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Homotopy Theory of Looped Polyhedral Products

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Abstract

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This thesis studies the pointed loop space of spaces known as polyhedral products and gives loop space decompositions in various cases as a product of well-studied spaces. It is a research paper thesis which contains the following papers:

[1] L. Stanton, *Loop space decompositions of moment-angle complexes associated to flag complexes*, Q. J. Math. **75** (2024), no. 2, 457–477

[2] L. Stanton, *Loop space decompositions of moment-angle complexes associated to two dimensional simplicial complexes*, (2024), to appear in Proceedings of the Edinburgh Mathematical Society, https://arxiv.org/abs/2407.10781

[3] L. Stanton and S. Theriault., *Polyhedral products associated to pseudomanifolds*, Int. Math. Res. Not. **2025** (2025), rnaf164

In [1], we show that the loop space of a moment-angle complex associated to the k-skeleton of a flag complex decomposes as a product of spheres and loops on spheres up to homotopy.

In [2], we show that the loop space of a moment-angle complex associated to a 2-dimensional simplicial complex decomposes as a product of spheres, loops on spheres and well-studied torsion spaces up to homotopy.

In [3], we study the homotopy theory of polyhedral products associated to a combinatorial generalisation of manifolds known as a pseudomanifold. We use this to show that the loop space of a moment-angle manifold associated to a connected, orientable surface, or a triangulation of S^3 decomposes as a product of spheres and loops on spheres up to homotopy.

Contents

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

[1] L. Stanton, *Loop space decompositions of moment-angle complexes associated to flag complexes*, Q. J. Math. **75** (2024), no. 2, 457–477

[2] L. Stanton, *Loop space decompositions of moment-angle complexes associated to two dimensional simplicial complexes*, (2024), to appear in Proceedings of the Edinburgh Mathematical Society, `https://arxiv.org/abs/2407.10781`

[3] L. Stanton and S. Theriault., *Polyhedral products associated to pseudomanifolds*, Int. Math. Res. Not. **2025** (2025), rnaf164

Signed:.. Date:..................

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Chapter 1

Introduction

The main goal of algebraic topology is to classify topological spaces up to continuous deformation. One of the main algebraic tools used to do this is the sequence of homotopy groups $\pi_*(X)$ of a topological space X. These groups are notoriously difficult to calculate, even in the case of spheres (Tod16). However, there are some global properties of the homotopy groups of spheres which are known. In particular, the torsion free parts of the homotopy groups were calculated by Serre (Ser51), and information about the odd primary torsion was obtained in celebrated work of Cohen, Moore and Neisendorfer (CMN79b; CMN79a; Nei81).

One approach to determining the homotopy groups of a space X is to find a homotopy equivalence of the form $X \simeq A \times B$, where A and B are spaces which are not contractible. This implies there is an isomorphism $\pi_*(X) \cong \pi_*(A) \times \pi_*(B)$, and so we may write the homotopy groups of X in terms of those of A and B. This may be difficult in general, and many spaces do not admit a product decomposition of this form.

One remedy to this is to consider the pointed loop space of X, $\Omega X := \mathrm{Map}_*(S^1, X)$, the space of pointed maps from the circle S^1 to X. There is an isomorphism $\pi_n(X) \cong \pi_{n-1}(\Omega X)$, and so the problem of determining the homotopy groups of X is equivalent to determining the homotopy groups of ΩX. The benefit of studying ΩX is that it has a multiplication up to homotopy given by concatenation of loops, and this makes it easier to find product decompositions of ΩX. This approach has been used to great effect in the context of Poincaré duality complexes (BT14; BB18; BT22**?**).

This thesis is a research paper thesis and the main aim of the papers contained within is to use this approach in the context of spaces known as polyhedral products. These papers greatly extend our understanding of the homotopy groups of these spaces, and verify major conjectures in homotopy theory for certain classes of polyhedral products. Other work completed by the author during the course of the PhD are (AHS24; ST25a).

Polyhedral products

Polyhedral products are a generalisation of spaces known as moment-angle complexes, which first were first constructed by Davis and Januszkiewicz (DJ91) in the context of toric topology. The definition was reformulated and generalised by Buchstaber and Panov (BP02) (and independently in unpublished notes of Strickland).

The definition of a polyhedral product is as follows. Let K be a simplicial complex on $[m] := \{1, \cdots, m\}$, and let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be a tuple of CW-pairs. For each $\sigma \in K$, define

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i \text{ where } Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is the space

$$(\underline{X}, \underline{A})^K := \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i.$$

The moment-angle complex and its closely associated Davis-Januszkiewicz space, denoted $\mathcal{Z}_K$ and DJ_K respectively, correspond to the cases where $(X_i, A_i) = (D^2, S^1)$, and $(X_i, A_i) = (\mathbb{C}P^\infty, *)$.

The first focused study of the homotopy theory of polyhedral products was conducted by Bahri, Bendersky, Cohen, and Gitler (BBCG10). They showed that the homotopy type of the suspension of any polyhedral product can be written in terms of a construction known as the polyhedral smash product. If either each X_i is contractible or each A_i is the basepoint of X_i, the polyhedral smash product has a homotopy decomposition as a wedge of spaces which can be written in terms of the combinatorics of K. This completely determines the homology groups of these polyhedral products in terms of the homology of the ingredient spaces and K. Focusing our attention on the case that each X_i is contractible, it was shown for various families of simplicial complexes that the decomposition proved by Bahri, Bendersky, Cohen and Gitler holds without suspension (GT13; IK13; GPTW16; IK19) and this has important consequences in combinatorics. Work is ongoing to further expand the families for which such a decomposition is known.

Conjectures in unstable homotopy theory

We now shift our focus onto the homotopy groups of polyhedral products. Before proceeding, we place the study of these in a broader context. To do this, we require some definitions. A CW-complex X is called *rationally elliptic* if it has finitely many rational homotopy groups, and X is called *rationally hyperbolic* otherwise. The *homotopy exponent* of X at a prime p is the least power of p which annihilates the p-torsion of the homotopy groups of X. A major driving force of research in homotopy theory is Moore's conjecture which asserts a deep connection between the rational and torsion parts of the homotopy groups.

Conjecture 1.1 (Moore's Conjecture)**.** *Let X be a finite, simply-connected CW-complex. The following are equivalent:*

1. *X is rationally elliptic,*
2. *X has a finite homotopy exponent at every prime p,*
3. *X has a finite homotopy exponent at some prime p.*

This conjecture has been verified for various families of spaces, including spheres (Jam56; Tod56), odd primary Moore spaces (Nei87), certain highly connected Poincaré duality complexes (BT14; BB18; BT22; ST25a) and moment-angle complexes (HST19). In the case of moment-angle complexes, this was proved by giving an explicit decomposition of its loop space in the case that it is rationally elliptic, and showing that a space with no homotopy exponent retracts off it in the rationally hyperbolic case. However, in the rationally hyperbolic case, this approach does not allow us to explicitly enumerate the homotopy groups that appear. The papers in this thesis will remedy this in certain cases, and expand the results to more general polyhedral products.

There are two related conjectures which can be seen as approximations to Moore's conjecture. Let X be a simply-connected CW-complex, and let p be a prime. Localisation at the prime p is a functor which outputs a space $X_{(p)}$ such that the homotopy groups of $X_{(p)}$ encode the p-torsion information of the homotopy groups of X. More precisely, there is an isomorphism $\pi_*(X_{(p)}) \cong \pi_*(X) \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the subring of $\mathbb{Q}$ consisting of fractions whose denominators are coprime to p.

Localised at a prime p, Huang and Wu (HW19) showed that if X is an H-space of finite type, then X decomposes uniquely (up to order and homotopy equivalence) as a product of indecomposable factors. In particular, if X is a simply-connected CW-complex, localised at a prime p, there is a homotopy equivalence

$$\Omega X \simeq \prod_{i \in \mathcal{I}} Y_i,$$

where $\mathcal{I}$ is some indexing set and each Y_i is an indecomposable H-space. Therefore to decompose ΩX, we first require candidates for indecomposable H-spaces which could appear in a decomposition for ΩX.

Localise at an odd prime p. The spaces S^{2n-1} and ΩS^{2n+1} for $n \geq 1$ are examples of indecomposable H-spaces. For integers $m \geq 1$ and $r \geq 1$, Cohen, Moore and

Neisendorfer (CMN79b; CMN79a; Nei87) defined spaces $S^{2m+1}\{p^r\}$ and $T^{2m+1}\{p^r\}$, whose homology consists purely of p^r summands. These are indecomposable H-spaces with the single exception of $T^3\{p\}$, for which there is a homotopy equivalence $T^3\{p\} \simeq T^{2p+1}\{p\} \times U^1$, where U^1 is an indecomposable H-space. Let $\overline{\mathcal{P}}$ be the collection of H-spaces which are homotopy equivalent to a finite type product of spheres S^{2n-1}, where $n \geq 1$, loops on simply connected spheres ΩS^{2m+1} and the indecomposable torsion spaces defined by Cohen, Moore and Neisendorfer. Anick conjectured that localised at all but finitely many primes, these spaces are enough to describe ΩX up to homotopy (Ani92).

Conjecture 1.2 (Anick's conjecture)**.** *Let X be a finite, simply-connected CW-complex. Localised at all but finitely many primes, $\Omega X \in \overline{\mathcal{P}}$.*

Anick's conjecture has been verified for rationally elliptic spaces (MW86) and certain two-cones (Ani89a). Decompositions of this form without localisation have been proven for certain highly connected Poincaré duality complexes (BT14; BB18; BT22; ST25a). It was shown by Panov and Theriault (PT19) that if K is a flag complex, then $\Omega \mathcal{Z}_K$ is homotopy equivalent to a product of spheres and loops on spheres without localisation. The papers in this thesis recover and greatly expand this result in various contexts.

Finally, let X be a space and p be a prime. The Steenrod algebra $\mathcal{A}_p$ consists of homomorphisms between the cohomology groups of X with coefficients in $\mathbb{Z}/p\mathbb{Z}$ which satisfy certain axioms (Hat02). McGibbon and Wilkerson conjectured the following (MW86).

Conjecture 1.3. *If X is a finite, simply-connected CW-complex, then for all but finitely many primes*

1. *p^{th} powers vanish in $\tilde{H}(\Omega X; \mathbb{Z}/p\mathbb{Z})$;*
2. *the Steenrod algebra acts trivially on $H^*(\Omega X; \mathbb{Z}/p\mathbb{Z})$.*

The first assertion was proved by Anick (Ani89b); however, the second assertion remains open. The Steenrod algebra acts trivially on spheres and loops on spheres, and so knowing a space is homotopy equivalent to a product of spheres and loops on spheres localised at an odd prime verifies the McGibbon and Wilkerson conjecture. In this thesis, we give loop space decompositions for various families of polyhedral products which verify all three of these conjectures for these spaces.

Summary of papers

The three papers (Sta24a; Sta24b; ST25b) contained in this thesis give loop space decompositions of various families of polyhedral products. The first two papers are single author papers, and the final paper is a joint paper with Stephen Theriault (my supervisor). In the joint paper, Stephen came up with the idea of introducing an auxillary simplicial complex and using this under certain conditions to show that a map induced by an inclusion of simplicial complexes has a right homotopy inverse. The application of this to pseudomanifolds and low dimensional triangulations of spheres was proved by myself. To describe the main results, we establish some notation. For a collection of spaces $\mathcal{X}$, let $\bigvee \mathcal{X}$ (resp. $\prod \mathcal{X}$) be the collection of spaces which are homotopy equivalent to a finite type wedge (resp. product) of spaces in $\mathcal{X}$. Let $\mathcal{P} := \{S^1, S^3, S^7, \Omega S^n \mid n \geq 2, n \notin \{2,4,8\}\}$. Recall by (HW19) that localised at a prime p, any finite type H-space decomposes into a product of indecomposable spaces which are unique up to order and homotopy equivalence. Let $\mathcal{T}$ be the collection of indecomposable spaces which appear in the decomposition of the loop space of a wedge of Moore spaces of the form $\bigvee_{i=1}^{m} P^{n_i}(p_i^{r_i})$, where $m \geq 2$, $n_i \geq 3$, p_i is a prime and $r_i \geq 1$. The collection $\mathcal{T}$ includes the spaces $T^{2m+1}\{p^r\}$, $S^{2m+1}\{p^r\}$ and U^1 defined by Cohen, Moore and Neisondorfer, and these are sufficient for odd primes. At the prime 2, for $r > 1$, Cohen (Coh89) defined an analogous space $T^{2m+1}\{2^r\}$. When $r = 1$, there is not an analogue of the Cohen, Moore and Neisendorfer space.

Let $\mathcal{W}$ be the collection of simply connected spheres, and $\mathcal{M}$ be the collection of Moore spaces of the form $P^n(p^r)$, where $n \geq 3$, p is a prime, and $r \geq 1$, and the indecomposable factors which appear as wedge summands in the unique 2-local wedge decomposition of spaces of the form $\Sigma((P^{n_1}(2) \wedge \cdots \wedge (P^{n_l}(2))$, where $l \geq 2$, and each $n_i \geq 3$ (which exist by (HW19)). The collection $\mathcal{M}$ contains Moore spaces of the form $P^n(p^r)$, and these spaces are sufficient when $p^r \neq 2$. Some progress on identifying the indecomposable spaces when $p^r = 2$ has been made by Wu (Wu03).

The first paper focuses on the case where K is the k-skeleton of a flag complex, generalising a result of Panov and Theriault (PT19). These simplicial complexes are obtained by starting with a graph, and gluing in all possible simplices of dimension $2 \leq l \leq k$. The main result is as follows.

Theorem 1.1. *Let K be the k-skeleton of a flag complex on $[m]$, and let $A_1, \cdots, A_m$ be CW-complexes such that $\Sigma A_i \in \bigvee \mathcal{W}$. Then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.*

A key technical result proved in the first paper is closure of the collection $\prod \mathcal{P}$ under retracts. This was a folklore result, but a proof did not appear in the literature.

The second paper extends the techniques of the first paper in order to include torsion spaces. Any graph G can be considered as the 1-skeleton of a flag complex, and so the previous result implies that $\Omega(\underline{CA}, \underline{A})^G \in \prod \mathcal{P}$, when each $\Sigma A_i \in \bigvee \mathcal{W}$. The second paper extends decompositions of this kind to the case of a 2-dimensional simplicial complex. One step in doing this is showing that $\prod(\mathcal{P} \cup \mathcal{T})$ is closed under retracts. The main result of this paper is as follows.

Theorem 1.2. *Let K be a 2-dimensional simplicial complex, and let $A_1, \cdots, A_m$ be CW-complexes such that $\Sigma A_i \in \bigvee(\mathcal{W} \cup \mathcal{M})$. Then $\Omega(\underline{CA}, \underline{A})^K \in \prod(\mathcal{P} \cup \mathcal{T})$.*

The third paper focuses on the case where K is a triangulation of a sphere. When K is a triangulation of a sphere, the moment-angle complex $\mathcal{Z}_K$ has the structure of a manifold (BP15). When K is a triangulation of S^{2n+1} for $n \geq 0$ such that every simplex of dimension n is in K, and K is the dual of the boundary of a simple polytope, it was shown by Gitler and López de Medrano (GdM13) that $\mathcal{Z}_K$ is diffeomorphic to a connected sum of products of two spheres. Outside of this case, not much is known about the homotopy type of moment-angle manifolds. The third paper starts the study of the homotopy type of the loop space of moment-angle manifolds. The following result is proved.

Theorem 1.3. *Let K be a triangulation of a connected, orientable closed surface or a triangulation of S^3. Then $\Omega\mathcal{Z}_K \in \prod \mathcal{P}$.*

Moment-angle manifolds are closely related to another family of manifolds known as quasitoric manifolds. A quasitoric manifold is a manifold of dimension $2n$ which has an action of a torus of dimension n with certain hypotheses. Quasitoric manifolds can be viewed as a quotient of a moment-angle complex under the action of a torus which acts freely on the moment-angle complex. The loop space of a quasitoric manifold can be related to that of the corresponding moment-angle complex. The second main result of the paper is the following.

Theorem 1.4. *Let M be a quasitoric manifold of dimension 4, 6 or 8. Then $\Omega M \in \prod \mathcal{P}$.*

Future work

The papers contained in this thesis greatly expand the families of polyhedral products for which Moore's conjecture, Anick's conjecture and the McGibbon-Wilkerson conjecture are verified. Previously, loop space decompositions which verified these conjectures were only known when K is a flag complex (PT19), with the proof relying on a folklore result which was proved in Paper 1 (Sta24a). Many of the decompositions are coarse, in the sense that the terms appearing in the decomposition are not explicitly enumerated. In principle, the techniques of the paper could be used to enumerate the terms, but it would be impractical to do so. In the case of a flag complex, following the release of Paper 1 (Sta24a), Vylegzhanin (Vyl24) enumerated the spheres and loops on spheres which appear in the decomposition of the associated moment-angle complex. This motivates the following problem.

Problem 1.5. If K is a simplicial complex such that $\Omega(\underline{CA}, \underline{A})^K \in \prod(\mathcal{P} \cup \mathcal{T})$, enumerate the factors which appear in the decomposition.

If X is a simply connected CW-complex and $\mathbb{F}$ is a field, then $H_*(\Omega X;\mathbb{F})$ has the structure of an algebra. The decompositions of the loop spaces of polyhedral products contained in this thesis are not splitting as H-spaces, and so these splittings only give you information about $H_*(\Omega(\underline{CA},\underline{A})^K;\mathbb{F})$ as a module. If K is a simplicial complex, let C_K be the set of full subcomplexes of K such that the 1-skeleton of K has no missing edges. A result in Paper 2 (Sta24b) shows that $\Omega(\underline{CA},\underline{A})^K \in \prod(\mathcal{P}\cup\mathcal{T})$ if and only if $\Omega(\underline{CA},\underline{A})^{K_I} \in \prod(\mathcal{P}\cup\mathcal{T})$ for all $K_I \in C_K$. This raises the question as to whether these decompositions give you information about the algebra structure of the loop homology.

Problem 1.6. Describe $H_*(\Omega(\underline{CA},\underline{A})^K)$ as an algebra in terms of $H_*(\Omega(\underline{CA},\underline{A})^{K_I})$ where $K_I \in C_K$.

Finally, the Hilton-Milnor theorem (Hil55) implies that $\Omega\left(\bigvee_{i=1}^m S^{n_i}\right) \in \prod\mathcal{P}$, where each $n_i \geq 2$. Moreover, the terms appearing in the product are explicitly identified in terms of a basis of the free Lie algebra, with the homotopy equivalence being given by a product of looped Whitehead products. This proof used the fact that $H_*\left(\Omega\left(\bigvee_{i=1}^m S^{n_i}\right)\right)$ is the universal enveloping algebra of the free Lie algebra. Polyhedral products of the form $(\underline{X},\underline{*})^K$ give a natural interpolation between $\bigvee_{i=1}^m X_i$ (when K is m disjoint points) and $\prod_{i=1}^m X_i$ (when K is the $(m-1)$-simplex). If K is flag, it was shown in Paper 1 (Sta24a) that if each $X_i = S^{n_i}$ with $n_i \geq 2$, then $\Omega(\underline{X},\underline{*})^K \in \prod\mathcal{P}$. In this case, Cai (Cai24) gave a presentation of $H_*(\Omega(\underline{X},\underline{*})^K)$ which recovers the presentation of $H_*(\Omega(\bigvee_{i=1}^m S^{n_i}))$ when K is a set of disjoint points. If each $X_i = S^{n_i}$ where $n_i \geq 2$, then $H_*(\Omega(\underline{X},\underline{*})^K)$ is a universal enveloping algebra of a Lie algebra. One may hope to use Cai's presentation in order to give an explicit homotopy equivalence for $\Omega(\underline{X},\underline{*})^K$ in terms of a basis of the underlying Lie algebra.

Problem 1.7. Let K be a flag complex and let $X_i = S^{n_i}$ where $n_i \geq 2$. Give an explicit homotopy equivalence for $\Omega(\underline{X},\underline{*})^K$ as a product of looped spheres in terms of looped Whitehead products.

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Chapter 2

Paper 1 - Loop space decompositions of moment-angle complexes associated to flag complexes

1 Introduction

Polyhedral products have attracted vast attention due to their many applications across mathematics (see (BBC)). A polyhedral product is a natural subspace of $\prod_{i=1}^{m} X_i$ defined as follows. Let K be a simplicial complex on the vertex set $[m] = \{1, 2, \cdots, m\}$. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed CW-subcomplex of X_i. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^{m}$ be the sequence of pairs. For each simplex $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^{m} Y_i \text{ where } Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^{m} X_i.$$

An important special case is when $(X_i, A_i) = (D^2, S^1)$ for all i. These polyhedral products are called *moment-angle complexes*, and are denoted $\mathcal{Z}_K$. More generally, when $(X_i, A_i) = (D^n, S^{n-1})$ for $n \geq 2$ and all i, the polyhedral products are called *generalised moment-angle complexes*. In this paper, we identify the homotopy type of the loop space of certain polyhedral products. One particular case is when K is a flag complex. When K is flag, certain polyhedral products give models for the classifying space of graph products of groups, implying that the loop space of these polyhedral products are graph products of groups. This geometric group theoretic framework has been generalised by Cai (Ca) to consider loops on a wider class of polyhedral products associated to flag complexes. For general simplicial complexes K, the loop space of the corresponding moment-angle complex is related to a certain diagonal subspace arrangement (D).

Let $\bigvee\mathcal{W}$ be the full subcategory of topological spaces which are homotopy equivalent to a finite type wedge of simply connected spheres, and let $\prod\mathcal{P}$ be the full subcategory of H-spaces which are homotopy equivalent to a finite type product of spheres and loops on simply connected spheres. Note that if $X\in\prod\mathcal{P}$, by the Hopf invariant one problem (Ad), the only spheres that can appear in a product decomposition for X are S^n for $n\in\{1,3,7\}$, and it will be assumed that the loops on spheres ΩS^n which appear are of dimension $n\geq 2$, $n\notin\{2,4,8\}$, as when $n\in\{2,4,8\}$, there is a homotopy equivalence $\Omega S^n\simeq S^{n-1}\times\Omega S^{2n-1}$. Relations between spaces in $\bigvee\mathcal{W}$ and spaces in $\prod\mathcal{P}$ will be used frequently throughout the paper. In particular, the Hilton-Milnor theorem (H; M) implies that looping sends spaces in $\bigvee\mathcal{W}$ to spaces in $\prod\mathcal{P}$, and decomposing the suspension of a product as a wedge and the James construction (J) implies that suspension sends spaces in $\prod\mathcal{P}$ to spaces in $\bigvee\mathcal{W}$.

Determining the homotopy type of polyhedral products in general is difficult, but in the special case of a moment-angle complex, progress has been made in showing that certain moment-angle complexes are in $\bigvee\mathcal{W}$. For example, moment-angle complexes associated with shifted complexes (GT2, Theorem 1.2), flag complexes with chordal 1-skeleton (GPTW, Theorem 4.6), or more generally, totally fillable simplicial complexes (IK2, Corollary 7.3) are in $\bigvee\mathcal{W}$. There is a wider range of moment-angle complexes (including the aforementioned ones) for which its loop space is in $\prod\mathcal{P}$. For example, moment-angle complexes associated to any flag complex are in $\prod\mathcal{P}$ (PT, Corollary 7.3). It is known that many moment-angle complexes are not in $\bigvee\mathcal{W}$ due to the existence of non-trivial cup products in cohomology. For example, when K is the boundary of a square, $\mathcal{Z}_K\simeq S^3\times S^3$, and it follows that for any simplicial complex L containing K as a full subcomplex, $\mathcal{Z}_L$ contains non-trivial cup products in cohomology, and so $\mathcal{Z}_L\notin\bigvee\mathcal{W}$.

In this paper, we specialise to the case where K is the k-skeleton of a flag complex. In particular, we prove the following result.

Theorem 1.1. *Let $k\geq 0$, and let K be the k-skeleton of a flag complex on the vertex set $[m]$ and $A_1,\cdots,A_m$ be path connected CW-complexes such that $\Sigma A_i\in\bigvee\mathcal{W}$ for all i. Then $\Omega(\underline{CA},\underline{A})^K\in\prod\mathcal{P}$.*

There are two cases of Theorem 1.1 which should be highlighted. The first important case is when k is the dimension of the flag complex. While not stated in this generality, the following result recovers (PT, Corollary 7.3) via a different method.

Corollary 1.2. *Let K be a flag complex on the vertex set $[m]$ and $A_1, \cdots, A_m$ be path connected CW-complexes such that $\Sigma A_i \in \bigvee \mathcal{W}$ for all i. Then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.* □

The second important case of Theorem 1.1 is when $k = 1$.

Corollary 1.3. *Let K be a graph on the vertex set $[m]$ and $A_1, \cdots, A_m$ be path connected CW-complexes such that $\Sigma A_i \in \bigvee \mathcal{W}$ for all i. Then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.* □

Loop spaces of moment-angle complexes associated to graphs in certain cases have been studied. In particular, explicit decompositions of the loops of moment-angle complexes associated to wheel graphs and certain generalisations of wheel graphs (T2), and certain classes of generalised book graph (St) have been given. This paper establishes that decompositions of this form exist for all graphs. While in principle an explicit decomposition could be obtained, in practice it would be difficult to do so.

Letting $A_i = S^{n-1}$ with $n \geq 2$ for all i in Theorem 1.1 has consequences for generalised moment-angle complexes and moment-angle complexes.

Corollary 1.4. *Let $k \geq 0$, and let K be the k-skeleton of a flag complex. Then $\Omega(D^n, S^{n-1})^K \in \prod \mathcal{P}$ where $n \geq 2$.* □

Corollary 1.5. *Let $k \geq 0$, and let K be the k-skeleton of a flag complex. Then $\Omega \mathcal{Z}_K \in \prod \mathcal{P}$.* □

It is interesting to note when the decomposition in Corollary 1.5 arises from the fact that $\mathcal{Z}_K \in \bigvee \mathcal{W}$. In the case of K^1, it is shown in (IK2, Theorem 11.8) that K^1 has $\mathcal{Z}_{K^1} \in \bigvee \mathcal{W}$ if and only if K^1 is chordal. In particular, if K^1 is not chordal, then $\mathcal{Z}_{K^1}$ is not in $\bigvee \mathcal{W}$. However, Corollary 1.5 implies that nevertheless, $\Omega \mathcal{Z}_{K^1}$ is still in $\prod \mathcal{P}$. In the case of K itself, a similar result is true (PT, Theorem 6.4), namely that $\mathcal{Z}_K \in \bigvee \mathcal{W}$ iff K^1 is chordal.

To prove Theorem 1.1, we will show that $\prod \mathcal{P}$ is closed under retracts. This result was stated in (PT, p. 224) without proof, so a proof is provided here. The main tool that is used in the proof of this is the atomicity of loops on spheres when localised at certain primes (see Theorem 2.4). Let K be a simplicial complex with a decomposition as $K = K_1 \cup_L K_2$ where L is a full subcomplex of both K_1 and K_2. Closedness of $\prod \mathcal{P}$ under retractions is applied to show that if $\Omega(\underline{CA}, \underline{A})^{K_1} \in \prod \mathcal{P}$ and $\Omega(\underline{CA}, \underline{A})^{K_2} \in \prod \mathcal{P}$, then $\Omega(\underline{CA}, \underline{A})^{K} \in \prod \mathcal{P}$. This then allows us to prove the main result by an inductive argument.

The decomposition in Theorem 1.1 fits into a wider story related to loop space decompositions of spaces. Localise at a prime p. Given a space X, one may wish to find a decomposition of ΩX into a product of spaces, where each space in the product is indecomposable. Spheres S^n where $n \in \{1, 3, 7\}$ and loops on simply connected spheres ΩS^{2m+1}, where $m \geq 1$ are examples of indecomposable H-spaces. In a series of papers (CMN1; CMN2; CMN3), Cohen, Moore and Neisendorfer defined spaces $S^{2m+1}\{p^r\}$ and $T^{2m+1}\{p^r\}$ for $r \geq 1$ and $m \geq 1$, for which the loop space of a Moore space decomposes as a finite type product of these spaces. The spaces $S^{2m+1}\{p^r\}$ are indecomposable, and the spaces $T^{2m+1}\{p^r\}$ are indecomposable except for $T^3\{p\}$, in which case there is a homotopy equivalence $T^3\{p\} \simeq T^{2p+1}\{p\} \times U^1$, where U^1 is some indecomposable space. Anick (An) conjectured that if X is a finite, connected CW-complex, then localised at almost all primes p, ΩX decomposes as a finite type product of indecomposable spaces consisting of spheres, loops on simply connected spheres, $S^{2m+1}\{p^r\}$, $T^{2m+1}\{p^r\}$ and U^1. Theorem 1.1 verifies Anick's conjecture for polyhedral products $(\underline{CA}, \underline{A})^K$ where $\Sigma A_i \in \bigvee \mathcal{W}$ and K is the k-skeleton of a flag complex, and does so without the need to localise.

In Section 2, some preliminary results in linear algebra and homotopy theory that will be required are introduced. In Section 3, we prove that the retract of a space in $\prod \mathcal{P}$ is in $\prod \mathcal{P}$. In Section 4, this is applied to polyhedral products to prove Theorem 1.1.

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2 Preliminary Material

2.1 Idempotent Matrices

In this section, we state and prove the basic properties of idempotent matrices that will be required in Section 3. Denote by $M_n(\mathbb{Z})$ the set of n x n matrices with integer entries. A matrix $A \in M_n(\mathbb{Z})$ is *idempotent* if $A^2 = A$. Let $N(A)$ and $C(A)$ denote the null space and column space of A respectively. Recall that the null space and column space of a matrix is the kernel and image of the corresponding linear map. The following result gives a decomposition of $\mathbb{Z}^n$ in terms of the null space and column space of an idempotent matrix. This lemma is given as an exercise in (L, p. 163), so we provide a proof here.

Lemma 2.1. *Let $A \in M_n(\mathbb{Z})$ be an idempotent matrix. Then $\mathbb{Z}^n \cong N(A) \oplus C(A)$.*

Proof. If A is the zero matrix, then $C(A) = \{0\}$ and $\mathbb{Z}^n = N(A)$, and so the result holds in this case. Now suppose A is non-trivial. First, we show that

$\mathbb{Z}^n = N(A) + C(A)$. Clearly, $N(A)$ and $C(A)$ are subspaces of $\mathbb{Z}^n$, and so $N(A) + C(A) \subseteq \mathbb{Z}^n$. For the opposite inclusion, let $x \in \mathbb{Z}^n$. Write x as $x = Ax - (Ax - x)$. Applying A to $Ax - x$ and using the fact that A is idempotent, we obtain

$$A(Ax - x) = A^2x - Ax = Ax - Ax = 0.$$

Therefore, since $Ax \in C(A)$ and $Ax - x \in N(A)$, $\mathbb{Z}^n \subseteq N(A) + C(A)$. Hence, $\mathbb{Z}^n = N(A) + C(A)$.

Now we show that $N(A) \cap C(A) = \{0\}$. The zero vector is contained in $N(A)$ and $C(A)$. Let $x \in N(A) \cap C(A)$. Since $x \in C(A)$, there exists $x' \in \mathbb{Z}^n$ such that $Ax' = x$. Applying A to x and using the fact that $x \in N(A)$, we obtain

$$0 = Ax = A^2x' = Ax' = x.$$

Therefore, $N(A) \cap C(A) = \{0\}$ and so $\mathbb{Z}^n = N(A) \oplus C(A)$. □

The next result describes how an idempotent matrix acts on an element of the column space. The proof is immediate from the definition of an idempotent matrix.

Lemma 2.2. *Let $A \in M_n(\mathbb{Z})$ be an idempotent matrix and let $x \in C(A)$. Then $Ax = x$.* □

The final result describes the properties of the components of a vector $v \in \mathbb{Z}^n$ which extends to a basis of $\mathbb{Z}^n$. The result is clear from the contrapositive.

Lemma 2.3. *Let $v = (v_1, \cdots, v_n)^T \in \mathbb{Z}^n$ be a vector which extends to a basis of $\mathbb{Z}^n$. Then the greatest common divisor of the non-zero components $v_1, \cdots, v_n$ is 1. Moreover, one of $v_1, \cdots, v_n$ is odd.* □

2.2 Atomicity of loops on spheres

In this section, we recall the notion of atomic spaces. A simply connected topological space X is *atomic* (CMN3, Section 4) if any self map $f : X \to X$ inducing an isomorphism in the lowest non-vanishing degree in homology is a homotopy equivalence. A space X is *decomposable* if it is homotopy equivalent to a product $A \times B$ where A and B are not contractible. A space is *indecomposable* if it is not decomposable. The study of atomic spaces is useful since atomic spaces are indecomposable. In Section 3, we will be interested in the atomicity properties of ΩS^n. In particular, the following result is from (CPS, Corollary 5.2).

Theorem 2.4. *Let p be a prime, and let f be a self-map of $\Omega^k S^{m+1}$, $k < m$, which induces an isomorphism on the (least non-vanishing) homology group $H_{m+1-k}(\Omega^k S^{m+1}; \mathbb{Z}/p\mathbb{Z})$. If $p > 2$, we suppose that m is even, and if $p = 2$, we suppose $m \notin \{1, 3, 7\}$. Then f is a p-local homotopy equivalence.* □

Theorem 2.4 implies that localised at any prime p, ΩS^n is atomic for n odd, and when n is even and $n \notin \{1,3,7\}$, ΩS^n is only atomic when localised at 2. The following result of Serre (Se) shows that, localised at an odd prime, the loop space of an even dimensional sphere is decomposable.

Theorem 2.5. *Let p be an odd prime. There is a p-local homotopy equivalence*

$$\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}. \qquad \square$$

2.3 James-Hopf maps and Hopf invariants

In this section, we introduce the James-Hopf maps and prove basic properties of their induced map on homology that will be required in Section 3. All homology groups will be assumed to have integer coefficients unless otherwise stated.

Let X be a path-connected CW-complex such that $H_*(X)$ is torsion free. Let $E : X \to \Omega\Sigma X$ be the suspension map. The Bott-Samelson theorem implies that $H_*(\Omega\Sigma X) \cong T(\tilde{H}_*(X))$ where T is the tensor algebra functor. Moreover, E_* induces the inclusion of $\tilde{H}_*(X)$ into $T(\tilde{H}_*(X))$. Let $e : \Sigma\Omega\Sigma X \xrightarrow{e} \bigvee_{k\geq 1} \Sigma X^{\wedge k}$ be the James decomposition (J), where $X^{\wedge k}$ is the k-fold smash product of X with itself. The James-Hopf map $h_k : \Omega\Sigma X \to \Omega(\Sigma X^{\wedge k})$ is the adjoint of the composite

$$\overline{h}_k : \Sigma\Omega\Sigma X \xrightarrow{\simeq} \bigvee_{k\geq 1} \Sigma X^{\wedge k} \to \Sigma X^{\wedge k}$$

where the righthand map is the pinch map. The case that is applicable to Section 3 is $X = S^{n-1}$ and $k = 2$. In this case, h_2 is a map from ΩS^n to ΩS^{2n-1}, and we can describe the induced map $(h_2)_*$ on homology in degree $2n-2$. Note that $H_{2n-2}(\Omega S^n) \cong H_{2n-2}(\Omega S^{2n-1}) \cong \mathbb{Z}$.

Lemma 2.6. *The map*

$$(h_2)_* : H_{2n-2}(\Omega S^n) \to H_{2n-2}(\Omega S^{2n-1})$$

is an isomorphism. In particular, the generator $\delta \in H_{2n-2}(\Omega S^n)$ maps to $\tau \in H_{2n-2}(\Omega S^{2n-1})$, where τ is a generator.

Proof. Consider the composite

$$h_2' : \Sigma\Omega S^n \xrightarrow{\Sigma h_2} \Sigma\Omega S^{2n-1} \xrightarrow{ev} S^{2n-1}$$

where ev is the evaluation map. The map h_2' is homotopic to $\overline{h}_2$ since it is the adjoint of h_2. By definition of $\overline{h}_2$ as the composite of a homotopy equivalence followed by the pinch map, $(\overline{h}_2)_*$ sends a generator $\sigma\delta \in H_{2n-1}(\Sigma\Omega S^n)$ to a generator

$\tau' \in H_{2n-1}(S^{2n-1})$. Therefore, $(h_2')_*$ also sends $\sigma\delta$ to τ'. Since $H_{2n-1}(\Sigma\Omega S^n)$, $H_{2n-1}(\Sigma\Omega S^{2n-1})$ and $H_{2n-1}(S^{2n-1})$ are isomorphic to $\mathbb{Z}$, the only possibility is that $(\Sigma h_2)_*$ and ev_* are isomorphisms in degree $2n-1$. Therefore, $(\Sigma h_2)_*(\sigma\delta) = \sigma\tau$ where τ is a generator of $H_{2n-2}(\Omega S^{2n-1})$. From the homology suspension isomorphism, we obtain that $(h_2)_*(\delta) = \tau$. $\square$

Now consider the case where $X = S^{2n-1}$. For $m, k \geq 1$ and maps $f : S^m \to Z$ and $g : S^k \to Z$, denote the Whitehead product of f and g by $[f, g] : S^{m+k+1} \to Z$, and denote its adjoint, the Samelson product, by $\langle \tilde{f}, \tilde{g} \rangle : S^{m+k} \to \Omega Z$, where $\tilde{f}$ and $\tilde{g}$ are the adjoints of f and g respectively. In particular, let $id : S^{2n} \to S^{2n}$ be the identity map.

Lemma 2.7. *The map*

$$(\Omega[id, id])_* : H_{4n-2}(\Omega S^{4n-1}) \to H_{4n-2}(\Omega S^{2n})$$

sends a generator $\tau \in H_{4n-2}(\Omega S^{4n-1})$ to $2\delta \in H_{4n-2}(\Omega S^{2n})$ where δ is a generator of $H_{4n-2}(\Omega S^{2n})$.

Proof. Consider the diagram

$$\begin{tikzcd}
S^{4n-2} \arrow[d, "E"] \arrow[dr, "\langle E,E\rangle"] & \\
\Omega S^{4n-1} \arrow[r, "\Omega[id,id]"] & \Omega S^{2n}
\end{tikzcd}$$

where $E : S^{2n-1} \to \Omega S^{2n}$ is the suspension map. The diagram homotopy commutes since $\langle E, E \rangle$ is the adjoint of $[id, id]$. Since E induces the inclusion of the generator $\tau \in H_{4n-2}(\Omega S^{4n-1})$, its image under $(\Omega[id, id])_*$ is determined by its image under $\langle E, E \rangle$. The Samelson product commutes with homology in the sense that $\langle E, E \rangle_* = \langle E_*, E_* \rangle$ where the bracket on the right is the commutator in $H_*(\Omega S^{2n}) \cong T(\gamma)$. The map E induces the inclusion of the generator $\gamma \in H_{2n-1}(\Omega S^{2n})$, and so by definition of the commutator, $\langle E_*(\gamma), E_*(\gamma) \rangle = 2\delta$, where δ is a generator of $H_{4n-2}(\Omega S^{2n})$. $\square$

2.4 Hurewicz images

Let X be a space. An element $x \in H_n(X)$ is said to be in the Hurewicz image if it is in the image of the Hurewicz homomorphism. We will require the following result about the Hurewicz image of even dimensional spheres in a certain degree.

Lemma 2.8. *The Hurewicz image $\pi_{4n-2}(\Omega S^{2n}) \to H_{4n-2}(\Omega S^{2n})$ when $n \notin \{1, 2, 4\}$ is $2\mathbb{Z}$.*

Proof. Suppose that $f : S^{4n-2} \to \Omega S^{2n}$ is a map with odd Hurewicz image. By the universal property of the James construction, there exists an H-map $\overline{f} : \Omega S^{4n-1} \to \Omega S^{2n}$ such that

$$\begin{tikzcd} S^{4n-2} \arrow[d, "E"'] \arrow[dr, "f"] & \\ \Omega S^{4n-1} \arrow[r, "\overline{f}"] & \Omega S^{2n} \end{tikzcd}$$

homotopy commutes, where E is the suspension map. Let τ be a generator of $H_{4n-2}(\Omega S^{4n-1})$ and δ be a generator of $H_{4n-2}(\Omega S^{2n})$. By commutativity of the diagram and the fact that f has odd Hurewicz image, $\overline{f}$ sends τ to $(2k+1)\delta$ for some k. Consider the composite

$$\phi : \Omega S^{4n-1} \xrightarrow{\Delta} \Omega S^{4n-1} \times \Omega S^{4n-1} \xrightarrow{p_{-k} \times \overline{f}} \Omega S^{4n-1} \times \Omega S^{2n} \xrightarrow{\Omega[id,id] \times id} \Omega S^{2n} \times \Omega S^{2n} \xrightarrow{\mu} \Omega S^{2n},$$

where p_{-k} is the $-k$'th power map. By Lemma 2.7 and definition of ϕ, $\phi_*(\tau) = \delta$. Now consider the composite

$$\psi : \Omega S^{4n-1} \xrightarrow{\phi} \Omega S^{2n} \xrightarrow{h_2} \Omega S^{4n-1}.$$

By Lemma 2.6 and definition of ϕ, $\psi_*(\tau) = \tau$, and so ψ induces an isomorphism on $H_{4n-2}(\Omega S^{4n-1})$. By Theorem 2.4, ΩS^{4n-1} is 2-locally atomic. Therefore, we obtain that ψ is a 2-local homotopy equivalence, implying that ΩS^{4n-1} retracts off ΩS^{2n} when localised at 2. However, by Theorem 2.4, ΩS^{2n} is 2-locally atomic, and therefore indecomposable. Hence, ΩS^{2n} has no non-trivial retracts localised at the prime 2. □

2.5 Preliminary loop space decompositions

In this section, we state and prove some initial loop space decompositions which will be applied in Section 4. Let K be a simplicial complex on $[m]$ and let L be a full subcomplex of K on $[n]$. It is well known (see for example (DS, Lemma 2.2.3)) that the projection map $\prod_{i=1}^{m} X_i \to \prod_{j=1}^{n} X_j$ restricts to a map $(\underline{X}, \underline{A})^K \to (\underline{X}, \underline{A})^L$, which is a right inverse for the map $(\underline{X}, \underline{A})^L \to (\underline{X}, \underline{A})^K$. Note that a full subcomplex L' of L is also a full subcomplex of K, and this fact will often be used without comment.

There are two main results which will be used in Section 4. The first result was proved in (GT1, Theorem 7.2). If X and Y have basepoints x_0 and y_0 respectively, the *right half-smash* is defined by $X \rtimes Y = X \times Y/(* \times Y)$ and the *left half-smash* is defined by $X \ltimes Y = X \times Y/(X \times *)$. The *reduced join* is defined by $X * Y = (X \times I \times Y)/ \sim$, where I is the unit interval, $(x, 0, y) \sim (x, 0, y')$, $(x, 1, y) \sim (x', 1, y)$ and $(x_0, t, y_0) \sim (x_0, 0, y_0)$ for all $x, x' \in X$, $y, y' \in Y$ and $t \in I$.

Proposition 2.9. *Let K_1 be a simplicial complex on the vertex set $\{1, \cdots m\}$, K_2 a simplicial complex on the vertex set $\{\ell+1, \cdots, n\}$, and τ be a common face of K_1 and K_2 on the vertex set $\{\ell+1, \cdots, m\}$, where $\ell < m < n$. Then there is a homotopy equivalence*

$$(\underline{CA}, \underline{A})^{K_1 \cup_\tau K_2} \simeq (\mathcal{A} * \mathcal{A}') \vee ((\underline{CA}, \underline{A})^{K_1} \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes (\underline{CA}, \underline{A})^{K_2})$$

where $\mathcal{A} = \prod_{i=1}^{\ell} A_i$ and $\mathcal{A}' = \prod_{i=m+1}^{n} A_i$. □

The next main result is from (T1, Theorem 1.1).

Proposition 2.10. *Let K_1 be a simplicial complex on the vertex set $\{1, \cdots, m\}$, K_2 a simplicial complex on the vertex set $\{\ell+1, \cdots, n\}$, and L a full subcomplex of both K_1 and K_2 on the vertex set $\{\ell+1, \cdots, m\}$, where $\ell < m < n$. Then there is a homotopy fibration*

$$(\mathcal{A} * \mathcal{A}') \vee (G \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes H) \to (\underline{CA}, \underline{A})^{K_1 \cup_L K_2} \to (\underline{CA}, \underline{A})^{L}$$

where $\mathcal{A} = \prod_{i=1}^{\ell} A_i$, $\mathcal{A}' = \prod_{j=m+1}^{n} A_i$, and G and H are the homotopy fibres of the retractions $(\underline{CA}, \underline{A})^{K_1} \to (\underline{CA}, \underline{A})^{L}$ and $(\underline{CA}, \underline{A})^{K_2} \to (\underline{CA}, \underline{A})^{L}$ respectively. Further, this fibration splits after looping to give a homotopy equivalence

$$\Omega(\underline{CA}, \underline{A})^{K_1 \cup_L K_2} \simeq \Omega(\underline{CA}, \underline{A})^{L} \times \Omega((\mathcal{A} * \mathcal{A}') \vee (G \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes H)).$$ □

Remark 2.11. The loop of the decomposition in Proposition 2.9 can be obtained from Proposition 2.10. However, the proof of Proposition 2.10 requires that M is non-empty, whereas in Proposition 2.9, τ can be the empty set.

The aim of Section 4 is to use the decompositions in Proposition 2.9 and Proposition 2.10 to show that the property of loop spaces of polyhedral products being in $\prod \mathcal{P}$ is closed under taking pushouts of simplicial complexes over a common full subcomplex. First, we give a decomposition of $\Omega(X \ltimes Y)$ for spaces X and Y. Observe there is a projection map $X \ltimes Y \to Y$ given by projecting onto Y.

Lemma 2.12. *Let X and Y be path-connected, CW-complexes. Then there exists a homotopy fibration*

$$X * \Omega Y \to X \ltimes Y \to Y.$$

Furthermore, this splits after looping to give a homotopy equivalence

$$\Omega(X \ltimes Y) \simeq \Omega(X * \Omega Y) \times \Omega Y.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\Omega Y \arrow[d] & X \times \Omega Y \arrow[l, "\pi_{\Omega Y}"'] \arrow[r, "\pi_X"] \arrow[d, "\pi_X"] & X \arrow[d, "i_X"] \\
* \arrow[d] & X \arrow[l] \arrow[r, "i_X"] \arrow[d, "*"] & X \times Y \arrow[d, "\pi_Y"] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

where the columns are homotopy fibrations, i_X is the inclusion and π_X, π_Y and $\pi_{\Omega Y}$ are the projections onto X, Y and ΩY respectively. Observe that the homotopy pushout of the top row is $X * \Omega Y$, the homotopy pushout of the middle row is $X \ltimes Y$ and the induced map from $X \ltimes Y$ to Y is the projection map. Therefore by (F, p.180), there is a homotopy fibration

$$X * \Omega Y \to X \ltimes Y \to Y.$$

Moreover, the projection $X \ltimes Y \to Y$ has a right homotopy inverse given by the inclusion map $Y \hookrightarrow X \ltimes Y$, which implies that there is a homotopy equivalence

$$\Omega(X \ltimes Y) \simeq \Omega(X * \Omega Y) \times \Omega Y. \qquad \square$$

Before determining conditions on X and Y for $\Omega(X \ltimes Y)$ to be in $\prod \mathcal{P}$, we prove some relations between spaces in $\bigvee \mathcal{W}$ and spaces in $\prod \mathcal{P}$.

Lemma 2.13. *Let X be a space such that $\Sigma X \in \bigvee \mathcal{W}$ and let $A_1, \cdots, A_m$ be spaces in $\prod \mathcal{P}$, then*

$$\Sigma(X \wedge A_1 \wedge \cdots \wedge A_m) \in \bigvee \mathcal{W}.$$

Proof. We proceed by induction. First consider the case $m = 1$. Since $A_1 \in \prod \mathcal{P}$, $\Sigma A_1 \in \bigvee \mathcal{W}$. There is a homeomorphism $\Sigma(X \wedge A_1) \cong X \wedge \Sigma A_1$. Therefore, distributing the wedge sum over the smash product implies $\Sigma(X \wedge A_1) \in \bigvee \mathcal{W}$.

Now suppose the result is true for $1 \leq m \leq k-1$ and consider the case $m = k$. There are homeomorphisms

$$\Sigma(X \wedge A_1 \wedge \cdots \wedge A_m) \cong \Sigma(A_1 \wedge X \wedge A_2 \wedge \cdots \wedge A_m) \cong A_1 \wedge \Sigma(X \wedge A_2 \wedge \cdots \wedge A_m).$$

The inductive hypothesis implies $\Sigma(X \wedge A_2 \wedge \cdots \wedge A_m) \in \bigvee \mathcal{W}$. Therefore, $\Sigma(X \wedge A_2 \wedge \cdots \wedge A_m) \simeq \Sigma W$ where W is a wedge of spheres. Hence, there is a homotopy equivalence

$$\Sigma(X \wedge A_1 \wedge \cdots \wedge A_m) \simeq X \wedge \Sigma W.$$

Since $\Sigma X \in \bigvee \mathcal{W}$ by assumption, shifting the suspension coordinate and distributing the smash product over the wedge sum implies $X \wedge \Sigma W \in \bigvee \mathcal{W}$. $\square$

Lemma 2.14. *Let X and Y be path-connected CW-complexes such that $\Sigma X \in \bigvee \mathcal{W}$ and $\Omega Y \in \prod \mathcal{P}$. Then*

$$\Omega(X \ltimes Y) \in \prod \mathcal{P}.$$

Proof. By Lemma 2.12, $\Omega(X \ltimes Y) \simeq \Omega(X * \Omega Y) \times \Omega Y$. Since $X * \Omega Y \simeq \Sigma(X \wedge \Omega Y)$, $\Sigma X \in \bigvee \mathcal{W}$ and $\Omega Y \in \prod \mathcal{P}$, Lemma 2.13 implies $\Sigma(X \wedge \Omega Y) \in \bigvee \mathcal{W}$. Therefore the Hilton-Milnor theorem (M) implies $\Omega(\Sigma(X \wedge \Omega Y)) \in \prod \mathcal{P}$, and so $\Omega(X \ltimes Y) \in \prod \mathcal{P}$. $\square$

Now we state a result of Porter (P, Theorem 1) which gives a loop space decomposition of a wedge of spaces. Let X be a pointed space. Denote by $X^{\vee k}$ the k-fold wedge sum of X with itself.

Lemma 2.15. *Let $X_1, \cdots X_m$ be path-connected CW-complexes. Then there exists a homotopy fibration*

$$\bigvee_{k=2}^{m} \bigvee_{1 \leq i_1 < \cdots < i_k \leq m} (\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)} \to \bigvee_{i=1}^{m} X_i \hookrightarrow \prod_{i=1}^{m} X_i.$$

Moreover, this splits after looping. $\square$

Lemma 2.15 can be applied to show that if there are spaces X_i such that $\Omega X_i \in \prod \mathcal{P}$, then the loop space of the wedge of the X_i's is in $\prod \mathcal{P}$.

Corollary 2.16. *Let $X_1 \cdots, X_n$ be spaces such that $\Omega X_i \in \prod \mathcal{P}$. Then $\Omega\left(\bigvee_{i=1}^{n} X_i\right) \in \prod \mathcal{P}$.*

Proof. By Lemma 2.15, there is a homotopy equivalence

$$\Omega\left(\bigvee_{i=1}^{m} X_i\right) \simeq \prod_{i=1}^{m} \Omega X_i \times \Omega\left(\bigvee_{k=2}^{m} \bigvee_{1 \leq i_1 < \cdots < i_k \leq m} (\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)}\right).$$

The product $\prod_{i=1}^{m} \Omega X_i$ is in $\prod \mathcal{P}$ since $\prod \mathcal{P}$ is closed under products, so consider the complimentary factor. Since $\Omega X_{i_1}, \Sigma \Omega X_{i_1} \in \bigvee \mathcal{W}$. Lemma 2.13 then implies

$$\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k} \in \bigvee \mathcal{W},$$

and so

$$\bigvee_{k=2}^{m} \bigvee_{1 \leq i_1 < \cdots < i_k \leq m} (\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)} \in \bigvee \mathcal{W}.$$

Therefore, by the Hilton-Milnor Theorem

$$\Omega\left(\bigvee_{k=2}^{m} \bigvee_{1 \leq i_1 < \cdots < i_k \leq m} (\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k})^{\vee(k-1)}\right) \in \prod \mathcal{P}.$$

□

3 Closure of $\prod \mathcal{P}$ under retracts

3.1 Setup

In this section, homology will be assumed to have integer coefficients unless otherwise stated. Let $X \in \prod \mathcal{P}$, and suppose there is a space A which retracts off X, that is, there exist maps $f : A \to X$ and $g : X \to A$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, equal] & X \arrow[d, "g"] \\ & A \end{tikzcd}$$

homotopy commutes. In this section, we will show that A is homotopy equivalent to a subproduct of X.

The product decomposition of X implies there is a coalgebra isomorphism of $H_*(X)$ as a tensor product of exterior algebras corresponding to the spheres, and single-variable polynomial rings corresponding to the loops on spheres. Consider the composite $\phi : X \xrightarrow{g} A \xrightarrow{f} X$. Observe that $f \circ g \circ f \circ g \simeq f \circ g$ which implies that the induced map ϕ_* is an idempotent. To show that A is homotopy equivalent to a subproduct of X, we will proceed in three stages. First, we consider the case where $H_*(A)$ contains a primitive generator in degree m for $m \in \{1, 2, 3, 6, 7, 14, 4m \mid m \geq 1\}$, then we will consider the case where $H_*(A)$ contains a primitive generator in the Hurewicz image in degree $4m + 2$, where $m \geq 2$, $m \neq 3$, and we will conclude by considering the case where $H_*(A)$ contains a primitive generator in degree m where m is odd and $m \notin \{1, 3, 7\}$.

Each case requires an adaptation of the same core idea, and the notation defined in each subsection will be reused to reflect where the argument is the same. There is some notation that will be universal which we now define. Let $Y = S^n$ for $n \in \{1, 3, 7\}$ or $Y = \Omega S^m$ for $m \notin \{2, 4, 8\}$. Let m_Y be the number of instances of Y in the product decomposition of X. In particular, write X as

$$X \simeq \prod_{i=1}^{m_Y} Y_i \times \prod_{\alpha' \in \mathcal{I}'} Z_{\alpha'}$$

where each Y_i is an instance of Y in the product decomposition of X, and each $Z_{\alpha'}$ are the spheres and loops on spheres that are not equal to Y. Denote by H the lowest non-vanishing homology group of Y, and H_i the lowest non-vanishing homology group of Y_i. Note that $H \cong H_i \cong \mathbb{Z}$ for all i. Let γ_i be the primitive generator of $H_*(X)$

which is the image of a generator $\gamma_i' \in H_i$ under the map induced by the inclusion $Y_i \hookrightarrow X$. We will define maps $\rho_v : Y \to X$ and $\rho_v' : X \to Y$ such that the composite $Y \xrightarrow{\rho_v} X \xrightarrow{\phi} X \xrightarrow{\rho_v'} Y$ is a homotopy equivalence. Since ϕ factors through A and A is a H-space, this will imply that we obtain a homotopy equivalence $A \simeq Y \times A'$ for some space A'. An iterative approach will then be used to conclude that $A \in \prod \mathcal{P}$.

3.2 Case 1

In this subsection, we implicitly fix Y to be either $Y = S^n$ for $n \in \{1,3,7\}$, $Y = \Omega S^{4m+1}$ for $m \geq 2$, or $Y = \Omega S^{4m+3}$ for $m \in \{0,1,3\}$. We show that if the homology of A contains a primitive generator in the same degree as H, then Y retracts off A.

Observe that in this case, the set $\{\gamma_1, \cdots, \gamma_{m_Y}\}$ forms a basis of primitives in $H_n(X)$ if $Y = S^n$, $H_{4m}(X)$ if $Y = \Omega S^{4m+1}$, or $H_{4m+2}(X)$ if $Y = \Omega S^{4m+3}$. Since ϕ_* is a graded coalgebra map, it maps primitive elements to primitive elements of the same degree, and so $\phi_*(\gamma_i) = \sum_{j=1}^{m_Y} z_{i,j}\gamma_j$, where $z_{i,j} \in \mathbb{Z}$ for all j. Let $B_Y \in M_{m_Y}(\mathbb{Z})$ be the matrix with entries $z_{i,j}$. Since ϕ_* is an idempotent map, B_Y is an idempotent matrix.

Suppose B_Y is not the zero matrix. Since B_Y is idempotent, Lemma 2.1 implies that there exists an element $v = (y_1, \cdots, y_{m_Y})^T \in C(B_Y)$ which extends to a basis of $\mathbb{Z}^{m_Y}$. Therefore, by Lemma 2.3, the greatest common divisor of the non-zero components $y_1, \cdots, y_{m_Y}$ is 1. By Bézout's Lemma, for $1 \leq i \leq m_Y$, there exists $c_i \in \mathbb{Z}$ such that $\sum_{i=1}^{m_Y} c_i y_i = 1$. Since $v \in C(B_Y)$, Lemma 2.2 implies $B_Y v = v$. Let the vector v correspond to the element $\sum_{i=1}^{m_Y} y_i\gamma_i$ in $H_*(X)$.

Let $d_k : S^n \to S^n$ be the degree k map, and let $p_k : \Omega S^n \to \Omega S^n$ be the k^{th} power map. Note that d_k and p_k both induce multiplication by k in H (in this case of p_k, this follows from the Hurewicz theorem). Let ψ_k be d_k if Y is a sphere or p_k if Y is the loops on a sphere. Define a map $\rho_v : Y \to X$ as the composite

$$\rho_v : Y \xrightarrow{\Delta} \prod_{i=1}^{m_Y} Y_i \xrightarrow{\prod_{i=1}^{m_Y} \psi_{y_i}} \prod_{i=1}^{m_Y} Y_i \hookrightarrow X$$

where Δ is the diagonal map, and the right map is the inclusion. Now define a map $\rho_v' : X \to Y$ as the composite

$$\rho_v' : X \xrightarrow{\pi} \prod_{i=1}^{m_Y} Y_i \xrightarrow{\prod_{i=1}^{m_Y} \psi_{c_i}} \prod_{i=1}^{m_Y} Y_i \xrightarrow{\mu} Y$$

where π is the projection, μ is some choice of m_Y-fold H-space multiplication on Y, and the c_i's have the property that $\sum_{i=1}^{m_Y} c_i y_i = 1$.

Lemma 3.1. *Suppose that B_Y is not the zero matrix. Then, the composite*

$$e : Y \xrightarrow{\rho_v} X \xrightarrow{\phi} X \xrightarrow{\rho'_v} Y$$

induces an isomorphism on H.

Proof. By definition, $(\rho_v)_*$ sends the generator $\gamma \in H$ to the element v in $H_*(X)$. Since v is in the column space of B_Y, Lemma 2.2 implies that v is fixed by ϕ_*. By definition of ρ'_v, $(\rho'_v)_*$ sends v to the generator $\gamma \in H$. Therefore, e_* is an isomorphism on H. □

Lemma 3.1 allows us to conclude that e is a homotopy equivalence.

Lemma 3.2. *Let Y be a sphere S^m for $m \in \{1,3,7\}$, loops on a sphere of the form ΩS^{4m+1} for $m \geq 1$, or ΩS^{4m+3} for $m \in \{0,1,3\}$. Suppose that $H_*(A)$ contains a primitive generator in degree m if Y is a sphere, in degree $4m$ if $Y = \Omega S^{4m+1}$, or in degree $4m+2$ if ΩS^{4m+3}. Then Y retracts off A.*

Proof. Since $H_n(A)$ contains a primitive generator, by definition of ϕ, the matrix B_Y is non-zero. If Y is a sphere, then H is the only non-vanishing homology group of Y. As e_* is an isomorphism on H by Lemma 3.1 and Y is an H-space, e is a homotopy equivalence. The map ϕ factors through A, and so Y retracts off A.

If $Y = \Omega S^{4m+1}$ or ΩS^{4m+3}, then e induces an isomorphism on $H_k(Y)$, where $k = 4m$ if $Y = \Omega S^{4m+1}$ and $k = 4m+2$ if $Y = \Omega S^{4m+3}$. This implies that e induces an isomorphism on $H_k(Y; \mathbb{Z}/p\mathbb{Z})$ for all primes p and rationally. Therefore when localised at p or rationally, Theorem 2.4 implies e is a homotopy equivalence. Since e is a homotopy equivalence localised at every prime and rationally, e is an integral homotopy equivalence. Hence, Y retracts off A. □

From the previous lemma, we obtain the following result.

Proposition 3.3. *Let $X \in \prod \mathcal{P}$ and A be a space which retracts off X. Suppose that $H_*(A)$ contains a primitive generator in degree m where $m \in \{1,2,3,6,7,14,4m \mid m \geq 1\}$. Then there is a homotopy equivalence*

$$A \simeq Y \times A'$$

where $Y = S^m$ if $m \in \{1,3,7\}$, or $Y = \Omega S^{m+1}$ otherwise. Moreover, A' retracts off X, and $H_m(A')$ contains one fewer primitive generator than $H_m(A)$.

Proof. Since there is a primitive generator of degree m, the matrix B_Y is non-zero, and so Lemma 3.2 implies that Y retracts off A. This implies there is a map $r : A \to Y$

which has a right homotopy inverse. Let F be the homotopy fibre of g and consider the homotopy fibration diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & X' \arrow[r] \arrow[d] & A' \arrow[d] \\
F \arrow[r] & X \arrow[r, "g"] \arrow[d, "g \circ r"] & A \arrow[d, "r"] \\
& Y \arrow[r, equal] & Y
\end{tikzcd}$$

where A' and X' are the homotopy fibres of r and $g \circ r$ respectively. Since A retracts off X, it is an H-space. The right homotopy inverse for r implies there is a homotopy equivalence $A \simeq Y \times A'$. Observe that A' has the same homology as A except with one less primitive generator in degree m. Moreover, since g has a right homotopy inverse and X is an H-space, there are homotopy equivalences $X \simeq A \times F \simeq Y \times A' \times F$. Hence, A' retracts off X. $\square$

3.3 Case 2

In this subsection, fix Y to be ΩS^{4n+3} for $n \geq 2$, $n \neq 3$. We show that if the homology of A contains a primitive generator in $H_{4n+2}(A)$ which is in the Hurewicz image, then ΩS^{4n+3} retracts off A. In this case, the set $\{\gamma_1, \cdots, \gamma_{m_Y}\}$ does not form a basis of primitives in $H_{4n+2}(X)$, since there may be ΩS^{2n+2} terms in the product decomposition for X. Let $\overline{Y} = \Omega S^{2n+2}$. Write X as

$$X \simeq \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{2n+2} \times \prod_{\alpha' \in \mathcal{I}'} Z_{\alpha'}$$

where each $Z_{\alpha'}$ are the spheres and loops on spheres that are not equal to ΩS^{4n+3} or ΩS^{2n+2}. Let $\overline{\gamma}_i$ be the primitive generator of $H_*(X)$ which is the image of a generator $\overline{\gamma}_i' \in H_{4n+2}(\Omega S^{2n+2})$ under the map induced by the inclusion $\Omega S_i^{2n+2} \hookrightarrow X$. The set $\{\gamma_1, \cdots, \gamma_{m_Y}, \overline{\gamma}_1, \cdots, \overline{\gamma}_{m_{\overline{Y}}}\}$ forms a basis of primitives in $H_{4n+2}(X)$.

Consider a primitive generator $a \in H_{4n+2}(A)$ such that a is in the Hurewicz image. Observe that $f_*(a)$ is primitive, in the Hurewicz image, and since f_* is injective, $f_*(a)$ is non-zero. By Lemma 2.8, $f_*(a) = \sum_{i=1}^{m_Y} y_i \gamma_i + \sum_{j=1}^{m_{\overline{Y}}} 2\overline{y}_j \overline{\gamma}_j$. Let $v = (y_1, \cdots, y_{m_Y}, 2\overline{y}_1, \cdots, 2\overline{y}_{m_{\overline{Y}}})$ correspond to the element $f_*(a)$. By definition of ϕ, $\mathrm{im}(\phi_*) = \mathrm{im}(f_*)$, and $f_*(a)$ is a generator of $\mathrm{im}(\phi_*)$. Therefore, by Lemma 2.3, the greatest common divisor of the components of v is 1. By Bézout's Lemma, for $1 \leq i \leq m_Y$ and $1 \leq j \leq m_{\overline{Y}}$, there exists $c_i, \overline{c}_j \in \mathbb{Z}$ such that $\sum_{i=1}^{m_Y} c_i y_i + \sum_{j=1}^{m_{\overline{Y}}} 2\overline{c}_j \overline{y}_j = 1$. Since $v \in \mathrm{im}(\phi)$, Lemma 2.2 implies $\phi_*(f_*(a)) = f_*(a)$.

Let λ_k be the composite

$$\lambda_k : \Omega S^{4n+3} \xrightarrow{\Omega[id,id]} \Omega S^{2n+2} \xrightarrow{p_k} \Omega S^{2n+2}.$$

By Lemma 2.7, λ_k maps γ to $2k\overline{\gamma}$. Define a map $\rho_v : \Omega S^{4n+3} \to X$ as the composite

$$\rho_v : \Omega S^{4n+3} \xrightarrow{\Delta} \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n+3} \xrightarrow{\prod_{i=1}^{m_Y} p_{y_i} \times \prod_{j=1}^{m_{\overline{Y}}} \lambda_{\overline{y}_j}} \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{2n+2} \hookrightarrow X$$

where Δ is the diagonal map, and the right map is the inclusion. Now define a map $\rho_v' : X \to \Omega S^{4n+3}$ as the composite

$$\rho_v' : X \xrightarrow{\pi} \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{2n+2} \xrightarrow{\prod_{i=1}^{m_Y} id \times \prod_{j=1}^{m_{\overline{Y}}} h_2} \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n+3}$$

$$\xrightarrow{\prod_{i=1}^{m_Y} p_{c_i} \times \prod_{j=1}^{m_{\overline{Y}}} p_{\overline{c}_j}} \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n+3} \xrightarrow{\mu} \Omega S^{4n+3}$$

where π is the projection, μ is some choice of m_Y-fold H-space multiplication on ΩS^{4n+3}, and the c_i's and $\overline{c}_j$'s have the property that $\sum_{i=1}^{m_Y} c_i y_i + \sum_{j=1}^{m_{\overline{Y}}} 2\overline{c}_j \overline{y}_j = 1$. By definition, $(\rho_v)_*$ sends the generator $\gamma \in H$ to the element v in $H_*(X)$, and by definition of ρ_v', $(\rho_v')_*$ sends v to the generator $\gamma \in H$. Therefore, arguing as in Lemma 3.1, Lemma 3.2 and Proposition 3.3, we obtain the following.

Proposition 3.4. *Let $X \in \prod \mathcal{P}$ and A be a space which retracts off X. Suppose $H_{4n+2}(A)$, $n \geq 2$, $n \neq 3$, contains a primitive generator in the Hurewicz image. Then there is a homotopy equivalence*

$$A \simeq \Omega S^{4n+3} \times A'$$

where A' retracts off X, and $H_{4n+2}(A')$ contains one fewer primitive generator in the Hurewicz image than $H_{4n+2}(A)$. □

3.4 Case 3

In this subsection, fix Y to be ΩS^{2n} for $n \notin \{1, 2, 4\}$. We show that if the homology of A contains a primitive generator in $H_{2n-1}(A)$, then ΩS^{2n} retracts off A. Observe that in this case, the set $\{\gamma_1, \cdots, \gamma_{m_Y}\}$ forms a basis of primitives in $H_{2n-1}(X)$. Therefore, if $H_{2n-1}(A)$ contains a primitive generator, arguing as in Subsection 3.2, we obtain

1. a generator of $\mathrm{im}(\phi_*)$, $\sum_{i=1}^{m_Y} y_i \gamma_i \in H_{2n-1}(X)$,
2. a vector $v = (y_1, \cdots, y_{m_Y})$ corresponding to the generator in (1) such that the greatest common divisor of the non-zero components is 1;

3. a composite
$$\rho_v : \Omega S^{2n} \xrightarrow{\Delta} \prod_{i=1}^{m_Y} \Omega S_i^{2n} \xrightarrow{p_{y_i}} \prod_{i=1}^{m_Y} \Omega S_i^{2n} \hookrightarrow X$$
such that a generator $\gamma \in H_{2n-1}(\Omega S^{2n})$ maps to v;

4. a composite
$$\rho_v' : X \xrightarrow{\pi} \prod_{i=1}^{m_Y} \Omega S_i^{2n} \xrightarrow{\prod_{i=1}^{m_Y} p_{c_i}} \prod_{i=1}^{m_Y} \Omega S_i^{2n} \xrightarrow{\mu} \Omega S^{2n}$$
where π is the projection, μ is some choice of m_Y-fold H-space multiplication on ΩS^{2n}, the c_i's have the property that $\sum_{i=1}^{m_Y} c_i y_i = 1$, and the map $(\rho_v')_*$ maps v to γ;

5. the composite $e : (\rho_v')_* \circ \phi_* \circ (\rho_v)_*$ is an isomorphism on $H_{2n-1}(\Omega S^{2n})$.

In Subsection 3.2, this was enough to conclude that the corresponding loop on sphere retracted off of A. In this case, e may not be a homotopy equivalence since ΩS^{2n} is not atomic at odd primes. However, we can adjust the maps ρ_v and ρ_v' to define a map $\overline{e}'$ which is a homotopy equivalence. In particular, these maps will agree with ρ_v and ρ_v' respectively in degree $2n-1$, but may differ in degree $4n-2$ so that $\overline{e}'$ is an isomorphism on both $H_{2n-1}(\Omega S^{2n})$ and $H_{4n-2}(\Omega S^{2n})$.

Since the generators of $H_*(\Omega S^{2n})$ in degree $2n-1$ and $4n-2$ are divisors of elements in the Hurewicz image, the map induced by the k^{th} power map sends a generator $\gamma \in H_{2n-1}(\Omega S^{2n})$ to $k\gamma$, and a generator $\delta \in H_{4n-2}(\Omega S^{2n})$ to $k\delta$. Recall that $v = (y_1, \cdots, y_{m_Y})^T$, and $(\rho_v)_*$ maps a generator $\gamma \in H_{2n-1}(\Omega S^{2n})$ to v in $H_*(X)$. Let δ_i be the primitive generator of $H_{4n-2}(X)$ which is the image of a generator $\delta_i' \in H_{4n-2}(\Omega S_i^{2n})$ under the map induced by the inclusion $\Omega S_i^{2n} \hookrightarrow X$. By definition of ρ_v, for a suitable choice of generator $\delta \in H_{4n-2}(\Omega S^{2n})$, $(\rho_v)_*$ maps δ to the element $\sum_{i=1}^{m_Y} y_i \delta_i$.

Let $\overline{Y} = \Omega S^{4n-1}$. Write X as
$$X \simeq \prod_{i=1}^{m_Y} \Omega S_i^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n-1} \times \prod_{\alpha' \in \mathcal{I}'} Z_{\alpha'}$$
where each $Z_{\alpha'}$ are the spheres and loops on spheres that are not equal to ΩS^{2n} or ΩS^{4n-1}. Let $\overline{\delta}_i$ be the primitive generator of $H_{4n-2}(X)$ which is the image of a generator $\overline{\delta}_i' \in H_{4n-2}(\Omega S_i^{4n-1})$ under the map induced by the inclusion $\Omega S_i^{4n-1} \hookrightarrow X$. Observe that the set $\{\delta_1, \cdots, \delta_{m_Y}, \overline{\delta}_1, \cdots, \overline{\delta}_{m_{\overline{Y}}}\}$ forms a basis of the primitives in $H_{4n-2}(X)$. Let $w = (y_1, \cdots, y_{m_Y}, 0, \cdots, 0)^T$ be the vector defined by taking the coefficients of $\sum_{i=1}^{m_Y} y_i \delta_i$. Since w is primitive, ϕ_* maps w to an element of the form $\sum_{i=1}^{m_Y} y_i' \delta_i + \sum_{j=1}^{m_{\overline{Y}}} \overline{y}_i \overline{\delta}_j$. Let $w' = (y_1', \cdots, y_{m_Y}', \overline{y}_1, \cdots, \overline{y}_{m_{\overline{Y}}})^T$ be the vector containing the

coefficients of $\sum_{i=1}^{m_Y} y_i'\delta_i + \sum_{j=1}^{m_{\overline{Y}}} \overline{y}_i\overline{\delta}_j$. The components of w' can be related to the components of v.

Lemma 3.5. *The components y_i are equal to y_i' modulo 2. Moreover, there exists $1 \leq j \leq m_Y$ such that y_j is odd.*

Proof. Since the vector v extends to a basis of $\mathbb{Z}^{m_Y}$, by Lemma 2.3, at least one of $y_1, \cdots, y_m$ must be odd. Consider a component y_i of v which is odd, and consider the composite

$$\psi : \Omega S^{2n} \xrightarrow{\rho_v} X \xrightarrow{\phi} X \xrightarrow{\pi_i} \Omega S_i^{2n}$$

where π_i is the projection map. By definition, ψ_* sends the generator $\gamma \in H_{2n-1}(\Omega S^{2n})$ to $y_i\gamma$, and the generator $\delta \in H_{4n-2}(\Omega S^{2n})$ to $y_i'\delta$. Since y_i is odd, ψ_* is an isomorphism in degree $2n-1$, and so by Theorem 2.4, ψ is a 2-local homotopy equivalence. This implies that integrally, y_i' must be odd and so $y_i' = y_i + 2k_i$ for some $k_i \in \mathbb{Z}$, and this holds for all i for which y_i is odd.

Now fix a component y_i of v which is odd, and consider a component y_j which is even. Consider the composite

$$\psi' : \Omega S^{2n} \xrightarrow{\rho_v} X \xrightarrow{\phi} X \xrightarrow{\pi_{i,j}} \Omega S_i^{2n} \times \Omega S_j^{2n} \xrightarrow{\mu} \Omega S^{2n}$$

where $\pi_{i,j}$ is the projection onto $\Omega S_i^{2n} \times \Omega S_j^{2n}$ and μ is the loop space multiplication. By definition, $(\psi')_*$ sends the generator $\gamma \in H_{2n-1}(\Omega S^{2n})$ to $(y_i + y_j)\gamma$ and the generator $\delta \in H_{4n-2}(\Omega S^{2n})$ to $(y_i' + y_j')\delta$. Since y_i is odd and y_j is even, $y_i + y_j$ is odd. Therefore, $(\psi')_*$ on $H_{2n-1}(\Omega S^{2n}; \mathbb{Z}/2\mathbb{Z})$ is an isomorphism, and so by Theorem 2.4, ψ' is a 2-local homotopy equivalence. Therefore, $y_i' + y_j'$ must be odd. As y_i is odd by assumption, y_i' is odd by the previous paragraph, which implies that y_j' must be even. This implies that $y_j' = y_j + 2k_j$ for some $k_j \in \mathbb{Z}$, and this holds for all j for which y_j is even. □

Lemma 3.5 implies that $w' = (y_1 + 2k_1, \cdots, y_{m_Y} + 2k_{m_Y}, \overline{y}_1, \cdots, \overline{y}_{m_{\overline{Y}}})$ for $k_1, \cdots, k_{m_Y} \in \mathbb{Z}$. The next result shows that there exists a vector $\overline{w} \in H_{4n-2}(X)$ with similar properties to v. For two vectors $u = (x_1, \cdots, x_n)^T$ and $u' = (x_1', \cdots, x_n')^T$, we say that u and u' are equal modulo 2 if $x_i \cong x_i'$ mod 2 for all i.

Lemma 3.6. *There exists a vector $\overline{w}$ which is equal to w' modulo 2, and whose non-zero components have greatest common divisor 1. Moreover, $\phi_*(\overline{w}) = \overline{w}$.*

Proof. Recall that $y_i + 2k_i = y_i'$. Let d be the greatest common divisor of the non-zero components of $y_1 + 2k_1, \cdots, y_{m_Y} + 2k_{m_Y}, \overline{y}_1, \cdots, \overline{y}_{m_{\overline{Y}}}$. Then $w' = d\overline{w}$ for some vector $\overline{w}$. Lemma 2.3 implies that one of $y_1, \cdots, y_{m_Y}$ is odd, and so it follows that d is odd. By definition of $\overline{w}$, the greatest common divisor of the non-zero components of $\overline{w}$ is 1. Since ϕ_* is an idempotent map,

$$d\overline{w} = w' = \phi_*(w') = d\phi_*(\overline{w}),$$

which implies that $\phi_*(\overline{w}) = \overline{w}$. Moreover, since d is odd, it follows that $\overline{w}$ is equal to w' modulo 2. $\square$

Lemma 3.6 implies there exists a vector
$\overline{w} = (y_1 + 2k'_1, \cdots, y_{m_Y} + 2k'_{m_Y}, \overline{y}_1 + 2\overline{k}_1, \cdots, \overline{y}_{m_{\overline{Y}}} + 2\overline{k}_{m_{\overline{Y}}})$ for
$k'_1, \cdots, k'_{m_Y}, \overline{k}_1, \cdots, \overline{k}_{m_{\overline{Y}}} \in \mathbb{Z}$ such that the greatest common divisor of

$$y_1 + 2k'_1, \cdots, y_{m_Y} + 2k'_{m_Y}, \overline{y}_1 + 2\overline{k}_1, \cdots, \overline{y}_{m_{\overline{Y}}} + 2\overline{k}_{m_{\overline{Y}}}$$

is 1. For an integer k, let $\lambda_k : \Omega S^{4n-1} \to \Omega S^{2n}$ be $p_k \circ \Omega[id, id]$, where p_k is the k^{th} power map and $id : S^{2n} \to S^{2n}$ is the identity map. By Lemma 2.7, $(\Omega[id, id])_*$ sends a generator $\tau \in H_{4n-2}(\Omega S^{4n-1})$ to the element $2\delta \in H_{4n-2}(\Omega S^{2n})$ where δ is a generator of $H_{4n-2}(\Omega S^{2n})$. Therefore by definition of λ_k, $(\lambda_k)_*$ sends τ to $2k\delta$. Let $h_2 : \Omega S^{2n} \to \Omega S^{4n-1}$ be the 2^{nd} James-Hopf invariant. By Lemma 2.6, $(h_2)_*$ sends δ to τ. For an integer k, let η_k be the composite

$$\eta_k : \Omega S^{2n} \xrightarrow{h_2} \Omega S^{4n-1} \xrightarrow{\lambda_k} \Omega S^{2n}.$$

The map $(\eta_k)_*$ is trivial on $H_{2n-1}(\Omega S^{2n})$ and sends δ to $2k\delta$. We adjust the map ρ_v by defining the map $\overline{\rho}_v$ as the composite

$$\overline{\rho}_v : \Omega S^{2n} \xrightarrow{\Delta} \prod_{i=1}^{m_Y} \Omega S^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S^{2n} \xrightarrow{\prod_{i=1}^{m_Y} \Delta \times \prod_{j=1}^{m_{\overline{Y}}} h_2} \prod_{i=1}^{m_Y} (\Omega S^{2n} \times \Omega S^{2n}) \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S^{4n-1}$$

$$\xrightarrow{\prod_{i=1}^{m_Y} (p_{y_i} \times \eta_{k'_i}) \times \prod_{j=1}^{m_{\overline{Y}}} p_{\overline{y}_j + 2\overline{k}_j}} \prod_{i=1}^{m_Y} (\Omega S^{2n} \times \Omega S^{2n}) \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S^{4n-1} \xrightarrow{\prod_{i=1}^{m_Y} \mu \times \prod_{j=1}^{m_{\overline{Y}}} id} \prod_{i=1}^{m_Y} \Omega S^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S^{4n-1} \hookrightarrow X.$$

By definition, $\overline{\rho}_v$ sends the generator $\gamma \in H_{2n-1}(\Omega S^{2n})$ to v, and the generator $\delta \in H_{4n-2}(\Omega S^{2n})$ to $\overline{w}$.

Lemma 3.7. *Suppose that* $H_{2n-1}(A)$, $n \notin \{1, 2, 4\}$ *contains a primitive generator. Then the composite*

$$\overline{e} : \Omega S^{2n} \xrightarrow{\overline{\rho}_v} X \xrightarrow{\phi} X \xrightarrow{\rho'_v} \Omega S^{2n}$$

induces an isomorphism on $H_{2n-1}(\Omega S^{2n})$. *Moreover,* $(\overline{e})_*$ *maps the generator* $\delta \in H_{4n-2}(\Omega S^{2n})$ *to an element of the form* $(2\overline{k}' + 1)\delta$ *for some* $\overline{k}' \in \mathbb{Z}$.

Proof. The map $(\eta_k)_*$ is trivial on H, and so $(\overline{\rho}_v)_*$ maps the generator $\gamma \in H$ to v. The element v is fixed by ϕ_*. The map $(\rho'_v)_*$ sends v to $\gamma \in H$. Therefore, $\overline{e}$ induces an isomorphism on $H_{2n-1}(\Omega S^{2n})$. By Theorem 2.4, the map $\overline{e}$ is a 2-local homotopy equivalence. Hence, $(\overline{e})_*$ maps $\delta \in H_{4n-2}(\Omega S^{2n})$ to an element of the form $(2\overline{k}+1)\delta$ for some $\overline{k} \in \mathbb{Z}$. $\square$

Now we adjust ρ'_v to obtain an isomorphism on the bottom two non-vanishing degrees in homology. Recall $\overline{w} = (y_1 + 2k'_1, \cdots, y_{m_Y} + 2k'_{m_Y}, \overline{y}_1 + 2\overline{k}_1, \cdots, \overline{y}_{m_{\overline{Y}}} + 2\overline{k}_{m_{\overline{Y}}})$, and the greatest common divisor of the non-zero components of $\overline{w}$ is 1. Therefore, by Bézout's Lemma, for $1 \leq i \leq m_Y$, and $1 \leq j \leq m_{\overline{Y}}$ there exist $c'_i, \overline{c}_j \in \mathbb{Z}$ such that $\sum_{i=1}^{m_Y} c'_i(y_i + 2k'_i) + \sum_{j=1}^{m_{\overline{Y}}} \overline{c}_j(\overline{y}_j + 2\overline{k}_j) = 1$. Let λ' be the composite

$$\lambda' : \prod_{i=1}^{m_Y} \Omega S^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S^{4n-1} \xrightarrow{\prod_{i=1}^{m_Y} h_2 \times \prod_{j=1}^{m_{\overline{Y}}} id} \prod_{i=1}^{m_Y} \Omega S^{4n-1} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S^{4n-1}$$

$$\xrightarrow{\prod_{i=1}^{m_Y} p_{c'_i} \times \prod_{j=1}^{m_{\overline{Y}}} p_{\overline{c}_j}} \prod_{i=1}^{m_Y} \Omega S^{4n-1} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S^{4n-1} \xrightarrow{\mu} \Omega S^{4n-1} \xrightarrow{\lambda_{-\overline{k}'}} \Omega S^{2n}.$$

By definition, λ' sends the element $\overline{w}$ in $H_{4n-2}\left(\prod_{i=1}^{m_Y} \Omega S_i^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n-1}\right)$ to $-2\overline{k}\delta$, and is trivial in degree $2n-1$. Now adjust the map ρ'_v by defining a map $\overline{\rho}'_v$ as the composite

$$\overline{\rho}'_v : X \xrightarrow{\pi} \prod_{i=1}^{m_Y} \Omega S_i^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n-1} \xrightarrow{\Delta} \left(\prod_{i=1}^{m_Y} \Omega S_i^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n-1}\right) \times \left(\prod_{i=1}^{m_Y} \Omega S_i^{2n} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{4n-1}\right)$$

$$\xrightarrow{\pi \times \lambda'} \left(\prod_{i=1}^{m_Y} \Omega S_i^{2n}\right) \times \Omega S^{2n} \xrightarrow{\rho'_v \times id} \Omega S^{2n} \times \Omega S^{2n} \xrightarrow{\mu} \Omega S^{2n}.$$

Using $\overline{\rho}_v$ and $\overline{\rho}'_v$, we can now conclude that ΩS^{2n} retracts off A when Y.

Lemma 3.8. *Suppose that $H_{2n-1}(A)$ contains a primitive generator where $n \notin \{1, 2, 4\}$. Then ΩS^{2n} retracts off A.*

Proof. By definition of $\overline{\rho}_v$, the induced map $(\overline{\rho}_v)_*$, sends the generator $\gamma \in H_{2n-1}(\Omega S^{2n})$ to v, and the generator $\delta \in H_{4n-2}(\Omega S^{2n})$ to the element $\overline{w}$. By construction, the induced map $\overline{\rho}'_v$ maps the element v to the generator $\gamma \in H_{2n-1}(\Omega S^{2n})$, and maps the element $\overline{w}$ to the generator $\delta \in H_{4n-2}(\Omega S^{2n})$.

Therefore since v and $\overline{w}$ are fixed by ϕ_*, the composite

$$\overline{e}' : \Omega S^{2n} \xrightarrow{\overline{\rho}_v} X \xrightarrow{\phi} X \xrightarrow{\overline{\rho}'_v} \Omega S^{2n}$$

induces an isomorphism on $H_{2n-1}(\Omega S^{2n})$ and $H_{4n-2}(\Omega S^{2n})$. By Theorem 2.4 at the prime 2, the map $\bar{e}'$ is a 2-local homotopy equivalence. Therefore, $\bar{e}'$ is also a rational homotopy equivalence. The splitting of ΩS^{2n} in Theorem 2.5 and atomicity of loops on odd spheres localised at an odd prime in Theorem 2.4 implies that $\bar{e}'$ is a homotopy equivalence when localised at any odd prime. Since $\bar{e}'$ is a homotopy equivalence localised at every prime and rationally, $\bar{e}'$ is an integral homotopy equivalence. Therefore, since ϕ factors through A, ΩS^{2n} retracts off A. □

Now arguing as in Proposition 3.3, we obtain the following result.

Proposition 3.9. *Let $X \in \prod \mathcal{P}$ and A be a space which retracts off X. Suppose that $H_{2n-1}(A)$ contains a primitive generator where $n \notin \{1,2,4\}$. Then there is a homotopy equivalence*

$$A \simeq \Omega S^{2n} \times A'$$

where A' retracts off X and $H_{2n-1}(A')$ contains one fewer primitive generator than $H_{2n-1}(A)$. □

3.5 Conclusion of proof

We can combine the work of the previous sections to conclude that $\prod \mathcal{P}$ is closed under retracts.

Theorem 3.10. *Let $X \in \prod \mathcal{P}$, and let A be a space which retracts off X. Then $A \in \prod \mathcal{P}$.*

Proof. Let n_0 be the degree of the lowest non-trivial homology group of A, and let k_0 be the rank of $H_{n_0}(A)$. Observe that since n_0 is the lowest non-trivial degree, each of the primitive generators of $H_{n_0}(A)$ are in the Hurewicz image. Let $Y = S^{n_0}$ if $n_0 \in \{1,3,7\}$ or $Y = \Omega S^{n_0+1}$ otherwise. We claim that $A \simeq \prod_{i=1}^{k_0} Y \times Z_0$ where Z_0 has no primitive generators in degree n_0, Z_0 retracts off X.

We proceed by induction. Suppose $k_0 = 1$. Then by Proposition 3.3, Proposition 3.4 or Proposition 3.9 (depending on the parity of n_0), there is a homotopy equivalence

$$A \simeq Y \times Z_0$$

where Z_0 retracts off X, and $H_{n_0}(Z_0)$ contains one fewer primitive generator than $H_{n_0}(Z_0)$. Since $k_0 = 1$, Z_0 contains no primitive generators in degree n_0.

Now suppose the result is true for $m-1$ and suppose $k_0 = m$. Then by Proposition 3.3, Proposition 3.4 or Proposition 3.9, there is a homotopy equivalence

$$A \simeq Y \times A'$$

where A' retracts off X, and $H_{n_0}(A')$ contains one fewer primitive generator than $H_{n_0}(A')$. Therefore, the inductive hypothesis implies that there is a homotopy equivalence $A' \simeq \prod_{i=1}^{m-1} Y \times Z_0$, where Z_0 retracts off X and has no primitive generators in degree n_0. Hence $A \simeq \prod_{i=1}^{m} Y \times Z_0$ as claimed.

Observe that Z_0 is more highly connected than A. Let n_1 be the degree of the lowest non-trivial homology group of Z, and let k_1 be the rank of $H_{n_1}(Z_0)$. We can repeat this argument to obtain a homotopy equivalence $Z_0 \simeq P \times Z_1$ where P is a product of spheres or loops on spheres whose lowest non-trivial homology group is n_1, Z_1 retracts off X, and Z_1 is more highly connected than Z_0. Since A is of finite type, we can iteratively repeat this argument for each degree of $H_*(A)$ containing a primitive generator. Therefore, we obtain that $A \in \prod \mathcal{P}$. □

4 Loop space decompositions of pushouts of polyhedral products as a product of spheres and loops on spheres

Recall from the introduction that $\bigvee \mathcal{W}$ is the collection of topological spaces which are homotopy equivalent to a finite type wedge of simply connected spheres, and $\prod \mathcal{P}$ is the collection of H-spaces which are homotopy equivalent to a finite type product of spheres and loops on simply connected spheres. The purpose of this section is to apply Theorem 3.10 to prove that under mild hypotheses, if a simplicial complex K can be decomposed as a pushout of simplicial complexes for which the loop space of the associated polyhedral product is in $\prod \mathcal{P}$, then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.

Theorem 4.1. *Let K be a simplicial complex defined as the pushout*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & K_1 \arrow[d] \\ K_2 \arrow[r] & K \end{tikzcd}$$

where either $L = \emptyset$ or L is a proper full subcomplex of K_1 and K_2. If $\Sigma A_i \in \bigvee \mathcal{W}$ for all i, $\Omega(\underline{CA}, \underline{A})^{K_1} \in \prod \mathcal{P}$ and $\Omega(\underline{CA}, \underline{A})^{K_2} \in \prod \mathcal{P}$, then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.

Proof. If L is the empty set, since $K = K_1 \cup_L K_2$, by Proposition 2.9, we obtain a homotopy equivalence

$$\Omega(\underline{CA}, \underline{A})^K \simeq \Omega((\mathcal{A} * \mathcal{A}') \vee ((\underline{CA}, \underline{A})^{K_1} \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes (\underline{CA}, \underline{A})^{K_2})) \tag{4.1}$$

where $\mathcal{A}$ and $\mathcal{A}'$ are a product of A_i's. If L is a full subcomplex of K_1 and K_2, since $K = K_1 \cup_L K_2$, the simplicial complex K satisfies the hypothesis of Proposition 2.10, so there is a homotopy equivalence

$$\Omega(\underline{CA},\underline{A})^K \simeq \Omega(\underline{CA},\underline{A})^L \times \Omega\left((\mathcal{A} * \mathcal{A}') \vee (G \rtimes \mathcal{A}') \vee (\mathcal{A} \ltimes H)\right) \tag{4.2}$$

where $\mathcal{A}$ and $\mathcal{A}'$ are a product of A_i's, and G and H are the homotopy fibres of the retractions $f_1 : (\underline{CA},\underline{A})^{K_1} \to (\underline{CA},\underline{A})^L$ and $f_2 : (\underline{CA},\underline{A})^{K_2} \to (\underline{CA},\underline{A})^L$ respectively. Since $\Omega(\underline{CA},\underline{A})^L$ retracts off $\Omega(\underline{CA},\underline{A})^{K_1}$, Theorem 3.10 implies that $\Omega(\underline{CA},\underline{A})^L \in \prod\mathcal{P}$. By Corollary 2.16, to show that the decompositions in (4.1) and (4.2) are in $\prod\mathcal{P}$, it suffices to show that each of $\Omega(\mathcal{A} * \mathcal{A}')$, $\Omega((\underline{CA},\underline{A})^{K_1} \rtimes \mathcal{A}')$, $\Omega(\mathcal{A} \ltimes (\underline{CA},\underline{A})^{K_2})$ are in $\prod\mathcal{P}$ for (4.1), and additionally $\Omega(G \rtimes \mathcal{A}')$ and $\Omega(\mathcal{A} \ltimes H)$ are in $\prod\mathcal{P}$ for (4.2).

Since $\mathcal{A}$ and $\mathcal{A}'$ are products of A's and $\Sigma A \in \bigvee\mathcal{W}$ by assumption, it follows that $\Sigma\mathcal{A} \in \bigvee\mathcal{W}$, $\Sigma\mathcal{A}' \in \bigvee\mathcal{W}$ and $\mathcal{A} * \mathcal{A}' \in \bigvee\mathcal{W}$. Therefore, the Hilton-Milnor theorem implies that $\Omega(\mathcal{A} * \mathcal{A}') \in \prod\mathcal{P}$. Since $\Omega(\underline{CA},\underline{A})^{K_1} \in \prod\mathcal{P}$ and $\Omega(\underline{CA},\underline{A})^{K_2} \in \prod\mathcal{P}$ by hypothesis, and $\Sigma\mathcal{A}$, $\Sigma\mathcal{A}' \in \bigvee\mathcal{W}$, by Lemma 2.14, $\Omega((\underline{CA},\underline{A})^{K_1} \rtimes \mathcal{A}') \in \prod\mathcal{P}$ and $\Omega(\mathcal{A} \ltimes (\underline{CA},\underline{A})^{K_2}) \in \prod\mathcal{P}$. Therefore, if L is the empty set, then $\Omega(\underline{CA},\underline{A})^K \in \prod\mathcal{P}$.

Now consider $G \rtimes \mathcal{A}'$. By Lemma 2.14, to show $\Omega(G \rtimes \mathcal{A}') \in \prod\mathcal{P}$, it suffices to show that $\Omega G \in \prod\mathcal{P}$. The map $f_1 : (\underline{CA},\underline{A})^{K_1} \to (\underline{CA},\underline{A})^L$ has a right homotopy inverse, which implies that there is a homotopy equivalence $\Omega(\underline{CA},\underline{A})^{K_1} \simeq \Omega(\underline{CA},\underline{A})^L \times \Omega G$. Therefore ΩG retracts off $\Omega(\underline{CA},\underline{A})^{K_1}$. Since $\Omega(\underline{CA},\underline{A})^{K_1} \in \prod\mathcal{P}$ by hypothesis, Theorem 3.10 implies $\Omega G \in \prod\mathcal{P}$. A similar argument shows that $\Omega(\mathcal{A} \ltimes H) \in \prod\mathcal{P}$. Hence $\Omega(\underline{CA},\underline{A})^K \in \prod\mathcal{P}$. □

The next result will be used to show that if K is the k-skeleton of a simplex, then the loop space of certain polyhedral products is in $\prod\mathcal{P}$. The following result was first proved by Porter (P, Theorem 1) in the $(\underline{C\Omega A},\underline{\Omega A})^K$ case, and was generalised independently by (GT1, Theorem 1.1) and (IK1, Theorem 1.7) for general polyhedral products of the form $(\underline{CA},\underline{A})^K$.

Proposition 4.2. *Let K be the k-skeleton of Δ^{m-1}. Then there is a homotopy equivalence*

$$(\underline{CA},\underline{A})^K \simeq \bigvee_{j=k+2}^{m}\left(\bigvee_{1\leq i_1<\cdots<i_j\leq m}(\Sigma^{k+1}A_{i_1}\wedge\cdots\wedge A_{i_j})^{\vee\binom{j-1}{k+1}}\right).$$

□

This proposition can be used to prove the following lemma.

Lemma 4.3. *Let K be the k-skeleton of Δ^{m-1} and $A_1,\cdots,A_m$ be spaces such that $\Sigma A_i \in \bigvee\mathcal{W}$ for all i, then $(\underline{CA},\underline{A})^K \in \bigvee\mathcal{W}$.*

Proof. Since $\Sigma A_i \in \bigvee\mathcal{W}$ for all i, by shifting the suspension coordinate it follows that $\Sigma^{k+1}A_{i_1}\wedge\cdots\wedge A_{i_j} \in \bigvee\mathcal{W}$. □

For a general simplicial complex K, a general decomposition of K will be required in order to apply Theorem 4.1. Let $V(K)$ be the vertex set of K and for a subset $S \subseteq V(K)$, let K_S be the full subcomplex of K on the vertices of S. For a vertex $v \in V(K)$, denote by $N(v)$ the set of vertices adjacent to v in the 1-skeleton of K.

Lemma 4.4. *Let K be a simplicial complex and $v \in V(K)$. Then K can be written as the pushout*

$$\begin{array}{ccc} K_{N(v)} & \longrightarrow & K_{v\cup N(v)} \\ \downarrow & & \downarrow \\ K_{V(K)\setminus\{v\}} & \longrightarrow & K. \end{array}$$

Moreover, $K_{N(v)}$ is a full subcomplex of both $K_{v\cup N(v)}$ and $K_{V(K)\setminus\{v\}}$.

Proof. Since $K_{V(K)\setminus\{v\}}$ contains every simplex which does not contain the vertex v and $K_{v\cup N(v)}$ contains every simplex containing v, $K_{v\cup N(v)} \cup K_{V(K)\setminus\{v\}} = K$.

Clearly, $K_{N(v)} \subseteq K_{v\cup N(v)} \cap K_{V(K)\setminus\{v\}}$, so let $\sigma \in K_{v\cup N(v)} \cap K_{V(K)\setminus\{v\}}$. Since $\sigma \in K_{v\cup N(v)}$, σ must have vertices in $v \cup N(v)$. However since $\sigma \in K_{V(K)\setminus\{v\}}$, none of the vertices can be v. Hence $K_{v\cup N(v)} \cap K_{V(K)\setminus\{v\}} \subseteq K_{N(v)}$, and so $K_{v\cup N(v)} \cap K_{V(K)\setminus\{v\}} = K_{N(v)}$.

By definition, the subcomplex $K_{N(v)}$ contains every simplex in K on the vertex set $N(v)$. Therefore, it is a full subcomplex of both $K_{v\cup N(v)}$ and $K_{V(K)\setminus\{v\}}$. □

We now prove Theorem 1.1. Recall that $k \geq 0$, and let K be the k-skeleton of a flag complex on the vertex set $[m]$. Let $A_1, \cdots, A_m$ be path connected CW-complexes such that $\Sigma A_i \in \bigvee \mathcal{W}$ for all i. Then we wish to prove that $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$. A *dominating vertex* of K is a vertex v such that $N(v) = V(K) \setminus \{v\}$. In other words, v is adjacent to every other vertex in the 1-skeleton of K.

Proof of Theorem 1.1. We proceed by strong induction. If K has one vertex, then $(\underline{CX}, \underline{X})^K$ is contractible, and so $\Omega(\underline{CX}, \underline{X})^K \in \prod \mathcal{P}$.

Now suppose K has m vertices, and the result is true for all $n < m$. Since K is the k-skeleton of a flag complex, if every vertex of K is a dominating vertex, then K is the k-skeleton of a simplex. In this case, Lemma 4.3 implies that $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$. Therefore, suppose there exists a vertex $v \in V(K)$ such that v is not a dominating vertex of K. By Lemma 5.4, K can be written as the pushout

$$\begin{array}{ccc} K_{N(v)} & \longrightarrow & K_{v\cup N(v)} \\ \downarrow & & \downarrow \\ K_{V(K)\setminus\{v\}} & \longrightarrow & K \end{array}$$

where $K_{N(v)}$ is a full subcomplex of both $K_{v\cup N(v)}$ and $K_{V(K)\setminus\{v\}}$. Since v is not a dominating vertex, $K_{v\cup N(v)}$ is not the whole of K, and so $K_{v\cup N(v)}$ and $K_{V(K)\setminus\{v\}}$ are simplicial complexes with strictly fewer vertices than K. Therefore, by the inductive hypothesis, $\Omega(\underline{CA},\underline{A})^{K_{v\cup N(v)}} \in \prod\mathcal{P}$ and $\Omega(\underline{CA},\underline{A})^{K_{V(K)\setminus\{v\}}} \in \prod\mathcal{P}$. Hence, Theorem 4.1 implies that $\Omega(\underline{CA},\underline{A})^K \in \prod\mathcal{P}$. $\square$

Remark 4.5. In principle, one could iteratively use Proposition 2.9, Proposition 2.10 and Lemma 5.4 to obtain an explicit decomposition for $(\underline{CX},\underline{X})^K$. However, in practice, this process would be unwieldy.

Theorem 1.1 also has consequences for other polyhedral products associated to the k-skeleton of flag complexes.

Lemma 4.6. *Let K be a simplicial complex on the vertex set $[m]$ and let $(\underline{X},\underline{A})$ be any sequence of pointed, path-connected CW-pairs. Denote by Y_i the homotopy fibre of the inclusion $A_i \to X_i$. Suppose $\Omega(\underline{CY},\underline{Y})^K \in \prod\mathcal{P}$ and for $1 \le i \le m$, $\Omega X_i \in \prod\mathcal{P}$ for all i. Then $\Omega(\underline{X},\underline{A})^K \in \prod\mathcal{P}$.*

Proof. By (HST, Theorem 2.1), there is a homotopy fibration

$$(\underline{CY},\underline{Y})^K \to (\underline{X},\underline{A})^K \to \prod_{i=1}^{m} X_i$$

which splits after looping. Therefore, there is a homotopy equivalence

$$\Omega(\underline{X},\underline{A})^K \simeq \prod_{i=1}^{m}\Omega X_i \times \Omega(\underline{CY},\underline{Y})^K.$$

By assumption, $\Omega X_i \in \prod\mathcal{P}$ for all i and $\Omega(\underline{CY},\underline{Y})^K \in \prod\mathcal{P}$, and so $\Omega(\underline{X},\underline{A})^K \in \prod\mathcal{P}$. $\square$

When K is the k-skeleton of a flag complex, Theorem 1.1 and Lemma 4.6 implies the following result.

Corollary 4.7. *Let K be the k-skeleton of a flag complex on the vertex set $[m]$. Let $(\underline{X},\underline{A})$ be any sequence of pointed, path-connected CW-pairs, and denote by Y_i the homotopy fibre of the inclusion $A_i \hookrightarrow X_i$. Suppose $\Omega X_i \in \prod\mathcal{P}$ for all i and $\Sigma Y_i \in \bigvee\mathcal{W}$ for all i. Then $\Omega(\underline{X},\underline{A})^K \in \prod\mathcal{P}$.*

Proof. Since $\Sigma Y_i \in \bigvee\mathcal{W}$ for all i, Theorem 1.1 implies that $\Omega(\underline{CY},\underline{Y})^K \in \prod\mathcal{P}$. By assumption $\Omega X_i \in \prod\mathcal{P}$ for all i, so Lemma 4.6 implies that $\Omega(\underline{X},\underline{A})^K \in \prod\mathcal{P}$. $\square$

Corollary 4.7 applies to more examples, as follows.

Corollary 4.8. *Let K be the k-skeleton of a flag complex on the vertex set $[m]$, and for $1 \leq i \leq m$, let $n_i \in \mathbb{N} \cup \{\infty\}$ and $m_i \in \mathbb{N}$ with $m_i < n_i$. Let $(X_i, A_i) = (\mathbb{C}P^{n_i}, \mathbb{C}P^{m_i})$ or $(X_i, A_i) = (\mathbb{C}P^{n_i}, *)$ for all i. Then $\Omega(\underline{X}, \underline{A})^K \in \prod \mathcal{P}$.*

Proof. There are homotopy equivalences $\Omega\mathbb{C}P^k \simeq S^1 \times \Omega S^{2k+1}$ and $\Omega\mathbb{C}P^\infty \simeq S^1$. Therefore, $\Omega\mathbb{C}P^{n_i} \in \prod \mathcal{P}$ for all i. First consider a pair of the form $(\mathbb{C}P^{n_i}, *)$. The homotopy fibre Y_i of the inclusion of the basepoint into $\mathbb{C}P^{n_i}$ is $\Omega\mathbb{C}P^{n_i}$, and so $Y_i \in \prod \mathcal{P}$. Hence, $\Sigma Y_i \in \bigvee \mathcal{W}$.

Now consider a pair of the form $(\mathbb{C}P^{n_i}, \mathbb{C}P^{m_i})$. Suppose $n_i = \infty$. In this case, there is a standard homotopy fibration

$$S^{2k+1} \to \mathbb{C}P^k \to \mathbb{C}P^\infty,$$

and $\Sigma S^{2k+1} \in \bigvee \mathcal{W}$. Now suppose $n_i \neq \infty$. Consider the homotopy fibration diagram

$$\begin{tikzcd}
S^{2m+1} \times \Omega S^{2n+1} \arrow[r] \arrow[d] & S^{2m+1} \arrow[r, "*"] \arrow[d] & S^{2n+1} \arrow[d] \\
Y_i \arrow[r] & \mathbb{C}P^{m_i} \arrow[r] \arrow[d] & \mathbb{C}P^{n_i} \arrow[d] \\
 & \mathbb{C}P^\infty \arrow[r, equal] & \mathbb{C}P^\infty
\end{tikzcd}$$

where the maps in the bottom square are all inclusions, and the top map in the top right square is null homotopic since $m < n$. The top right square is a homotopy pullback, implying that $Y_i \simeq S^{2m+1} \times \Omega S^{2n+1}$. Therefore, $Y_i \in \prod \mathcal{P}$, and so $\Sigma Y_i \in \bigvee \mathcal{W}$. Hence, $\Sigma Y_i \in \bigvee \mathcal{W}$ for all i, and Lemma 4.7 implies that $\Omega(\underline{X}, \underline{A})^K \in \prod \mathcal{P}$. $\square$

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Chapter 3

Paper 2 - Loop space decompositions of moment-angle complexes associated to two dimensional simplicial complexes

1 Introduction

Polyhedral products are a natural subspace of the Cartesian product which are indexed by the face poset of a simplicial complex. They have generated much interest due to their far reaching applications across mathematics (see (BBC)). Let K be a simplicial complex on the vertex set $[m]$, and for $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed CW-subcomplex of X_i. The *polyhedral product associated to* K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} \left(\prod_{i=1}^{m} Y_i^{\sigma} \right),$$

where $Y_i^{\sigma} = X_i$ if $i \in \sigma$, and $Y_i^{\sigma} = A_i$ if $i \notin \sigma$. An important special case, which appears in toric topology, is when $(X_i, A_i) = (D^2, S^1)$ for all i. These polyhedral products are called *moment-angle complexes*, and are denoted $\mathcal{Z}_K$.

One particular problem associated to moment-angle complexes is understanding their loop spaces. In the case that K is a flag complex, the loop homology of moment-angle complexes models commutator subalgebras of algebraic analogues of right angled Coxeter groups (GPTW). More generally, for any simplicial complex K, the loop space of the corresponding moment-angle complex is related to a certain diagonal subspace arrangement (D). When K is flag, most homotopical and homological information about $\Omega\mathcal{Z}_K$ is known. In particular, a coarse description of $\Omega\mathcal{Z}_K$ was given in (S), which was upgraded to an explicit decomposition in (V). This allowed for a complete description of $H_*(\Omega\mathcal{Z}_K; R)$ as an algebra, where R is any commutative ring with unit (V). Another interesting case is when K is a 1-dimensional simplicial complex (a graph). In this case, it was shown in (S) that $\Omega\mathcal{Z}_K$ decomposes as a finite type product of spheres and loops on spheres. In particular, this implies that the homology of $\Omega\mathcal{Z}_K$ is torsion free. In this paper, we study the case of a 2-dimensional simplicial complex, and give a coarse description of $\Omega\mathcal{Z}_K$ in this case. The homology of $\Omega\mathcal{Z}_K$ in this case can contain torsion, and this will require the introduction of certain indecomposable torsion spaces which have been considered in (CMN1; CMN2; C). We also give a coarse description of $\Omega\mathcal{Z}_K$ as a product of spheres and loops on spheres after localising away from a finite set of primes, under conditions on the rational cohomology of certain full subcomplexes of K.

It is useful to identify two families of H-spaces. For a collection of topological spaces $\mathcal{X}$, let $\prod \mathcal{X}$ be the collection of spaces homotopy equivalent to a finite type product of spaces in $\mathcal{X}$. Let $\mathcal{P} := \{S^1, S^3, S^7, \Omega S^n \mid n \geq 2, n \notin \{2,4,8\}\}$. In (S), it was shown that $\prod \mathcal{P}$ is closed under retracts, and this was the key ingredient in proving coarse descriptions of $\Omega\mathcal{Z}_K$, in the case that K is the k-skeleton of a flag complex. In this paper, we extend this to include torsion spaces. Denote the mod p^r Moore space by $P^n(p^r)$, which is the mapping cone of the degree p^r map on S^{n-1}. By (HW, Theorem 1.1), for a finite type H-space X localised at a prime p, there is a unique decomposition of X, up to homotopy, as a finite type product of indecomposable spaces. Let $\mathcal{T}$ be the collection of indecomposable spaces which appear in the decomposition of the loop space of a wedge of Moore spaces of the form $\bigvee_{i=1}^m P^{n_i}(p_i^{r_i})$, where $m \geq 2$, $n_i \geq 3$, p_i is a prime and $r_i \geq 1$. Through an adaptation of the argument in (S), we will show in Section 4 that $\prod(\mathcal{P} \cup \mathcal{T})$ is also closed under retracts. This is the key technical result which is required to prove the main result of this paper.

Theorem 1.1 (Theorem 6.4 in the text)**.** *Let K be a 2-dimensional simplicial complex. Then $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$.*

In Section 3, a collection of co-H spaces, $\bigvee(\mathcal{W} \cup \mathcal{M})$, related to $\prod(\mathcal{P} \cup \mathcal{T})$ will be defined, and it will be shown that this collection is also closed under retracts. An important ingredient of the proof of Theorem 1.1 is a generalisation of (S, Theorem 1.1). For a simplicial complex K, let C_K be the collection of full subcomplexes of K whose 1-skeleton has no missing edges. The generalisation states $\Omega\mathcal{Z}_K$ being in $\prod(\mathcal{P} \cup \mathcal{T})$ depends only on $\Omega\mathcal{Z}_{K_I}$ being in $\prod(\mathcal{P} \cup \mathcal{T})$ for each full subcomplex $K_I \in C_K$ (see Theorem 5.5). This result can also be used to prove a localised result, which gives conditions on the rational cohomology of each $K_I \in C_K$ in K, for which after localising away from a finite set of primes (controlled by these full subcomplexes), $\Omega\mathcal{Z}_K \in \prod \mathcal{P}$. A simplicial complex K is called k-neighbourly if any $I \subseteq [m]$ with $|I| \leq k+1$ spans a simplex.

Theorem 1.2 (Theorem 7.1 in the text). *Let K be a simplicial complex such that all cup products and higher Massey products in $H^*(|K_I|;\mathbb{Q})$ are trivial, for all $K_I \in C_K$. For $K_I \in C_K$, suppose that K_I is k_I-neighbourly, and let $a = \max_{K_I \in C_K}\{|I| + dim(K_I) - 2k_I\}$. Localise away from primes $p \leq \frac{1}{2}a$ and primes p appearing as p-torsion in $H_*(|K_I|;\mathbb{Z})$ for any $K_I \in C_K$. Then $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$.*

Theorem 1.2 has consequences for a question posed by McGibbon and Wilkerson. A space X is called *rationally elliptic* if it has finitely many rational homotopy groups. Otherwise, it is called *rationally hyperbolic*. It was shown by McGibbon and Wilkerson (MW) that if X is rationally elliptic then at almost all primes p, the Steenrod algebra acts trivially on $H^*(\Omega X;\mathbb{Z}/p\mathbb{Z})$. They asked to what extent this holds for rationally hyperbolic spaces. We will show in Section 7 that Theorem 1.2 gives infinitely many examples for which this question has an affirmative answer.

Theorem 1.1 and Theorem 1.2 also verifies a conjecture of Anick (A). Anick conjectured that if X is a finite, simply connected CW-complex, then at all but finitely many primes, $\Omega X \in \prod(\mathcal{P} \cup \mathcal{T})$. Theorem 1.1 and Theorem 1.2 shows that such a decomposition holds for a family of moment-angle complexes.

We remark that the proofs in this paper hold more generally for polyhedral products of the form $(\underline{CX}, \underline{X})^K$, where each ΣX_i is homotopy equivalent to a finite type wedge of spheres and Moore spaces, however, we work only with moment-angle complexes to ease notation.

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2 Preliminary results

2.1 Unique decomposition of H-spaces and co-H spaces

In this subsection, we show a cancellation result after localisation that will be required to show that $\prod(\mathcal{P} \cup \mathcal{T})$ is closed under retracts. We will also require analogous results for wedges of spaces. We first state a result of (HW, Theorem 1.1) showing that after localising at a prime p, there is a unique decomposition of H-spaces and co-H spaces into indecomposable spaces.

Proposition 2.1. *The following hold:*

1. *Let X be a connected, finite type, p-local H-space. Then X can be uniquely decomposed into a weak product of indecomposable factors up to order and homotopy equivalence.*

2. *Let X be a connected, finite type, p-local co-H-space. Then X can be uniquely decomposed into a finite type wedge of indecomposable factors up to order and homotopy equivalence.* □

We now apply Proposition 2.1 to show a cancellation result after localising at a prime p. Let X and Y be spaces. We say that X *retracts off* Y if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to the identity on X.

Proposition 2.2. *The following hold:*

1. *Let X be a connected, finite type H-space, which is p-locally homotopy equivalent to a product*
$$X \simeq \prod_{i \in \mathcal{I}} X_i,$$
where each X_i is indecomposable. If A is a space which retracts off X, then there is a p-local homotopy equivalence
$$A \simeq \prod_{j \in \mathcal{J}} X_j$$
where $\mathcal{J} \subseteq \mathcal{I}$.

2. *Let X be a connected, finite type co-H-space, which is p-locally homotopy equivalent to a wedge*
$$X \simeq \bigvee_{i \in \mathcal{I}} X_i,$$
where each X_i is indecomposable. If A is a space which retracts off X, then there is a p-local homotopy equivalence
$$A \simeq \bigvee_{j \in \mathcal{J}} X_j$$
where $\mathcal{J} \subseteq \mathcal{I}$

Proof. We prove part (1), and part (2) follows by arguing dually. Localise at a prime p. Since A retracts off X, there exists a map $g : X \to A$ which has a right homotopy inverse. Proposition 2.1 implies that there is a unique p-local decomposition $A_i \simeq \prod_{k \in \mathcal{K}} A_k$, where each A_k is indecomposable. Since g has a right homotopy inverse and X is an H-space, there is a p-local homotopy equivalence $X \simeq A \times F$, where F is the homotopy fibre of g. The space F retracts off X, and so F is an H-space, implying by Proposition 2.1 that there is a p-local homotopy equivalence $F \simeq \prod_{k' \in \mathcal{K}'} F_{k'}$, where each $F_{k'}$ is indecomposable. Hence,
$$\prod_{i \in \mathcal{I}} X_i \simeq X \simeq A \times F \simeq \prod_{k \in \mathcal{K}} A_k \times \prod_{k' \in \mathcal{K}'} F_{k'}.$$
Since the product decomposition of X is unique, there exists an indexing set $\mathcal{J} \subseteq \mathcal{I}$ such that $A \simeq \prod\limits_{j \in \mathcal{J}} X_j$. □

2.2 Rational and p-local decompositions of moment-angle complexes

In this subsection, we state some preliminary localised decompositions of spaces. The first states conditions under which a finite CW-complex decomposes as a wedge of spheres after localising away from sufficiently many primes. This result is a mild generalisation of a result proved in (HT, Lemma 5.1), however, the proof goes through unchanged.

Lemma 2.3. *Let X be a simply-connected, finite CW-complex of dimension d and connectivity s. Suppose that X is rationally homotopy equivalent to a wedge of spheres. Let p be a prime such that $p > \frac{1}{2}(d - s + 1)$, and $H_*(X;\mathbb{Z})$ is p-torsion free. Then X is p-locally homotopy equivalent to a wedge of spheres.* □

The next result relates to a rational decomposition for certain moment-angle complexes. Let K be a simplicial complex. The simplicial complex K is said to be *Golod over a ring R* if all products and higher Massey products in $H^*(\mathcal{Z}_K; R)$ are trivial. In this case, there is a relation between the rational Golodness of K and $\mathcal{Z}_K$ being a suspension. The following result is attributed to Berglund, however, the reference now appears to be unavaliable. Therefore, we provide an alternative proof.

Proposition 2.4. *Let K be a simplicial complex. Then K is rationally Golod if and only if $\mathcal{Z}_K$ is rationally a co-H space.*

Proof. Rationally, any co-H space is homotopy equivalent to a wedge of spheres. Therefore, if $\mathcal{Z}_K$ is rationally a co-H space then K is rationally Golod. Suppose K is rationally Golod. Then by (K, Proposition 3.6), K is Golod over $\mathbb{Z}/p\mathbb{Z}$, when p is a sufficiently large prime. Moreover, by (BG, Theorem 3.1), localised at a sufficiently large prime p, $\mathcal{Z}_K$ is a co-H space if and only if $\mathcal{Z}_K$ is Golod over $\mathbb{Z}/p\mathbb{Z}$. Therefore, $\mathcal{Z}_K$ is rationally a co-H space. □

A counterexample to the integral analogue of Proposition 2.4 was constructed in (IY). Since any co-H space is rationally homotopy equivalent to a wedge of spheres, a rational decomposition of $\mathcal{Z}_K$ in this case can be recovered from its homology. For any moment-angle complex, a suspension splitting was proved in (BBCG1, Theorem 2.21).

Proposition 2.5. *Let K be a simplicial complex. There is a homotopy equivalence*

$$\Sigma\mathcal{Z}_K \simeq \bigvee_{I \notin K} \Sigma^{2+|I|}|K_I|.$$ □

Finally, we require a result which relates the rational homotopy type and the p-local homotopy type of an H-space. This result is known as the Sullivan arithmetic square (see (MP, Theorem 8.1.3) for a modern presentation). For a prime p, denote by $X_{(p)}$ the localisation of X at p, and let $X_{\mathbb{Q}}$ denote the rationalisation of X.

Theorem 2.6. *Let X be an H-space. Then there is a homotopy pullback*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_{(p)} \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \prod_p X_{\mathbb{Q}}. \end{tikzcd}$$

In particular, if X is rationally trivial, there is a homotopy equivalence

$$X \simeq \prod_p X_{(p)}. \qquad \square$$

3 Closure of $\bigvee(\mathcal{W} \cup \mathcal{M})$ under retracts

To prove Theorem 1.1, we will need to consider wedge decompositions of certain spaces. For a collection of topological spaces $\mathcal{X}$, let $\bigvee \mathcal{X}$ denote the collection of spaces homotopy equivalent to a finite type wedge of spaces in $\mathcal{X}$. Let $\mathcal{W}$ be the collection of simply connected spheres. By Proposition 2.1, when localised at a prime, there is a unique decomposition of any co-H space, up to homotopy, as a finite type wedge of indecomposable spaces. Let $\mathcal{M}$ be the collection of Moore spaces of the form $P^n(p^r)$, where $n \geq 3$, p is a prime, and $r \geq 1$, and the indecomposable factors which appear as wedge summands in the unique 2-local wedge decomposition of spaces of the form $\Sigma((P^{n_1}(2) \wedge \cdots \wedge (P^{n_l}(2))$, where $l \geq 2$, and each $n_i \geq 3$. Note that we do not require smash products of Moore spaces of the form $P^n(p^r)$ when $p^r \neq 2$, since in this case by (N3, Corollary 6.6), there is a homotopy equivalence

$$P^n(p^{r_1}) \wedge P^m(p^{r_2}) \simeq P^{n+m}(p^{\min\{r_1,r_2\}}) \vee P^{n+m-1}(p^{\min\{r_1,r_2\}})$$

when $p^{r_1}, p^{r_2} \neq 2$. In general, the indecomposable wedge summands that appear in the decomposition of spaces of the form $\Sigma((P^{n_1}(2) \wedge \cdots \wedge (P^{n_l}(2)))$ are unknown, but some progress has been made in (W). In this section, we will show that $\bigvee(\mathcal{W} \cup \mathcal{M})$ is closed under retracts. This is well known for spaces in $\bigvee \mathcal{W}$, and it was shown for a wedge of Moore spaces of a fixed odd prime power in (N2, Lemma 4.2). The introduction of the 2-torsion spaces necessitates a more technical argument which we complete here.

Let $X \in \bigvee(\mathcal{W} \cup \mathcal{M})$, and let A be a space which retracts off X. The strategy to show that $A \in \bigvee(\mathcal{W} \cup \mathcal{M})$ is to retract a sphere off A for every $\mathbb{Z}$ summand which appears in the homology of A. This will give us a homotopy equivalence $A \simeq W \vee A'$, where $W \in \bigvee \mathcal{W}$, and the homology of A' is torsion. We can then use Theorem 2.6 and Proposition 2.2 to show that $A' \in \bigvee \mathcal{M}$.

To retract spheres off A, we argue similarly to (S, Section 3). Since A retracts off X, there exists maps $f : A \to X$ and $g : X \to A$ such that $g \circ f \simeq id_A$. Define $\phi : X \to X$ as the composite

$$\phi : X \xrightarrow{g} A \xrightarrow{f} X.$$

Note that ϕ is an idempotent. Let W be the wedge of spheres that appear in the wedge decomposition of X. Define ϕ' to be the composite

$$\phi' : W \hookrightarrow X \xrightarrow{\phi} X \xrightarrow{p} W,$$

where the lefthand map is the inclusion, and p is the pinch map. While ϕ' may not be an idempotent, the following shows that the induced map $(\phi')_*$ is an idempotent on homology, which suffices for our purposes. This follows from the following technical lemma.

Lemma 3.1. *Let G be a finitely generated abelian group, G_{free} be the free part of G, and G_{tor} be the torsion part of G. Let $\phi : G \to G$ be an idempotent. Then the composite*

$$\phi' : G_{free} \xrightarrow{i} G \xrightarrow{\phi} G \xrightarrow{\pi} G_{free},$$

where i is the inclusion and π is the projection, is an idempotent. Moreover, if $\phi(g, t) = (g', t')$ where $g, g' \in G_{free}$ and $t, t' \in G_{tor}$, then $g' \in Im(\phi')$.

Proof. Consider $\phi' \otimes \mathbb{Q}$. Since G_{free} is free, the maps $i \otimes \mathbb{Q}$ and $\pi \otimes \mathbb{Q}$ are both the identity map. By assumption, ϕ is idempotent implying that $\phi \otimes \mathbb{Q}$ is idempotent and so $\phi' \otimes \mathbb{Q}$ is idempotent. Hence, ϕ' is idempotent. A similar argument shows that the second part is true. □

The inclusion $W \to X$ and the pinch map $X \xrightarrow{p} W$ induce the inclusion and projection respectively of the free part of $H_*(X)$, and so by Lemma 3.1, $(\phi')_*$ is an idempotent on homology. Let $a \in H_n(A)$ be a generator of a $\mathbb{Z}$ summand. We aim to show that S^n retracts off A. First, we show that $(\phi'_*)_n : H_n(W) \to H_n(W)$ is non-zero.

Lemma 3.2. *Let $a \in H_n(A)$ be a generator of a $\mathbb{Z}$ summand. Then the induced map $(\phi'_*)_n$ in homology is non-zero.*

Proof. Recall that $f : A \to X$ and $g : X \to A$ are maps such that $g \circ f \simeq id_A$, and ϕ is the composite

$$\phi : X \xrightarrow{g} A \xrightarrow{f} X.$$

Since f_n is injective, $f_n(a)$ is non-zero and not torsion. Moreover, the composite $g_n \circ f_n$ is the identity, and so $g_n \circ f_n(a)$ is non-zero. Hence, $\phi_n(f_n(a)) = (f_n \circ g_n \circ f_n)(a) = f_n(a)$ is non-zero, and so Lemma 3.1 implies that $(\phi'_*)_n$ is non-zero. □

Let x be a generator of $\mathrm{Im}((\phi'_*)_n)$. Let $H_n(W) = \bigoplus_{i=1}^m \mathbb{Z}$, and write x as $x = (x_1, \cdots, x_m)$. Since $(\phi'_*)_n$ is an idempotent of free abelian groups, x must extend to a basis of $\mathbb{Z}^m$, and so the greatest common divisor of $x_1, \cdots, x_m$ is 1 (see (S, Lemma 2.3) for example). Bézout's Lemma implies that there exists $y_1, \cdots, y_m \in \mathbb{Z}$ such that $\sum_{i=1}^m y_i x_i = 1$. Let $\rho : S^n \to W$ be the composite

$$\rho : S^n \xrightarrow{\sigma} \bigvee_{i=1}^m S^n \xrightarrow{\bigvee_{i=1}^m p_{x_i}} \bigvee_{i=1}^m S^n \hookrightarrow W,$$

where σ is a choice of m-fold suspension comultiplication, and p_{x_i} is the x_i^{th} degree map. Let γ be a generator of $H_n(S^n)$. By definition of ρ and the fact that p_{x_i} induces multiplication by x_i in homology, ρ_n sends γ to $x \in H_n(W)$. Now let ρ' be the composite

$$\rho' : W \xrightarrow{p} \bigvee_{i=1}^m S^n \xrightarrow{\bigvee_{i=1}^m p_{y_i}} \bigvee_{i=1}^m S^n \xrightarrow{\nabla} S^n,$$

where p is the pinch map, and ∇ is the fold map. By definition of ρ', since the degree map p_{y_i} induces multiplication by y_i in homology, ρ'_n sends x to γ. Since $(\phi'_*)_n$ is idempotent, it fixes its image, and so the composite

$$e : S^n \xrightarrow{\rho} W \xrightarrow{\phi'} W \xrightarrow{\rho'} S^n,$$

is an isomorphism in homology. This implies that e is a homotopy equivalence. Since ϕ' factors through A, S^n retracts off A. Arguing dually to (S, Proposition 3.3, Theorem 3.10) for each generator of a $\mathbb{Z}$ summand in $H_*(A)$, we obtain the following.

Proposition 3.3. *Let $X \in \bigvee(\mathcal{W} \cup \mathcal{M})$, and let A be a space which retracts off X. Then there is a homotopy equivalence*

$$A \simeq S \vee A',$$

where $S \in \bigvee \mathcal{W}$, and the homology of A' is torsion. □

It now suffices to show that A' in Proposition 3.3 is in $\mathcal{M}$.

Proposition 3.4. *Let $X \in \bigvee(\mathcal{W} \cup \mathcal{M})$, and let A' be a space which retracts off X, such that the homology of A' is torsion. Then $A' \in \bigvee \mathcal{M}$.*

Proof. By assumption, $H_*(A')$ is torsion, and so A' is rationally trivial. By Theorem 2.6, there is a homotopy equivalence

$$A' \simeq \prod_p A'_{(p)}.$$

For each prime p, $A'_{(p)}$ retracts off $X_{(p)}$. Proposition 2.2(2) therefore implies that each $A'_{(p)}$ is homotopy equivalent to a wedge of p-torsion spaces in $\mathcal{M}$.

Let $i : \bigvee_p A'_{(p)} \to \prod_p A'_{(p)}$ be the inclusion. Localised at a prime p, the map i is the identity map $A'_{(p)} \to A'_{(p)}$ and so i is a homotopy equivalence localised at any prime p. Rationally each $A'_{(p)}$ is contractible, and so i is a homotopy equivalence rationally. Hence, i is a homotopy equivalence integrally. Putting this all together, we obtain a homotopy equivalence $A' \simeq \bigvee_p A'_{(p)}$, where each $A'_{(p)} \in \bigvee \mathcal{M}$. □

Combining Proposition 3.3 and Proposition 3.4, we obtain the following.

Theorem 3.5. *Let $X \in \bigvee(\mathcal{W} \cup \mathcal{M})$, and let A be a space which retracts off X. Then $A \in \bigvee(\mathcal{W} \cup \mathcal{M})$.* □

4 Closure of $\prod(\mathcal{P} \cup \mathcal{T})$ under retracts

4.1 Special cases

Recall that for a collection of topological spaces $\mathcal{X}$, $\prod \mathcal{X}$ is the collection of spaces homotopy equivalent to a finite type product of spaces in $\mathcal{X}$. Moreover, recall the collections $\mathcal{P} := \{S^1, S^3, S^7, \Omega S^n \mid n \geq 2, n \notin \{2, 4, 8\}\}$, and $\mathcal{T}$, which is the collection of indecomposable spaces which appear in the decomposition of the loop space of a wedge of Moore spaces of the form $\bigvee_{i=1}^{m} P^{n_i}(p_i^{r_i})$, where $m \geq 2$, $n_i \geq 3$, p_i is a prime and $r_i \geq 1$. In this section, we show that $\prod(\mathcal{P} \cup \mathcal{T})$ is closed under retracts.

We start with some special cases. First, we have the following result from (S, Theorem 3.10).

Theorem 4.1. *Let $X \in \prod \mathcal{P}$, and A be a space which retracts off X. Then $A \in \prod \mathcal{P}$.* □

We can also prove a similar result in the case of a space A retracting off $X \in \prod(\mathcal{P} \cup \mathcal{T})$, where the homology of A is torsion.

Theorem 4.2. *Let $X \in \prod(\mathcal{P} \cup \mathcal{T})$, and A be a space which retracts off X, such that the homology of A is torsion. Then $A \in \prod \mathcal{T}$.*

Proof. Since the homology of A is torsion, A is rationally trivial. Therefore by Theorem 2.6, there is a homotopy equivalence

$$A \simeq \prod_p A_{(p)}.$$

For each prime p, $A_{(p)}$ retracts off $X_{(p)}$. Proposition 2.2 implies that each $A_{(p)} \in \prod \mathcal{T}$, and so $A \in \prod \mathcal{T}$. □

4.2 Review of the proof of Theorem 4.1

First, we recall the strategy from (S, Section 3) which was used to prove Theorem 4.1. The strategy is similar to the one used in Section 3. Let $X \in \prod \mathcal{P}$, and let A be a space which retracts off X. This implies there are maps $f : A \to X$, and $g : X \to A$ such that $g \circ f$ is homotopic to the identity on A. The first ingredient of the proof is the idempotent $\phi : X \xrightarrow{g} A \xrightarrow{f} X$. The key property that is used here is that ϕ_* is an idempotent on homology, and so ϕ_* fixes its image. Let n be an integer such that $H_n(A)$ contains a primitive generator. The proof is split into three cases, the first is where $n \in \{1, 2, 3, 6, 7, 14, 4m \mid m \geq 1\}$, the second is where $n = 4m + 2$, $m \geq 2$, $m \neq 3$, and the third is where $n = 2m - 1$, where $m \notin \{1, 2, 4\}$.

Consider the first and third case (see (S, Subsections 3.2 and 3.4)), where

$$n \in \{1, 2, 3, 6, 7, 14, 4m, 2l - 1 \mid m, l \geq 1, l \notin \{1, 2, 4\}\}.$$

Fix such an n and write X as

$$X \simeq \prod_{i=1}^{m_Y} Y_i \times \prod_{\alpha' \in \mathcal{I}'} Z_{\alpha'},$$

where each Y_i is an instance of S^n if $n \in \{1, 3, 7\}$, or $Y_i = \Omega S^{n+1}$ otherwise, and each $Z_{\alpha'}$ are the spheres and loops on spheres not equal to Y_i. Let $Y = S^n$ if $n \in \{1, 3, 7\}$, or $Y = \Omega S^{n+1}$ otherwise. In this case, the bottom non-vanishing degree of each Y_i gives a basis of primitives $\{\gamma_1, \cdots, \gamma_{m_Y}\}$ of $H_n(X)$. It was shown that there exists a non-zero element $x = \sum_{i=1}^{m_Y} y_i \gamma_i \in \mathrm{Im}(\phi_*)$ such that the greatest common divisor of $y_1, \cdots, y_{m_Y}$ is 1. Let γ be a generator of the lowest non-vanishing degree in the homology of Y. Two maps $\rho : Y \to X$, and $\rho' : X \to Y$ were defined such that $\rho_*(\gamma) = x$, and $\rho'_*(x) = \gamma$. Since ϕ_* fixes its image, the composite

$$e : Y \xrightarrow{\rho} X \xrightarrow{\phi} X \xrightarrow{\rho'} Y$$

is an isomorphism on the lowest non-vanishing degree in homology. If Y is a sphere, then e is a homotopy equivalence by Whitehead's theorem. If Y is the loops on a sphere, localisation and atomicity properties of the loops on spheres (with a slight adjustment to the maps ρ and ρ' in the case that $n = 2l - 1$) are used to show that e is a homotopy equivalence, implying that Y retracts off X.

Now consider the second case (see (S, Subsection 3.3)), where $H_{4n+2}(A)$ contains a primitive generator, $n \geq 2$, $n \neq 3$. In this case, write X as

$$X \simeq \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{2n+2} \times \prod_{\alpha' \in \mathcal{I}'} Z_{\alpha'}$$

where each $Z_{\alpha'}$ are the spheres and loops on spheres that are not equal to ΩS^{4n+3} or ΩS^{2n+2}. A basis of primitives $\{\gamma_1, \cdots, \gamma_{m_Y}, \overline{\gamma}_1, \cdots, \overline{\gamma}_{m_{\overline{Y}}}\}$ was obtained of $H_{4n+2}(X)$, where γ_i is a generator of $H_{4n+2}(\Omega S_i^{4n+3})$, and $\overline{\gamma}_i$ is a generator of $H_{4n+2}(\Omega S_i^{2n+2})$. It was shown that there exists a non-zero element $\sum_{i=1}^{m_Y} y_i\gamma_i + \sum_{i=1}^{m_{\overline{Y}}} 2\overline{y}_i\overline{\gamma}_i \in \mathrm{Im}(\phi_*)$ such that the greatest common divisor of $y_1, \cdots, y_{m_Y}, 2\overline{y}_1, \cdots, 2\overline{y}_{m_{\overline{Y}}}$ is 1, and as in the previous case, this element was used to define maps $\rho : Y \to X$, and $\rho' : X \to Y$ such that the composite

$$Y \xrightarrow{\rho} X \xrightarrow{\phi} X \xrightarrow{\rho'} Y$$

is a homotopy equivalence, implying that Y retracts off X. Therefore, for each primitive generator in $H_*(A)$, we can retract a sphere or the loops on a sphere off A. Iterating this, we obtain a product decomposition for A as a product of spheres and loops on spheres (see (S, Theorem 3.10)).

To generalise to the case where $X \in \prod(\mathcal{P} \cup \mathcal{T})$, we first retract off all the spheres and loops on spheres that we expect to obtain in a decomposition for A, by analysing the coalgebra structure of $H_*(A;\mathbb{Q})$. This will give us a homotopy equivalence $A \simeq P \times A'$, where $P \in \prod \mathcal{P}$, and the homology of A' is torsion. We can then use Theorem 2.6 to obtain that $A' \in \prod \mathcal{T}$.

4.3 Defining ϕ'

From now on, homology will be assumed to be taken to be taken with integral coefficients unless otherwise stated. The coalgebra structure on the homology of a space is defined whenever the Künneth isomorphism holds. In this case however, the homology of X and A may contain torsion, and so we can not appeal to the coalgebra structure in order to repeat the argument for Theorem 4.1. However, we will adjust the map ϕ to obtain a self map ϕ' of the spheres and loops on spheres that appear in the product decomposition of X which is idempotent in homology. This will allow us to find the required elements in the image of ϕ' in order to appeal to the argument in (S). Let $X \in \prod(\mathcal{P} \cup \mathcal{T})$, and let A be a space which retracts off X, such that $H_*(A;\mathbb{Q})$ is non-trivial. In particular there exists maps $f : A \to X$ and $g : X \to A$ such that $g \circ f$ is homotopic to the identity on A. Observe that the map $\phi = f \circ g$ is an idempotent. Write X as $X \simeq S \times M$, where $S \in \prod \mathcal{P}$ and $M \in \prod \mathcal{T}$. Define the map $\phi' : S \to S$ as the composite

$$\phi' : S \hookrightarrow X \xrightarrow{\phi} X \xrightarrow{\pi} S,$$

where the left map is the inclusion of S into X, and π is the projection. We would like ϕ' to also be an idempotent in order to emulate the map ϕ in the case where $X \in \prod \mathcal{P}$. This may not be true for the map itself, however, the inclusion $S \to X$ induces the inclusion of the free part of $H_*(X)$ and the projection $X \to S$ induces the projection onto the free part of $H_*(X)$. Therefore, Lemma 3.1 implies that $(\phi')_*$ is an idempotent on homology.

To appeal to the argument in (S, Section 3), we require a primitive element $x \in \mathrm{Im}(\phi')$ with the properties as described in Subsection 4.2. First, we need to show that in certain degrees, $(\phi')_*$ is non-zero when restricted to the submodule of primitives in $H_*(S)$. In the case that $X \in \prod \mathcal{P}$, this was done by showing that ϕ_n is non-zero when restricted to the submodule of primitives whenever there is a primitive generator in $H_n(A)$. However, in this case, we can not appeal to the coalgebra structure in integral homology as $H_*(A)$ may contain torsion. However, rational homology $H_*(A;\mathbb{Q})$ does have a coalgebra structure. We can use this to show that if $a \in H_n(A)$ is an element which reduces to a primitive generator in rational homology, then $(\phi'_*)_n$ is non-zero when restricted to the submodule of primitives in $H_n(S)$.

Lemma 4.3. *Suppose that $a \in H_*(A)$ is a generator of a $\mathbb{Z}$ summand in degree n, which reduces to a primitive generator in rational homology. Then $f_*(a)$ maps to an element in $H_n(X)$ whose free part reduces to an element in the submodule of primitives in rational homology, and $(\phi'_*)_n$ is non-zero when restricted to the submodule of primitives in $H_*(S)$.*

Proof. Let $\overline{a} \in H_*(A;\mathbb{Q})$ be the primitive generator which is the reduction of a. Recall that ϕ' is the composite

$$\phi' : S \hookrightarrow X \xrightarrow{\phi} X \xrightarrow{\pi} S,$$

where S is the product of spheres and loops on spheres that appear in X, ϕ is the composite $f \circ g$ which is an idempotent, the left map is the inclusion of S into X, and π is the projection. By the naturality of the universal coefficient theorem with respect to coefficients, there is a commutative diagram

$$\begin{array}{ccc} H_n(A) & \xrightarrow{f_n} & H_n(X) \\ \downarrow & & \downarrow \\ H_n(A)\otimes\mathbb{Q} & \xrightarrow{f_n\otimes\mathbb{Q}} & H_n(X)\otimes\mathbb{Q}. \end{array}$$

In particular, since $H_n(A)\otimes\mathbb{Q}$ is a coalgebra, and $\overline{a}$ is a primitive generator, $f_n(a)$ must map to an element $x \in H_n(X)$ such that the free part of x, which we will denote by x', reduces to a primitive element in $H_n(X)\otimes\mathbb{Q}$. This proves the first part of the lemma. Since $f_n(a)$ is injective, x is non-zero and has infinite order, and so x' is also non-zero. By definition of ϕ, the image of ϕ is equal to the image of f, and so $x \in \mathrm{Im}(\phi)$. Lemma 3.1 implies that x' is in the image of $(\phi'_*)_n$. Since $S \in \prod \mathcal{P}$ and x' reduces to a primitive element in rational homology, $(\phi'_*)_n(x')$ is contained in the submodule of primitives of $H_n(S)$ integrally. Hence $(\phi'_*)_n$ is non-zero when restricted to the submodule of primitives in $H_*(S)$. □

4.4 Case 1

In this subsection, fix $n \in \{1, 2, 3, 6, 7, 14, 4m, 2l - 1 \mid m, l \geq 1, l \notin \{1, 2, 4\}\}$ such that A contains a generator $a \in H_n(A)$ which reduces to a primitive generator in rational homology. Write S as

$$S \simeq \prod_{i=1}^{m_Y} Y_i \times \prod_{\alpha' \in \mathcal{I}'} Z_{\alpha'},$$

where each Y_i is an instance of S^n if $n \in \{1, 3, 7\}$, or $Y_i = \Omega S^{n+1}$ otherwise, and the factors $Z_{\alpha'}$ are the spheres, and loops on spheres that are not equal to Y_i. In this case, the bottom non-vanishing degree of each Y_i gives a basis of primitives $\{\gamma_1, \cdots, \gamma_{m_Y}\}$ of $H_n(S)$. As in Subsection 4.2, we require an element $\sum_{i=1}^{m_Y} y_i \gamma_i \in \operatorname{Im}(\phi')$ such that the greatest common divisor of $y_1, \cdots, y_{m_Y}$ is 1. By Lemma 4.3, $(\phi'_*)_n$ is a non-zero idempotent map in this degree. Let $\lambda = \sum_{i=1}^{m_Y} z_i \gamma_i$ be a generator of $\operatorname{Im}(\phi')$. Since $(\phi'_*)_n$ is an idempotent, the greatest common divisor of $z_1, \cdots, z_{m_Y}$ is 1 (see for example (S, Section 2)). Hence λ has the desired properties. Therefore, we can argue as in (S, Subsection 3.2) in the case where $n \in \{1, 2, 3, 6, 7, 14, 4m \mid m \geq 1\}$, and (S, Subsection 3.4) in the case where $n = 2l - 1$, $l \notin \{1, 2, 4\}$ to retract the following spheres and loops on spheres off A: for each generator of a $\mathbb{Z}$ summand in $H_n(A)$ which reduces to a primitive generator in rational homology, an S^n or ΩS^{n+1} retracts off A. We obtain the following.

Lemma 4.4. *Let $X \in \prod(\mathcal{P} \cup \mathcal{T})$, and A be a space which retracts off X. There is a homotopy equivalence*

$$A \simeq P \times A',$$

where P is a product of spheres of the form S^n, where $n \in \{1, 3, 7\}$, and loops on spheres of the form ΩS^{m+1}, where $m \in \{2, 6, 14, 4m, 2l - 1 \mid m \geq 1, l \notin \{1, 2, 4\}\}$. Moreover, the only generators in $H_*(A')$ which reduce to primitive generators in rational homology are in degrees of the form $4k + 2$, where $k \geq 2$, $k \neq 3$. □

4.5 Case 2

By Lemma 4.4, it suffices to consider a space A which retracts off $X \in \prod(\mathcal{P} \cup \mathcal{T})$, such that the only generators of $H_*(A)$ which reduce to primitive generators in rational homology are in degrees of the form $4n + 2$, where $n \geq 2$, and $n \neq 3$. Fixing n, write X as

$$X \simeq \prod_{i=1}^{m_Y} \Omega S_i^{4n+3} \times \prod_{j=1}^{m_{\overline{Y}}} \Omega S_j^{2n+2} \times \prod_{\alpha' \in \mathcal{I}'} Z_{\alpha'}$$

where the factors $Z_{\alpha'}$ are the spheres, loops on spheres and indecomposable torsion spaces that are not equal to ΩS^{4n+3} or ΩS^{2n+2}. For $1 \leq i \leq m_Y$, let γ_i be the generator of $H_{4n+2}(X)$ corresponding to a generator of $H_{4n+2}(\Omega S_i^{4n+3})$, and for $1 \leq j \leq m_{\overline{Y}}$, let $\overline{\gamma}_j$ be the generator of $H_{4n+2}(X)$ corresponding to a generator of $H_{4n+2}(\Omega S_j^{2n+2})$.

Observe that by definition of X, the generators γ_i and $\overline{\gamma}_j$ form a basis for the submodule of primitives of rational homology in degree $4n+2$. Let $a \in H_{4n+2}(A)$ be a generator of a $\mathbb{Z}$ summand which reduces to a primitive generator in rational homology. By Lemma 3.1, $f_*(a) = \sum_{i=1}^{m_Y} y_i\gamma_i + \sum_{j=1}^{m_{\overline{Y}}} \overline{y}_j\overline{\gamma}_j + t$, where t has finite order. Recall ϕ' is the composite

$$\phi' : S \hookrightarrow X \xrightarrow{\phi} X \xrightarrow{\pi} S,$$

where the left map is the inclusion of S into X, $\phi = f \circ g$ and π is the projection. Since f_* is injective, the image of ϕ_* is equal to the image of f_*. Therefore, by Lemma 4.3, $\sum_{i=1}^{m_Y} y_i\gamma_i + \sum_{j=1}^{m_{\overline{Y}}} \overline{y}_j\overline{\gamma}_j$ is in the image of $(\phi')_{4n+2}$. As described in Subsection 4.2, to retract ΩS^{4n+2} off A, it suffices to show that the greatest common divisor of $y_1, \cdots, y_{m_Y}, \overline{y}_1, \cdots, \overline{y}_{m_{\overline{Y}}}$ is 1, and that each $\overline{y}_j$ is even.

Lemma 4.5. *The greatest common divisor of $y_1, \cdots, y_{m_Y}, \overline{y}_1, \cdots, \overline{y}_{m_{\overline{Y}}}$ is 1.*

Proof. Let $y = (y_1, \cdots, y_{m_Y}, \overline{y}_1, \cdots, \overline{y}_{m_{\overline{Y}}})$, and suppose $y = dy'$ for some $d > 1$. In this proof, we work with homology with rational coefficients and use the same notation for each element and map to mean its reduction in rational homology.

By definition of y, $f_*(a) = y$. Since $g_* \circ f_* = id_*$, $g_*(y) = a$. Let $g_*(y') = a'$ for some $a' \in H_{4n+2}(A;\mathbb{Q})$. By definition of y', $da' = dg_*(y') = g_*(y) = a$. However, integrally, a is a generator of a $\mathbb{Z}$ summand, and so $d = 1$ and $a' = a$, which is a contradiction. □

Now we must show that each $\overline{y}_1, \cdots, \overline{y}_{m_{\overline{Y}}}$ is even. To do this, we first determine the rational homotopy type of A.

Lemma 4.6. *Let A be a space which retracts off $X \in \prod(\mathcal{P} \cup \mathcal{T})$. Suppose that the only generators in $H_*(A)$ which reduce to primitive generators in rational homology are in degrees of the form $4n+2$, where $n \geq 2$, $n \neq 3$. Then A is rationally homotopy equivalent to a finite type product of loops on spheres of the form ΩS^{4n+3}, where $n \geq 2$, $n \neq 3$.*

Proof. Since A retracts off X, A is an H-space. Rationally, every H-space is homotopy equivalent to a product of spheres and loops on odd dimensional spheres. Since A only contains primitive generators in degrees of the form $4n+2$, $n \geq 2$, $n \neq 3$, there is a rational homotopy equivalence as claimed. □

Using this lemma, we can now show that each $\overline{y}_j$ must be even.

Lemma 4.7. *For $1 \leq j \leq m_{\overline{Y}}$, $\overline{y}_j$ is even.*

Proof. By Lemma 4.6, there is a rational homotopy equivalence $A_{\mathbb{Q}} \simeq \prod_{i\in\mathcal{I}} \Omega S_{\mathbb{Q}}^{4n_i+3}$ for some indexing set $\mathcal{I}$. Now localise at the prime 2, by Proposition 2.2, there is a 2-local homotopy equivalence

$$A_{(2)} \simeq \prod_{i=1}^{l} \Omega S^{4n+3}_{(2)} \times S' \times T,$$

where $l \geq 1$, S' is a product of 2-local loops on spheres of the form ΩS^m, where $m \neq 4n+3$, and $T \in \mathcal{T}$ is a product of indecomposable, 2-torsion spaces. Recall that $a \in H_{4n+2}(A)$ is a generator reducing to a primitive generator in rational homology, and $f_*(a) = \sum_{i=1}^{m_Y} y_i \gamma_i + \sum_{j=1}^{m_{\overline{Y}}} \overline{y}_j \overline{\gamma}_j + t$, where t has finite order. The Hurewicz theorem implies that there is a map $\nu : S^{4n+2} \to A_{(2)}$ such that in homology, ν sends a generator λ of $H_{4n+2}(S^{4n+2})$ to a. By the universal property of the James construction (J), this extends to a map $\nu' : \Omega S^{4n+3} \to A_{(2)}$, which sends a generator λ' of $H_{4n+2}(\Omega S^{4n+3})$ to a. Finally, by the universal property of localisation, there exists a map $\overline{\nu}' : \Omega S^{4n+3}_{(2)} \to A_{(2)}$ which sends λ' to a.

Consider the composite

$$\psi : \Omega S^{4n+3}_{(2)} \xrightarrow{\overline{\nu}'} A_{(2)} \xrightarrow{f_{(2)}} X_{(2)} \xrightarrow{\pi_j} \Omega S^{2n_j+2}_{(2)},$$

where π_j is the projection onto the loops on a sphere corresponding to $\overline{\gamma}_j$, and $n = n_j$. Suppose that $\overline{y}_j$ is odd. By definition, ψ_* sends the generator λ' to $\overline{y}_j \overline{\gamma}_j$. Let $J : \Omega S^{2n_j+2} \to \Omega S^{4n+3}$ be the 2^{nd} James-Hopf invariant. Consider the composite $\psi' = J_{(2)} \circ \psi : \Omega S^{4n+3}_{(2)} \to \Omega S^{4n+3}_{(2)}$. As recounted in (S, Lemma 2.6) for example, in homology, J_* induces an isomorphism in degree $4n+2$, and so ψ'_* sends λ' to $\pm \overline{y}_j \lambda'$. Since $\overline{y}_j$ is odd, ψ'_* is an isomorphism localised at 2. By (CPS, Corollary 5.2), ΩS^{4n+3} is atomic localised at the prime 2, meaning that any self-map which is an isomorphism in the bottom non-trivial degree in homology localised at the prime 2 is a 2-local homotopy equivalence. The map ψ' factors through $\Omega S^{2n_j+2}_{(2)}$, implying that $\Omega S^{4n+3}_{(2)}$ retracts off $\Omega S^{2n_j+2}_{(2)}$. However, since $n_j \notin \{0,1,3\}$, (CPS, Corollary 5.2) implies that $\Omega S^{2n_j+2}_{(2)}$ is also atomic, and atomic spaces are indecomposable. Hence, $\overline{y}_j$ must be even. $\square$

Combining Lemma 4.5 and Lemma 4.7, $\sum_{i=1}^{m_Y} y_i \gamma_i + \sum_{j=1}^{m_{\overline{Y}}} \overline{y}_j \overline{\gamma}_j$ is an element in the image of $(\phi')_{4n+2}$ as described in Subsection 4.2. Therefore, we can use the argument in (S, Subsection 3.3) to show the following.

Lemma 4.8. *Let $X \in \prod(\mathcal{P} \cup \mathcal{T})$, and A be a space which retracts off X. Suppose that the only generators in $H_*(A)$ which reduce to primitive generators in rational homology are in degrees $4n+2$, $n \geq 2$, $n \neq 3$. There is a homotopy equivalence*

$$A \simeq P \times T,$$

where P is a product of loops on spheres of the forms ΩS^{4n+3}, where $n \geq 2$, $n \neq 3$, and the homology of T is torsion. $\square$

4.6 Conclusion of proof

Returning to the general case, let $X \in \prod(\mathcal{P} \cup \mathcal{T})$, and A be a space which retracts off X. By Lemma 4.4, there is a homotopy equivalence $A \simeq P \times A'$, where P is a product of spheres and loops on spheres of the forms S^n, where $n \in \{1, 3, 7\}$, and ΩS^{m+1}, where $m \in \{2, 6, 14, 4m, 2l - 1 \mid m \geq 1, l \notin \{1, 2, 4\}\}$. Moreover, the only generators in $H_*(A')$ which reduce to primitive generators in rational homology are in degrees of the form $4k + 3$, where $k \geq 2$ and $k \neq 3$. Lemma 4.8 then implies there is a homotopy equivalence $A' \simeq P' \times T$ where P' is a product of loops on spheres of the forms ΩS^{4n+3}, where $n \geq 2$, $n \neq 3$, and the homology of T is torsion. Combining these, we obtain a homotopy equivalence

$$A \simeq P \times P' \times T,$$

where $P, P' \in \prod \mathcal{P}$, and the homology of T is torsion. Since A retracts off X, T retracts off X, and Theorem 4.2 implies that $T \in \prod \mathcal{T}$. Summarising, we have obtained the following result.

Theorem 4.9. *Let $X \in \prod(\mathcal{P} \cup \mathcal{T})$ and A be a space which retracts off X. Then $A \in \prod(\mathcal{P} \cup \mathcal{T})$.* □

5 Preliminary decompositions of Moment-angle complexes

In this section, we prove some relations between spaces in $\bigvee(\mathcal{W} \cup \mathcal{M})$ and $\prod(\mathcal{P} \cup \mathcal{T})$. These will be generalisations of the relations between spaces in $\bigvee \mathcal{W}$ and $\prod \mathcal{P}$ shown in (S, Subsection 2.5). Before proving these relations, we require a result of (N3, Corollary 6.6), which gives a wedge decomposition for certain smash products of Moore spaces.

Lemma 5.1. *Let p and q be primes, and $r, s \geq 1$ such that $\max\{p^r, q^s\} > 2$. If $p \neq q$, then $P^n(p^r) \wedge P^m(q^s)$ is contractible. If $p = q$, there is a homotopy equivalence*

$$P^n(p^r) \wedge P^m(p^s) \simeq P^{n+m}(p^{\min\{r,s\}}) \vee P^{n+m-1}(p^{\min\{r,s\}}).$$

□

Lemma 5.2. *The following hold:*

1. *let A be a space in $\prod \mathcal{T}$, then $\Sigma A \in \bigvee \mathcal{M}$;*
2. *let $A \in \bigvee(\mathcal{W} \cup \mathcal{M})$, then $\Omega A \in \prod(\mathcal{P} \cup \mathcal{T})$;*
3. *let $A \in \prod(\mathcal{P} \cup \mathcal{T})$, then $\Sigma A \in \bigvee(\mathcal{W} \cup \mathcal{M})$;*

4. *let X be a space such that $\Sigma X \in \bigvee(\mathcal{W} \cup \mathcal{M})$, and let $A_1, \cdots, A_m$ be spaces in $\prod(\mathcal{P} \cup \mathcal{T})$, then $\Sigma(X \wedge A_1 \wedge \cdots A_m) \in \bigvee(\mathcal{W} \cup \mathcal{M})$;*

5. *let X and Y are spaces such that $\Sigma X \in \bigvee(\mathcal{W} \cup \mathcal{M})$, and $\Omega Y \in \prod(\mathcal{P} \cup \mathcal{T})$, then we obtain $\Omega(X \ltimes Y) \in \prod(\mathcal{P} \cup \mathcal{T})$;*

6. *let $X_1, \cdots, X_m$ be spaces such that $\Omega X_i \in \prod(\mathcal{P} \cup \mathcal{T})$, then $\Omega(\bigvee_{i=1}^m X_i) \in \prod(\mathcal{P} \cup \mathcal{T})$.*

Proof. For part (1), since $A \in \prod \mathcal{T}$, there is a homotopy equivalence $A \simeq \prod_{i \in \mathcal{I}} T_i$, where each T_i is an indecomposable space in the loop space decomposition of $\Omega(\bigvee_{j \in \mathcal{J}} P^{n_i}(p_i^{r_i}))$. In particular, ΣT_i retracts off $\Sigma\Omega(\bigvee_{j \in \mathcal{J}} P^{n_i}(p_i^{r_i}))$. Since each $P^{n_i}(p^r)$ is a suspension, the James splitting implies that there is a homotopy equivalence

$$\Sigma\Omega(\bigvee_{j \in \mathcal{J}} P^{n_i}(p_i^{r_i})) \simeq \bigvee_{j \geq 1}(\bigvee_{j \in \mathcal{J}} P^{n_i}(p_i^{r_i}))^{\wedge j}. \tag{5.1}$$

Distributing the wedge sum over the smash, we obtain a wedge of spaces which are smashes of Moore spaces, where by Lemma 5.1, each wedge summand consists of Moore spaces of a fixed prime. By Lemma 5.1, if $p^r \neq 2$, then the smash products decompose further as a wedge of Moore spaces. If $p^r = 2$, then by Proposition 2.1, this decomposes as a finite type wedge of indecomposable spaces which appear in the unique decomposition of smashes of mod 2 Moore spaces. Therefore, by Theorem 3.5 and by definition of $\mathcal{M}$, $\Sigma T_i \in \bigvee(\mathcal{W} \cup \mathcal{M})$. Hence, iterating the homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$, and shifting the suspension coordinate, we obtain that $\Sigma A \in \bigvee \mathcal{M}$.

For part (2), write A as

$$A \simeq \bigvee_{i \in \mathcal{I}} S^{n_i} \vee \bigvee_{j \in \mathcal{J}} P^{m_j}(p_j^{r_j}).$$

The Hilton Milnor theorem implies there is a homotopy equivalence

$$\Omega A \simeq \prod_{k \in \mathcal{K}} \Omega\Sigma(A_1 \wedge \cdots \wedge A_k),$$

where $\mathcal{K}$ is some indexing set, and each A_i is either a sphere or a Moore space.

Consider the term $A'_k = \Omega\Sigma(A_1 \wedge \cdots \wedge A_k)$. If each A_l is a sphere, then A'_k is homeomorphic to the loops on a sphere. If there exists an A_l which is a Moore space, then by Lemma 5.1, A'_k is either contractible, the loops on a wedge of Moore spaces of the form $P^{n_j}(p^r)$ for $n_j \geq 3$, p a fixed prime, and $r \geq 1$ fixed, or the smash product of mod 2 Moore spaces. In the first two cases, it is clear by definition of $\mathcal{T}$ that $A'_k \in \prod \mathcal{T}$. For the latter case, consider $\Omega\Sigma(P^{n_1}(2) \wedge \cdots \wedge P^{n_l}(2))$, where $l \geq 2$, and $n_i \geq 2$. The Hilton-Milnor thoerem implies there is a homotopy equivalence

$$\Omega\Sigma\left(\bigvee_{i=1}^{l} P^{n_i}(2)\right) \simeq \prod_{i \in B} \Omega\Sigma(P^{n_1}(2)^{\wedge b(1)} \wedge \cdots \wedge P^{n_l}(2)^{\wedge b(l)}),$$

where B is a Hall basis of the free ungraded Lie algebra on $\{1, \cdots, l\}$ over $\mathbb{Z}$, and $b(i)$ is the number of times i appears in the bracket b. By construction of a Hall basis (see (N1, p.120) for example), the smash product $\Omega\Sigma(P^{n_1}(2) \wedge \cdots \wedge P^{n_l}(2))$ appears as a product term, and so $\Omega\Sigma(P^{n_1}(2) \wedge \cdots \wedge P^{n_l}(2))$ retracts off $\Omega\Sigma(\bigvee_{i=1}^{l} P^{n_i}(2))$. Therefore, Theorem 4.2 implies $\Omega\Sigma(P^{n_1}(2) \wedge \cdots \wedge P^{n_l}(2)) \in \prod \mathcal{T}$. Hence, each $A'_k \in \prod \mathcal{T}$, and so we obtain that $\Omega A \in \prod(\mathcal{P} \cup \mathcal{T})$.

For part (3), write A as

$$A \simeq \prod_{i \in \mathcal{I}} S^{n_i} \times \prod_{j \in \mathcal{J}} \Omega S^{m_i} \times \prod_{k \in \mathcal{K}} T_i,$$

where $T_i \in \mathcal{T}$. Iterating the homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \vee Y)$, we obtain a homotopy equivalence

$$\Sigma A \simeq \bigvee_{l \in \mathcal{L}} \Sigma(A_1 \wedge \cdots \wedge A_l), \tag{5.2}$$

where each A_i is either a sphere, the loops on a sphere, or some T_j. Consider $A'_l = \Sigma(A_1 \wedge \cdots \wedge A_l)$. By the James construction, $\Sigma\Omega S^n \in \bigvee \mathcal{W}$, and by part (a), each $\Sigma T_k \in \bigvee \mathcal{M}$. By shifting the suspension coordinate, we can decompose A'_l as a wedge of spaces $W'_{l'}$, where each $W'_{l'}$ is the suspension of a smash product of spheres and Moore spaces. If each space is a sphere, then W'_l is a sphere. If there is a Moore space, then by Lemma 5.1, $W_{l'}$ can be decomposed further as a wedge of spaces, each of which is either a Moore space, or a smash of mod 2 Moore spaces. In either case, by definition of $\mathcal{M}$, $W_{l'} \in \bigvee \mathcal{M}$, and so $A'_l \in \bigvee(\mathcal{W} \cup \mathcal{M})$. Therefore, by (5.2), $A \in \bigvee(\mathcal{W} \cup \mathcal{M})$.

The remaining parts follow by arguing as in (S, Lemma 2.13, Lemma 2.14, Corollary 2.16) respectively. □

The next result shows the property of $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$ is closed under pushouts of simplicial complexes over full subcomplexes.

Theorem 5.3. *Let K be a simplicial complex defined as the pushout*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & K_1 \arrow[d] \\ K_2 \arrow[r] & K \end{tikzcd}$$

where either $L = \emptyset$ or L is a proper full subcomplex of K_1 and K_2. If $\Sigma A_i \in \bigvee(\mathcal{W} \cup \mathcal{M})$ for all i, $\Omega(\underline{CA}, \underline{A})^{K_1} \in \prod(\mathcal{P} \cup \mathcal{T})$ and $\Omega(\underline{CA}, \underline{A})^{K_2} \in \prod(\mathcal{P} \cup \mathcal{T})$, then $\Omega(\underline{CA}, \underline{A})^{K} \in \prod(\mathcal{P} \cup \mathcal{T})$.

Proof. The case where $\Sigma A_i \in \bigvee \mathcal{W}$, and each loop space is in $\prod \mathcal{P}$ was proved in (S, Theorem 4.1). The proof depended on two results: closure of $\prod \mathcal{P}$ under retracts and the $\bigvee \mathcal{W}$ analogue of Lemma 5.2. The same argument holds for $\prod(\mathcal{P} \cup \mathcal{T})$ using Theorem 4.9, and Lemma 5.2. $\square$

Using Theorem 2.9, we can obtain a generalisation of (S, Theorem 1.1). To do this, we require the following pushout decomposition for a simplicial complex K from (S, Lemma 4.4).

Lemma 5.4. *Let K be a simplicial complex and $v \in V(K)$. Then K can be written as the pushout*

$$\begin{array}{ccc} K_{N(v)} & \longrightarrow & K_{v \cup N(v)} \\ \downarrow & & \downarrow \\ K_{V(K)\setminus\{v\}} & \longrightarrow & K. \end{array}$$

Moreover, $K_{N(v)}$ is a full subcomplex of both $K_{v\cup N(v)}$ and $K_{V(K)\setminus\{v\}}$. $\square$

For a simplicial complex K, let C_K be the collection of full subcomplexes of K whose 1-skeleton has no missing edges. For a vertex $v \in V(K)$, denote by $N(v)$ the set of vertices adjacent to v in the 1-skeleton of K. A *dominating vertex* of K is a vertex v such that $N(v) = V(K) \setminus \{v\}$. In other words, v is adjacent to every other vertex in the 1-skeleton of K. The following result is a generalisation of (S, Theorem 1.1)

Theorem 5.5. *If K is a simplicial complex, then $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$ if and only if $\Omega\mathcal{Z}_{K_I} \in \prod(\mathcal{P} \cup \mathcal{T})$ for all $K_I \in C_K$. Moreover, $\Omega\mathcal{Z}_K \in \prod \mathcal{P}$ if and only if $\Omega\mathcal{Z}_{K_I} \in \prod \mathcal{P}$ for all $K_I \in C_K$.*

Proof. Suppose $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$ but there exists $K_I \in C_K$ such that $\Omega\mathcal{Z}_{K_I} \notin \prod(\mathcal{P} \cup \mathcal{T})$. By (DS), $\mathcal{Z}_{K_I}$ retracts off $\mathcal{Z}_K$, and so $\Omega\mathcal{Z}_{K_I}$ retracts off $\Omega\mathcal{Z}_K$. Theorem 4.9 then implies that $\Omega\mathcal{Z}_{K_I} \in \prod(\mathcal{P} \cup \mathcal{T})$ which is a contradiction.

Now suppose that $\Omega\mathcal{Z}_{K_I} \in \prod(\mathcal{P} \cup \mathcal{T})$ for all $K_I \in C_K$. We proceed by strong induction. If K has one vertex, then $\mathcal{Z}_K$ is contractible, and so $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$.

Now suppose K has m vertices, and the result is true for all $n < m$. If every vertex of K is a dominating vertex, then every vertex in K is adjacent to every other vertex in K. Hence, $K \in C_K$ so by assumption, $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$. Therefore, suppose there exists a vertex $v \in V(K)$ such that v is not a dominating vertex of K. By Lemma 5.4, K can be written as the pushout

$$\begin{array}{ccc} K_{N(v)} & \longrightarrow & K_{v \cup N(v)} \\ \downarrow & & \downarrow \\ K_{V(K)\setminus\{v\}} & \longrightarrow & K \end{array}$$

where $K_{N(v)}$ is a full subcomplex of both $K_{v\cup N(v)}$ and $K_{V(K)\setminus\{v\}}$. Since v is not a dominating vertex, $v \cup N(v)$ is not the whole vertex set of K, so $K_{v\cup N(v)}$ and $K_{V(K)\setminus\{v\}}$ are simplicial complexes with strictly fewer vertices than K. Therefore, by the inductive hypothesis, $\Omega\mathcal{Z}_{K_{v\cup N(v)}} \in \prod(\mathcal{P}\cup\mathcal{T})$ and $\Omega\mathcal{Z}_{K_{V(K)\setminus\{v\}}} \in \prod(\mathcal{P}\cup\mathcal{T})$. Hence, Theorem 2.9 implies that $\Omega\mathcal{Z}_K \in \prod(\mathcal{P}\cup\mathcal{T})$.

In the special case where $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$, the same proof follows through. □

6 Loop spaces of moment-angle complexes associated to 2-dimensional simplicial complexes

In this section, we consider moment-angle complexes associated to 2-dimensional simplicial complexes. We start with a more general statement. To prove this, we require the following from (IK, Theorem 1.6). Recall that a simplicial complex K is called k-neighbourly if any $I \subseteq [m]$ with $|I| \leq k+1$ spans a simplex. For $x \in \mathbb{R}$, define $\lceil x \rceil$ to be the smallest integer z such that $z \geq x$.

Lemma 6.1. *Let K be a simplicial complex on $[m]$. If K is $\lceil \frac{\dim(K)}{2} \rceil$-neighbourly, then there is a homotopy equivalence*

$$\mathcal{Z}_K \simeq \bigvee_{I \notin K} \Sigma^{1+|I|}|K_I|.$$ □

We also require a wedge decomposition of $(n-1)$ connected, $(n+1)$-dimensional CW-complexes, where $n \geq 2$. This is proved in (H, Example 4C.2) for example.

Lemma 6.2. *Let X be an $(n-1)$ connected, $(n+1)$-dimensional CW-complex, where $n \geq 2$. Then X is homotopy equivalent to a wedge of spheres and Moore spaces. If $H_*(X)$ is torsion free, then $X \in \bigvee\mathcal{W}$.* □

Theorem 6.3. *If K is a simplicial complex on $[m]$ such that each $K_I \in C_K$ is $\dim(K_I)-1$-neighbourly, then $\Omega\mathcal{Z}_K \in \prod(\mathcal{P}\cup\mathcal{T})$. If $H_*(|K_I|;\mathbb{Z})$ is torsion free for all $K_I \in C_K$, then $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$.*

Proof. By Theorem 5.5, it suffices to show that $\Omega Z_{K_I} \in \prod(\mathcal{P}\cup\mathcal{T})$, where $K_I \in C_K$. If $\dim(K_I) = 1$, then K_I is the 1-skeleton of a simplex, and so by (GT, Theorem 9.1), $\mathcal{Z}_{K_I}$ is homotopy equivalent to a wedge of simply connected spheres. The Hilton-Milnor theorem then implies that $\Omega\mathcal{Z}_{K_I} \in \prod\mathcal{P}$. If $\dim(K_I) > 1$, then $\dim(K_I) - 1 \geq \lceil \frac{\dim(K_I)}{2} \rceil$, and so Lemma 6.1 implies that there is a homotopy equivalence

$$\mathcal{Z}_{K_I} \simeq \bigvee_{J \notin K_I} \Sigma^{1+|J|}|K_J|.$$

Since K_I is $\dim(K_I)-1$-neighbourly, each K_J is $\dim(K_I)-1$-neighbourly. Moreover, each K_J is a full subcomplex of K_I, and so $\dim(K_J) \leq \dim(K_I)$. Hence, each K_J is at least $\dim(K_J) - 1$ neighbourly. By (IK, Lemma 10.8) for example, this implies that each K_J is at least $(\dim(K_J) - 2)$-connected. Since $\Sigma^{1+|J|}|K_J|$ is simply connected, by Lemma 6.2, each $\Sigma^{1+|J|}|K_J|$ is homotopy equivalent to a wedge of spheres and Moore spaces. Hence part (2) of Lemma 5.2 implies that $\Omega\Sigma^{1+|J|}|K_J| \in \prod(\mathcal{P} \cup \mathcal{T})$, and therefore part (6) of Lemma 5.2 implies that $\Omega\mathcal{Z}_{K_I} \in \prod(\mathcal{P} \cup \mathcal{T})$. Hence, $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$.

If $H_*(|K_I|;\mathbb{Z})$ is torsion free for all $K_I \in C_K$, then by Lemma 6.2, each $\Sigma^{1+|J|}|K_J|$ is homotopy equivalent to a wedge of spheres. The Hilton-Milnor theorem implies that $\Omega\Sigma^{1+|J|}|K_J| \in \prod\mathcal{P}$, and so Theorem 5.5 implies that $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$. □

Theorem 5.5 and Theorem 6.3 can be applied to give a coarse decomposition of the loops on a moment-angle complex associated to any 2-dimensional simplicial complex.

Theorem 6.4. *Let K be a 2-dimensional simplicial complex. Then $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$.*

Proof. If $K_I \in C_K$ is 1-dimensional, then by (GT, Theorem 9.1), $\mathcal{Z}_{K_I}$ is homotopy equivalent to a wedge of simply connected spheres. The Hilton-Milnor theorem implies that $\Omega\mathcal{Z}_{K_I} \in \prod\mathcal{P}$, which is contained in $\prod(\mathcal{P} \cup \mathcal{T})$. If $K_I \in C_K$ is 2-dimensional, then it is 1-neighbourly and Theorem 6.3 implies that $\Omega\mathcal{Z}_{K_I} \in \prod(\mathcal{P} \cup \mathcal{T})$. Since each $K_I \in C_K$ is 1-dimensional or 2-dimensional, Theorem 5.5 implies that $\Omega\mathcal{Z}_K \in \prod(\mathcal{P} \cup \mathcal{T})$. □

As an example of Theorem 6.4 which contains torsion, let K be the 6-vertex triangulation of $\mathbb{R}P^2$. In this case, K is a 1-neighbourly simplicial complex whose homology contains 2-torsion, and so the decomposition in Theorem 6.4 will contain indecomposable 2-torsion spaces. In (LMR), 1-neighbourly, 2-dimensional simplicial complexes which have arbitrarily large torsion in homology are constructed. Therefore, the loop space of a moment-angle complex associated to a 2 dimensional simplicial complex can contain arbitrarily large torsion in homology. In the case that each $|K_I|$ has torsion free homology, where $K_I \in C_K$, we obtain the following.

Corollary 6.5. *Let K be a 2-dimensional simplicial complex. Suppose that $H_*(|K_I|;\mathbb{Z})$ is torsion free for all $K_I \in C_K$. Then $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$.* □

7 Loop space decompositions of certain moment-angle complexes after localisation

In this section, we use the argument in Theorem 6.3 to show that for a simplicial complex K under certain conditions on the full subcomplexes $K_I \in C_K$, there is a finite set of primes P for which $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$ localised away from P. This set of primes is controlled purely by the underlying simplicial complex. Recall that a simplicial complex K is rationally Golod if all products and higher Massey products in $H^*(\mathcal{Z}_K;\mathbb{Q})$ are trivial.

Theorem 7.1. *Let K be a simplicial complex such that K_I is rationally Golod for all $K_I \in C_K$. For $K_I \in C_K$, let K_I be k_I-neighbourly, and let $M = \max_{K_I \in C_K}\{|I| + dim(K_I) - 2k_I\}$. Localise away from primes $p \leq \frac{1}{2}M$ and primes p appearing as p-torsion in $H_*(|K_I|;\mathbb{Z})$ for any $K_I \in C_K$. Then $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$.*

Proof. First, note that Theorem 5.5 holds p-locally for any prime p. By Theorem 5.5, it suffices to show $\Omega\mathcal{Z}_{K_I} \in \prod\mathcal{P}$ for each $K_I \in C_K$, after localisation away from primes $p \leq \frac{1}{2}M$ and primes p appearing as p-torsion in $H_*(|K_I|;\mathbb{Z})$ for any $K_I \in C_K$. Consider $\mathcal{Z}_{K_I}$, where $K_I \in C_K$. By assumption, K_I is rationally Golod. Therefore, by Proposition 2.4 and Proposition 2.5, there is a rational homotopy equivalence

$$\mathcal{Z}_{K_I} \simeq \bigvee_{J \subseteq I} \Sigma^{1+|J|}|K_J|.$$

Since each wedge summand is a suspension, $\mathcal{Z}_{K_I}$ is rationally a wedge of spheres. It can be shown using Proposition 2.5 that $\mathcal{Z}_{K_I}$ is $(2k_I + 2)$-connected, and has dimension $1 + |I| + dim(K_I)$. Therefore, by Lemma 2.3, localised away from

$$p \leq \frac{1}{2}(1 + |I| + dim(K_I) - (2k_I + 2) + 1) = \frac{1}{2}(|I| + dim(K_I) - 2k_I) \leq \frac{1}{2}M$$

and those primes p appearing as p-torsion in $H_*(|K_J|;\mathbb{Z})$, $\mathcal{Z}_{K_I}$ is homotopy equivalent to a wedge of spheres. The Hilton-Milnor theorem then implies that $\Omega\mathcal{Z}_{K_I} \in \prod\mathcal{P}$. Repeating this argument for each $K_I \in C_K$, by Lemma 5.5, $\Omega\mathcal{Z}_{K_I} \in \prod\mathcal{P}$ when localised away from primes $p \leq \frac{1}{2}M$ and any primes appearing as p-torsion in $H_*(|K_I|;\mathbb{Z})$ for each $K_I \in C_K$. □

Theorem 7.1 has interesting connections to a problem posed by McGibbon and Wilkerson. Recall from the Introduction that a space X is called *rationally elliptic* if it has finitely many rational homotopy groups. Otherwise, it is called *rationally hyperbolic*. McGibbon and Wilkerson (MW) showed that any finite, simply connected, rationally elliptic space X has $\Omega X \in \prod\mathcal{P}$, after localising at a sufficiently large prime. A consequence of a decomposition of this form pointed out is that the Steenrod algebra acts trivially on $H_*(\Omega X;\mathbb{Z}/p\mathbb{Z})$ at almost all primes p. They asked the extent to which this holds for rationally hyperbolic spaces. Theorem 7.1 gives an analogous result for the moment-angle complexes in Theorem 7.1.

Corollary 7.2. *Let K be a simplicial complex such that K_I is rationally Golod for all $K_I \in C_K$. Then at all but finitely many p, the Steenrod algebra acts trivially on $H^*(\Omega\mathcal{Z}_K;\mathbb{Z}/p\mathbb{Z})$.* □

One particular family of examples is when K is a 2-dimensional simplicial complex, Lemma 6.1 implies that every $K_I \in C_K$ is rationally Golod, and so we obtain the following.

Corollary 7.3. *Let K be a 2-dimensional simplicial complex. Then at all but finitely many primes p, the Steenrod algebra acts trivially on $H^*(\Omega\mathcal{Z}_K;\mathbb{Z}/p\mathbb{Z})$.* □

It was shown in (BBCG2) that the moment-angle complex associated to K is rationally elliptic if and only if the minimal missing faces of K are mutually disjoint. In particular, if K is a simplicial complex such that there is a vertex which is not adjacent to two other vertices, then $\mathcal{Z}_K$ is rationally hyperbolic. Corollary 7.3 can be used to generate infinitely many rationally hyperbolic examples for which the answer to the question of McGibbon and Wilkerson is affirmative. One other point to note is that the set of primes for which the decomposition of rationally elliptic spaces given by McGibbon and Wilkerson holds is not explicit. The conditions on K in Theorem 7.1 give an explicit set of primes for which such a decomposition holds for $\Omega\mathcal{Z}_K$. Also, this set of primes can certainly be enlarged depending on the combinatorics of K. For example, if K is a 2-dimensional simplicial complex such that $H_*(|K_I|;\mathbb{Z})$ is torsion free for all $K_I \in C_K$, Corollary 6.5 implies that such a decomposition holds integrally.

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Chapter 4

Paper 3 - Polyhedral products associated to pseudomanifolds

1 Introduction

Polyhedral products are subspaces of the Cartesian product, the properties of which are governed by an underlying simplicial complex. They unify various constructions across mathematics, such as complements of complex coordinate subspace arrangements in combinatorics, intersections of quadrics in complex geometry and classifying spaces of graph products of groups in geometric group theory. Understanding their homotopy theory has implications in all these areas. In this paper, we study the homotopy theory of polyhedral products associated to a family of simplicial complexes known as pseudomanifolds.

Let K be a simplicial complex on the vertex set $[m] = \{1, 2, \cdots, m\}$. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed CW-subcomplex of X_i. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be the sequence of pairs. For each simplex $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i \text{ where } Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i.$$

A particularly important special case is when $(X_i, A_i) = (D^2, S^1)$ for all i. These polyhedral products are called *moment-angle complexes*, and are denoted $\mathcal{Z}_K$.

One aspect of the homotopy theory of polyhedral products that has been the subject of intense study recently is the homotopy type of their based loop spaces. Let $\prod \mathcal{P}$ be the collection of H-spaces homotopy equivalent to a finite type product of spheres and loops on spheres. If X is a simply-connected space, there are advantages to knowing that $\Omega X \in \prod \mathcal{P}$: it means the homology of ΩX is torsion-free, the Steenrod algebra acts trivially on the mod-p cohomology of ΩX for any prime p, and if the factors in the decomposition of ΩX are explicit, then the homotopy groups of X are known to the same extent as the homotopy groups of spheres. Various families of polyhedral products have been shown to have their loop space in $\prod \mathcal{P}$, including flag complexes (PT; V), graphs (St1), 2-dimensional simplicial complexes with torsion free homology (St2) and certain polyhedral join products (E). In this paper, we focus on the case when K is the triangulation of a sphere.

If K is the triangulation of a sphere, then $\mathcal{Z}_K$ has the structure of a manifold, and is known as a *moment-angle-manifold*. The diffeomorphism type of $\mathcal{Z}_K$ is known for an important family of triangulations. If P is a simple polytope obtained from a simplex by iterated vertex cuts and K is the dual of the boundary of P, then $\mathcal{Z}_K$ is diffeomorphic to a connected sum of products of two spheres. This statement originated in work of MacGavran (M) pre-dating polyhedral products, took a spectacular leap forward in work of Bosio and Meersseman (BM) and Gitler and López de Medrano (GLdM) on intersections of quadrics, and culminated in the solution of a conjecture in (GLdM) by Chen, Fan and Wang (CFW). However, very little is known about even the homotopy type of moment-angle manifolds for triangulations of spheres outside this family.

In this paper we develop new methods to study the homotopy type of $\Omega\mathcal{Z}_K$ for a combinatorial generalisation of triangulations of spheres known as pseudomanifolds. The collection of pseudomanifolds include triangulations of manifolds. We establish criteria for when a polyhedral product of the form $(\underline{CA}, \underline{A})^K$ with K a pseudomanifold has $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$. In particular, we prove the following.

Theorem 1.1. *If K is the triangulation of a connected, orientable, closed surface on $[m]$, then $\Omega\mathcal{Z}_K \in \prod \mathcal{P}$.*

This includes the case when K is a triangulation of S^2. We also obtain an analogous result when K is a triangulation of S^3.

Theorem 1.2. *Let K be a triangulation of S^3 on $[m]$. Then $\Omega\mathcal{Z}_K \in \prod \mathcal{P}$.*

The argument proving Theorem 1.2 breaks into two cases, one of which can be generalised to certain triangulations of any odd dimensional sphere. A simplicial complex is *k-neighbourly* if every set of $k+1$ vertices spans a simplex. A triangulation K of S^{2n+1} is *neighbourly* if K is n-neighbourly.

Theorem 1.3. *If K is a neighbourly triangulation of S^{2n+1} on $[m]$ with $n \geq 1$, then $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$.*

The methods used to prove Theorems 1.1, 1.2 and 1.3 involve a new approach to studying how the homotopy type of a polyhedral product is affected by the removal of a maximal face. This is likely to be of wider use. It is inspired by how certain simply-connected manifolds (not necessarily moment-angle manifolds) have their loop spaces retracting off the loops of the associated punctured manifold (T), and makes use of key properties proved in (St1; St2) that generate retractions of looped polyhedral products with respect to $\prod\mathcal{P}$.

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2 Preliminary material

This section collects some preliminary combinatorial and homotopy theoretic information.

Pseudomanifolds. A simplicial complex K of dimension n is called *pure* if every simplex is contained in at least one n-simplex. To any pure simplicial complex K of dimension n, there is an associated graph $D(K)$ called the *dual graph* of K. The vertices of $D(K)$ are given by the n-simplices of K, and two vertices in $D(K)$ are adjacent if and only if their respective faces in K intersect over a face of codimension one.

A simplicial complex K of dimension n is called a *weak pseudomanifold with boundary* if every face of codimension one is contained in either one or two maximal faces. The *boundary* of a weak pseudomanifold K is the simplicial complex whose maximal faces are given by the codimension one faces of K which are contained in exactly one maximal face. If the boundary is empty then K is a *weak pseudomanifold*.

A simplicial complex K is a *pseudomanifold of dimension n* if: (i) it is a pure simplicial complex of dimension n, (ii) it is a weak pseudomanifold, and (iii) $D(K)$ is a connected graph. The definition of a *pseudomanifold with boundary* is analogous. Triangulations of manifolds are examples of pseudomanifolds.

Two combinatorial statements. We first describe a general graph theoretic result. For a graph G, let $V(G)$ be the vertex set of G. For a vertex v in a graph G, the *degree* of v, denoted $\deg_G(v)$, is the number of edges incident to v.

Lemma 2.1. *Let G be a connected simple graph on m vertices, and let $n \in \mathbb{N}$. Suppose $\deg_G(v) \leq n$ for all $v \in V(G)$ and there exists a vertex $w \in V(G)$ such that $\deg_G(w) < n$. Then there exists an ordering of the vertices $v_1, \cdots, v_m$ such that $\deg_G(v_1) < n$ and $\deg_{G \setminus \{v_1, \cdots, v_{i-1}\}}(v_i) < n$ for $2 \leq i \leq m$.*

Proof. The proof is by induction on the number of vertices. Suppose $|V(G)| \leq n$. The maximum degree of a vertex in such a graph is $n-1$, and so any ordering of the vertices suffices in this case.

Suppose that $|V(G)| = k > n$ and the result is true for all connected graphs H with $|V(H)| < k$. Let v_1 be a vertex of G such that $\deg_G(v_1) < n$. Since G is connected, $\deg_G(v_1) \geq 1$. Consider $G \setminus v_1$. By hypothesis, $\deg_G(v) \leq n$ for each vertex $v \in G \setminus v_1$, so $\deg_{G \setminus v_1}(v) \leq n$. Moreover, since $\deg_G(v_1) \geq 1$, there exists a vertex $v \in G \setminus v_1$ which is adjacent to v_1 in G. It follows that $\deg_{G \setminus v_1}(v) < n$. There are two cases to consider. If $G \setminus v_1$ is connected, then the inductive hypothesis implies there is an ordering of the vertices $v_2, \cdots, v_m$ of $G \setminus v_1$ such that the statement holds. Therefore, the ordering $v_1, \cdots, v_m$ implies the result is true for G.

If $G \setminus v_1$ is disconnected, denote by $C_1, \cdots, C_l$ the connected components of $G \setminus v_1$, and let C_i have d_i vertices. Let $x \in C_i$ and $y \in C_j$, where $i \neq j$. Since G is connected and $G \setminus v_1$ is disconnected, any path between x and y must pass through v_1. It follows that in G, for each $1 \leq i \leq l$, the vertex v_1 must be adjacent to some vertex $c_i \in C_i$, implying that $\deg_{G \setminus v_1}(c_i) < n$. Therefore, each C_i is a connected graph with strictly less vertices than G which satisfies the hypotheses in the statement of the lemma. The inductive hypothesis implies that there is an ordering of the vertices $c_{i_1}, \cdots, c_{i_{d_i}}$ of C_i such that the result holds for C_i. The ordering of the vertices $v_1, c_{1_1}, \cdots, c_{1_{d_1}}, \cdots, c_{l_1}, \cdots, c_{l_{d_k}}$ therefore implies the result is true for G. □

Next, we describe a pushout decomposition for certain simplicial complexes K. Let $\sigma \in K$ be a maximal face and let $\partial\sigma$ be the boundary of σ. Let $K \backslash \sigma$ be the simplicial complex K with the interior of the face σ removed.

Lemma 2.2. *Let K be a simplicial complex and let σ be a maximal face of K. There exists a subcomplex L of K such that there is a pushout*

$$\begin{array}{ccc} \partial\sigma \cap L & \longrightarrow & \partial\sigma \\ \downarrow & & \downarrow \\ L & \longrightarrow & K \setminus \sigma \end{array}$$

with $\partial\sigma \cap L \neq \partial\sigma$ if and only if there exists a face $\tau \in \partial\sigma$ with $|\tau| = |\sigma| - 1$ which is maximal in $K \setminus \sigma$. Moreover, L can be chosen to be $K \setminus \{\sigma, \tau\}$ when it exists.

Proof. Suppose that L exists but all maximal faces τ with respect to $\partial\sigma$ are not maximal in $K \setminus \sigma$. This implies there is a face $\gamma_\tau \in K \setminus \sigma$ such that $\tau \subset \gamma_\tau$. Since τ is maximal with respect to $\partial\sigma$, the pushout implies that γ_τ must be contained in L, which in turn implies that $\tau \in L$. This is true for all maximal faces $\tau \in \partial\sigma$, so $\partial\sigma \cap L = \partial\sigma$, which is a contradiction.

Conversely, let τ be a maximal face with respect to $\partial\sigma$ which is also maximal with respect to $K \setminus \sigma$. Define $L = K \setminus \{\sigma, \tau\}$. By definition, $\partial\sigma \cap L = \partial\sigma \setminus \tau \neq \partial\sigma$. Now let $\gamma \in K \setminus \sigma$ be such that $\gamma \notin \partial\sigma$. Since τ is maximal in $K \setminus \sigma$, we have $\tau \notin \gamma$, implying that $\gamma \in L$. □

Remark 2.3. It is worth noting for what comes in the next section that if σ has dimension $d \geq 1$ then $K \setminus \sigma$ has the same vertex set as K. If σ has dimension $d \geq 2$ then $L = K \setminus \{\sigma, \tau\}$ also has the same vertex set as K.

Spaces in $\prod \mathcal{P}$. Recall that $\prod \mathcal{P}$ is the collection of H-spaces that are homotopy equivalent to a finite type product of spheres and loops on spheres. We state some properties of the collection $\prod \mathcal{P}$ that will be needed. In (St1, Theorem 3.10) it was shown that the property of being in $\prod \mathcal{P}$ is preserved by retractions.

Theorem 2.4. *Let $X \in \prod \mathcal{P}$ and A be a space which retracts off X. Then $A \in \prod \mathcal{P}$.* □

One source of retractions in the context of polyhedral products come from the following result from (DS). If K is a simplicial complex on the vertex set $[m]$ and $I \subseteq [m]$ then the *full subcomplex* K_I of K on I is the subcomplex of K consisting of the faces of K whose vertices are all in I.

Lemma 2.5. *If K is a simplicial complex, and K_I is a full subcomplex of K, then $(\underline{X}, \underline{A})^{K_I}$ retracts off $(\underline{X}, \underline{A})^K$.* □

We next describe two collections of polyhedral products that are in $\prod \mathcal{P}$. Let $\mathcal{W}$ be the collection of topological spaces that are homotopy equivalent to a finite type wedge of spheres. The first result was proved in (St1, Theorem 1.1) and the second in (St2, Corollary 6.5).

Theorem 2.6. *Let K be a 1-dimensional simplicial complex on $[m]$. Let $A_1, \cdots, A_m$ be spaces such that $\Sigma A_i \in \mathcal{W}$. Then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.* □

Theorem 2.7. *Let K be a 2-dimensional simplicial complex on $[m]$. Let $A_1, \cdots, A_m$ be spaces such that $\Sigma A_i \in \mathcal{W}$. If $H_*(|L|)$ is torsion free for all full subcomplexes L of K with complete 1-skeleton, then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.* □

If a space $X \in \mathcal{W}$, then the Hilton-Milnor theorem implies $\Omega X \in \prod \mathcal{P}$. A result we will use to show that a space is in $\mathcal{W}$ is the following from (H, Example 4C.2).

Lemma 2.8. *If X is a simply connected space with cells in two consecutive dimensions and $H_*(X)$ is torsion free, then $X \in \mathcal{W}$.* □

Finally, in (St1, Theorem 4.1) it was shown that $\prod \mathcal{P}$ is closed under pushouts over full subcomplexes.

Theorem 2.9. *Let K be a simplicial complex defined as the pushout*

$$\begin{tikzcd} L \arrow[r] \arrow[d] & K_1 \arrow[d] \\ K_2 \arrow[r] & K \end{tikzcd}$$

where either $L = \emptyset$ or L is a proper full subcomplex of K_1 and K_2. If $\Sigma A_i \in \mathcal{W}$ for all i, $\Omega(\underline{CA}, \underline{A})^{K_1} \in \prod \mathcal{P}$ and $\Omega(\underline{CA}, \underline{A})^{K_2} \in \prod \mathcal{P}$, then $\Omega(\underline{CA}, \underline{A})^{K} \in \prod \mathcal{P}$. □

A homotopy pushout. It will be important to identify the homotopy type of a certain homotopy pushout. For spaces X and Y, the *right half-smash* of X and Y, denoted $X \rtimes Y$, is the quotient $(X \times Y)/(* \times Y)$.

Lemma 2.10. *Suppose that there is a homotopy pushout*

$$\begin{tikzcd} A \times B \arrow[r, "* \times 1"] \arrow[d, "f"] & D \times B \arrow[d] \\ C \arrow[r] & Q \end{tikzcd}$$

where the restriction of f to B is null homotopic. Let $f' \colon A \rtimes B \longrightarrow C$ be the quotient map and let E be its homotopy cofibre. Then there is a homotopy equivalence $Q \simeq (D \rtimes B) \vee E$.

Proof. Since the restriction of f to B is null homotopic, and the map $* \times 1$ is the identity on B, the space B can be collapsed out of the diagram to give a homotopy pushout

$$\begin{tikzcd} A \rtimes B \arrow[r, "* \rtimes 1"] \arrow[d, "f"] & D \rtimes B \arrow[d] \\ C \arrow[r] & Q. \end{tikzcd}$$

The map $* \rtimes 1$ is null homotopic. Thus the previous homotopy pushout can be expanded to a diagram of iterated homotopy pushouts

$$\begin{tikzcd} A \rtimes B \arrow[r] \arrow[d, "f'"] & * \arrow[r] \arrow[d] & D \rtimes B \arrow[d] \\ C \arrow[r] & E \arrow[r] & Q. \end{tikzcd}$$

Here, in the left square the homotopy pushout of f' and the constant map is the homotopy cofibre of f', which is E, and in the right square, the homotopy pushout can be identified as Q since the outer rectangle is also a homotopy pushout. The right square itself now identifies Q as $(D \rtimes B) \vee E$. $\square$

3 The effect on polyhedral products of removing certain maximal faces

Let K be a simplicial complex on the vertex set $[m]$ and let $\sigma \in K$ be a maximal face. Let $K \backslash \sigma$ be K with the interior of the face σ removed. Observe that $\partial\sigma \subseteq K \backslash \sigma$ and there is a pushout of simplicial complexes

$$\begin{array}{ccc} \partial\sigma & \longrightarrow & \sigma \\ \downarrow & & \downarrow \\ K \setminus \sigma & \longrightarrow & K. \end{array} \tag{3.1}$$

The inclusion $K \setminus \sigma \longrightarrow K$ induces a map of polyhedral products $(\underline{CA}, \underline{A})^{K \setminus \sigma} \longrightarrow (\underline{CA}, \underline{A})^K$. In this section, conditions are given for when this map has a right homotopy inverse. Moreover, $(\underline{CA}, \underline{A})^K$ is shown to be a wedge summand of $(\underline{CA}, \underline{A})^{K \setminus \sigma}$ and the complementary wedge summand is identified.

Suppose that σ has dimension $d \geq 2$ and there exists a face $\tau \in \partial\sigma$ with $|\tau| = |\sigma| - 1$ which is maximal in $K \setminus \sigma$. Let $L = K \setminus \{\sigma, \tau\}$ and note that $\partial\sigma \cap L \neq \partial\sigma$. Combining Lemma 2.2 and (3.1), there is an iterated pushout of simplicial complexes

$$\begin{array}{ccccc} \partial\sigma \cap L & \longrightarrow & \partial\sigma & \longrightarrow & \sigma \\ \downarrow & & \downarrow & & \downarrow \\ L & \longrightarrow & K \setminus \sigma & \longrightarrow & K. \end{array} \tag{3.2}$$

As the dimension of σ is at least 2, by Remark 2.3, L, $K \setminus \sigma$ and K all have the same vertex set. If $\sigma \neq K$ then σ has a smaller vertex set than K, and we regard both σ and $\partial\sigma$ as simplicial complexes on the vertex set $[m]$, giving ghost vertices which we denote by $1 \leq i \leq m$ with $i \notin \sigma$. By (GT, Proposition 3.1), the iterated pushout of simplicial complexes in (3.2) implies that there is an iterated pushout of polyhedral products

$$\begin{array}{ccccc} (\underline{CA}, \underline{A})^{\partial\sigma \cap L} \times \prod_{i \notin \sigma} A_i & \xrightarrow{i' \times 1} & (\underline{CA}, \underline{A})^{\partial\sigma} \times \prod_{i \notin \sigma} A_i & \xrightarrow{i \times 1} & (\underline{CA}, \underline{A})^{\sigma} \times \prod_{i \notin \sigma} A_i \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\ (\underline{CA}, \underline{A})^{L} & \longrightarrow & (\underline{CA}, \underline{A})^{K \setminus \sigma} & \longrightarrow & (\underline{CA}, \underline{A})^{K} \end{array} \tag{3.3}$$

where i' is induced by the inclusion $\partial\sigma \cap L \to \partial\sigma$, i is induced by the inclusion $\partial\sigma \longrightarrow \sigma$ and f is induced by the inclusion $\partial\sigma \cap L \to \partial\sigma$. We first show that i' is null homotopic.

Lemma 3.1. *Let σ be a maximal face of K of dimension $d \geq 2$. Suppose there exists a face $\tau \in \partial\sigma$ with $|\tau| = |\sigma| - 1$ which is maximal in $K \setminus \sigma$ and let $L = K \setminus \{\sigma, \tau\}$. Then the map of polyhedral products $(\underline{CA}, \underline{A})^{\partial\sigma\cap L} \xrightarrow{i'} (\underline{CA}, \underline{A})^{\partial\sigma}$ induced by the inclusion $\partial\sigma \cap L \to \partial\sigma$ is null homotopic.*

Proof. By definition of L, $\partial\sigma \cap L = \partial\sigma \setminus \tau$. Therefore, we show that the map of polyhedral products

$$(\underline{CA}, \underline{A})^{\partial\sigma\setminus\tau} \to (\underline{CA}, \underline{A})^{\partial\sigma}$$

induced by $\partial\sigma \setminus \tau \to \partial\sigma$ is null homotopic.

Let v be the vertex of $\partial\sigma$ not contained in τ. Then $\partial\sigma \setminus \tau = v * \partial\tau$. By definition of the polyhedral product, there are homotopy equivalences,

$$(\underline{CA}, \underline{A})^{\partial\sigma\setminus\tau} \cong (\underline{CA}, \underline{A})^{\partial\tau} \times CA_v \simeq (\underline{CA}, \underline{A})^{\partial\tau},$$

and the map $(\underline{CA}, \underline{A})^{\partial\sigma\setminus\tau} \to (\underline{CA}, \underline{A})^{\partial\sigma}$, up to these homotopy equivalences, becomes the map induced by the inclusion $\partial\tau \to \partial\sigma$. However, $\tau \in \partial\sigma$, and so this map factors as $\partial\tau \to \tau \to \partial\sigma$. By definition, $(\underline{CA}, \underline{A})^{\tau}$ is contractible, and so the map induced by $\partial\tau \to \partial\sigma$ is null homotopic. □

This allows us to give a decomposition of $(\underline{CA}, \underline{A})^{K\setminus\sigma}$ in terms of $(\underline{CA}, \underline{A})^{K}$.

Theorem 3.2. *Let K be a simplicial complex and σ be a maximal face of K. Suppose there exists a face $\tau \in \partial\sigma$ with $|\tau| = |\sigma| - 1$ which is maximal in $K \setminus \sigma$. Then the map $(\underline{CA}, \underline{A})^{K\setminus\sigma} \longrightarrow (\underline{CA}, \underline{A})^{K}$ has a right homotopy inverse and there is a homotopy equivalence*

$$(\underline{CA}, \underline{A})^{K\setminus\sigma} \simeq \left((\underline{CA}, \underline{A})^{\partial\sigma} \rtimes \prod_{i\notin\sigma} A_i\right) \vee (\underline{CA}, \underline{A})^{K}.$$

Proof. From the left square of (3.3), there is a pushout of polyhedral products

$$\begin{array}{ccc}
(\underline{CA}, \underline{A})^{\partial\sigma\cap L} \times \prod_{i\notin\sigma} A_i & \xrightarrow{i'\times 1} & (\underline{CA}, \underline{A})^{\partial\sigma} \times \prod_{i\notin\sigma} A_i \\
\Big\downarrow f & & \Big\downarrow \\
(\underline{CA}, \underline{A})^{L} & \longrightarrow & (\underline{CA}, \underline{A})^{K\setminus\sigma}.
\end{array}$$

Since $\partial\sigma \cap L \neq \partial\sigma$, Lemma 3.1 implies that i' is null homotopic. Since L and $K \setminus \sigma$ have the same vertex set, L has no ghost vertices. Therefore, by (GT, Proposition 3.4), the restriction of f to $\prod_{i\notin\sigma} A_i$ is null homotopic. Thus Lemma 2.10 implies that there is a homotopy equivalence

$$(\underline{CA},\underline{A})^{K\setminus\sigma} \simeq \Big((\underline{CA},\underline{A})^{\partial\sigma} \rtimes \prod_{j\notin\sigma} A_j\Big) \vee E \tag{3.4}$$

where E is the homotopy cofibre of $f'\colon (\underline{CA},\underline{A})^{\partial\sigma\cap L} \rtimes \prod_{i\notin\sigma} A_i \longrightarrow (\underline{CA},\underline{A})^{L}$.

The next step is to identify E. By (3.3), there is an iterated diagram of pushouts of polyhedral products

$$\begin{tikzcd}
(\underline{CA},\underline{A})^{\partial\sigma\cap L} \times \prod\limits_{i\notin\sigma} A_i \arrow[r, "i'\times 1"] \arrow[d, "f"] & (\underline{CA},\underline{A})^{\partial\sigma} \times \prod\limits_{i\notin\sigma} A_i \arrow[r, "i\times 1"] \arrow[d] & (\underline{CA},\underline{A})^{\sigma} \times \prod\limits_{i\notin\sigma} A_i \arrow[d] \\
(\underline{CA},\underline{A})^{L} \arrow[r] & (\underline{CA},\underline{A})^{K\setminus\sigma} \arrow[r] & (\underline{CA},\underline{A})^{K}.
\end{tikzcd}$$

Since $(\underline{CA},\underline{A})^{\sigma}$ is contractible, this diagram of iterated pushouts is equivalent up to homotopy to the iterated diagram of homotopy pushouts

$$\begin{tikzcd}
(\underline{CA},\underline{A})^{\partial\sigma\cap L} \times \prod\limits_{i\notin\sigma} A_i \arrow[r, "i'\times 1"] \arrow[d, "f"] & (\underline{CA},\underline{A})^{\partial\sigma} \times \prod\limits_{i\notin\sigma} A_i \arrow[r, "\pi_2"] \arrow[d] & \prod\limits_{i\notin\sigma} A_i \arrow[d] \\
(\underline{CA},\underline{A})^{L} \arrow[r] & (\underline{CA},\underline{A})^{K\setminus\sigma} \arrow[r] & (\underline{CA},\underline{A})^{K}
\end{tikzcd}$$

where π_2 is the projection. As noted above, the restriction of f to $\prod_{i\notin\sigma} A_i$ is null homotopic, so all the vertical maps restrict trivially to $\prod_{i\notin\sigma} A_i$, implying that this factor may be collapsed out to give an iterated diagram of homotopy pushouts

$$\begin{tikzcd}
(\underline{CA},\underline{A})^{\partial\sigma\cap L} \rtimes \prod\limits_{i\notin\sigma} A_i \arrow[r, "i'\rtimes 1"] \arrow[d, "f'"] & (\underline{CA},\underline{A})^{\partial\sigma} \rtimes \prod\limits_{i\notin\sigma} A_i \arrow[r] \arrow[d] & * \arrow[d] \\
(\underline{CA},\underline{A})^{L} \arrow[r] & (\underline{CA},\underline{A})^{K\setminus\sigma} \arrow[r] & (\underline{CA},\underline{A})^{K}
\end{tikzcd} \tag{3.5}$$

In particular, all three vertical maps have the same homotopy cofibre. By definition, the homotopy cofibre of f' is E, while the right vertical map clearly has $(\underline{CA},\underline{A})^{K}$ as its homotopy cofibre. Thus $E \simeq (\underline{CA},\underline{A})^{K}$ and therefore from (3.4) there is a homotopy equivalence

$$(\underline{CA},\underline{A})^{K\setminus\sigma} \simeq \Big((\underline{CA},\underline{A})^{\partial\sigma} \rtimes \prod_{j\notin\sigma} A_j\Big) \vee (\underline{CA},\underline{A})^{K}.$$

Further, the right homotopy inverse for $(\underline{CA},\underline{A})^{K\setminus\sigma} \longrightarrow E$, together with the bottom row of (3.5), gives a composite $E \longrightarrow (\underline{CA},\underline{A})^{K\setminus\sigma} \longrightarrow (\underline{CA},\underline{A})^{K}$ that is a homotopy equivalence. Thus the map $(\underline{CA},\underline{A})^{K\setminus\sigma} \longrightarrow (\underline{CA},\underline{A})^{K}$ has a right homotopy inverse. $\square$

4 Polyhedral products associated to pseudomanifolds with boundary

In order to study polyhedral products associated to pseudomanifolds, we first consider the case with non-trivial boundary. The results will be used frequently in subsequent sections when the boundary is empty. For a simplicial complex K, and an integer $t \geq 0$, let K^t be the t-skeleton of K. If K has dimension n, we apply the results from the previous section in order to show that $(\underline{CA}, \underline{A})^K$ retracts off $(\underline{CA}, \underline{A})^{K^{n-1}}$ under certain hypotheses.

Theorem 4.1. *Let K be an n-dimensional, pure, weak pseudomanifold with boundary having ℓ maximal faces $\sigma_1, \cdots, \sigma_\ell$. Suppose that each connected component of $D(K)$ contains a vertex of degree strictly less than $n+1$. Then there is a homotopy equivalence*

$$(\underline{CA}, \underline{A})^{K^{n-1}} \simeq \bigvee_{i=1}^{\ell} \left((\underline{CA}, \underline{A})^{\partial\sigma_i} \rtimes \prod_{j \notin \sigma_i} A_j \right) \vee (\underline{CA}, \underline{A})^K$$

and the map of polyhedral products $(\underline{CA}, \underline{A})^{K^{n-1}} \to (\underline{CA}, \underline{A})^K$ induced by the inclusion $K^{n-1} \to K$ has a right homotopy inverse.

Proof. Applying Lemma 2.1 to each connected component of $D(K)$ and relabelling the maximal faces if necessary, we can assume σ_1 has degree strictly less than $n+1$ in $D(K)$, and for $2 \leq i \leq \ell$, σ_i has degree strictly less than $n+1$ in $D(K) \setminus \{\sigma_1, \cdots, \sigma_{i-1}\}$. Define $K_0 = K$, and for $1 \leq i \leq \ell$, define $K_i = K \setminus \{\sigma_1, \cdots, \sigma_i\}$. Observe that by definition, $K_\ell = K^{n-1}$. There is a sequence of inclusions

$$K^{n-1} = K_\ell \to K_{\ell-1} \to \cdots \to K_1 \to K_0 = K,$$

which factors the inclusion of K^{n-1} into K.

We show that for each i, the map of polyhedral products $(\underline{CA}, \underline{A})^{K_i} \to (\underline{CA}, \underline{A})^{K_{i-1}}$ induced by the inclusion $K_i \to K_{i-1}$ has a right homotopy inverse. Since σ_i has degree strictly less than $n+1$ in $D(K) \setminus \{\sigma_1, \cdots, \sigma_{i-1}\}$, there exists a face $\tau \in \partial\sigma_i$ with $|\tau| = |\sigma_i| - 1$ which is contained in only one maximal face, namely σ_i. In particular, τ is maximal in $K_i = K_{i-1} \setminus \sigma_i$. Hence, Theorem 3.2 implies there is a homotopy equivalence

$$(\underline{CA}, \underline{A})^{K_i} \simeq \left((\underline{CA}, \underline{A})^{\partial\sigma_i} \rtimes \prod_{j \notin \sigma_i} A_j \right) \vee (\underline{CA}, \underline{A})^{K_{i-1}},$$

and the map of polyhedral products $(\underline{CA}, \underline{A})^{K_i} \to (\underline{CA}, \underline{A})^{K_{i-1}}$ induced by the inclusion $K_i \to K_{i-1}$ has a right homotopy inverse. The homotopy equivalence asserted by the theorem then follows by induction. $\square$

Theorem 4.1 has a homological consequence that will be important in Section 6.

Proposition 4.2. *Let K be an n-dimensional, pure, weak pseudomanifold with boundary. Suppose that each connected component of $D(K)$ contains a vertex of degree strictly less than $n+1$. Let $A_1, \cdots, A_m$ be spaces such that $H_*(A_i)$ is torsion free for all i. Then $H_*((\underline{CA}, \underline{A})^K)$ is torsion free if and only if $H_*((\underline{CA}, \underline{A})^{K^{n-1}})$ is torsion free.*

Proof. If $H_*((\underline{CA}, \underline{A})^{K^{n-1}})$ is torsion free then, by Theorem 4.1, $(\underline{CA}, \underline{A})^K$ retracts off $(\underline{CA}, \underline{A})^{K^{n-1}}$, implying that $H_*((\underline{CA}, \underline{A})^K)$ is torsion free.

Now suppose that $H_*((\underline{CA}, \underline{A})^K)$ is torsion free. By Theorem 4.1, there is a homotopy equivalence

$$(\underline{CA}, \underline{A})^{K^{n-1}} \simeq \bigvee_{i=1}^{\ell} \left((\underline{CA}, \underline{A})^{\partial\sigma_i} \rtimes \prod_{j \notin \sigma_i} A_j \right) \vee (\underline{CA}, \underline{A})^K,$$

where $\sigma_1, \ldots, \sigma_\ell$ are maximal faces of K. By assumption, $H_*((\underline{CA}, \underline{A})^K)$ is torsion free, so to show that $H_*((\underline{CA}, \underline{A})^{K^{n-1}})$ is torsion free it suffices to show that $H_*((\underline{CA}, \underline{A})^{\partial\sigma_i} \rtimes \prod_{j\notin\sigma_\ell} A_j)$ is torsion free for $1 \leq i \leq \ell$. Let $\sigma_i = \{j_1, \cdots, j_n\}$. By (IK3, Theorem 1.7) or (GT, Theorem 1.1), there is a homotopy equivalence

$$(\underline{CA}, \underline{A})^{\partial\sigma_i} \simeq \Sigma^{n-1} A_{j_1} \wedge \cdots \wedge A_{j_n}.$$

In particular, $(\underline{CA}, \underline{A})^{\partial\sigma_i}$ is a suspension. In general, if A is a suspension then there is a homotopy equivalence $A \rtimes B \simeq A \vee (A \wedge B)$, so in our case there is a homotopy equivalence

$$(\underline{CA}, \underline{A})^{\partial\sigma_i} \rtimes \prod_{j\notin\sigma_i} A_j \simeq (\underline{CA}, \underline{A})^{\partial\sigma_i} \vee ((\underline{CA}, \underline{A})^{\partial\sigma_i} \wedge \prod_{j\notin\sigma_i} A_j).$$

By hypothesis, each $H_*(A_i)$ is torsion free, so the reduced Künneth theorem implies that both $H_*((\underline{CA}, \underline{A})^{\partial\sigma_i})$ and $H_*((\underline{CA}, \underline{A})^{\partial\sigma_i} \wedge \prod\limits_{j\notin\sigma_i} A_j)$ are torsion free, and hence $H_*((\underline{CA}, \underline{A})^{\partial\sigma_i} \rtimes \prod\limits_{j\notin\sigma_i} A_j)$ is torsion free. □

Theorem 4.1 can also be used to give coarse decompositions of the loop spaces of polyhedral products associated to pseudomanifolds with boundary in low dimensions.

Theorem 4.3. *Let K be a pure, weak pseudomanifold with boundary of dimension n on $[m]$, and let $A_1, \cdots, A_m$ be spaces such that $\Sigma A_i \in \mathcal{W}$ for all i. Suppose that each connected component of $D(K)$ contains a vertex of degree strictly less than $n+1$. If $n=1$, $n=2$, or $n=3$ and $H_*(|L|)$ is torsion free for all subcomplexes L of K with complete 1-skeleton, then $\Omega(\underline{CA}, \underline{A})^K \in \prod\mathcal{P}$.*

Proof. If $n = 1$, then Theorem 2.6 implies $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$, so assume $n \geq 2$. By Theorem 4.1, $(\underline{CA}, \underline{A})^K$ retracts off $(\underline{CA}, \underline{A})^{K^{n-1}}$, and so $\Omega(\underline{CA}, \underline{A})^K$ retracts off $\Omega(\underline{CA}, \underline{A})^{K^{n-1}}$. By Theorem 2.4, to show $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$ it suffices to show that $\Omega(\underline{CA}, \underline{A})^{K^{n-1}} \in \prod \mathcal{P}$. But Theorem 2.6 when $n = 2$ and Theorem 2.7 when $n = 3$ imply that $\Omega(\underline{CA}, \underline{A})^{K^{n-1}} \in \prod \mathcal{P}$. □

5 Polyhedral products associated to pseudomanifolds

The results from the previous section are applied to certain classes of pseudomanifolds. In particular, we show that loop spaces of certain polyhedral products associated to surfaces are in $\prod \mathcal{P}$. We start with a general statement giving conditions for when a polyhedral product has its loop space in $\prod \mathcal{P}$.

Theorem 5.1. *Let K be a simplicial complex on $[m]$ that does not have a complete 1-skeleton. Let $A_1, \cdots, A_m$ be spaces such that $\Sigma A_i \in \mathcal{W}$ for all $i \in [m]$. If $\Omega(\underline{CA}, \underline{A})^{K \setminus i} \in \prod \mathcal{P}$ for all $i \in [m]$ then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.*

Proof. For a vertex $v \in K$, let $N(v)$ be the set of vertices adjacent to v in the 1-skeleton of K. Since K does not have a complete 1-skeleton, there exists a vertex v such that $v \cup N(V) \neq K^0$. By (St1, Lemma 4.4), there is a pushout of simplicial complexes

$$\begin{array}{ccc} K_{N(v)} & \longrightarrow & K_{v \cup N(v)} \\ \downarrow & & \downarrow \\ K \setminus v & \longrightarrow & K. \end{array}$$

If $\Omega(\underline{CA}, \underline{A})^{K \setminus v} \in \prod \mathcal{P}$ and $\Omega(\underline{CA}, \underline{A})^{K_{v \cup N(v)}} \in \prod \mathcal{P}$ then Theorem 2.9 implies that $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.

By assumption, $\Omega(\underline{CA}, \underline{A})^{K \setminus v} \in \prod \mathcal{P}$. For $\Omega(\underline{CA}, \underline{A})^{K_{v \cup N(v)}}$, since $v \cup N(v) \neq K^0$, there exists a vertex w such that $v \cup N(v)$ is a full subcomplex of $K \setminus w$. By Lemma 2.5, $(\underline{CA}, \underline{A})^{v \cup N(v)}$ retracts off $(\underline{CA}, \underline{A})^{K \setminus w}$, and so $\Omega(\underline{CA}, \underline{A})^{v \cup N(v)}$ retracts off $\Omega(\underline{CA}, \underline{A})^{K \setminus w}$. By assumption, $\Omega(\underline{CA}, \underline{A})^{K \setminus w} \in \prod \mathcal{P}$, and so Theorem 2.4 implies that $\Omega(\underline{CA}, \underline{A})^{v \cup N(v)} \in \prod \mathcal{P}$. □

Theorem 5.1 will be used to show that low dimensional pseudomanifolds which do not have a complete 1-skeleton have their associated polyhedral products in $\prod \mathcal{P}$. To do this, we first show that if K is a pseudomanifold, then $K \setminus i$ satisfies the hypotheses of Theorem 4.1.

Lemma 5.2. *Let K be a pseudomanifold of dimension n on $[m]$. For any $i \in [m]$, $K \setminus i$ is a pure simplicial complex of dimension n, a weak pseudomanifold with boundary, and each connected component of $D(K \setminus i)$ contains a vertex of degree strictly less than $n + 1$.*

Proof. First, we show that $K \setminus i$ is pure of dimension n. Suppose σ is a maximal face of $K \setminus i$ of dimension $k < n$. By assumption, K is pure of dimension n so σ must be contained in some maximal simplex $\sigma' \in K$ with $i \in \sigma'$. Since $i \notin \sigma$, there must exist a codimension one face $\tau \subset \sigma'$ such that $\sigma \subseteq \tau$ and $i \notin \tau$. Moreover K is a pseudomanifold, and so in K, τ is contained in two maximal faces, σ' and σ''. However, since σ' contains i and τ is of codimension one, σ'' does not contain i, and therefore $\sigma'' \in K \setminus i$. Since $\sigma \subseteq \tau$, this implies $\sigma \subset \sigma''$, which is a contradiction. Thus every maximal face of $K \setminus i$ has dimension n, implying that $K \setminus i$ is pure of dimension n.

Next, we show that $K \setminus i$ is a weak pseudomanifold with boundary. Let τ be a codimension one face of $K \setminus i$. In K, since K is a pseudomanifold, τ is contained in two maximal faces, σ and σ'. At most one of σ and σ' contains the vertex i, otherwise $\sigma = \tau \cup \{i\} = \sigma'$. Therefore one of σ and σ' is in $K \setminus i$. Hence, τ is contained in either one or two maximal faces in $K \setminus i$. We now show that the boundary of $K \setminus i$ is non-empty. Since K is pure, the vertex i must be contained in at least one maximal face σ'' in K. Hence, if τ' is the codimension one face of σ'' which does not contain i, then it follows that τ' is contained in the boundary of $K \setminus i$.

Finally, we show that each connected component of $D(K \setminus i)$ contains a vertex of degree strictly less than $n+1$. Since K is a pseudomanifold of dimension n, each maximal face contains $n+1$ codimension one faces, each of which is contained in two distinct maximal faces. Therefore, each vertex in $D(K)$ has degree $n+1$. The graph $D(K \setminus i)$ is obtained from $D(K)$ by removing vertices corresponding to maximal faces of K containing the vertex i. If $D(K \setminus i)$ is connected, then since $D(K)$ is connected, there must exist a vertex in $D(K \setminus i)$ which is adjacent to at least one of the vertices removed from $D(K)$. Therefore, there must exist a vertex in $D(K \setminus i)$ with degree strictly less than $n+1$. Now suppose $D(K \setminus i)$ is disconnected, and let $x, y \in D(K \setminus i)$ be two vertices in different connected components. Since $D(K)$ is connected and $D(K \setminus i)$ is disconnected, any path in $D(K)$ between x and y must pass through one of the vertices removed from $D(K)$ to obtain $D(K \setminus i)$. Therefore, for each connected component of $D(K \setminus i)$, there must exist a vertex v such that v is adjacent to at least one of the vertices removed from $D(K)$. Hence, v has degree strictly less than $n+1$ in $D(K \setminus i)$. □

Theorem 5.3. *Let K be either a 2-dimensional pseudomanifold or a 3-dimensional pseudomanifold such that $H_*(|L|)$ is torsion free for all full subcomplexes L of K with complete 1-skeleton. Suppose that K is on the vertex set $[m]$ and $A_1, \cdots, A_m$ are spaces such that $\Sigma A_i \in \mathcal{W}$ for all $i \in [m]$. If K does not have complete 1-skeleton then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.*

Proof. For all $i \in [m]$, Lemma 5.2 implies $K \setminus i$ satisfies the hypotheses of Theorem 4.3. Therefore, $\Omega(\underline{CA}, \underline{A})^{K \setminus i} \in \prod \mathcal{P}$ for all $i \in [m]$. Since K does not have a complete 1-skeleton, Theorem 5.1 implies that $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$. □

A special case of pseudomanifolds of dimension 2 are connected, orientable, closed surfaces. In this case, we can give a complete picture of $\Omega(\underline{CA}, \underline{A})^K$ without the assumption on the 1-skeleton.

Theorem 5.4. *Let K be the triangulation of a connected, orientable, closed surface on $[m]$. Let $A_1, \cdots, A_m$ be spaces such that $\Sigma A_i \in \mathcal{W}$. Then $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$.*

Proof. Since K is the triangulation of a connected, orientable, closed surface, for each $I \subseteq [m]$, $|K_I|$ embeds into $\mathbb{R}^3$. By (H, Corollary 3.46), this implies that $H_*(|K_I|)$ is torsion free. Therefore, Theorem 2.7 implies that $\Omega(\underline{CA}, \underline{A})^K \in \prod \mathcal{P}$. □

A special case of Theorem 5.4 proves Theorem 1.1.

Proof of Theorem 1.1. Take each pair (CA_i, A_i) in Theorem 5.4 to be (D^2, S^1). □

6 Loop space decompositions of moment-angle manifolds

In this section, we specialise to moment-angle complexes associated to triangulations of spheres, all of which are pseudomanifolds. If K is a triangulation of S^2 then $\Omega\mathcal{Z}_K \in \prod \mathcal{P}$ by Theorem 1.1. We will prove an analogous result if K is a triangulation of S^3. To start, we consider more general properties of a family of odd dimensional sphere triangulations called neighbourly triangulations. Let K be a triangulation of S^n on $[m]$. In this case, $\mathcal{Z}_K$ has the structure of a manifold of dimension $m+n+1$ (BP, Theorem 4.1.4) which is 2-connected.

Pseudomanifolds and the minimally non-Golod property. An important algebraic condition on simplicial complexes is the notion of Golodness. A simplicial complex K on $[m]$ is called *Golod* if all cup products and higher Massey products in $H^*(\mathcal{Z}_K)$ are trivial, and K is *minimally non-Golod* if $K \setminus i$ is Golod for all $i \in [m]$. For example, if $\mathcal{Z}_K$ is a suspension, or a co-H-space, then all cup products and higher Massey products vanish in $H^*(\mathcal{Z}_K)$, implying that K is Golod.

We focus our attention on a special family of odd dimensional sphere triangulations. Recall from the Introduction that a simplicial complex K is called *k-neighbourly* if every set of $k+1$ vertices spans a simplex. A triangulation K of a sphere S^{2n+1} is called *neighbourly* if K is n-neighbourly. It was shown in (L, Proposition 3.6) that if K is the boundary of a dual polytope and neighbourly, then K is minimally non-Golod. Gitler and Lopez de Medrano (GLdM, Theorem 1.3) showed that in this case the corresponding $\mathcal{Z}_K$ is diffeomorphic to a connected sum of sphere products, with two spheres in each product. We give an anaologue of Limonchenko's result that holds for any n-neighbourly $(2n+1)$-dimensional pseudomanifold. This requires a suspension splitting of moment-angle complexes from (BBCG, Corollary 2.23), known as the BBCG decomposition.

Theorem 6.1. *Let K be a simplicial complex. There is a homotopy equivalence*

$$\Sigma\mathcal{Z}_K \simeq \bigvee_{I\notin K} \Sigma^{2+|I|}|K_I|$$

that is natural for inclusions of simplicial complexes. □

The BBCG decomposition for $\Sigma\mathcal{Z}_K$ "desuspends" if there is a homotopy equivalence

$$\mathcal{Z}_K \simeq \bigvee_{I\notin K} \Sigma^{1+|I|}|K_I|.$$

Observe that if the BBCG decomposition desuspends then $\mathcal{Z}_K$ is a suspension, and so is Golod.

Theorem 6.2. *Let K be a pseudomanifold on $[m]$ of dimension $2n+1$. If K is n-neighbourly then the BBCG decomposition for $\Sigma\mathcal{Z}_{K\setminus i}$ desuspends for all $i \in [m]$. Consequently, K is either Golod or minimally non-Golod.*

Proof. By (IK2, Theorem 1.3), for any simplicial complex K, $\mathcal{Z}_K$ is a co-H space if and only if the BBCG decomposition desuspends. Hence, it suffices to show that $\mathcal{Z}_{K\setminus i}$ is a co-H space for all $i \in [m]$. Since K is a pseudomanifold, Lemma 5.2 implies $K \setminus i$ satisfies the hypotheses of Theorem 4.1, implying that $\mathcal{Z}_{K\setminus i}$ retracts off $\mathcal{Z}_{(K\setminus i)^{2n}}$. The simplicial complex $(K\setminus i)^{2n}$ is an n-neighbourly, $2n$-dimensional simplicial complex, so by (IK2, Theorem 1.6), the BBCG decomposition for $\Sigma\mathcal{Z}_{(K\setminus i)^{2n}}$ desuspends. Thus $\mathcal{Z}_{(K\setminus i)^{2n}}$ is a suspension. As $\mathcal{Z}_{K\setminus i}$ retracts off $\mathcal{Z}_{(K\setminus i)^{2n}}$, $\mathcal{Z}_{K\setminus i}$ is therefore a co-H space. □

If K is a triangulation of S^n, we can characterise when K is Golod. If $K = \partial\Delta^{n+1}$, then Theorem 6.1 implies that $\mathcal{Z}_K$ has one non-trivial homology group, and therefore has no nontrivial cup products or Massey products, implying that K is Golod. If $K \neq \partial\Delta^{n+1}$, then Theorem 6.1 implies that a minimal missing face corresponds to a $\mathbb{Z}$ summand in $H^i(\mathcal{Z}_K)$, where $i < m+n+1$. If $x \in H^i(\mathcal{Z}_K)$ generated this summand, then as $\mathcal{Z}_K$ is a manifold, Poincaré duality implies there is a class $y \in H^{m+n+1-i}(\mathcal{Z}_K)$ such that $x \cup y \neq 0$. Thus $H^*(\mathcal{Z}_K)$ has non-trivial cup products, implying that K is not Golod. Therefore, we obtain the following.

Lemma 6.3. *If K is a triangulation of S^n then K is Golod if and only if $K = \partial\Delta^{n+1}$.* □

Neighbourly trianglulations of S^{2n+1}. To start, let K be a triangulation of S^n on $[m]$. Let $\overline{\mathcal{Z}_K}$ be the $(m+n)$-skeleton of $\mathcal{Z}_K$. There is a homotopy cofibration

$$S^{n+m} \xrightarrow{f} \overline{\mathcal{Z}_K} \to \mathcal{Z}_K$$

where f attaches the $(m+n+1)$-cell to $\mathcal{Z}_K$. We aim for a decomposition of $\overline{\mathcal{Z}_K}$ under certain hypotheses. These hypotheses will be satisfied when K is a neighbourly triangulation of an odd dimensional sphere. First, we determine the homology of $\overline{\mathcal{Z}_K}$.

Proposition 6.4. *Let K be a triangulation of S^n on $[m]$. There are isomorphisms*

$$H_*(\mathcal{Z}_K) \cong \bigoplus_{I \notin K} H_*(\Sigma^{1+|I|}|K_I|) \qquad H_*(\overline{\mathcal{Z}_K}) \cong \bigoplus_{I \notin K, I \neq [m]} H_*(\Sigma^{1+|I|}|K_I|).$$

Proof. The first isomorphism follows from Theorem 6.1. For the second, one summand has been deleted, corresponding to $I = [m]$. When $I = [m]$ then $K_I = K$. Since K is a triangulation of a sphere, $|K| = S^n$, so $\Sigma^{1+|[m]|}|K| \simeq S^{m+n+1}$. This accounts for the generator in $H_{m+n+1}(\mathcal{Z}_K)$. As $\overline{\mathcal{Z}_K}$ is the $(m+n)$-skeleton of $\mathcal{Z}_K$, the second isomorphism follows. □

In case the BBCG decomposition for $\Sigma\mathcal{Z}_{K\setminus i}$ desuspends for each $i \in [m]$ we can decompose $\overline{\mathcal{Z}_K}$.

Proposition 6.5. *Let K be a triangulation of S^n on $[m]$. If the BBCG decomposition for $\Sigma\mathcal{Z}_{K\setminus i}$ desuspends for all $i \in [m]$, then K is Golod when $K = \partial\Delta^{n-1}$ or minimally non-Golod when $K \neq \partial\Delta^{n-1}$, and there is a homotopy equivalence*

$$\overline{\mathcal{Z}_K} \simeq \bigvee_{I \notin K, I \neq [m]} \Sigma^{1+|I|}|K_I|.$$

Proof. The BBCG decomposition for $\Sigma\mathcal{Z}_K$ is

$$\Sigma\mathcal{Z}_K \simeq \bigvee_{I \notin K} \Sigma^{2+|I|}|K_I|.$$

Consider the map $\mathcal{Z}_{K\setminus i} \longrightarrow \mathcal{Z}_K$ induced by the inclusion $K \setminus i \longrightarrow K$. The naturality of the BBCG decomposition implies that the decomposition of $\Sigma\mathcal{Z}_{K\setminus i}$ may be obtained by restricting the decomposition for $\Sigma\mathcal{Z}_K$ to those full subcomplexes K_I with $I \notin K$ and $i \notin I$. As the BBCG decomposition for $\Sigma\mathcal{Z}_{K\setminus i}$ desuspends by hypothesis, we obtain a homotopy equivalence

$$\mathcal{Z}_{K\setminus i} \simeq \bigvee_{I \notin K, i \notin I} \Sigma^{1+|I|}|K_I|.$$

Taking the wedge sum of the inclusion maps $\mathcal{Z}_{K\setminus i} \longrightarrow \mathcal{Z}_K$ over all $i \in [m]$ then gives a map

$$\bigvee_{i=1}^{m} \left(\bigvee_{I \notin K, i \notin I} \Sigma^{1+|I|}|K_I| \right) \longrightarrow \mathcal{Z}_K.$$

Observe that the index set on the left accounts for all $I \notin K$ except for an instance of I that contains each $i \in [m]$, of which there is only one, $I = [m]$. However, the index set may include multiple copies of the same wedge summand. Restricting to a single copy for each instance of $I \notin K$, $I \neq [m]$, we obtain a map

$$g\colon \bigvee_{I\notin K, I\neq[m]} \Sigma^{1+|I|}|K_I| \longrightarrow \mathcal{Z}_K$$

whose suspension induces the inclusion of all wedge summands in the BBCG decomposition of $\mathcal{Z}_K$ except for the $I = [m]$ summand. In particular, g induces an injection in homology. As each wedge summand $\Sigma^{1+|I|}|K_I|$ has dimension $< m+n+1$ for $I \neq [m]$, the map g factors through the $(m+n)$-skeleton $\overline{\mathcal{Z}_K}$ of $\mathcal{Z}_K$ to give a map

$$g'\colon \bigvee_{I\notin K, I\neq[m]} \Sigma^{1+|I|}|K_I| \longrightarrow \overline{\mathcal{Z}_K}.$$

Since g induces an injection in homology, so does g'. The description of $H_*(\overline{\mathcal{Z}_K})$ in Proposition 6.4 therefore implies that g' must induce an isomorphism in homology, and hence g' is a homotopy equivalence by Whitehead's Theorem. $\square$

We will show that Proposition 6.5 holds when K is a neighbourly triangulation of S^{2n+1}. In this case, the decomposition of $\overline{\mathcal{Z}_K}$ can be refined. The following argument is essentially due to Gitler and Lopez de Medrano (GLdM), and the authors thank a referee for pointing out the following result holds for all neighbourly triangulations of S^{2n+1}, rather than just S^3.

Theorem 6.6. *If K is a neighbourly triangulation of S^{2n+1} on $[m]$ with $n \geq 1$ then the simplicial complex K is Golod when $K = \partial\Delta^{2n+2}$, or minimally non-Golod when $K \neq \partial\Delta^{2n+2}$. Moreover, $\overline{\mathcal{Z}_K} \in \mathcal{W}$.*

Proof. Consider the real moment-angle complex $\mathbb{R}\mathcal{Z}_K := (D^1, S^0)^K$ associated to K, which is a closed topological manifold of dimension $2n+2$ (BP, Theorem 4.1.7). By (BBCG, Corollary 2.24), there is a homotopy equivalence

$$\Sigma\mathbb{R}\mathcal{Z}_K \simeq \bigvee_{I\notin K} \Sigma^2|K_I|. \tag{6.6}$$

Since K is a neighbourly triangulation of S^{2n+1}, each full subcomplex K_I has $H_k(|K_I|) = 0$ for all $k < n$. It follows from (6.6) that $\mathbb{R}\mathcal{Z}_K$ is n-connected. By Poincaré duality, the reduced homology of $\mathbb{R}\mathcal{Z}_K$ is non-trivial only in degrees $n+1$ and $2n+2$. Therefore since $H_{2n+2}(\mathbb{R}\mathcal{Z}_K) \cong \mathbb{Z}$ and $K = S^{2n+1}$, (6.6) implies that for all $I \subseteq [m]$ with $I \neq [m]$, $\widetilde{H}_k(|K_I|)$ can be non-trivial if and only if $k = n$. Hence either $|K_I|$ is contractible or homotopy equivalent to a wedge of S^n's. In particular, each such $\Sigma|K_I| \in \mathcal{W}$.

Combining Theorem 6.2, Lemma 6.3, and Proposition 6.5, we then obtain the desired result. □

Now we can prove Theorem 1.3, which states that if K is a neighbourly triangulation of S^{2n+1} then $\Omega\mathcal{Z}_K \in P$.

Proof of Theorem 1.3. Theorem 6.6 implies that $\overline{\mathcal{Z}_K} \in \mathcal{W}$. The Hilton-Milnor theorem then implies that $\Omega\overline{\mathcal{Z}_K} \in \prod\mathcal{P}$. Using the fact that $\mathcal{Z}_K$ is a manifold, by (T, Example 5.4), the inclusion $\overline{\mathcal{Z}_K} \to \mathcal{Z}_K$ has a right homotopy inverse after looping. Hence, Theorem 2.4 implies that $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$. □

Triangulations of S^3. Now we specialise to any triangulation K of S^3 and prove Theorem 1.2, which states that $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$. This splits into two cases, the first where K has a complete 1-skeleton, and the second where it does not. The first case follows from Theorem 1.3 and the second requires a preliminary homological result from (Si, Lemma 3.4.12) on the homology of $\mathcal{Z}_K$. We provide a proof for completeness.

Lemma 6.7. *Let K be a triangulation of S^3 on $[m]$. Then $H_*(\mathcal{Z}_K)$ is torsion free.* □

Proof. Since K is a triangulation of S^3, $H_*(|K|)$ is torsion free. If $I \subseteq [m]$ with $I \neq [m]$, $|K_I|$ embeds into $S^3 \setminus \{pt\} \cong \mathbb{R}^3$, and so by (H, Corollary 3.46), $H_*(|K_I|)$ is torsion free. Therefore, Proposition 6.4 implies that $H_*(\mathcal{Z}_K)$ is torsion free. □

We can now prove Theorem 1.2.

Proof of Theorem 1.2. By Lemma 6.7, $H_*(\mathcal{Z}_K)$ is torsion free, so Theorem 6.1 implies that $H_*(|K_I|)$ is torsion free for all $I \subseteq [m]$. If the 1-skeleton of K is not a complete graph, then Theorem 5.3 implies that $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$.

If the 1-skeleton is a complete graph, then Theorem 1.3 implies $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$. □

Remark 6.8. By a result of Cai (C, Corollary 2.10), $\mathcal{Z}_K$ is a manifold if and only if K is a generalised homology sphere. It would be interesting to know if these results also hold when K is a generalised homology sphere, but not a triangulation of a sphere.

Remark 6.9. Not every triangulation K of a sphere will result in $\Omega\mathcal{Z}_K \in P$. For example, let L be the 6-vertex triangulation of $\mathbb{R}P^2$. By (GPTW, Example 3.3), there is a homotopy equivalence

$$\mathcal{Z}_L \simeq W \vee \Sigma^7\mathbb{R}P^2, \tag{6.7}$$

where $W \in \mathcal{W}$. As in (LW, Theorem 3.2), one can construct a triangulation of S^4 containing L as a full subcomplex by applying certain stellar subdivisions to $\partial\Delta^5$. Let K be such a triangulation. By Lemma 2.5 and (6.7), $\Sigma^7\mathbb{R}P^2$ retracts off $\mathcal{Z}_K$, and so

$\Omega\Sigma^7\mathbb{R}P^2$ retracts off $\Omega\mathcal{Z}_K$. This implies that $H_*(\Omega\mathcal{Z}_K)$ contains 2-torsion and so $\Omega\mathcal{Z}_K \notin \prod\mathcal{P}$.

Quasitoric manifolds. Theorem 1.2 will be applied in Proposition 6.11 to show similar results for certain manifolds known as quasitoric manifolds. As in (DJ), a $2n$-dimensional manifold has a *locally standard* T^n-action if locally it is the standard action of T^n on $\mathbb{C}^n$. A *quasitoric manifold* over an n-dimensional simple polytope P is a closed, smooth $2n$-dimensional manifold M that has a smooth locally standard T^n-action for which the orbit space M/T^n is homeomorphic to P as a manifold with corners.

Let P be an n-dimensional simple polytope with m facets, and let $K = \partial P^*$ be the dual of the boundary of P. The simplicial complex K is a triangulation of S^{n-1}, and therefore $\mathcal{Z}_K$ is a moment-angle manifold. By (BP, Proposition 7.3.12), a quasitoric manifold M of dimension $2n$ arises as a quotient $M \cong \mathcal{Z}_K/T^{m-n}$ for some subtorus $T^{m-n} \subseteq T^m$ that acts freely on the corresponding moment-angle complex $\mathcal{Z}_K$. The quotient description of M implies that there is a principal T^{m-n}-fibration

$$T^{m-n} \longrightarrow \mathcal{Z}_K \longrightarrow M. \tag{6.8}$$

The following lemma is well known to experts in the area.

Lemma 6.10. *Let M be a quasi-toric manifold of dimension $2n$ associated to a polytope P of dimension n. Let $K = \partial P^*$. Then there is a homotopy equivalence $\Omega M \simeq T^{m-n} \times \Omega\mathcal{Z}_K$.*

Proof. Consider the homotopy fibration $\Omega M \xrightarrow{r} T^{m-n} \longrightarrow \mathcal{Z}_K$ induced by (6.8). By (BP, Proposition 4.3.5 (a)), $\mathcal{Z}_K$ is 2-connected. Therefore r induces an isomorphism on π_1. Each $\mathbb{Z}$ generator of $\pi_1(\Omega M)$ is the Hurewicz image of a map $S^1 \longrightarrow \Omega M$, and the loop space structure allows these to be multiplied together to obtain a map $s\colon T^{m-n} \longrightarrow \Omega M$. The composite $r \circ s$ therefore induces an isomorphism on π_1. As T^{m-n} is an Eilenberg-Mac Lane space, this implies $r \circ s$ is a homotopy equivalence. Thus $\Omega M \simeq T^{m-n} \times \Omega\mathcal{Z}_K$. □

Proposition 6.11. *If M is a quasitoric manifold of dimension 4, 6 or 8, then $\Omega M \in \prod\mathcal{P}$.*

Proof. If M is $2n$-dimensional with m facets then, by Lemma 6.10, there is a homotopy equivalence $\Omega M \simeq T^{m-n} \times \Omega\mathcal{Z}_K$, where K is the dual of the boundary of an n-dimensional polytope. To show that $\Omega M \in \prod\mathcal{P}$, it therefore suffices to show that $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$. But the hypotheses that M has dimension 4, 6 or 8 implies that K is a triangulation of S^1, S^2 or S^3 respectively. Theorem 2.6 in the first case, Theorem 1.1 in the second case, and Theorem 1.2 in the third case imply that $\Omega\mathcal{Z}_K \in \prod\mathcal{P}$, as required. □

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