The massive one-loop four-point string amplitude in pure spinor superspace

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The open- and closed-string three- and four-point one-loop amplitudes involving massless states and one first-level massive state are computed in pure spinor superspace. For the open string, we show that their one-loop correlators can be rewritten in terms of tree-level kinematic factors. We then analyze the closed string. For three points, this is immediate. For four points, we show that it is possible to rewrite the one-loop closed-string correlator using tree-level kinematic factors, but only for certain combinations of massive and massless states (different for type IIA and IIB).

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1. Introduction

The main goal of this paper is to calculate the open-string one-loop correlators of the three- and four-point string amplitudes with a single massive state of $(\text{mass})^2 = 1/\alpha'$ using the pure spinor formalism [1,2]. In order to do this, we will employ the same BRST-cohomology techniques that were used in [3,4,5] to obtain the string one-loop amplitudes up to seven massless external legs.

The answers in pure spinor superspace [6] can be expressed in different ways, depending on which feature is emphasized among BRST invariance, single-valuedness and locality. For instance, the three- and four-point open-string massive correlators at one loop

$$\begin{aligned} \mathcal{K}_3 &= C_{\underline{1}|2,3}\,, \\ \mathcal{K}_4 &= s_{23} f^{(1)}_{23} C_{\underline{1}|23,4} + s_{24} f^{(1)}_{24} C_{\underline{1}|24,3} + s_{34} f^{(1)}_{34} C_{\underline{1}|34,2}\,, \end{aligned} \tag{1.1}$$

manifest BRST invariance and single-valuedness, where the massive state is labelled by $\underline{1}$ while the other labels $2, 3, 4$ represent the massless super-Yang-Mills states. In the above,

$C_{\underline{1}|\ldots}$ are massive BRST invariants defined below and $f^{(1)}_{ij}$ denote single-valued worldsheet functions on a genus-one surface [7,4].

A BRST cohomology proof that the three-point massive one-loop *open-string* correlator is proportional to the open-string kinematic factor $\mathcal{K}^{\text{tree}}_{\underline{1}|2,3}$ at *tree level* is presented in the Appendix A. Using the explicit polarizations and momenta extracted from its pure spinor superspace expression[1] [9,10], the four-point *open-string* correlator at one loop is also shown to be rewritten in terms of open-string kinematic factor $\mathcal{K}^{\text{tree}}_{\underline{1}|2,3,4}$ at tree level, for the full supermultiplet; this extends the earlier RNS analysis of [11].

We also investigate whether the *closed string* massive one-loop correlators can be rewritten in terms of *tree-level* kinematic factors. For three points, this can be clearly done. For four points, a vectorial contraction between left- and right-movers present at one-loop and absent at tree-level has the potential to prevent this rewriting. Rather surprisingly, it turns out that for certain combinations of massive and massless states the one-loop correlator can be rewritten in terms of its tree-level counterpart. The details are in section 5.1. Finally, Appendix B reviews the mapping between $SO(n)$ Dynkin labels and Young diagrams [12], their tensorial description as well as their symmetries.

Conventions. We use $s_{ij} = (k_i \cdot k_j)$ as a shorthand; these are not proportional to the usual Mandelstam variables when either i or j represents a massive leg. In addition, the pure spinor bracket $\langle . \rangle$ [1] that extracts the component expansions from the pure spinor superspace expressions is frequently omitted throughout. Its presence is based on context; sometimes we emphasize the BRST variations of expressions, sometimes we extract their components. And finally, the word *massive* in this paper refers to a single massive state of the first mass level.

2. Review

In this section we will briefly review the discussion of section 2.1 in [3], and refer the reader to it for any missing details.

[1] Available to download in [8].

2.1. The pure spinor amplitude prescription

The one-loop amplitude prescription in the minimal pure spinor formalism is [2],

$$\mathcal{A}_n = \sum_{\rm top} C_{\rm top} \int_{D_{\rm top}} d\tau\, \langle (\mu, b)\, \mathcal{Z}\, V_1(z_1) \prod_{j=2}^{n} \int dz_j\, U_j(z_j) \rangle\,, \tag{2.1}$$

where the Beltrami differential μ and the modulus τ encode the topological information of the genus-one surface. The sum over the different genus-one topologies (planar cylinder, Möbius strip, and non-planar cylinder) implies different integration domains $D_{\rm top}$ and color factors $C_{\rm top}$ for each, but in this paper we will not focus on this aspect (see [13] for details). Rather, we concentrate on the CFT aspect of evaluating the correlation function of the picture-changing operators $\mathcal{Z}$, the b ghost, and the vertex operators. More specifically, after integrating out all the non-zero modes as well as the zero modes in (2.1), we obtain an expression of the form

$$\mathcal{A}_n = \sum_{\rm top} C_{\rm top} \int_{D_{\rm top}} d\tau\, dz_2\, dz_3\, \ldots\, dz_n \int d^D\ell\, |\mathcal{I}_n(\ell)|\, \langle \mathcal{K}_n(\ell) \rangle\,, \tag{2.2}$$

where $\mathcal{K}_n(\ell)$ is called the *correlator*, and $|\mathcal{I}_n(\ell)|$ denotes the Koba-Nielsen factor arising from the plane waves of the external vertices whose explicit form is not relevant for the purpose of this paper but can be looked at in [3,4,5]. In addition, $\ell^m = \oint_A dz \Pi^m(z)$ is the *loop momentum* of the chiral splitting formalism [14,15,16] and A denotes the A-cycle of the genus-one surface under consideration.

Vertex operators. For n-point amplitudes involving one first-level massive state and $(n-1)$ massless states, we place the massive leg $\underline{1}$ in the unintegrated massive vertex and use [17]

$$V_{\underline{1}} = [\lambda^\alpha [\partial\theta^\beta B^1_{\alpha\beta}]_0]_0 + [\lambda^\alpha [\Pi^m H^m_{1\,\alpha}]_0]_0 + 2\alpha' [\lambda^\alpha [d_\beta C^{\beta}_{1\;\alpha}]_0]_0 + \alpha' [\lambda^\alpha [N^{mn} F^1_{\alpha mn}]_0]_0\,, \tag{2.3}$$

where the superfields $B_{\alpha\beta}$, H^m_α, $C^\alpha{}_\beta$, $F_{\alpha mn}$ and $G^{mn} = -\frac{1}{144}\big[(D\gamma^m H^n) + (D\gamma^n H^m)\big]$ describe the open-string massive supermultiplet at the first mass level. This is composed of the symmetric traceless g_{mn} and the totally antisymmetric b_{mnp} bosonic fields and a fermionic field ψ^m_α comprising $128+128$ degrees of freedom. They are subject to the transversality constraint $k^m g_{mn} = k^m b_{mnp} = k_m \psi^m_\alpha = 0$ [17]. For the remaining massless integrated vertices $U_i(z_i)$ we use [1],

$$U_i(z_i) = [\partial\theta^\alpha A^i_\alpha]_0 + [\Pi^m A^i_m]_0 + 2\alpha' [d_\alpha W^\alpha_i]_0 + \alpha' [N^{mn} F^i_{mn}]_0\,, \tag{2.4}$$

where A_α, A^m, W^α and F^{mn} are the super-Yang-Mills superfields of [18] (for a review, see [19]). The normal-ordering bracket $[\ldots]_0$ is reviewed in [20].

Zero-mode integrations. We refer the reader to the discussion in section 2.1 of [3] for more details; the summary is that the zero-mode saturation of the pure spinor variables imply two different contributions from the external vertices: terms proportional to $d_\alpha d_\beta N^{mn}$ and terms proportional to $d_\alpha d_\beta d_\gamma d_\delta$. The resulting contribution of the zero-mode integration can be determined by a group-theory analysis [3],

$$\int d_\alpha d_\beta N^{mn} \to (\lambda\gamma^{[m})_\alpha(\lambda\gamma^{n]})_\beta \,, \tag{2.5}$$

$$\int d_\alpha d_\beta d_\gamma d_\delta \to \ell_m(\lambda\gamma^a)_{[\alpha}(\lambda\gamma^b)_\beta(\gamma^{abm})_{\gamma\delta]} \,, \tag{2.6}$$

where ℓ_m represents the loop momentum and the integral sign represents the integration over the different contributions from the b ghost and picture-changing operators using the zero-mode measures of [2].

Worldsheet functions. Two families of worldsheet functions $f^{(n)}(z,\tau)$ and $g^{(n)}(z,\tau)$ indexed by the integer n appearing in one-loop amplitudes are discussed at length in [4]. In this paper, however, only two such functions make an appearance: the meromorphic $g^{(1)}(z,\tau) = \partial\log\theta_1(z,\tau)$ which captures the simple-pole OPE singularity of a bc system of conformal weights $(1,0)$, where $\theta_1(z,\tau)$ is the odd Jacobi theta function and the doubly periodic but non-holomorphic $f^{(1)}(z,\tau) = g^{(1)}(z,\tau) + 2\pi i(\operatorname{Im} z/\operatorname{Im}\tau)$, with modular weight one.

Equations of motion and BRST charge. The (linearized) massless $[A_\alpha, A^m, W^\alpha, F^{mn}]$ and massive $[G^{mn}, B_{\alpha\beta}, H^m_\alpha, C^\alpha{}_\beta, F_{\alpha mn}]$ superfields satisfy the following equations of motion under the supersymmetric derivative $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}(\gamma^m\theta)_\alpha\partial_m$,

$$\begin{aligned} D_\alpha A_\beta + D_\beta A_\alpha &= (\gamma^m)_{\alpha\beta}A_m \,, & D_\alpha W^\beta &= \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta F_{mn}, \\ D_\alpha A_m &= (\gamma^m W)_\alpha + \partial_m A_\alpha \,, & D_\alpha F_{mn} &= \partial_m(\gamma_n W)_\alpha - \partial_n(\gamma_m W)_a \,, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} D_\alpha G^{mn} &= -\frac{1}{18}\partial_p(\gamma^{pm}H^n)_\alpha - \frac{1}{18}\partial_p(\gamma^{pn}H^m)_\alpha \,, \\ D_\alpha B_{mnp} &= -\frac{1}{18}(\gamma^{mn}H^p)_\alpha + \frac{\alpha'}{18}\partial_a\partial_m\Big((\gamma^{an}H^p)_\alpha - (\gamma^{ap}H^n)_\alpha\Big) + \text{cyc}(mnp) \,, \\ D_\alpha H^m_\beta &= -\frac{9}{2}G_{mn}\gamma^n_{\alpha\beta} - \frac{3}{2}\partial_a B_{bcm}\gamma^{abc}_{\alpha\beta} + \frac{1}{4}\partial_a B_{bcd}\gamma^{mabcd}_{\alpha\beta} \,, \end{aligned} \tag{2.8}$$

$$D_\alpha C^\gamma{}_\beta = -\frac{1}{24}(\gamma^{mnpq})^\gamma{}_\beta \partial_m(\gamma^{np}H^q)_\alpha\,,$$

$$\begin{aligned}D_\alpha F_{\beta mn} &= \frac{3}{8\alpha'}B_{amn}\gamma^a_{\alpha\beta} + \frac{1}{32\alpha'}B_{abc}\gamma^{mnabc}_{\alpha\beta}\\ &+ \frac{3}{64}\Big[15B_{amb}\partial_n\partial_c\gamma^{cba}_{\alpha\beta} - \frac{1}{\alpha'}B_{amb}\gamma^{nba}_{\alpha\beta} - 3\partial_m\partial_d B_{abc}\gamma^{dnbca}_{\alpha\beta}\\ &\quad - 6\partial_b G_{am}\gamma^{bna}_{\alpha\beta} + 42\partial_n G_{am}\gamma^a_{\alpha\beta} - (m\leftrightarrow n)\Big]\end{aligned}$$

where $B_{mnp} = \gamma^{\alpha\beta}_{mnp}B_{\alpha\beta}$. If we define

$$\lambda^\alpha B_{\alpha\beta} = (\lambda B)_\beta, \quad \lambda^\alpha H^m_\alpha = (\lambda H^m)\,, \quad C^\beta{}_\alpha\lambda^\alpha = (C\lambda)^\beta\,, \quad \lambda^\alpha F_{\alpha mn} = (\lambda F)_{mn}\,, \tag{2.9}$$

and use the BRST charge Q

$$Q = \lambda^\alpha D_\alpha \tag{2.10}$$

the equations of motion (2.8) simplify drastically. In this case we get

$$\begin{aligned}Q(\lambda B)_\alpha &= (\lambda\gamma^m)_\alpha(\lambda H)_m\,, & Q(C\lambda)^\alpha &= \frac{1}{4}(\lambda\gamma^{mn})^\alpha(\lambda F)_{mn} = \frac{1}{4}(\lambda\gamma^{mn})^\alpha\partial_m(\lambda H_n)\,,\\ Q(\lambda H^m) &= (\lambda\gamma^m C\lambda)\,, & Q(\lambda F)_{mn} &= \frac{1}{2}\partial_{[m}(\lambda\gamma_{n]}C\lambda) - \frac{1}{16}\partial^p(\lambda\gamma_{[m}C\gamma_{n]p}\lambda)\,,\end{aligned} \tag{2.11}$$

Alternatively, the equation of motion of $(\lambda F)_{mn}$ can also be written as

$$Q(\lambda F)_{mn} = -\frac{1}{32\alpha'}(\lambda\gamma^{mn})^\beta(\lambda B)_\beta + \frac{9}{64}\big[\partial_n\partial_p(\lambda\gamma^{mp})^\beta(\lambda B)_\beta - (m\leftrightarrow n)\big]\,. \tag{2.12}$$

One can also show that $\partial_m\gamma^m_{\alpha\beta}C^\beta{}_\gamma = \frac{1}{4\alpha'}B_{\alpha\gamma}$.

2.2. Tree-level amplitudes with one massive state

The open string n-point amplitudes at tree level with one first-level massive state and $n-1$ massless states were obtained in [21] in terms of Berends-Giele component currents. Their pure spinor superspace expressions were subsequently found in [22]. We briefly review the pure spinor results below.

2.2.1. Three points

At tree-level, the three-point open-string amplitude of one first-level massive state labelled by $\underline{1}$ and two massless states labelled $2,3$ is given in pure spinor superspace by [20,22,23]

$$\mathcal{K}^{\text{tree}}_{\underline{1}|2,3} = (\lambda H^m_1)V_2(\lambda\gamma^m W_3)\,, \tag{2.13}$$

where $V_2 = \lambda^\alpha A^2_\alpha$. Using the equations of motion (2.7) and (2.8) one can show that (2.13) is BRST closed. Computing its component expansion also shows that it is not BRST exact[2] and therefore it is in the cohomology of the BRST charge.

[2] It would be BRST exact if the momentum phase space is such that $(k_1+k_2)^2 \neq 0$ [24].

2.2.2. Four points

At tree-level, the four-point open-string amplitude of one first-level massive state labeled by $\underline{1}$ and three massless states labeled by $2,3,4$ can be written in terms of its kinematic factor in pure spinor superspace [22][3],

$$\mathcal{K}^{\text{tree}}_{\underline{1}|2,3,4} = (\lambda H^m_1) C^m_{2|34} \,. \tag{2.14}$$

In the above, $C^m_{2|34}$ denotes the BRST-closed combination [22]

$$C^m_{2|34} = M_{23}(\lambda\gamma^m \mathcal{W}_4) + M_2(\lambda\gamma^m \mathcal{W}_{34}) - M_{24}(\lambda\gamma^m \mathcal{W}_3) \,, \tag{2.15}$$

where M_P and $\mathcal{W}^\alpha_P$ are the multiparticle Berends-Giele currents of the multiparticle superfields V_P and W^α_P, see [19] for a review.

3. Massive multiparticle superfields

It is clear that the massless unintegrated vertex given by $V_1 = \lambda^\alpha A^1_\alpha(x,\theta)$ does not contribute any d_α or N^{mn} zero modes simply because these variables are absent in the vertex. However, this is no longer the case when the unintegrated vertex represents a *massive* string state. The consequence is that all vertices in $V_{\underline{1}} U_2 \dots U_n$ can possibly contribute those zero modes. Comparing the unintegrated massive vertex (2.3) with the massless integrated vertex (2.4), it is easy to see that (λH^m_1), $(C_1\lambda)^\alpha$ and $(\lambda F_1)_{mn}$ play an analogous role as the super-Yang-Mills superfields A^m_i, W^α_i and F^{mn}_i,

$$A^m \leftrightarrow (\lambda H^m), \quad W^\alpha \leftrightarrow (C\lambda)^\alpha, \quad F_{mn} \leftrightarrow (\lambda F)_{mn} \,. \tag{3.1}$$

We will use this observation in the construction of BRST-covariant objects below.

When the saturation of zero modes admits a prior OPE contraction among the vertices, their contribution is summarized by *multiparticle* superfields. They are defined as the coefficients of the remaining conformal weight-one variables in a suitable pole of the OPE. When the vertices involved in the OPE are massless, this is encoded in the massless multiparticle superfields of [25] (for a review see [19]). In the present case, the OPE may

[3] See [21] for the full string amplitude, including the Beta function.

involve the massive vertex $V_{\underline{1}}$. For example, a multiparticle superfield $(C_{12}\lambda)^\alpha$ can be read off from the coefficient of d_α in the OPE bracket $[V_{\underline{1}}U_2]_1$,

$$
\begin{aligned}
(C_{12}\lambda)^\alpha &= -(\lambda H_1^m)W_2^\alpha ik_2^m - (C_1\lambda)^\alpha(ik_1\cdot A_2) + \big(D_\beta(C_1\lambda)^\alpha\big)W_2^\beta \\
&\quad - \frac{1}{4}\big[(\gamma^{mn}C_1\lambda)^\alpha F_2^{mn} - (C_1\gamma^{mn}\lambda)^\alpha F_2^{mn}\big]\,.
\end{aligned}
\tag{3.2}
$$

After contraction with $(\lambda\gamma^m)_\alpha$, its BRST variation can be shown to be

$$
Q(\lambda\gamma^m C_{12}\lambda) = s_{12}V_2(\lambda\gamma^m C_1)\,, \tag{3.3}
$$

which has the desired properties for our purposes, see below. Unfortunately, the definition of the multiparticle superfield $(\lambda F_{12})_{mn}$ is not so straightforward as reading off the coefficient of N^{mn} in the OPE $[V_{\underline{1}}U_2]_1$. The constraint identity $[N^{mn}\lambda^\beta]_0\gamma^m_{\beta\gamma} = \frac{1}{2}[J\lambda^\beta]_0\gamma^n_{\beta\gamma} + 2(\gamma^n\partial\lambda)_\gamma$ leads to a non-unique definition of the coefficient of N^{nm}. Luckily, BRST covariance can be used to the rescue and we define $(\lambda F_{12})_{mn}$ such that

$$
Q\big[(\lambda F_{12})_{mn}(\lambda\gamma^m W_3)(\lambda\gamma^n W_4)\big] = s_{12}V_2(\lambda F_1)_{mn}(\lambda\gamma^m W_3)(\lambda\gamma^n W_4)\,. \tag{3.4}
$$

A solution to (3.4) is given by

$$
(\lambda F_{12})_{mn} = -\big(D_\alpha(\lambda F_1)_{mn}\big)W_2^\alpha - (\lambda F_1)_{mn}(ik_1\cdot A_2) - (\lambda\gamma^{pq}F_1)_{mn}F^2_{pq}\,. \tag{3.5}
$$

In doing the above calculations, it is necessary to use that there is no 4-form irrep in the decomposition of $\lambda^3 W$ [26] to conclude $(\lambda\gamma^{[m}W)(\lambda\gamma^{npqrs]}\lambda) = 0$. In particular

$$
\begin{aligned}
(\lambda\gamma^q W^4)B^1_{mnp}A^2_q(\lambda\gamma^{mnpk^1k^2}\lambda) &= -(\lambda\gamma^{k^1}W^4)B^1_{mnp}A^2_q(\lambda\gamma^{qmnpk^2}\lambda) \\
&\quad + (\lambda\gamma^{k^2}W^4)B^1_{mnp}A^2_q(\lambda\gamma^{qmnpk^1}\lambda) \\
&\quad + 3(\lambda\gamma^m W^4)B^1_{mnp}A^2_q(\lambda\gamma^{qnpk^1k^2}\lambda)
\end{aligned}
\tag{3.6}
$$

Scalar BRST blocks. Since the massless W^α and F^{mn} appear in schematic form as $(\lambda\gamma^m W)(\lambda\gamma^n W)F_{mn}$ as the result of the zero-mode integration (2.5), we can use the observation (3.1) to define massive superfield building blocks that are analogous to the massless $T_{A,B,C}$ in [3]

$$
\begin{aligned}
T_{\underline{1},2,3} &= -(\lambda\gamma^m C_1\lambda)(\lambda\gamma^n W_2)F_3^{mn} - (\lambda\gamma^m C_1\lambda)F_2^{mn}(\lambda\gamma^n W_3) \\
&\quad - (\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3) \\
T_{\underline{12},3,4} &= -(\lambda\gamma^m C_{12}\lambda)(\lambda\gamma^n W_3)F_4^{mn} - (\lambda\gamma^m C_{12}\lambda)F_3^{mn}(\lambda\gamma^n W_4) \\
&\quad - (\lambda F_{12})_{mn}(\lambda\gamma^m W_3)(\lambda\gamma^n W_4) \\
T_{\underline{1},23,4} &= -(\lambda\gamma^m C_1\lambda)(\lambda\gamma^n W_{23})F_4^{mn} - (\lambda\gamma^m C_1\lambda)F_{23}^{mn}(\lambda\gamma^n W_4) \\
&\quad - (\lambda F_1)_{mn}(\lambda\gamma^m W_{23})(\lambda\gamma^n W_4)\,.
\end{aligned}
\tag{3.7}
$$

Their BRST variations can be readily computed to be

$$
\begin{aligned}
QT_{\underline{1},2,3} &= 0 \\
QT_{\underline{12},3,4} &= s_{12}V_2T_{\underline{1},3,4} \\
QT_{\underline{1},23,4} &= s_{23}\big(V_3T_{\underline{1},2,4} - V_2T_{\underline{1},3,4}\big)\,.
\end{aligned}
\tag{3.8}
$$

The scalar BRST blocks $T_{\underline{1i},j,k}$ and $T_{\underline{1},ij,k}$ for different labels i,j,k are obtained from relabeling the expressions above.

Vectorial BRST block. Similarly, the zero-mode integration of four d_α given in (2.6) results in a vector constructed of three massless W^α and one massive $(C\lambda)^\alpha$. Inspired by the massless expression of $W^m_{A,B,C,D}$ given in [3] and the correspondence (3.1), we propose

$$
\begin{aligned}
W^m_{\underline{1},2,3,4} = &-\frac{5}{12}\,(\lambda\gamma^a C_1\lambda)(\lambda\gamma^b W^2)(W^3\gamma^{mab}W^4) \\
&-\frac{5}{12}\,(\lambda\gamma^a C_1\lambda)(\lambda\gamma^b W^4)(W^2\gamma^{mab}W^3) \\
&-\frac{5}{12}\,(\lambda\gamma^a C_1\lambda)(\lambda\gamma^b W^3)(W^4\gamma^{mab}W^2) \\
&-\frac{1}{4}\,(\lambda\gamma^a W^2)(\lambda\gamma^b W^3)(W^4\gamma^{mab}C_1\lambda) \\
&-\frac{1}{4}\,(\lambda\gamma^a W^4)(\lambda\gamma^b W^2)(W^3\gamma^{mab}C_1\lambda) \\
&-\frac{1}{4}\,(\lambda\gamma^a W^3)(\lambda\gamma^b W^4)(W^2\gamma^{mab}C_1\lambda)\,.
\end{aligned}
\tag{3.9}
$$

The relative coefficients in (3.9) are determined from the requirement of BRST covariance; it follows from the massive and massless superfield equations of motion that

$$
QW^m_{\underline{1},2,3,4} = -(\lambda\gamma^m C_1\lambda)T_{2,3,4} - \big[(\lambda\gamma^m W_2)T_{\underline{1},3,4} + (2\leftrightarrow 3,4)\big]\,. \tag{3.10}
$$

As discussed in [3], considerations of BRST covariance involving a mixture of left- and right-movers require a vectorial object whose BRST variation carries the vector index exclusively in momenta. Inspired by the massless case of [3], one is led to consider the following vectorial BRST block

$$
T^m_{\underline{1},2,3,4} = -(\lambda H^m_1)T_{2,3,4} - \big[A^m_2 T_{\underline{1},3,4} + 2\leftrightarrow 3,4\big] - W^m_{\underline{1},2,3,4} \tag{3.11}
$$

with BRST variation

$$
QT^m_{\underline{1},2,3,4} = -ik^m_2 V_2 T_{\underline{1},3,4} + (2\leftrightarrow 3,4)\,. \tag{3.12}
$$

Berends-Giele. For convenience, we introduce the Berends-Giele non-local counterparts of the above BRST blocks as

$$M_{\underline{1}2,3,4} = \frac{1}{s_{12}} T_{\underline{1}2,3,4}, \quad M_{\underline{1},23,4} = \frac{1}{s_{23}} T_{\underline{1},23,4}, \quad M_{\underline{1}3,2,4} = \frac{1}{s_{13}} T_{\underline{1}3,2,4}, \tag{3.13}$$
$$M^m_{\underline{1},2,3,4} = T^m_{\underline{1},2,3,4}\,, \quad M_{\underline{1},2,3} = T_{\underline{1},2,3}$$

which satisfy

$$QM_{\underline{1}2,3,4} = M_2 M_{\underline{1},3,4}, \; QM_{\underline{1}3,2,4} = M_3 M_{\underline{1},2,4}, \; QM_{\underline{1},23,4} = M_3 M_{\underline{1},2,4} - M_2 M_{\underline{1},3,4}, \tag{3.14}$$
$$QM^m_{\underline{1},2,3,4} = -ik_2^m M_2 M_{\underline{1},3,4} + (2 \leftrightarrow 3,4), \quad QM_{\underline{1},2,3} = 0\,.$$

4. The massive 1-loop amplitude

In this section we will use the multiparticle massive superfields defined in section 3 to construct a single-valued and BRST-invariant expression for the four-point open string amplitude at one loop involving one first-level massive state. As a warm-up we will derive, for the first time, the three-point massive amplitude at one loop.

4.1. Three points

Using the pure spinor prescription (2.1) with a massive $V_{\underline{1}}$ yields a unique saturation of the zero modes $d_\alpha d_\beta N^{mn}$ as there is only three external vertices. More precisely,

$$\Big[V_{\underline{1}} U_2 U_3\Big]_{ddN} = d_\alpha d_\beta N^{mn}\Big((C_1\lambda)^\alpha W_2^\beta F_3^{mn} + (C_1\lambda)^\alpha F_2^{mn} W_3^\beta + (\lambda F_1)_{mn} W_2^\alpha W_3^\beta\Big). \tag{4.1}$$

In particular, there are not enough external vertices to produce a contribution with the loop momentum, so the three-point correlator defined in (2.2) does not depend on ℓ^m. Integrating out $d_\alpha d_\beta N^{mn}$ using (2.5) yields

$$\mathcal{K}_3 = -(\lambda\gamma^m C_1\lambda)(\lambda\gamma^n W_2)F_3^{mn} - (\lambda\gamma^m C_1\lambda)F_2^{mn}(\lambda\gamma^n W_3) - (\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3). \tag{4.2}$$

As will be demonstrated in (A.6), this becomes

$$\mathcal{K}_3 = (\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\,. \tag{4.3}$$

The component expansion of (4.3) is straightforward to calculate [9,10] using the pure spinor bracket $\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)\rangle = 1$ [1] and the theta expansions of the various superfields [27,19,28,22]. It yields

$$\langle\mathcal{K}_3\rangle = \frac{1}{640\alpha'}\Big(g_{1\,mn} f_2^{ma} f_3^{na} - \frac{1}{2\alpha'} b_{1\,mnp} f_2^{mn} e_3^p\Big) + \text{ fermions} \tag{4.4}$$

where $g_{1\,mn}$ and $b_{1\,mnp}$ are the bosonic massive polarizations [17], and $f_j^{mn} = ik_j^m e_j^n - ik_j^n e_j^m$ are the linearized field strengths and e_i^m are the gluons.

In the appendix A, we will use BRST cohomology manipulations similar to the ones used in [29] to show the relation between the one-loop correlator (4.3) and tree-level amplitude (2.13):

$$\langle (\lambda F_1)_{mn} (\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\rangle = \frac{1}{2\alpha'}\langle (\lambda H_1^m) V_2 (\lambda\gamma^m W_3)\rangle = \frac{1}{2\alpha'} A(\underline{1}, 2, 3)\,, \tag{4.5}$$

where $1/2\alpha' = -(k_2 \cdot k_3)$. For convenience, we define the BRST invariant

$$C_{\underline{1}|2,3} = T_{\underline{1},2,3}\,, \tag{4.6}$$

and write the one-loop three-point correlator as $\mathcal{K}_3 = C_{\underline{1}|2,3}$.

4.2. Four points

The four-point amplitude with one massive state admits two types of zero-mode saturation: contributions with $d_\alpha d_\beta N^{mn}$ or $d_\alpha d_\beta d_\gamma d_\delta$. The first kind leads to OPE contractions among the vertices leading to the massive multiparticle superfields discussed in section 3 as well as $\ell_m (\lambda H_1^m) T_{2,3,4}$ or $\ell_m A_2^m T_{\underline{1},3,4} + (2 \leftrightarrow 3, 4)$. The zero-mode integration of the second kind of contribution leads to $\ell_m W^m_{\underline{1},2,3,4}$. These various contributions are organized according to a desired property of BRST covariance of the individual blocks of superfields resulting in an overall BRST-invariant correlator. As the massive four-point amplitude is analogous to the massless five-point amplitude, the five-point correlator of [5] leads to the following proposal,

$$\begin{aligned}\mathcal{K}_4(\ell) = \ell_m T^m_{\underline{1},2,3,4} &+ g^{(1)}_{12} T_{\underline{1}2,3,4} + g^{(1)}_{13} T_{\underline{1}3,2,4} + g^{(1)}_{14} T_{\underline{1}4,2,3} \\ &+ g^{(1)}_{23} T_{\underline{1},23,4} + g^{(1)}_{24} T_{\underline{1},24,3} + g^{(1)}_{34} T_{\underline{1},34,2}\,.\end{aligned} \tag{4.7}$$

Its BRST variation is a total worldsheet derivative,

$$Q\mathcal{K}_4(\ell) = V_2 T_{\underline{1},3,4}\Big(-(ik_2\cdot\ell) - s_{21} g^{(1)}_{21} - s_{23} g^{(1)}_{23} - s_{24} g^{(1)}_{24}\Big) + (2 \leftrightarrow 3, 4)\,. \tag{4.8}$$

To see this, one uses the worldsheet derivative of the Koba-Nielsen factor [4],

$$\frac{\partial}{\partial z_i}\mathcal{I}_n(\ell) = \big(\ell\cdot ik_i + \sum_{j\neq i}^{n} s_{ij} g^{(1)}_{ij}\big)\mathcal{I}_n(\ell) \tag{4.9}$$

is proportional to the right-hand side of (4.8). Therefore, once the integration over the worldsheet insertion points is carried out, the amplitude (2.2) is BRST invariant as the boundary terms vanish using the canceled propagator argument.

4.2.1. Manifesting BRST invariance

Similarly to the discussion of BRST invariance in the massless one-loop amplitudes of [5], one can use the integration by parts identities

$$
\begin{aligned}
s_{12}g^{(1)}_{12} &\sim (\ell.ik_2) + s_{23}g^{(1)}_{23} + s_{24}g^{(1)}_{24}\,,\\
s_{13}g^{(1)}_{13} &\sim (\ell.ik_3) - s_{23}g^{(1)}_{23} + s_{34}g^{(1)}_{34}\,,\\
s_{14}g^{(1)}_{14} &\sim (\ell.ik_4) - s_{24}g^{(1)}_{24} - s_{34}g^{(1)}_{34}\,,
\end{aligned}
\tag{4.10}
$$

to obtain the BRST-invariant correlator

$$
\mathcal{K}_4(\ell) \sim \ell_m C^m_{\underline{1}|2,3,4} + s_{23}g^{(1)}_{23}C_{\underline{1}|23,4} + s_{24}g^{(1)}_{24}C_{\underline{1}|24,3} + s_{34}g^{(1)}_{34}C_{\underline{1}|34,2}\,. \tag{4.11}
$$

In the above we defined the scalar BRST invariants

$$
C^m_{\underline{1}|2,3,4} = M^m_{\underline{1},2,3,4} + ik^m_2 M_{\underline{1}2,3,4} + ik^m_3 M_{\underline{1}3,2,4} + ik^m_4 M_{\underline{1}4,2,3} \tag{4.12}
$$

$$
C_{\underline{1}|23,4} = M_{\underline{1}2,3,4} + M_{\underline{1},23,4} - M_{\underline{1}3,2,4} \tag{4.13}
$$

$$
C_{\underline{1}|24,3} = M_{\underline{1}2,3,4} + M_{\underline{1},24,3} - M_{\underline{1}4,2,3} \tag{4.14}
$$

$$
C_{\underline{1}|34,2} = M_{\underline{1}3,2,4} + M_{\underline{1},34,2} - M_{\underline{1}4,2,3}, \tag{4.15}
$$

in terms of the Berends-Giele currents defined in (3.13). Using the BRST variations (3.14), one can show that the above combinations are BRST closed,

$$
QC^m_{\underline{1}|2,3,4} = QC_{\underline{1}|23,4} = QC_{\underline{1}|24,3} = QC_{\underline{1}|34,2} = 0. \tag{4.16}
$$

The above four-point BRST invariants with one massive state have the same structure as the five-point BRST invariants in the massless one-loop amplitudes of [3,5].

4.2.2. Manifesting single valuedness

It turns out that the massive BRST invariants (4.12)-(4.15) obey the analog momentum-contraction identities as their massless counterparts, i.e.,

$$
\begin{aligned}
ik^m_2\langle C^m_{\underline{1}|2,3,4}\rangle - s_{23}\langle C_{\underline{1}|23,4}\rangle - s_{24}\langle C_{\underline{1}|24,3}\rangle &= 0,\\
ik^m_1\langle C^m_{\underline{1}|2,3,4}\rangle &= 0
\end{aligned}
\tag{4.17}
$$

As discussed in [5], these identities are sufficient to show that the correlator (4.11) is single-valued and can be rewritten in a manifestly single-valued manner as

$$
\mathcal{K}_4 = s_{23}f^{(1)}_{23}C_{\underline{1}|23,4} + s_{24}f^{(1)}_{24}C_{\underline{1}|24,3} + s_{34}f^{(1)}_{34}C_{\underline{1}|34,2}\,, \tag{4.18}
$$

after integrating out the loop momentum.

4.2.3. Relation to the four-point tree amplitude

We know that the pure spinor superspace expressions of one-loop and tree-level correlators are proportional in the massive three-point (4.5) and massless four-point [29] cases. It is natural to ask what happens in the case of massive four-point correlators. Indeed, one can show directly via their component expansions that (note $s_{ij} = (k_i \cdot k_j)$)

$$\begin{aligned}
\langle C_{\underline{1}|23,4}\rangle &= s_{24}s_{34}\Big(\frac{1}{s_{12}} + \frac{1}{s_{13}}\Big)\mathcal{K}^{\rm tree}_{\underline{1}|2,3,4} \\
\langle C_{\underline{1}|24,3}\rangle &= -s_{23}s_{34}\Big(\frac{1}{s_{12}} + \frac{1}{s_{14}}\Big)\mathcal{K}^{\rm tree}_{\underline{1}|2,3,4} \\
\langle C_{\underline{1}|34,2}\rangle &= s_{23}s_{24}\Big(\frac{1}{s_{13}} + \frac{1}{s_{14}}\Big)\mathcal{K}^{\rm tree}_{\underline{1}|2,3,4}
\end{aligned} \tag{4.19}$$

where the tree-level kinematic factor is given by (2.14) and the one-loop BRST invariant is given by (4.13). Note that (4.19) has been checked to hold for both bosonic massive states in the first mass level (g_{mn} and b_{mnp}). By supersymmetry, also the fermionic components will match on both sides. Similar identities were found with the RNS formalism in [11], but the analysis was restricted to the bosonic state g_{mn} (where they agree).

The above calculations use the convention $s_{ij} = k_i \cdot k_j$, and therefore

$$\begin{aligned}
s_{13} &= \frac{1}{2\alpha'} - s_{12} - s_{23}, \qquad & s_{24} &= -\frac{1}{2\alpha'} + s_{13} = -s_{12} - s_{23}, \\
s_{14} &= \frac{1}{2\alpha'} + s_{23} & s_{34} &= -\frac{1}{2\alpha'} + s_{12}
\end{aligned} \tag{4.20}$$

where $k_1^2 = -1/\alpha'$ and $k_2^2 = k_3^2 = k_4^2 = 0$. In addition, the relations

$$\begin{aligned}
\frac{s_{12}}{s_{13}} &= \frac{1}{2\alpha' s_{13}} - \frac{s_{23}}{s_{13}} - 1 \qquad & \frac{s_{12}}{s_{24}} &= -1 - \frac{s_{23}}{s_{24}} \qquad & \frac{s_{23}}{s_{13}s_{34}} &= -\frac{1}{s_{23}} - \frac{1}{s_{34}} \\
\frac{s_{23}}{s_{14}} &= 1 - \frac{1}{2\alpha' s_{14}} & \frac{s_{12}}{s_{34}} &= \frac{1}{2\alpha' s_{34}} + 1 & \frac{s_{23}}{s_{12}s_{24}} &= -\frac{1}{s_{12}} - \frac{1}{s_{24}}
\end{aligned} \tag{4.21}$$

follow from (4.20) and are useful in checking (4.19).

5. Closed strings

We have seen in (4.3) and (4.18) that the *open-string* one-loop three- and four-point correlators with one massive state can be written in terms of tree-level kinematic factors. This section is motivated by the desire to know to which extent the *closed-string* four-point[4] correlator admits a rewriting in terms of tree-level kinematic data, if at all or for which combination of massive and massless closed-string states.

[4] From (4.3) it immediately follows that the holomorphic square of the open-string three-point correlator can be written in terms of tree-level amplitudes.

5.1. The four-point correlator

According to the chiral splitting formalism [14,15,16], the correlator M_4 of the closed string amplitude is obtained after integrating the loop momentum in the holomorphic square of the open-string correlator (4.11). Using the techniques described in [5] this leads to the four-point closed-string correlator M_4

$$M_4 = |s_{23} f^{(1)}_{23} C_{\underline{1}|23,4} + (2,3|2,3,4)|^2 + \frac{\pi}{\operatorname{Im}\tau} C^m_{\underline{1},2,3,4} \tilde{C}^m_{\underline{1},2,3,4} \,. \tag{5.1}$$

The four-point closed string amplitude at genus-one is then written as

$$\mathcal{M}_4 = \int_{\mathcal{F}} d^2\tau \int d^2 z_2 d^2 z_3 d^2 z_4 \hat{\mathcal{I}}_4 M_4 \tag{5.2}$$

where $\mathcal{F}$ is the fundamental domain of the genus-one moduli space and $\hat{\mathcal{I}}_4$ is the genus-one Koba-Nielsen factor [5]

5.2. One-loop versus tree-level

The form of the correlator (5.1) featuring two distinct contributions, one with a holomorphic square and another with a vectorial contraction between the left- and right-movers, is reminiscent of the one-loop correlator for five *massless* closed-string states considered in [30]. In that case, the analogous vectorial contraction (see their equation (3.35)) was shown to be rewritten in terms of the massless five-point tree-level amplitudes for type IIB states. Moreover, that analysis had an immediate impact on the S-duality symmetry of type IIB strings. While the massive amplitudes considered in this paper have no direct relation to the S-duality property of massless amplitudes, it is still interesting to check whether the massive correlator (5.1) at one-loop can be written in terms of its massive counterpart (2.14) at tree level[5]. Surprisingly, we will show below that, for certain combinations of massive and massless states, this is indeed possible.

First-level massive closed-string spectrum. The closed-string states in the first mass level of bosonic amplitudes can be characterized by the dimension of their $SO(9)$ irrep:

$$\mathbf{1}, \mathbf{36}, \mathbf{44}, \mathbf{84}, \mathbf{126}, \mathbf{231}, \mathbf{450}, \mathbf{495}, \mathbf{594}, \mathbf{910}, \mathbf{924}, \mathbf{1980}, \mathbf{2457}, \mathbf{2772}. \tag{5.3}$$

[5] I thank Oliver Schlotterer for raising this question.

An explicit component expansion using the closed-string state decompositions listed in the Appendix B, reveals that the left-right contraction term in (5.1) can be written in terms of the tree-level amplitude (2.14),

$$s_{12}s_{13}s_{14}C^m_{\underline{1}|2,3,4}\tilde{C}^m_{\underline{1}|2,3,4} = 4s^2_{23}s^2_{24}s^2_{34}\mathcal{K}^{\text{tree}}_{\underline{1}|2,3,4}\tilde{\mathcal{K}}^{\text{tree}}_{\underline{1}|2,3,4}\,, \tag{5.4}$$

for the following combination of external states in type IIB

$$\begin{aligned} gh^3, g\phi h^2: &\qquad g \in \{\mathbf{1}, \mathbf{44}, \mathbf{84}, \mathbf{450}, \mathbf{495}, \mathbf{1980}, \mathbf{2457}\} \\ gh^2\phi: &\qquad g \in \{\mathbf{126}, \mathbf{231}\} \\ g\phi^3: &\qquad g \in \{\mathbf{231}\} \\ gb\phi^2: &\qquad g \in \{\mathbf{2772}\} \end{aligned} \tag{5.5}$$

where the massive state is represented by g, the graviton by h, the NS-NS 2-form by b and the dilaton by ϕ. Similarly, equation (5.4) is satisfied in type IIA for[6]

$$\begin{aligned} gh^3, gh\phi^2: &\qquad g \in \{\mathbf{84}, \mathbf{450}, \mathbf{2457}\} \\ gh^2\phi: &\qquad g \in \{\mathbf{126}, \mathbf{231}\} \\ g\phi^3: &\qquad g \in \{\mathbf{231}\} \end{aligned} \tag{5.6}$$

Note that the type IIB result (5.5) for the states **44** and **495** is only achieved if their decompositions coming from $g_{mn} \otimes \tilde{g}_{pq}$ and $b_{mnp} \otimes \tilde{b}_{qrs}$ are fine-tuned with a relative coefficient. More explicitly, for **44**,

$$\begin{aligned} g_{mn} \otimes \tilde{g}_{pq} &= \Pi^{mn}_{am_1}\Pi^{pq}_{am_2}g^{m_1m_2}_{20000}\,, \\ b_{mnp} \otimes \tilde{b}_{qrs} &= \frac{13}{2}\alpha'\Pi^{mnp}_{abm_1}\Pi^{pqr}_{abm_2}g^{m_1m_2}_{20000} \end{aligned} \tag{5.7}$$

while for **495** we have,

$$\begin{aligned} g_{mn} \otimes \tilde{g}_{pq} &= g^{mpqn}_{02000} + g^{npqm}_{02000} + g^{mqpn}_{02000} + g^{nqpm}_{02000}, \\ b_{mnp} \otimes \tilde{b}_{qrs} &= -\frac{27}{5}\alpha'\Pi^{mnp}_{am_1m_2}\Pi^{pqr}_{am_3m_4}g^{m_1m_2m_3m_4}_{02000} \end{aligned} \tag{5.8}$$

see the appendix B for the definitions and the complete list.

It is interesting to note that a linear combination of massive first-level states in the scalar representation **1** was considered in [31]. However the scalar **1** in $\mathbf{44} \otimes \mathbf{44}$ and the scalar **1** in $\mathbf{84} \otimes \mathbf{84}$ individually satisfy (5.4) in the type IIB theory when paired with h^3 or $\phi^2 h$, so no linear combination of the scalars is necessary in this case.

[6] A natural question to ask, in view of the results of [30], is whether different states satisfy a modified version of (5.4) where the relative coefficient 4 is replaced by a rational number. However, we have not found such a case.

6. Conclusions

In this paper we used arguments based on pure spinor BRST covariance and single-valuedness to propose expressions for one-loop correlators of the open string for three and four points when one of the external legs is a first-level massive state.

We then integrated the loop momentum to obtain the closed-string correlator (5.1) of the one-loop four-point massive amplitude. Its similarity with the massless five-point one-loop amplitude in [30] led us to consider whether it can be rewritten in terms of tree-level massive kinematic factors. We found certain combinations of massive and massless states in (5.5) for the type IIB and (5.6) for the type IIA string theory for which this rewriting can be done. It is worth noting that the analogous analysis of the closed-string five-point massless one-loop correlator [30] had implications for the S-duality of type IIB string theory. It is unclear at the moment whether the results of section 5.1 are an indication of something else than a curiosity. It will be interesting to see if similar results hold for higher-point and/or higher-loop amplitudes. We leave these investigations for the future.

Acknowledgements: I thank Oliver Schlotterer for asking a question which provided the motivation for this work, for pointing out the RNS calculations of [11], and for suggesting the closed-string analysis. I also thank him for comments on the draft.

Appendix A. Proof of tree-level and one-loop relation at three points

Lemma. *In on-shell pure spinor superspace we have*

$$\langle(\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\rangle = \frac{1}{2\alpha'}\langle(\lambda H_1^m)V_2(\lambda\gamma^m W_3)\rangle\,. \qquad (A.1)$$

Proof. To see this, we start from the expression (4.3) and observe that $(\lambda\gamma^m C_1\lambda) = Q(\lambda H_1^m)$ [22]. Integrating the BRST charge by parts, using $QF^{mn} = \partial^m(\lambda\gamma^n W) - \partial^n(\lambda\gamma^m W)$ and the Dirac equation, leads to

$$\begin{aligned} &-(\lambda\gamma^m C_1\lambda)(\lambda\gamma^n W_2)F_3^{mn} - (\lambda\gamma^m C_1\lambda)F_2^{mn}(\lambda\gamma^n W_3) = \\ &\qquad -\big(\partial_1^n(\lambda H_1^m) - \partial_1^m(\lambda H_1^n)\big)(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\,. \end{aligned} \qquad (A.2)$$

On the other hand, in the Berkovits-Chandia gauge [17] we have

$$(\lambda F_1)_{mn} = \frac{1}{16}\Big(7\partial_m(\lambda H_1^n) - 7\partial_n(\lambda H_1^m) + (\lambda\gamma^{k^1 m}H_1^n) - (\lambda\gamma^{k^1 n}H_1^m)\Big)\,, \qquad (A.3)$$

and $(\lambda H_1^{k^1}) = 0$. After a short calculation using the pure spinor constraint, this implies

$$2(\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3) = \big(\partial_1^m(\lambda H_1^n) - \partial_1^n(\lambda H_1^m)\big)(\lambda\gamma^m W_2)(\lambda\gamma^n W_3) \tag{A.4}$$

Plugging this into (A.2) yields

$$-(\lambda\gamma^m C_1\lambda)(\lambda\gamma^n W_2)F_3^{mn} - (\lambda\gamma^m C_1\lambda)F_2^{mn}(\lambda\gamma^n W_3) = 2(\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\,, \tag{A.5}$$

which means that

$$T_{\underline{1},2,3} = (\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\,. \tag{A.6}$$

Therefore,

$$\begin{aligned}
T_{\underline{1},2,3} &= (\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\\
&= \frac{1}{2}\Big(\partial_1^m(\lambda H_1^n) - \partial_1^n(\lambda H_1^m)\Big)(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\\
&= \frac{1}{2}(\lambda H_1^m)(\lambda\gamma^m W_2)\partial_2^n(\lambda\gamma^n W_3)\\
&- \frac{1}{2}(\lambda H_1^n)\partial_3^m(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)
\end{aligned} \tag{A.7}$$

where we used momentum conservation $\partial_1^m = -\partial_2^m - \partial_3^m$ and the Dirac equation to arrive at the third equality. Using $(\lambda\gamma^n W_3) = QA_3^n - \partial_3^n V_3$ in the third line of (A.7) and $(\lambda\gamma^m W_2) = QA_2^m - \partial_2^m V_2$ in the fourth yields

$$\begin{aligned}
T_{\underline{1},2,3} &= \frac{1}{2}(\partial_2\cdot\partial_3)(\lambda H_1^m)\Big(V_2(\lambda\gamma^m W_3) + V_3(\lambda\gamma^m W_2)\Big)\\
&= \frac{1}{2\alpha'}(\lambda H_1^m)V_2(\lambda\gamma^m W_3)
\end{aligned} \tag{A.8}$$

where we dropped BRST-exact terms such as $(\lambda H_1^m)(\lambda\gamma^m W_2)\partial_2^n QA_3^n$, used equation (3.9) of [20] showing that the two terms inside the parenthesis in (A.8) are equal in the BRST cohomology, and $(\partial_2\cdot\partial_3) = -s_{23} = \frac{1}{2\alpha'}$. Finally, this means

$$\langle(\lambda F_1)_{mn}(\lambda\gamma^m W_2)(\lambda\gamma^n W_3)\rangle = \frac{1}{2\alpha'}\langle(\lambda H_1^m)V_2(\lambda\gamma^m W_3)\rangle \tag{A.9}$$

or equivalently, $\langle K_{100}^{\text{1loop}}\rangle = \frac{1}{2\alpha'}\langle K_{100}^{\text{tree}}\rangle$, finishing the proof. □

Appendix B. Tensorial description of first-level massive closed-string states

In this appendix we briefly review the *tensorial* description of irreps of $SO(9)$ and $SO(10)$ using the mapping between Dynkin labels and Young diagrams [12] (see also [32]). We also review the symmetries of the associated tensors that are not manifestly encoded in the Young diagrams [33,34]. Finally, we list the explicit tensorial expressions of how the closed-string states at the first massive level are constructed from the holomorphic square of open-string states.

$SO(9)$ tensor irreps. The irreps of $SO(9)$ are labelled by four Dynkin labels $(a_1a_2a_3a_4)$. They correspond to a tensor (as opposed to a spinor) when a_4 is *even.* In this case, the irrep $(a_1a_2a_3a_4)$ encodes a Young diagram with a_i columns with i boxes for $i = 1, 2, 3$ and $a_4/2$ columns with 4 boxes. For example (2200) maps to $\ydiagram{4,2}$ because there are two columns with one box and two columns with two boxes.

SO(10) tensor irreps. The irreps of SO(10) labelled by five Dynkin labels $(a_1a_2a_3a_4a_5)$ correspond to a tensor when $a_4 + a_5$ is *even.* Their mapping to Young diagrams works the same way as in $SO(9)$ for the labels a_1, a_2, a_3, while the mapping of the labels a_4 and a_5 to columns require a separate analysis depending on two cases:

1. $a_4 \leq a_5$: there are a_4 columns of 4 boxes and $(a_5 - a_4)/2$ columns of 5 boxes
2. $a_4 > a_5$: this is the conjugate representation, there are a_5 columns of 4 boxes and $(a_4 - a_5)/2$ columns of 5 boxes

For example, (00011) is a tensor that maps to the 4-form $\ydiagram{1,1,1,1}$. Similarly, both $(00002) = \ydiagram{1,1,1,1,1}$ and $(00020) = \ydiagram{1,1,1,1,1}$ constitute 5-forms (self-dual and anti-self-dual).

Young tableaux and tensors. We associate traceless tensors to Young tableaux of $SO(n)$ by populating the columns with the tensor indices according to the antisymmetric basis scheme [33]. For example,

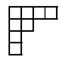
$$\begin{ytableau} 1 & 5 & 7 & 8 \\ 2 & 6 \\ 3 \\ 4 \end{ytableau} \qquad \leftrightarrow \qquad T_{12345678} = T_{[1234][56](78)} \tag{B.1}$$

where the labels j in the tensor are a shorthand for vector indices m_j. In (B.1) we also display the explicit (anti)symmetries of the associated tensor. Being traceless irreps of $SO(n)$ means that the trace with respect to any two indices vanish

$$\delta^{m_i m_j} T_{\dots m_i \dots m_j \dots} = 0\,, \quad \forall i, j\,. \tag{B.2}$$

Garnir symmetries. The manifest symmetries in (B.1) do not describe all the symmetries of the tensor. In general, tensors obey additional symmetries described by the Garnir relations among Young tableaux [35,36,34]. Suppose column j of the Young diagram mapped to a tensor T has b_j boxes and let $M = m_1 m_2 \ldots m_p$ and $N = n_1 n_2 \ldots n_q$ be indices from columns j and $j+1$. If $p + q > b_j$ then

$$T_{\ldots[MN]\ldots} = 0\,. \tag{B.3}$$

For example, the Garnir symmetries of the tensor associated to the Young diagram  (see (B.1)) can be written as

$$T_{[12345]678} = 0\,, \quad T_{1234[567]8} = 0\,, \quad T_{1[23456]78} = 0\,. \tag{B.4}$$

Note that if columns j and $j+1$ have the same number q of boxes, then it is also true that swapping these two columns is a symmetry:

$$T_{\ldots m_1 \ldots m_q n_1 \ldots n_q \ldots} = T_{\ldots n_1 \ldots n_q m_1 \ldots m_q \ldots} \tag{B.5}$$

The symmetry (B.5) is not independent of the symmetries (B.3). This is easier to see with the alternative description of the Garnir symmetries reviewed in section B.1.1.

B.1. open⊗open=closed for first-level massive states

The *massive* states of the superstring combine to representations of $SO(9)$ but the amplitude calculations are done in the Wick-rotated ten-dimensional spacetime, where the states are described by $SO(10)$ irreps.

Consider all the $SO(10)$ irreps in the tensor products [26] of $g_{mn} = (20000)$ and $b_{mnp} = (00100)$ appearing in the open⊗open description of the closed-string states:

$$\begin{aligned}
(20000) \otimes (00100) &= (00100) + (10011) + (11000) + (20100) \qquad \text{(B.6)}\\
(20000) \otimes (20000) &= (00000) + (01000) + (02000) + (20000) + (21000) + (40000)\\
(00100) \otimes (00100) &= (00000) + 2(00011) + (00200) + (01000) + (01011) + (02000)\\
&\quad + (10002) + (10020) + (10100) + (20000)
\end{aligned}$$

To each irrep $(a_1 a_2 a_3 a_4 a_5)$ we associate a transverse $SO(10)$ tensor $g^{m\ldots}_{a_1 a_2 a_3 a_4 a_5}$. Their precise ranks and symmetries are determined from the mapping to Young diagrams described

Dim	Dynkin	Young	$44 \otimes 44$	$84 \otimes 84$	$84 \otimes 44$
1	(0000)	$\cdot$	$\surd$	$\surd$	$\times$
36	(0100)	$\yng(1,1)$	$\surd$	$\surd$	$\times$
44	(2000)	$\yng(2)$	$\surd$	$\surd$	$\times$
84	(0010)	$\yng(1,1,1)$	$\times$	ϵ_9	$\surd$
126	(0002)	$\yng(1,1,1,1)$	$\times$	$\surd$	$\times$
231	(1100)	$\yng(2,1)$	$\times$	$\times$	$\surd$
450	(4000)	$\yng(4)$	$\surd$	$\times$	$\times$
495	(0200)	$\yng(2,2)$	$\surd$	$\surd$	$\times$
594	(1010)	$\yng(2,1,1)$	$\times$	$\surd$	$\times$
910	(2100)	$\yng(3,1)$	$\surd$	$\times$	$\times$
924	(1002)	$\yng(2,1,1,1)$	$\times$	ϵ_9	$\surd$
1980	(0020)	$\yng(2,2,2)$	$\times$	$\surd$	$\times$
2457	(2010)	$\yng(3,1,1)$	$\times$	$\times$	$\surd$
2772	(0102)	$\yng(2,2,1,1)$	$\times$	$\surd$	$\times$

Table 1. The first-level massive closed-string states characterized by their dimension, $SO(9)$ Dynkin labels and Young diagram. The columns dim $\otimes$ dim indicate the presence ($\surd$) or absence ($\times$) of the closed-string state in the tensor product of open-string states given in (B.25). The entries ϵ_9 indicate that the state contains a 9-dimensional Levi-Civita tensor.

above. For example, in $SO(10)$ we have $\mathbf{320} = (11000) \leftrightarrow \yng(2,1)$ and therefore the associated tensor g^{mnp}_{11000} has three indices with the symmetries of $\yng(2,1)$

$$k_m g^{mnp}_{11000} = 0, \quad g^{mnp}_{11000} = -g^{nmp}_{11000}, \quad g^{mnp}_{11000} + g^{npm}_{11000} + g^{pmn}_{11000} = 0\,. \tag{B.7}$$

The branching to $SO(9)$ irreps is given by $\mathbf{320} = \mathbf{9} + \mathbf{36} + \mathbf{44} + \mathbf{231}$ [37]. The transverse and traceless conditions remove the lower dimensional irreps and we are left with $\mathbf{231}$. In this sense we will call the tensor g^{mnp}_{11000} the $\mathbf{231}$ of $SO(9)$. This leads to the following decompositions:

- The closed-string state $\mathbf{1}$:

$$\begin{aligned} g_{mn} \otimes \tilde{g}_{pq} &= \Pi^{mn}_{pq} g_{00000} \\ b_{mnp} \otimes \tilde{b}_{qrs} &= \alpha' \Pi^{mnp}_{qrs} g_{00000} \end{aligned} \tag{B.8}$$

- The closed-string state $\mathbf{36} = \yng(1,1)$

$$\begin{aligned} g_{mn} \otimes \tilde{g}_{pq} &= \Pi^{mn}_{am_1} \Pi^{pq}_{am_2} g^{m_1 m_2}_{01000} \\ b_{mnp} \otimes \tilde{b}_{qrs} &= \alpha' \Pi^{mnp}_{abm_1} \Pi^{pqr}_{abm_2} g^{m_1 m_2}_{01000} \end{aligned} \tag{B.9}$$

- The closed-string state $\mathbf{44} = \ydiagram{2}$

$$g_{mn} \otimes \tilde{g}_{pq} = \Pi^{mn}_{am_1} \Pi^{pq}_{am_2} g^{m_1 m_2}_{20000} \tag{B.10}$$
$$b_{mnp} \otimes \tilde{b}_{qrs} = \frac{13}{2} \alpha' \Pi^{mnp}_{abm_1} \Pi^{pqr}_{abm_2} g^{m_1 m_2}_{20000}$$

- The closed-string state $\mathbf{84} = \ydiagram{1,1,1}$

$$g_{mn} \otimes \tilde{b}_{qrs} = \tilde{g}_{mn} \otimes b_{qrs} = \Pi^{mn}_{am_1} \Pi^{qrs}_{am_2 m_3} g^{m_1 m_2 m_3}_{00100} \tag{B.11}$$

- The closed-string state $\mathbf{126} = \ydiagram{1,1,1,1}$

$$b_{mnp} \otimes \tilde{b}_{qrs} = \alpha' \Pi^{mnp}_{am_1 m_2} \Pi^{pqr}_{am_3 m_4} g^{m_1 m_2 m_3 m_4}_{00011} + \hat{\epsilon}_{10}^{mnpqrsabcd} g^{abcd}_{00011} \tag{B.12}$$

- The closed-string state $\mathbf{231} = \ydiagram{2,1}$

$$g_{mn} \otimes \tilde{b}_{qrs} = \Pi^{mn}_{am_1} \Pi^{qrs}_{am_2 m_3} g^{m_2 m_3 m_1}_{11000} \tag{B.13}$$

- The closed-string state $\mathbf{450} = \ydiagram{4}$

$$g_{mn} \otimes \tilde{g}_{pq} = g^{mnpq}_{40000} \tag{B.14}$$

- The closed-string state $\mathbf{495} = \ydiagram{2,2}$

$$g_{mn} \otimes \tilde{g}_{pq} = g^{mpqn}_{02000} + g^{npqm}_{02000} + g^{mqpn}_{02000} + g^{nqpm}_{02000} \tag{B.15}$$
$$b_{mnp} \otimes \tilde{b}_{qrs} = -\frac{27}{5} \alpha' \Pi^{mnp}_{am_1 m_2} \Pi^{pqr}_{am_3 m_4} g^{m_1 m_2 m_3 m_4}_{02000}$$

- The closed-string state $\mathbf{594} = \ydiagram{2,1,1}$

$$b_{mnp} \otimes \tilde{b}_{qrs} = \alpha' \Pi^{mnp}_{am_1 m_2} \Pi^{pqr}_{am_3 m_4} g^{m_1 m_2 m_3 m_4}_{10100} \tag{B.16}$$

- The closed-string state $\mathbf{910} = \ydiagram{3,1}$

$$g_{mn} \otimes \tilde{g}_{pq} = g^{mpqn}_{21000} + g^{npqm}_{21000} + g^{mqpn}_{21000} + g^{nqpm}_{21000} \tag{B.17}$$

- The closed-string state $\mathbf{924} = \ydiagram{2,1,1,1}$

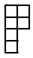
$$g_{mn} \otimes \tilde{b}_{pqr} = \tilde{g}_{mn} \otimes b_{pqr} = g^{mpqrn}_{10011} + g^{npqrm}_{10011} \tag{B.18}$$

- The closed-string state **1980** = 

$$b_{mnp} \otimes \tilde{b}_{abc} = g_{00200}^{mnpabc} \tag{B.19}$$

- The closed-string state **2457** = 

$$g_{mn} \otimes \tilde{b}_{qrs} = \tilde{g}_{mn} \otimes b_{qrs} = g_{20100}^{qrsmn} \tag{B.20}$$

- The closed-string state **2772** = 

$$b_{mnp} \otimes \tilde{b}_{qrs} = g_{01011}^{mnpqrs} + g_{01011}^{mnpsqr} + g_{01011}^{mnprsq} \tag{B.21}$$

and we ignore the $SO(9)$ states associated with (10002) and (10020) as their $SO(10)$ Young diagrams have columns with five boxes which imply that the $SO(9)$ states are proportional to a nine-dimension Levi-Civita. In the above the various tensors $g_{\dots}$ have the same symmetries of their associated Young diagrams and Π_{ab}^{mn} and Π_{abc}^{mnp} are the Young projectors [38] (see also [33,39,40]):

$$\begin{aligned} \Pi_{pq}^{mn} &= \frac{1}{2}(\hat{\delta}_{mp}\hat{\delta}_{nq} + \hat{\delta}_{mq}\hat{\delta}_{np}) - \frac{1}{9}\hat{\delta}_{mn}\hat{\delta}_{pq} \\ \Pi_{qrs}^{mnp} &= \hat{\delta}_q^{[m}\hat{\delta}_r^{n}\hat{\delta}_s^{p]} \end{aligned} \tag{B.22}$$

where

$$\hat{\delta}_{mn} = \delta_{mn} - \frac{k_m k_n}{(k \cdot k)} \tag{B.23}$$

satisfies $k^m \hat{\delta}_{mn} = 0$ for the first-level massive condition $\alpha' k^2 = -1$. Both projectors (B.22) are traceless and transverse if $\alpha' k^2 = -1$. These conditions are necessary in order for the above decompositions to be traceless and transverse, since g_{mn} and b_{mnp} are both traceless and transverse (w.r.t k_1). Furthermore

$$\hat{\epsilon}_{10}^{mnpqrsabcd} = \epsilon_{10}^{mnpqrsabcd} - 10!\alpha' k_1^{[m} \epsilon_{10}^{npqrsabcd]k_1} \tag{B.24}$$

satisfies $k_1^m \hat{\epsilon}_{10}^{mnpqrsabcd} = 0$. If we denote g_{mn} and b_{mnp} by their open-string $SO(9)$ irrep dimensions **44** and **84**, the decompositions above can be summarized by the following tensor products [26]

$$\begin{aligned} &\mathbf{44} \otimes \mathbf{84} = \mathbf{84} \oplus \mathbf{231} \oplus \mathbf{924} \oplus \mathbf{2457} \\ &\mathbf{44} \otimes \mathbf{44} = \mathbf{1} \oplus \mathbf{36} \oplus \mathbf{44} \oplus \mathbf{450} \oplus \mathbf{495} \oplus \mathbf{910} \\ &\mathbf{84} \otimes \mathbf{84} = \mathbf{1} \oplus \mathbf{36} \oplus \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{126} \oplus \mathbf{495} \oplus \mathbf{594} \oplus \mathbf{924} \oplus \mathbf{1980} \oplus \mathbf{2772} \end{aligned} \tag{B.25}$$

B.1.1. Alternative description of the Garnir symmetries

In this subsection we give an alternative description of the Garnir symmetries following [41], in which all the signs are positive. More precisely, for a given tableau, summing over all ways of exchanging the top k elements of one column with k elements of the preceding column while preserving the vertical orders within each set of k elements gives back the original tableau.

As an illustration, consider the tableau in (B.1) and its associated tensor $T_{12345678}$. To find all its non-trivial symmetries, we start by considering the $k = 1$ relation for the second column. We get the corresponding identity

$$\begin{array}{|c|c|c|c|}\hline 1&5&7&8\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 5&1&7&8\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&7&8\\\hline 5&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&3&7&8\\\hline 2&6\\\cline{1-2} 5\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&4&7&8\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 5\\\cline{1-1}\end{array}$$

$$T_{12345678} = T_{52341678} + T_{15342678} + T_{12543678} + T_{12354678} \tag{B.26}$$

Similarly, $k = 1$ for the third column yields

$$\begin{array}{|c|c|c|c|}\hline 1&5&7&8\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&7&5&8\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&5&6&8\\\hline 2&7\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$$

$$T_{12345678} = T_{12347658} + T_{12345768}\,, \tag{B.27}$$

It is easy to see that $k = 1$ for the fourth column leads to the trivial symmetry (78) that can be read off from the tableau itself.

Now we consider the $k = 2$ relation with the top two elements $\{5, 6\}$ of the second column in (B.1). This yields

$$\begin{array}{|c|c|c|c|}\hline 1&5&7&8\\\hline 2&6\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} = \begin{array}{|c|c|c|c|}\hline 5&1&7&8\\\hline 6&2\\\cline{1-2} 3\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 5&1&7&8\\\hline 2&3\\\cline{1-2} 6\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 5&1&7&8\\\hline 2&4\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&7&8\\\hline 5&3\\\cline{1-2} 6\\\cline{1-1} 4\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&2&7&8\\\hline 5&4\\\cline{1-2} 3\\\cline{1-1} 6\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|}\hline 1&3&7&8\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1} 6\\\cline{1-1}\end{array}$$

$$T_{12345678} = T_{56341278} + T_{52641378} + T_{52361478} + T_{15642378} + T_{15362478} + T_{12563478}\,. \tag{B.28}$$

It is not difficult to show that the above symmetries are equivalent to[7]

$$T_{[12345]678} = 0\,, \quad T_{1234[567]8} = 0\,, \quad T_{1[23456]78} = 0\,. \tag{B.29}$$

[7] To show the third equation, start from (B.28) and use (B.26) to rewrite $T_{5\ldots} = \sum T_{1\ldots}$.

In this description of the Garnir symmetries it is easy to see that if two adjacent columns have the same number of boxes, then swapping the corresponding columns in the Young tableau is a symmetry (choose k to be the total number of boxes).

Young projections. There is a way to experimentally verify the above Garnir symmetries. To see this for the specific example in (B.1), define the tensor as

$$T(1,\ldots,8) = W_{1\ldots8} + (1578) + (26) + [1234] + [56] \tag{B.30}$$

where $W_{\ldots}$ denote words and the (anti)symmetrizations must follow the specified order: first symmetrize over $1,5,7,8$, then symmetrize over $2,6$, then antisymmetrize over $1,2,3,4$ and then antisymmetrize over $5,6$. This yields 2304 words in the right-hand side of (B.30), and the expression is available to download in [8]. The symmetries (B.29) are then satisfied by (B.30).

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