{matrix}1\\ -\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right)\Bigg|_{i=0} = {}_3F_2\left(\begin{matrix}5,\ 1,\ 1\\ \frac{9}{2},\ 2\end{matrix};\frac{1}{\cosh^2\tau}\right), \tag{5.143}$$

$$= \frac{35\pi}{128\tau} + \frac{14}{15} + \mathcal{O}(\tau), \tag{5.144}$$

and then for $i > 0$, safely expanding around $\tau = 0$

$$
\begin{aligned}
F^{2,2,1}_{2,1,0}&\left(\begin{matrix}5,\ 1-i\\ \frac{9}{2},\ 2\end{matrix};\begin{matrix}1,\ -i\\ 1-i\end{matrix};\begin{matrix}1\\ -\end{matrix};\frac{1}{\cosh^2\tau},\frac{1}{\cosh^2\tau}\right)\Bigg|_{i>0}\\
&=\sum_{l=0}^{i}\frac{(5)_l(-i)_l}{\left(\frac{9}{2}\right)_l(2)_l}\,{}_3F_2\left(\begin{matrix}5+l,\ l-i+1,\ 1\\ \frac{9}{2}+l,\ 2+l\end{matrix};1\right)+\mathcal{O}(\tau)\,,\\
&=-\frac{105\,\Gamma\left(-\frac{1}{2}+i\right)^2(1+i)\left[4(1+i)^2-5\right]}{128\,\Gamma\left(\frac{7}{2}+i\right)^2}+\mathcal{O}(\tau)\,,
\end{aligned}
\tag{5.145}
$$

where we computed the value of the hypergeometric ${}_3F_2$ of unit argument using the identity (B.19). Putting everything together, this results in the τ-expansion of the coefficients c_i and d_i

$$
c_0(\tau)=\frac{1}{128\pi\tau}-\frac{29}{600\pi^2}+\mathcal{O}(\tau)\,, \tag{5.146}
$$

$$
\begin{aligned}
c_{i>0}(\tau)&=\frac{3\,\Gamma(3+i)\Gamma\left(-\frac{1}{2}+i\right)(3+2i)}{256\pi^2\Gamma\left(\frac{7}{2}+i\right)i!}\\
&\quad-\frac{3\,\Gamma(3+i)^2\Gamma\left(-\frac{1}{2}+i\right)^2(1+i)\left[4(1+i)^2-5\right]}{512\pi^2\Gamma\left(\frac{7}{2}+i\right)^2 i!^2}+\mathcal{O}(\tau)\,,
\end{aligned}
\tag{5.147}
$$

$$
d_i(\tau)=\frac{3\,\Gamma(3+i)^2\Gamma\left(-\frac{1}{2}+i\right)}{512\pi^2\Gamma\left(\frac{7}{2}+i\right)i!^2}+\mathcal{O}(\tau)\,, \tag{5.148}
$$

with $\chi^{4,2}_{3,3,3}(X;\tau)$ in the form of (5.98)

$$
\chi^{4,2}_{3,3,3}(X;\tau)=\sum_{i=0}^{\infty}\left[c_i(\tau)+d_i(\tau)\ln X\right]X^i\,. \tag{5.149}
$$

In contrast to the previous example $d = 3,\ \Delta = 2$ where all non-vanishing contributions in the limit $\tau \to 0$ are concentrated in the coefficients $i = 0$, in the current case one has non-vanishing contributions at each $i \geq 0$. In the example of $d = 5,\ \Delta = 4$ a quick analysis suggests the non-vanishing contributions are again concentrated in the first coefficients, in this case in $i = 0$ and $i = 1$, possibly indicating that the cases where Δ is an even number are special. It would be interesting to investigate this further.

Chapter 6

Example: Φ^4 theory

As an example of holographic renormalization at loop order, in this chapter we work out in detail the case of a scalar Φ^4 theory on AdS, where as a first approximation the backreaction with the background metric may be ignored. More interesting theories, for instance those coming from the low-energy limit of string theory, also include other type of fields and interactions, however this toy model suffices to show many of the interesting physics that occurs in the AdS/CFT duality once subleading corrections are taken into account, such as the renormalization of the boundary CFT data due to the bulk loops. We begin this study by constructing the renormalized on-shell 1PI effective action for the $\lambda\Phi^4$ theory in Section 6.1, to then in Section 6.2 solve the resulting exact equation of motion perturbatively in the coupling λ. In Section 6.3 we analyze the first effects of loop corrections to order λ, to then in Section 6.4 embrace all the loop corrections appearing in the bulk theory to order λ^2. Holographic renormalization at loop order is carried out for arbitrary values of the bulk mass and dimension, and we end this chapter by studying a concrete case in Section 6.5.

Parts of this chapter have been previously published in [15], and parts to appear in [23].

6.1 Renormalized 1PI effective action

Consider a scalar Φ^4 theory with Dirichlet boundary conditions on AdS, described by the gravitational path integral

$$Z_{\text{AdS}}[\varphi_{B(0)}] = \int_{\Phi\sim\varphi_{B(0)}} \mathrm{D}\Phi\, e^{-S_{\text{AdS}}[\Phi]}\,, \tag{6.1}$$

$$S_{\text{AdS}}[\Phi] = \int \mathrm{d}^{d+1}x\sqrt{g}\left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \frac{m_B^2}{2}\Phi^2 + \frac{\lambda_B}{4!}\Phi^4\right)\,. \tag{6.2}$$

Our goal is to arrive at the renormalized 1PI effective action for this theory as defined in (4.36). But before working out this object directly, with the intention to illustrate the methodology involved we will first compute its bare value. Once these steps are understood, we will go back and repeat this process introducing the different regulators and counterterms, to finally obtain its renormalized version. As discussed in Section 4.3, in the background field method the effective action is derived from the decomposition $\Phi = \phi[\varphi_{B(0)}] + h$, integrating out h and restricting to 1PI terms in ϕ. For the case of (6.1), this decomposition results in

$$Z_{\text{AdS}}[\varphi_{B(0)}] = \int \mathrm{D}h\, e^{-S_{\text{AdS}}[\phi+h]} = e^{-S_{\text{AdS}}[\phi]} \int \mathrm{D}h\, e^{-S_{\text{AdS}}[h]} e^{-\lambda_B \int_x \left(\frac{1}{4}\phi^2 h^2 + \frac{1}{6}\phi h^3\right)}, \tag{6.3}$$

where $S_{\text{AdS}}[\phi]$ is the original action evaluated at $\Phi \to \phi$ (and similarly for h), and where for brevity we called $\int_x \equiv \int \mathrm{d}^{d+1}x\sqrt{g}$. In the last equality, terms linear in h have been omitted as they lead to tadpoles and non-1PI contributions for ϕ. The resulting path integral in h can be evaluated perturbatively in the bare coupling λ_B, in terms of the bulk n-point functions

$$G_n(x_1, \ldots, x_n) = \frac{\int \mathrm{D}h\, h(x_1)\cdots h(x_n) e^{-S_{\text{AdS}}[h]}}{\int \mathrm{D}h\, e^{-S_{\text{AdS}}[h]}}. \tag{6.4}$$

By expanding the last exponential in (6.3), one finds for instance to order λ_B^2

$$\begin{aligned}\frac{Z_{\text{AdS}}[\varphi_{B(0)}]}{Z_{\text{AdS}}[0]} = e^{-S_{\text{AdS}}[\phi]} \Bigg(1 &- \lambda_B \int_x \left[\frac{1}{4}\phi^2(x) G_2(x,x) + \frac{1}{6}\phi(x) G_3(x,x,x)\right] \\ &+ \frac{\lambda_B^2}{2}\int_x\int_y \left[\frac{1}{16}\phi^2(x)\phi^2(y) G_4(x,x,y,y) + \frac{1}{12}\phi^2(x)\phi(y) G_5(x,x,y,y,y)\right. \\ &\left. + \frac{1}{36}\phi(x)\phi(y) G_6(x,x,x,y,y,y)\right] + \mathcal{O}(\lambda_B^3)\Bigg). \end{aligned} \tag{6.5}$$

These bulk n-point functions G_n are simply those of a h^4 theory, and thus computable using standard perturbative QFT manipulations: define the path integral Z_h of the theory $S_{\text{AdS}}[h]$, add a source J_h coupled to the field h, write interaction terms in h as derivatives of J_h and move them outside the path integral, and perform the remaining Gaussian integral in terms of the inverse of the differential operator $-\Box + m_B^2$, corresponding to the bare bulk-to-bulk propagator $G_{\Delta_B}(x,y)$

$$Z_h[J_h] = \int \mathrm{D}h\, e^{-S_{\text{AdS}}[h] + \int_x hJ_h} = N e^{-\frac{\lambda_B}{4!}\int_x \left(\frac{1}{\sqrt{g}}\frac{\delta}{\delta J_h}\right)^4} e^{\frac{1}{2}\int_x\int_y J_h(x) G_{\Delta_B}(x,y) J_h(y)}, \tag{6.6}$$

where N is an unimportant constant. The quantities G_n are then computed from the expression above by normalizing, functionally differentiating with respect to J_h, and setting the sources to 0

$$G_n(x_1,\dots,x_n) = \frac{1}{\sqrt{g_1}}\frac{\delta}{\delta J_h(x_1)}\cdots\frac{1}{\sqrt{g_n}}\frac{\delta}{\delta J_h(x_n)}\frac{Z_h[J_h]}{Z_h[0]}\Bigg|_{J_h=0}, \tag{6.7}$$

obtaining to the relevant order in λ_B

$$G_2(x,x) = G_{\Delta_B}(x,x) - \frac{\lambda_B}{2}\int_y G^2_{\Delta_B}(x,y)G_{\Delta_B}(y,y) + \mathcal{O}(\lambda_B^2)\,, \tag{6.8}$$

$$G_4(x,x,y,y) = G_{\Delta_B}(x,x)G_{\Delta_B}(y,y) + 2\,G^2_{\Delta_B}(x,y) + \mathcal{O}(\lambda_B)\,, \tag{6.9}$$

$$G_6(x,x,x,y,y,y) = 9\,G_{\Delta_B}(x,x)G_{\Delta_B}(x,y)G_{\Delta_B}(y,y) + 6\,G^3_{\Delta_B}(x,y) + \mathcal{O}(\lambda_B)\,, \tag{6.10}$$

with the odd-point functions $G_{2n+1} = 0$, as expected for a $\mathbb{Z}_2$-invariant theory. Replacing these in the expression above, to order λ_B^2 the resulting expansion may be resummed back into an exponential, which restricted to 1PI terms allows for the identification of the bare effective action as defined in (4.34)

$$\begin{aligned}\Gamma_{\text{AdS}}[\varphi_{B(0)}] = S_{\text{AdS}}[\phi] &+ \frac{\lambda_B}{4}\int_x \phi^2(x)G_{\Delta_B}(x,x) - \frac{\lambda_B^2}{8}\int_x\int_y \phi^2(x)G^2_{\Delta_B}(x,y)G_{\Delta_B}(y,y)\\ &- \frac{\lambda_B^2}{12}\int_x\int_y \phi(x)\phi(y)G^3_{\Delta_B}(x,y) - \frac{\lambda_B^2}{16}\int_x\int_y \phi^2(x)\phi^2(y)G^2_{\Delta_B}(x,y) + \mathcal{O}(\lambda_B^3)\,,\end{aligned} \tag{6.11}$$

with $\phi = \phi[\varphi_{B(0)}]$. Tree-level correlators come from S_{AdS}, while the rest of terms in Γ_{AdS} constitute the loop corrections computed perturbatively in λ_B. Now, this bare object is clearly ill-defined, not only due to the usual IR divergences at tree-level present in S_{AdS}, but also at loop-level due to the IR divergences of the integrals and the UV divergences coming from the short-distance singularities of the bulk propagator. The effective action needs to be renormalized, and we will follow the recipe of Chapter 4: divergences are regularized by restricting the radial coordinate to $z \geq \varepsilon$ and replacing the propagator by its regularized version $G_{\tau,\Delta}$, and these divergences are renormalized by adding a boundary counterterm B at $z = \varepsilon$ and Z-factors for the bulk parameters. To order λ^2, the required Z-factors to absorb all bulk divergences are

$$\varphi_{B(0)} = Z_\varphi\varphi_{(0)}\,,\qquad m_B^2 = Z_m m^2 = m^2 + \delta m^2\,,\qquad \lambda_B = Z_\lambda\lambda = \lambda + \delta\lambda\,, \tag{6.12}$$

of orders $Z_\varphi = 1 + \mathcal{O}(\lambda)$, $\delta m^2 = \mathcal{O}(\lambda)$ and $\delta\lambda = \mathcal{O}(\lambda^2)$. Surprisingly, no wavefunction renormalization is required to absorb the divergences to this order. This is very different from the case of Φ^4 theory in flat space, where this counterterm is needed to absorb one of the UV divergences in the sunset diagram proportional to p^2 [93]. In our case, the sunset diagram turns out to be proportional to the mass-shift diagram thanks to the property (5.65) of the Witten diagrams (later on used to compute the sunset in (6.51)), being able to renormalize all its UV divergences with δm^2.

The computation of the renormalized effective action $\Gamma^{\text{Ren}}_{\text{AdS}}$ follows from the same starting point (6.1) as the bare case, but now in the presence of these regulators and counterterms, and the perturbative problem set in terms of the finite coupling λ rather than its bare value λ_B. As before, decomposing $\Phi = \phi + h$, ignoring terms linear in h, and evaluating the resulting path integral in h perturbatively in λ in terms of the bulk n-point functions (6.4), one finds

$$\begin{aligned}\frac{Z^{\text{Sub}}_{\text{AdS}}[\varphi_{(0)}]}{Z^{\text{Sub}}_{\text{AdS}}[0]} = e^{-S_{\text{AdS}}[\phi]-B[\phi]}\Bigg(1 &- (\lambda+\delta\lambda)\int_x \left[\frac{1}{4}\phi^2(x)G_2(x,x) + \frac{1}{6}\phi(x)G_3(x,x,x)\right] \\ &+ \frac{\lambda^2}{2}\int_x\int_y \left[\frac{1}{16}\phi^2(x)\phi^2(y)G_4(x,x,y,y) + \frac{1}{12}\phi^2(x)\phi(y)G_5(x,x,y,y,y)\right. \\ &\left. + \frac{1}{36}\phi(x)\phi(y)G_6(x,x,x,y,y,y)\right] + \mathcal{O}(\lambda^3)\Bigg)\,, \end{aligned} \tag{6.13}$$

where now bulk integrals correspond to the regularized volume element $\int_x \equiv \int_{z\geq\varepsilon} \mathrm{d}^{d+1}x\sqrt{g}$, and where the bulk n-point functions G_n are computed from

$$Z_h[J_h] = N e^{-\frac{\delta m^2}{2}\int_x \left(\frac{1}{\sqrt{g}}\frac{\delta}{\delta J_h}\right)^2} e^{-\frac{\lambda+\delta\lambda}{4!}\int_x \left(\frac{1}{\sqrt{g}}\frac{\delta}{\delta J_h}\right)^4} e^{\frac{1}{2}\int_x\int_y J_h(x)G_{\tau,\Delta}(x,y)J_h(y)}\,, \tag{6.14}$$

obtaining to the relevant order in λ

$$G_2(x,x) = G_{\tau,\Delta}(x,x) - \int_y G^2_{\tau,\Delta}(x,y)\left[\delta m^2 + \frac{\lambda}{2}G_{\tau,\Delta}(y,y)\right] + \mathcal{O}(\lambda^2)\,, \tag{6.15}$$

$$G_4(x,x,y,y) = G_{\tau,\Delta}(x,x)G_{\tau,\Delta}(y,y) + 2\,G^2_{\tau,\Delta}(x,y) + \mathcal{O}(\lambda)\,, \tag{6.16}$$

$$G_6(x,x,x,y,y,y) = 9\,G_{\tau,\Delta}(x,x)G_{\tau,\Delta}(x,y)G_{\tau,\Delta}(y,y) + 6\,G^3_{\tau,\Delta}(x,y) + \mathcal{O}(\lambda)\,, \tag{6.17}$$

with odd-point functions vanishing. Calling the regularized number $G_{\tau,\Delta}(x,x) \equiv G_{\tau,\Delta}(1)$, replacing these values in the expression above, to order λ^2 the resulting expansion may be resummed back into an exponential, which restricted to 1PI terms allows for the identification of the subtracted effective action. Then the renormalized effective action is simply obtained under the limit of vanishing regulators

$$\begin{aligned}\Gamma^{\text{Ren}}_{\text{AdS}} = \lim_{\varepsilon\to 0}\lim_{\tau\to 0}\, & S_{\text{AdS}}[\phi] + B[\phi] + \frac{\lambda+\delta\lambda}{4}G_{\tau,\Delta}(1)\int_x \phi^2(x) \\ & - \frac{\lambda}{4}\left[\delta m^2 + \frac{\lambda}{2}G_{\tau,\Delta}(1)\right]\int_x\int_y \phi^2(x)G^2_{\tau,\Delta}(x,y) \\ & - \frac{\lambda^2}{12}\int_x\int_y \phi(x)\phi(y)G^3_{\tau,\Delta}(x,y) - \frac{\lambda^2}{16}\int_x\int_y \phi^2(x)\phi^2(y)G^2_{\tau,\Delta}(x,y)\,,\end{aligned} \tag{6.18}$$

with $\phi = \phi[Z_\varphi \varphi_{(0)}]$. This result followed from repeating the same steps as the derivation of the bare effective action, but with the introduction of regulators and counterterms right from the beginning to have a well-defined starting point. Similarly, we could have used the result for the bare effective action (6.11) as our starting point, and introduce at this stage the same regulators and counterterms to try constructing directly its renormalized version. By doing so, upon use of the results in Section 5.2 to write the λ-expansion of the bare propagator as

$$G_{\Delta_B}(x,y) = G_\Delta(x,y) - \delta m^2 \int_w G_\Delta(x,w) G_\Delta(w,y) + \mathcal{O}(\lambda^2)\,, \tag{6.19}$$

one can check this alternative derivation exactly reproduces the same expression for the renormalized effective action.

In the decomposition $\Phi = \phi + h$, the ϕ only appear as external lines with internal loops run by the fluctuations h and thus responsible of the UV divergences. In the computation of Γ_{AdS} these internal lines of h are represented by the bulk n-point functions G_n, which are computed from the path integral Z_h of the theory $S_{\text{AdS}}[h]$. Once Z_h has been expressed in terms of the bulk propagator G_Δ, the UV regularization scheme has been implemented as discussed at the beginning of Subsection 4.1.1 with the prescription $G_\Delta \to G_{\tau,\Delta}$, leading to regularized internal lines G_n constructed from $G_{\tau,\Delta}$ but unregularized external lines of ϕ constructed from K_Δ. As discussed in 5.1, since external K_Δ do not contribute with UV divergences in the bulk, this scheme suffices to regularize the UV of all loop integrals on AdS. Similarly, we could have chosen a more symmetric picture and implement the regularization scheme discussed at the end of 4.1.1, replacing the bare kinetic term of $S_{\text{AdS}}[\Phi]$ with the regularized term (4.26) that has the regularized propagator $G_{\tau,\Delta}$ as its inverse. In this scheme, one obtains the same expression for $\Gamma^{\text{Ren}}_{\text{AdS}}$ as (6.18) but with the kinetic term in $S_{\text{AdS}}[\phi]$ replaced by the regularized one, leading also to the regularization of external lines now constructed by some $K_{\tau,\Delta}$. Note however this scheme only differs by subleading, scheme-dependent terms of τ, and the previous scheme can be directly recovered by simply evaluating the regularized kinetic term for ϕ at $\tau = 0$.

6.2 Exact solution

The variation of the renormalized effective action (6.18) consists only in a boundary term

$$\delta\Gamma^{\text{Ren}}_{\text{AdS}} = \lim_{\varepsilon\to 0}\lim_{\tau\to 0}\ \delta B[\phi] - \int \mathrm{d}^d x\ \sqrt{g}\,\partial^z\phi\delta\phi\big|_{z=\varepsilon}\,, \tag{6.20}$$

as the bulk term vanishes given the on-shell equation for the field ϕ

$$\begin{aligned}(-\Box + m^2)\phi(x) = &\\ -\left[\delta m^2 + \frac{\lambda+\delta\lambda}{2}G_{\tau,\Delta}(1)\right]\phi(x) &+ \frac{\lambda}{2}\left[\delta m^2 + \frac{\lambda}{2}G_{\tau,\Delta}(1)\right]\phi(x)\int_y G^2_{\tau,\Delta}(x,y) \\ + \frac{\lambda^2}{6}\int_y \phi(y) G^3_{\tau,\Delta}(x,y) &- \frac{\lambda+\delta\lambda}{6}\phi^3(x) + \frac{\lambda^2}{4}\phi(x)\int_y \phi^2(y) G^2_{\tau,\Delta}(x,y) + \mathcal{O}(\lambda^3)\,.\end{aligned} \tag{6.21}$$

This is a non-linear integral equation for the field. It may be solved perturbatively in the coupling λ by expanding the field and the counterterms in this parameter

$$\phi = \phi_{\{0\}} + \lambda\phi_{\{1\}} + \lambda^2\phi_{\{2\}} + \mathcal{O}(\lambda^3)\,, \tag{6.22}$$

$$\delta m^2 = \lambda\delta m^2_{\{1\}} + \lambda^2\delta m^2_{\{2\}} + \mathcal{O}(\lambda^3)\,, \tag{6.23}$$

$$\delta\lambda = \lambda^2\delta\lambda_{\{2\}} + \mathcal{O}(\lambda^3)\,, \tag{6.24}$$

where the subscript $\{n\}$ denotes the nth-order component in the λ-expansion. Replacing these expansions back in the equation of motion, since the equation must hold at each order in λ this leads to a set of equations for each $\phi_{\{n\}}$ in terms of the previous $n-1$ components

$$\mathcal{O}(\lambda^0):\ (-\Box + m^2)\phi_{\{0\}}(x) = 0\,, \tag{6.25}$$

$$\mathcal{O}(\lambda^1):\ (-\Box + m^2)\phi_{\{1\}}(x) = -\left[\delta m^2_{\{1\}} + \frac{1}{2}G_{\tau,\Delta}(1)\right]\phi_{\{0\}}(x) - \frac{1}{6}\phi^3_{\{0\}}(x)\,, \tag{6.26}$$

$$\begin{aligned}\mathcal{O}(\lambda^2):\ (-\Box + m^2)\phi_{\{2\}}(x) &= -\left[\delta m^2_{\{1\}} + \frac{1}{2}G_{\tau,\Delta}(1)\right]\phi_{\{1\}}(x) \\ &- \left[\delta m^2_{\{2\}} + \frac{\delta\lambda_{\{2\}}}{2}G_{\tau,\Delta}(1)\right]\phi_{\{0\}}(x) + \frac{1}{2}\left[\delta m^2_{\{1\}} + \frac{1}{2}G_{\tau,\Delta}(1)\right]\phi_{\{0\}}(x)\int_y G^2_{\tau,\Delta}(x,y) \\ &+ \frac{1}{6}\int_y \phi_{\{0\}}(y)G^3_{\tau,\Delta}(x,y) - \frac{1}{2}\phi^2_{\{0\}}(x)\phi_{\{1\}}(x) - \frac{\delta\lambda_{\{2\}}}{6}\phi^3_{\{0\}}(x) \\ &+ \frac{1}{4}\phi_{\{0\}}(x)\int_y \phi^2_{\{0\}}(y)G^2_{\tau,\Delta}(x,y)\,.\end{aligned} \tag{6.27}$$

The solution to (6.25) with Dirichlet boundary condition $\phi \sim \varphi_{B(0)} = Z_\varphi\varphi_{(0)}$, that is also regular in the bulk interior is given by

$$\phi_{\{0\}}(x) = \int \mathrm{d}^d y\, K_\Delta(x,\vec{y})Z_\varphi\varphi_{(0)}(\vec{y})\,, \tag{6.28}$$

with $K_\Delta(x,\vec{y})$ the bulk-to-boundary propagator. Replacing this in (6.26), the component $\phi_{\{1\}}$ is found inverting the operator $-\Box + m^2$ using the bulk-to-bulk propagator $G_\Delta(x,y)$. This process is then repeated to find $\phi_{\{2\}}$ and all higher-order components. In this way, the exact solution to (6.21) is constructed iteratively in λ starting from the leading component $\phi_{\{0\}}$. The resulting expression for $\phi(x)$ is long as it explicitly contains, to a given order in the bulk loop expansion, all connected contributions to every holographic n-point function of the dual theory. Once the expression for the field has been obtained, it is then convenient to write it in powers of the source $\varphi_{(0)}$

$$\phi(x) = \phi_{[1]}(x) + \phi_{[3]}(x) + \cdots\,, \qquad \phi_{[n]}(x) = \mathcal{O}(\varphi^n_{(0)})\,, \tag{6.29}$$

with the data of each $(n+1)$-point function contained in the term $\phi_{[n]}$ (not to be confused with $\phi_{\{n\}}$). To order λ^2 in bulk loops and to order $\varphi^3_{(0)}$ in the source (relevant up to the 4-point function) we find

$$
\begin{aligned}
\phi_{[1]}(x) = {} & \int_{\vec{y}} K_\Delta(x,\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y}) \\
& - \left[\lambda \delta m^2_{\{1\}} + \lambda^2 \delta m^2_{\{2\}} + \frac{\lambda + \lambda^2 \delta\lambda_{\{2\}}}{2} G_{\tau,\Delta}(1)\right] \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] \\
& + \frac{\lambda}{2}\left[\lambda \delta m^2_{\{1\}} + \frac{\lambda}{2} G_{\tau,\Delta}(1)\right] \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] \int_{x''} G^2_{\tau,\Delta}(x',x'') \\
& + \frac{\lambda^2}{6} \int_{x'} G_\Delta(x,x') \int_{x''} G^3_{\tau,\Delta}(x',x'') \left[\int_{\vec{y}} K_\Delta(x'',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] \\
& + \left[\lambda \delta m^2_{\{1\}} + \frac{\lambda}{2} G_{\tau,\Delta}(1)\right]^2 \int_{x'} G_\Delta(x,x') \int_{x''} G_\Delta(x',x'') \left[\int_{\vec{y}} K_\Delta(x'',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right],
\end{aligned}
\tag{6.30}
$$

and

$$
\begin{aligned}
\phi_{[3]}(x) = {} & -\frac{\lambda + \lambda^2 \delta\lambda_{\{2\}}}{6} \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right]^3 \\
& + \frac{\lambda^2}{4} \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] \int_{x''} G^2_{\tau,\Delta}(x',x'') \left[\int_{\vec{y}} K_\Delta(x'',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right]^2 \\
& + \frac{\lambda}{2}\left[\lambda \delta m^2_{\{1\}} + \frac{\lambda}{2} G_{\tau,\Delta}(1)\right] \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right]^2 \\
& \qquad \times \int_{x''} G_\Delta(x',x'') \left[\int_{\vec{y}} K_\Delta(x'',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] \\
& + \frac{\lambda}{6}\left[\lambda \delta m^2_{\{1\}} + \frac{\lambda}{2} G_{\tau,\Delta}(1)\right] \int_{x'} G_\Delta(x,x') \int_{x''} G_\Delta(x',x'') \left[\int_{\vec{y}} K_\Delta(x'',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right]^3,
\end{aligned}
\tag{6.31}
$$

where we called $\int_{\vec{y}} \equiv \int \mathrm{d}^d y$. Diagrammatically, the Witten diagrams contributing to every correlator can be read directly from the expressions for $\phi_{[n]}(x)$ by representing the propagators K and G as lines on AdS, and with the bulk point x extended all the way to the boundary. By doing so, the terms contained in $\phi_{[1]}$ and $\phi_{[3]}$ are seen to correspond precisely to all connected contributions, including those coming from the counterterms δm^2 and $\delta\lambda$, to the 2-point function (Fig. 6.1) and the 4-point function (Fig. 6.2) of a scalar quartic theory to order λ^2. Holographic renormalization through the counterterms B in (6.20) and $Z_\varphi, \delta m^2, \delta\lambda$ in $\phi(x)$ requires the identification of the different divergent pieces, and those coming from the loop integrals will be determined using the results of Chapter 5.

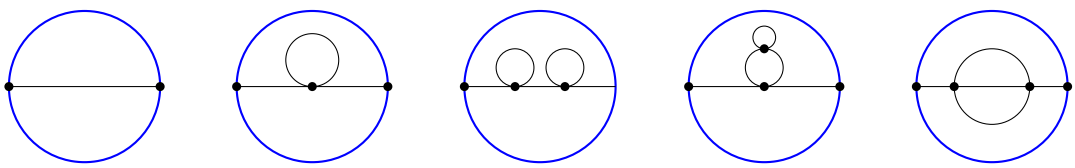

FIGURE 6.1: Witten diagrams contained in $\phi_{[1]}$ contributing to the 2-point function.

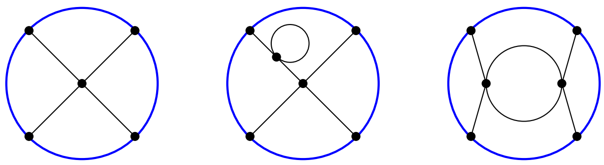

FIGURE 6.2: Witten diagrams contained in $\phi_{[3]}$ contributing to the 4-point function.

6.3 Holographic renormalization at order λ

6.3.1 Bulk renormalization

Before dealing with all the loops to order λ^2, for simplicity first we will deal with those at order λ. To this order, the value of the field computed in (6.29) differs from the tree-level case only from the terms linear in the source

$$\begin{aligned}\phi_{[1]}(x) = &\int_{\vec{y}} K_\Delta(x,\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y}) \\ &- \left[\lambda \delta m^2_{\{1\}} + \frac{\lambda}{2} G_{\tau,\Delta}(1)\right] \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] , \end{aligned} \tag{6.32}$$

with $\phi_{[3]}$ being already of order λ, the same as tree-level. All UV divergences come from the regularized propagator at coincident points $G_{\tau,\Delta}(1)$, which are renormalized through the component $\delta m^2_{\{1\}}$ of the mass counterterm. Indeed, decomposing the propagator in its divergent and convergent parts

$$G_{\tau,\Delta}(1) = \text{Div}\left[G_{\tau,\Delta}(1)\right] + \text{Con}\left[G_{\tau,\Delta}(1)\right] , \tag{6.33}$$

fixes $\delta m^2_{\{1\}}$ to the value

$$\delta m^2_{\{1\}} = -\frac{1}{2}\text{Div}\left[G_{\tau,\Delta}(1)\right] + F_{m,1} , \tag{6.34}$$

where $F_{m,1}$ is an arbitrary constant that captures the scheme dependence of such subtraction, which in turn is fixed by renormalization conditions. With UV divergences renormalized, one may safely take the limit of vanishing regulator τ, allowing us to write $\phi_{[1]}$ as

$$\phi_{[1]}(x) = \int_{\vec{y}} K_\Delta(x,\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y}) - \Pi \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] , \tag{6.35}$$

where we defined the finite mass correction coefficient

$$\Pi \equiv \lim_{\tau\to 0} \lambda \left(\frac{1}{2}\text{Con}\left[G_{\tau,\Delta}(1)\right] + F_{m,1}\right) . \tag{6.36}$$

The remaining IR divergences in the field come from the $\int GK$ integral in the second term of $\phi_{[1]}$. This integral was studied in Subsection 5.3.2, with its value derived in

(5.46). Using this, the resulting expressions can be resummed to order λ thanks to the series expansion

$$\frac{(a)_x}{(b)_x} z^x = 1 + x \ln\left[z e^{\psi(a)-\psi(b)}\right] + \frac{x^2}{2}\left(\ln^2\left[z e^{\psi(a)-\psi(b)}\right] + \psi'(a) - \psi'(b)\right) + \mathcal{O}(x^3)\,, \tag{6.37}$$

where $(a)_x$ is the Pochhammer symbol and $\psi(a)$ the digamma function, leading to

$$\phi_{[1]}(x) = \int_{\vec{y}} K_{\Delta_{\mathcal{R}}}(x,\vec{y}) \left(\varepsilon\, e^{-\frac{1}{2\nu}}\right)^{\Delta_{\mathcal{R}}-\Delta} Z_\varphi \varphi_{(0)}(\vec{y})\,, \tag{6.38}$$

where we defined the renormalized mass and renormalized conformal dimension

$$m_{\mathcal{R}}^2 \equiv m^2 + \Pi\,, \qquad \Delta_{\mathcal{R}} \equiv \frac{d}{2} + \sqrt{\frac{d^2}{4} + m_{\mathcal{R}}^2} = \Delta + \frac{\Pi}{2\nu} + \mathcal{O}(\lambda^2)\,. \tag{6.39}$$

Thus, IR divergences in $\phi(x)$ are renormalized through the source counterterm Z_φ, fixing its value to

$$Z_\varphi = F_\varphi\, \varepsilon^{\Delta-\Delta_{\mathcal{R}}}\,, \tag{6.40}$$

with the scheme-dependence captured by the arbitrary factor $F_\varphi = 1 + \mathcal{O}(\lambda)$. The freedom of changing the source by a constant factor is already present at tree-level, allowing us to fix the normalization of the 2-point function. We will use this freedom to define a renormalized source

$$\varphi_{\mathcal{R}(0)}(\vec{x}) \equiv F_\varphi \left(e^{-\frac{1}{2\nu}}\right)^{\Delta_{\mathcal{R}}-\Delta} \varphi_{(0)}(\vec{x})\,, \tag{6.41}$$

allowing us to write the renormalized expression for $\phi_{[1]}$ as

$$\phi_{[1]}(x) = \int_{\vec{y}} K_{\Delta_{\mathcal{R}}}(x,\vec{y}) \varphi_{\mathcal{R}(0)}(\vec{y})\,. \tag{6.42}$$

One can check the renormalized source $\varphi_{\mathcal{R}(0)}$, or equivalently $\varphi_{(0)}$ (as they only differ by finite numeric factors), transforms as the source of an operator with dimension $\Delta_{\mathcal{R}}$, thanks to Z_φ carrying the precise factors of ε: under a bulk rescaling $x^\mu \to \lambda x^\mu$, $\varepsilon \to \lambda\varepsilon$, using that the bare source $\varphi_{B(0)} = Z_\varphi \varphi_{(0)}$ has conformal weight $d - \Delta$

$$\begin{aligned} \varphi_{(0)}(\lambda\vec{x}) &= Z_\varphi^{-1}(\lambda\varepsilon)\varphi_{B(0)}(\lambda\vec{x}) = \lambda^{\Delta_{\mathcal{R}}-\Delta} Z_\varphi^{-1}(\varepsilon) \lambda^{-(d-\Delta)} \varphi_{B(0)}(\vec{x}) \\ &= \lambda^{-(d-\Delta_{\mathcal{R}})} \varphi_{(0)}(\vec{x})\,, \end{aligned} \tag{6.43}$$

which is the correct transformation.

With the bulk field completely renormalized to order λ, one may write it then to this order as

$$\phi(x) = \int_{\vec{y}} K_{\Delta_{\mathcal{R}}}(x,\vec{y}) \varphi_{\mathcal{R}(0)}(\vec{y}) - \frac{\lambda}{6} \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) \varphi_{(0)}(\vec{y})\right]^3. \tag{6.44}$$

From this point on, we can continue the analysis in complete analogy with the classical case: for generic values of $\nu_{\mathcal{R}} = \Delta_{\mathcal{R}} - d/2$ the field has a near-boundary expansion of the form

$$\phi(x) = z^{d-\Delta_{\mathcal{R}}}\varphi_{\mathcal{R}(0)}(\vec{x}) + \cdots + z^{\Delta_{\mathcal{R}}}\varphi_{(2\nu_{\mathcal{R}})}(\vec{x}) + \cdots, \tag{6.45}$$

with the normalizable mode $\varphi_{(2\nu_{\mathcal{R}})}$ as a functional of the source

$$\varphi_{(2\nu_{\mathcal{R}})}(\vec{x}) = \int_{\vec{y}} \frac{c_{\Delta_{\mathcal{R}}}}{|\vec{x}-\vec{y}|^{2\Delta_{\mathcal{R}}}}\varphi_{\mathcal{R}(0)}(\vec{y}) - \frac{\lambda}{12\nu}\int_{x'} K_{\Delta}(x',\vec{x})\left[\int_{\vec{y}} K_{\Delta}(x',\vec{y})\varphi_{(0)}(\vec{y})\right]^3 . \tag{6.46}$$

Since this is the same asymptotics as tree-level but for a shifted value of the bulk mass, the IR divergences coming from the boundary term (6.20) are renormalized with the same structure of boundary counterterms as (2.18) but with $\Delta \to \Delta_{\mathcal{R}}$

$$B[\phi] = \int_{z=\varepsilon} \mathrm{d}^d x\sqrt{\gamma}\left[\frac{(d-\Delta_{\mathcal{R}})}{2}\phi^2(x) + \frac{1}{2}\sum_{n=1}^{\lfloor \nu_{\mathcal{R}}\rfloor} c_n(\nu_{\mathcal{R}})\phi(x)\Box_{\gamma}^n\phi(x)\right], \tag{6.47}$$

and with ϕ the renormalized bulk field to loop order λ, leading to the finite variation

$$\delta\Gamma^{\mathrm{Ren}}_{\mathrm{AdS}}[\varphi_{\mathcal{R}(0)}] = -\int \mathrm{d}^d x\; 2\nu_{\mathcal{R}}\varphi_{(2\nu_{\mathcal{R}})}(\vec{x})\delta\varphi_{\mathcal{R}(0)}(\vec{x})\,. \tag{6.48}$$

6.3.2 Renormalized correlators

Differentiating with respect to the renormalized sources $\varphi_{\mathcal{R}(0)}$ and setting them to 0 results in the quantum corrected holographic 2- and 4-point functions to order λ

$$\langle \mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_1)\mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_2)\rangle = \frac{2\nu_{\mathcal{R}}c_{\Delta_{\mathcal{R}}}}{|\vec{y}_1-\vec{y}_2|^{2\Delta_{\mathcal{R}}}}, \tag{6.49}$$

$$\langle \mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_1)\mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_2)\mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_3)\mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_4)\rangle = -\lambda c_{\Delta}^4 D_{\Delta,\Delta,\Delta,\Delta}\,. \tag{6.50}$$

To this loop order, the effect of the bulk quantum corrections are seen to renormalize the dimension of the boundary source and operator from Δ to $\Delta_{\mathcal{R}}$. Note however, this renormalization is not reflected by the expression of the 4-point function. This is expected, since this term being already of order λ does not see the effect of loops until λ^2. As we will see next, this is precisely what happens to the correlator when one starts considering the contributions from the next loop order.

6.4 Holographic renormalization at order λ^2

6.4.1 Bulk renormalization

Having studied the effects of loops to lowest order in λ in Section 6.3, we will move to the next order in the loop expansion with the full solution for the bulk field derived in (6.29), expressed in terms of the quantities $\phi_{[1]}$ and $\phi_{[3]}$ in (6.30) and (6.31), respectively. We will being the analysis by simplifying the expression for $\phi_{[1]}$ noting that the integrals at the bulk point x'' of the terms represented by the eight and sunset diagrams (3rd and 4th lines of (6.30)), being IR convergent for the Dirichlet case $\Delta > d/2$, can be computed in terms of the coefficients μ and η introduced in Subsections 5.4.2 and 5.4.3

$$\int_{x''} G^2_{\tau,\Delta}(x',x'') = \mu^{d,2}_{\Delta}(\tau)\,, \qquad \int_{x''} G^3_{\tau,\Delta}(x',x'')K_\Delta(x'',\vec{y}) = \eta^{d,3}_{\Delta,\Delta}(\tau)K_\Delta(x',\vec{y})\,, \qquad (6.51)$$

properties that were proved for these bulk loop integrals exploiting only the AdS covariance of the measure and the propagators. In terms of these coefficients and the component $\delta m^2_{\{1\}}$ of the mass counterterm fixed from the UV renormalization at order λ in (6.34), making a similar decomposition in divergent and convergent parts as in (6.33) the remaining UV divergences in the 2-point function of order λ^2 are renormalized by the component $\delta m^2_{\{2\}}$, fixing it to the value

$$\begin{aligned}\delta m^2_{\{2\}} =& -\frac{1}{2}\mathrm{Div}\left[\delta\lambda_{\{2\}}G_{\tau,\Delta}(1)\right] + \frac{1}{2}\mathrm{Div}\left[\left(\frac{1}{2}\mathrm{Con}\left[G_{\tau,\Delta}(1)\right] + F_{m,1}\right)\mu^{d,2}_{\Delta}(\tau)\right] \\ &+ \frac{1}{6}\mathrm{Div}\left[\eta^{d,3}_{\Delta,\Delta}(\tau)\right] + F_{m,2}\,,\end{aligned} \qquad (6.52)$$

with $F_{m,2}$ the arbitrary constant that captures the scheme dependence of the subtraction to this order. With UV divergences in $\phi_{[1]}$ renormalized, one may safely take in this quantity the limit of vanishing regulator τ, allowing us to write it to order λ^2 as

$$\begin{aligned}\phi_{[1]}(x) =& \int_{\vec{y}} K_\Delta(x,\vec{y})Z_\varphi\varphi_{(0)}(\vec{y}) - \Pi\int_{x'} G_\Delta(x,x')\left[\int_{\vec{y}} K_\Delta(x',\vec{y})Z_\varphi\varphi_{(0)}(\vec{y})\right] \\ &+ \Pi^2\int_{x'} G_\Delta(x,x')\int_{x''} G_\Delta(x',x'')\left[\int_{\vec{y}} K_\Delta(x'',\vec{y})Z_\varphi\varphi_{(0)}(\vec{y})\right]\,,\end{aligned} \qquad (6.53)$$

where we defined the finite mass correction coefficient

$$\begin{aligned}\Pi = \lim_{\tau\to 0} \lambda& \left(\frac{1}{2}\mathrm{Con}\left[G_{\tau,\Delta}(1)\right] + F_{m,1}\right) \\ &+ \lambda^2\left(\frac{1}{2}\mathrm{Con}\left[\delta\lambda_{\{2\}}G_{\tau,\Delta}(1)\right] - \frac{1}{2}\mathrm{Con}\left[\left(\frac{1}{2}\mathrm{Con}\left[G_{\tau,\Delta}(1)\right] + F_{m,1}\right)\mu^{d,2}_{\Delta}(\tau)\right]\right. \\ &\left. -\frac{1}{6}\mathrm{Con}\left[\eta^{d,3}_{\Delta,\Delta}(\tau)\right] + F_{m,2}\right)\,.\end{aligned} \qquad (6.54)$$

IR divergences in $\phi_{[1]}$ come from the bulk integrals of the last 2 terms, which from the study at order λ are expected to be renormalized by the source counterterm Z_φ. These integrals can be evaluated using the results of Sections 5.2 and 5.3.2, and the resulting expressions can be resummed to order λ^2 thanks to the series expansion (6.37), leading to

$$\phi_{[1]}(x) = \int_{\vec{y}} K_{\Delta_\mathcal{R}}(x,\vec{y}) \left(\varepsilon\, e^{-\frac{1}{2\nu}}\right)^{\Delta_\mathcal{R}-\Delta} Z_\varphi \varphi_{(0)}(\vec{y}) \,, \tag{6.55}$$

where we defined the renormalized mass and renormalized conformal dimension

$$m_\mathcal{R}^2 \equiv m^2 + \Pi \,, \qquad \Delta_\mathcal{R} \equiv \frac{d}{2} + \sqrt{\frac{d^2}{4} + m_\mathcal{R}^2} = \Delta + \frac{\Pi}{2\nu} - \frac{\Pi^2}{8\nu^3} + \mathcal{O}(\lambda^3) \,. \tag{6.56}$$

Note that this resummed expression has the same form as the one obtained at order λ in (6.38), where now $m_\mathcal{R}^2$, or equivalently $\Delta_\mathcal{R}$, has been explicitly computed to order λ^2. Thus, IR divergences are renormalized with the same form (6.40) for the source counterterm Z_φ as before, leading to the same definition for the renormalized source $\varphi_{\mathcal{R}(0)}$ as (6.41), and consequently to its correct transformation rule as a source now to order λ^2, as ensured from the analysis in (6.43). One then obtains the renormalized expression

$$\phi_{[1]}(x) = \int_{\vec{y}} K_{\Delta_\mathcal{R}}(x,\vec{y}) \varphi_{\mathcal{R}(0)}(\vec{y}) \,. \tag{6.57}$$

Moving now to $\phi_{[3]}$, the UV divergences coming from $G_{\tau,\Delta}(1)$ of the integrals represented by the tadpole diagrams (last 2 terms of (6.31)) are renormalized with the mass counterterm fixed from the analysis of the 2-point function. Then writing the remaining UV finite factors in terms of Π, computing the $\int GK$ and $\int GG$ integrals allows for a resummation to order λ^2 of the tadpole diagrams and the contact diagram, with the IR divergences renormalized by the source counterterm fixed previously. In terms of Δ_R and $\varphi_{\mathcal{R}(0)}$, this resummation reads

$$\begin{aligned}
&-\frac{\lambda}{6}\int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right]^3 \\
&+\frac{\lambda}{2}\left[\lambda\delta m_{\{1\}}^2 + \frac{\lambda}{2} G_{\tau,\Delta}(1)\right] \int_{x'} G_\Delta(x,x') \left[\int_{\vec{y}} K_\Delta(x',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right]^2 \\
&\qquad\qquad \times \int_{x''} G_\Delta(x',x'') \left[\int_{\vec{y}} K_\Delta(x'',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right] \\
&+\frac{\lambda}{6}\left[\lambda\delta m_{\{1\}}^2 + \frac{\lambda}{2} G_{\tau,\Delta}(1)\right] \int_{x'} G_\Delta(x,x') \int_{x''} G_\Delta(x',x'') \left[\int_{\vec{y}} K_\Delta(x'',\vec{y}) Z_\varphi \varphi_{(0)}(\vec{y})\right]^3 \\
&\qquad = -\frac{\lambda}{6}\int_{x'} G_{\Delta_\mathcal{R}}(x,x') \left[\int_{\vec{y}} K_{\Delta_\mathcal{R}}(x',\vec{y}) \varphi_{\mathcal{R}(0)}(\vec{y})\right]^3 + \mathcal{O}(\lambda^3) \,.
\end{aligned} \tag{6.58}$$

This is the renormalization of Δ in the 4-point function as a λ^2 effect, not seen by the previous analysis at order λ. At this order, one also has the contribution to $\phi_{[3]}$ coming from the double exchange/bubble diagram (2nd line of (6.31)), which being IR convergent can be computed in terms of the coefficient χ introduced in Subsection

5.4.4

$$\int_{x''} G^2_{\tau,\Delta}(x',x'')K_\Delta(x'',\vec{y}_3)K_\Delta(x'',\vec{y}_4) = \chi^{d,2}_{\Delta,\Delta,\Delta}(x',\vec{y}_3,\vec{y}_4;\tau)K_\Delta(x',\vec{y}_3)K_\Delta(x',\vec{y}_4)\,, \quad (6.59)$$

which is represented as the infinite series

$$\chi^{d,2}_{\Delta,\Delta,\Delta}(x',\vec{y}_3,\vec{y}_4;\tau) = \sum_{i=0}^{\infty}\left[c_i(\tau) + d_i(\tau)\ln X\right] X^i\,, \quad (6.60)$$

with $X = K(x',\vec{y}_3)K(x',\vec{y}_4)|\vec{y}_{34}|^2$, and $K(x',\vec{y}_i) = z'/[z'^2 + (\vec{x}' - \vec{y}_i)^2]$. UV divergences of χ come from the coefficient $c_i(\tau)$, which in the limit $\tau \to 0$ converges for $d < 3 + 2i$: $c_0(\tau)$ converges for $d < 3$, $c_1(\tau)$ converges for $d < 5$, and so on. For instance, for $d < 5$ with divergences coming only from $c_0(\tau)$, these being of order λ^2 and proportional to the contact diagram are renormalized through the component $\delta\lambda_{\{2\}}$ of the coupling counterterm, fixing it to the value

$$\delta\lambda_{\{2\}} = \frac{3}{2}\mathrm{Div}\left[c_0(\tau)\right] + F_{\lambda,2} \qquad (d < 5)\,, \quad (6.61)$$

where $F_{\lambda,2}$ is the scheme dependent constant. With UV divergences in $\phi_{[3]}$ renormalized, one can safely evaluate the limit $\tau \to 0$. The bulk field completely renormalized to order λ^2 may then be written as

$$\begin{aligned}
\phi(x) = &\int_{\vec{y}} K_{\Delta_\mathcal{R}}(x,\vec{y})\varphi_{\mathcal{R}(0)}(\vec{y}) - \frac{\lambda}{6}\int_{x'} G_{\Delta_\mathcal{R}}(x,x')\left[\int_{\vec{y}} K_{\Delta_\mathcal{R}}(x',\vec{y})\varphi_{\mathcal{R}(0)}(\vec{y})\right]^3 \qquad (6.62)\\
&- \frac{\lambda^2 F_{\lambda,2}}{6}\int_{x'} G_\Delta(x,x')\left[\int_{\vec{y}} K_\Delta(x',\vec{y})\varphi_{(0)}(\vec{y})\right]^3\\
&+ \frac{\lambda^2}{4}\int_{x'} G_\Delta(x,x')\left[\int_{\vec{y}} K_\Delta(x',\vec{y})\varphi_{(0)}(\vec{y})\right]\left[\int_{\vec{y}_3} K_\Delta(x',\vec{y}_3)\varphi_{(0)}(\vec{y}_3)\right]\left[\int_{\vec{y}_4} K_\Delta(x',\vec{y}_4)\varphi_{(0)}(\vec{y}_4)\right]\\
&\quad\times\sum_{i=0}^{\infty}\left(\mathrm{Con}\left[c_i\right] + d_i\ln\left[K(x',\vec{y}_3)K(x',\vec{y}_4)|\vec{y}_{34}|^2\right]\right)\left[K(x',\vec{y}_3)K(x',\vec{y}_4)|\vec{y}_{34}|^2\right]^i\,.
\end{aligned}$$

The proper renormalization of the bulk field allows for the correct identification of its near-boundary expansion, taking the form of (6.45) where now $\Delta_\mathcal{R}$ has been computed to order λ^2 in bulk perturbation theory. As such, the remaining boundary divergences of the effective action which can be seen from the variation (6.20) are renormalized with the same structure of boundary counterterms as (6.47), corresponding to the standard counterterms of a scalar theory with Dirichlet boundary conditions under the replacement $\Delta \to \Delta_\mathcal{R}$, and with ϕ the renormalized bulk field to loop order λ^2, leading to a finite variation in the form of (6.48).

6.4.2 Renormalized correlators

Differentiation with respect to the renormalized sources $\varphi_{\mathcal{R}(0)}$ leads to the 2-point function

$$\langle \mathcal{O}_{\Delta_\mathcal{R}}(\vec{y}_1)\mathcal{O}_{\Delta_\mathcal{R}}(\vec{y}_2)\rangle = \frac{2\nu_\mathcal{R} c_{\Delta_\mathcal{R}}}{|\vec{y}_1-\vec{y}_2|^{2\Delta_\mathcal{R}}}\,. \tag{6.63}$$

For the case of the 4-point function, it can be expressed as a sum of contact terms represented by the D-functions defined in (2.37), upon writing

$$\ln\left[K(x,\vec{y}_3)K(x,\vec{y}_4)|\vec{y}_{34}|^2\right] = \partial_\alpha\left[K(x,\vec{y}_3)K(x,\vec{y}_4)|\vec{y}_{34}|^2\right]^\alpha\Big|_{\alpha=0}, \tag{6.64}$$

leading to the expression

$$\begin{aligned}\langle \mathcal{O}_{\Delta_\mathcal{R}}(\vec{y}_1)\mathcal{O}_{\Delta_\mathcal{R}}(\vec{y}_2)\mathcal{O}_{\Delta_\mathcal{R}}(\vec{y}_3)\mathcal{O}_{\Delta_\mathcal{R}}(\vec{y}_4)\rangle &= -\lambda c^4_{\Delta_\mathcal{R}} D_{\Delta_\mathcal{R},\Delta_\mathcal{R},\Delta_\mathcal{R},\Delta_\mathcal{R}} - \lambda^2 F_{\lambda,2}\, c^4_\Delta D_{\Delta,\Delta,\Delta,\Delta}\\ &+\frac{\lambda^2}{2}c^4_\Delta\sum_{i=0}^{\infty}\left[\mathrm{Con}\,[c_i]\,D_{\Delta,\Delta,\Delta+i,\Delta+i}|\vec{y}_{34}|^{2i} + d_i\partial_\alpha\left(D_{\Delta,\Delta,\Delta+i+\alpha,\Delta+i+\alpha}|\vec{y}_{34}|^{2i+2\alpha}\right)_{\alpha=0}\right]\times 3\,,\end{aligned} \tag{6.65}$$

where the factor $\times 3$ at the end denotes the 3 permutations (12,34), (13,24) and (14,23) of the double exchange diagram (s-, t- and u-channels). To this loop order, the effect of the bulk quantum corrections are seen to renormalize the conformal dimension of the boundary source and operator from Δ to $\Delta_\mathcal{R}$, computed to order λ^2 in the bulk coupling. This renormalization is now also reflected by the 4-point function through the resummation of the contact diagram, which being already of order λ only sees $\Delta_\mathcal{R}$ renormalized to order λ. The scheme-dependence $F_{\lambda,2}$ and the double exchange being of order λ^2 do not see the renormalization of Δ, however the latter renormalizes instead the OPE coefficients by introducing a new dependence on the external points through a function of the cross-ratios. One way to see this is by conveniently writing the D-functions appearing in (6.65) as

$$D_{\Delta,\Delta,\Delta+\beta,\Delta+\beta}|\vec{y}_{34}|^{2\beta} = \frac{\pi^{\frac{d}{2}}}{2\,\Gamma(\Delta)^2}\frac{(uv)^{\frac{\Delta}{3}}}{\prod_{i<j}(y^2_{ij})^{\frac{\Delta}{3}}}\hat{H}_\beta(u,v)\,, \tag{6.66}$$

where we defined a normalized version of the function H

$$\hat{H}_\beta(u,v) \equiv \frac{\Gamma\left(2\Delta-\frac{d}{2}+\beta\right)}{\Gamma(\Delta+\beta)^2}H(\Delta,\Delta,1-\beta,2\Delta;u,v)\,. \tag{6.67}$$

This is obtained by evaluating in (2.38) $\Delta_1=\Delta_2=\Delta$, $\Delta_3=\Delta_4=\Delta+\beta$, and rearranging. Performing the $\mathcal{O}(\lambda^3)$ manipulation of replacing every Δ in (6.65) by $\Delta_\mathcal{R}$ and using the representation above for the D-functions, allows us to express the resummed 4-point function in the expected CFT form

$$\langle \mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_1)\mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_2)\mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_3)\mathcal{O}_{\Delta_{\mathcal{R}}}(\vec{y}_4)\rangle = \frac{F(u,v)}{\prod_{i<j}(y_{ij}^2)^{\frac{\Delta_{\mathcal{R}}}{3}}}\,, \tag{6.68}$$

with the function of cross-ratios $F(u,v)$ determined to order λ^2 in bulk loops

$$F(u,v) = \frac{\pi^{\frac{d}{2}}}{2}\frac{c_{\Delta_{\mathcal{R}}}^4}{\Gamma(\Delta_{\mathcal{R}})^2}(uv)^{\frac{\Delta_{\mathcal{R}}}{3}}\Bigg(-(\lambda+\lambda^2 F_{\lambda,2})\hat{H}_0 + \frac{\lambda^2}{2}\sum_{i=0}^{\infty}\Big[\mathrm{Con}[c_i]\hat{H}_i + d_i\partial_\alpha\left(\hat{H}_{i+\alpha}\right)_{\alpha=0}\Big]\times 3\Bigg)\,, \tag{6.69}$$

where $\hat{H}_0(u,v)$ is the tree-level structure of cross-ratios.

The fact that the UV divergences of the 4-point function depend on the coefficients $c_i(\tau)$, and the divergence of these in turn depend in the number of spacetime dimensions, highly constrain the renormalizability of the bulk theory and hence in the construction of a possible dual theory. A priori, it seems that only the divergences of $c_0(\tau)$ can be absorbed in the coupling counterterm $\delta\lambda$, being able to renormalize the theory only up to $d+1=5$ bulk dimensions. However, the D-functions satisfy many nice identities among which is found [45]

$$D_{\Delta,\Delta,\Delta+1,\Delta+1}|\vec{y}_{34}|^2 + D_{\Delta,\Delta+1,\Delta,\Delta+1}|\vec{y}_{24}|^2 + D_{\Delta,\Delta+1,\Delta+1,\Delta}|\vec{y}_{23}|^2 = \frac{(4\Delta-d)}{2\Delta}D_{\Delta,\Delta,\Delta,\Delta}\,, \tag{6.70}$$

allowing us to also write the contributions from $c_1(\tau)$ proportional to the contact term and therefore absorb its divergences through $\delta\lambda$, extending the renormalizability of the theory up to $d+1=7$ bulk dimensions, upon fixing

$$\delta\lambda_{\{2\}} = \frac{3}{2}\mathrm{Div}\left[c_0(\tau)\right] + \frac{(4\Delta-d)}{4\Delta}\mathrm{Div}\left[c_1(\tau)\right] + F_{\lambda,2} \qquad (d<7)\,. \tag{6.71}$$

The contributions from the coefficients $c_2(\tau)$ and higher are no longer expressible only in terms of $D_{\Delta,\Delta,\Delta,\Delta}$ and hence their divergences cannot be absorbed through $\delta\lambda$, rendering the theory for bulk dimensions greater than 7 non-renormalizable. This is consistent with the computation of the double exchange/bubble diagram from the conformal bootstrap equations, where for $d\geq 7$ it is found that additional counterterms are required to renormalize its divergences [1].

The holographic renormalization of the theory to loop order has been carried out in a very general way, thanks to the possibility of encoding all the contributions from bulk loops in the abstract coefficients μ, η and χ, the latter through c_i and d_i. We will end this chapter by considering a concrete case of this procedure, discussing one of the examples at the end of Chapter 5 where the values of these coefficients have been explicitly computed.

6.5 Example: $d = 3$, $\Delta = 2$

Consider the example of a scalar $\lambda\Phi^4$ theory with a mass $m^2 = -2$ in a fixed AdS$_4$ background, describing in the CFT$_3$ at the boundary a relevant single trace operator of leading dimension $\Delta = 2$ in the $1/N^2$ expansion (for a previous treatment of this case at loop order, see [21, 22]). At order λ^2 in bulk perturbation theory, the bulk loop data is encoded in the value of the propagator at coincident points $G_{\tau,2}(1)$ representing the contribution from the tadpole diagram

$$G_{\tau,2}(1) = \frac{1}{4\pi^2\tau^2} - \frac{1}{12\pi^2} + \mathcal{O}(\tau)\,, \tag{6.72}$$

together with the value of the coefficients $\mu_2^{3,2}(\tau)$, $\eta_{2,2}^{3,3}(\tau)$ and $\chi_{2,2,2}^{3,2}(X;\tau)$ ($c_i(\tau)$ and $d_i(\tau)$) representing the contributions from the eight, sunset and double exchange diagrams, which have been computed in Subsection 5.5.1

$$\begin{aligned}
\mu_2^{3,2}(\tau) &= -\frac{\ln\tau}{8\pi^2} + \frac{2\ln 2 - 1}{16\pi^2} + \mathcal{O}(\tau)\,, && (6.73)\\
\eta_{2,2}^{3,3}(\tau) &= \frac{1}{128\pi^4\tau^2} + \frac{3\ln\tau}{128\pi^4} + \frac{5 - 9\ln 2}{384\pi^4} + \mathcal{O}(\tau)\,, && (6.74)\\
c_i(\tau) &= -\frac{1}{8\pi^2}(\ln\tau + 1)\delta_{i,0} + \mathcal{O}(\tau)\,, && (6.75)\\
d_i(\tau) &= -\frac{1}{16\pi^2}\delta_{i,0} + \mathcal{O}(\tau)\,. && (6.76)
\end{aligned}$$

From these, one can read directly the renormalization of the UV divergences of the bulk theory from the expressions for the mass and coupling counterterms derived in (6.34), (6.52) and (6.61)

$$\begin{aligned}
\delta m^2 &= \lambda\left(-\frac{1}{8\pi^2\tau^2} + F_{m,1}\right) && (6.77)\\
&\quad + \lambda^2\left[\frac{3\ln\tau}{128\pi^4\tau^2} + \frac{1}{8\pi^2\tau^2}\left(\frac{1}{96\pi^2} - F_{\lambda,2}\right) - \frac{\ln\tau}{16\pi^2}\left(\frac{1}{48\pi^2} + F_{m,1}\right) + F_{m,2}\right],\\
\delta\lambda &= \lambda^2\left(-\frac{3\ln\tau}{16\pi^2} + F_{\lambda,2}\right), && (6.78)
\end{aligned}$$

leading, from (6.54) and (6.56), to the renormalized value of the bulk mass

$$m_{\mathcal{R}}^2 = -2 + \lambda\left(-\frac{1}{24\pi^2} + F_{m,1}\right) + \lambda^2\left(\frac{15\ln 2 - 8}{2304\pi^4} + \frac{F_{m,1}(1 - 2\ln 2)}{32\pi^2} - \frac{F_{\lambda,2}}{24\pi^2} + F_{m,2}\right), \tag{6.79}$$

and consequently, to the renormalized conformal dimension of the boundary operator

$$\Delta_{\mathcal{R}} = 2 + \lambda\left(-\frac{1}{24\pi^2} + F_{m,1}\right) + \lambda^2\left(\frac{5\ln 2 - 4}{768\pi^4} + \frac{F_{m,1}(11 - 6\ln 2)}{96\pi^2} - \frac{F_{\lambda,2}}{24\pi^2} - F_{m,1}^2 + F_{m,2}\right). \tag{6.80}$$

In turn, renormalization of the IR divergences of the theory can be read from the expressions for the source and boundary counterterms derived in (6.40) and (6.47), which are written in terms of $\Delta_{\mathcal{R}}$. With the counterterms taking these values to make the bulk theory finite, the resulting renormalized boundary correlators are given by the 2-point function (6.63) and the 4-point function (6.68), precisely corresponding to CFT correlation functions for an operator of dimension $\Delta_{\mathcal{R}}$, the latter in terms of the function of cross-ratios (6.69)

$$F(u,v) = -\frac{\pi^{\frac{3}{2}}}{2}\frac{c^4_{\Delta_{\mathcal{R}}}}{\Gamma(\Delta_{\mathcal{R}})^2}(uv)^{\frac{\Delta_{\mathcal{R}}}{3}}\left[\left(\lambda + \lambda^2 F_{\lambda,2} + \frac{3\lambda^2}{16\pi^2}\right)\hat{H}_0 + \frac{\lambda^2}{32\pi^2}\partial_\alpha\left(\hat{H}_\alpha\right)_{\alpha=0} \times 3\right]. \tag{6.81}$$

Determining the running of the bulk couplings, and their effect on the boundary correlators, requires a more deeper analysis of the renormalization group equations for the bulk theory under the regularization schemes chosen. Nevertheless, since the UV should be independent of the curvature of spacetime at large scales, for energy scales much larger than the scale set by the AdS radius one expects to reproduce the standard beta functions from flat space. For instance, the divergent part of (6.78) suggests

$$\beta_\lambda = \frac{3\lambda^2}{16\pi^2}, \tag{6.82}$$

which is the expected 1-loop beta function for the quartic coupling in $d + 1 = 4$ dimensions.

Chapter 7

Discussion

In this thesis, progress has been made in the context of holography and the AdS/CFT correspondence. More concretely, the previous dictionary that related the quantities on both sides of the correspondence at leading order in the UV/IR duality has been extended to systematically incorporate all the subleading corrections. This has been achieved in a way that mimics the prescription in the classical approximation, with the role of the renormalized action played by the renormalized effective action. This allowed us to identify the dual of the boundary operators to be not the fields minimizing the classical theory but the full quantum theory in the bulk. The near-boundary expansion of these fields also decomposes in non-normalizable and normalizable modes identified with sources and VEVs in the dual theory, and these may be written in terms of the renormalized bulk parameters computed perturbatively in AdS loops around their classical values. This leads to the renormalization of the usual infrared divergences present in holographic theories through the standard set of boundary counterterms, but expressed in terms of the renormalized bulk fields and couplings. The resulting renormalized holographic correlators manifestly obey the expected conformal Ward identities with the CFT data renormalized order by order in the bulk loop perturbation, providing further evidence for holography and in particular for the AdS/CFT correspondence.

Our methods have been applied to the example of a scalar Φ^4 theory, finding perfect agreement with the expectations from the general methodology developed in Chapter 4. Interestingly, no wavefunction counterterm is required to renormalize the UV divergences appearing in the holographic 2-point functions up to two loops, as opposed to the case in flat space. This might be understood from the resemblance of Witten diagrams as scattering amplitudes, which are invariant under field redefinitions thanks to the equivalence theorem. In the case of AdS/CFT, since the external legs are extended all the way to the boundary of the spacetime where the bulk fields have a prescribed value due to the holographic boundary conditions, one is computing the scattering of states which are well-defined asymptotically, analogous to the computation of scattering amplitudes. Since this computation is indifferent to such counterterms, this may hint to the un-necessity of wavefunction renormalization perhaps to all orders in the bulk loop expansion. It would be interesting to understand this better and investigate its implications. Instead of wavefunction renormalization, what is found to be required is source renormalization to absorb the IR divergences that appear on AdS loops as the external legs are pushed to the boundary. This is completely expected for bulk fields receiving mass renormalization, since the value of the mass is directly identified with the conformal dimension of the dual operator, sources must also renormalize in a specific way to account for the corrected dimensions. Consistency here of AdS/CFT at loop order intimately links source renormalization with mass renormalization in a non-trivial way, and this is indeed found to be the case for the example studied.

In this work we have initiated a more rigorous study of the subleading corrections in AdS/CFT with a special focus on scalar fields, however for gauge fields there are additional subtleties (gauge choices, ghosts, etc) that we have not addressed. For instance, already in the example of a scalar Φ^4 theory the backreaction with the background metric would involve the propagation of gravitons in the bulk, and these would contribute to the energy-momentum tensor of the dual theory and to the dimensions and OPE coefficients of the dual scalar operators through graviton exchanges and loops. A more complete description of the bulk theory must account for these and it would be interesting to spell out the details. Another interesting direction is to look at bulk fields with different boundary conditions. In the example of the scalar field, we focused exclusively on the case of Dirichlet boundary conditions with leading dimensions $\Delta > d/2$, however for $\Delta = d/2$ or for scalars with Neumann boundary conditions $\Delta < d/2$, additional IR divergences appear at loop order in the bulk, and a priori it is not clear how these are renormalized by the current set of counterterms nor their implications for the boundary theory.

Appendices

Appendix A

Bulk-to-bulk Propagator

The objective of this appendix is to work out the properly normalized solution to the deformed Green's equation defining the regularized bulk-to-bulk propagator on AdS. But before doing this, let us first remember how this is properly done for the unregulated case.

A.1 Bare propagator

The equation for the bare propagator was given in (4.22)

$$(-\Box_\xi + m^2)G(\xi(x,x')) = \frac{1}{\sqrt{g}}\delta(x-x')\,, \tag{A.1}$$

with the explicit form of $\Box_\xi$ in (4.20). Its functional form can be determined away from the coincident point $x = x'$. In this region, the delta at the RHS of the equation vanishes and one is left to solve the ordinary differential equation

$$\left[\xi^2(1-\xi^2)\partial_\xi^2 + (1-d-2\xi^2)\xi\partial_\xi - m^2\right] G(\xi) = 0\,. \tag{A.2}$$

Modulo a factor of ξ to some power, this is the hypergeometric differential equation in the variable ξ^2. To see this, rewrite the propagator as $G(\chi) = \chi^{\frac{\Delta}{2}}F(\chi)$ in the variable $\chi = \xi^2$. Then the function $F(\chi)$ satisfies

$$\left[\chi(1-\chi)\partial_\chi^2 + \left[\Delta - \frac{d}{2} + 1 - \left(\Delta + \frac{3}{2}\right)\chi\right]\partial_\chi - \frac{\Delta(\Delta+1)}{4}\right] F(\chi) = 0\,. \tag{A.3}$$

Compare this with the hypergeometric equation

$$z(1-z)y'' + \left[c - (a+b+1)z\right] y' - aby = 0\,. \tag{A.4}$$

We make the identification

$$a = \frac{\Delta}{2}\,, \quad b = \frac{\Delta+1}{2}\,, \quad c = \Delta - \frac{d}{2} + 1\,, \quad z = \chi\,, \quad y = F\,. \tag{A.5}$$

The equation has 2 independent solutions, however we still need to impose appropriate boundary conditions. Demanding regularity of the propagator at the interior of AdS, and a Dirichlet fall-off near the conformal boundary, picks in the interval $0 < z < 1$ the single solution $y = {}_2F_1(a, b; c; z)$. In terms of the original variable ξ, this then implies for the propagator

$$G(\xi) = C\,\xi^{\Delta}\,{}_2F_1\left(\begin{matrix}\frac{\Delta}{2} & \frac{\Delta+1}{2} \\ \Delta - \frac{d}{2} + 1 & \end{matrix}; \xi^2\right), \tag{A.6}$$

up to some normalization constant C. This remaining constant is determined by the discontinuity introduced by the delta, which can be captured for instance integrating the Green's equation in a region that contains the coincident point $x = x'$. The integral of the delta just becomes 1, and on the LHS of the equation one has to integrate $(-\Box_{\xi} + m^2)G_{\Delta}(\xi)$. The computation of this integral however is subtle for the following reason: away from the coincident point, the integrand for the solution found is just 0, so the only region that contributes to the integral is the infinitesimal one that encloses the point $x = x'$. In this region, the integral proportional to m^2 vanishes given the continuity of $G(x, x')$, and one is left to integrate its Laplacian. Now, since this is a total derivative one is tempted to use Stokes' theorem, however the volume region being integrated contains the non-regular point $x = x'$, and one first has to assert whether there is an extra contribution coming from this point. It turns out, ignoring the singular point and assuming that the only contribution to the integral comes from the boundary of the infinitesimal region, a naive use of Stokes' theorem leads to the correct normalization constant. This is, however, unsatisfactory as this unjustified assumption is only validated by prior knowledge of the constant. As soon as we deform the equation, this assumption may no longer be valid and one would be led to conclude an incorrect normalization.

The correct way to proceed is acknowledging the fact that one is dealing with a distributional equation, and thus translating the problem to the language of distribution theory. One only needs the very basic ingredients, so let us introduce them briefly. By definition, a distribution d is a continuous linear functional on the set of test functions $f \in C_c^{\infty}$, that is, functions that are bounded, have compact support, and are infinitely differentiable in the whole domain. For our purposes, we will be interested in the linear mapping $\langle d, f\rangle : C_c^{\infty} \to \mathbb{R}$, and we will be thinking in the distribution $DG_{x'}$, subject to the distributional Green's equation

$$DG_{x'} = \frac{1}{\sqrt{g}}\delta_{x'}\,, \quad D \equiv -\Box + m^2\,, \tag{A.7}$$

where $\delta_{x'}$ is the Dirac delta distribution with support at x'. Derivatives of distributions are defined by their action on the test functions f. Given the Hermicity of the differential operator D, the action of $DG_{x'}$ on f is defined by

$$\langle DG_{x'}, f\rangle \equiv \langle G_{x'}, Df\rangle \,, \tag{A.8}$$

where the boundary terms are discarded given the nice properties of f. For solutions of the Green's equation, the distribution on the LHS is equal to a delta, and one obtains the test function evaluated at x'. Thus, explicitly the expression above becomes

$$f(x') = \int \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')Df(x)\,. \tag{A.9}$$

Consistency with this formula for any well-behaved test function f is what properly fixes the undetermined constant in $G(x,x')$. Let us then proceed to this calculation. We would like to use Stokes' theorem, but as we argued, the integrand contains the singular point $x = x'$, or $\xi = 1$ in geodesic coordinates. Separate then the integral in 2 regions, one "ball" of chordal radius $1-\delta \le \xi \le 1$ containing the singular point, and the rest of the AdS volume $0 \le \xi < 1-\delta$

$$f(x') = \int_{1-\delta\le\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')Df(x) + \int_{\xi<1-\delta} \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')Df(x)\,. \tag{A.10}$$

Focus on the first integral. Since by definition the test function f is bounded and infinitely differentiable

$$\left|\int_{1-\delta\le\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')Df(x)\right| \le \sup_{1-\delta\le\xi}|Df(x)|\left|\int_{1-\delta\le\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')\right|\,, \tag{A.11}$$

where $\sup_{1-\delta\le\xi}|Df(x)| < \infty$ denotes the maximum value the function $|Df(x)|$ takes in $1-\delta \le \xi$. Writing the integral in geodesic coordinates and performing the angular integrals

$$\left|\int_{1-\delta\le\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')Df(x)\right| \le \sup_{1-\delta\le\xi}|Df(x)|\Omega_d\left|\int_{1-\delta}^{1} \mathrm{d}\xi\sqrt{g_\xi}\, G(\xi)\right|\,. \tag{A.12}$$

The divergent factor of the propagator can be extracted from the hypergeometric after an Euler's transformation (see (B.12))

$$G(\xi) = C\frac{\xi^\Delta}{(1-\xi^2)^{\frac{d-1}{2}}}\,{}_2F_1\left(\begin{matrix}\frac{\Delta-d}{2}+1 & \frac{\Delta-d+1}{2}\\ \Delta-\frac{d}{2}+1 & \end{matrix};\xi^2\right)\,. \tag{A.13}$$

Then to leading order in δ

$$\left|\int_{1-\delta\le\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')Df(x)\right| \le \sup_{1-\delta\le\xi}|Df(x)|\Omega_d|C|\left|{}_2F_1\left(\begin{matrix}\frac{\Delta-d}{2}+1 & \frac{\Delta-d+1}{2}\\ \Delta-\frac{d}{2}+1 & \end{matrix};1\right)\right|\delta\,. \tag{A.14}$$

We see then that the contribution from the singular point is proportional to δ, and it vanishes as we take the limit $\delta \to 0$. This explains why the naive computation of the constant C described previously leads to the correct value. In this limit then the first integral in (A.10) vanishes and one is left with

$$f(x') = \lim_{\delta\to 0} \int_{\xi<1-\delta} \mathrm{d}^{d+1}x\sqrt{g}\, G(x,x')Df(x)\,. \tag{A.15}$$

Now with the singular point removed, one can safely use Stokes' theorem. This is done by noticing that, since $DG(x,x') = 0$ in the region $\xi < 1-\delta$, the integrand corresponds to a total derivative

$$G(x,x')Df(x) = \nabla_\mu \left[f(x)\nabla^\mu G(x,x') - G(x,x')\nabla^\mu f(x)\right]\,, \quad x \neq x'\,. \tag{A.16}$$

The contribution from the boundary $\xi = 0$ at infinity of AdS is zero given the compact support of f. Thus, Stokes' theorem only picks up the contribution from the boundary at $\xi = 1-\delta$

$$f(x') = \lim_{\delta\to 0} \int_{\xi=1-\delta} \mathrm{d}^{d}x\sqrt{\gamma}\, n_\mu \left[f(x)\nabla^\mu G(x,x') - G(x,x')\nabla^\mu f(x)\right]\,. \tag{A.17}$$

In geodesic coordinates this expression becomes

$$f(x') = \lim_{\delta\to 0} \int \mathrm{d}\Omega_d\, \xi^{1-d}(1-\xi^2)^{\frac{d+1}{2}} \left[f(x)\partial_\xi G(\xi) - G(\xi)\partial_\xi f(x)\right]\Big|_{\xi=1-\delta}. \tag{A.18}$$

Without loss of generality, from the set of test functions we can choose bump functions that are constant inside the ball $\xi' \leq \xi$, for some $\xi' < 1-\delta$, which smoothly transition to 0 away from this region. Under this choice, the second term above vanishes. For the first term, a direct computation for the derivative of the propagator yields

$$\partial_\xi G(\xi) = C\Delta \frac{\xi^{\Delta-1}}{(1-\xi^2)^{\frac{d+1}{2}}}\, {}_2F_1\left(\begin{matrix}\frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta-\frac{d}{2}+1 & \end{matrix};\xi^2\right)\,, \tag{A.19}$$

leading to

$$f(x') = \lim_{\delta\to 0} \int \mathrm{d}\Omega_d\, f(x)C\Delta\xi^{\Delta-d}\, {}_2F_1\left(\begin{matrix}\frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta-\frac{d}{2}+1 & \end{matrix};\xi^2\right)\Big|_{\xi=1-\delta}. \tag{A.20}$$

All these terms are regular at $\xi = 1$, thus the limit $\delta \to 0$ can be evaluated directly

$$f(x') = \Omega_d f(x')C\Delta\, {}_2F_1\left(\begin{matrix}\frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta-\frac{d}{2}+1 & \end{matrix};1\right)\,. \tag{A.21}$$

This fixes the normalization constant C to the value

$$C = \frac{1}{\Omega_d \Delta}\, {}_2F_1\left(\begin{matrix} \frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta - \frac{d}{2}+1 & \end{matrix};1\right)^{-1}. \tag{A.22}$$

A more familiar expression can be obtained using

$$\Omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}, \quad {}_2F_1\left(\begin{matrix} \frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta - \frac{d}{2}+1 & \end{matrix};1\right) = \frac{2^{\Delta}\Gamma\left(\Delta-\frac{d}{2}+1\right)\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\,\Gamma(\Delta+1)}, \tag{A.23}$$

resulting in

$$C = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} 2^{\Delta+1}\Gamma\left(\Delta-\frac{d}{2}+1\right)}. \tag{A.24}$$

A.2 Regularized propagator

In the previous section we reviewed the derivation of the bare bulk-to-bulk propagator. Now we want to repeat this calculation for the regularized propagator. It is the solution of the deformed Green's equation constructed in (4.25)

$$D_\tau G_\tau(\xi(x,x')) = \frac{1}{\sqrt{g}}\delta(x-x'), \quad D_\tau \equiv c(\tau)\sqrt{\frac{g_{\xi_\tau}}{g_\xi}}(-\Box_{\xi_\tau}+m^2). \tag{A.25}$$

As discussed in the bare case, the functional form of the propagator is determined away from the coincident point. Repeating this calculation for $G_\tau(\xi)$, the equation to solve is exactly the same as before, but with every ξ replaced by ξ_τ. Then after imposing the same boundary conditions, the solution found is:

$$G_\tau(\xi) = C_\tau \xi_\tau^{\Delta}\, {}_2F_1\left(\begin{matrix} \frac{\Delta}{2} & \frac{\Delta+1}{2} \\ \Delta - \frac{d}{2}+1 & \end{matrix};\xi_\tau^2\right), \tag{A.26}$$

for some normalization constant C_τ. As we argued, the proper way to determine it is in the language of distributions. The regularized Green's equation in the sense of distributions is

$$D_\tau G_{\tau,x'} = \frac{1}{\sqrt{g}}\delta_{x'}, \tag{A.27}$$

where, since by construction the operator D_τ is self-adjoint, its action on the set of test functions is defined as

$$\langle D_\tau G_{\tau,x'}, f\rangle \equiv \langle G_{\tau,x'}, D_\tau f\rangle. \tag{A.28}$$

For solutions of the Green's equation, explicitly the formula above is

$$f(x') = \int \mathrm{d}^{d+1}x\sqrt{g}\; G_\tau(\xi) D_\tau f(x). \tag{A.29}$$

In the undeformed case, the bare propagator is non-regular at $\xi = 1$ and one has to treat the contribution coming from this point carefully. In the current case however, the propagator has been regularized at coincident points (as long as τ is non-zero) and thus this point is no longer singular. This would seem to suggest there are no longer issues at $\xi = 1$ in the expression being integrated and we can safely use Stokes' theorem. This is of course not true, and it can be seen from the fact that the deformed Green's equation is still equal to a Dirac delta, divergent at $x = x'$. The previous divergence of the bare propagator has been moved to the factor of $1/\sqrt{g_\xi}$ in the expression for the regularized operator D_τ. In fact, it is precisely this factor what saves the day and leads to a non-vanishing constant C_τ. Thus, in the current case one still has to be careful with this point. As before then, separate the integral in a chordal ball containing this point, and the rest of the AdS volume

$$f(x') = \int_{1-\delta\leq\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G_\tau(\xi) D_\tau f(x) + \int_{\xi<1-\delta} \mathrm{d}^{d+1}x\sqrt{g}\, G_\tau(\xi) D_\tau f(x)\,. \tag{A.30}$$

Focusing on the first integral, writing it in geodesic coordinates

$$\int_{1-\delta\leq\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G_\tau(\xi) D_\tau f(x) = \int \mathrm{d}\Omega_d \int_{1-\delta}^{1} \mathrm{d}\xi\sqrt{g_\xi}\, G_\tau(\xi) D_\tau f(x)\,. \tag{A.31}$$

Since the propagator $G_\tau(\xi)$ has been regularized, it is bounded inside the ball $1-\delta \leq \xi$. Moreover, since $f \in C_c^\infty$ and $\sqrt{g_\xi}\, D_\tau \sim 1$, then $\sqrt{g_\xi}\, D_\tau f(x)$ is also bounded in this region. Then, to linear order in δ

$$\left|\int_{1-\delta\leq\xi} \mathrm{d}^{d+1}x\sqrt{g}\, G_\tau(\xi) D_\tau f(x)\right| \leq \sup_{1-\delta\leq\xi} \left|\sqrt{g_\xi}\, G_\tau(\xi) D_\tau f(x)\right| \Omega_d\, \delta\,. \tag{A.32}$$

As in the bare case, the contribution from the coincident point is proportional to δ, and it vanishes as we take $\delta \to 0$. In this limit then

$$f(x') = \lim_{\delta\to 0} \int_{\xi<1-\delta} \mathrm{d}^{d+1}x\sqrt{g}\, G_\tau(\xi) D_\tau f(x)\,. \tag{A.33}$$

Since $D_\tau G_\tau(\xi) = 0$ in the region $\xi < 1-\delta$, the integrand corresponds to a total derivative

$$G_\tau(\xi) D_{\xi_\tau} f(x) = \frac{c(\tau)}{\sqrt{g_\xi}} \partial_{\xi_\tau} \left(\sqrt{g_{\xi_\tau}}\, \xi_\tau^2 (1-\xi_\tau^2) \left[f(x)\partial_{\xi_\tau} G_\tau(\xi) - G_\tau(\xi)\partial_{\xi_\tau} f(x)\right]\right)\,, \quad x \neq x'\,. \tag{A.34}$$

Then after using Stokes' theorem and ignoring the contributions at $\xi = 0$ given the compact support of f, the boundary term at $\xi = 1-\delta$ in geodesic coordinates is

$$f(x') = \lim_{\delta\to 0} c(\tau) \int \mathrm{d}\Omega_d\, \sqrt{g_{\xi_\tau}}\, \xi\xi_\tau (1-\xi_\tau^2) \left[f(x)\partial_{\xi_\tau} G_\tau(\xi) - G_\tau(\xi)\partial_{\xi_\tau} f(x)\right] \Big|_{\xi=1-\delta}\,. \tag{A.35}$$

For the same class of test functions used in the previous section, the second term vanishes. The first term is completely regular at $\xi = 1$ and the limit can be evaluated directly. In terms of the derivative (A.19), the resulting expression is

$$f(x') = \frac{c(\tau)\Omega_d f(x') C_\tau \Delta}{(\cosh\tau)^{\Delta-d-1}} \, {}_2F_1\left(\begin{matrix} \frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta-\frac{d}{2}+1 & \end{matrix} ; \frac{1}{\cosh^2\tau}\right), \tag{A.36}$$

fixing the constant C_τ to the value

$$C_\tau = \frac{(\cosh\tau)^{\Delta-d-1}}{c(\tau)\Omega_d \Delta} \, {}_2F_1\left(\begin{matrix} \frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta-\frac{d}{2}+1 & \end{matrix} ; \frac{1}{\cosh^2\tau}\right)^{-1}. \tag{A.37}$$

Note that for $\tau = 0$, $C_0 = C$, and one recovers the undeformed normalization constant. By conveniently choosing $c(\tau)$, one can fix $C_\tau = C$ for all values of τ. Under this choice, the regularized propagator is just the bare propagator in the new variable ξ_τ: $G_\tau(\xi) = G(\xi_\tau)$. This is achieved by picking

$$c(\tau) = (\cosh\tau)^{\Delta-d-1} \, {}_2F_1\left(\begin{matrix} \frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta-\frac{d}{2}+1 & \end{matrix} ; 1\right) \Big/ \, {}_2F_1\left(\begin{matrix} \frac{\Delta-d}{2} & \frac{\Delta-d+1}{2} \\ \Delta-\frac{d}{2}+1 & \end{matrix} ; \frac{1}{\cosh^2\tau}\right). \tag{A.38}$$

Appendix B

Useful formulae

B.1 Modified Bessel functions

The modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ are the independent solutions to the second-order ODE

$$\left[z^2\partial_z^2 + z\partial_z - (z^2+\nu^2)\right] f(z) = 0\,. \tag{B.1}$$

The modified Bessel function of the first kind $I_\nu(z)$ has a series expansion of the form

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{1}{(1+\nu)_k\, k!}\left(\frac{z}{2}\right)^{2k}\,, \tag{B.2}$$

while the modified Bessel function of the second kind $K_\nu(z)$ has a series expansion for $\nu \neq \mathbb{Z}$

$$K_\nu(z) = \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^{\infty}\frac{1}{(1-\nu)_k\, k!}\left(\frac{z}{2}\right)^{2k} + \frac{\Gamma(-\nu)}{2}\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{1}{(1+\nu)_k\, k!}\left(\frac{z}{2}\right)^{2k}\,. \tag{B.3}$$

For $\nu = n \in \mathbb{N}_0$, the series expansion of $K_n(z)$ is given instead by

$$\begin{aligned} K_n(z) = &\frac{1}{2}\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{(-1)^k(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k} \\ &+ \frac{(-1)^n}{2}\left(\frac{z}{2}\right)^{n}\sum_{k=0}^{\infty}\frac{1}{k!(k+n)!}\left(\frac{z}{2}\right)^{2k}\left[-2\ln\left(\frac{z}{2}\right) + \psi(k+1) + \psi(k+n+1)\right]\,, \end{aligned} \tag{B.4}$$

where for $n = 0$ the first term is discarded. From these, one can read the asymptotic series of the functions for small argument $z \ll 1$

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}\left[1+\mathcal{O}(z^2)\right]\,, \tag{B.5}$$

$$K_\nu(z) = \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu}\left[1+\mathcal{O}(z^2)\right] + \frac{\Gamma(-\nu)}{2}\left(\frac{z}{2}\right)^{\nu}\left[1+\mathcal{O}(z^2)\right]\,, \tag{B.6}$$

where in the latter, a $\ln(z)$ term appears at order z^ν for $\nu = n$. In turn, the asymptotic series of the functions for large argument $z \gg 1$ correspond to

$$I_\nu(z) = \frac{1}{\sqrt{2\pi z}} e^z \left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right] , \quad K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right] . \tag{B.7}$$

B.2 Hypergeometric series

Hypergeometric functions are represented by the series

$${}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!} , \tag{B.8}$$

with $(a)_k = \Gamma(a+k)/\Gamma(a)$ the Pochhammer symbol. When one of the upper coefficients is a non-positive integer $a_i = -n$, $n \in \mathbb{N}_0$, the series terminates at $k = n$. Instead, when one of the lower coefficients is a non-positive integer $b_i = -n$, the denominator becomes 0 and the series is ill-defined. Outside these cases, for $p < q+1$ the series converges for all finite values of z, while for $p > q+1$ it only converges at $z = 0$. When $p = q+1$ the series converges for $|z| < 1$, diverges for $|z| > 1$, and at the unit value $|z| = 1$, for real parameters it converges for $\sum b_i - \sum a_j > 0$. From the latter case, of special interest are the binomial series

$${}_1F_0(a; z) = (1-z)^{-a} , \tag{B.9}$$

and Gauss' hypergeometric function ${}_2F_1(a, b; c; z)$. It can be shown to satisfy many relations, of which relevant for this work are Pfaff's and Euler's transformations

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) , \tag{B.10}$$

$$= (1-z)^{-b} {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right) , \tag{B.11}$$

$$= (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z) , \tag{B.12}$$

and also its expansion in $1-z$

$$\begin{aligned} {}_2F_1(a, b; c; z) = & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; 1+a+b-c; 1-z) \\ & + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-z)^{c-a-b} {}_2F_1(c-a, c-b; 1+c-a-b; 1-z) . \end{aligned} \tag{B.13}$$

If $c-a-b \in \mathbb{N}_0$, this last one becomes

$$\begin{aligned}{}_2F_1(a,b;c;z) = & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{i=0}^{c-a-b-1} \frac{(a)_i (b)_i}{(1+a+b-c)_i\, i!} (1-z)^i \\ & + \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} (z-1)^{c-a-b} \sum_{i=0}^{\infty} \frac{(c-a)_i (c-b)_i}{(c-a-b+i)!\, i!} \left[-\ln(1-z) + \psi(i+1) \right. \\ & \left. + \psi(i+1+c-a-b) - \psi(i+c-a) - \psi(i+c-b) \right] (1-z)^i ,\end{aligned} \tag{B.14}$$

where for $c-a-b=0$, the first series is omitted. At $z=1$, for $c-a-b>0$ which precisely corresponds to the convergent region of ${}_2F_1(a,b;c;z)$, from these last 2 formulas the Gauss' hypergeometric function can be seen to take the value

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} . \tag{B.15}$$

In the main text also is required the formula at $z=1$ for the generalized hypergeometric function ${}_3F_2$ when one of the 3 upper coefficients has value 1 and when one of the other 2 is related to one of the lower coefficients by an integer: ${}_3F_2(e+n,b,1;d,e;1)$, $n \in \mathbb{N}_0$, with convergence requiring $d-1-b-n>0$. Its value can be derived from the known relation [55]

$${}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix} ;1\right) = \frac{c(e-a)}{de} {}_3F_2\left(\begin{matrix} a,\ b+1,\ c+1 \\ d+1,\ e+1 \end{matrix} ;1\right) + \frac{d-c}{d} {}_3F_2\left(\begin{matrix} a,\ b+1,\ c \\ d+1,\ e \end{matrix} ;1\right) , \tag{B.16}$$

which for $b=0$ where the LHS is just 1, after a relabeling

$$\frac{d-1-b}{d-1} {}_3F_2\left(\begin{matrix} a,\ b,\ 1 \\ d,\ e \end{matrix} ;1\right) = 1 - \frac{b(e-a)}{e(d-1)} {}_3F_2\left(\begin{matrix} a,\ b+1,\ 1 \\ d,\ e+1 \end{matrix} ;1\right) . \tag{B.17}$$

This formula relates ${}_3F_2(a,b,1;d,e;1)$ to ${}_3F_2(a,b+1,1;d,e+1;1)$. The relation to ${}_3F_2(a,b+n,1;d,e+n;1)$ is obtained after iterating it $n-1$ times

$$\begin{aligned}\frac{d-1-b}{d-1} {}_3F_2\left(\begin{matrix} a,\ b,\ 1 \\ d,\ e \end{matrix} ;1\right) = & \sum_{k=0}^{n-1} \frac{(b)_k (e-a)_k}{(e)_k (b+2-d)_k} \\ & + \frac{(b)_n (e-a)_n}{(e)_n (b+2-d)_n} \frac{d-1-b-n}{d-1} {}_3F_2\left(\begin{matrix} a,\ b+n,\ 1 \\ d,\ e+n \end{matrix} ;1\right) .\end{aligned} \tag{B.18}$$

When $a=e+n$ with $n=0,\ 1,\ 2,\ \ldots$, the generalized hypergeometric on the RHS becomes a Gauss' hypergeometric ${}_2F_1$ at $z=1$ whose value is given above, resulting in the nice identity

$${}_3F_2\left(\begin{matrix} e+n,\ b,\ 1 \\ d,\ e \end{matrix} ;1\right) = \frac{d-1}{d-1-b} {}_3F_2\left(\begin{matrix} -n,\ b,\ 1 \\ b+2-d,\ e \end{matrix} ;1\right) . \tag{B.19}$$

This expresses the quantity we were after in terms of a terminating series of $n+1$ terms. For instance, for the cases $n=0$ and $n=1$

$$ {}_3F_2\left(\begin{matrix} e+n,\ b,\ 1 \\ d,\ e \end{matrix};1\right) = \frac{d-1}{d-1-b} \qquad (n=0)\,, \tag{B.20}$$

$$ = \frac{d-1}{d-1-b}\left[1-\frac{b}{(b+2-d)e}\right] \qquad (n=1)\,. \tag{B.21}$$

Generalizations of the hypergeometric series to 2 variables are known as Kampé de Fériet functions, represented by the double series

$$ F^{p,q,r}_{s,t,u}\left(\begin{matrix}\vec{a}_p \\ \vec{d}_s\end{matrix};\begin{matrix}\vec{b}_q \\ \vec{e}_t\end{matrix};\begin{matrix}\vec{c}_r \\ \vec{f}_u\end{matrix};x\,,y\right) = \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\vec{a}_p)_{l+k}(\vec{b}_q)_l(\vec{c}_r)_k}{(\vec{d}_s)_{l+k}(\vec{e}_t)_l(\vec{f}_u)_k}\frac{x^l y^k}{l!\,k!}\,, \tag{B.22}$$

with Appell hypergeometric functions being special cases. For compactness it is understood $\vec{a}_p = a_1,\ldots,a_p$, $(\vec{a}_p)_{l+k} = (a_1)_{l+k}\cdots(a_p)_{l+k}$, and similarly for the other coefficients. From exchanging the order of the 2 series, the function is seen to satisfy

$$ F^{p,q,r}_{s,t,u}\left(\begin{matrix}\vec{a}_p \\ \vec{d}_s\end{matrix};\begin{matrix}\vec{b}_q \\ \vec{e}_t\end{matrix};\begin{matrix}\vec{c}_r \\ \vec{f}_u\end{matrix};x\,,y\right) = F^{p,r,q}_{s,u,t}\left(\begin{matrix}\vec{a}_p \\ \vec{d}_s\end{matrix};\begin{matrix}\vec{c}_r \\ \vec{f}_u\end{matrix};\begin{matrix}\vec{b}_q \\ \vec{e}_t\end{matrix};y\,,x\right)\,. \tag{B.23}$$

Moreover, by computing one of the series it can be expressed as a single sum involving generalized hypergeometric functions

$$ F^{p,q,r}_{s,t,u}\left(\begin{matrix}\vec{a}_p \\ \vec{d}_s\end{matrix};\begin{matrix}\vec{b}_q \\ \vec{e}_t\end{matrix};\begin{matrix}\vec{c}_r \\ \vec{f}_u\end{matrix};x\,,y\right) = \sum_{l=0}^{\infty}\frac{(\vec{a}_p)_l(\vec{b}_q)_l}{(\vec{d}_s)_l(\vec{e}_t)_l}\frac{x^l}{l!}\;{}_{p+r}F_{s+u}\left(\begin{matrix}\vec{a}_p+l,\ \vec{c}_r \\ \vec{d}_s+l,\ \vec{f}_u\end{matrix};y\right)\,. \tag{B.24}$$

From this expression, one can see the Kampé de Fériet function reduces to a generalized hypergeometric when either $x=0$ or one of the coefficients in $\vec{b}_q$ is 0

$$ F^{p,q,r}_{s,t,u}\left(\begin{matrix}\vec{a}_p \\ \vec{d}_s\end{matrix};\begin{matrix}\vec{b}_q \\ \vec{e}_t\end{matrix};\begin{matrix}\vec{c}_r \\ \vec{f}_u\end{matrix};0\,,y\right) = F^{p,q,r}_{s,t,u}\left(\begin{matrix}\vec{a}_p \\ \vec{d}_s\end{matrix};\begin{matrix}0,\ \vec{b}_{q-1} \\ \vec{e}_t\end{matrix};\begin{matrix}\vec{c}_r \\ \vec{f}_u\end{matrix};x\,,y\right) = {}_{p+r}F_{s+u}\left(\begin{matrix}\vec{a}_p,\ \vec{c}_r \\ \vec{d}_s,\ \vec{f}_u\end{matrix};y\right)\,. \tag{B.25}$$

By the symmetry of (B.23), same can be said for $y=0$ and $\vec{c}_r = (0,\ \vec{c}_{r-1})$. Meanwhile, when one of the coefficients in $\vec{a}_p$ is 0, the Kampé function simply evaluates to 1

$$ F^{p,q,r}_{s,t,u}\left(\begin{matrix}0,\ \vec{a}_{p-1} \\ \vec{d}_s\end{matrix};\begin{matrix}\vec{b}_q \\ \vec{e}_t\end{matrix};\begin{matrix}\vec{c}_r \\ \vec{f}_u\end{matrix};x\,,y\right) = 1\,. \tag{B.26}$$

There are many other cases where the Kampé de Fériet function reduces to hypergeometric functions. A particularly useful case is

$$ F^{1,1,2}_{1,0,1}\left(\begin{matrix}a \\ d\end{matrix};\begin{matrix}b \\ -\end{matrix};\begin{matrix}c,\ d-b \\ f\end{matrix};x\,,x\right) = (1-x)^{-a}\,{}_3F_2\left(\begin{matrix}a,\ d-b,\ f-c \\ d,\ f\end{matrix};\frac{x}{x-1}\right)\,, \tag{B.27}$$

which for the special case $f = a$, using (B.11) it further reduces to

$$F^{1,1,2}_{1,0,1}\begin{pmatrix} a & b & c,\ d-b \\ d ; & - ; & a \end{pmatrix} ; x\,,\, x \Bigg) = (1-x)^{-c}{}_2F_1(a-c, b; d; x)\,. \tag{B.28}$$

For a more comprehensive list of properties and special cases for the Kampé de Fériet function, we refer the reader to [79].

B.3 Differentiation

Consider the function in d dimensions

$$\frac{(m^2x^2)^a}{x^\alpha}\,, \tag{B.29}$$

where $x \equiv |\vec{x}| > 0$. Acting on it with $\Box$ results in

$$\Box\left[\frac{(m^2x^2)^a}{x^\alpha}\right] = \frac{(m^2x^2)^a}{x^{\alpha+2}}\left[4a^2 + 2(d-2\alpha-2)a + \alpha(\alpha+2-d)\right]\,. \tag{B.30}$$

Expanding both sides in a and matching orders leads to

$$\Box\left(\frac{1}{x^\alpha}\right) = \frac{\alpha(\alpha+2-d)}{x^{\alpha+2}}\,, \tag{B.31}$$

$$\Box\left[\frac{\ln(m^2x^2)}{x^\alpha}\right] = \frac{1}{x^{\alpha+2}}\left[\alpha(\alpha+2-d)\ln(m^2x^2) + 2(d-2\alpha-2)\right]\,, \tag{B.32}$$

$$\begin{aligned}\Box\left[\frac{\ln^k(m^2x^2)}{x^\alpha}\right] = \frac{1}{x^{\alpha+2}}\Big[&\alpha(\alpha+2-d)\ln^k(m^2x^2) + 2k(d-2\alpha-2)\ln^{k-1}(m^2x^2) \\ &+4k(k-1)\ln^{k-2}(m^2x^2)\Big] \qquad (k>1)\,.\end{aligned} \tag{B.33}$$

The first formula may be iterated to give

$$\Box^n\left(\frac{1}{x^\alpha}\right) = \frac{4^n\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma\left(\frac{\alpha-d}{2}+1+n\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-d}{2}+1\right)}\frac{1}{x^{\alpha+2n}}\,. \tag{B.34}$$

The second one may be iterated more easily by first rewriting it as

$$\Box\left[\frac{\ln(m^2x^2)}{x^\alpha}\right] = \frac{\alpha(\alpha+2-d)}{x^{\alpha+2}}\ln\left[\frac{m^2x^2e^{\psi(\frac{\alpha}{2})+\psi(\frac{\alpha-d}{2}+1)}}{e^{\psi(\frac{\alpha}{2}+1)+\psi(\frac{\alpha-d}{2}+2)}}\right]\,, \tag{B.35}$$

where we used the property of the digamma function $\psi(z+1) - \psi(z) = 1/z$. Then

$$\Box^n\left[\frac{\ln(m^2x^2)}{x^\alpha}\right] = \frac{4^n\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma\left(\frac{\alpha-d}{2}+1+n\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-d}{2}+1\right)x^{\alpha+2n}}\ln\left[\frac{m^2x^2e^{\psi(\frac{\alpha}{2})+\psi(\frac{\alpha-d}{2}+1)}}{e^{\psi(\frac{\alpha}{2}+n)+\psi(\frac{\alpha-d}{2}+1+n)}}\right]\,. \tag{B.36}$$

The case $k=2$ on the third formula can also be iterated more easily if we first rewrite it as

$$\Box\left[\frac{\ln^2(m^2x^2)}{x^\alpha}\right]=\frac{\alpha(\alpha+2-d)}{x^{\alpha+2}}\Bigg(\ln^2\left[\frac{m^2x^2e^{\psi(\frac{\alpha}{2})+\psi(\frac{\alpha-d}{2}+1)}}{e^{\psi(\frac{\alpha}{2}+1)+\psi(\frac{\alpha-d}{2}+2)}}\right]+\psi'\left(\frac{\alpha}{2}+1\right)-\psi'\left(\frac{\alpha}{2}\right)$$
$$+\psi'\left(\frac{\alpha-d}{2}+2\right)-\psi'\left(\frac{\alpha-d}{2}+1\right)\Bigg)\,, \tag{B.37}$$

where we completed squares for the logarithmic term, and used the property above for the digamma together with the property for the polygamma function $\psi'(z+1)-\psi'(z)=-1/z^2$. Then

$$\Box^n\left[\frac{\ln^2(m^2x^2)}{x^\alpha}\right]=\frac{4^n\Gamma\left(\frac{\alpha}{2}+n\right)\Gamma\left(\frac{\alpha-d}{2}+1+n\right)}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha-d}{2}+1\right)x^{\alpha+2n}}\Bigg(\ln^2\left[\frac{m^2x^2e^{\psi(\frac{\alpha}{2})+\psi(\frac{\alpha-d}{2}+1)}}{e^{\psi(\frac{\alpha}{2}+n)+\psi(\frac{\alpha-d}{2}+1+n)}}\right] \tag{B.38}$$
$$+\psi'\left(\frac{\alpha}{2}+n\right)-\psi'\left(\frac{\alpha}{2}\right)+\psi'\left(\frac{\alpha-d}{2}+1+n\right)-\psi'\left(\frac{\alpha-d}{2}+1\right)\Bigg)\,.$$

The rest of the iterations of (B.33) for $k>2$ can be obtained in a similar manner.

B.4 Integrals

In here we list a number of useful integral formulas used in our work:

- Solid angle integral in d dimensions

$$\int \mathrm{d}\Omega_{d-1}=\frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}\,,\qquad (d\in\mathbb{N})\,. \tag{B.39}$$

- Integral 5 in 3.915 of [69]

$$\int_0^\pi \mathrm{d}\theta\,\sin^{2\nu}(\theta)e^{i\beta\cos\theta}=\sqrt{\pi}\left(\frac{2}{\beta}\right)^\nu\Gamma\left(\nu+\frac{1}{2}\right)J_\nu(\beta)\,,\qquad \left[\mathrm{Re}(\nu)>-\frac{1}{2}\right]\,. \tag{B.40}$$

- Integral 14 in 6.561 of [69]

$$\int_0^\infty \mathrm{d}x\,x^\mu J_\nu(ax)=2^\mu a^{-\mu-1}\frac{\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)}\,,\qquad\left[-\mathrm{Re}(\nu)-1<\mathrm{Re}(\mu)<\frac{1}{2},\,a>0\right]\,. \tag{B.41}$$

- Integral 3 in 6.576 of [69]

$$\int_0^\infty \mathrm{d}x\; x^{-\lambda} K_\mu(ax) J_\nu(bx) = \frac{b^\nu \Gamma\left(\frac{\nu-\lambda+\mu+1}{2}\right)\Gamma\left(\frac{\nu-\lambda-\mu+1}{2}\right)}{2^{\lambda+1} a^{\nu-\lambda+1}\Gamma(1+\nu)} \times {}_2F_1\left(\begin{matrix}\frac{\nu-\lambda+\mu+1}{2},\; \frac{\nu-\lambda-\mu+1}{2} \\ \nu+1\end{matrix}\; ; -\frac{b^2}{a^2}\right),$$
$$[\mathrm{Re}(a \pm ib) > 0,\; \mathrm{Re}(\nu - \lambda + 1) > |\mathrm{Re}(\mu)|] \;. \quad \text{(B.42)}$$

- Integral 11 in 6.578 of [69]

$$\int_0^\infty \mathrm{d}x\; x^{\nu+1} K_\mu(ax) I_\mu(bx) J_\nu(cx) = \frac{(ab)^{-\nu-1} c^\nu e^{-\left(\nu+\frac{1}{2}\right)\pi i} Q_{\mu-\frac{1}{2}}^{\nu+\frac{1}{2}}(u)}{\sqrt{2\pi}(u^2-1)^{\frac{\nu}{2}+\frac{1}{4}}}\,, \quad \text{(B.43)}$$
$$\left[2abu = a^2 + b^2 + c^2,\; \mathrm{Re}(a) > |\mathrm{Re}(b)| + |\mathrm{Im}(c)|,\; \mathrm{Re}(\nu) > -1,\; \mathrm{Re}(\mu+\nu) > -1\right] \;.$$

- Page 303 of [76]

$$\int_0^z \mathrm{d}x\; \frac{\ln(a+bx)}{c+ex} = \frac{1}{e}\ln\left(\frac{ae-bc}{e}\right)\ln\left(\frac{c+ez}{c}\right) - \frac{1}{e}\mathrm{Li}_2\left[\frac{b(c+ez)}{bc-ae}\right] + \frac{1}{e}\mathrm{Li}_2\left(\frac{bc}{bc-ae}\right) \;. \quad \text{(B.44)}$$

Appendix C

Fourier transforms

We are interested in computing integrals of the following form

$$F_{\nu,k} = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^{2\nu} \ln^k\left(\frac{p^2}{M^2}\right) e^{-i\vec{p}\cdot\vec{x}}\,, \tag{C.1}$$

for $k \in \mathbb{N}_0$, and $\nu \geq 0$, including the special values $\nu = n \in \mathbb{N}_0$. All these cases are contained in the integral

$$F(a) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^{2\nu} \left(\frac{p^2}{M^2}\right)^a e^{-i\vec{p}\cdot\vec{x}}\,, \tag{C.2}$$

as a Taylor expansion in the parameter a

$$F(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!} F_{\nu,k}\,. \tag{C.3}$$

Our goal is to find a closed form expression for $F(a)$, and then expand it in a to obtain the results for $F_{\nu,k}$. We will start by considering a coordinate system for $\vec{p}$ where the z-axis is aligned with $-\vec{x}$. In this frame the resulting integral to compute is

$$F(a) = \frac{1}{(2\pi)^d (M^2)^a} \int_0^\infty \mathrm{d}p\; p^{2\nu+d-1+2a} \int_0^\pi \mathrm{d}\theta\, (\sin\theta)^{d-2} e^{ipx\cos\theta} \int \mathrm{d}\Omega_{d-2}\,. \tag{C.4}$$

where we have introduced spherical coordinates. Using formulas (B.39), (B.40), and (B.41), the integrals can be computed in closed form to give

$$F(a) = \frac{4^\nu}{\pi^{\frac{d}{2}} x^{d+2\nu}} \left(\frac{4}{M^2 x^2}\right)^a \frac{\Gamma\left(\frac{d}{2}+\nu+a\right)}{\Gamma(-\nu-a)}\,. \tag{C.5}$$

To expand this result in the parameter a, the following expression for the gamma function becomes useful

$$\Gamma(x+a) = \Gamma(x) \exp\left[\sum_{k=1}^{\infty} \frac{a^k}{k!} \psi^{(k-1)}(x)\right] , \tag{C.6}$$

where $\psi^{(n)}(x)$ is the polygamma function of order n. This expression can be easily derived from the exponentiation of the series for $\ln[\Gamma(x)]$, together with the definition of the polygamma. With this, we can represent $F(a)$ as

$$F(a) = \frac{4^\nu \Gamma\left(\frac{d}{2}+\nu\right)}{\pi^{\frac{d}{2}} \Gamma(-\nu) x^{d+2\nu}} \exp\left(-a \ln(\overline{M}^2 x^2) + \sum_{k=2}^{\infty} \frac{a^k}{k!} \left[\psi^{(k-1)}\left(\frac{d}{2}+\nu\right) - (-1)^k \psi^{(k-1)}(-\nu)\right]\right) , \tag{C.7}$$

where we defined

$$\overline{M}^2 \equiv \frac{M^2}{4e^{\psi(\frac{d}{2}+\nu)+\psi(-\nu)}} . \tag{C.8}$$

Taking the series of the exponential, the Fourier transforms $F_{\nu,k}$ are found by matching orders in a with (C.3). Here we list the first four

$$F_{\nu,0} = \frac{4^\nu \Gamma\left(\frac{d}{2}+\nu\right)}{\pi^{\frac{d}{2}} \Gamma(-\nu) x^{d+2\nu}} , \tag{C.9}$$

$$F_{\nu,1} = -\frac{4^\nu \Gamma\left(\frac{d}{2}+\nu\right) \ln(\overline{M}^2 x^2)}{\pi^{\frac{d}{2}} \Gamma(-\nu) x^{d+2\nu}} , \tag{C.10}$$

$$F_{\nu,2} = \frac{4^\nu \Gamma\left(\frac{d}{2}+\nu\right)}{\pi^{\frac{d}{2}} \Gamma(-\nu) x^{d+2\nu}} \left[\ln^2(\overline{M}^2 x^2) + \psi'\left(\frac{d}{2}+\nu\right) - \psi'(-\nu)\right] , \tag{C.11}$$

$$F_{\nu,3} = -\frac{4^\nu \Gamma\left(\frac{d}{2}+\nu\right)}{\pi^{\frac{d}{2}} \Gamma(-\nu) x^{d+2\nu}} \left[\ln^3(\overline{M}^2 x^2) + 3\left[\psi'\left(\frac{d}{2}+\nu\right) - \psi'(-\nu)\right] \ln(\overline{M}^2 x^2) - \psi''\left(\frac{d}{2}+\nu\right) - \psi''(-\nu)\right] . \tag{C.12}$$

While the derivation of these formulas relies on using the integrals listed in B.4, which only converge for parameters satisfying specific inequalities, one may extend the validity of the formulas by using analytic continuation. With this understanding then, the formulas (C.9)-(C.12) are valid for any ν such that $\nu \neq -d/2 - n$, with $n \in \mathbb{N}_0$, which always holds in unitary QFTs.

We would also like to understand these formulas when $\nu = n$ is a non-negative integer. At these values, the gamma function $\Gamma(-\nu)$ has a pole and naively (C.9)-(C.12) appear to go to zero. However, we will show that the function $1/x^{d+2\nu}$ also has a pole at these values, and both poles conspire to give a finite, non-vanishing value for the Fourier transform. The pole of $1/x^{d+2\nu}$ when $\nu = n$ is directly related to the divergence of its Fourier transform, reflecting the fact that it is not well-defined as a distribution.

We want to determine the expansion of $1/x^{d+2\nu}$ around $\nu = n + \epsilon$, with $n \in \mathbb{N}_0$ and $0 < \epsilon \ll 1$, in the whole domain $x \geq 0$. At leading order, this seems to be given by $1/x^{d+2\nu} = 1/x^{d+2n} + \mathcal{O}(\epsilon)$, however it turns out this expansion only holds for $x > 0$. The correct expansion with the point $x = 0$ included can be obtained with the help of (B.34), which for $n = \lfloor \nu \rfloor + 1$ and $\alpha = d - 2 + 2\{\nu\}$ allows us to express the function as:

$$\frac{1}{x^{d+2\nu}} = \frac{\Gamma\left(\frac{d}{2} - 1 + \{\nu\}\right)\Gamma\left(\{\nu\}\right)}{4^{\lfloor \nu \rfloor + 1}\Gamma\left(\frac{d}{2} + \nu\right)\Gamma\left(\nu + 1\right)} \Box^{\lfloor \nu \rfloor + 1}\left(\frac{1}{x^{d-2+2\{\nu\}}}\right) . \tag{C.13}$$

Here $\nu = \lfloor \nu \rfloor + \{\nu\}$ has been decomposed in its integer and fractional parts. The above expression is valid in the whole $x \geq 0$ region, as long as $\{\nu\} \neq 0$. In particular for $\lfloor \nu \rfloor = n$ and $\{\nu\} = \epsilon$, the expansion of the function becomes

$$\begin{aligned} \frac{1}{x^{d+2n+2\epsilon}} = &- \frac{1}{\epsilon}\frac{\pi^{\frac{d}{2}}}{4^n\Gamma\left(\frac{d}{2} + n\right) n!}\Box^n \delta(\vec{x}) \\ &- \frac{\Gamma\left(\frac{d}{2} - 1\right)}{4^{n+1}\Gamma\left(\frac{d}{2} + n\right) n!}\Box^{n+1}\left(\frac{1}{x^{d-2}} \ln\left[x^2 \frac{e^{\psi(\frac{d}{2}+n)+\psi(n+1)}}{e^{\psi(\frac{d}{2}-1)+\psi(1)}}\right]\right) + \mathcal{O}(\epsilon) , \end{aligned} \tag{C.14}$$

where we used the known result:

$$\Box\left(\frac{1}{x^{d-2}}\right) = -\frac{4\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2} - 1\right)}\delta(\vec{x}) . \tag{C.15}$$

We see from the first term in (C.14) that the correct expansion of the function has a pole proportional to a Dirac delta with support at $x = 0$. We also see from the identity above that acting with boxes on the second term is subtle. Of course, for $x > 0$ the delta vanishes and now we can easily act with the boxes recovering the naive expansion:

$$\frac{1}{x^{d+2n+2\epsilon}} = \frac{1}{x^{d+2n}} + \mathcal{O}(\epsilon), \qquad (x > 0) . \tag{C.16}$$

The Fourier transform $F_{n,0}$ is then constructed from $F_{\nu,0}$, formula (C.9), by taking the limit $\nu = n + \epsilon$ properly:

$$F_{n,0} = \lim_{\epsilon \to 0} F_{n+\epsilon,0} = \lim_{\epsilon \to 0} \frac{4^{n+\epsilon}\Gamma\left(\frac{d}{2} + n + \epsilon\right)}{\pi^{\frac{d}{2}}\Gamma(-n-\epsilon)x^{d+2n+2\epsilon}} = (-\Box)^n \delta(\vec{x}) . \tag{C.17}$$

This result is consistent with the integral definition of $F_{n,0}$, where each factor of p^2 in its integrand can be moved outside as $(-\Box)$, with the remaining integral being the representation of the delta:

$$F_{n,0} = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^{2n} e^{-i\vec{p}\cdot\vec{x}} = (-\Box)^n \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, e^{-i\vec{p}\cdot\vec{x}} = (-\Box)^n \delta(\vec{x})\,. \tag{C.18}$$

Let us now move to understand $F_{n,1}$. As we mentioned, the presence of the pole in $1/x^{d+2\nu}$ at $\nu = n$ is the failure to understand this quantity as a distribution. This motivates to define a renormalized version of the function, where the pole has been subtracted [88] (the divergence being local may be renormalized using a local counterterm, see for instance [26]):

$$\mathcal{R}\left(\frac{1}{x^{d+2n}}\right) \equiv \lim_{\epsilon\to 0}\left[\frac{1}{x^{d+2n+2\epsilon}} + \frac{1}{\epsilon}\frac{\pi^{\frac{d}{2}}}{4^n\Gamma\left(\frac{d}{2}+n\right)n!}\Box^n\delta(\vec{x})\right]\,, \tag{C.19}$$

$$= -\frac{\Gamma\left(\frac{d}{2}-1\right)}{4^{n+1}\Gamma\left(\frac{d}{2}+n\right)n!}\Box^{n+1}\left(\frac{1}{x^{d-2}}\ln\left[x^2\frac{e^{\psi(\frac{d}{2}+n)+\psi(n+1)}}{e^{\psi(\frac{d}{2}-1)+\psi(1)}}\right]\right)\,, \tag{C.20}$$

where in the second line the limit has been evaluated. This renormalized function has two key properties: it reduces to the bare function away from $x = 0$, and it is well-behaved as a distribution, i.e., it has a Fourier transform. To find what Fourier transform it corresponds to, one simply has to replace in its definition above the function written in momentum space:

$$\frac{1}{x^{d+2n+2\epsilon}} = \frac{\pi^{\frac{d}{2}}\Gamma(-n-\epsilon)}{4^{n+\epsilon}\Gamma\left(\frac{d}{2}+n+\epsilon\right)}F_{n+\epsilon,0}\,, \tag{C.21}$$

$$= -\frac{1}{\epsilon}\frac{\pi^{\frac{d}{2}}}{4^n\Gamma\left(\frac{d}{2}+n\right)n!}\Box^n\delta(\vec{x}) + \frac{\pi^{\frac{d}{2}}(-1)^{n+1}}{4^n\Gamma\left(\frac{d}{2}+n\right)n!}\int\frac{d^d p}{(2\pi)^d}\, p^{2n}\ln\left[\frac{p^2}{4e^{\psi(\frac{d}{2}+n)+\psi(n+1)}}\right]e^{-i\vec{p}\cdot\vec{x}} + \mathcal{O}(\epsilon)\,. \tag{C.22}$$

Plugging this expansion in (C.19) and introducing an arbitrary scale M, one obtains the expression for $F_{n,1}$ in terms of a renormalized function:

$$F_{n,1} = \int\frac{\mathrm{d}^d p}{(2\pi)^d}\, p^{2n}\ln\left(\frac{p^2}{M^2}\right)e^{-i\vec{p}\cdot\vec{x}} = (-1)^{n+1}4^n\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}+n\right)n!\,\mathcal{R}_M\left(\frac{1}{x^{d+2n}}\right)\,, \tag{C.23}$$

where we defined

$$\mathcal{R}_M\left(\frac{1}{x^{d+2n}}\right) \equiv -\frac{\Gamma\left(\frac{d}{2}-1\right)}{4^{n+1}\Gamma\left(\frac{d}{2}+n\right)n!}\Box^{n+1}\left(\frac{1}{x^{d-2}}\ln\left[\frac{M^2x^2}{4e^{\psi(\frac{d}{2}-1)+\psi(1)}}\right]\right) \tag{C.24}$$

The same analysis can be done for the Fourier transforms $F_{n,2}$ and higher. In general, what one finds is that they can be expressed in terms of a renormalized function of the schematic form:

$$F_{n,k} = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^{2n} \ln^k\left(\frac{p^2}{M^2}\right) e^{-i\vec{p}\cdot\vec{x}} = \mathcal{R}_M\left[\sum_{i=0}^{k-1} c_i \frac{\ln^i\left(M^2x^2\right)}{x^{d+2n}}\right] \qquad (k>0)\,, \tag{C.25}$$

for some coefficients c_i, and where the renormalized term is understood similarly as in (C.24) in the sense of differential regularization [58], with the property that $\mathcal{R}_M[f(x)] = f(x)$, for $x > 0$. Now, we can continue in the same way as before to work out the explicit form of the next case, $F_{n,2}$, however there is an easier (and equivalent) way to proceed. It comes from the simple observation that $(-\Box)F_{\nu,k} = F_{\nu+1,k}$. For $d > 2$, the Fourier transform $F_{-1,k}$ is well-defined, thus we can construct all the $F_{n,k}$ from it by acting with enough boxes:

$$F_{n,k} = (-\Box)^{n+1} F_{-1,k}\,. \tag{C.26}$$

Then acting on (C.9)-(C.12) with $(-\Box)^{n+1}$ and evaluating at $\nu = -1$ we obtain:

$$F_{n,0} = \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}}(-\Box)^{n+1}\left(\frac{1}{x^{d-2}}\right)\,, \tag{C.27}$$

$$F_{n,1} = -\frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}}(-\Box)^{n+1}\left[\frac{\ln(\mathcal{M}^2x^2)}{x^{d-2}}\right]\,, \tag{C.28}$$

$$F_{n,2} = \frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}}(-\Box)^{n+1}\left(\frac{1}{x^{d-2}}\left[\ln^2(\mathcal{M}^2x^2) + \psi'\left(\frac{d}{2}-1\right) - \psi'(1)\right]\right)\,, \tag{C.29}$$

$$F_{n,3} = -\frac{\Gamma\left(\frac{d}{2}-1\right)}{4\pi^{\frac{d}{2}}}(-\Box)^{n+1}\left(\frac{1}{x^{d-2}}\left[\ln^3(\mathcal{M}^2x^2) + 3\left[\psi'\left(\frac{d}{2}-1\right) - \psi'(1)\right]\ln(\mathcal{M}^2x^2) - \psi''\left(\frac{d}{2}-1\right) - \psi''(1)\right]\right)\,, \tag{C.30}$$

where

$$\mathcal{M}^2 = \overline{M}^2(\nu=-1) = \frac{M^2}{4e^{\psi(\frac{d}{2}-1)+\psi(1)}}\,. \tag{C.31}$$

Once again, for $F_{n,0}$ using the identity (C.15) in (C.27), we recover the previous result (C.18). Similarly, for $F_{n,1}$ we identify in (C.28) the definition of the renormalized function (C.24), recovering (C.23) as well. Now, the expressions for $F_{n,2}$ and $F_{n,3}$ are also explicitly given. To what renormalized quantities do (C.29) and (C.30) correspond to? This can be determined from the resulting functions after the action of the boxes in the region $x > 0$. For $F_{n,2}$, upon using (B.31) and (B.33) for $x \neq 0$, one obtains after the action of 1 box

$$\Box\left(\frac{1}{x^{d-2}}\left[\ln^2(\mathcal{M}^2x^2) + \psi'\left(\frac{d}{2}-1\right) - \psi'(1)\right]\right) = -\frac{4(d-2)}{x^d}\ln\left[\frac{\mathcal{M}^2x^2e^{\psi(\frac{d}{2}-1)}}{e^{\psi(\frac{d}{2})}}\right]\,. \tag{C.32}$$

The rest of the boxes can be acted with the help of (B.36). This leads to the definition

$$\mathcal{R}_M\left[\frac{\ln\left(M^2x^2\right)}{x^{d+2n}}\right]\equiv-\frac{1}{2}\frac{\Gamma\left(\frac{d}{2}-1\right)}{4^{n+1}\Gamma\left(\frac{d}{2}+n\right)n!}\Box^{n+1}\left(\frac{1}{x^{d-2}}\left[\ln^2\left(\mu^2x^2\right)+\psi'\left(\frac{d}{2}-1\right)-\psi'(1)\right]\right)\,, \tag{C.33}$$

where for simplicity we called

$$\mu^2\equiv\frac{e^{\psi(\frac{d}{2}+n)+\psi(n+1)}}{e^{\psi(\frac{d}{2}-1)+\psi(1)}}M^2\,. \tag{C.34}$$

In terms of (C.33), the Fourier transform $F_{n,2}$ can be expressed as

$$F_{n,2}=\int\frac{\mathrm{d}^dp}{(2\pi)^d}\,p^{2n}\ln^2\left(\frac{p^2}{M^2}\right)e^{-i\vec{p}\cdot\vec{x}}=2(-1)^n4^n\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}+n\right)n!\times\mathcal{R}_M\left(\frac{1}{x^{d+2n}}\ln\left[\frac{M^2x^2}{4e^{\psi(\frac{d}{2}+n)+\psi(n+1)}}\right]\right)\,. \tag{C.35}$$

We can repeat the same analysis for $F_{n,3}$. Using (B.31)-(B.33) for $x>0$,

$$\Box\left(\frac{1}{x^{d-2}}\left[\ln^3(\mathcal{M}^2x^2)+3\left[\psi'\left(\frac{d}{2}-1\right)-\psi'(1)\right]\ln(\mathcal{M}^2x^2)-\psi''\left(\frac{d}{2}-1\right)-\psi''(1)\right]\right)=-\frac{6(d-2)}{x^d}\left(\ln^2\left[\frac{\mathcal{M}^2x^2e^{\psi(\frac{d}{2}-1)}}{e^{\psi(\frac{d}{2})}}\right]+\psi'\left(\frac{d}{2}\right)-\psi'(1)\right)\,. \tag{C.36}$$

Then with the help of (B.38), one can define

$$\mathcal{R}_M\left(\frac{1}{x^{d+2n}}\left[\ln^2\left(M^2x^2\right)+\psi'\left(\frac{d}{2}+n\right)+\psi'(n+1)-2\psi'(1)\right]\right) \tag{C.37}$$

$$\equiv-\frac{1}{3}\frac{\Gamma\left(\frac{d}{2}-1\right)}{4^{n+1}\Gamma\left(\frac{d}{2}+n\right)n!}\Box^{n+1}\left(\frac{1}{x^{d-2}}\left[\ln^3\left(\mu^2x^2\right)+3\left[\psi'\left(\frac{d}{2}-1\right)-\psi'(1)\right]\ln\left(\mu^2x^2\right)-\psi''\left(\frac{d}{2}-1\right)-\psi''(1)\right]\right)\,.$$

In terms of (C.37), $F_{n,3}$ takes the form:

$$F_{n,3}=\int\frac{\mathrm{d}^dp}{(2\pi)^d}\,p^{2n}\ln^3\left(\frac{p^2}{M^2}\right)e^{-i\vec{p}\cdot\vec{x}}=3(-1)^{n+1}4^n\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2}+n\right)n!\times\mathcal{R}_M\left[\frac{1}{x^{d+2n}}\left(\ln^2\left[\frac{M^2x^2}{4e^{\psi(\frac{d}{2}+n)+\psi(n+1)}}\right]+\psi'\left(\frac{d}{2}+n\right)+\psi'(n+1)-2\psi'(1)\right)\right]\,. \tag{C.38}$$

The rest of the Fourier transforms $F_{n,k}$ can be obtained in a similar manner.

Appendix D

Meijer G-function

A general definition of the Meijer G-function is given in terms of a Mellin-Barnes type integral

$$G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ \vec{b}_q\end{matrix}\,\middle|\, x\right) \equiv \frac{1}{2\pi i}\int_L \mathrm{d}s\, \frac{\prod_{i=1}^{m}\Gamma(b_i - s)\prod_{i=1}^{n}\Gamma(1-a_i+s)}{\prod_{i=n+1}^{p}\Gamma(a_i - s)\prod_{i=m+1}^{q}\Gamma(1-b_i+s)}\, x^s\,, \tag{D.1}$$

where $\vec{a}_p = (a_1, \ldots, a_n; a_{n+1}, \ldots, a_p)$ and $\vec{b}_q = (b_1, \ldots, b_m; b_{m+1}, \ldots, b_q)$ are p- and q-dimensional lists of real or complex entries, respectively. The integration path L separates the poles of $\Gamma(b_i - s)$ from the poles of $\Gamma(1 - a_i + s)$. There are three possible paths, see [51, Figure 16.17.1]. When more than one of these paths lead to a convergent integral, they all agree on its value (for more details, see for instance [17, 18]).

The Meijer G-function satisfy many properties. For instance, from its definition we see when one of the a_i in $1 \le i \le n$ is equal to one of the b_i in $m+1 \le i \le q$, or similarly when one of the a_i in $n+1 \le i \le p$ is equal to one of the b_i in $1 \le i \le m$, then the respective Gamma functions cancel and the Meijer reduce to another Meijer:

$$G^{m,n}_{p,q}\left(\begin{matrix}c,\ \vec{a}_{p-1}\\ \vec{b}_{q-1},\ c\end{matrix}\,\middle|\, x\right) = G^{m,n-1}_{p-1,q-1}\left(\begin{matrix}\vec{a}_{p-1}\\ \vec{b}_{q-1}\end{matrix}\,\middle|\, x\right), \quad G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_{p-1},\ c\\ c,\ \vec{b}_{q-1}\end{matrix}\,\middle|\, x\right) = G^{m-1,n}_{p-1,q-1}\left(\begin{matrix}\vec{a}_{p-1}\\ \vec{b}_{q-1}\end{matrix}\,\middle|\, x\right). \tag{D.2}$$

More involved manipulations of (D.1), together with known properties of the Gamma function, lead to a large number of identities for the Meijer G-function. The ones that will be relevant to us are:

$$x^\sigma G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ \vec{b}_q\end{matrix}\,\middle|\,x\right) = G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p+\sigma\\ \vec{b}_q+\sigma\end{matrix}\,\middle|\,x\right)\,, \tag{D.3}$$

$$G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ \vec{b}_q\end{matrix}\,\middle|\,x\right) = (2\pi)^{\frac{p+q}{2}-m-n}2^{\frac{p-q}{2}+1+\Sigma b_i-\Sigma a_j}G^{2m,2n}_{2p,2q}\left(\begin{matrix}\frac{\vec{a}_p}{2},\ \frac{\vec{a}_p+1}{2}\\ \frac{\vec{b}_q}{2},\ \frac{\vec{b}_q+1}{2}\end{matrix}\,\middle|\,4^{p-q}x^2\right)\,, \tag{D.4}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ \vec{b}_q\end{matrix}\,\middle|\,x\right) = -G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ b_1+1,\ b_2,\ \ldots,\ b_q\end{matrix}\,\middle|\,x\right) + b_1 G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ \vec{b}_q\end{matrix}\,\middle|\,x\right) \qquad (m\geq 1)\,, \tag{D.5}$$

$$= G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ b_1,\ \ldots,\ b_{q-1},\ b_q+1\end{matrix}\,\middle|\,x\right) + b_q G^{m,n}_{p,q}\left(\begin{matrix}\vec{a}_p\\ \vec{b}_q\end{matrix}\,\middle|\,x\right) \qquad (m<q)\,. \tag{D.6}$$

Many functions can be represented in terms of Meijer G-functions. In our particular case, we will be interested in the representation of the incomplete gamma function, and the product of two modified Bessel functions:

$$\Gamma(\alpha,x) = G^{2,0}_{1,2}\left(\begin{matrix}1\\ 0,\ \alpha\end{matrix}\,\middle|\,x\right)\,, \tag{D.7}$$

$$x^\sigma K_\nu(x)K_\mu(x) = \frac{\sqrt{\pi}}{2}G^{4,0}_{2,4}\left(\begin{matrix}\frac{\sigma}{2},\ \frac{\sigma+1}{2}\\ \frac{\sigma+\nu+\mu}{2},\ \frac{\sigma+\nu-\mu}{2},\ \frac{\sigma-\nu+\mu}{2},\ \frac{\sigma-\nu-\mu}{2}\end{matrix}\,\middle|\,x^2\right)\,, \tag{D.8}$$

$$x^\sigma I_\nu(x)K_\mu(x) = \frac{1}{2\sqrt{\pi}}G^{2,2}_{2,4}\left(\begin{matrix}\frac{\sigma}{2},\ \frac{\sigma+1}{2}\\ \frac{\sigma+\nu+\mu}{2},\ \frac{\sigma+\nu-\mu}{2},\ \frac{\sigma-\nu+\mu}{2},\ \frac{\sigma-\nu-\mu}{2}\end{matrix}\,\middle|\,x^2\right)\,. \tag{D.9}$$

For $\sigma = 0$ and $\mu = \nu$, the last two reduce to:

$$K^2_\nu(x) = \frac{\sqrt{\pi}}{2}G^{3,0}_{1,3}\left(\begin{matrix}\frac{1}{2}\\ \nu,\ 0,\ -\nu\end{matrix}\,\middle|\,x^2\right)\,, \tag{D.10}$$

$$I_\nu(x)K_\nu(x) = \frac{1}{2\sqrt{\pi}}G^{2,1}_{1,3}\left(\begin{matrix}\frac{1}{2}\\ \nu,\ 0,\ -\nu\end{matrix}\,\middle|\,x^2\right)\,. \tag{D.11}$$

In our study of loops in AdS/CFT, we encounter two Meijer G-functions in the computation of the $\int KK$ and $\int GK$ integrals. These are of the form:

$$G^{4,0}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1\\ \nu,\ 0,\ 0,\ -\nu\end{matrix}\,\middle|\,x\right)\,,\quad G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1\\ \nu,\ 0,\ 0,\ -\nu\end{matrix}\,\middle|\,x\right)\,. \tag{D.12}$$

where $\nu \geq 0$. From (D.5), (D.10) and (D.11), we see these are related to the product of 2 Bessel functions via:

$$\frac{K_\nu^2(x)}{x} = -\frac{\sqrt{\pi}}{4}\frac{\mathrm{d}}{\mathrm{d}x}G_{2,4}^{4,0}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x^2\right), \tag{D.13}$$

$$\frac{I_\nu(x)K_\nu(x)}{x} = -\frac{1}{4\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}x}G_{2,4}^{3,1}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x^2\right). \tag{D.14}$$

We would like to know their series representation in x and, if possible, their expression in terms of elementary functions. With this objective in mind, let us start analyzing the first one.

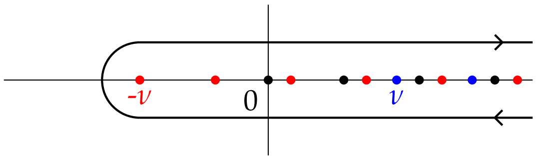

FIGURE D.1: Contour L for the computation of the Meijer G-function in (D.15) for an arbitrary value of ν. The red dots are position of poles integer spaced from $-\nu$, the blue ones integer spaced from ν and the **black** ones integer spaced from 0.

By definition:

$$G_{2,4}^{4,0}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right) = \frac{1}{2\pi i}\int_L \mathrm{d}s\, \frac{\Gamma(-s)^2\Gamma(\nu-s)\Gamma(-\nu-s)}{\Gamma\left(\frac{1}{2}-s\right)\Gamma(1-s)}x^s\,, \tag{D.15}$$

where the path L starts at infinity on a line parallel to the positive real axis, encircles once in the negative direction the poles of the Gamma functions in the numerator and returns to infinity on another line parallel to the positive real axis, see Fig. D.1. In virtue of the residue theorem:

$$G_{2,4}^{4,0}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right) = -\sum_k \mathrm{Res}\left[f(s), s_k\right]\,, \tag{D.16}$$

where $\mathrm{Res}\left[f(s), s_k\right]$ is the residue of $f(s)$ around the pole s_k (inside the path L) of order n,

$$\mathrm{Res}\left[f(s), s_k\right] = \frac{1}{(n-1)!}\lim_{s\to s_k}\frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}}\left[(s-s_k)^n f(s)\right]\,, \tag{D.17}$$

where in our case the function $f(s)$ is given by

$$f(s) = \frac{\Gamma(-s)^2\Gamma(\nu-s)\Gamma(-\nu-s)}{\Gamma\left(\frac{1}{2}-s\right)\Gamma(1-s)}x^s\,. \tag{D.18}$$

The pole structure of $f(s)$ depends whether ν is: (1) $\nu > 0$ real but not an integer, (2) $\nu = n \in \mathbb{N}$, and (3) $\nu = 0$. Even though the Gamma functions in the denominator of $f(s)$ do not contribute with poles, they can (and in fact, they will) change the order of the poles contained inside the contour L.

In case (1), the poles are

$$\begin{aligned} s_k &= 0 && \to \text{double}, \\ s_k &= 1+k,\ k \in \mathbb{N}_0 && \to \text{simple}, \\ s_k &= \nu+k,\ k \in \mathbb{N}_0 && \to \text{simple}, \\ s_k &= -\nu+k,\ k \in \mathbb{N}_0 && \to \text{simple}, \end{aligned} \tag{D.19}$$

with the corresponding residues:

$$\operatorname{Res}\left[f(s), 0\right] = \frac{\Gamma(\nu)\Gamma(-\nu)}{\sqrt{\pi}} \ln\left[\frac{x}{4e^{\psi(\nu)+\psi(-\nu)}}\right], \tag{D.20}$$

$$\operatorname{Res}\left[f(s), -\nu+k\right] = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2\nu-k)\Gamma(\nu-k)^2(-1)^{k+1}}{\Gamma(2\nu+1-2k)k!} \left(\frac{x}{4}\right)^{-\nu+k}, \tag{D.21}$$

The other 2 residues yield a contribution that vanishes as $x \to 0$. This implies that the small x series representation of the Meijer G-function is given by

$$\begin{aligned} G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix} \middle|\, x\right) &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\lfloor \nu \rfloor} \frac{\Gamma(2\nu-k)\Gamma(\nu-k)^2(-1)^k}{\Gamma(2\nu+1-2k)k!} \left(\frac{x}{4}\right)^{-\nu+k} \\ &\quad - \frac{\Gamma(\nu)\Gamma(-\nu)}{\sqrt{\pi}} \ln\left[\frac{x}{4e^{\psi(\nu)+\psi(-\nu)}}\right] + \mathcal{O}(x^{0<}) \quad (\nu > 0,\ \nu \neq \mathbb{N}), \end{aligned} \tag{D.22}$$

where $\lfloor \nu \rfloor$ denotes the greatest integer less than or equal to ν.

In case (2) ($\nu = n$ is a positive integer), the pole structure is instead

$$\begin{aligned} s_k &= -n+k,\ k \in \{0, \dots, n-1\} && \to \text{simple}, \\ s_k &= 0 && \to \text{triple}, \\ s_k &= 1+k,\ k \in \{0, \dots, n-2\} && \to \text{double}, \\ s_k &= n+k,\ k \in \mathbb{N}_0 && \to \text{triple}, \end{aligned} \tag{D.23}$$

and the corresponding residues are given by

$$\operatorname{Res}\left[f(s), -n+k\right] = \frac{1}{\sqrt{\pi}} \frac{\Gamma(2n-k)\Gamma(n-k)^2(-1)^{k+1}}{\Gamma(2n+1-2k)k!} \left(\frac{x}{4}\right)^{-n+k}, \tag{D.24}$$

$$\operatorname{Res}\left[f(s), 0\right] = \frac{(-1)^{n+1}}{2n^3\sqrt{\pi}} \left(1 + n^2 \ln^2\left[\frac{x}{4e^{\psi(n)+\psi(n+1)}}\right]\right), \tag{D.25}$$

with the other two residues yielding a contribution that vanishes as $x \to 0$. Thus, in this case the small x series representation of the Meijer G-function is given by

$$\begin{aligned} G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ n,\ 0,\ 0,\ -n \end{matrix} \middle|\, x\right) &= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{\Gamma(2n-k)\Gamma(n-k)^2(-1)^k}{\Gamma(2n+1-2k)k!} \left(\frac{x}{4}\right)^{-n+k} \\ &\quad + \frac{(-1)^n}{2n^3\sqrt{\pi}} \left(1 + n^2 \ln^2\left[\frac{x}{4e^{\psi(n)+\psi(n+1)}}\right]\right) + \mathcal{O}(x^{0<}) \quad (n \in \mathbb{N}). \end{aligned} \tag{D.26}$$

Finally, for case (3) (*i.e.* $\nu = 0$) the pole structure of $f(s)$ is

$$\begin{aligned} s_k &= 0 && \rightarrow \text{ quadruple,} \\ s_k &= 1+k,\ k \in \mathbb{N}_0 && \rightarrow \text{ triple,} \end{aligned} \tag{D.27}$$

and the corresponding residues are

$$\text{Res}\left[f(s), 0\right] = -\frac{4\zeta(3)}{3\sqrt{\pi}} + \frac{1}{6\sqrt{\pi}} \ln^3\left[\frac{x}{4e^{2\psi(1)}}\right], \tag{D.28}$$

with the other one yielding a contribution that vanishes as $x \to 0$. Thus this case the small x series representation of the Meijer is given by

$$G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ 0,\ 0,\ 0,\ 0 \end{matrix} \middle| x\right) = \frac{4\zeta(3)}{3\sqrt{\pi}} - \frac{1}{6\sqrt{\pi}} \ln^3\left[\frac{x}{4e^{2\psi(1)}}\right] + \mathcal{O}(x^{0<}). \tag{D.29}$$

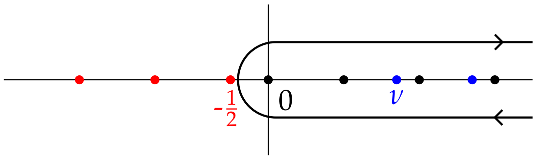

FIGURE D.2: Contour L for the computation of the Meijer G-function in (D.30) for an arbitrary value of ν. The blue dots are position of poles integer spaced from ν, the **black** ones integer spaced from 0 and the red ones, outside of L, integer spaced from $-\frac{1}{2}$.

Moving now to the second Meijer G-function of interest, we will perform an identical analysis. By its definition:

$$G^{3,1}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix} \middle| x\right) = \frac{1}{2\pi i}\int_L \mathrm{d}s\, \frac{\Gamma(-s)^2\Gamma(\nu-s)\Gamma\left(\frac{1}{2}+s\right)}{\Gamma(1-s)\Gamma(1+\nu+s)} x^s, \tag{D.30}$$

where the path L starts at infinity on a line parallel to the positive real axis, encircles once in the negative direction the poles of all the gamma functions in the numerator except those of $\Gamma\left(\frac{1}{2}+s\right)$ and returns to infinity on another line parallel to the positive real axis, see Fig. D.2. The function to analyze in this case is

$$f(s) = \frac{\Gamma(-s)^2\Gamma(\nu-s)\Gamma\left(\frac{1}{2}+s\right)}{\Gamma(1-s)\Gamma(1+\nu+s)} x^s. \tag{D.31}$$

Its pole structure will depend whether ν is: (1) $\nu > 0$ real but not an integer, (2) $\nu = n \in \mathbb{N}$, or (3) $\nu = 0$, and as before, the order of the poles will be affected by the Gamma functions in the denominator of $f(s)$.

In case (1), the poles are

$$\begin{aligned} s_k &= 0 && \rightarrow \text{ double,} \\ s_k &= 1+k,\ k\in\mathbb{N}_0 && \rightarrow \text{ simple,} \\ s_k &= \nu+k,\ k\in\mathbb{N}_0 && \rightarrow \text{ simple,} \end{aligned} \tag{D.32}$$

and the corresponding residue are

$$\mathrm{Res}\left[f(s),0\right] = -\frac{\sqrt{\pi}}{\nu^2}\left(1-\nu\ln\left[\frac{x}{4e^{2\psi(\nu)}}\right]\right)\ , \tag{D.33}$$

with the other 2 residues being subleading in x in the $x\rightarrow 0$ limit. The resulting series representation for the Meijer G-function is

$$G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu\end{matrix}\ \middle|\ x\right) = \frac{\sqrt{\pi}}{\nu^2}\left(1-\nu\ln\left[\frac{x}{4e^{2\psi(\nu)}}\right]\right) + \mathcal{O}(x^{0<}) \quad (\nu>0,\ \nu\neq\mathbb{N})\ . \tag{D.34}$$

In case (2) ($\nu = n$ is a positive integer), the pole structure is instead

$$\begin{aligned} s_k &= 0 && \rightarrow \text{ double,} \\ s_k &= 1+k,\ k\in\{0,\dots,n-2\} && \rightarrow \text{ simple,} \\ s_k &= n+k,\ k\in\mathbb{N}_0 && \rightarrow \text{ double,} \end{aligned} \tag{D.35}$$

and the corresponding residues are

$$\mathrm{Res}\left[f(s),0\right] = -\frac{\sqrt{\pi}}{n^2}\left(1-n\ln\left[\frac{x}{4e^{2\psi(n)}}\right]\right)\ , \tag{D.36}$$

with the other 2 residues being subleading in x in the $x\rightarrow 0$ limit. In this case, the series representation for the Meijer is

$$G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\ 1 \\ n,\ 0,\ 0,\ -n\end{matrix}\ \middle|\ x\right) = \frac{\sqrt{\pi}}{n^2}\left(1-n\ln\left[\frac{x}{4e^{2\psi(n)}}\right]\right) + \mathcal{O}(x^{0<}) \quad (n\in\mathbb{N})\ . \tag{D.37}$$

Finally, for case (3) (*i.e.* $\nu = 0$) the pole structure of $f(s)$ is

$$\begin{aligned} s_k &= 0 && \rightarrow \text{ triple,} \\ s_k &= 1+k,\ k\in\mathbb{N}_0 && \rightarrow \text{ double,} \end{aligned} \tag{D.38}$$

and the corresponding residue are

$$\mathrm{Res}\left[f(s),0\right] = -\frac{\sqrt{\pi}}{2}\left(4\psi'(1)+\ln^2\left[\frac{x}{4e^{2\psi(1)}}\right]\right)\ , \tag{D.39}$$

with the other one being subleading in x in the $x \to 0$ limit. In this case then, the series representation of the Meijer is:

$$G^{3,1}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ 0,\ 0,\ 0,\ 0 \end{matrix}\,\middle|\, x\right) = \frac{\sqrt{\pi}}{2}\left(4\psi'(1) + \ln^2\left[\frac{x}{4e^{2\psi(1)}}\right]\right) + \mathcal{O}(x^{0<})\,. \tag{D.40}$$

Having answered the question about the series representation for the Meijer G-functions, we would now like to address whether it is possible to write them in terms of known, simpler functions. It turns out that the simple pole structure of the Meijers in the case where ν is not an integer, allow us to perform the sums of residues in terms of hypergeometric functions:

$$\begin{aligned} G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right) &= \frac{\Gamma(\nu)^2}{2\nu\sqrt{\pi}}\left(\frac{x}{4}\right)^{-\nu} {}_2F_3\left(\begin{matrix} -\nu,\ -\nu+\frac{1}{2} \\ -\nu+1,\ -\nu+1,\ -2\nu+1 \end{matrix};x\right) + (\nu \to -\nu) \\ &- \frac{\Gamma(\nu)\Gamma(-\nu)}{\sqrt{\pi}}\left[\ln\left(\frac{x}{4e^{\psi(\nu)+\psi(-\nu)}}\right) + \frac{x}{2(1-\nu^2)}{}_3F_4\left(\begin{matrix} 1,\ 1,\ \frac{3}{2} \\ 2,\ 2,\ 2+\nu,\ 2-\nu \end{matrix};x\right)\right], \end{aligned} \tag{D.41}$$

and

$$\begin{aligned} G^{3,1}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right) = \frac{\sqrt{\pi}}{\nu}\Bigg[&\frac{1}{\nu} - \ln\left(\frac{x}{4e^{2\psi(\nu)}}\right) - \frac{x}{2(1-\nu^2)}{}_3F_4\left(\begin{matrix} 1,\ 1,\ \frac{3}{2} \\ 2,\ 2,\ 2+\nu,\ 2-\nu \end{matrix};x\right) \\ &-\frac{\Gamma(-\nu)}{\Gamma(\nu+1)}\left(\frac{x}{4}\right)^{\nu} {}_2F_3\left(\begin{matrix} \nu,\ \nu+\frac{1}{2} \\ \nu+1,\ \nu+1,\ 2\nu+1 \end{matrix};x\right)\Bigg]. \end{aligned} \tag{D.42}$$

For integer ν however, the pole structures become more complicated and this is no longer possible. Nevertheless, given the nice identities satisfied by the Meijers (in particular (D.5) and (D.6)), one can relate the Meijers of parameter ν with those with $\nu+1$:

$$\begin{aligned} (\nu+1)G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu+1,\ 0,\ 0,\ -\nu-1 \end{matrix}\,\middle|\, x\right) &= G^{3,0}_{1,3}\left(\begin{matrix} \frac{1}{2} \\ \nu+1,\ 0,\ -\nu-1 \end{matrix}\,\middle|\, x\right) - G^{3,0}_{1,3}\left(\begin{matrix} \frac{1}{2} \\ \nu,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right) \\ &- \nu G^{4,0}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right), \end{aligned} \tag{D.43}$$

$$\begin{aligned} (\nu+1)G^{3,1}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu+1,\ 0,\ 0,\ -\nu-1 \end{matrix}\,\middle|\, x\right) &= G^{2,1}_{1,3}\left(\begin{matrix} \frac{1}{2} \\ \nu+1,\ 0,\ -\nu-1 \end{matrix}\,\middle|\, x\right) + G^{2,1}_{1,3}\left(\begin{matrix} \frac{1}{2} \\ \nu,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right) \\ &+ \nu G^{3,1}_{2,4}\left(\begin{matrix} \frac{1}{2},\ 1 \\ \nu,\ 0,\ 0,\ -\nu \end{matrix}\,\middle|\, x\right). \end{aligned} \tag{D.44}$$

Iterating these $n-1$ times and using (D.10) and (D.11) one obtains

$$G^{4,0}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ \nu+n,\,0,\,0,\,-\nu-n\end{matrix}\,\middle|\,x\right) = \frac{\nu(-1)^n}{\nu+n}G^{4,0}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ \nu,\,0,\,0,\,-\nu\end{matrix}\,\middle|\,x\right) + \frac{2}{(\nu+n)\sqrt{\pi}}\sum_{i=0}^{n-1}(-1)^i\left[K^2_{\nu+n-i}(\sqrt{x}) - K^2_{\nu+n-1-i}(\sqrt{x})\right]\,, \tag{D.45}$$

$$G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ \nu+n,\,0,\,0,\,-\nu-n\end{matrix}\,\middle|\,x\right) = \frac{\nu}{\nu+n}G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ \nu,\,0,\,0,\,-\nu\end{matrix}\,\middle|\,x\right) + \frac{2\sqrt{\pi}}{\nu+n}\sum_{i=0}^{n-1}\left[I_{\nu+n-i}(\sqrt{x})K_{\nu+n-i}(\sqrt{x}) + I_{\nu+n-1-i}(\sqrt{x})K_{\nu+n-1-i}(\sqrt{x})\right]\,. \tag{D.46}$$

Thus, one may restrict the study of the Meijer G-functions to the window $0 \leq \nu < 1$, as the other cases can be brought to this window through the use of these relations. For instance, the lower bound $\nu = 0$ relates the Meijers of positive integer parameter n to the sum of product of Bessels of the form:

$$G^{4,0}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ n,\,0,\,0,\,-n\end{matrix}\,\middle|\,x\right) = \frac{2}{n\sqrt{\pi}}\sum_{i=0}^{n-1}(-1)^i\left[K^2_{n-i}(\sqrt{x}) - K^2_{n-1-i}(\sqrt{x})\right]\,, \tag{D.47}$$

$$G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ n,\,0,\,0,\,-n\end{matrix}\,\middle|\,x\right) = \frac{2\sqrt{\pi}}{n}\sum_{i=0}^{n-1}\left[I_{n-i}(\sqrt{x})K_{n-i}(\sqrt{x}) + I_{n-1-i}(\sqrt{x})K_{n-1-i}(\sqrt{x})\right]\,. \tag{D.48}$$

As a last remark on Meijer G-functions, it turns out that the two Meijers of interest are closely related to $\partial_\nu K_\nu(x)$. This can be seen by writing down the equation satisfied by this function:

$$\left[\partial_x^2 + \frac{1}{x}\partial_x - \left(1 + \frac{\nu^2}{x^2}\right)\right]\partial_\nu K_\nu(x) = \frac{2\nu}{x^2}K_\nu(x)\,, \tag{D.49}$$

obtained from the Bessel equation for $K_\nu(x)$ by differentiating it with respect to ν. Solving this equation using variation of parameters, one finds

$$\partial_\nu K_\nu(x) = aK_\nu(x) + bI_\nu(x) - 2\nu K_\nu(x)\int \mathrm{d}x\,\frac{I_\nu(x)K_\nu(x)}{x} + 2\nu I_\nu(x)\int \mathrm{d}x\,\frac{K_\nu^2(x)}{x}\,. \tag{D.50}$$

Matching the asymptotic of both sides fixes the integration constants $a = b = 0$, while the integrals of Bessels are solved precisely in terms of these Meijers through (D.13) and (D.14), resulting in the identity:

$$\frac{2}{\nu}\partial_\nu K_\nu(x) = \frac{1}{\sqrt{\pi}}K_\nu(x)G^{3,1}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ \nu,\,0,\,0,\,-\nu\end{matrix}\,\middle|\,x^2\right) - \sqrt{\pi}I_\nu(x)G^{4,0}_{2,4}\left(\begin{matrix}\frac{1}{2},\,1\\ \nu,\,0,\,0,\,-\nu\end{matrix}\,\middle|\,x^2\right)\,, \tag{D.51}$$

valid for any $\nu > 0$. For $\nu = 0$, the RHS has a well-defined value, however the LHS becomes $0/0$. Then using L'Hopital's rule:

$$2\partial_\nu^2 K_\nu(x)\big|_{\nu=0} = \frac{1}{\sqrt{\pi}} K_0(x) G_{2,4}^{3,1}\left(\begin{matrix} \frac{1}{2},\ 1 \\ 0,\ 0,\ 0,\ 0 \end{matrix} \,\middle|\, x^2\right) - \sqrt{\pi} I_0(x) G_{2,4}^{4,0}\left(\begin{matrix} \frac{1}{2},\ 1 \\ 0,\ 0,\ 0,\ 0 \end{matrix} \,\middle|\, x^2\right). \tag{D.52}$$

The same result may be obtained by taking one more derivative of the inhomogeneous Bessel equation (D.49) with respect to ν, and repeating the same steps as we did when $\nu > 0$ but now for $\nu = 0$.

Appendix E

Master formulas

In this paper, our strategy to solve many of the bulk loop integrals is to deal with them directly in position space, by writing the bulk-to-bulk propagators in their series representations, and expressing the resulting integrals in terms of the fundamental vertices

$$V_{\Delta,\Delta_1,\ldots,\Delta_n}(x_1,\vec{y}_1,\ldots,\vec{y}_n) = \int \mathrm{d}^{d+1}x_2\sqrt{g_2}\,\xi^{\Delta}(x_1,x_2)K^{\Delta_1}(x_2,\vec{y}_1)\cdots K^{\Delta_n}(x_2,\vec{y}_n)\,, \quad \text{(E.1)}$$

where $\xi(x_1,x_2)$ is the chordal distance and $K(x_2,\vec{y}_i) = z_2/[z_2^2+(\vec{x}_2-\vec{y}_i)^2]$. It turns

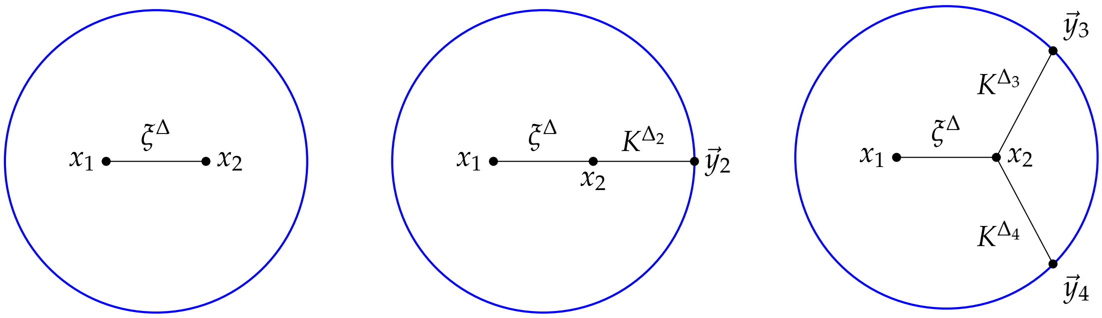

FIGURE E.1: Fundamental vertices V_{Δ}, V_{Δ,Δ_2} and $V_{\Delta,\Delta_3,\Delta_4}$ with 0, 1 and 2 external legs extended to the boundary, respectively. The bulk point x_1 and the boundary points $\vec{y}_i$ are fixed, while the bulk vertex point x_2 is integrated in the whole AdS space.

out, for vertices involving any number of internal legs between the same 2 points in the bulk, and at most 2 external legs extended to the boundary, can be solved in terms of the master integral

$$I_{a,b,c}(w,\vec{y}_1,\vec{y}_2) = \int \mathrm{d}^{d+1}x\sqrt{g}\,\frac{z^a}{[z^2+w^2+(\vec{x}-\vec{y}_1)^2]^b[z^2+(\vec{x}-\vec{y}_2)^2]^c}\,. \quad \text{(E.2)}$$

We will proceed then to give a closed form for $I_{a,b,c}$, to then compute $V_{\Delta,\Delta_1,\ldots,\Delta_n}$ for the cases $n = 0,1,2$ (see Fig. E.1). With the use of Feynman parametrization

$$\frac{1}{\prod A_i^{\alpha_i}} = \frac{\Gamma\left(\sum \alpha_i\right)}{\prod \Gamma(\alpha_i)} \int_0^1 \frac{\delta\left(\sum u_i - 1\right) \prod u_i^{\alpha_i - 1} \mathrm{d}u_i}{\left(\sum u_i A_i\right)^{\sum \alpha_i}}\,, \tag{E.3}$$

after completing squares in $\vec{x}$ and translating, one obtains

$$\begin{aligned} I_{a,b,c} =& \frac{\Gamma(b+c)}{\Gamma(b)\Gamma(c)} \int_0^1 \mathrm{d}^2 u\, \delta(u_1 + u_2 - 1) u_1^{b-1} u_2^{c-1} \\ &\times \int \mathrm{d}^{d+1} x \sqrt{g}\, \frac{z^a}{(z^2 + |\vec{x}|^2 + u_1 u_2 |\vec{y}_{12}|^2 + u_1 w^2)^{b+c}}\,. \end{aligned} \tag{E.4}$$

The integral in $\vec{x}$ can be performed in spherical coordinates after a rescaling $\vec{x} \mapsto \vec{x}\sqrt{z^2 + u_1 u_2 |\vec{y}_{12}|^2 + u_1 w^2}$

$$\begin{aligned} I_{a,b,c} =& \pi^{\frac{d}{2}} \frac{\Gamma\left(b + c - \frac{d}{2}\right)}{\Gamma(b)\Gamma(c)} \int_0^1 \mathrm{d}^2 u\, \delta(u_1 + u_2 - 1) u_1^{b-1} u_2^{c-1} \\ &\times \int_0^\infty \mathrm{d}z\, \frac{z^{a-d-1}}{(z^2 + u_1 u_2 |\vec{y}_{12}|^2 + u_1 w^2)^{b+c-\frac{d}{2}}}\,. \end{aligned} \tag{E.5}$$

The integral in z can be done in a similar fashion after a rescaling $z \mapsto z\sqrt{u_1 u_2 |\vec{y}_{12}|^2 + u_1 w^2}$

$$I_{a,b,c} = \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\left(\frac{a-d}{2}\right) \Gamma\left(b + c - \frac{a}{2}\right)}{\Gamma(b)\Gamma(c)} \int_0^1 \mathrm{d}^2 u\, \delta(u_1 + u_2 - 1) \frac{u_1^{\frac{a}{2} - c - 1} u_2^{c-1}}{(u_2 |\vec{y}_{12}|^2 + w^2)^{b+c-\frac{a}{2}}}\,. \tag{E.6}$$

Evaluating the Dirac delta in $u_2 = 1 - u_1$, the resulting integral in u_1 can be identified in terms of the integral representation for the hypergeometric function ${}_2F_1$

$$\int_0^1 \mathrm{d}x\, x^{b-1} (1-x)^{c-b-1} (1 - zx)^{-a} = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z)\,, \tag{E.7}$$

finally leading to the master formula

$$\begin{aligned} I_{a,b,c}(w, \vec{y}_1, \vec{y}_2) =& \frac{\pi^{\frac{d}{2}}}{2} \frac{\Gamma\left(\frac{a-d}{2}\right) \Gamma\left(b + c - \frac{a}{2}\right) \Gamma\left(\frac{a}{2} - c\right)}{\Gamma(b) \Gamma\left(\frac{a}{2}\right) (w^2 + |\vec{y}_{12}|^2)^{b+c-\frac{a}{2}}} \\ &\times {}_2F_1 \left(\begin{matrix} b + c - \frac{a}{2},\ \frac{a}{2} - c \\ \frac{a}{2} \end{matrix} ; 1 - \frac{w^2}{w^2 + |\vec{y}_{12}|^2} \right)\,. \end{aligned} \tag{E.8}$$

Consider the fundamental vertex (E.1) with no external legs to the boundary. Writing the chordal distance explicitly, it can be expressed in terms of the master formula (E.8) as

$$V_\Delta(x_1) = (2z_1)^\Delta I_{\Delta,\Delta,0}(z_1, \vec{x}_1, \vec{0})\,. \tag{E.9}$$

Using the binomial series ${}_1F_0(a;z) = (1-z)^{-a}$, and Legendre duplication formula $\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$, $I_{\Delta,\Delta,0}$ can be evaluated to the simple form

$$I_{\Delta,\Delta,0}(z_1,\vec{x}_1,\vec{0}) = \frac{\pi^{\frac{d+1}{2}}}{(2z_1)^{\Delta}} \frac{\Gamma\left(\frac{\Delta-d}{2}\right)}{\Gamma\left(\frac{\Delta+1}{2}\right)}\,, \tag{E.10}$$

resulting in the vertex

$$V_{\Delta}(x_1) = \pi^{\frac{d+1}{2}} \frac{\Gamma\left(\frac{\Delta-d}{2}\right)}{\Gamma\left(\frac{\Delta+1}{2}\right)}\,. \tag{E.11}$$

Notice it is independent of the point x_1, i.e., a constant. Moving now to the fundamental vertex with 1 external leg to the boundary, written in terms of $I_{a,b,c}$

$$V_{\Delta,\Delta_2}(x_1,\vec{y}_2) = (2z_1)^{\Delta} I_{\Delta+\Delta_2,\Delta,\Delta_2}(z_1,\vec{x}_1,\vec{y}_2)\,. \tag{E.12}$$

Again, using the binomial series and Legendre duplication formula, the vertex is evaluated to

$$V_{\Delta,\Delta_2}(x_1,\vec{y}_2) = \pi^{\frac{d+1}{2}} \frac{\Gamma\left(\frac{\Delta+\Delta_2-d}{2}\right)\Gamma\left(\frac{\Delta-\Delta_2}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\frac{\Delta+1}{2}\right)} K^{\Delta_2}(x_1,\vec{y}_2)\,. \tag{E.13}$$

Lastly, for the fundamental vertex with 2 external legs to the boundary, it can be expressed in terms of (E.8) by first translating by $\vec{y}_4$ and then inverting every point

$$V_{\Delta,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4) = (2z_1'')^{\Delta} |\vec{y}_{34}'|^{2\Delta_3} I_{\Delta+\Delta_3+\Delta_4,\Delta,\Delta_3}(z_1'',\vec{x}_1'',\vec{y}_{34}')\,, \tag{E.14}$$

where $x_1'' = x_1'/x_1'^2$, $x_1' = x_1 - \vec{y}_4$, and $\vec{y}_{34}' = \vec{y}_{34}/|\vec{y}_{34}|^2$. Using Legendre duplication formula and Euler's transformation (B.12), $I_{\Delta+\Delta_3+\Delta_4,\Delta,\Delta_3}$ takes the form

$$\begin{aligned} I_{\Delta+\Delta_3+\Delta_4,\Delta,\Delta_3}(z_1'',\vec{x}_1'',\vec{y}_{34}') &= \frac{\pi^{\frac{d+1}{2}}}{(2z_1'')^{\Delta}} \frac{\Gamma\left(\frac{\Delta+\Delta_3+\Delta_4-d}{2}\right)\Gamma\left(\frac{\Delta+\Delta_3-\Delta_4}{2}\right)\Gamma\left(\frac{\Delta+\Delta_4-\Delta_3}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\frac{\Delta+1}{2}\right)\Gamma\left(\frac{\Delta+\Delta_3+\Delta_4}{2}\right)} \\ &\times K^{\Delta_3}(x_1'',\vec{y}_{34}') z_1''^{\Delta_4}\, {}_2F_1\left(\begin{matrix}\Delta_3,\ \Delta_4 \\ \frac{\Delta+\Delta_3+\Delta_4}{2}\end{matrix}; 1 - \frac{z_1''^2}{z_1''^2 + (\vec{x}_1'' - \vec{y}_{34}')^2}\right)\,. \end{aligned} \tag{E.15}$$

Replacing this result in the vertex and expressing it back in terms of the original coordinates

$$\begin{aligned} V_{\Delta,\Delta_3,\Delta_4}(x_1,\vec{y}_3,\vec{y}_4) &= \pi^{\frac{d+1}{2}} \frac{\Gamma\left(\frac{\Delta+\Delta_3+\Delta_4-d}{2}\right)\Gamma\left(\frac{\Delta+\Delta_3-\Delta_4}{2}\right)\Gamma\left(\frac{\Delta+\Delta_4-\Delta_3}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\frac{\Delta+1}{2}\right)\Gamma\left(\frac{\Delta+\Delta_3+\Delta_4}{2}\right)} \\ &\times K^{\Delta_3}(x_1,\vec{y}_3)K^{\Delta_4}(x_1,\vec{y}_4)\, {}_2F_1\left(\begin{matrix}\Delta_3,\ \Delta_4 \\ \frac{\Delta+\Delta_3+\Delta_4}{2}\end{matrix}; 1 - K(x_1,\vec{y}_3)K(x_1,\vec{y}_4)|\vec{y}_{34}|^2\right)\,. \end{aligned} \tag{E.16}$$

These vertices suffice to perform the study of Chapter 6, however at higher-loop order, for higher-point functions or for other type of interactions, generalizations of the master integral (E.2) are required. Useful extensions include, for instance, replacing its second factor in the denominator by $[z^2 + w'^2 + (\vec{x} - \vec{y}_2)^2]^c$, or adding a third factor in the denominator of the form $[z^2 + (\vec{x} - \vec{y}_3)^2]^e$. These possibilities have not been explored, and it would be interesting to do so in future work.

Appendix F

Φ^4 in six flat dimensions

In Chapter 6 we found that four-point functions in ϕ^4 theory in AdS can be renormalized by renormalizing the parameters in the action up to $d = 6$. While this result up to $d = 4$ is expected, the $d = 6$ case is surprising. The UV structure of the theory should be the same as that of the same theory in flat space, and while ϕ^4 theory in flat space is renormalizable up to $d = 4$, it is not renormalizable when $d > 4$. The purpose of this appendix is to show that 4-point one-loop scattering amplitudes in $d > 4$ are indeed renormalizable with only renormalizing the parameters of the ϕ^4 theory in any odd dimension and in $d = 6$. The result for odd dimensions follows trivially from the fact that we may regulate the theory with dimensional regularization and with this regulator there are no divergences at one-loop order, thus there is no need to renormalize. In the following we will focus in the case of even d.

Let us consider the action of a ϕ^4 theory in a flat d dimensions in euclidean signature.

$$S = \int d^d x \left[\frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{4!}(\mu^2)^{2-\frac{d}{2}}\Phi^4\right] \tag{F.1}$$

where λ is dimensionless and μ has dimensions of mass.

There are three diagrams that contribute to the 4-point function at 1-loop, see Fig. F.1. Each diagram gives an identical Feynman integral,

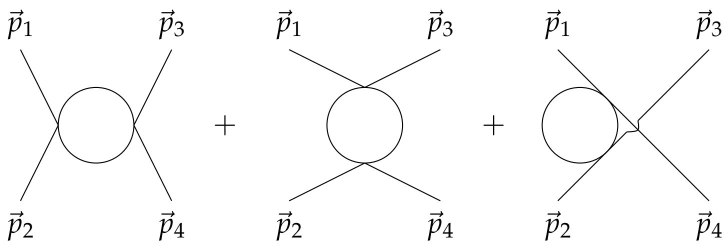

FIGURE F.1: One-loop 4-point functions of Φ^4 theory.

$$I = \frac{1}{2}\lambda^2(\mu^2)^{4-d}\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+m^2)((\vec{l}-\vec{p})^2+m^2)} \tag{F.2}$$

where $\vec{p}$ stands for either $\vec{p}_1+\vec{p}_2$, $\vec{p}_1+\vec{p}_3$ or $\vec{p}_1+\vec{p}_4$, depending on the diagram. $\vec{p}_1, \vec{p}_2, \vec{p}_3, \vec{p}_4$ are the external momenta (which we take them all as incoming) and $\vec{l}$ is the momentum flowing in the loop. After standard manipulations (see, for example, [93] for a pedagogical account of this computation) we arrive at

$$I = \frac{\lambda^2}{2}(\mu^2)^{4-d}\int_0^1 dx\left[\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\frac{1}{[m^2+p^2x(1-x)]^{2-\frac{d}{2}}}\right] \tag{F.3}$$

where x is a Feynman parameter. The tree-level diagram has the dimension of $(\mu^2)^{2-\frac{d}{2}}$, then we will keep this term unchanged. So, we have

$$I = \lambda(\mu^2)^{2-\frac{d}{2}}\left\{\frac{\lambda}{2}(\mu^2)^{2-\frac{d}{2}}\int_0^1 dx\left[\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\frac{1}{[m^2+p^2x(1-x)]^{2-\frac{d}{2}}}\right]\right\} \tag{F.4}$$

In d even there is a pole in the Gamma function, which we regulate using dimensional regularization. In $d=6-2\epsilon$ and expanding in $\epsilon\to 0$ we find

$$\begin{aligned} I = \lambda(\mu^2)^{\epsilon-1}\Bigg(\frac{\lambda}{128\pi^3\mu^2}\Bigg(&-\frac{1}{\epsilon}\int_0^1 dx(m^2+p^2x(1-x)) \\ &+(\gamma_E-1)\int_0^1 dx(m^2+p^2x(1-x)) \\ &+\int_0^1 dx(m^2+p^2x(1-x))\log\left(\frac{(m^2+p^2x(1-x))}{4\pi\mu^2}\right)\Bigg)\Bigg) \end{aligned} \tag{F.5}$$

The integrals are given by

$$\int_0^1 dx(m^2+p^2x(1-x)) = m^2+\frac{p^2}{6} \tag{F.6}$$

$$\begin{aligned} A(m^2,p^2,\mu^2) &= \int_0^1 dx(m^2+p^2x(1-x))\log\left(\frac{(m^2+p^2x(1-x))}{4\pi\mu^2}\right) \\ &= 4\pi\mu^2\frac{d}{da}\left[\int_0^1 dx\left(\frac{m^2+p^2x(1-x)}{4\pi\mu^2}\right)^a\right]\Bigg|_{a=1} \end{aligned} \tag{F.7}$$

Then,

$$I = \lambda(\mu^2)^{\epsilon-1}\frac{\lambda}{128\pi^3\mu^2}\left(-\frac{1}{\epsilon}\left(m^2+\frac{p^2}{6}\right)+(\gamma_E-1)\left(m^2+\frac{p^2}{6}\right)+A(m^2,p^2,\mu^2)\right) \tag{F.8}$$

Thus, (as one may have expected based on power-counting) there is a p^2 divergence. which suggests that we would need a counterterm of the form $\Phi^2\partial^\mu\Phi\partial_\mu\Phi$ to remove this divergence, making the theory non-renormalizable already at this order.

However, upon adding all three channels we obtain,

$$\tilde{\Gamma}^{(4)}(p_1,\ldots,p_4) = -\mu^{2\epsilon-2}\lambda\left[1-\frac{\lambda}{128\pi^3\mu^2}\left(-\frac{1}{\epsilon}\left(3m^2+\frac{s+t+u}{6}\right)\right.\right.$$
$$\left.\left.+(\gamma_E-1)\left(3m^2+\frac{s+t+u}{6}\right)+A(m^2,s,\mu^2)+A(m^2,t,\mu^2)+A(m^2,u,\mu^2)\right)\right] \quad \text{(F.9)}$$

where

$$s=(\vec{p}_1+\vec{p}_2)^2, \quad t=(\vec{p}_1+\vec{p}_3)^2, \quad u=(\vec{p}_1+\vec{p}_4)^2 \quad \text{(F.10)}$$

In scattering amplitudes external momenta are on-shell, $p_i^2=-m^2$, where $p_i=\sqrt{\vec{p}_i^{\,2}}$, thus

$$s=(\vec{p}_1+\vec{p}_2)^2=p_1^2+p_2^2+2\vec{p}_1\cdot\vec{p}_2=-2m^2+2\vec{p}_1\cdot\vec{p}_2\,. \quad \text{(F.11)}$$

A similar relation holds for t and s, and we find the (well-known) relation

$$s+t+u=-4m^2 \quad \text{(F.12)}$$

where we used momentum conservation, $\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4=0$. Thus,

$$\tilde{\Gamma}^{(4)}(p_1,\ldots,p_4) = -\,\mu^{2\epsilon-2}\lambda\left[1-\frac{\lambda}{128\pi^3\mu^2}\left(\frac{7}{3\epsilon}m^2+(\gamma_E-1)\left(2m^2+\frac{p_1(p_2+p_3+p_4)}{3}\right)\right.\right.$$
$$\left.\left.+A(m^2,s,\mu^2)+A(m^2,t,\mu^2)+A(m^2,u,\mu^2)\right)\right] \quad \text{(F.13)}$$

and the divergence is now independent of the momenta and may be renormalized by renormalizing the parameters of the original Lagrangian.

A different way to understand this result is to note that the counterterm $L_{\text{ct}}=\Phi^2\partial^\mu\Phi\partial_\mu\Phi$ may be removed by a field redefinition. Indeed, up to a total derivative,

$$L_{\text{ct}}=-\frac{1}{3}\Phi^3\Box\Phi \quad \text{(F.14)}$$

and the redefinition

$$\Phi=\Phi'-\frac{1}{3}\Phi'^3 \quad \text{(F.15)}$$

removes L_{ct} (while modifying the coefficient of the Φ^4 term and adding higher order terms in Φ). Thus, since scattering amplitudes are invariant under field redefinitions the one-loop 4-point scattering amplitude may be renormalized without the need of this counterterm.

Note that this conclusion is special to the one-loop 4-point scattering amplitudes. The ϕ^4 theory in $d=6$ is non-renormalizable – there are other scattering amplitudes that do require adding new counterterms to the action. If we consider $d=8$ (and higher) then even the one-loop 4-point scattering amplitude would require require adding new counterterms to the action.

References

[1] Ofer Aharony, Luis F. Alday, Agnese Bissi, and Eric Perlmutter. Loops in AdS from Conformal Field Theory. *JHEP*, 07:036, 2017.

[2] Soner Albayrak, Chandramouli Chowdhury, and Savan Kharel. New relation for Witten diagrams. *JHEP*, 10:274, 2019.

[3] Soner Albayrak, Chandramouli Chowdhury, and Savan Kharel. Study of momentum space scalar amplitudes in AdS spacetime. *Phys. Rev. D*, 101(12):124043, 2020.

[4] Soner Albayrak and Savan Kharel. Towards the higher point holographic momentum space amplitudes. *JHEP*, 02:040, 2019.

[5] Soner Albayrak and Savan Kharel. Spinning loop amplitudes in anti–de Sitter space. *Phys. Rev. D*, 103(2):026004, 2021.

[6] Luis F. Alday and Agnese Bissi. Loop Corrections to Supergravity on $AdS_5 \times S^5$. *Phys. Rev. Lett.*, 119(17):171601, 2017.

[7] Luis F. Alday, Agnese Bissi, and Eric Perlmutter. Holographic Reconstruction of AdS Exchanges from Crossing Symmetry. *JHEP*, 08:147, 2017.

[8] Luis F. Alday, Agnese Bissi, and Xinan Zhou. One-loop gluon amplitudes in AdS. *JHEP*, 02:105, 2022.

[9] Luis F. Alday and Simon Caron-Huot. Gravitational S-matrix from CFT dispersion relations. *JHEP*, 12:017, 2018.

[10] Luis F. Alday, Vasco Gonçalves, and Xinan Zhou. Supersymmetric Five-Point Gluon Amplitudes in AdS Space. *Phys. Rev. Lett.*, 128(16):161601, 2022.

[11] Luis F. Alday and Xinan Zhou. Simplicity of AdS Supergravity at One Loop. *JHEP*, 09:008, 2020.

[12] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul. Loop corrections for Kaluza-Klein AdS amplitudes. *JHEP*, 05:056, 2018.

[13] F. Aprile, J. M. Drummond, P. Heslop, and H. Paul. Quantum Gravity from Conformal Field Theory. *JHEP*, 01:035, 2018.

[14] G. Arutyunov and S. Frolov. Four point functions of lowest weight CPOs in N=4 SYM(4) in supergravity approximation. *Phys. Rev. D*, 62:064016, 2000. arXiv hep-th/0002170.

[15] Máximo Bañados, Ernesto Bianchi, Iván Muñoz, and Kostas Skenderis. Bulk renormalization and the AdS/CFT correspondence. *Phys. Rev. D*, 107(2):L021901, 2023.

[16] Maximo Bañados, Ernesto Bianchi, Ivan Muñoz, and Kostas Skenderis. AdS amplitudes as CFT correlators. *JHEP*, 01:101, 2025.

[17] H. Bateman and A. Erdélyi. *Higher Transcendental Functions, Vol. 1*. McGraw-Hill, New York, 1953.

[18] H. Bateman and A. Erdélyi. *Tables of Integral Transforms, Vol. 2*. McGraw-Hill, New York, 1954.

[19] Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev, and Charlotte Sleight. Quartic AdS Interactions in Higher-Spin Gravity from Conformal Field Theory. *JHEP*, 11:149, 2015.

[20] Xavier Bekaert, Johanna Erdmenger, Dmitry Ponomarev, and Charlotte Sleight. Towards holographic higher-spin interactions: Four-point functions and higher-spin exchange. *JHEP*, 03:170, 2015.

[21] Igor Bertan and Ivo Sachs. Loops in Anti–de Sitter Space. *Phys. Rev. Lett.*, 121(10):101601, 2018.

[22] Igor Bertan, Ivo Sachs, and Evgeny D. Skvortsov. Quantum ϕ^4 Theory in AdS_4 and its CFT Dual. *JHEP*, 02:099, 2019.

[23] Ernesto Bianchi, Ivan Muñoz, and Kostas Skenderis. Bulk renormalization and AdS/CFT at loop order. (to appear), 2025.

[24] Massimo Bianchi, Daniel Z. Freedman, and Kostas Skenderis. How to go with an RG flow. *JHEP*, 08:041, 2001.

[25] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Implications of conformal invariance in momentum space. *Journal of High Energy Physics*, 2014(3), mar 2014.

[26] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies. *JHEP*, 03:066, 2016.

[27] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Renormalised 3-point functions of stress tensors and conserved currents in CFT. *Journal of High Energy Physics*, 2018(11), nov 2018.

[28] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Renormalised CFT 3-point functions of scalars, currents and stress tensors. *Journal of High Energy Physics*, 2018(11), nov 2018.

[29] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Conformal n-point functions in momentum space. *Phys. Rev. Lett.*, 124(13):131602, 2020.

[30] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Conformal correlators as simplex integrals in momentum space. *JHEP*, 01:192, 2021.

[31] Adam Bzowski, Paul McFadden, and Kostas Skenderis. Renormalisation of IR divergences and holography in de Sitter. *JHEP*, 05:053, 2024. arXiv 2312.17316.

[32] Carlos Cardona. Mellin-(Schwinger) representation of One-loop Witten diagrams in AdS. 8 2017.

[33] Dean Carmi. Loops in AdS: From the Spectral Representation to Position Space. *JHEP*, 06:049, 2020.

[34] Dean Carmi. Loops in AdS: from the spectral representation to position space. Part II. *JHEP*, 07:186, 2021.

[35] Dean Carmi. Loops in AdS: from the spectral representation to position space. Part III. *JHEP*, 08:193, 2024.

[36] Simon Caron-Huot and Anh-Khoi Trinh. All tree-level correlators in $AdS_5 \times S_5$ supergravity: hidden ten-dimensional conformal symmetry. *JHEP*, 01:196, 2019.

[37] Heng-Yu Chen, En-Jui Kuo, and Hideki Kyono. Towards Spinning Mellin Amplitudes. *Nucl. Phys. B*, 931:291–323, 2018.

[38] Chandramouli Chowdhury, Pratyusha Chowdhury, Radu N. Moga, and Kajal Singh. Loops, recursions, and soft limits for fermionic correlators in (A)dS. *JHEP*, 10:202, 2024.

[39] Chandramouli Chowdhury and Kajal Singh. Analytic results for loop-level momentum space Witten diagrams. *JHEP*, 12:109, 2023.

[40] Jinwei Chu and Savan Kharel. Mellin amplitude for n-gluon scattering in anti–de Sitter spacetime. *Phys. Rev. D*, 109(10):L101901, 2024.

[41] Miguel S. Costa, Vasco Gonçalves, and João Penedones. Spinning AdS Propagators. *JHEP*, 09:064, 2014.

[42] Miguel S. Costa, Joao Penedones, David Poland, and Slava Rychkov. Spinning Conformal Correlators. *JHEP*, 11:071, 2011. arXiv 1107.3554.

[43] Alexandria Costantino and Sylvain Fichet. Opacity from Loops in AdS. *JHEP*, 02:089, 2021.

[44] Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis. Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence. *Commun. Math. Phys.*, 217:595–622, 2001.

[45] E. D'Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli. Graviton exchange and complete four-point functions in the ads/cft correspondence. 562 (1-2), 353-394, 1999.

[46] E. D'Hoker, D. Z. Freedman, and L. Rastelli. Ads/cft four-point functions: How to succeed at z-integrals without really trying. 562 (1-2), 395-411, 1999.

[47] Eric D'Hoker and Daniel Z. Freedman. Gauge boson exchange in AdS(d+1). *Nucl. Phys. B*, 544:612–632, 1999.

[48] Eric D'Hoker and Daniel Z. Freedman. General scalar exchange in AdS(d+1). *Nucl. Phys. B*, 550:261–288, 1999.

[49] Eric D'Hoker and Daniel Z. Freedman. Supersymmetric gauge theories and the AdS / CFT correspondence. In *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2001): Strings, Branes and EXTRA Dimensions*, pages 3–158, 1 2002. arXiv hep-th/0201253.

[50] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal field theory*. Springer, N.Y., 1997.

[51] *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.2.1 of 2024-06-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[52] F. A. Dolan and H. Osborn. Conformal four point functions and the operator product expansion. *Nucl. Phys. B*, 599:459–496, 2001.

[53] F. A. Dolan and H. Osborn. Implications of N=1 superconformal symmetry for chiral fields. *Nucl. Phys. B*, 593:599–633, 2001.

[54] Francis A. Dolan, Laurent Gallot, and Emery Sokatchev. On four-point functions of 1/2-BPS operators in general dimensions. *JHEP*, 09:056, 2004. arXiv hep-th/0405180.

[55] G. Donovan, J. Geronimo, and D. Hardin. Orthogonal polynomials and the construction of piecewise polynomial smooth wavelets. *Siam Journal on Mathematical Analysis - SIAM J MATH ANAL*, 30, 06 1999.

[56] A. Liam Fitzpatrick and Jared Kaplan. Unitarity and the Holographic S-Matrix. *JHEP*, 10:032, 2012.

[57] A. Liam Fitzpatrick, Jared Kaplan, Joao Penedones, Suvrat Raju, and Balt C. van Rees. A Natural Language for AdS/CFT Correlators. *JHEP*, 11:095, 2011.

[58] Daniel Z. Freedman, Kenneth Johnson, and Jose I. Latorre. Differential regularization and renormalization: A New method of calculation in quantum field theory. *Nucl. Phys. B*, 371:353–414, 1992.

[59] Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Comments on 4 point functions in the CFT / AdS correspondence. *Phys. Lett. B*, 452:61–68, 1999. arXiv 9808006.

[60] Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Comments on 4-point functions in the CFT/AdS correspondence. *Physics Letters B*, 452(1-2):61–68, apr 1999.

[61] Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Correlation functions in the CFT(d) / AdS(d+1) correspondence. *Nucl. Phys. B*, 546:96–118, 1999. arXiv hep-th/9804058.

[62] Daniel Z. Freedman, Samir D. Mathur, Alec Matusis, and Leonardo Rastelli. Correlation functions in the CFT(d) / AdS(d+1) correspondence. *Nucl. Phys. B*, 546:96–118, 1999.

[63] R. Fukuda, M. Komachiya, and M. Ukita. On-shell Expansion of the Effective Action: S Matrix and the Ambiguity - Free Stability Criterion. *Phys. Rev. D*, 38:3747–3754, 1988.

[64] Kausik Ghosh. Polyakov-Mellin Bootstrap for AdS loops. *JHEP*, 02:006, 2020.

[65] Steven B. Giddings. The Boundary S matrix and the AdS to CFT dictionary. *Phys. Rev. Lett.*, 83:2707–2710, 1999.

[66] Paul H. Ginsparg. APPLIED CONFORMAL FIELD THEORY. In *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena*, 9 1988.

[67] Simone Giombi, Charlotte Sleight, and Massimo Taronna. Spinning AdS Loop Diagrams: Two Point Functions. *JHEP*, 06:030, 2018.

[68] Rajesh Gopakumar, Apratim Kaviraj, Kallol Sen, and Aninda Sinha. Conformal Bootstrap in Mellin Space. *Phys. Rev. Lett.*, 118(8):081601, 2017.

[69] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, New York, eighth edition, 2015. Edited by D. Zwillinger and V. Moll.

[70] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett. B*, 428:105–114, 1998.

[71] Till Heckelbacher, Ivo Sachs, Evgeny Skvortsov, and Pierre Vanhove. Analytical evaluation of AdS_4 Witten diagrams as flat space multi-loop Feynman integrals. *JHEP*, 08:052, 2022. arXiv 2201.09626.

[72] M. Henningson and K. Skenderis. The Holographic Weyl anomaly. *JHEP*, 07:023, 1998.

[73] Mans Henningson and Kostas Skenderis. Holography and the Weyl anomaly. *Fortsch. Phys.*, 48:125–128, 2000.

[74] Johan Henriksson and Tomasz Lukowski. Perturbative Four-Point Functions from the Analytic Conformal Bootstrap. *JHEP*, 02:123, 2018.

[75] M. Komachiya, M. Ukita, and R. Fukuda. On-shell expansion of the effective action. 2: Coherent state and S matrix. *Phys. Rev. D*, 42:2792–2805, 1990.

[76] Leonard Lewin. *Polylogarithms and Associated Functions*. North-Holland, New York, first edition, 1981.

[77] Yue-Zhou Li and Jiajie Mei. Bootstrapping Witten diagrams via differential representation in Mellin space. *JHEP*, 07:156, 2023.

[78] Hong Liu and Arkady A. Tseytlin. On four point functions in the CFT / AdS correspondence. *Phys. Rev. D*, 59:086002, 1999.

[79] Hongmei Liu and Weiping Wang. Transformation and summation formulae for kampé de fériet series. *Journal of Mathematical Analysis and Applications*, 409(1):100–110, 2014.

[80] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[81] David Meltzer, Eric Perlmutter, and Allic Sivaramakrishnan. Unitarity Methods in AdS/CFT. *JHEP*, 03:061, 2020.

[82] Dhritiman Nandan, Anastasia Volovich, and Congkao Wen. On Feynman Rules for Mellin Amplitudes in AdS/CFT. *JHEP*, 05:129, 2012.

[83] H. Osborn and A. C. Petkou. Implications of conformal invariance in field theories for general dimensions. *Annals Phys.*, 231:311–362, 1994. arXiv hep-th/9307010.

[84] Hugh Osborn. Lectures on conformal field theories, 2019.

[85] Ioannis Papadimitriou and Kostas Skenderis. AdS / CFT correspondence and geometry. *IRMA Lect. Math. Theor. Phys.*, 8:73–101, 2005.

[86] Miguel F. Paulos. Towards Feynman rules for Mellin amplitudes. *JHEP*, 10:074, 2011.

[87] Joao Penedones. Writing CFT correlation functions as AdS scattering amplitudes. *JHEP*, 03:025, 2011.

[88] Anastasios Petkou and Kostas Skenderis. A Nonrenormalization theorem for conformal anomalies. *Nucl. Phys. B*, 561:100–116, 1999.

[89] Alexander M. Polyakov. Conformal symmetry of critical fluctuations. *JETP Lett.*, 12:381–383, 1970.

[90] Dmitry Ponomarev. From bulk loops to boundary large-N expansion. *JHEP*, 01:154, 2020.

[91] Suvrat Raju. BCFW for Witten Diagrams. *Phys. Rev. Lett.*, 106:091601, 2011.

[92] Suvrat Raju. Recursion Relations for AdS/CFT Correlators. *Phys. Rev. D*, 83:126002, 2011.

[93] Pierre Ramond. *Field theory: a modern primer*, volume 51. 1981.

[94] Leonardo Rastelli, Konstantinos Roumpedakis, and Xinan Zhou. $\mathbf{AdS_3} \times \mathbf{S^3}$ Tree-Level Correlators: Hidden Six-Dimensional Conformal Symmetry. *JHEP*, 10:140, 2019.

[95] Leonardo Rastelli and Xinan Zhou. Mellin amplitudes for $AdS_5 \times S^5$. *Phys. Rev. Lett.*, 118(9):091602, 2017.

[96] Leonardo Rastelli and Xinan Zhou. How to Succeed at Holographic Correlators Without Really Trying. *JHEP*, 04:014, 2018.

[97] Slava Rychkov. EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions, 1 2016. arXiv 1601.05000.

[98] Kostas Skenderis. Lecture notes on holographic renormalization. *Class. Quant. Grav.*, 19:5849–5876, 2002.

[99] Charlotte Sleight and Massimo Taronna. Higher Spin Interactions from Conformal Field Theory: The Complete Cubic Couplings. *Phys. Rev. Lett.*, 116(18):181602, 2016.

[100] Leonard Susskind. The World as a hologram. *J. Math. Phys.*, 36:6377–6396, 1995.

[101] Leonard Susskind and Edward Witten. The Holographic bound in anti-de Sitter space. 5 1998.

[102] Gerard 't Hooft. Dimensional reduction in quantum gravity. *Conf. Proc. C*, 930308:284–296, 1993.

[103] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.

[104] Ellis Ye Yuan. Loops in the Bulk. 10 2017.

[105] Ellis Ye Yuan. Simplicity in AdS Perturbative Dynamics, 1 2018. arXiv 1801.07283.