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Decision Support

Optimization models for cumulative prospect theory under incomplete preference information

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ABSTRACT

Prospect stochastic dominance conditions can be used to compare pairs of uncertain decision alternatives when the decision makers' choice behavior is characterized by cumulative prospect theory, but their preferences are not precisely specified. This paper extends the use of prospect stochastic dominance conditions to decision settings in which the use of pairwise comparisons is not possible due to large or possibly infinite number of decision alternatives (e.g., financial portfolio optimization). In particular, we first establish equivalence results between these conditions and the existence of solutions to a specific system of linear inequalities. We then utilize these results to develop stochastic optimization models whose feasible solutions are guaranteed to dominate a pre-specified benchmark distribution. These models can be used to identify if there exists a decision alternative within a set that is preferred to a given benchmark by all decision makers with an S-shaped value function and a pair of inverse S-shaped probability weighting functions. Thus, the models offer a flexible tool to analyze choice behavior in decision settings that can be modeled as optimization problems. We demonstrate the use of the developed models with two empirical applications in financial portfolio diversification and procurement optimization.

1. Introduction

Stochastic Dominance (SD) offers a family of well-established criteria for ranking decision alternatives with uncertain outcomes under incomplete information on the decision maker's (DM's) risk preferences (see, e.g., Levy, 2016, for a recent overview). SD-criteria have strong decision-theoretic foundations in expected utility theory (EUT; Neumann & Morgenstern, 1944): For instance, if an alternative dominates another in the sense of First-order Stochastic Dominance (FSD; Hadar & Russell, 1969; Hanoch & Levy, 1969; Quirk & Saposnik, 1962), then any expected utility maximizing DM with a non-decreasing utility function would prefer the dominating alternative. Similarly, no risk-averse expected utility maximizing DM with a non-decreasing concave utility function would choose a decision alternative that is dominated in the sense of Second-order Stochastic Dominance (SSD; Hadar & Russell, 1969; Hanoch & Levy, 1969; Rothschild & Stiglitz, 1970). Recent research efforts have focused on extending the use of SD-criteria from pairwise comparisons of decision alternatives to full-fledged stochastic optimization. These efforts have resulted in several stochastic optimization models in which the outcome distribution of the optimal solution is guaranteed to stochastically dominate some pre-specified benchmark distribution[1] (see, e.g., Armbruster & Delage, 2015; Bruni et al., 2017; Cesarone & Puerto, 2025; Dentcheva & Ruszczyński, 2003; Kopa et al., 2018; Kuosmanen, 2004; Liesiö et al., 2020; Post, 2003; Post & Kopa, 2017; Xu, 2024).

However, it is widely known that actual decision behavior systematically deviates from that predicted by EUT, which suggests that traditional SD-criteria might not accommodate all empirically observed preferences. Thus, while traditional SD-criteria offer a solid foundation for prescriptive models geared towards decision support, they might not be well-suited for behavioral models seeking to describe decision behavior. This has motivated the research by Baucells and Heukamp (2006) and Levy and Wiener (1998) to develop dominance criteria that capture preferences through cumulative prospect theory (CPT; Tversky & Kahneman, 1992) rather than EUT. In CPT, the decision maker's risk preferences are captured jointly by (i) an S-shaped value function that encodes outcomes into gains and losses with regard to a certain reference point, and (ii) a pair of inverse S-shaped probability weighting functions (PWFs) that transforms cumulative probabilities

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[1] This requires that a dominating solution exists in the set of feasible solutions. Moreover, there can be several dominating solutions in which case the optimal solution might not be unique.

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of the gains and losses into subjective decision weights. The resulting Prospect Stochastic Dominance (PSD; Levy & Wiener, 1998) compares the outcome distributions of two decision alternatives (prospects) to determine if one is preferred to the other by all decision makers with an S-shaped value function. In turn, the more general Weighted Prospect Stochastic Dominance (PWSD; Baucells & Heukamp, 2006) establishes if a prospect is preferred to another by all decision makers with an S-shaped value function and a pair of inverse S-shaped PWFs.

Although these existing methods can determine if PSD or PWSD holds between a given pair of prospects, they cannot be directly utilized in stochastic optimization problems. However, in many decision problems there does not exist a full list of relevant prospects, but instead prospects are implicitly defined as feasible solutions to a system of constraints (cf. project combinations satisfying resource constraints). In such problems, the number of feasible solutions is often very large or infinite, which makes pairwise comparisons of all prospects time-consuming or even impossible. This gap in the current literature is especially significant in view of the fact that a plethora of methods have been developed to enable the utilization of EUT-based SD-criteria (e.g., FSD and SSD) in stochastic optimization models, and these methods have found applications in several domains, including financial portfolio optimization (e.g., Armbruster & Delage, 2015; Kuosmanen, 2004), project scheduling (e.g., Gutjahr, 2015) and project portfolio selection (e.g., Liesiö et al., 2023).

We address this gap by developing novel stochastic optimization models that capture preferences through CPT with incomplete information, thereby enhancing the applicability of SD-criteria in analyzing choice behavior. In particular, we first develop results for a discrete state-space that allow to establish PSD and PWSD between two prospects by examining their probability distributions in a finite number of outcome levels. Based on these results, we then develop mixed-integer linear programming (MILP) models such that any feasible solution to these models is guaranteed to dominate a given benchmark prospect (or an outcome distribution) in the sense of PSD or PWSD. These models make it possible to identify if there exists a prospect within a feasible set which would be preferred to a given benchmark prospect by all decision makers with an S-shaped utility function and a pair of inverse S-shaped probability weighting functions. Moreover, as the models do not impose restrictions on the objective function, they can be readily utilized in identifying, for instance, the expected value maximizing prospect among those that dominate the benchmark.

The optimization models developed in this paper can be used to analyze choice behavior from the perspective of CPT in a broad range of problems in finance, operations management, and economics, in which prospects (decision alternatives) correspond to feasible solutions of a stochastic optimization problem and uncertainties are captured with a finite state-space (see, e.g., Cinfrignini et al., 2025; Gustafsson & Salo, 2005; Gutjahr, 2015; Harris & Mazibas, 2022; Kopa et al., 2018; Kuosmanen, 2004; Sillanpää et al., 2021). We demonstrate the practical relevance of our methodological contributions with two empirical applications in financial portfolio diversification and procurement optimization based on real-world data. In these applications, the developed models are used to analyze if the decision makers' choice of particular decision alternatives can be explained by CPT.

The rest of the paper is organized as follows. Section 2 introduces the notation and standard definitions required for modeling incomplete preference information in decision making under uncertainty. Section 3 derives conditions for PSD and PWSD in discrete state-space. Section 4 employs these conditions to develop stochastic optimization models with PSD and PWSD constraints and Section 5 presents the two empirical applications. Section 6 provides some concluding remarks.

2. Decision making under uncertainty and incomplete preference information

Let X denote a risky prospect, technically a random variable, whose support is a subset of the interval $[a,b]$ and whose cumulative distribution function (CDF) is denoted by F_X. Under expected utility theory (EUT), a decision maker with a utility function u prefers prospect X if its expected utility, given by

$$\mathbb{E}[u(X)] = \int_a^b u(t)\,dF_X(t), \tag{1}$$

is higher than the expected utility of other available prospects. Thus, the decision maker's preferences in EUT are captured solely by some utility function $u : [a,b] \to \mathbb{R}$, which maps each outcome onto a utility scale.

Suppose the information on the decision maker's preferences is incomplete in the sense that it is known only that the decision maker prefers higher outcomes to lower ones. In this case, these preferences cannot be modeled by a single utility function but rather by the set of all non-decreasing functions, which we denote by U^1. Even under such incomplete preference information, it can be possible to infer the decision maker's choice between a pair of prospects only by comparing their CDFs. Specifically, prospect X is said to (weakly) dominate[2] Y in the sense of First-order Stochastic Dominance (FSD) if

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]\ \forall\ u \in U^1 \Leftrightarrow F_X(t) \leq F_Y(t)\ \forall\ t \in [a,b].$$

If the decision maker is also risk-averse, then his/her utility function must be concave. The choice behavior of such decision makers agrees with the Second-order Stochastic Dominance (SSD) criterion: Prospect X dominates Y in the sense of SSD if

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)]\ \forall\ u \in U^2 \Leftrightarrow \int_a^{t'} F_X(t)dt \leq \int_a^{t'} F_Y(t)dt\ \forall\ t' \in [a,b],$$

where $U^2 \subset U^1$ is the set of all non-decreasing concave utility functions.

Although EUT and SSD provide a solid theory of rational decision making on top of which decision support models and tools can be built, observed decision behavior often deviates from that predicted by EUT (see, e.g., Starmer, 2000). This has led to the development of prospect theory (Kahneman & Tversky, 1979), and later cumulative prospect theory (CPT; Tversky & Kahneman, 1992), which are known to provide a more accurate description of choice behavior particularly in small-scale decision settings compared to EUT (Levy & Levy, 2021; Rabin, 2000). CPT can be considered as an extension of EUT to incorporate a richer representation of decision makers' risk preferences. First, CPT introduces the concept of a reference outcome that divides the outcomes into gains and losses, for which the decision makers exhibit different preferences. Specifically, the utility function is convex over losses and concave over gains. Second, CPT uses a pair of probability weighting functions to capture the decision makers' tendency to distort probabilities of outcomes. These probability weighting functions are applied on the prospect's cumulative distribution function prior to evaluating the expectation of the prospect's utility.

More formally, CPT assumes that $u \in U^S$, where set U^S depends on the reference outcome $r \in [a,b]$ and consists of all non-decreasing utility functions $u : [a,b] \to \mathbb{R}$ that are convex for losses $t \in [a,r]$ and concave for gains $t \in [r,b]$. Moreover, let w^- and w^+ denote a pair of probability weighting functions, i.e., increasing mappings from $[0,1]$ to $[0,1]$ which satisfy $w^-(0) = w^+(0) = 0$ and $w^-(1) = w^+(1) = 1$. Then, under CPT the decision maker prefers prospect X whose value

$$V_{w^-}^{w^+}[u(X)] = \int_a^r u(t)\,d[w^-(F_X(t))] + \int_r^b u(t)\,d[-w^+(1-F_X(t))] \tag{2}$$

is the highest among all available prospects. Expected utility (1) can be viewed as a special case of CPT value (2) since under linear probability weighting functions (i.e., $w^+(p) = w^-(p) = p$) it holds that

$$\begin{aligned} V_{w^-}^{w^+}[u(X)] &= \int_a^r u(t)\,d[F_X(t)] + \int_r^b u(t)\,d[-1+F_X(t)] \\ &= \int_a^b u(t)\,dF_X(t) = \mathbb{E}[u(X)]. \end{aligned}$$

[2] To be concise we use the term 'dominance' in this paper when referring to weak dominance.

Suppose the information on the decision makers' preferences is limited to knowing that they are risk-averse about gains and risk-seeking about losses, and that they do not distort the probabilities. This corresponds to considering all S-shaped utility functions ($u \in U^S$) but only linear weighting functions ($w^+(q) = w^-(q) = q$). If any such decision maker prefers one prospect to another, we say that the former dominates the latter in the sense of Prospect Stochastic Dominance (PSD).

Definition 1 (*PSD*). Prospect X dominates Y by PSD, denoted by $X \succeq Y$, if

$$V_{w^-}^{w^+}[u(X)] \geq V_{w^-}^{w^+}[u(Y)] \ \forall\ u \in U^S, \tag{3}$$

where $w^+(q) = w^-(q) = q$ for all $q \in [0, 1]$.

Establishing PSD between two prospects can be based on the comparison of their integrated CDFs for all pairs of gains and losses. This result, introduced by Levy and Wiener (1998), is formalized in the following proposition.

Proposition 1 (Levy & Wiener, 1998). *Let X and Y be prospects. Then*

$$X \succeq Y \Leftrightarrow \int_{t^-}^{t^+} F_X(t)dt \leq \int_{t^-}^{t^+} F_Y(t)dt \ \forall\ a \leq t^- \leq r \leq t^+ \leq b.$$

Levy and Wiener (1998) also show that allowing all convex probability weighting functions in Definition 1 would result in a dominance condition equivalent to that in Proposition 1 (see also Yang, 2019). However, empirical evidence from studies estimating the parameters of CPT models suggests that the probability weighting functions are not convex, but rather inverse S-shaped, i.e., concave for small probabilities and convex for large probabilities (see, e.g., Wakker, 2010). This motivates the use of Weighted Prospect Stochastic Dominance (PWSD; Baucells & Heukamp, 2006) to compare prospects as it enables to determine if a prospect is preferred to another by any decision maker with an S-shaped utility function and a pair of inverse S-shaped probability weighting functions.

Definition 2 (*PWSD*). Prospect X dominates Y by PWSD, denoted by $X \succeq_{c^-}^{c^+} Y$, if

$$V_{w^-}^{w^+}[u(X)] \geq V_{w^-}^{w^+}[u(Y)] \ \forall\ u \in U^S,\ w^+ \in W^{c^+},\ w^- \in W^{c^-},$$

where W^c is the set of all strictly increasing functions $w : [0, 1] \to [0, 1]$, which are convex on $[c, 1]$.

Note that set W^c includes also probability weighting functions that are concave on some interval $[0, d]$, although this is not explicitly required by the definition. The choice to omit this requirement is motivated by the fact that when all S-shaped utility functions are allowed, considering only those weighting functions in W^c that are concave on some interval $[0, d]$ would result in exactly the same dominance relation as the one in Definition 2 (see Definition 3 and Proposition 5 of Baucells & Heukamp, 2006).

Increasing the values of c^+ and c^- enlarges the sets of feasible probability weighting functions W^{c^+} and W^{c^-}, respectively. Thus, if dominance between prospects X and Y is established for some values of parameters c^+ and c^-, then it will hold also for any smaller values. More formally, for all values $c^+ \geq \tilde{c}^+$ and $c^- \geq \tilde{c}^-$ it holds that

$$X \succeq_{c^-}^{c^+} Y \Rightarrow X \succeq_{\tilde{c}^-}^{\tilde{c}^+} Y.$$

Prospect X dominates Y in the sense of PWSD (Definition 2) if and only if (i) X dominates Y in the sense of PSD (Definition 1) and (ii) X dominates Y is the sense of FSD in the tails of the distributions, the length of which depends on the values of parameters c^- and c^+. This result established by Baucells and Heukamp (2006) is formally presented in the following proposition.

Proposition 2 (Baucells & Heukamp, 2006). *Let X and Y be prospects. Then $X \succeq_{c^-}^{c^+} Y$ if and only if $X \succeq Y$ and*

$$F_X(t) \leq F_Y(t) \ \forall\ t \in [a, t^L) \cup [t^R, b], \tag{4}$$

where $t^L = \inf(\{t \leq r \mid F_X(t) \geq c^-,\ F_Y(t) \geq c^-\} \cup \{r\})$ and $t^R = \sup(\{t \geq r \mid F_X(t) \leq 1 - c^+,\ F_Y(t) \leq 1 - c^+\} \cup \{r\})$.

A direct implication of Proposition 2 is that if convexity of the probability weighting functions is required on the entire interval $[0, 1]$, then PWSD becomes equivalent to PSD. This is because setting $c^+ = c^- = 0$ yields $t^L = a$ and $t^R = b$, and hence inequality (4) is trivially satisfied. In turn, if there are no convexity requirements (i.e., $c^+ = c^- = 1$), then $t^L = t^R = r$ and as a result, inequality (4) holds only if prospect X dominates Y in the sense of FSD. More formally, for any prospects X and Y it holds that

$$X \succeq_0^0 Y \Leftrightarrow X \succeq Y \text{ and}$$
$$X \succeq_1^1 Y \Leftrightarrow F_X(t) \leq F_Y(t) \ \forall\ t \in [a, b].$$

3. Prospect stochastic dominance in discrete state-space

Utilizing PSD and PWSD in stochastic optimization requires developing conditions for these dominance criteria, in which the integrated CDFs of the prospects need to be evaluated only at a finite number of different outcome levels. To achieve this we consider a discrete state-space $S = \{s_i \mid i \in N\}$, where $N = \{1, \ldots, n\}$, consisting of n mutually exclusive and collectively exhaustive states with the state probabilities $p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n p_i = 1$. Prospects thus correspond to discrete random variables $X : S \to [a, b] \subset \mathbb{R}$, where the interval $[a, b]$ includes possible outcomes of all prospects under consideration. We use $x_i = X(s_i)$ to the denote the state-specific outcome of prospect X in the ith state. Under a discrete state-space, the expected outcome of prospect X is given by $\mathbb{E}[X] = \sum_{i=1}^n p_i x_i$ and its cumulative density function (CDF) by

$$F_X(t) = \mathbb{P}\left(\left\{ s_i \in S \,\middle|\, X(s_i) \leq t \right\}\right) = \sum_{\substack{i \text{ s.t.} \\ x_i \in [a,t]}} p_i. \tag{5}$$

Moreover, the integral of the CDF of prospect X becomes

$$\begin{aligned} F_X^2(t) &= \int_{-\infty}^{t} F_X(t')\, dt' = \int_{a}^{t} F_X(t')\, dt' = \sum_{\substack{i \text{ s.t.} \\ x_i \in [a,t]}} p_i(t - x_i) \\ &= \sum_{i=1}^{n} p_i \max\{t - x_i,\ 0\}. \end{aligned} \tag{6}$$

With this notation, we formulate the necessary and sufficient conditions for prospect X to dominate prospect Y by PSD. Under a discrete state-space, these conditions are formally established by the following theorem.

Theorem 1. *Consider two prospects X and Y and suppose the state-space is discrete. Then $X \succeq Y$ if and only if the following conditions hold:*

$$F_Y^2(t^+) - F_Y^2(r) \geq F_X^2(t^+) - F_X^2(r) \text{ for all } t^+ \in \{y_i \mid y_i \geq r\} \text{ and} \tag{7}$$

$$F_Y^2(r) - F_Y^2(t^-) \geq F_X^2(r) - F_X^2(t^-) \text{ for all } t^- \in \{x_i \mid x_i < r\}. \tag{8}$$

All proofs are presented in Appendix A. Theorem 1 shows that in order to establish whether X dominates Y by PSD in a discrete state-space, it suffices to evaluate their integrated CDFs only for a finite number of outcomes. Specifically, these integrated CDFs need to be evaluated at the reference outcome and at those outcomes that correspond either to the gains of the dominated prospect Y (cf. condition (7)) or to the losses of the dominating prospect X (cf. condition (8)). This result follows from the fact that in a discrete state-space the prospects' integrated CDFs are convex non-decreasing piece-wise linear functions.

Based on Proposition 2, the conditions of Theorem 1 are necessary also for PWSD to hold between two prospects. Thus, establishing sufficient conditions for PWSD requires incorporating the comparison of the

CDFs across the outcomes in their tails $[a, t^L)$ and $[t^R, b]$ (see inequality (4)). Moreover, for these conditions to be relevant in optimization models, they should evaluate the CDFs only at a finite number of these outcomes. The following theorem formalizes such conditions.

Theorem 2. *Consider two prospects X and Y and suppose the state-space is discrete. Denote $\tilde{F}_X(t) = \sum_{x_i<t} p_i$ and let $(\tilde{y}_1, \dots, \tilde{y}_{n+1})$ be a permutation of $(y_1, \dots, y_n, r)$ satisfying $\tilde{y}_1 \le \tilde{y}_2 \le \cdots \le \tilde{y}_{n+1}$. Then $X \succeq_{c^-}^{c^+} Y$ if and only if $X \succeq Y$ and*

$$\tilde{F}_X(\tilde{y}_i) \le F_Y(\tilde{y}_{i-1}) \ \forall\ i \in \{1, \dots, n+1\} \text{ s.t. } F_Y(\tilde{y}_{i-1}) < c^- \text{ and } \tilde{y}_i \le r \tag{9}$$

$$\tilde{F}_X(\tilde{y}_i) \le \max\left\{F_Y(\tilde{y}_{i-1}),\ F_Y(r),\ 1-c^+\right\} \ \forall\ i \in \{1, \dots, n+1\}, \tag{10}$$

where $F_Y(\tilde{y}_0) = 0$.

Note that the conditions of Theorem 2 require evaluating the CDFs of the two prospects only at outcome levels corresponding to the state-specific outcomes of the dominated prospect Y and the reference outcome r. This is beneficial for developing optimization models that identify a prospect X dominating a given benchmark Y by PWSD, because in this case the state-specific outcomes y_i are fixed.

To illustrate the PSD and PWSD conditions in Theorems 1 and 2, consider $n = 10$ equally likely states ($p_i = 0.1$) as well as two prospects X and Y with the state-specific outcomes $x = (75, 94, 99, 21, 22, 36, 38, 58, 61, 65)$ and $y = (10, 12, 14, 15, 45, 62, 63, 68, 88, 96)$. Moreover, suppose that the reference outcome is $r = 50$. Fig. 1(a) shows the resulting CDFs of these two prospects and Fig. 1(b) the integrated CDFs evaluated by conditions (7) and (8) of Theorem 1. Based on Fig. 1(b), these conditions are satisfied and thus prospect X dominates prospect Y in the sense of PSD (i.e., $X \succeq Y$). To establish if PWSD holds for the sets of feasible probability weighting functions defined by $c^- = 0.15$ and $c^+ = 0.25$, it is sufficient to check if the conditions of Theorem 2 are satisfied based on the information given in Fig. 1(a). Adding the reference outcome $r = 50$ to the vector of state-specific outcomes y results in $\tilde{y} = (10, 12, 14, 15, 45, 50, 62, 63, 68, 88, 96)$. Since $F_Y(\tilde{y}_2) = F_Y(12) = 0.2 > c^- = 0.15$, condition (9) is equivalent to the two inequalities $\tilde{F}_X(\tilde{y}_1) = \sum_{x_i<\tilde{y}_1} p_i = 0 \le F_Y(\tilde{y}_0) = 0$ and $\tilde{F}_X(\tilde{y}_2) = \sum_{x_i<\tilde{y}_2} p_i = 0 \le F_Y(\tilde{y}_1) = 0.1$, which both hold. To confirm that condition (10) is satisfied, note that $\tilde{F}_X(\tilde{y}_i) \nleq F_Y(\tilde{y}_{i-1})$ only when $i = 7$ (i.e., $\tilde{F}_X(\tilde{y}_7) = \tilde{F}_X(62) = 0.6 > F_Y(\tilde{y}_6) = F_Y(50) = 0.5$), but in this case $\tilde{F}_X(62) = 0.6 < 1 - c^+ = 0.75$. Thus, it holds that $X \succeq_{0.15}^{0.25} Y$.

4. Optimization models for prospect stochastic dominance

In many decision problems, the set of available prospects is not explicitly given as a finite list of decision alternatives, but implicitly defined as a set of solutions satisfying a specific system of constraints. For instance, potential investment portfolios of financial assets can be characterized by a set of vectors of asset weights whose elements sum to one. In turn, feasible portfolios of R&D projects correspond to project subsets that do not consume more resources than are available. In such applications the use of Theorem 1 is often impractical, as it would require enumerating all available prospects, the number of which can be infinite.

To address this shortcoming, we develop stochastic optimization problems to identify if a set of prospects

$$\mathbb{X} \subset \left\{X : S \to [a, b]\right\} \tag{11}$$

contains some prospect $X \in \mathbb{X}$ that stochastically dominates a pre-specified benchmark prospect Y in the sense of PSD or PWSD. We also show that these stochastic optimization models can be used to identify the prospect that optimizes a suitable objective function among those that dominate Y. As an example, consider the portfolio selection problem with m base assets whose returns are captured by random variables $X_1, \dots, X_m$. In this context, the set of portfolio returns would correspond to the set of prospects $\mathbb{X} = \left\{\sum_{j=1}^m \lambda_j X_j \mid \lambda \in \mathbb{R}_+^m,\ \sum_{j=1}^m \lambda_j = 1\right\}$ when short-selling is not allowed. Consequently, identifying the optimal portfolio that dominates a given benchmark portfolio Y by PSD (PWSD) and yields the highest expected return would thus correspond to solving the stochastic optimization problem $\max_{X\in\mathbb{X}}\left\{\mathbb{E}[X] \mid X \succeq Y\right\}$ $\left(\max_{X\in\mathbb{X}}\left\{\mathbb{E}[X] \mid X \succeq_{c^-}^{c^+} Y\right\}\right)$.

4.1. Formulating prospect stochastic dominance constraints

To utilize Theorem 1 in stochastic optimization with PSD constraints, we first present two lemmas formulating its conditions through linear constraints. In particular, condition (7) of Theorem 1 states that for prospect X to dominate prospect Y, it must hold that $F_X^2(y_i) - F_X^2(r) \le F_Y^2(y_i) - F_Y^2(r)$ for each state-specific outcome y_i that is a gain. The following lemma shows that this condition is satisfied if and only if there exists a feasible solution to a specific system of constraints, each of which is linear in the state-specific outcomes x_k of prospect X.

Lemma 1. *Consider two prospects X and Y. Then condition (7) of Theorem 1 holds if and only if there exist $h^+ \in \mathbb{R}_+^{n^+\times n}$ and $z^+ \in \{0,1\}^n$ that satisfy the constraints*

$$h_{ik}^+ \ge y_i - x_k - M(1 - z_k^+) \ \forall\ i \in N^+,\ k \in N \tag{12}$$

$$h_{ik}^+ \ge y_i - r - M z_k^+ \ \forall\ i \in N^+,\ k \in N \tag{13}$$

$$\sum_{k=1}^n p_k h_{ik}^+ \le F_Y^2(y_i) - F_Y^2(r) \ \forall\ i \in N^+, \tag{14}$$

where M is a sufficiently large positive constant, $N^+ = \left\{i \in N \mid y_i \ge r\right\}$ and $n^+ = |N^+|$.

While a detailed proof of this lemma is presented in Appendix A, we provide a brief summary of its logic here: For an arbitrary $y_i \ge r$, it can be shown that there always exists h^+ satisfying constraints (12)–(13) such that $\sum_{i=1}^n p_k h_{ik}^+$ is equal to $F_X^2(y_i) - F_X^2(r)$, i.e., the RHS of condition (7) in Theorem 1. Note that since $F_X^2(y_i) - F_X^2(r)$ is non-convex with respect to the outcomes of prospect X, there is a need to introduce an auxiliary binary variable z_k^+ for each state to indicate if the state-specific outcome x_k is greater than the reference outcome r. Moreover, it can be shown that for any h^+ satisfying constraints (12)–(13), $\sum_{i=1}^n p_k h_{ik}^+$ provides an upper bound for $F_X^2(y_i) - F_X^2(r)$. As the choice of y_i was arbitrary, it is relatively straightforward to establish the lemma from these two results.

Next we develop the second system of linear constraints based on condition (8) of Theorem 1. In particular, this condition states that for prospect X to dominate prospect Y, it must hold that $F_X^2(r) - F_X^2(x_i) \le F_Y^2(r) - F_Y^2(x_i)$ for each state-specific outcome x_i that is a loss. A key difference here compared to condition (7) is that the integrated CDFs are evaluated at outcome levels corresponding to the state-specific outcomes of the dominating prospect X rather than to those of the dominated prospect Y. The resulting system of linear constraints is formalized by the following lemma.

Lemma 2. *Consider two prospects X and Y. Then condition (8) of Theorem 1 holds if and only if there exist $d^- \in \mathbb{R}_+^n$, $g^- \in \mathbb{R}_+^{n\times n}$, $h^- \in \mathbb{R}_+^{n\times n}$, and $z^- \in \{0,1\}^{n\times n}$ that satisfy the constraints*

$$d_k^- \ge r - x_k \ \forall\ k \in N \tag{15}$$

$$g_{ik}^- \ge x_i - y_k \ \forall\ i, k \in N \tag{16}$$

$$h_{ik}^- \le x_i - x_k + M(1 - z_{ik}^-) \ \forall\ i, k \in N \tag{17}$$

$$h_{ik}^- \le M z_{ik}^- \ \forall\ i, k \in N \tag{18}$$

$$\sum_{k=1}^n p_k d_k^- + \sum_{k=1}^n p_k g_{ik}^- - \sum_{k=1}^n p_k h_{ik}^- \le F_Y^2(r) \ \forall\ i \in N \text{ s.t. } x_i < r, \tag{19}$$

where M is a sufficiently large positive constant.

The overall proof strategy here is similar to that of Lemma 1: For an arbitrary $x_i < r$, rearranging condition (8) of Theorem 1 gives the

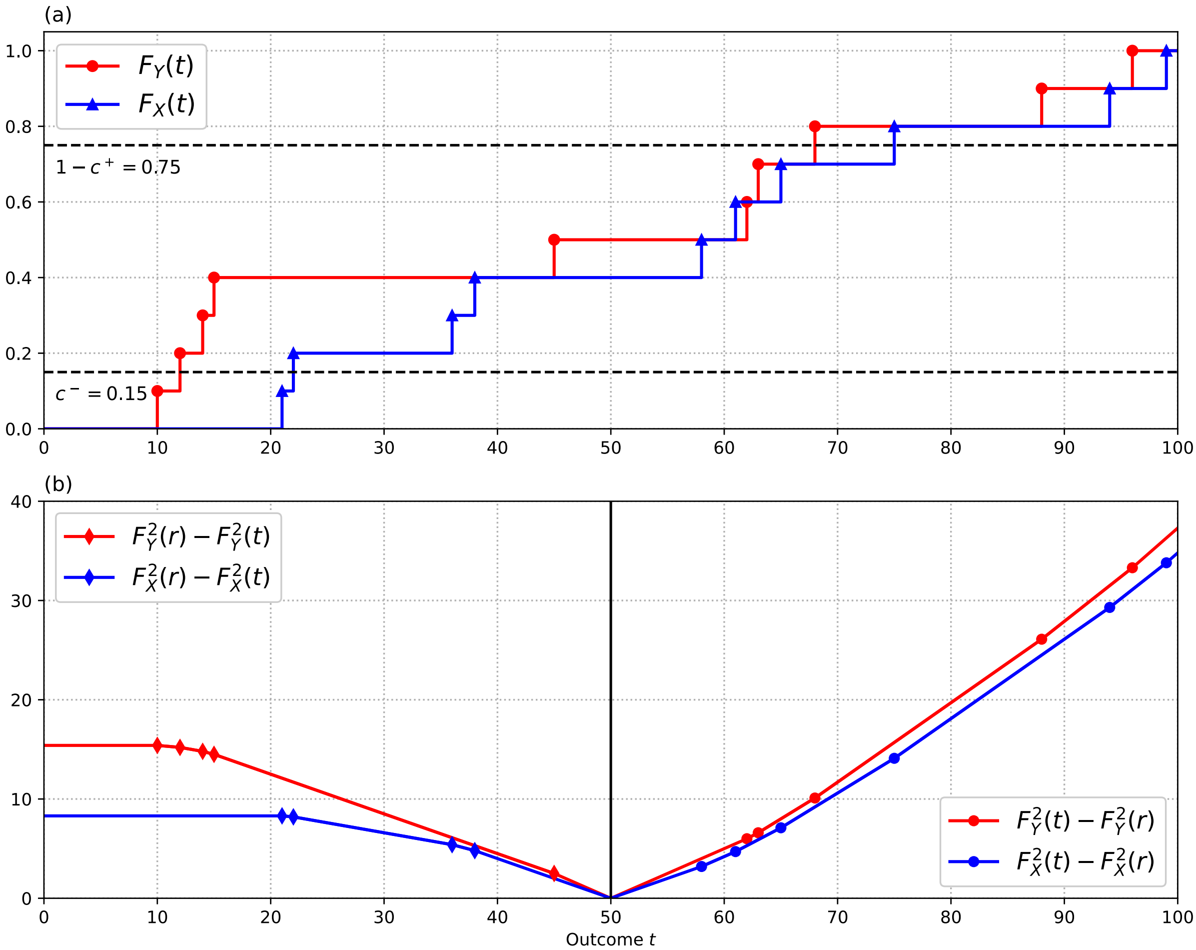

Fig. 1. Illustration of dominance conditions in Theorems 1 and 2. The state-specific outcomes of prospects X (blue) and Y (red) are marked with points, triangles, and diamonds.

constraint $F_X^2(r)+F_Y^2(x_i)-F_X^2(x_i)\le F_Y^2(r)$. Then, it can be shown that $\sum_{k=1}^n p_k d_k^- + \sum_{k=1}^n p_k g_{ik}^- - \sum_{k=1}^n p_k h_{ik}^-$ provides an upper bound for the LHS of this constraint for any feasible values of d^-, g^-, h^-, z^- satisfying constraints (15)–(19), and also that there exist feasible values for which this upper bound is tight. The binary decision variables $z_{i1}^-,\dots,z_{in}^-$ are needed here to indicate which of the state-specific outcomes $x_1,\dots,x_n$ are below the state-specific outcome x_i. As these results hold for any $x_i<r$, they imply the lemma.

Note that the constraints of Lemmas 1 and 2 are linear in the state-specific outcomes of prospect X. Thus, we can readily treat x_i as a decision variable and identify if there exists some prospect in set $\mathbb{X}$ (11) whose state-specific outcomes satisfy all these constraints when appropriate values for the auxiliary decision variables $(h^+, z^+, d^-, g^-, h^-, z^-)$ are chosen. This result is formally established by the following theorem, which uses $\mathcal{X}\subset\mathbb{R}^n$ to denote the set of all vectors of state-specific outcomes generated by the prospects in set $\mathbb{X}$, i.e.,

$$\mathcal{X}=\left\{(x_1,\dots,x_n)=(X(s_1),\dots,X(s_n))\in\mathbb{R}^n \mid X\in\mathbb{X}\right\}. \tag{20}$$

Theorem 3. *Consider a set of prospects $\mathbb{X}$ and prospect Y. Then the following statements hold:*

(i) If there exists $X\in\mathbb{X}$ such that $X\succeq Y$, then there exist $h^+\in\mathbb{R}_+^{n^+\times n}$, $z^+\in\{0,1\}^n$, $d^-\in\mathbb{R}_+^n$, $g^-\in\mathbb{R}_+^{n\times n}$, $h^-\in\mathbb{R}_+^{n\times n}$, $z^-\in\{0,1\}^{n\times n}$, and $\zeta^-\in\{0,1\}^n$ that satisfy a system of linear constraints

$$x_k + M(1-z_k^+) + h_{ik}^+ \ge y_i \;\forall\, i\in N^+,\; k\in N \tag{21}$$

$$r + M z_k^+ + h_{ik}^+ \ge y_i \;\forall\, i\in N^+,\; k\in N \tag{22}$$

$$\sum_{k=1}^n p_k h_{ik}^+ \le F_Y^2(y_i) - F_Y^2(r)\;\forall\, i\in N^+ \tag{23}$$

$$x_k + d_k^- \ge r \;\forall\, k\in N \tag{24}$$

$$x_i - g_{ik}^- \le y_k \;\forall\, i,k\in N \tag{25}$$

$$x_i - x_k + M(1-z_{ik}^-) - h_{ik}^- \ge 0 \;\forall\, i,k\in N \tag{26}$$

$$M z_{ik}^- - h_{ik}^- \ge 0 \;\forall\, i,k\in N \tag{27}$$

$$r - x_i - M\zeta_i^- \le 0 \;\forall\, i\in N \tag{28}$$

$$\sum_{k=1}^n p_k d_k^- + \sum_{k=1}^n p_k g_{ik}^- - \sum_{k=1}^n p_k h_{ik}^- - M(1-\zeta_i^-) \le F_Y^2(r)\;\forall\, i\in N, \tag{29}$$

where M is a sufficiently large positive constant, $N^+=\{i\in N \mid y_i\ge r\}$ and $n^+=|N^+|$.

(ii) Conversely, if there exist $x\in\mathcal{X}$, $h^+\in\mathbb{R}_+^{n^+\times n}$, $z^+\in\{0,1\}^n$, $d^-\in\mathbb{R}_+^n$, $g^-\in\mathbb{R}_+^{n\times n}$, $h^-\in\mathbb{R}_+^{n\times n}$, $z^-\in\{0,1\}^{n\times n}$, and $\zeta^-\in\{0,1\}^n$ that satisfy the system of linear constraints (21)–(29), then there exists some prospect $X'\in\mathbb{X}$ such that $X'(s_i)=x_i$ for all $i\in N$, and $X'\succeq Y$.

The theorem follows quite directly from the two systems of linear constraints presented in Lemmas 1 and 2. However, it also introduces constraint (28) and the binary variables $\zeta_1,\dots,\zeta_n$ to indicate which of the state-specific outcomes $x_1,\dots,x_n$ of prospect X are losses, i.e., below the reference outcome r. These binary variables are also incorporated into constraint (29) to make it redundant for those outcomes x_i that are gains.

The linear system of constraints in Theorem 3 can be readily used to formulate stochastic optimization models with PSD constraints. Moreover, since these constraints are linear, they do not give rise to nonlinearities in an optimization model, although they do necessitate the introduction of binary decision variables. For instance, if a stochastic optimization model has (i) an objective function which is linear in the state-specific outcomes $x = (x_1, \dots, x_n)$, and (ii) a set of feasible outcomes $\mathcal{X}$ (20) defined through linear constraints, then implementing PSD constraints results in a MILP formulation. The following corollary demonstrates how the constraints of Theorem 3 can be used to formulate a stochastic optimization model that identifies, among all prospects dominating the benchmark by PSD, the one yielding the highest expected outcome.

Corollary 1. *X^* is an optimal solution to the optimization problem $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \geq Y\}$ if and only if $x^* = (x_1^*, \dots, x_n^*) = (X^*(s_1), \dots, X^*(s_n))$ is an optimal solution to the optimization problem*

$$\max \sum_{i=1}^{n} p_i x_i \tag{30}$$

$$\text{s.t. } (x, h^+, z^+, d^-, g^-, h^-, z^-, \zeta^-) \text{ satisfies constraints (21)–(29)}$$

$$x \in \mathcal{X}, \text{where } \mathcal{X} \text{ is given by (20).}$$

4.2. Incorporating weighted prospect stochastic dominance constraints

In this section, we develop an approach for incorporating PWSD constraints into stochastic optimization models. Recall that according to Theorem 2, prospect X dominates prospect Y by PWSD, if X dominates Y by PSD and the CDFs of these prospects satisfy two additional conditions. Thus, implementing PWSD constraints requires only formulating these additional conditions as linear constraints, because PSD can be guaranteed by enforcing the constraints given by Theorem 3.

In order to keep the presentation clear, we assume throughout the remainder of this section that the states in $S = \{s_1, \dots, s_n\}$ are indexed in an ascending order of the state-specific outcomes of prospect Y, i.e., $y_1 \leq y_2 \leq \dots \leq y_n$ and that one of these state-specific outcomes is equal to the reference outcome, i.e., $r \in \{y_1, y_2, \dots, y_n\}$. Note that this assumption does not lead to a loss of generality, since the state indexing is arbitrary and S can be readily augmented with an additional zero-probability state in which the state-specific outcome of prospect Y is equal to the reference outcome r. Yet, the assumption avoids the need to introduce separate notation for the ordered outcomes as well as to duplicate listing of the constraints that are required to hold also for the reference outcome. This is exemplified in the following lemma, which formalizes the linear constraints that need to hold, in addition to PSD, for a prospect to dominate another in the sense of PWSD.

Lemma 3. *Consider two prospects X and Y. Then $X \succeq_{c^-}^{c^+} Y$ if and only if $X \geq Y$ and there exists $z \in \{0,1\}^{n\times n}$ that satisfies the constraints*

$$x_k + M z_{ik} \geq y_i \;\; \forall\, i, k \in N \tag{31}$$

$$\sum_{k=1}^{n} p_k z_{ik} \leq F_Y(y_{i-1}) \;\; \forall\, i \in N \text{ s.t. } F_Y(y_{i-1}) < c^- \text{ and } y_i \leq r \tag{32}$$

$$\sum_{k=1}^{n} p_k z_{ik} \leq \max\left\{F_Y(y_{i-1}),\; F_Y(r),\; 1 - c^+\right\} \;\; \forall\, i \in N, \tag{33}$$

where $F_Y(y_0) = 0$ and M is a sufficiently large positive constant.

A detailed proof of this lemma is given in Appendix A, but its logic can be elaborated as follows: The binary variables $z_{i1}, \dots, z_{in}$ satisfying constraint (31) have the property that $z_{ik} = 1$ for all $x_k < y_i$. Therefore, $\sum_{k=1}^{n} p_k z_{ik}$ provides an upper bound for $\sum_{x_k<y_i} p_i$, and moreover, this upper bound is tight when $z_{ik} = 0$ for all $x_k \geq y_i$. Since $\sum_{x_k<y_i} p_i$ is equal to the LHSs of both constraints (9) and (10), the lemma follows from Theorem 2.

Lemma 3 together with the results of Section 4.1 can be used to determine if a set of prospects $\mathbb{X}$ contains some prospect X dominating a given benchmark prospect Y by PWSD. Specifically, such a PWSD dominating prospect exists if and only if the system of linear constraints that comprises those in Lemma 3 and in Theorem 3 has a feasible solution. This result is formalized by the following theorem.

Theorem 4. *Consider a set of prospects $\mathbb{X}$ and prospect Y. Then the following statements hold:*

(i) If there exists $X \in \mathbb{X}$ such that $X \succeq_{c^-}^{c^+} Y$, then there exist $h^+ \in \mathbb{R}_+^{n^+\times n}$, $z^+ \in \{0,1\}^n$, $d^- \in \mathbb{R}_+^n$, $g^- \in \mathbb{R}_+^{n\times n}$, $h^- \in \mathbb{R}_+^{n\times n}$, $z^- \in \{0,1\}^{n\times n}$, $\zeta^- \in \{0,1\}^n$, and $z \in \{0,1\}^{n\times n}$ that satisfy the system of linear constraints (21)–(29) and (31)–(33).

(ii) Conversely, if there exist $x \in \mathcal{X}$, $h^+ \in \mathbb{R}_+^{n^+\times n}$, $z^+ \in \{0,1\}^n$, $d^- \in \mathbb{R}_+^n$, $g^- \in \mathbb{R}_+^{n\times n}$, $h^- \in \mathbb{R}_+^{n\times n}$, $z^- \in \{0,1\}^{n\times n}$, $\zeta^- \in \{0,1\}^n$, and $z \in \{0,1\}^{n\times n}$ that satisfy the system of linear constraints (21)–(29) and (31)–(33), then there exists some prospect $X' \in \mathbb{X}$ such that $X'(s_i) = x_i$ for all $i \in N$, and $X' \succeq_{c^-}^{c^+} Y$.

Since the constraints in Theorem 4 are linear in all decision variables, they can be used in stochastic optimization models to implement PWSD constraints without introducing non-linearities. Corollary 2 provides a formulation for a stochastic optimization model which identifies the expected outcome maximizing prospect among all those in set $\mathbb{X}$ that dominate the benchmark prospect Y by PWSD.

Corollary 2. *X^* is an optimal solution to the optimization problem $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \succeq_{c^-}^{c^+} Y\}$ if and only if $x^* = (x_1^*, \dots, x_n^*) = (X^*(s_1), \dots, X^*(s_n))$ is an optimal solution to the optimization problem*

$$\max \sum_{i=1}^{n} p_i x_i$$

$$\text{s.t. } (x, h^+, z^+, d^-, g^-, h^-, z^-, \zeta^-, z) \text{ satisfies constraints (21)–(29) and (31)–(33)}$$

$$x \in \mathcal{X}, \text{where } \mathcal{X} \text{ is given by (20).}$$

5. Empirical applications

5.1. Application to industry portfolio optimization

This section applies the developed stochastic optimization models to financial data to investigate the efficiency of the market portfolio. Specifically, we test if diversification across industries makes it possible to construct a portfolio that dominates the market portfolio by PSD or PWSD. If such PSD dominating portfolios exist, they would be preferred over the benchmark market portfolio by all investors with an S-shaped utility function. Moreover, if there exists a portfolio dominating the benchmark by PWSD, then it would be preferred by all CPT investors with an S-shaped utility function and inverse S-shaped probability weighting functions contained in sets W^{c^+} and W^{c^-}. However, if such benchmark dominating portfolios cannot be identified, then holding the market portfolio is a justified optimal investment decision for at least some CPT preferences.

We use monthly returns of the Fama–French 49 value-weighted industry portfolios as the base assets $X_1, \dots, X_{49}$ and the all-share index 'Bench' from the Center for Research in Security Prices (CRSP) to proxy the benchmark market portfolio Y. 'Bench' is a tracking index of the value-weighted return of all CRSP firms incorporated in the US and listed on the NYSE, AMEX, or NASDAQ exchanges. Therefore, the data set includes all monthly asset and benchmark return observations from January 1927 to December 2021, spanning a sample period of 95 years or 1140 trading months. The descriptive statistics of this data set can be found in the online supplementary material.

We deploy a rolling estimation approach with an estimation window of 36 months that is shifted forward 12-month at a time. With this estimation approach, our data set yields a total of 93 overlapping 3-year estimation periods 01/1927–12/1929, 01/1928–12/1930, …, 01/2019–12/2021. For each estimation period, we solve three stochastic optimization models in which the state-space is constructed using the monthly returns of the base assets $X_1, \dots, X_{49}$ and the benchmark market portfolio Y. The first optimization model identifies the expected return maximizing portfolio among those that dominate the benchmark

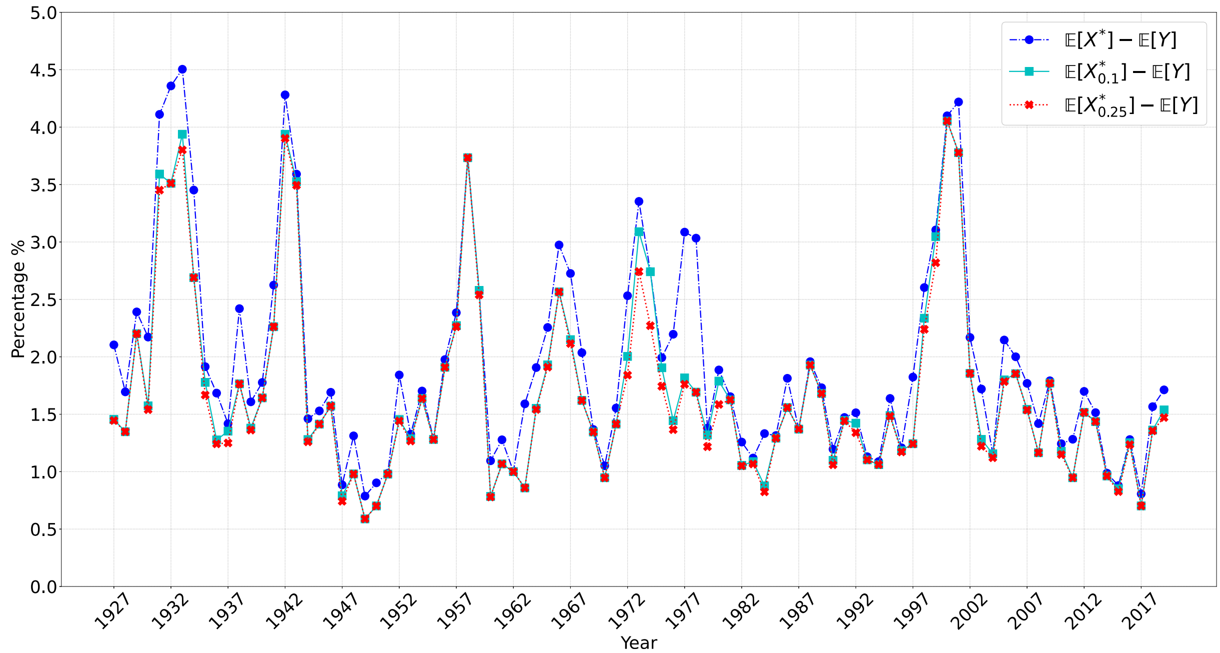

Fig. 2. Inefficiency of the market portfolio 1927–2021.
Note: Excess returns of the optimized portfolios X^*, $X^*_{0.1}$, and $X^*_{0.25}$ over the benchmark market portfolio Y. Portfolio X^* dominates the benchmark Y in the sense of PSD and portfolios $X^*_{0.1}$ and $X^*_{0.25}$ in the sense of PWSD.

Y in the sense of PSD, when short-selling is not allowed. Formally, this model can be expressed as $X^* \in \arg\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \succeq Y\}$, where $\mathbb{X} = \{\sum_{j=1}^{49} \lambda_j X_j \mid \lambda \in \mathbb{R}^{49}_+,\ \sum_{j=1}^{49} \lambda_j = 1\}$ (see Corollary 1 for the MILP formulation). The second and third optimization models identify the expected return maximizing portfolios among those that dominate the benchmark Y in the sense of PWSD with two threshold parameter values $c^- = c^+ = 0.1$ and $c^- = c^+ = 0.25$. Formally, these portfolios are defined as $X^*_{0.1} \in \arg\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \succeq^{0.1}_{0.1} Y\}$ and $X^*_{0.25} \in \arg\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \succeq^{0.25}_{0.25} Y\}$ (see Corollary 2 for the MILP formulation). For each estimation period, the geometric mean of the monthly risk-free T-bill rates is used as the reference rate r. The expected returns of the three optimal portfolios satisfy $\mathbb{E}[X^*] \geq \mathbb{E}[X^*_{0.1}] \geq \mathbb{E}[X^*_{0.25}]$, since $\{X \in \mathbb{X} \mid X \succeq Y\} \supseteq \{X \in \mathbb{X} \mid X \succeq^{0.1}_{0.1} Y\} \supseteq \{X \in \mathbb{X} \mid X \succeq^{0.25}_{0.25} Y\}$.

Fig. 2 shows the excess returns of the optimized portfolios X^*, $X^*_{0.1}$, and $X^*_{0.25}$ over the benchmark market portfolio Y for the years 1927–2021. Fig. 3 presents the number of base assets included in the optimal PSD (X^*) and PWSD ($X^*_{0.1}, X^*_{0.25}$) portfolios for each year from 1927 to 2021. Moreover, Fig. 4 illustrates the range of asset weights $\lambda_j \in [0,1]$ for each of the $m = 49$ base assets in the optimal PSD (X^*) and PWSD ($X^*_{0.1}, X^*_{0.25}$) portfolios over the period 1927–2021 (see online supplementary material for visualizations of asset compositions).

These results suggest that the market portfolio is inefficient, as for each estimation period it is possible to identify another portfolio that dominates the market portfolio in the sense of PSD and PWSD. On average, the excess return of portfolio X^* is approximately 2% while that of portfolios $X^*_{0.1}$ and $X^*_{0.25}$ is around 1.6%–1.7% (see Table 1). Overall, we observe a consistent pattern that these excess returns are considerably higher during economic or financial downturns in history such as the Wall Street Crash (1929), World War II (1939–1945), the Eisenhower Recession (1958), the OPEC Oil Price Shock (1973), the Energy Crisis (1979), the Doc-com Bubble (2000), and the September 11 Attacks (2001). Surprisingly, the excess returns do not soar to a new record high level during the Subprime Mortgage Crisis (2007–2008), which is known to have triggered a devastating worldwide financial crisis impacting global economies. In most periods, PSD dominating portfolios (X^*) lead to only marginally greater excess returns over the benchmark than those dominating it in the sense of PWSD ($X^*_{0.1}, X^*_{0.25}$),

Table 1
Summary statistics of excess returns of benchmark dominating portfolios by PSD (X^*) and PWSD ($X^*_{0.1}, X^*_{0.25}$).

Excess returns over benchmark (%)	Statistics				
	Mean	Std.	Min.	Median	Max.
$\mathbb{E}[X^*]-\mathbb{E}[Y]$	1.949	0.888	0.787	1.702	4.504
$\mathbb{E}[X^*_{0.1}]-\mathbb{E}[Y]$	1.706	0.804	0.589	1.454	4.053
$\mathbb{E}[X^*_{0.25}]-\mathbb{E}[Y]$	1.671	0.783	0.589	1.445	4.053

although some notably higher returns are observed in late 1920s and 1970s. Moreover, the larger set of feasible probability weighting functions ($c^- = c^+ = 0.25$) generally results in only a modest decrease in the excess returns, but requires a slightly broader diversification across the asset universe compared to the smaller set ($c^- = c^+ = 0.1$). In contrast, PSD (X^*) portfolios are in general the least diversified compared to PWSD ($X^*_{0.1}, X^*_{0.25}$) portfolios. Interestingly, industries ‘BldMt’ and ‘Mach’ are not included in any of the optimal PSD (X^*) or PWSD ($X^*_{0.1}$) portfolios. In addition, the optimal PSD (X^*) portfolios do not include industries ‘Hshld’, ‘Chems’, ‘Steel’, and ‘Trans’ either.

5.2. Application to a multi-period newsvendor problem

This section demonstrates how the developed models can be utilized in other application areas apart from financial portfolio optimization to analyze if observed decision behavior can be explained by CPT. For this purpose we utilize the real-world procurement optimization application of Sillanpää et al. (2021), which can be viewed as a multi-period newsvendor problem in the area of operations management.

In this application a pulp & paper company decides on the order quantities of natural gas to satisfy uncertain demand with minimal costs. This decision is complicated by multiple time periods and a piecewise linear pricing scheme of the procurement contracts. Originally, Sillanpää et al. (2021) developed a prescriptive stochastic optimization model to identify ordering policies that minimize the expected cost. This model was deployed to support decision making at the case company, which had previously been relying on heuristic ordering policies. Here, however, we analyze the decision setting from a behavioral

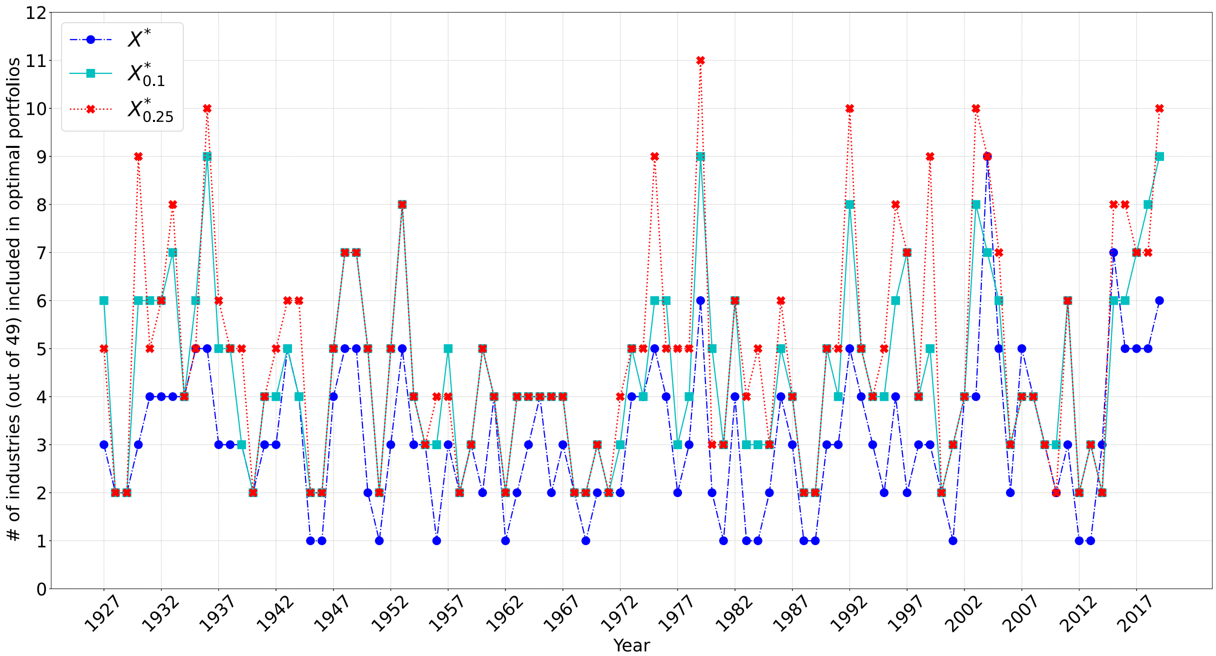

Fig. 3. Number of industries (out of $m = 49$) included in the optimal PSD (X^*) and PWSD ($X^*_{0.1}, X^*_{0.25}$) portfolios for each year 1927–2021.

perspective and examine whether these heuristic ordering policies are optimal under CPT. This is achieved by utilizing the models developed in Section 4 to identify if there exists an ordering policy with a cost distribution that dominates the one generated by a specific heuristic policy in the sense of PSD or PWSD. If such a dominating policy exists, then the heuristic ordering policy cannot be justified by behavioral arguments based on CPT, since any CPT value maximizer would prefer the dominating policy over the heuristic one. Conversely, if no such dominating policy exists, then the heuristic policy would be consistent with CPT decision behavior for some S-shaped utility functions and inverse S-shaped probability weighting functions.

To formally present the decision setting and related stochastic optimization models, let $D_{\tau\eta}$ denote the random variable capturing the demand at the η:th hour of month $\tau \in T = \{1,\ldots,12\}$. There are two types of procurement contracts: First, the case company can commit to ordering a constant fixed quantity q_F for all hours of the upcoming year at a unit price γ^F_τ, which varies from month to month. Second, the case company can commit to ordering quantity q_τ for each hour of month $\tau \in T$ with the unit price $\gamma^0_\tau > c^F_\tau$.

The realized costs are contingent on the realized gas demand. Specifically, if on a given hour η of month τ the total ordered quantity exceeds the gas demand ($q_F + q_\tau > D_{\tau\eta}$), the supplier compensates at a rate of $\gamma^-_\tau < \gamma^F_\tau$ per unit of unused gas. On the contrary, if the demand exceeds the total order, i.e., ($q_F + q_\tau < D_{\tau\eta}$), then any additional gas request will be supplied at a higher price. However, this higher price depends on the magnitude of the shortage: If the demand exceeds the total order by at most $(\alpha_\tau - 1)\times 100\%$, the unit price is $\gamma^+_\tau > \gamma^0_\tau$, beyond which the unit price is $\gamma^*_\tau > \gamma^+_\tau$. Thus, for month τ the total cost is captured by the random variable

$$C_\tau(q_F,q_\tau) = \sum_{\eta\in H_\tau} \max \Big\{ \gamma^F_\tau q_F + \gamma^0_\tau q_\tau + \gamma^-_\tau(D_{\tau\eta} - (q_F+q_\tau)), \quad (34)$$
$$\gamma^F_\tau q_F + \gamma^0_\tau q_\tau + \gamma^+_\tau(D_{\tau\eta} - (q_F+q_\tau)),$$
$$\gamma^F_\tau q_F + \gamma^0_\tau q_\tau + \gamma^+_\tau(D_{\tau\eta} - (q_F+q_\tau)) + (\gamma^*_\tau - \gamma^+_\tau)$$
$$\times (D_{\tau\eta} - \alpha_\tau(q_F+q_\tau)) \Big\},$$

where H_τ is the index set of hours in month $\tau \in T$.

The expected cost minimizing order quantities are thus obtained by solving the optimization problem

$$\max_{q_F,q_\tau\geq 0} \left\{ \mathbb{E}[X] \;\middle|\; X = -\sum_{\tau\in T} C_\tau(q_F,q_\tau) \right\}. \quad (35)$$

This problem has an LP formulation when the uncertain demand is captured by a discrete state-space (see online supplementary material for details). Augmenting this formulation with the constraints and decision variables from Theorem 4 gives a MILP formulation for the optimization problem

$$\max_{q_F,q_\tau\geq 0} \left\{ \mathbb{E}[X] \;\middle|\; X = -\sum_{\tau\in T} C_\tau(q_F,q_\tau),\ X \succeq^{c^+}_{c^-} Y \right\}. \quad (36)$$

The resulting MILP model makes it possible to identify the expected cost minimizing cost distribution X that dominates a chosen benchmark Y in the sense of PWSD.

To estimate the parameters of the stochastic optimization problem (36), we utilize the data from year 2015 as reported in Sillanpää et al. (2021). Specifically, for the unit prices $\gamma^-_\tau, \gamma^F_\tau, \gamma^0_\tau, \gamma^+_\tau, \gamma^*_\tau$ we use the cost forecasts from Table 1 of Sillanpää et al. (2021) and the value $\alpha_\tau = 1.15$. Furthermore, for each month $\tau \in T$, all hourly demands $D_{\tau\eta}$, $\eta \in H_\tau$ are assumed to be identically distributed and to follow a lognormal distribution D_τ. These distributions are fitted to the expected demands and the 95%–confidence intervals presented in Figure 4 of Sillanpää et al. (2021). The resulting lognormal distributions have the expected values $(\mathbb{E}[D_1],\ldots,\mathbb{E}[D_{12}]) = (5.6, 4.6, 4.0, 4.3, 3.4, 4.8, 5.6, 4.3, 5.5, 4.0, 4.2, 4.4)$ and standard deviations $(\mathrm{Std}[D_1],\ldots,\mathrm{Std}[D_{12}]) = (1.1, 1.3, 0.8, 1.4, 1.1, 1.9, 2.6, 3.1, 1.1, 0.7, 0.9, 1.1)$. A random sample of $n = 50$ hourly demand time-series (each with a length of $\sum_{\tau\in T}|H_\tau| = 8760$) is drawn from these distributions to construct the state-space.

As the benchmark we use the cost distribution generated by the heuristic ordering policy that the case company had been practicing before deploying stochastic optimization for decision support. Specifically, in this heuristic policy, the fixed order quantity is given by $\tilde{q}_F = \arg\min_{q_F} \sum_{\tau\in T}(\gamma^F_\tau q_F + \gamma^0_\tau \max\{0,\ \mathbb{E}[D_\tau] - q_F\})$ (see online supplementary material for the LP formulation) and the monthly order quantities by $\tilde{q}_\tau = \max\{0,\ \mathbb{E}[D_\tau] - \tilde{q}_F\}$. The resulting benchmark cost

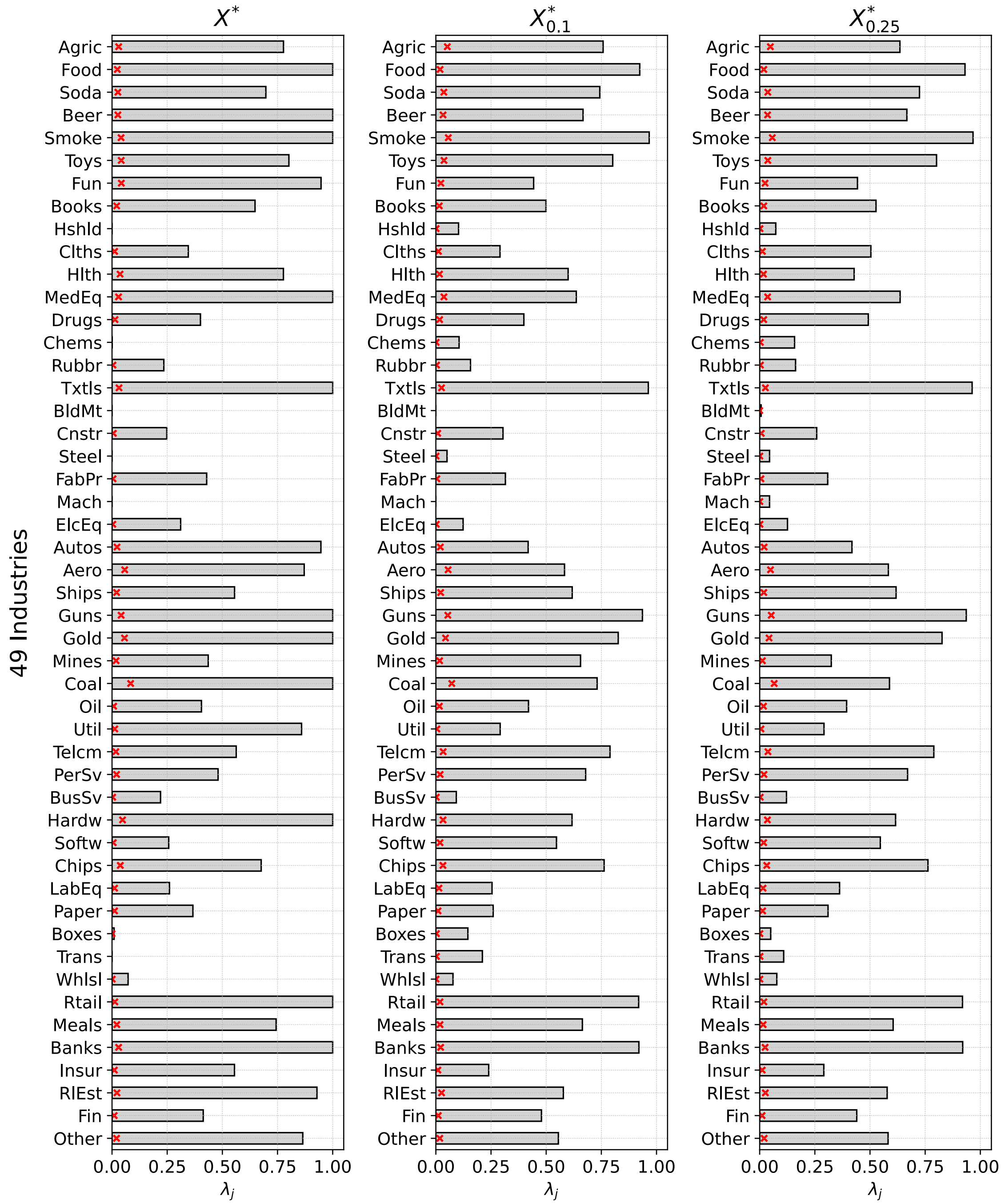

Fig. 4. Asset compositions of the optimal PSD (X^*) and PWSD ($X^*_{0.1}, X^*_{0.25}$) portfolios. The gray bars show the minimum and maximum values for λ_j, $j \in \{1,\ldots,49\}$, across the years 1927–2021, and the red crosses denote the average values.

distribution from the heuristic order quantities is thus given by $Y = -\sum_{\tau\in T} C_\tau(\tilde{q}_F, \tilde{q}_\tau)$, where $C_\tau(\cdot)$ is given by (34).

We solve the stochastic optimization problem (36) using three different sets of probability weighting functions ($c^- = c^+ \in \{0, 0.1, 0.25\}$) and two reference outcomes ($r \in \{0, \text{median}(Y)\}$). Surprisingly, the six problems all produce exactly the same optimal ordering policy $q_F^* = 4.2$, $q^* = (1.1, 0, 0, 0, 0, 0, 0.4, 0, 0.9, 0, 0, 0)$. This optimal policy offers a 0.5% reduction in the expected costs compared to the heuristic policy $\tilde{q}_F = 3.4$, $\tilde{q} = (2.2, 1.2, 0.6, 0.9, 0, 1.4, 2.2, 0.9, 2.1, 0.6, 0.8, 1.0)$. Moreover, we find that this optimal policy (q_F^*, q_τ^*) results in a cost distribution (X^*) which first-order stochastically dominates the cost distribution (Y) produced by the heuristic policy $(\tilde{q}_F, \tilde{q}_\tau)$ (i.e., $F_{X^*}(t) \le F_Y(t)\ \forall\ t \in \mathbb{R}$, see Fig. 5). Consequently, the heuristic policy is not optimal for any CPT decision maker as the optimal policy (q_F^*, q_τ^*) yields a higher CPT value for any S-shaped utility function and any pair of inverse S-shaped probability weighting functions. More broadly, all decision makers whose risk preferences are characterized by non-decreasing utility functions would prefer the optimal ordering policy (q_F^*, q_τ^*) over the heuristic one.

6. Discussion and conclusions

In this research, we have developed stochastic optimization models for cumulative prospect theory (CPT) that allow incomplete preference information. This incomplete information is modeled by utilizing the PSD and PWSD criteria which accommodate sets of utility functions and probability weighting functions. The developed optimization models enable to identify an optimal decision alternative that is preferred to a

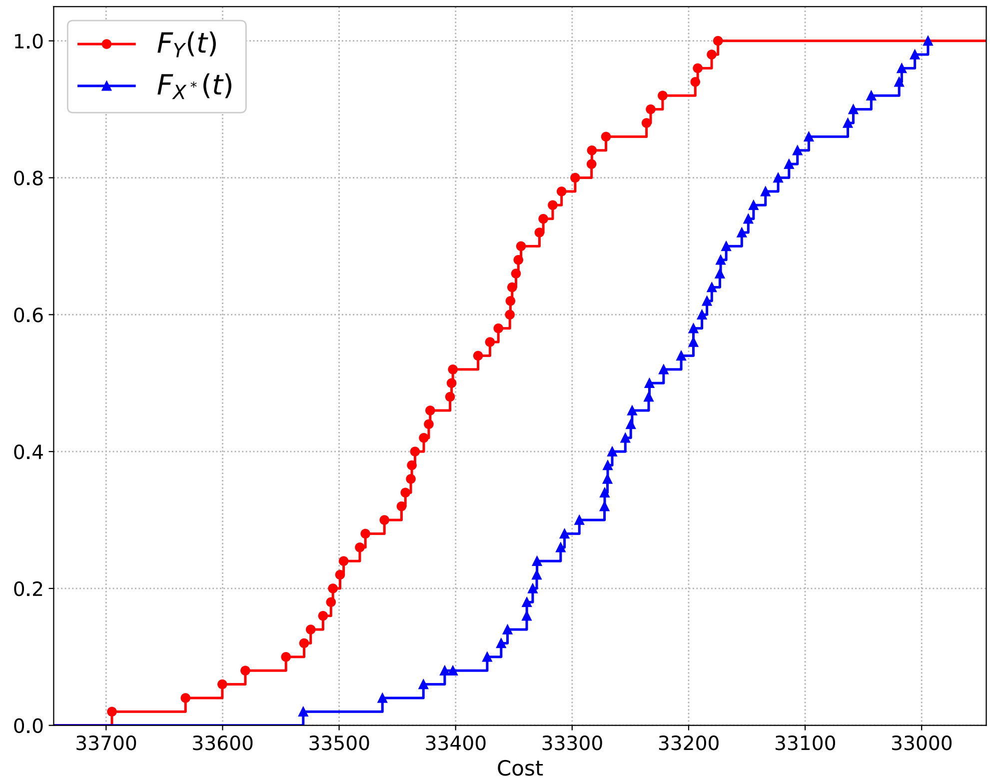

Fig. 5. Cost distributions of the heuristic ordering policy (Y) and the optimized ordering policy (X^*).

pre-specified benchmark by all CPT decision makers with an S-shaped utility function and a pair of inverse S-shaped probability weighting functions. These contributions complement existing prescriptive stochastic optimization models that utilize EUT-based dominance criteria (e.g., SSD), and make it possible to utilize stochastic optimization in descriptive behavioral analyses in which CPT describes empirically observed decision behavior.

The two reported applications in portfolio diversification and procurement optimization demonstrate that the developed optimization models are suitable for analyzing decision settings in which it is impractical or even impossible to enumerate all feasible decision alternatives. In both applications, the developed optimization models identified a decision alternative that would be preferred to the benchmark alternative by a large group of decision makers whose choice behavior agrees with CPT. As a result, these benchmark alternatives are not optimal for decision makers whose decision behavior is characterized by CPT.

This research opens up several avenues for future research. Firstly, the methods developed here should be tested in other empirical applications in areas such as finance, operations management, and economics to analyze if observed decision behavior is consistent with CPT. Secondly, the potential usefulness of the developed models in prescriptive decision support could be explored. For instance, in financial portfolio selection, recommending the expected utility maximizing portfolio (for some reasonable choice of a utility function) among those that dominate the market portfolio in the sense of PSD or PWSD might strike a practical balance between rationality and acceptability from a viewpoint of an investor whose preferences are consistent with CPT. Thirdly, the optimal solution given by any stochastic optimization model utilizing stochastic dominance constraints can be sensitive to the benchmark selection. Although the choice of a benchmark depends on the application context and the aims of the specific empirical analysis, it would be useful to explore further if some general properties of a suitable benchmark solution can be identified. Finally, another attractive approach would be to develop optimization models that avoid the need of specifying a benchmark solution by solving the entire set of efficient solutions that are non-dominated in the sense of PSD or PWSD. This approach seems promising since recently multiobjective optimization models have been successfully applied to identify solutions to stochastic optimization problems that are not dominated by any other feasible solutions in the sense of Second- and Third-order Stochastic Dominance (Liesiö et al., 2023).

CRediT authorship contribution statement

Peng Xu: Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Funding acquisition, Formal analysis, Conceptualization. **Juuso Liesiö:** Writing – review & editing, Writing – original draft, Visualization, Software, Methodology, Formal analysis, Conceptualization.

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Appendix A. Proofs

Proof of Theorem 1. By Proposition 1, $X \succeq Y$ if and only if for any $t^- \le r$ and $t^+ \ge r$

$$0 \le \int_{t^-}^{t^+} F_Y(\tau) - F_X(\tau) d\tau = \underbrace{\int_{t^-}^{r} F_Y(\tau) - F_X(\tau) d\tau}_{=\kappa} + \underbrace{\int_{r}^{t^+} F_Y(\tau) - F_X(\tau) d\tau}_{=\eta} . \tag{A.1}$$

Note that if $\kappa < 0$ ($\eta < 0$) from some value of t^- (t^+), then setting $t^+ = r$ ($t^- = r$) would imply that $\kappa + \eta < 0$. In turn, if $\kappa + \eta < 0$ for some values of t^- and t^+, then it holds that $\kappa < 0$ for t^+ or $\eta < 0$ for t^+. Thus, $X \succeq Y$ if and only if $\kappa \geq 0$ and $\eta \geq 0$. Substituting $\int_\alpha^\beta F_{(\cdot)}(t)dt = F^2_{(\cdot)}(\beta) - F^2_{(\cdot)}(\alpha)$ into (A.1) implies that $X \succeq Y$ if and only if

$$F^2_Y(t^+) - F^2_Y(r) \geq F^2_X(t^+) - F^2_X(r) \text{ for all } t^+ \geq r \tag{A.2}$$

$$F^2_Y(r) - F^2_Y(t^-) \geq F^2_X(r) - F^2_X(t^-) \text{ for all } t^- < r. \tag{A.3}$$

Since $\{y_i \mid y_i \geq r\} \subseteq \{t^+ | t^+ \geq r\}$ and $\{x_i \mid x_i < r\} \subseteq \{t^- | t^- \leq r\}$, conditions (A.2) and (A.3) imply the theorem's conditions

$$\underbrace{F^2_Y(t^+) - F^2_Y(r)}_{=\gamma^+(t^+)} \geq \underbrace{F^2_X(t^+) - F^2_X(r)}_{=\phi^+(t^+)} \text{ for all } t^+ \in \{y_i \mid y_i \geq r\}$$

$$\underbrace{F^2_Y(r) - F^2_Y(t^-)}_{=\gamma^-(t^-)} \geq \underbrace{F^2_X(r) - F^2_X(t^-)}_{=\phi^-(t^-)} \text{ for all } t^- \in \{x_i \mid x_i < r\}.$$

Thus, what remains to be proven is that in a discrete state-space conditions (7) and (8) imply conditions (A.2) and (A.3), respectively. These proofs make use of the fact that in a discrete state-space the integrated CDFs are non-decreasing convex piece-wise linear functions (cf. Eq. (6)).

In the gains domain, we show that if (A.2) does not hold, then condition (7) not hold. Assume condition (A.2) does not hold for some $t^* \geq r$, i.e., $\gamma^+(t^*) < \phi^+(t^*)$. First, if $t^* < y^+_{min}$, where $y^+_{min} = \min\{y_i | y_i > r\}$, we have $\gamma^+(y^+_{min}) - \phi^+(y^+_{min}) < \gamma^+(t^*) - \phi^+(t^*) < 0$ as both γ^+ and ϕ^+ are non-decreasing convex piece-wise linear functions. Then, condition (7) does not hold for $t^+ = y^+_{min}$. Second, if $y^+_{min} \leq t^* \leq y^+_{max}$, where $y^+_{max} = \max\{y_i | y_i \geq r\}$, there exist y_j and y_k such that $t^* \in [y_j, y_k]$ and γ^+ is linear on $[y_j, y_k]$. Then, $\gamma^+(t^*) < \phi^+(t^*)$ implies that either $\gamma^+(y_j) < \phi^+(y_j)$ or $\gamma^+(y_k) < \phi^+(y_k)$, since ϕ^+ is a non-decreasing convex function. Hence, condition (7) does not hold for $t^+ = y_j$ or $t^+ = y_k$. Finally, in case $t^* > y^+_{max}$, it holds that $\gamma^+(y^+_{max}) < \phi^+(y^+_{max})$, since $\frac{\partial}{\partial t}\gamma^+(t) = F_Y(t) = 1 \geq F_X(t) = \frac{\partial}{\partial t}\phi^+(t)$ for all $t > y^+_{max}$.

In the domain of losses, we show that if (A.3) does not hold, then condition (8) does not hold. Assume (A.3) does not hold for some $t^* < r$, i.e., $\gamma^-(t^*) < \phi^-(t^*)$. First, if $t^* > x^-_{max}$, where $x^-_{max} = \max\{x_i | x_i < r\}$, we have $\gamma^-(x^-_{max}) - \phi^-(x^-_{max}) < \gamma^-(t^*) - \phi^-(t^*) < 0$ as both γ^- and ϕ^- are non-increasing concave piece-wise linear functions. Then, condition (8) does not hold for $t^- = x^+_{max}$. Second, if $x^-_{min} \leq t^* \leq x^-_{max}$, where $x^-_{min} = \min\{x_i | x_i < r\}$, there exist x_j and x_k such that $t^* \in [x_j, x_k]$ and ϕ^- is linear on $[x_j, x_k]$. Then, $\gamma^-(t^*) < \phi^-(t^*)$ implies that either $\gamma^-(x_j) < \phi^-(x_j)$ or $\gamma^-(x_k) < \phi^-(x_k)$, and since ϕ^- is a non-increasing concave function. Hence, condition (8) does not hold for $t^- = x_j$ or $t^- = x_k$. Finally, in case $t^* < x^-_{min}$, it holds that $\gamma^-(x^-_{min}) < \phi^-(x^-_{min})$, since $\frac{\partial}{\partial t}\phi^-(t) = -F_X(t) = 0 \geq -F_Y(t) = \frac{\partial}{\partial t}\gamma^-(t)$ for all $t < x^-_{min}$. □

Proof of Theorem 2. To prove the theorem we need to show that inequality (4) of Proposition 2 holds if and only if inequalities (9) and (10) hold.

We first prove the 'if' part by contra-positive, i.e., if inequality (4) of Proposition 2 does not hold, then inequalities (9)–(10) do not hold. Assume that inequality (4) of Proposition 2 does not hold, which implies that $\exists\ t^* \in [a, t^L) \cup [t^R, b]$ such that $F_X(t^*) > F_Y(t^*)$, where $t^L = \inf(\{t \leq r \mid F_X(t) \geq c^-,\ F_Y(t) \geq c^-\} \cup \{r\})$ and $t^R = \sup(\{t \geq r \mid F_X(t) \leq 1 - c^+,\ F_Y(t) \leq 1 - c^+\} \cup \{r\})$. Suppose first that $t^* \in [a, t^L)$. This implies that $t^* < r$ and $\min\{F_X(t^*), F_Y(t^*)\} < c^-$, which together with $F_X(t^*) > F_Y(t^*)$, gives $F_Y(t^*) < c^-$. Choose $l \in \{1, \ldots, n+1\}$ such that $t^* \in [\tilde{y}_{l-1}, \tilde{y}_l)$, then $F_Y(\tilde{y}_{l-1}) = F_Y(t^*) < c^-$. Moreover, since $t^* < r$, we have $\tilde{y}_{l-1} < r$, which together with the assumption that $r = \tilde{y}_i$ for some $i \in N$, implies that $\tilde{y}_l \leq r$. Thus, we obtain that $F_Y(\tilde{y}_{l-1}) < c^-$ and $\tilde{y}_l \leq r$, but evaluating the LHS of inequality (9) for index $i = l$ gives

$$\tilde{F}_X(\tilde{y}_l) = \sum_{\substack{k \text{ s.t.} \\ x_k < \tilde{y}_l}} p_k \geq \sum_{\substack{k \text{ s.t.} \\ x_k < t^*}} p_k = F_X(t^*) > F_Y(t^*) = F_Y(\tilde{y}_{l-1}),$$

which implies that inequality (9) is not satisfied.

Now suppose that $t^* \in [t^R, b]$. Then, $\max\{F_X(t^*), F_Y(t^*)\} > 1 - c^+$, which together with $F_X(t^*) > F_Y(t^*)$, implies $F_X(t^*) > 1 - c^+$. Moreover, $t^* \geq t^R \geq r$ implies $F_X(t^*) > F_Y(t^*) \geq F_Y(r)$. Choose $l \in \{1, \ldots, n+1\}$ such that $t^* \in [\tilde{y}_{l-1}, \tilde{y}_l)$. Then, $F_Y(\tilde{y}_{l-1}) = F_Y(t^*) < F_X(t^*)$. Together these inequalities imply that $F_X(t^*) > \max\{F_Y(\tilde{y}_{l-1}), F_Y(r), 1 - c^+\}$. Evaluating the LHS of inequality (10) for index $i = l$ gives

$$\tilde{F}_X(\tilde{y}_l) = \sum_{\substack{k \text{ s.t.} \\ x_k < \tilde{y}_l}} p_k \geq \sum_{\substack{k \text{ s.t.} \\ x_k < t^*}} p_k = F_X(t^*) > \max\{F_Y(\tilde{y}_{l-1}), F_Y(r), 1 - c^+\},$$

which implies that inequality (10) does not hold.

Next, we prove the 'only if' part by assuming that inequality (4) of Proposition 2 holds. To show that inequality (9) is satisfied we evaluate its LHS for an arbitrary $i \in \{1, \ldots, n+1\}$ such that $F_Y(\tilde{y}_{i-1}) < c^-$ and $\tilde{y}_i \leq r$ to obtain

$$\tilde{F}_X(\tilde{y}_i) = \sum_{\substack{k \text{ s.t.} \\ x_k < \tilde{y}_i}} p_k = F_X(x^*),$$

where $x^* = \max_k\{x_k | x_k < \tilde{y}_i\}$. By inequality (4) of Proposition 2, it holds that $F_X(t) \leq F_Y(t)\ \forall\ t \in [a, t^L)$, where $t^L = \inf(\{t \leq r \mid F_X(t) \geq c^-,\ F_Y(t) \geq c^-\} \cup \{r\})$. To confirm that $x^* \in [a, t^L)$, first suppose that $t^L = r$. Then, this implies that $t^L = r \geq \tilde{y}_i > x^*$ and thus, $x^* \in [a, t^L)$. Second, suppose that $t^L \neq r$, which implies $t^L = \min\{t \leq r \mid F_X(t) \geq c^-,\ F_Y(t) \geq c^-\}$. If $t^L < \tilde{y}_i$, this would imply that $F_Y(t^L) \leq F_Y(\tilde{y}_{i-1}) < c^-$, which contradicts the required condition that $F_Y(t^L) \geq c^-$. Hence, it must hold that $t^L \geq \tilde{y}_i > x^*$ and thus, $x^* \in [a, t^L)$. Evaluating inequality (4) of Proposition 2 at $t = x^*$ yields

$$F_X(x^*) \leq F_Y(x^*) \leq \sup_{t \in [x^*, \tilde{y}_i)} F_Y(t) \leq F_Y(y_{i-1}),$$

which is the right-hand side of inequality (9).

To show that inequality (10) is satisfied we evaluate its LHS for an arbitrary $i \in \{1, \ldots, n+1\}$ to obtain

$$\tilde{F}_X(\tilde{y}_i) = \sum_{\substack{k \text{ s.t.} \\ x_k < \tilde{y}_i}} p_k = F_X(x^*),$$

where $x^* = \max_k\{x_k | x_k < \tilde{y}_i\}$. Since inequality (4) of Proposition 2 holds, we have $F_X(t) \leq F_Y(t)\ \forall\ t \in [t^R, b]$, where $t^R = \sup(\{t \geq r \mid F_X(t) \leq 1 - c^+,\ F_Y(t) \leq 1 - c^+\} \cup \{r\})$. Suppose $x^* < t^R$. If $t^R > r$, then $\{t \geq r \mid F_X(t) \leq 1 - c^+,\ F_Y(t) \leq 1 - c^+\} \neq \emptyset$ and thus, there exists $x' \in (x^*, t^R]$ such that $F_X(x') \leq 1 - c^+$. Then,

$$F_X(x^*) \leq F_X(x') \leq 1 - c^+ \leq \max\{F_Y(\tilde{y}_{i-1}), F_Y(r), 1 - c^+\}.$$

In turn, if $t^R = r$, then (4) implies that $F_X(r) \leq F_Y(r)$, which gives

$$F_X(x^*) \leq F_X(t^R) = F_X(r) \leq F_Y(r) \leq \max\{F_Y(\tilde{y}_{i-1}), F_Y(r), 1 - c^+\}.$$

Finally, if $x^* \geq t^R$, then (4) implies that $F_X(x^*) \leq F_Y(x^*)$, which gives

$$F_X(x^*) \leq F_Y(x^*) \leq \sup_{t \in [x^*, \tilde{y}_i)} F_Y(t) \leq F_Y(\tilde{y}_{i-1}) \leq \max\{F_Y(\tilde{y}_{i-1}), F_Y(r), 1 - c^+\}. \quad \square$$

Proof of Lemma 1. We first show that the value of the right-hand side (RHS) of condition (7) in Theorem 1 is equal to the optimal objective function value of a MILP problem. Specifically, for any $y_i \in [r, b]$, the RHS of condition (7) in Theorem 1 is equal to

$$\begin{aligned} F^2_Y(y_i) - F^2_Y(r) \geq F^2_X(y_i) - F^2_X(r) &= \sum_{x_k \leq y_i} p_k\big(y_i - x_k\big) - \sum_{x_k \leq r} p_k\big(r - x_k\big) \\ &= \sum_{k=1}^n p_k \max\big\{y_i - x_k,\ 0\big\} \\ &\quad - \sum_{k=1}^n p_k \max\big\{r - x_k,\ 0\big\} \\ &= \sum_{k=1}^n p_k\ \Big(\max\big\{y_i - x_k,\ 0\big\} \\ &\quad - \max\big\{r - x_k,\ 0\big\}\Big). \end{aligned} \tag{A.4}$$

The value of the kth term in (A.4) falls into one of the three cases: (i) If $x_k \geq y_i \geq r$, then both max operators in (A.4) give zeros and hence, the value of the kth term in (A.4) equals zero; (ii) If $y_i \geq x_k \geq r$, then the first and second max operators yield $y_i - x_k$ and 0, respectively, and therefore the value of the kth term in (A.4) equals $y_i - x_k$; (iii) If $y_i \geq r \geq x_k$, the value of the kth term in (A.4) is then equal to $y_i - r$. As a consequence, Eq. (A.4) reduces to the compact form

$$F_X^2(y_i) - F_X^2(r) = \sum_{k=1}^{n} p_k \left(\max\left\{ \min\left\{ y_i - x_k,\ y_i - r \right\},\ 0 \right\} \right),$$

which can then be modeled as a MILP problem by introducing non-negative continuous decision variables $h_{i1}^+, \dots, h_{in}^+$ as well as binary decision variables $z_1^+, \dots, z_n^+$. In particular, we formulate two constraints $h_{ik}^+ \geq y_i - x_k - M(1 - z_k^+)$ and $h_{ik}^+ \geq y_i - r - M z_k^+$. In each state k, only one of these two constraints binds while the other becomes redundant such that $z_k^+ = 1$ if x_k is greater than r or $z_k^+ = 0$ otherwise, for $k \in N$, where $N = \{1, \dots, n\}$. Specifically, for any $y_i \geq r$, the value of $F_X^2(y_i) - F_X^2(r)$ equals to the optimal objective function value of the MILP problem

$$\min_{\substack{(h_{i1}^+,\dots,h_{in}^+)\in\mathbb{R}_+^n \\ z^+\in\{0,1\}^n}} \sum_{k=1}^{n} p_k h_{ik}^+ \tag{A.5}$$

$$\text{s.t. } h_{ik}^+ \geq y_i - x_k - M(1 - z_k^+)\ \forall\ k \in N \tag{A.6}$$

$$h_{ik}^+ \geq y_i - r - M z_k^+\ \forall\ k \in N, \tag{A.7}$$

where M is a sufficiently large positive constant.

We now prove the 'if' part. Assume that (h^+, z^+) satisfies constraints (12)–(14). Then, (h^+, z^+) is a feasible solution to MILP problem (A.5)–(A.7). Thus, the value of the objective function (A.5) evaluated at (h^+, z^+) is greater than (or equal to) the value of $F_X^2(y_i) - F_X^2(r)$. Since (h^+, z^+) satisfies (14), then the value of the objective function (A.5) is less than (or equal to) the value of $F_Y^2(y_i) - F_Y^2(r)$. Together, for an arbitrary $i \in N^+$, where $N^+ = \{i \in N \mid y_i \geq r\}$, we obtain

$$F_Y^2(y_i) - F_Y^2(r) \geq F_X^2(y_i) - F_X^2(r).$$

Hence, condition (7) of Theorem 1 holds.

Finally, we prove the 'only if' part. Assume that condition (7) of Theorem 1 holds, i.e., $F_Y^2(y_i) - F_Y^2(r) \geq F_X^2(y_i) - F_X^2(r)$ for all $y_i \geq r$. Take any $y_i \geq r$ and let (h^{+*}, z^{+*}) be the optimal solution to MILP problem (A.5)–(A.7). Then, the solution clearly satisfies constraints (12)–(13), as they are identical to constraints (A.6)–(A.7) and the value of the objective function (A.5) is

$$\sum_{k=1}^{n} p_k h_{ik}^{+*} = F_X^2(y_i) - F_X^2(r) \leq F_Y^2(y_i) - F_Y^2(r),$$

which implies that constraint (14) is satisfied. □

Proof of Lemma 2. We first show that the value of the right-hand side (RHS) of condition (8) in Theorem 1 is equal to the optimal objective function value of a MILP problem. Specifically, for any $x_i \in [a, r]$, the RHS of condition (8) of Theorem 1 equals, after rearrangement,

$$\begin{aligned} F_Y^2(r) &\geq F_X^2(r) + F_Y^2(x_i) - F_X^2(x_i) \\ &= \sum_{x_k \leq r} p_k (r - x_k) + \sum_{y_k \leq x_i} p_k (x_i - y_k) - \sum_{x_k \leq x_i} p_k (x_i - x_k) \\ &= \sum_{k=1}^{n} p_k \max\{r - x_k,\ 0\} + \sum_{k=1}^{n} p_k \max\{x_i - y_k,\ 0\} \\ &\quad - \sum_{k=1}^{n} p_k \max\{x_i - x_k,\ 0\}. \end{aligned} \tag{A.8}$$

Obviously, the value of the kth term in (A.8) depends on the joint outputs from each of the three max operators. Therefore, in order to determine its outcome, Eq. (A.8) can be modeled as a MILP problem by introducing non-negative continuous decision variables $d_1^-, \dots, d_n^-$, $g_{i1}^-, \dots, g_{in}^-$, $h_{i1}^-, \dots, h_{in}^-$, and binary decision variables $z_{i1}^-, \dots, z_{in}^-$. In particular, for each decision variable d_k^-, we construct the constraint $d_k^- \geq r - x_k$ such that $d_k^- > 0$ if r is greater than x_k or $d_k^- = 0$ otherwise, for $k \in N$, where $N = \{1, \dots, n\}$. Then, for each decision variable g_{ik}^-, we establish the constraint $g_{ik}^- \geq x_i - y_k$ such that g_{ik}^- is strictly positive if x_i is larger than y_k or otherwise g_{ik}^- gets zero, for $k \in N$. Finally, for each decision variable h_{ik}^-, two constraints $h_{ik}^- \leq x_i - x_k + M(1 - z_{ik}^-)$ and $h_{ik}^- \leq M z_{ik}^-$ alternate to bind each state k such that $h_{ik}^- > 0$ and $z_{ik}^- = 1$ if x_i is greater than x_k or $h_{ik}^- = 0$ and $z_{ik}^- = 0$ otherwise, for $k \in N$. Specifically, for any $x_i < r$, the value of $F_X^2(r) + F_Y^2(x_i) - F_X^2(x_i)$ is equal to the optimal objective function value of the MILP problem

$$\min_{\substack{d^-\in\mathbb{R}_+^n,\ (g_{i1}^-,\dots,g_{in}^-)\in\mathbb{R}_+^n \\ (h_{i1}^-,\dots,h_{in}^-)\in\mathbb{R}_+^n,\ (z_{i1}^-,\dots,z_{in}^-)\in\{0,1\}^n}} \sum_{k=1}^{n} p_k d_k^- + \sum_{k=1}^{n} p_k g_{ik}^- - \sum_{k=1}^{n} p_k h_{ik}^- \tag{A.9}$$

$$\text{s.t. } d_k^- \geq r - x_k\ \forall\ k \in N \tag{A.10}$$

$$g_{ik}^- \geq x_i - y_k\ \forall\ k \in N \tag{A.11}$$

$$h_{ik}^- \leq x_i - x_k + M(1 - z_{ik}^-)\ \forall\ k \in N \tag{A.12}$$

$$h_{ik}^- \leq M z_{ik}^-\ \forall\ k \in N, \tag{A.13}$$

where M is a sufficiently large positive constant.

We now prove the 'if' part. Assume that (d^-, g^-, h^-, z^-) satisfies constraints (15)–(19). Then, (d^-, g^-, h^-, z^-) is a feasible solution to MILP problem (A.9)–(A.13). Thus, the value of the objective function (A.9) evaluated at (d^-, g^-, h^-, z^-) is no less than the value of $F_X^2(r) + F_Y^2(x_i) - F_X^2(x_i)$. Since (d^-, g^-, h^-, z^-) satisfies (19), then the value of the objective function (A.9) is no more than $F_Y^2(r)$. For an arbitrary $i \in \{i \in N \mid x_i < r\}$, combining these results yields then

$$F_Y^2(r) \geq F_X^2(r) + F_Y^2(x_i) - F_X^2(x_i) \Leftrightarrow F_Y^2(r) - F_Y^2(x_i) \geq F_X^2(r) - F_X^2(x_i).$$

Therefore, condition (8) of Theorem 1 holds.

Finally, we prove the 'only if' part. Assume that condition (8) of Theorem 1 holds, i.e., $F_Y^2(r) - F_Y^2(x_i) \geq F_X^2(r) - F_X^2(x_i)$ for all $x_i < r$. Take any $x_i < r$ and let $(d^{-*}, g^{-*}, h^{-*}, z^{-*})$ be the optimal solution to MILP problem (A.9)–(A.13). Then, the solution clearly satisfies constraints (15)–(18), as they are identical to constraints (A.10)–(A.13) and the value of the objective function (A.9) is

$$\begin{aligned} \sum_{k=1}^{n} p_k d_k^{-*} + \sum_{k=1}^{n} p_k g_{ik}^{-*} - \sum_{k=1}^{n} p_k h_{ik}^{-*} &= F_X^2(r) + F_Y^2(x_i) - F_X^2(x_i) \leq F_Y^2(r) \\ &\Leftrightarrow F_X^2(r) - F_X^2(x_i) \leq F_Y^2(r) - F_Y^2(x_i), \end{aligned}$$

which implies that constraint (19) is satisfied. □

Proof of Theorem 3. Consider any prospects X and Y. Theorem 1 together with Lemmas 1 and 2 imply that $X \succeq Y$ if and only if there exist $h^+ \in \mathbb{R}_+^{n^+ \times n}$, $z^+ \in \{0,1\}^n$, $d^- \in \mathbb{R}_+^n$, $g^- \in \mathbb{R}_+^{n \times n}$, $h^- \in \mathbb{R}_+^{n \times n}$, and $z^- \in \{0,1\}^{n \times n}$ that satisfy constraints (12)–(14), (15)–(18), and constraint (19), i.e.,

$$\sum_{k=1}^{n} p_k d_k^- + \sum_{k=1}^{n} p_k g_{ik}^- - \sum_{k=1}^{n} p_k h_{ik}^- \leq F_Y^2(r)\ \forall\ i \in \{i \in N \mid x_i < r\}.$$

Since this constraint needs to hold for losses only, we modify it by adding an extra term to obtain

$$\sum_{k=1}^{n} p_k d_k^- + \sum_{k=1}^{n} p_k g_{ik}^- - \sum_{k=1}^{n} p_k h_{ik}^- - M(1 - \zeta_i^-) \leq F_Y^2(r)\ \forall\ i \in N, \tag{A.14}$$

where the new binary variables $\zeta_1^-, \dots, \zeta_n^-$ indicate which of the state-specific outcomes $x_1, \dots, x_n$ are losses (i.e., below the reference outcome r). This can be implemented by introducing the additional constraints

$$r - x_i \leq M \zeta_i^-\ \forall\ i \in \{1, \dots, n\}, \tag{A.15}$$

which ensure that $\zeta_i = 1$ if $x_i < r$. The set of constraints (12)–(14), (15)–(18), (A.14) and (A.15) is equivalent to constraints (21)–(29). This equivalence directly implies that statement (i) holds. Together with the fact that for any $x \in \mathcal{X}$, there exists $X' \in \mathbb{X}$ such that $(X'(s_1), \dots, X'(s_n)) = (x_1, \dots, x_n)$ (see (20)), the equivalence implies that statement (ii) holds. □

Proof of Corollary 1. We first prove the 'if' part. If $x = (x_1, \dots, x_n) = (X(s_1), \dots, X(s_n))$ is an optimal solution to optimization problem (30), then random variable X by Theorem 3 is a feasible solution to $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \geq Y\}$. Assume now by contradiction that X is not optimal to $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \geq Y\}$. Then, there must exist another feasible solution X' that yields a higher objective function value such that $\mathbb{E}\left[X'\right] > \mathbb{E}\left[X\right]$. Then, by Theorem 3, $x' = (x'_1, \dots, x'_n) = (X'(s_1), \dots, X'(s_n))$ is a feasible solution to (30), and moreover, it yields the objective function value $\sum_{i=1}^n p_i x'_i = \mathbb{E}[X'] > \mathbb{E}[X] = \sum_{i=1}^n p_i x_i$, which is contradictory to the assumption that x is the optimal solution to (30). Hence, X is an optimal solution to $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \geq Y\}$.

Then, we prove the 'only if' part. Assume that random variable X is an optimal solution to $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \geq Y\}$. Then, by Theorem 3, $x = (x_1, \dots, x_n) = (X(s_1), \dots, X(s_n))$ is a feasible solution to optimization problem (30). Now assume by contradiction that x is not optimal to (30), then there must exist another feasible solution x' such that $\sum_{i=1}^n p_i x'_i > \sum_{i=1}^n p_i x_i$. Then, by Theorem 3, random variable X' such that $(X'(s_1), \dots, X'(s_n)) = (x'_1, \dots, x'_n)$, is also a feasible solution to $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \geq Y\}$ and $\mathbb{E}[X'] = \sum_{i=1}^n p_i x'_i > \sum_{i=1}^n p_i x_i = \mathbb{E}[X]$, which contradicts to the assumption that X is the optimal solution to $\max_{X\in\mathbb{X}}\{\mathbb{E}[X] \mid X \geq Y\}$. Thus, x is an optimal solution to (30). □

Proof of Lemma 3. Based on Theorem 2 it is sufficient to prove that its inequalities (9) and (10) hold if and only if there exists $z \in \{0,1\}^{n\times n}$ that satisfies constraints (31)–(33). Since we have assumed that $r \in \{y_1, \dots, y_n\}$ and $y_1 \leq y_2 \leq, \dots, \leq y_{n-1} \leq y_n$, we have $\tilde{y} = y$, which implies that the right-hand sides of constraints (32) and (33) are equal to those of inequalities (9) and (10), respectively.

First, assume (9) and (10) hold. Then, construct $z \in \{0,1\}^{n\times n}$ such that

$$z_{i,k} = \begin{cases} 1, & \text{if } x_k < y_i \\ 0, & \text{otherwise} \end{cases} \quad \forall\, i,k \in N.$$

Clearly, z satisfies constraint (31). Moreover, $\sum_{k=1}^n p_k z_{ik} = \sum_{x_k<y_i} p_k = \tilde{F}_X(y_i)$ for all $i \in N$. This implies that the left-hand sides of (32) and (33) are equal to those of constraints (9) and (10). Thus, z satisfies constraints (32) and (33).

Second, assume that there exists $z \in \{0,1\}^{n\times n}$ that satisfies constraints (31)–(33). Satisfying constraint (31) requires that $z_{ik} = 1$ for each $i,k \in N$ such that $x_k < y_i$. This implies that for each $i \in N$, $\sum_{k=1}^n p_k z_{ik} \geq \sum_{x_k<y_i} p_k = \tilde{F}_X(y_i)$ and therefore the left-hand sides of (32) and (33) are greater than those of inequalities (9) and (10). Thus, inequalities (9) and (10) are satisfied. □

Proof of Theorem 4. (i) Assume there exists $X \in \mathbb{X}$ such that $X \succeq_{c^-}^{c^+} Y$. Lemma 3 implies that there exists $z \in \{0,1\}^{n\times n}$ satisfying constraints (31)–(33) and $X \geq Y$. Based on statement (i) of Theorem 3, $X \geq Y$ implies that there exist $h^+ \in \mathbb{R}_+^{n^+\times n}$, $z^+ \in \{0,1\}^n$, $d^- \in \mathbb{R}_+^n$, $g^- \in \mathbb{R}_+^{n\times n}$, $h^- \in \mathbb{R}_+^{n\times n}$, $z^- \in \{0,1\}^{n\times n}$, and $\zeta^- \in \{0,1\}^n$ that satisfy constraints (21)–(29). Together these imply that constraints (21)–(29) and (31)–(33) are satisfied.

(ii) Assume there exist $x \in \mathcal{X}$, $h^+ \in \mathbb{R}_+^{n^+\times n}$, $z^+ \in \{0,1\}^n$, $d^- \in \mathbb{R}_+^n$, $g^- \in \mathbb{R}_+^{n\times n}$, $h^- \in \mathbb{R}_+^{n\times n}$, $z^- \in \{0,1\}^{n\times n}$, $\zeta^- \in \{0,1\}^n$, and $z \in \{0,1\}^{n\times n}$ that satisfy constraints (21)–(29) and (31)–(33). Based on statement (ii) of Theorem 3, there exists $X' \in \mathbb{X}$ such that $X' \geq Y$ and $(X'(s_1), \dots, X'(s_n)) = (x_1, \dots, x_n)$. Lemma 3 then implies that $X' \succeq_{c^-}^{c^+} Y$. □

Proof of Corollary 2. The proof of Corollary 2 is omitted for brevity as it is similar to that of Corollary 1. □

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.ejor.2025.08.013.

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