Toward null-state equations in d > 2

Kuo-Wei Huang

Mathematical Sciences, University of Southampton,
Highfield, Southampton SO17 1BJ, U.K.

E-mail: K-W.Huang@soton.ac.uk

ABSTRACT: In two-dimensional CFTs with a large central charge, the level-two BPZ equation governs the heavy-light scalar four-point correlator when the light probe scalar has dimension $h = -\frac{1}{2}$; the corresponding linear ordinary differential equation can be recast into a schematic form $x^2 u_{xx} + u = 0$. In this paper, we make an observation that in a class of four-dimensional CFTs with a large central charge, the heavy-light scalar correlator in the near-lightcone limit obeys a similar equation, $x^3 u_{xxxy} + u = 0$, when the light scalar has dimension $\Delta = -1$. We focus on the multi-stress tensor sector of the theory and also discuss the corresponding equations for the cases with $\Delta = -2, -3$. The solutions to these linear partial differential equations in higher dimensions are shown, after a suitable change of variables, to reproduce the near-lightcone correlators previously obtained via holography and the conformal bootstrap.

Contents

1   Introduction and summary

It is safe to say that our understanding of generic Quantum Field Theory (QFT) is still in its infancy. In particular, computing physical observables at strong coupling presents a primary difficulty. As a special class of QFTs that describe the critical phenomena at second-order phase transitions, Conformal Field Theory (CFT) has the convergent expansions in conformal blocks, allowing rigorous computations. The extra symmetries in CFT lead to non-trivial constraints, sometimes powerful enough to solve the theory. Almost forty years ago, Belavin, Polyakov, and Zamolodchikov (BPZ) discovered that the $d = 2$ conformal correlators in Virasoro minimal models obey a set of null-state linear differential equations [1], beyond the usual Ward identities. As a consequence of the enhanced infinite-dimensional Virasoro symmetry, these linear differential equations allow us to compute the correlators without using perturbative Lagrangian techniques. Although the landscape of $d = 2$ CFTs remains largely uncharted, the present paper aims to search for BPZ-type equations in higher dimensional CFTs at large central charge (large N), focusing on a scalar four-point correlator in the *near-lightcone* regime.

Certainly, in $d > 2$ the conformal group is finite dimensional and the factorization between a holomorphic and an anti-holomorphic part is a special feature in $d = 2$. The Virasoro-algebra related techniques cannot be extended to higher dimensions in general. There is no *a priori* reason either to expect that large-N CFT correlators in a kinematical (near-lightcone) limit would satisfy any BPZ-type equations in $d > 2$, unless an effective symmetry emerges in a system with large degrees of freedom or an effective action near the lightcone can be used. Very often when a question is difficult to address due to a lack of clues, trying to identify the final answer directly may help reconstruct an underlying reason afterward — this is the approach we adopt in the present work.

We will discuss a set of linear differential equations in $d = 4$, inspired by crossing symmetry and similarity with the level-2 BPZ equation, and verify their agreement with

results obtained using other techniques. Furthermore, we will demonstrate the effectiveness of these new equations by presenting results that were previously unattainable. Our approach is not entirely satisfactory due to the lack of a first-principle derivation for these equations.[1] There is, therefore, no fundamentally new organizing principle that will be established in this paper. However, it is always interesting to try to infer the final answer via pattern recognition, and to observe that simple equations govern intricate recursive structure. As a part of our motivation, we hope that these $d = 4$ differential equations we observe may serve as an impetus to develop a rigorous framework in the future.

Studying four-point conformal correlators in the near-lightcone limit is a very useful starting point toward a better understanding of higher dimensional CFTs more generally. In such an effective limit, a vast simplification occurs, yet preserving the richness and intricacy of CFT dynamics. One can analytically solve the corresponding crossing equations, i.e., the lightcone bootstrap equations, and crossing symmetry requires the existence of large spin double-twist operators [2, 3]. See [4] for an earlier discussion and a recent work [5] for a rigorous treatment. Recall that a local operator can be characterized by its twist $\tau = \Delta - J$, where Δ, J are the dimension and spin. The near-lightcone correlator encodes information about the leading-twist sector of the theory. In this work, we are interested in the primaries formed by the product of n stress tensors, denoted schematically as T^n, with the lowest twist $\tau_{\min} = n(d-2)$. These leading-twist multi-stress tensors are built from stress tensors and possible derivatives with no contracted Lorentz indices, ensuring the twist to be minimal. In two dimensions, the Virasoro algebra determines the resummation of leading-twist T^n operators which is given by the Virasoro vacuum block, whose structure and various applications have been extensively studied in the literature, e.g., [6–52].

In this paper, we restrict our attention to a class of $d = 4$ CFTs at large central charge C_T with a large gap in the spectrum of higher-spin single-trace operators. These conditions are required for a CFT to have a local gravitational dual description in Anti-de Sitter space (AdS), as conjectured in [53]. See, i.e., [54–65] for related discussions. At large C_T, a natural observable is the "heavy-light" four-point correlator, $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$. The heavy scalars have dimensions Δ_H of order C_T which, importantly, compensate for the large C_T suppression. In such a "semiclassical" limit, the Δ_H dependence enters the correlator only through the fixed ratio $\mu \sim \frac{\Delta_H}{C_T}$ as $C_T \to \infty$. The light probe scalars, on the other hand, have dimensions $\Delta_L \equiv \Delta$ of order one. Focusing on the contributions from multi-stress tensors, the $d = 4$ correlator in the near-lightcone limit $\bar{z} \to 0$ can be represented as

$$F(z, \bar{z}) \equiv \lim_{\bar{z} \to 0} \frac{\langle \mathcal{O}_H(\infty) \mathcal{O}_L(z, \bar{z}) \mathcal{O}_L(0) \mathcal{O}_H(1) \rangle}{\langle \mathcal{O}_H(\infty) \mathcal{O}_H(1) \rangle \langle \mathcal{O}_L(z, \bar{z}) \mathcal{O}_L(0) \rangle} \bigg|_{T^n} = \sum_{n=0}^{\infty} G_n(\Delta, z)(\mu \bar{z})^n \,. \tag{1.1}$$

The functions $G_n(\Delta, z)$ are generally challenging to compute for $n \geq 2$. The power of the parameter μ counts the number of exchanged stress-tensor operators. When $n = 0$, we have $G_0 = 1$ for the vacuum correlator. The $n = 1$ case corresponds to the single-stress

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[1]By a first-principle approach, we are referring to a CFT derivation that begins with TT and $T\mathcal{O}$ operator product expansions (OPEs).

tensor exchange which is universally fixed by the Ward identity. Note that the parameter μ can be absorbed into $\bar{z}$ in the near-lightcone expansion (1.1). We will scale $\bar{z}$ so that the $d = 4$ expressions do not show an explicit μ. In $d = 2$, μ is the expansion parameter since the holomorphic function $F(z)$ has no $\bar{z}$ dependence.

Using the AdS/CFT correspondence [66–68], one can compute the heavy-light four-point correlator at large C_T as a thermal two-point scalar correlator in a black-hole background created by the heavy operators. The Hawking temperature of a black hole can be related to the CFT parameter μ. The multi-stress tensor exchanges are dual to multi-graviton exchanges in AdS. In the context of $\text{AdS}_3/\text{CFT}_2$, the bulk-to-boundary propagator with the Banados-Teitelboim-Zanelli (BTZ) black hole [69] has a closed-form expression [70], allowing for various analytic computations. Moreover, due to the Virasoro symmetry in the dual $d = 2$ CFTs, it is anticipated that higher-curvature corrections to the AdS_3 gravitational action do not affect the boundary correlators. In higher dimensions, the correlators depend on the specific theory. It is, however, interesting to investigate whether there is a subsector of the theory that exhibits a more robust structure. In [71], it was observed that the leading-twist sector of multi-stress tensors in the dual $d > 2$ CFTs is insensitive to higher-curvature corrections. This observation was made by considering a minimally coupled probe scalar in a black-hole background in $\text{AdS}_{D>3}$. The near-lightcone correlators at large C_T take the same form as that computed in pure Einstein gravity. This statement goes beyond the geodesic approximation.[2] Furthermore, it was observed that certain simplifications of the correlator occur at special negative integers of Δ [71]

$$\Delta = -1, -2, \cdots, -\frac{d+2}{2}, \qquad d = 4, 6, 8, \cdots \tag{1.2}$$

where closed-form expressions summing over *all* multi-stress tensor exchanges can be found (based on a computation with a planar black hole). The radius of convergence becomes infinite at these special values of Δ. The motivation of the present work stems from these curious simplifications in higher-dimensional CFTs.[3] For concreteness we consider $d = 4$, where the corresponding special values are $\Delta = -1, -2, -3$.

We have excluded the $d = 2$ case from (1.2). The reason is as follows: it appears that the corresponding computation in AdS_3 is somewhat "too simple" to detect any special conformal dimensions, and the closed form of the Virasoro vacuum block at large c with a general conformal dimension can be obtained directly. In the $d = 2$ case, where we denote $\Delta \equiv 2h_L$, we assume that the corresponding special value is $h_L = -\frac{1}{2}$. This value, as we will review, can be derived using the Virasoro algebra.

By adding non-minimally coupled bulk interactions, some explicit corrections to the leading-twist OPEs in $d = 4$ were found in [73], showing that the near-lightcone correlator

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[2]In addition to the T^n operators, there are contributions from the double-trace operators made from two light scalars and derivatives. Their contributions depend on an interior boundary condition [71]. In this work we focus on the contributions from multi-stress tensors.

[3]For a connection to gravitational shockwaves, see [72], where a further analysis based on a spherical black hole can be found. Note that a planar black hole limit suppresses derivatives inserted into stress tensors, resulting in just one leading-twist T^n primary at each n. The differential equations discussed in this work, as we will verify, apply to the holographic computation with a spherical black hole where an *infinite* number of leading-twist T^n primaries contribute at each $n \geq 2$.

at large C_T can depend on the higher-spin gap scale. In this work, we observe that the corrections due to the quartic derivative, non-minimally coupled interactions between the graviton and the light bulk scalar computed in [73] vanish precisely at the values $\Delta = -1, -2, -3$. These are the required values in the $d = 4$ linear differential equations we will discuss below. It is thus tempting to speculate that the differential equations considered in this work might apply to a broader class of conformal theories.

The resummations of the leading-twist double- and triple-stress tensors were obtained in [74–76]. These results correspond to a holographic computation utilizing a spherical black hole. It was shown that the heavy-light correlator in $d = 4$ (and similarly in other even dimensions) has a general structure

$$F(z, \bar{z}) = 1 + \sum_{n=1}^{\infty} \left[\sum_{i_p=1} A_{i_1 \cdots i_n}(\Delta) \, f_{i_1}(z) \cdots f_{i_n}(z) \right] (\mu \bar{z})^n, \quad \sum_{p=1}^{n} i_p = 3n \qquad (1.3)$$

where $p = 1, 2, \cdots, n$, and $f_a(z) \equiv z^a \, _2F_1(a, a, 2a, z)$ involves the Gaussian hypergeometric function.[4] The coefficients $A_{i_1 \cdots i_n}(\Delta)$ depend on the number of exchanged stress tensors. The indices $i_p \geq 1$ are integers subjected to the constraint above. Unrestricted to the special values (1.2), the expression for the $n = 2$ case was found in [74] based on the multi-stress tensor OPEs computed from holography. Subsequently, both $n = 2$ and $n = 3$ expressions were computed in [75, 76] by employing the lightcone bootstrap, treating (1.3) as an ansatz. Certain similarity between such a $d = 4$ structure and the $d = 2$ $\mathcal{W}_3$ vacuum blocks was discussed in [77, 78]. On the other hand, the Lorentzian inversion formula [79, 80] also allows one to compute the OPE coefficients of multi-stress tensor exchanges at large C_T [81, 82]. To compute the OPEs, it appears that, starting from triple-stress tensor exchanges, the ansatz (1.3) combined with the inversion formula or crossing symmetry is more efficient than only using the inversion formula, as remarked in [76].[5] In this work, we are interested in an approach via BPZ-type linear differential equations for analyzing the near-lightcone correlators at large C_T in $d > 2$.

Let us comment on the required negative values of Δ. Recall that many remarkable results in $d = 2$ CFTs are consequences of shortening conditions of the conformal algebra at certain values of conformal dimensions and the central charge. At large central charge, the corresponding dimensions of the light scalars take *non-unitary* values, i.e., negative integers. This may appear to be a serious limitation, but, as emphasized in [24], since the conformal blocks are analytic functions of their defining quantum numbers, the Virasoro blocks with general conformal dimensions can be obtained by an analytical continuation of the correlators involving degenerate operators. We may take the same perspective in the $d = 4$ case. The focus of this paper, however, is to identify and analyze a set of differential

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[4]Our $(z, \bar{z})$ corresponds to $(1 - z, 1 - \bar{z})$ in [74–76].

[5]Further advancements have been made. For instance, a closed-form expression of the $d = 4$ heavy-light correlator at large Δ was obtained in [83] based on a holographic computation with a planar black hole. The thermalization of multi-stress tensor operators was discussed in [84]. Another recent study [85] adopted a formal connection to the Nekrasov-Shatashvili partition function [86] of a supersymmetric gauge theory to compute the $d = 4$ thermal scalar two-point correlator through holography.

equations in $d = 4$ and we will not delve into an additional investigation of the analytical continuation.

In contrast to the $d = 2$ case where the large-c BPZ equations are ordinary differential equations, the $d = 4$ equations discussed below are *partial* differential equations. This is not surprising, given that the correlator in the near-lightcone regime depends on the coordinates z and $\bar{z}$ in a conformal frame. While solving the $d = 4$ equations is more challenging than the $d = 2$ case, we find them straightforward to solve in a small $\bar{z}$ expansion with appropriate boundary conditions. The resulting solutions, valid for $\Delta = -1, -2, -3$, are consistent with those previously derived via holography and the lightcone bootstrap. These involve expressions that resum *infinite* numbers of leading-twist multi-stress tensors. These new differential equations also enable us to make predictions about the near-lightcone correlator.

We now provide a summary of the main equations.

Two dimensions. To compare to the $d = 4$ case, it is useful to first present the corresponding differential equation in $d = 2$. The level-2 BPZ null-state equation [1], when applied to the heavy-light correlator at large central charge, can be expressed as

$$Q''(z) + 6\eta \frac{Q(z)}{(1-z)^2} = 0, \qquad Q(z) \equiv zF(z) \tag{1.4}$$

which is valid at $h_L = -\frac{1}{2} + \mathcal{O}\left(\frac{1}{c}\right)$. In this context, the function F corresponds to the definition in (1.1), but the $\bar{z}$ dependence is neglected as the focus here is on the holomorphic part of the correlator. Instead of using the notations μ and C_T, we adopt $\eta = \frac{h_H}{c}$ in the $d = 2$ case where h_H represents the weight of the heavy scalars. The central charge c can be defined through the stress-tensor two-point function: $\langle T(z)T(0)\rangle = \frac{c}{2z^4}$. The function $Q(z)$ in the $d = 2$ case satisfies a slightly more concise equation compared to the equation of $F(z)$. This type of simplification will be particularly useful in identifying and representing the linear differential equations in $d = 4$. In the $d = 2$ case, the factor z in the relation $Q = zF$ corresponds to the scalar two-point function, $z^{-2h_L} = z$. The relation between functions Q and F will vary slightly in $d = 4$.

It is straightforward to solve the equation (1.4). The solution

$$F(z) = \frac{1}{z} \frac{(1-z)^{\frac{1}{2}-\frac{1}{2}\sqrt{1-24\eta}}\left(1 - (1-z)^{\sqrt{1-24\eta}}\right)}{\sqrt{1-24\eta}} \tag{1.5}$$

corresponds to the large-c heavy-light Virasoro vacuum block when $h_L = -\frac{1}{2}$ [10, 14]. One may extract specific contributions from exchanged T^n operators via the expansion $F(z) = \sum_{n=0} G_n(z)\eta^n$. (Incorporating an $\mathcal{O}(\frac{1}{c})$ correction requires the level-3 BPZ equation. See a related discussion in [24].)

Four dimensions. To take the story further, we hypothesize that at the special values (1.2), certain $d = 4$ generalized primaries at large C_T have null descendants, thereby necessitating their correlators to obey linear differential equations resembling (1.4).

As mentioned, the parameter μ can be absorbed into $\bar{z}$ in the near-lightcone expansion. We perform such a rescaling to remove the explicit dependence on μ in the following $d = 4$ expressions.

We introduce

$$Q_{\Delta^*}(z, \bar{z}) \equiv \frac{z}{(z\bar{z})^\Delta} F(z, \bar{z})\big|_{\Delta = \Delta^*}, \qquad \Delta^* = -1, -2, -3, \tag{1.6}$$

where $F(z, \bar{z})$ is defined in (1.1). We propose the following linear differential equations to be the $d = 4$ generalizations to the BPZ equation (1.4) at large central charge:

$$Q_{-1}^{(3,1)} + \frac{Q_{-1}}{(1 - z)^3} = 0, \tag{1.7}$$

$$Q_{-2}^{(4,1)} + 6\frac{Q_{-2}^{(1,0)}}{(1 - z)^3} + 9\frac{Q_{-2}}{(1 - z)^4} = 0, \tag{1.8}$$

$$Q_{-3}^{(5,1)} + 21\frac{Q_{-3}^{(2,0)}}{(1 - z)^3} + 63\frac{Q_{-3}^{(1,0)}}{(1 - z)^4} + 72\frac{Q_{-3}}{(1 - z)^5} = 0 \tag{1.9}$$

where $Q_{\Delta^*}^{(m,n)} = \partial_z^m \partial_{\bar{z}}^n Q_{\Delta^*}(z, \bar{z})$.

Some preliminary remarks are in order:

1. Crossing symmetry, due to the presence of two identical light scalars, imposes a strong constraint on the possible forms of the equations. By translating these $d = 4$ equations to the ones using the original four-point function $F(z, \bar{z})$, the resulting equations transform covariantly under $z \to \frac{z}{z-1}$, $\bar{z} \to \frac{\bar{z}}{\bar{z}-1} \approx -\bar{z}$, where a small $\bar{z}$ is assumed. Note that, however, crossing symmetry along is insufficient to fully determine the equations.

2. The equation (1.7) is formulated by drawing upon the BPZ equation in the form of the Schrödinger equation (1.4), but instead of a second-order derivative we consider a higher-derivative structure. Assuming the equation takes the form $Q^{(\alpha,\beta)} + \kappa Q/(1 - z)^\gamma = 0$, the consistency with the $d = 4$ universal single-stress tensor exchange in the near-lightcone expansion fixes α, β, γ and κ (note that the relative coefficient, κ, depends on the normalization of the single-stress tensor exchange which has a canonical convention). In fact, with the $Q^{(3,1)}$ structure, crossing symmetry fixes the power of $(1 - z)$. The resemblance between the BPZ equation (1.4) and the $d = 4$ equation (1.7), as we will demonstrate below, leads to analogous recursive integral formulas in both $d = 2$ and $d = 4$, which provide a systematic and efficient method for computing the correlators.

3. The equation form (1.8) is obtained by applying one additional z-derivative to (1.7), with different constant coefficients. Likewise, the form (1.9) is derived by applying two more z-derivatives to (1.7). Treating the universal single-stress tensor exchange as an input enables completely fixing the constant coefficients — the unexpected bonus is that the higher-order solutions in the near-lightcone expansion, which correspond

to the exchanges of multi-stress tensors, automatically reproduce the previously computed results via holography and bootstrap. We find that this "derivation" of the equations does *not* apply to other values of Δ, in agreement with the assumed conditions of (1.2).[6] We emphasize that these equations are not derived by fitting with the known results of *multi*-stress tensor contributions (although in a first-principle approach the single-stress tensor exchange contribution should also be an output). As we get more out of these equations than was originally put into them, in this paper we explore their consequences. Further examination of these equations is certainly warranted.

4. An interesting property of these $d = 4$ "Q-equations" (1.7)–(1.9), which are linear partial differential equations, is that the coefficients do not depend on $\bar{z}$. Moreover, the derivative of $\bar{z}$ appears with an order no greater than one. Expressing these equations in terms of $F(z, \bar{z})$ results in longer equations with coefficients that explicitly depend on $\bar{z}$.[7] To our knowledge, there is no literature linking this type of partial differential equations to any known physical phenomenon.

5. The BPZ equations can be derived using the Virasoro algebra. We do not have a similar derivation of these $d = 4$ equations. First-principle derivations, based on a symmetry or not, of these equations are left for the future. We hope that this work may stimulate one to think of possible symmetry enhancements in $d > 2$ CFTs.

This article is organized as follows. Section 2 provides a brief review of the $d = 2$ case and presents a recursive integral formula for the heavy-light correlator. In section 3, we examine the $d = 4$ differential equations and verify that their solutions in the near-lightcone expansion are consistent with the results obtained via holography and the lightcone bootstrap. Corresponding integral formulas are also obtained. Using these formulas, we present some predictions of the $d = 4$ equations. We conclude with additional remarks in section 4.

2 $d = 2$ BPZ at large c

One can derive the null-state equation (1.4) starting with the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}. \qquad (2.1)$$

We are interested in Virasoro descendants with a zero norm. As is well-known, the simplest, non-trivial null state is the level-2 descendant, which can be represented as

$$|\chi = \left(L_{-1}^2 + b^2 L_{-2}\right)|h, \qquad c = 1 + 6\left(b + \frac{1}{b}\right)^2. \qquad (2.2)$$

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[6] A source of intuition comes from $d = 2$, where the large-c correlator with negative dimensions other than $h_L = -\frac{1}{2}$ satisfies higher-order BPZ equations. However, in the $d = 4$ case, there exist three special values of Δ in the *leading* large C_T limit, whereas in $d = 2$, there is only one value.

[7] In the abstract, we let $Q = u$, $z = 1 - x$, $\bar{z} = -y$ as a way to represent a $d = 4$ partial differential equation, and we drop the η dependence in the $d = 2$ equation as a schematic form.

The vanishing determinant of the level-2 Gram matrix requires the weight h to be

$$h = -\frac{1}{2} - \frac{3}{4b^2}\,. \tag{2.3}$$

A null state is orthogonal to any state in the theory. This property translates, using the mode operator L_{-m} together with the Ward identities, into the level-2 BPZ differential equation [1]:

$$\left(\partial_z^2 + \left(\frac{b^{-2}}{1-z} + \frac{2 + 2b^{-2}}{z}\right)\partial_z + \frac{h_H b^{-2}}{(1-z)^2}\right)F(z) = 0\,. \tag{2.4}$$

Here we focus on the heavy-light correlator $F(z) = \frac{\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_h(z)\mathcal{O}_h(0)\rangle}{\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\rangle\langle \mathcal{O}_h(z)\mathcal{O}_h(0)\rangle}$ with the dimension of the light scalars given by (2.3). By taking the dimensions of the heavy scalars to be of order c, $h_H \sim \mathcal{O}(c)$, the null-state equation at large c reduces to

$$\left(\partial_z^2 + \frac{2}{z}\partial_z\right)F(z) + \frac{6\eta}{(1-z)^2}F(z) = 0\,, \qquad \eta = \frac{h_H}{c}\,. \tag{2.5}$$

Note that $h \to -\frac{1}{2}$ at large c, where the parameter $b \to \infty$. This equation transforms covariantly under the exchange of the two identical light scalars, i.e., $z \to \frac{z}{z-1}$.[8] The Schrödinger-type equation (1.4) is obtained by using $Q(z) = zF(z)$.

The equation (2.5) has the following general solution:

$$F(z) = \frac{1}{z}\left(c_1 + c_2(1-z)^{\sqrt{1-24\eta}}\right)(1-z)^{\frac{1}{2} - \frac{1}{2}\sqrt{1-24\eta}}\,. \tag{2.6}$$

The coefficients c_1 and c_1 can be fixed by a small z expansion:

$$\lim_{z \to 0} F(z) = \frac{c_1 + c_2}{z} + \left(\frac{1}{2}(c_1 + c_2)\left(\sqrt{1-24\eta} - 1\right) - c_2\sqrt{1-24\eta}\right) + \mathcal{O}(z)\,. \tag{2.7}$$

The regularity at $z \to 0$ requires $c_1 = -c_2$. The remaining coefficient can be fixed by the normalization $F(0) = 1$, giving $c_2 = -\frac{1}{\sqrt{1-24\eta}}$. The result, presented in (1.5), is the large-c heavy-light Virasoro vacuum block at $h = -\frac{1}{2}$. The same consideration applies to the anti-holomorphic part of the correlator.

Although the closed-form solution to the $d = 2$ equation (2.5) can be obtained straightforwardly, performing a perturbative analysis in a μ-expansion will be useful for the discussions in $d = 4$. This will enable us to observe common patterns among the $d = 2$ and $d = 4$ perturbative solutions.

Consider a near-lightcone expansion

$$F(z) = 1 + \eta G_1(z) + \eta^2 G_2(z) + \eta^3 G_3(z) + \mathcal{O}(\eta^4) \tag{2.8}$$

which has incorporated the normalization $F(0) = G_0(0) = 1$. The BPZ equation in such an expansion exhibits a recursive pattern:

$$0 = \left(zG_1''(z) + 2G_1'(z) + \frac{6z}{(1-z)^2}\right)\eta \tag{2.9}$$

$$+ \left(zG_2''(z) + 2G_2'(z) + \frac{6zG_1(z)}{(1-z)^2}\right)\eta^2 + \left(zG_3''(z) + 2G_3'(z) + \frac{6zG_2(z)}{(1-z)^2}\right)\eta^3 + \mathcal{O}(\eta^4)\,.$$

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[8]We note that crossing symmetry fixes the ratio $1 : 2$ in the derivative part of the large-c BPZ equation (2.5), but is not enough to fully determine the equation.

The general solution of G_1 is

$$G_1 = \frac{1}{z}\Big(c_1 + c_2 z - 6(z - 2)\ln(1 - z)\Big) \tag{2.10}$$

where constants c_1 and c_2 can be fixed by the $z \to 0$ behavior. Requiring $G_1(0) = 0$ gives $c_1 = 0$, $c_2 = 12$, yielding the correct result for the single-stress tensor exchange near the lightcone

$$G_1 = -\frac{1}{z}\Big(6(z - 2)\ln(1 - z) - 12z\Big) = -f_2 \tag{2.11}$$

where $f_a \equiv f_a(z) = z^a \, {}_2F_1(a, a, 2a, z)$. Similarly, the higher-order solutions can be obtained:

$$\begin{aligned} G_2 &= \frac{18}{z}\Big(z\ln^2(1 - z) - 6(z - 2)\ln(1 - z) + 12z\Big) \\ &= -\frac{6}{5}f_1 f_3 + \frac{3}{2}f_2 f_2 \,, \end{aligned} \tag{2.12}$$

$$\begin{aligned} G_3 &= -\frac{36}{z}\Big((z - 2)\ln^3(1 - z) - 12z\ln^2(1 - z) + 60(z - 2)\ln(1 - z) - 120z\Big) \\ &= -\frac{54}{35}f_1 f_1 f_4 + 4f_1 f_2 f_3 - \frac{5}{2}f_2 f_2 f_2 \,. \end{aligned} \tag{2.13}$$

We note in passing that a pattern similar to (1.3) appears: $G_n \sim \sum_{i_p} f_{i_1} \ldots f_{i_p}$ with $\sum_{p=1} i_p = 2n$. This pattern in $d = 2$ has an origin in the Virasoro algebra (2.1).

Let us introduce an integral formula to organize the perturbative solutions. By plugging the expansion

$$Q(z) = \sum_{n=0} R_n(z)\eta^n \,, \quad R_0(z) = z \tag{2.14}$$

into the equation (1.4) and matching the power of η, we deduce that

$$R_{n+1}(z) = -6 \int_0^z ds_2 \int_0^{s_2} ds_1 \, \frac{R_n(s_1)}{(1 - s_1)^2} \,. \tag{2.15}$$

The double integrals arise from the second-order differential equation. This formula reproduces the G_1, G_2, G_3 results (using $R_n = zG_n$) and presents a streamlined approach to determine the correlator at higher orders. While this approach is somewhat redundant in $d = 2$, the similar formulas in $d = 4$, which we will present below, provide an efficient way to analyze the near-lightcone structure of the correlator.

These $d = 2$ correlator results can be obtained using holography by considering a bulk scalar two-point function with a BTZ black hole in AdS$_3$. In higher dimensions, a similar gravity computation is also possible, as mentioned in the introduction. The motivation of this work is to find a way to calculate $d > 2$ correlators at large central charge by utilizing a method similar to the $d = 2$ approach discussed in this section. In higher dimensions, we will focus on the correlator structure near the lightcone.

3 $d = 4$ equations at large C_T

In this section, we examine the $d = 4$ linear differential equations (1.7)–(1.9), which are proposed as the generalizations to the $d = 2$ equation (1.4) or, equivalently, (2.5).

3.1 $\Delta = -1$ case

It may be illuminating to first take a look at the equation (1.7) in a different form, adopting the function $F(z, \bar{z})$ with the parameter μ restored:

$$(1 + \bar{z}\partial_{\bar{z}}) \left(\partial_z^3 + \frac{6}{z}\partial_z^2 + \frac{6}{z^2}\partial_z \right) F(z, \bar{z}) + \frac{\mu\bar{z}}{(1-z)^3} F(z, \bar{z}) = 0. \tag{3.1}$$

Compared to the $d = 2$ equation (2.5), a notable difference is that this $d = 4$ equation includes both z- and $\bar{z}$-derivatives. A factorization between the z-derivative part and the $\bar{z}$-derivative part of the equation can be observed. The equation transforms covariantly under $z \to \frac{z}{z-1}$, $\bar{z} \to -\bar{z}$ (in the near-lightcone limit).[9] Crossing symmetry is expected as two light scalars are identical. Similar to the $d = 2$ case, the crossing relation fixes the ratio $1 : 6 : 6$ in the bracket of the equation (3.1) involving z-derivatives, but is insufficient to uniquely determine the full equation. It would be interesting to construct certain mode operators, possibly with regularization in the near-lightcone regime, as well as a null state to provide a derivation of the $d = 4$ equation.

The partial differential equation (3.1) is non-trivial to solve directly. Here we adopt an approximate analysis. Consider an expansion

$$F(z, \bar{z}) = 1 + G_1(z)\bar{z} + G_2(z)\bar{z}^2 + G_3(z)\bar{z}^3 + \mathcal{O}(\bar{z}^4) \tag{3.2}$$

where we have fixed a normalization and absorbed the parameter μ into $\bar{z}$. This near-lightcone expansion (3.2) will be used for all cases with $\Delta = -1, -2, -3$.

The differential equation (3.1) leads to

$$
\begin{aligned}
0 = {} & \left(2z^2 G_1^{(3)}(z) + 12z G_1''(z) + 12 G_1'(z) + \frac{z^2}{(1-z)^3} \right) \bar{z} \\
& + \left(3z^2 G_2^{(3)}(z) + 18z G_2''(z) + 18 G_2'(z) + \frac{z^2 G_1(z)}{(1-z)^3} \right) \bar{z}^2 \\
& + \left(4z^2 G_3^{(3)}(z) + 24z G_3''(z) + 24 G_3'(z) + \frac{z^2 G_2(z)}{(1-z)^3} \right) \bar{z}^3 + \mathcal{O}\left(\bar{z}^4 \right).
\end{aligned}
\tag{3.3}
$$

The general solution of G_1 is

$$G_1 = \frac{1}{4z^2} \left(c_1 + c_2 z + c_3 z^2 + (6 + (z-6)z)\ln(1-z) \right). \tag{3.4}$$

---

[9] Here z and $\bar{z}$ are two independent real variables and we explore the near-lightcone limit, $\bar{z} \to 0$. In Euclidean signature, z is a complex number and $\bar{z}$ is its conjugate.

We can consider a small z limit and require $G_1(0) = 0$ to determine that $(c_1, c_2, c_3) = (0, 6, -3)$. The result is the universal single-stress tensor exchange contribution near the lightcone:

$$G_1 = \frac{1}{4z^2}\Big((6 + (z-6)z)\ln(1-z) - 3(z-2)z\Big) = -\frac{f_3}{120}. \qquad (3.5)$$

Proceeding to solve for G_2 and G_3, which correspond to double- and triple-stress tensor exchanges, respectively, we find

$$\begin{aligned}
G_2 &= \frac{1}{48z^2}\Big[(z(z+6)-6)\ln^2(1-z) - 9(z-2)z\ln(1-z) + 24z^2\Big] \\
&= -\frac{f_2 f_4}{4480} + \frac{f_3 f_3}{4320}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.6) \\
G_3 &= \frac{1}{1152z^2}\Big[(z(z-18)+18)\ln^3(1-z) - 18(z-2)z\ln^2(1-z) \\
&\qquad\qquad + (3(41z-210)z + 630)\ln(1-z) - 315(z-2)z\Big] \\
&= -\frac{5f_1 f_2 f_6}{6386688} + \frac{f_1 f_3 f_5}{1209600} - \frac{f_2 f_2 f_5}{2903040} + \frac{f_2 f_3 f_4}{483840} - \frac{11 f_3 f_3 f_3}{6220800}. \qquad (3.7)
\end{aligned}$$

In this $\Delta = -1$ case, a possible $f_1 f_5$ term in (3.6) and an $f_1 f_1 f_7$ term in (3.7) both have a vanishing coefficient.[10] The G_2 result obtained here using the differential equation is in agreement with that obtained in [74] based on multi-stress tensor OPEs computed from holography. The G_3 result is in agreement with [75, 76] which employ the lightcone bootstrap.

To compute higher-order results, let us derive an integral formula using the equation (1.7). The derivation is essentially identical to the $d = 2$ case. Here we take

$$Q(z, \bar{z}) = \sum_{n=0} R_n(z)\bar{z}^{n+1}, \quad R_0(z) = z^2, \qquad (3.8)$$

where the subscript of Q is suppressed. By matching the power of $\bar{z}$, we obtain

$$R_{n+1}(z) = \frac{-1}{n+2}\int_0^z ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \frac{R_n(s_1)}{(1-s_1)^3}. \qquad (3.9)$$

This formula reproduces the G_1, G_2, G_3 results (recall $R_n = z^2 G_n$ in this case) and provides an efficient way to compute higher-order contributions.[11]

---

[10]The expressions written in terms of hypergeometric functions are not unique due to relations among them. For instance, $9f_2 f_4 - \frac{28}{3}f_3 f_3 = 140(f_1 f_3 - f_2 f_2)$.

[11]For instance,

$$\begin{aligned}
G_4 &= \frac{1}{46080z^2}\Big((z^2 + 30z - 30)\ln^4(1-z) - 30(z-2)z\ln^3(1-z) \\
&\qquad + 15(25z^2 + 78z - 78)\ln^2(1-z) - 2295(z-2)z\ln(1-z) + 5760z^2\Big) \\
&= -\frac{f_1 f_2 f_2 f_7}{1014773760} - \frac{17 f_1 f_1 f_3 f_7}{1353031680} + \frac{5 f_2 f_2 f_2 f_6}{204374016} + \frac{643 f_1 f_2 f_3 f_6}{224811417600} \\
&\quad + \frac{433 f_1 f_1 f_4 f_6}{58284441600} + \frac{3 f_1 f_1 f_5 f_5}{551936000} - \frac{7 f_2 f_2 f_3 f_5}{182476800} - \frac{7 f_1 f_3 f_3 f_5}{3649536000} + \frac{7 f_3 f_3 f_3 f_3}{513216000}.
\end{aligned}$$

The expression written as hypergeometric functions allow varying forms.

Let us test the equation (1.7) further. By utilizing the formula (3.9) to compute higher-order terms, we may focus on the leading small-z piece of each function G_n. We observe a pattern that allows for the resummation over n:

$$\sum_{n=0}^{\infty} \frac{2\left(-z^3\bar{z}\right)^n}{\Gamma(n+2)\Gamma(3n+3)} = {}_0F_3\left(\left\{\frac{4}{3}, \frac{5}{3}, 2\right\}, -\frac{z^3\bar{z}}{27}\right). \tag{3.10}$$

Note that this is an expression for $F(z, \bar{z})$ where the identity contribution due to the $n = 0$ term is included in the resummation. We observe that (3.10) agrees with the corresponding correlator result obtained holographically by solving a bulk scalar equation of motion with a planar black hole in AdS$_5$ [71].[12] This result corresponds to a resummation of *all* leading-twist multi-stress tensors without derivatives inserted.

Another related observation is that the expression $(z^2\bar{z})\, {}_0F_3\left(\left\{\frac{4}{3}, \frac{5}{3}, 2\right\}, -\frac{z^3\bar{z}}{27}\right)$, which was first obtained via holography, is an *exact* solution to the partial differential equation $Q^{(3,1)}(z, \bar{z}) + Q(z, \bar{z}) = 0$. This equation corresponds to (1.7) with a replacement of the factor $(1-z)^3 \to 1$.

The differential equation (1.7) (or equivalently (3.1)) allows us to make predictions on the correlator structure beyond the leading small-z (or planar black hole) limit. Specifically, by using higher-order solutions generated from the formula (3.9) and collecting both sub-leading and sub-sub-leading small-z contributions (at each order of $\bar{z}$), we can identify the patterns allowing for the resummation. The corresponding results for the function $F(z, \bar{z})$ are given by

$$\sum_{n=1}^{\infty} \frac{3n}{\Gamma(n+2)\Gamma(3n+3)}\left(-z^3\bar{z}\right)^n z = -\frac{z^4\bar{z}}{80}\, {}_0F_3\left(\left\{\frac{7}{3}, \frac{8}{3}, 3\right\}, -\frac{z^3\bar{z}}{27}\right), \tag{3.11}$$

$$\sum_{n=1}^{\infty} \frac{\pi 3^{-3n-\frac{5}{2}}(9n^2 + 17n + 6)}{2\Gamma(n)\Gamma\left(n+\frac{5}{3}\right)\Gamma(n+2)\Gamma\left(n+\frac{7}{3}\right)}\left(-z^3\bar{z}\right)^n z^2$$

$$= -\frac{z^5\bar{z}}{2240}\left[6\, {}_0F_3\left(\left\{\frac{8}{3}, 3, \frac{10}{3}\right\}, -\frac{z^3\bar{z}}{27}\right) + 17\, {}_1F_4\left(\{2\}, \left\{1, \frac{8}{3}, 3, \frac{10}{3}\right\}, -\frac{z^3\bar{z}}{27}\right)\right.$$

$$\left. + 9\, {}_2F_5\left(\{2, 2\}, \left\{1, 1, \frac{8}{3}, 3, \frac{10}{3}\right\}, -\frac{z^3\bar{z}}{27}\right)\right]. \tag{3.12}$$

More higher-order results can be worked out. Obtaining these new results through holography may be a non-trivial task as it requires a *spherical* black hole.

3.2 $\Delta = -2$ case

The equation (1.8) written in terms of the function $F(z, \bar{z})$ is given by

$$0 = (2 + \bar{z}\partial_{\bar{z}})\left(\partial_z^4 + \frac{12}{z}\partial_z^3 + \frac{36}{z^2}\partial_z^2 + \frac{24}{z^3}\partial_z\right)F(z, \bar{z}) + \frac{6\mu\bar{z}}{(1-z)^3}\left(\partial_z + \frac{3(2-z)}{2z(1-z)}\right)F(z, \bar{z}). \tag{3.13}$$

---

[12]Specifically, we are referring to eq. (4.27) in that paper.

In this expression we have rescaled $\bar{z}$ to make the parameter μ explicit. This equation is also consistent with crossing symmetry. Solving this equation via the near-lightcone expansion, using the same boundary conditions discussed above, is straightforward.

To derive an integral formula, we adopt the equation (1.8) and write

$$Q(z,\bar{z}) = \sum_{n=0} R_n(z)\bar{z}^{n+2}, \quad R_0(z) = z^3. \tag{3.14}$$

(The subscript of Q is suppressed and μ is absorbed into $\bar{z}$.) We find

$$R_{n+1}(z) = \frac{-1}{n+3} \int_0^z ds_4 \int_0^{s_4} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 \left[\frac{9R_n(s_1)}{(1-s_1)^4} + \frac{6R_n'(s_1)}{(1-s_1)^3} \right]. \tag{3.15}$$

Using $R_n = z^3 G_n$, where G_n is defined via (3.2), in this $\Delta = -2$ case we have verified that the formula (3.15) reproduces the universal G_1, and the known G_2, G_3 expressions obtained in [74–76]. Higher-order contributions can be obtained readily.

Although we shall not provide an exhaustive list of higher-order solutions, let us mention that the resummation over n, similar to those in the $\Delta = -1$ case, can be obtained in a small z limit. At the first three orders in the $z \to 0$ expansion, we have (for the function $F(z,\bar{z})$ in the near-lightcone limit)

$$\sum_{n=0}^{\infty} \frac{2^{n+2}3^{n+1}}{\Gamma(n+3)\Gamma(3n+4)} \left(-z^3\bar{z}\right)^n = \frac{1}{z^3\bar{z}} \left[2 - {}_0F_3\left(\left\{\frac{1}{3},\frac{2}{3},2\right\}, -\frac{2z^3\bar{z}}{9}\right) \right], \tag{3.16}$$

$$\sum_{n=1}^{\infty} \frac{2^{n+1}3^{n+2}n}{\Gamma(n+3)\Gamma(3n+4)} \left(-z^3\bar{z}\right)^n z = -\frac{3}{z^2\bar{z}} \left[1 + {}_0F_3\left(\left\{\frac{1}{3},\frac{2}{3},1\right\}, -\frac{2z^3\bar{z}}{9}\right) \right.$$
$$\left. - 2\,{}_0F_3\left(\left\{\frac{1}{3},\frac{2}{3},2\right\}, -\frac{2z^3\bar{z}}{9}\right) \right], \tag{3.17}$$

$$\sum_{n=1}^{\infty} \frac{2^{n-1}3^{n+2}n\left(3n(9n^2+32n+34)+31\right)}{\Gamma(n+3)\Gamma(3n+6)} \left(-z^3\bar{z}\right)^n z^2 =$$
$$\frac{1}{8960z\bar{z}} \left[\frac{54}{55}z^9\bar{z}^3\,{}_0F_3\left(\left\{4,\frac{13}{3},\frac{14}{3}\right\}, -\frac{2z^3\bar{z}}{9}\right) + 84\left(12z^3\bar{z}\,{}_0F_3\left(\left\{2,\frac{7}{3},\frac{8}{3}\right\}, -\frac{2z^3\bar{z}}{9}\right)\right.\right.$$
$$\left.+ 320\,{}_0F_3\left(\left\{1,\frac{4}{3},\frac{5}{3}\right\}, -\frac{2z^3\bar{z}}{9}\right) - 400\,{}_0F_3\left(\left\{\frac{4}{3},\frac{5}{3},2\right\}, -\frac{2z^3\bar{z}}{9}\right)\right.$$
$$\left.- 3z^6\bar{z}^2\,{}_0F_3\left(\left\{3,\frac{10}{3},\frac{11}{3}\right\}, -\frac{2z^3\bar{z}}{9}\right) + 80\right) \right]. \tag{3.18}$$

The leading-order result (3.16), which includes the identity operator, is in perfect agreement with the holographic calculation with a black brane.[13] The remaining two expressions that go beyond the leading small-z limit are predictions of (1.8).

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[13] See [71], eq. (4.25), for verification.

3.3 $\Delta = -3$ case

The equation (1.9) expressed in terms of $F(z, \bar{z})$ can be written as follows:

$$0 = (3 + \bar{z}\partial_{\bar{z}}) \left(\partial_z^5 + \frac{20}{z}\partial_z^4 + \frac{120}{z^2}\partial_z^3 + \frac{240}{z^3}\partial_z^2 + \frac{120}{z^4}\partial_z \right) F(z, \bar{z})$$

$$+ \frac{21\mu\bar{z}}{(1-z)^3} \left(\partial_z^2 + \frac{8 - 5z}{(1-z)z}\partial_z + \frac{24z^2 - 84z + 84}{7(1-z)^2 z^2} \right) F(z, \bar{z}). \tag{3.19}$$

The equation transforms covariantly under the crossing relation.

Adopting the function $Q(z, \bar{z})$ and its equation, in this case we have

$$Q(z, \bar{z}) = \sum_{n=0} R_n(z)\bar{z}^{n+3}, \quad R_0(z) = z^4, \tag{3.20}$$

$$R_{n+1}(z) = \frac{-1}{n+4} \int_0^z ds_5 \int_0^{s_5} ds_4 \int_0^{s_4} ds_3 \int_0^{s_3} ds_2 \int_0^{s_2} ds_1$$

$$\times \left[\frac{72R_n(s_1)}{(1-s_1)^5} + \frac{63R_n'(s_1)}{(1-s_1)^4} + \frac{21R_n''(s_1)}{(1-s_1)^3} \right] \tag{3.21}$$

where, as before, we rescale $\bar{z}$ to absorb the parameter μ. Using $R_n = z^4 G_n$, (3.21) yields the results consistent with the universal G_1, as well as the known expressions for G_2, G_3 [74–76]. After computing the higher-order terms of G_n (whose expressions will not be listed here), we again observe patterns in a small z approximation. At the first three orders, the corresponding $F(z, \bar{z})$ resummation results are given by

$$\sum_{n=0}^{\infty} \frac{32\pi 3^{-2n-\frac{5}{2}} 7^n \left(-z^3\bar{z}\right)^n}{\Gamma\left(n + \frac{5}{3}\right)\Gamma(n+2)\Gamma\left(n + \frac{7}{3}\right)\Gamma(n+4)} = \frac{24}{7z^3\bar{z}} \left(1 - {}_0F_3\left(\left\{ \frac{2}{3}, \frac{4}{3}, 3 \right\}, \frac{-7z^3\bar{z}}{9} \right) \right), \tag{3.22}$$

$$\sum_{n=1}^{\infty} \frac{16\pi 3^{-2n-\frac{3}{2}} 7^n n\, z \left(-z^3\bar{z}\right)^n}{\Gamma\left(n + \frac{5}{3}\right)\Gamma(n+2)\Gamma\left(n + \frac{7}{3}\right)\Gamma(n+4)}$$

$$= -\frac{36}{7z^2\bar{z}} \left[1 - 2\, {}_0F_3\left(\left\{ \frac{2}{3}, \frac{4}{3}, 3 \right\}, \frac{-7z^3\bar{z}}{9} \right) + {}_1F_4\left(\{2\}, \left\{ \frac{2}{3}, 1, \frac{4}{3}, 3 \right\}, \frac{-7z^3\bar{z}}{9} \right) \right], \tag{3.23}$$

$$\sum_{n=1}^{\infty} \frac{2\, 3^{n+4} 7^{n-1} n \left(7n(9n^2 + 38n + 47) + 110\right) z^2 \left(-z^3\bar{z}\right)^n}{\Gamma(n+4)\Gamma(3n+7)}$$

$$= -\frac{18z^2}{5}\, {}_0F_3\left(\left\{ \frac{7}{3}, \frac{8}{3}, 3 \right\}, \frac{-7z^3\bar{z}}{9} \right) - \frac{27z^5\bar{z}}{320}\, {}_0F_3\left(\left\{ \frac{10}{3}, \frac{11}{3}, 4 \right\}, \frac{-7z^3\bar{z}}{9} \right)$$

$$- \frac{36}{343z^4\bar{z}^2} \left[288\, {}_0F_3\left(\left\{ \frac{1}{3}, \frac{2}{3}, 1 \right\}, \frac{-7z^3\bar{z}}{9} \right) + 539z^3\bar{z}\, {}_0F_3\left(\left\{ \frac{4}{3}, \frac{5}{3}, 2 \right\}, \frac{-7z^3\bar{z}}{9} \right) \right.$$

$$\left. - 276\, {}_0F_3\left(\left\{ \frac{1}{3}, \frac{2}{3}, 2 \right\}, \frac{-7z^3\bar{z}}{9} \right) - 14z^3\bar{z} - 12 \right]. \tag{3.24}$$

The leading-order result (3.22) is in agreement with the holographic computation [71] with a planar black hole. The other two expressions are predictions.

More generally, examining other limits of z also reveal patterns that allow the re-summation. For instance, by considering $z \to 1$ instead of $z \to 0$, it is also possible to perform the resummation for all $\Delta = -1, -2, -3$ cases. However, here we refrain from listing excessive expressions based on varying approximations of z.

Without making any approximations of z, upon computing the higher-order terms we find that the $d = 4$ differential equations suggest a general structure: ($X \equiv 1 - z$)

$$R_n(z) = \sum_{j=0,1,2,\ldots}^{n} \left(a_{nj}(\Delta) + b_{nj}(\Delta)X + c_{nj}(\Delta)X^2 + d_{nj}(\Delta)X^3 + e_{nj}(\Delta)X^4 \right)(\ln X)^j$$

where $\Delta = -1, -2, -3$ and the coefficients $a_{nj}, b_{nj}, \ldots, e_{nj}$ can be computed to higher orders efficiently. Integers n, j satisfy $n \geq j \geq 0$. The power of X is restricted. Moreover, we find

$$d_{nj}(\Delta = -1) = 0, \qquad e_{nj}(\Delta = -1, -2) = 0. \tag{3.25}$$

Note these coefficients are subjected to the initial conditions (3.8), (3.14), (3.20). We have not yet fully analyzed these coefficients to see if there are any interrelated patterns that would allow a resummation and obtain the closed-form expression without needing to approximate z, i.e., a result may be viewed as a closed-form generalization to (1.5). Obtaining such an expansion would be interesting. As mentioned, information about the Virasoro blocks of general conformal dimensions can be obtained by analytically continuing the correlators involving degenerate operators. It may be possible to generalize the story to $d = 4$ by utilizing the differential equations we consider here.

4 Discussion

In this paper, we have proposed a system of $d = 4$ equations (1.7)–(1.9), valid in the near-lightcone regime, and verified that the solutions to these equations are in agreement with the existing results obtained via holography and the lightcone bootstrap. The physical observable we are interested in is the heavy-light scalar four-point correlator at large central charge. Our approach is guided by the similarity of the $d = 2$ BPZ equation in the form of (1.4) and crossing symmetry. It is worth emphasizing that this work is made possible by the advancements in holography and bootstrap. Let us conclude with additional remarks and speculate on what might lie ahead.

What is the origin of these $d = 4$ differential equations? Could there be an approximate symmetry in the near-lightcone regime? No attempt is made in this work to provide a first-principle derivation, whether through an algebraic approach or otherwise, of the equations that we observe. We do not presume that a rigorous CFT derivation starting with TT and $T\mathcal{O}$ OPEs can be easily provided. Rather, we suspect that new insights and methods may be needed to fill this logical chasm. Similar to the $d = 2$ case, the Ward identities should play a role, but additional ingredients are likely required to rigorously derive these $d = 4$ equations.

There is a clue that arises from a tension. The structure of $d = 2$ multi-stress tensors is directly connected to the stress-tensor commutators (which leads to the Virasoro algebra

after adopting the Virasoro mode operator). In higher dimensions, however, it is known that the *c*-number term (the Schwinger term) of the stress-tensor commutators is divergent, i.e., [87, 88]. Reconciling this tension may be essential to develop a possible algebraic derivation of these $d = 4$ differential equations.[14]

It was discovered that certain $d > 2$ superconformal field theories exhibit a chiral algebra [97–99]. (See also [100, 101], which considers extended operators intersecting along a codimension-one surface.) In this work, we focus on the multi-stress tensor sector of the near-lightcone correlator at large C_T without explicitly assuming supersymmetry. It could be interesting to investigate whether supersymmetry is necessary for these linear differential equations to hold.

Alternatively, it might be possible to derive the null-state equations without prior knowledge of a symmetry. In $d = 2$, if one already knows about the fusion rule, one may still derive the level-2 BPZ equation, which corresponds to a rigid Fuchsian system, without knowing the Virasoro algebra. (Such a derivation, however, cannot be extended to the higher-order BPZ equations. See, e.g., [102, 103] for related remarks.) Another possible derivation of the null-state equations without relying on an algebra is to properly extract the degenerate-operator contributions utilizing only $T\mathcal{O}$ OPE and the Ward identities. Such an "indirect" derivation in higher dimensions could be thorny, but valuable.

Since the most significant obstacles arise when transitioning from $d = 2$ to any higher dimensions, if the computation can be uplifted from two dimensions to four dimensions, the similar extensions to other $d > 2$ dimensions should pose no difficulty. For instance, again by drawing upon the $d = 2$ BPZ equation in the form of (1.4), crossing symmetry, and the input from the universal single-stress tensor exchange, we find that in $d = 6$, the corresponding equations are

$$Q_{-1}^{(4,2)} + 9\frac{Q_{-1}}{(1-z)^4} = 0, \quad (4.1)$$

$$Q_{-2}^{(5,2)} + 72\frac{Q_{-2}^{(1,0)}}{(1-z)^4} + 144\frac{Q_{-2}}{(1-z)^5} = 0, \quad (4.2)$$

$$Q_{-3}^{(6,2)} + 324\frac{Q_{-3}^{(2,0)}}{(1-z)^4} + 1296\frac{Q_{-3}^{(1,0)}}{(1-z)^5} + 1800\frac{Q_{-3}}{(1-z)^6} = 0, \quad (4.3)$$

$$Q_{-4}^{(7,2)} + 1080\frac{Q_{-4}^{(3,0)}}{(1-z)^4} + 6480\frac{Q_{-4}^{(2,0)}}{(1-z)^5} + 18000\frac{Q_{-4}^{(1,0)}}{(1-z)^6} + 21600\frac{Q_{-4}}{(1-z)^7} = 0 \quad (4.4)$$

where $Q_{\Delta^*} = \frac{z^2}{(z\bar{z})^\Delta}F|_{\Delta=\Delta^*}$ with the special values of Δ^* in $d = 6$ given by $\Delta^* = -1, -2, -3, -4$. We adopt the same convention where the parameter μ is properly absorbed into the coordinate $\bar{z}$ in the near-lightcone expansion. By performing an identical

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[14]It has been established that the commutator of the lightray operator in $d = 4$ has a divergent central term [87, 89–94]. A regularization procedure was proposed in [77] which isolates an algebra-like structure using the integrals over directions transverse to the lightcone (see also [88, 95, 96]). One challenge in utilizing such a formulation is that the effective algebra is made of line operators integrated along the lightcone and their action becomes nonlocal. It is unclear what the right rules for the nonlocal terms should be. I thank L. Fitzpatrick for this remark.

analysis as for the $d = 4$ case, we have verified that these equations are consistent with the existing results, obtained via holography and bootstrap, for the leading-twist multi-stress tensor contributions.

However, we would like to mention that the structure of heavy-light correlators in odd dimensions is underinvestigated. One may observe that there is an increase in the power of the z-derivative by one and two when comparing the level-2 BPZ equation with the simplest equations in $d = 4$ and $d = 6$, respectively, for the $\Delta = -1$ case. This perhaps suggests that techniques from fractional calculus may assist in organizing the equations in odd dimensions.

Further clarification is needed to establish the validity of the results in this work. These results apply to a class of large-N CFTs that have a holographic dual description as a minimally coupled theory in AdS with arbitrary higher-curvature corrections, including Einstein gravity as the simplest case. What about non-minimally coupled bulk interactions? Here we remark that the corrections to the leading-twist multi-stress tensor OPEs due to the quartic derivative bulk interactions computed in [73] vanish at $\Delta = -1, -2, -3$, which coincide with the required conformal dimensions in the $d = 4$ linear differential equations. (To be precise, we are referring to the corrections computed in eq. (71) and eqs. (86)–(87) in that paper.) A possible future investigation is to search for a mechanism that could explain this phenomenon.

Acknowledgments

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