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Supersymmetric indices, defects and holography

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Abstract

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This thesis explores the use of holography in the study of codimension-2 and codimension-4 defects of the 6d $\mathcal{N} = (2, 0)$ theories. By means of the Ryu-Takayanagi formula, we successfully determine various central charges associated to these defects, which play a crucial role in their full characterization. We also establish connections between different families of supergravity solutions, delivering a more comprehensive picture of their landscape.

Another focus of this thesis is that of background conformal supergravity configurations for 4d $\mathcal{N} = 2$ and 4d $\mathcal{N} = 4$ SCFTs. The departure from the holographic tools is nevertheless an interesting one, as we are able to find such backgrounds that engineer various topological twists of the theories. Our results are novel in the context of $\mathcal{N} = 4$, where we are able to construct the supercharges of the Vafa-Witten, Kapustin-Witten and half-twists, all valuable non-perturbative tools in the study of four-dimensional theories. Backgrounds for the $\mathcal{N} = 2$ indices were already known, however, we provide a novel family of indices that interpolates exactly between the twisted index and a point on the moduli of Coulomb-branch indices; allowing us to equate the two.

Finally, we introduce a novel realisation of defect-conformal-field-theories as finite-dimensional integrals over neural-networks. This construction extends the known formalism for CFTs to the defect case, providing a systematic framework for generating dCFT data from neural-network concatenations.

Contents

List of Figures ix

List of Tables xi

Declaration of Authorship xiii

Acknowledgements xv

Introduction xix

I Background Material 1

1 Conformal Field Theory 3
- 1.1 The Conformal Group . . . 4
- 1.2 The Conformal Algebra . . . 6
- 1.3 Correlation Functions . . . 10
  - 1.3.1 One-point functions . . . 12
  - 1.3.2 Two-point functions . . . 12
  - 1.3.3 Three-point functions . . . 12
  - 1.3.4 Four-point function . . . 13
- 1.4 The Conformal Blocks and the OPE . . . 13
- 1.5 The Embedding Space Formalism . . . 14
  - 1.5.1 Two-point function . . . 17
  - 1.5.2 Three-point function . . . 17
- 1.6 Defect CFTs and Boundary CFTs . . . 18
  - 1.6.1 One-point functions . . . 21
  - 1.6.2 Two-point functions . . . 21

2 11D supergravity 25
- 2.1 The action principle . . . 27
- 2.2 Its toroidal compactifications . . . 29
- 2.3 BPS solutions . . . 30
- 2.4 Its susy algebra . . . 34
- 2.5 Further Reading . . . 36

3 Supersymmetry on curved spaces 37
- 3.1 Rigid Supergravity and Festuccia-Seiberg . . . 38
- 3.2 Conformal Supergravity . . . 41

3.2.1 The Weyl multiplet of 4d $\mathcal{N} = 2$ conformal supergravity 42
3.2.2 Weyl multiplet of 4d $\mathcal{N} = 4$ conformal supergravity 45

II Holographic Defects 51

4 Codimension-4 defect in 6d SCFT 53
4.1 Introduction 53
4.2 Review: small $\boldsymbol{\mathcal{N} = (4,4)}$ surface defects 57
4.3 From large to small $\boldsymbol{\mathcal{N} = (4,4)}$ surface defects 59
4.3.1 Supergravity solutions for general γ 62
4.3.2 The $\gamma \to -\infty$ limit 65
4.3.3 Inclusion of KK-monopoles and recovering known solutions 67
4.4 New small $\boldsymbol{\mathcal{N} = (4,4)}$ surface defects 68
4.4.1 Asymptotic local behaviour 70
4.4.2 Single-pole vacuum 71
4.4.3 Partition data 73
4.5 Entanglement entropy of small $\boldsymbol{\mathcal{N} = (4,4)}$ surface defects 77
4.6 Summary and Outlook 80

5 Codimension-2 defect in 6d SCFT 83
5.1 Introduction 83
5.2 Review 85
5.2.1 Defect Weyl anomalies 86
5.2.2 11d SUGRA solutions 88
5.3 Holographic stress-energy tensor one-point function 94
5.3.1 Two-charge solutions 96
5.3.2 Electrostatic solutions 99
5.4 Defect sphere EE and the defect A-type anomaly 101
5.4.1 Two-charge solutions 102
5.4.2 Electrostatic solutions 104
5.5 Discussion 108
5.5.1 Future directions and open questions 112

III Supersymmetry and Indices 115

6 Interpolating Index 117
6.1 Introduction 117
6.2 The Coulomb-branch index 120
6.3 The twisted index 124
6.4 Current multiplet transformations 126
6.5 Interpolating index 128
6.5.1 Interpolating background 128
6.5.2 Supersymmetric exactness of the interpolation 130
6.6 Conclusion/Remarks 134

7 The $\mathcal{N} = 4$ Ω-deformations 137

7.1 Introduction . . . 138
7.2 Euclidean Weyl multiplet . . . 140
7.3 The twists of $\mathcal{N} = 4$ . . . 143
7.4 Ω-deformed twists . . . 147
7.5 Conclusion/Remarks . . . 150

IV Neural networks 153

8 nn-dCFT 155
8.1 The nn-FT correspondence . . . 156
8.2 Neural-network-CFTs . . . 158
8.3 Neural-network-defect-CFTs . . . 160
8.3.1 Example: monomial nn-dCFTs . . . 161
8.3.2 Second example: reciprocal nn-dCFT . . . 164
8.4 Spinning nn-dCFT . . . 165
8.5 Conclusion/Remarks . . . 166

9 Conclusion 169

V Appendices 173

Appendix A Appendix to Chapter 4 175
Appendix A.1 Fefferman-Graham parametrization . . . 175
Appendix A.2 Area of the Ryu-Takayanagi hypersurface . . . 177

Appendix B Appendix to Chapter 5 181
Appendix B.1 Fefferman-Graham coordinates . . . 181
Appendix B.1.1 Two-charge solutions . . . 181
Appendix B.1.2 Electrostatic solutions . . . 183
Appendix B.2 On-shell action . . . 187
Appendix B.2.1 Two-charge solutions . . . 187
Appendix B.3 Regulating the on-shell action . . . 190
Appendix B.3.1 $\mathrm{AdS}_7 \times S^4$ . . . 191
Appendix B.3.2 Renormalised AdS_5 volume . . . 192

Appendix C Useful identities 195
Appendix C.1 Differential Geometry Identities . . . 195
Appendix C.1.1 The ϵ-δ identities (3d Cartesian) . . . 197
Appendix C.1.2 The ϵ-δ identities (4d Lorentzian) . . . 197
Appendix C.1.3 General identities . . . 197
Appendix C.1.3.1 Cartan structure equations . . . 199
Appendix C.2 Geometric conventions of Chapter 6 . . . 200
Appendix C.3 Complexification of the equivariant parameters . . . 201
Appendix C.4 Young Tableau . . . 203
Appendix C.4.1 Calculating the dimension . . . 203
Appendix C.4.2 Tensor product of Young tableaux . . . 204
Appendix C.5 Hypergeometric functions . . . 207

Appendix C.5.1 Special values 208
Appendix C.5.2 Integral formulae 209
Appendix C.6 Lemmas of Chapter 8 211

Appendix D Spinor Conventions **215**
Appendix D.1 Clifford Algebras 215
Appendix D.1.1 Relation to the Physics language 217
Appendix D.1.2 A basis in any dimension and any signature 218
Appendix D.2 Spin(4) conventions 220
Appendix D.3 Killing Spinors 224
Appendix D.4 Mapping $S^3 \times S^1$ KS to $\mathbb{R}^4$ KS 227
Appendix D.5 Supergroups and Lie superalgebras 230
Appendix D.5.1 Classification of classical Lie superalgebras 231

References **233**

List of Figures

1 A visual representation of AdS_{d+1} together with its conformal boundary. We denoted in blue the CFT that lives on the conformal boundary, at a fixed time t. . . . xix

2 The Ryu-Takayanagi prescription. On the right-hand-side, a spatial region in the CFT, $\mathcal{B}$, together with its entropy $S_{\mathcal{B}}$. On the left-hand-side, the asymptotically AdS bulk with the minimal surface ζ, homologous (i.e. shares the same boundary) to $\mathcal{B}$ on the boundary. The holographic entanglement entropy is equal to the entanglement entropy $S_{\mathrm{EE}} = S_{\mathcal{B}}$. . . xxvi

3 The Ryu-Takayanagi prescription with a defect insertion. The conformal defect Y manifests itself as AdS_n fibre within the bulk AdS geometry. . . xxvii

4 A flowchart illustrating the chapter dependencies within this thesis. Chapters marked in blue mainly contain supergravity material. Chapters marked in green are mainly focused on conformal field theory and related topics. Finally, the appendices for each chapter are indicated below them. xxxii

1.1 A visualisation of crossing symmetry in the four-point function. Black circles denote operator insertions, blue circles their decomposition through the OPE. . . . 14

1.2 A visual representation of the null cone $\mathrm{NC} = \{X \in \mathbb{R}^{1,D+1} \,|\, X^2 = 0\}$. The Poincaré section (PS) $X_+ = 1$ and its antipodal locus $X_+ = -1$ are shown in red; they select a unique representative from each projective orbit $X \sim \lambda X$ within each section. An example of such a null ray is shown in blue. . . . 16

1.3 A visualisation of constraints imposed on the coefficients in the defect OPE. Black circles denote operator insertions, blue circles their decomposition through the OPE and the solid line represents the defect. . . . 20

2.1 The field content of 11d supergravity and its $\mathfrak{so}(1,10)$ representations. . . 26

4.1 A diagram representing a constructive approach to building the brane intersection in [1]. One can start with either M2 or M5 branes (top) and consider adding the other branes successively, working downwards. Under each brane setup is the corresponding near horizon geometry. *This content was added for the purpose of this thesis, and does not appear in the original article.* . . . 60

4.2 A visual representation of the solutions in [2] — a foliation of AdS_3 and two S^3 over a Riemann surface Σ_2. At the singularities on the boundary of the Riemann surface, ξ_i, either one of the two three-spheres in the foliation collapses. . . . 63

4.3 A particular choice of collapse points $\nu_j \in \partial\Sigma_2$ and the behaviour of the singular points $\xi_i \in \partial\Sigma_2$ as $\gamma \to -\infty$ under pairwise collapse. The collapse points ν_j can be chosen arbitrarily, while maintaining $\nu_j < \nu_{j+1}$, but for clarity in the figure have been depicted at the midpoint in the region $[\xi_j, \xi_{j+1}]$ to demonstrate pairwise collapse. 68

4.4 The 1PV solution described in section 4.4.2 is characterised by two singular points $\xi_{\{1,2\}}$, which collapse to the same $\nu \in \partial\Sigma_2$ in the $\gamma \to -\infty$ limit. As discussed in section 4.4.3, the basis of non-contractible four-cycles for this solution is one-dimensional. The profile of a representative cycle $\mathfrak{C}_a$ along Σ_2 is shown; note that the same 3-sphere collapses at both of its endpoints on $\partial\Sigma_2$. On the right, the 1PV on the upper half plane is mapped to a semi-infinite strip. 71

4.5 The profile of the non-contractible four-cycles $\mathfrak{C}_a$ and $\mathfrak{C}'_a$ in Σ_2. Every point on the red and blue curves is a 3-sphere. 74

4.6 (a) Young diagram corresponding to the partition specifying a $\gamma \to -\infty$ solution with 5 distinct ν_a constructed by pairwise collapse of 10 different ξ_j. (b) Young diagram obtained by "multiwise" collapse of 10 different ξ_j to 4 distinct ν_a. Another way to realise the construction of (b) starts from the partition in (a) and collapses $\nu_2 = \nu_3$. 76

5.1 **(Top)** A generic line charge distribution $\varpi(\eta)$, with n kinks at positions η_a along the axis of cylindrical symmetry, specifying a solution to the axially symmetric Laplace equation in $\mathbb{R}^3$. The $\mathrm{AdS}_7 \times S^4$ vacuum corresponds to the single-kink ($n = 1$) charge distribution with $k_1 = 1$; the location of the kink is then given by $\eta_1 = N/2$. **(Bottom)** The Young Tableau corresponding to the partition $N = 2\sum_{a=1}^{n} k_a\eta_a = \sum_{a=1}^{n} N_a$. The height and width of the a-th block are given by the location $\eta_a \in \mathbb{Z}$ and slope change $k_a \in \mathbb{Z}$ of the a-th kink in $\varpi(\eta)$, respectively. The $\mathrm{AdS}_7 \times S^4$ vacuum is associated to the $\mathbf{1}$ of $\mathfrak{su}(N)$ determined by $n = k_1 = 1$ and $\eta_1 = N/2$. 94

5.2 The solutions to the constraint in (5.15) for $\hat{n} = 1$ (red), $\hat{n} = 2$ (blue), and $\hat{n} = 3$ (green) on the (q_1, q_2) plane, reproduced from [3, 4]. The shaded regions correspond to the two-charge configurations for which $Q(y) = 0$ admits no real solutions (region I) or which violate the defect ANEC (region II). 98

6.1 Decomposition of four-dimensional superconformal multiplets for various amounts of supersymmetry. On the right-hand-side, we denote the corresponding index, illustrating how many parameters it depends on. 119

6.2 The various limits of the sci considered in [5]. Note that the Schur index, while explicitly dependent on two parameters, p and q, only actually dependents on the latter. 123

8.1 Illustration of the large-width limit (left) and large-depth limit (right) of the neural network (middle). 157

8.2 A pictorial representation of all possible Wick contractions that build the k-point expectation value $\mathbb{E}[(X_1 \bullet \theta)^{n_1} \cdots (X_k \bullet \theta)^{n_k}]$. 164

List of Tables

1 A summary of the neural-network-CFT (nnCFT) data, and their standard CFT counterparts. ϕ is a conformal primary, and obeys the conformal primary condition displayed above. In the nnCFT, this translate to requiring a homogenous neuron Φ. The partition function is given as a finite integral over neuron parameters, θ, with distribution $\mathcal{P}(\theta)$. The distribution plays the role of the action in the standard CFT. Correlators are given as standard expectation values of probability variables. xxxi

1.1 A detailed breakdown of the different generators of the conformal group of $\mathbb{R}^{t,s}$. For each generator, the corresponding conformal Killing vector is included. 8

3.1 Field content of the 4d $\mathcal{N} = 2$ Weyl multiplet on an oriented spin Riemannian manifold $\mathcal{M}$. Throughout, $\Gamma(P)$ denotes the space of sections over a bundle $P \longrightarrow \mathcal{M}$, $S\mathcal{M}$ is the spin bundle over $\mathcal{M}$, $\mathcal{K}_R$ is the total space of the associated bundle to the principal $SU(2)_R$ R-symmetry bundle over $\mathcal{M}$, and $\Omega^k(\mathcal{M})$ is the space of k-forms on $\mathcal{M}$. Wherever present, $\pm$ denotes either (self-)anti-self duality or positive/negative chirality. The latter also entails that the spinor is a section of one of the two parts of the spin bundle $\mathcal{S}\mathcal{M} \cong \mathcal{S}^+\mathcal{M} \otimes \mathcal{S}^-\mathcal{M}$ only. 43

3.2 The (Grassmann even) parameters of Q- and S-supersymmetries, and their quantum numbers. 44

3.3 Field content of the Lorentzian $\mathcal{N} = 4$ Weyl multiplet. The double line separates the four fermionic fields (top) from the bosonic ones (bottom). The third row details all constraints obeyed by the field, including chirality/self-duality (if any). We de not detail those for the auxiliary fields. The final row displays the $SU(4)_R$ representation. 47

4.1 The 1/8-BPS brane setup of [1], with M2-M5 defect branes intersecting orthogonal M5′-branes, and with both stacks of 5-branes probing A-type singularities. In our conventions, $-$, $\cdot$ and $\sim$ denote directions along which a brane is extended, localised, and smeared, respectively, while ISO(metric) denotes the compact direction of the KK-monopoles. 57

6.1 The functional dependence of the supersymmetric variations of the fields in the Weyl multiplet on the dilatino χ and $\partial_\mu\chi$. 131

6.2 The functional dependence of the supersymmetric variations of the fields in the Weyl multiplet on the gravitino ψ and $\partial_\mu\psi$. 132

7.1 List of invariants computed by the Donaldson-Witten twist for various $\mathcal{N} = 2$ gauge theories. Taken from [6]. 139

7.2 List of the fermionic components of the Euclidean $\mathcal{N} = 4$ Weyl multiplet. In the second column, any remaining constraints are detailed. The chirality is given in the third row, while the final row contains their $SU(4)_R$ representation. 141

7.3 List of the bosonic components of the Euclidean $\mathcal{N} = 4$ Weyl multiplet. In the second column, any remaining constraints are detailed. The self-duality is given in the third row, while the final row contains their $SU(4)_R$ representation. Note that the vierbeins e^a, spin connection ω^{ab} and $SU(4)$ connection $V^i{}_j$ can be taken as complex through this procedure. However, we will only focus on backgrounds where they are real. 142

7.4 The topological (anti-)twists of $\mathcal{N} = 4$ theories on oriented Riemannian (but otherwise arbitrary) manifolds, as defined by their respective homomorphic injection φ of the Euclidean rotations into the $SU(4)_R$ R-symmetries. For each twist, we provide the data Σ defining the $SU(4)_R$ connection via equation (7.6) and the positive- (negative-)chirality Killing spinor(s) ξ^i (ξ_i). I_2 is the unit 2×2 matrix. 145

Appendix C.1 List of common structures on a differential manifold $\mathcal{M}$, together with their corresponding G-structure group. The map is not one-to-one as many structures can require additional integrability conditions which are not made explicit from the G-structures. Almost complex and complex structures for instance share the same G-structures but the former requires the vanishing of the Nijenhuis tensor. 196

Appendix D.1 Description of the various index labels used in Chapters 6 and 7. 220

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as: Pietro Capuozzo, John Estes, Brandon Robinson, and Benjamin Suzzoni. Holographic Weyl anomalies for 4d defects in 6d SCFTs. *JHEP*, 04:120, 2024
   Pietro Capuozzo, John Estes, Brandon Robinson, and Benjamin Suzzoni. From large to small $\mathcal{N} = (4, 4)$ superconformal surface defects in holographic 6d SCFTs. *JHEP*, 08:094, 2024
   Pietro Capuozzo, Brandon Robinson, and Benjamin Suzzoni. Conformal Defects from Neural Networks. (to appear soon) 2025

Signed:.. Date:..................

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To Valerie and Ronald

Introduction

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [10–12] has introduced novel ways of studying strongly coupled superconformal field theories (SCFTs)[1]. In its prototypical example, this conjectured correspondence states that Type IIB string theory on $\mathrm{AdS}_5 \times S^5$, with N units of 5-form flux through the S^5, is equivalent to $\mathcal{N}=4$ super-Yang-Mills (SYM) in four dimensions, with gauge group $SU(N)$ and coupling related to the string coupling via $g_{\mathrm{YM}}^2 = g_s$. When assumed to hold for any value of N and any string coupling g_s, this is the conjecture's *strong form*. However, most treatments look at the large 't Hooft coupling limit, $\lambda = g_{\mathrm{YM}}^2 N \to \infty$, together with the large-N limit. This, in turn, reduces the string theory to a classical supergravity theory on $\mathrm{AdS}_5 \times S^5$, which is conjectured to be dual to a strongly coupled $\mathcal{N}=4$ SYM, with gauge group $SU(N)$ and $N \to \infty$.

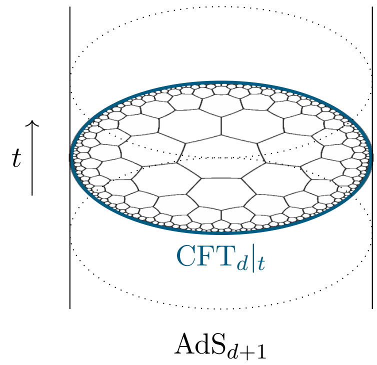


FIGURE 1: A visual representation of AdS_{d+1} together with its conformal boundary. We denoted in blue the CFT that lives on the conformal boundary, at a fixed time t.

More generally, the AdS/CFT correspondence states that a quantum theory of gravity with asymptotic AdS geometry is dual to a CFT in one dimension lower. The 'quantum field theory of gravity' in question is typically a Type II string theory on $\mathrm{AdS}_{d+1} \times X^{9-d}$ or M-theory on $\mathrm{AdS}_{d+1} \times X^{10-d}$ and the CFT is one on $\mathbb{R}^{1,d-1}$. Given how this correspondence relates the degrees of freedom of a gauge theory in d dimensions to that of a gravitational one in one dimension higher, it is often called gauge/gravity duality and is an example of a *holographic* correspondence, or *holography* for short. Its weak

[1]See the reviews [13–25].

form then relates a strongly coupled CFT in d dimensions to a classical supergravity theory with AdS_{d+1} asymptotics, such as the Type II supergravities or 11d supergravity.

Even the weakest form of this correspondence has its uses. Indeed, standard perturbative techniques are not effective when studying strongly coupled systems. In the weak-string-coupling and large-N limit we are able to probe the strongly coupled regimes of holographic CFTs using classical supergravities. In other words, the AdS/CFT correspondence becomes an excellent non-perturbative tool for the study of CFTs, and very often recasts CFT computations into purely geometric ones within the bulk AdS space.

For holographic theories that do not admit a weakly coupled description the AdS/CFT toolset becomes paramount; this is the case of the elusive 6d $\mathcal{N} = (2,0)$ SCFTs. Indeed, these are special in many ways, one of which is their lack of a known Lagrangian description. These theories contain a multiplet (B, λ, Φ), transforming in the $(\mathbf{1}, \mathbf{4}, \mathbf{5})$ of the R-symmetry $\mathfrak{so}(5)_R$, where B is a self-dual 2-form gauge field, λ a spinor and Φ a scalar. Since B is a self-dual two-form, it is not known how to construct interacting terms for it in six dimensions. These 6d theories are also special due to their unique positioning within the landscape of supersymmetric CFTs. Thanks to Nahm's classification [26], we know that six is the maximal number of dimensions in which the superconformal algebra exists; and $\mathcal{N} = (2,0)$ is the maximal amount of supersymmetry therein. We can explain this by decomposing the Lie superalgebra into its bosonic and fermionic components. For a superalgebra to be conformal, its bosonic subalgebra must contain the spacetime conformal algebra. Above six dimensions, this is no longer possible; and while the superalgebra can be defined, it is not a suitable conformal one. These unique properties don't stop there, as many CFTs in lower dimensions can be engineered from twist-compactifications of the 6d theory on compact manifolds — the 6d $\mathcal{N} = (2,0)$ is in many ways a *master theory*[2].

The holographic correspondence described above isn't the only tool one can use to probe the strongly coupled regimes of theories. Those which display some amount of supersymmetry contain a special set of observables which are protected under certain continuous deformations. Cancellations between bosonic and fermionic degrees of freedom render some quantities one-loop exact, such as the anomaly polynomials, and others invariant under certain renormalisation-group (RG) flows, such as supersymmetric indices. The motivation behind this thesis lies in the latter — by constructing RG-flow-invariant observables, one can study the strong-coupling regime of a supersymmetric field theory from the knowledge of other supersymmetric theories along the flow.

[2] See [27] for a great introduction to the 6d $(2,0)$ theories and its class-$\mathcal{S}$ constructions of 4d theories.

For superconformal field theories in four dimensions, the superconformal index counts protected states in the quantum field theory, such that it is invariant under any continuous deformation that preserves the conformal structure [28, 29]. It contains all the information, one can extract from group theory alone, about local operators in the theory. For holographic CFTs, it is related to the microstate counting of AdS black holes [29, 30]. In essence, it is an important observable that can aid in identifying dualities between SCFTs and is equally important in the study of AdS black holes. A better understanding of this index is thus essential for the study of SCFTs, and is one of the objectives of this thesis.

Furthermore, a complete understanding of a generic quantum field theory comes with more than the knowledge of its strong-coupling behaviour. We wish to emphasise that "complete" refers to the set of all data that must be specified in order to uniquely identify a QFT: global symmetry, field content, Hilbert space, etc. One such data most QFTs are adorned with is the possible existence of extended operators. Whenever they exist, these extended operators hold important information about the global properties of the theory[3]. They also are acted on by so-called *higher-form symmetries*, leading to important constraints in their dynamics [33]. So, for instance, a complete understanding of the 6d $\mathcal{N} = (2, 0)$ SCFTs necessarily comes with the knowledge of its extended operators.

The study of arbitrary extended operators in an arbitrary QFT is, so far, out of reach. We can hope to extract interesting statements about these for more symmetric cases. Indeed, a lot more is known about conformally-invariant extended operators, or conformal defects, within CFTs. For instance, the 6d $\mathcal{N} = (2, 0)$ SCFTs possess two types of superconformal defect — one of dimension two, and the other of dimension four[4]. It is the context of AdS/CFT that one can most simply extract information about these defects. Indeed, the 6d $\mathcal{N} = (2, 0)$ theories are dual to asymptotically (locally) AdS_7 geometries[5], and introducing a defect can be done in a couple of different ways. In the probe-brane picture, another M-brane is inserted in the bulk AdS space, and extends all the way to the boundary, without affecting the geometry. In the backreacted setup, the AdS geometry breaks down, locally on the boundary, into an AdS_3 or AdS_5 foliation; the boundaries of which describe the defect location. The latter treatment will be the focus of Part II of this thesis, where we extract universal information about the defects for different families of dual AdS geometries.

[3] For instance, line defects in four dimensions can tell apart $SU(2)$ from $SO(3)$ gauge theories [31, 32].

[4] Their existence can be inferred from the possible central extensions of the 6d supersymmetry algebra, or equivalently by looking at the allowed M-brane intersections.

[5] Which can be seen as the near-horizon limit of M5-branes, which are extensions of strings to one time direction and five spatial ones.

We expand slightly upon those key concepts in the succeeding pages, and highlight the contributions brought forth in this thesis. The remainder of the thesis is structured as follows.

Part I contains the necessary background material to understand our work. Therein, Chapter 1 presents the basics of conformal symmetry and CFT. We give a brief presentation on the conformal group in dimensions greater than two, and its corresponding algebra. After introducing the embedding space formalism, we put forward the most general form of the one-, two- and three-point functions of conformal primaries. We close off the chapter by presenting the extension to defect CFTs of this embedding space formalism, together with the dCFT's various correlators. On the other hand, Chapter 2 gives a succinct presentation of 11d supergravity. The action, equations of motion and supersymmetry transformations are enumerated, and its most common BPS solutions are listed. We also briefly mention its algebra and its relation to the extended objects of M-theory. In Chapter 3 we introduce the reader to the Festuccia-Seiberg formalism, essential to our study of supersymmetric partition functions on curved spaces. We detail the specifics of the conformal supergravities in four dimensions with both eight ($\mathcal{N} = 2$) and sixteen ($\mathcal{N} = 4$) Poincaré supercharges.

Part II pertains to the study of superconformal defects of the 6d $\mathcal{N} = (2, 0)$ theories, from a holographic perspective. Chapters 4 and 5 are a reprint of our publications [8] and [7]. There, we use asymptotically AdS_7 solutions to supergravity to study universal information about the two types of superconformal defects in the 6d theories; namely their Weyl anomaly coefficients. In passing, we also identify one family of supergravity solutions as being contained within another.

Part III is dedicated to the construction of background supergravity solutions that still preserve some amount of supersymmetry on $S^3 \times S^1$. In Chapter 6 we construct such a solution that interpolates between the twisted index and a point on the moduli space of Coulomb-branch indices. We prove that the interpolation is exact via a formula which relates the supersymmetry transformation of a current multiplet to that of the background supergravity fields. In essence, this allows us to prove the equality between the indices for any 4d $\mathcal{N} = 2$ SCFT, be it Lagrangian or not. On the other hand, in Chapter 7 we propose a formulation of conformal supergravity with sixteen Poincaré supercharges in four Euclidean dimensions and identify a number of supersymmetric configurations that engineer the various twists of 4d $\mathcal{N} = 4$ theories. We successfully reproduce the half-twists, Kapustin-Witten and Vafa-Witten twists. We further our search by looking for Ω-deformations of these configurations.

Finally, Part IV, which only contains Chapter 8, presents novel results in the field of neural-network-CFTs. There, we extend the known formalism of neural-network-CFTs to defect CFTs by constructing explicit examples of correlation functions between scalar conformal primaries in the presence of a defect. We also propose a formulation which includes symmetric traceless tensors. This chapter is the subject of an upcoming paper [9].

Weyl anomaly coefficients

Placing a conformal field theory on a curved manifold generally incurs a breaking of the conformal symmetry. For an even number of dimensions, this comes as an anomaly in the tracelessness of the stress-energy tensor, $T^{\mu\nu}$, coined Weyl anomaly. The coefficients that dictate the various terms in the anomaly are called *Weyl anomaly coefficients*[6].

In two dimensions, the Weyl anomaly gives the standard result wherein the trace of the stress-energy tensor is proportional to the Ricci scalar $\langle T^\mu{}_\mu\rangle = \frac{c^{(2d)}}{24\pi}R$. The coefficient $c^{(2d)}$ is the central charge that appears in the Virasoro algebra, and is essential in characterising 2d CFTs. For unitary theories this central charge obeys a strong monotonic property under RG flows — for a CFT in the UV connected to another in the IR, $c^{(2d)}_{\text{UV}} \geq c^{(2d)}_{\text{IR}}$.[7] This is the celebrated c-theorem [36].

For four-dimensional theories, the Weyl anomaly contains more terms, since the Weyl tensor doesn't vanish,

$$\langle T^\mu{}_\mu\rangle = \frac{1}{(4\pi)^2}\left(a^{(4d)}\mathcal{E}_4 + c^{(4d)}W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} + \tilde{c}^{(4d)}\mathcal{P}_4\right), \tag{1}$$

where $\mathcal{E}_4$ is the 4d Euler density, W the Weyl tensor and $\mathcal{P}_4$ the Pontryagin density. Here too, the A-type anomaly coefficient $a^{(4d)}$ obeys an a-theorem, whereby $a^{(4d)}_{\text{UV}} \geq a^{(4d)}_{\text{IR}}$ for unitary theories [37–40].

In six dimensions, we can play the same game and look at the Weyl anomaly

$$\langle T^\mu{}_\mu\rangle = \frac{1}{(4\pi)^3}\left(a^{(6d)}\mathcal{E}_6 + b_1^{(6d)}W_{\mu\lambda\nu\rho\nu}W^{\lambda\sigma\tau\rho}W_\sigma{}^\mu{}_\tau{}^\nu + \cdots\right), \tag{2}$$

where $\mathcal{E}_6$ is the 6d Euler density and $\cdots$ denotes other terms built from the Weyl and Riemann tensors [41]. Here too the A-type anomaly coefficient $a^{(6d)}$ obeys an a-theorem [42–47].

The important takeaway here lies in the fact that the Weyl anomaly coefficients, a, b_1, c, etc, contain universal information about the CFT at hand[8], and constitute a generalisation of the notion of central charge of a two-dimensional CFT. A comprehensive analysis of these is required for a full understanding of the landscape of CFTs.

Let us now briefly outline what happens when we introduce a conformal defect, Y, in the CFT. Here too the stress-tensor will pick up a Weyl anomaly, however, there is a

[6] See [34] for a brief overview of Adam Chalabi's thesis [35] for a deeper presentation.

[7] There is an understanding in which the central charge counts the number of degrees of freedom. The monotonic behaviour is in agreement with the standard picture whereby an IR theory is given by integrating out certain UV degrees of freedom.

[8] In other words, they appear in the n-point functions of various operators, such as the stress-energy tensor.

distinct separation to its contributions — one from the ambient CFT, $T^{(\text{ambient})}$, and the other localised on the defect, $T^{(\text{defect})}$. Schematically, we write

$$T = T^{(\text{ambient})} + \delta(\mathrm{Y}) T^{(\text{defect})}, \tag{3}$$

where $\delta(\mathrm{Y})$ is a Dirac delta function that localises to the defect submanifold. Naturally, the defect-localised contribution displays its own set of Weyl anomaly coefficients, however, these are generally more numerous than their ambient counterparts.

For instance, a 2d conformal defect embedded in a 6d ambient CFT engenders a Weyl anomaly of the form

$$\langle T^{(\text{defect})\mu}{}_{\mu} \rangle = \frac{1}{24\pi} \left(a_{\mathrm{Y}} \tilde{R} + d_1 \mathring{\Pi}^2 + d_2 W_{ab}{}^{ab} \right) + \cdots, \tag{4}$$

where $\tilde{R}$ is the Ricci scalar of the induced metric on Y, $\mathring{\Pi}$ is the traceless second fundamental form and W is the pullback of the Weyl tensor onto Y. A similar expression for a four-dimensional conformal defect exists, but is a lot more complicated [34].

The defect Weyl anomaly coefficients also appear in various observables of the defect CFT (dCFT), and as such contain universal information about the dCFT. Their study is essential in understanding the landscape of dCFTs.

> Part of this thesis' aim lies in calculating a subset of these coefficient for two- and four-dimensional defect within the 6d $\mathcal{N} = (2, 0)$ SCFTs of holographic type. This is the subject of Part II and appears in our published works [7, 8].

Holographic entanglement entropy

In the study of holographic theories, for which a static, asymptotically AdS filling exists, the Ryu-Takayanagi formula [48–50] poses a relation between the entanglement entropy of a spatial region within the CFT and a Plateau, or minimisation, problem for a surface within the bulk. In other words, it recasts a field theory computation (at large-N) into a purely geometric one. The entanglement entropy calculated in this way is coined *holographic entanglement entropy*.

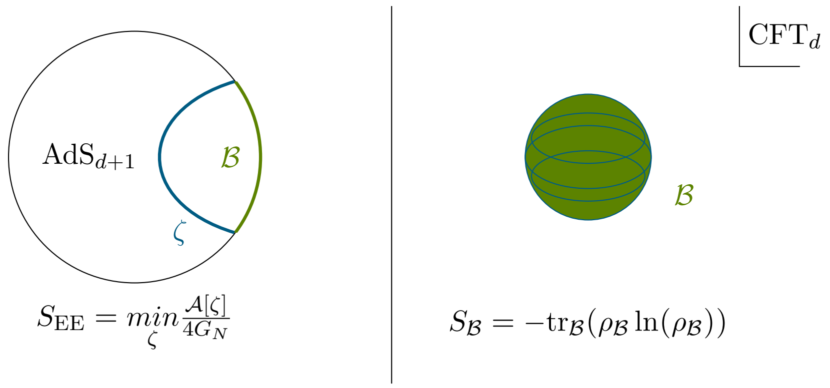


FIGURE 2: The Ryu-Takayanagi prescription. On the right-hand-side, a spatial region in the CFT, $\mathcal{B}$, together with its entropy $S_{\mathcal{B}}$. On the left-hand-side, the asymptotically AdS bulk with the minimal surface ζ, homologous (i.e. shares the same boundary) to $\mathcal{B}$ on the boundary. The holographic entanglement entropy is equal to the entanglement entropy $S_{\text{EE}} = S_{\mathcal{B}}$.

Naturally, due to the infinite volume of AdS, this entanglement entropy is divergent and one must regulate it. In that effect, let us introduce a cutoff, ϵ, in the holographic direction, such that as ϵ tends to zero, we approach the boundary of AdS, ∂AdS_{d+1}. Furthermore, if $\mathcal{B}$ is a spatial ball of radius l within the d-dimensional CFT, then the holographic entanglement entropy admits an expansion in terms of ϵ of the form

$$S_{\text{EE}} = \begin{cases} p_1 \left(\frac{l}{\epsilon}\right)^{d-2} + \cdots + p_{d-2}\frac{l}{\epsilon} + p_{d-1} + o(1) & d\text{: odd,} \\ p_1 \left(\frac{l}{\epsilon}\right)^{d-2} + \cdots + p_{d-3}\left(\frac{l}{\epsilon}\right)^2 + q \ln\left(\frac{l}{\epsilon}\right) + o(1) & d\text{: even.} \end{cases} \tag{5}$$

In the above, most coefficients of the Laurent polynomials are renormalisation-scheme dependent. In other words, for any change in the cutoff parameter $\epsilon = \epsilon(\eta)$, these would see their numerical value affected. Those which are not[9], p_{d-1} and q, contain information about the CFT specifics. For example, in two dimensions the universal coefficient q contains information of the central charge of the Virasoro algebra, $S_{EE} = \frac{c}{3} \ln\left(\frac{l}{a}\right)$.

More generally, the universal coefficients contain information about the Weyl anomaly coefficients of the given holographic CFT.

[9]There is a sense in which the change in cutoff must be "sufficiently" regular. One can see that changing ϵ via a logarithmic transformation would affect the universal coefficients, which is no desirable.

Let us now illustrate how this construction is affected by the insertion of a defect within the CFT. Let Y denote an arbitrary such (conformal) defect. Due to the potential interactions between Y and the ambient CFT, the entanglement entropy of the region $\mathcal{B}$ will, in general, contain information about both the defect and the ambient theory. Untangling their contributions is generally regarded as a difficult problem. However, in simple symmetric cases we have a stronger handle on things — as the dependence within the universal coefficients is known.

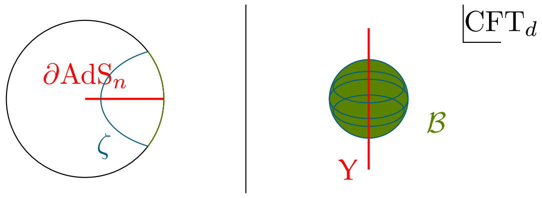


FIGURE 3: The Ryu-Takayanagi prescription with a defect insertion. The conformal defect Y manifests itself as AdS_n fibre within the bulk AdS geometry.

Take for example a 2d conformal defect within a 6d holographic CFT. We consider the entanglement entropy of a spatial ball $\mathcal{B}$, of radius l, centred around the defect Y. In this case, the universal coefficient q is a simple linear combination of a defect, q_{Y}, and an ambient term, $q_{6\mathrm{d}}$, [51]; and one can isolate the defect contribution by performing so-called *vacuum subtraction*. In other words, we evaluate the entanglement entropy S_{EE} both with and without the defect Y, and take their difference,

$$q_{\mathrm{Y}} = l\frac{d}{dl}(S_{\mathrm{EE}}[\mathrm{Y}] - S_{\mathrm{EE}}[]). \tag{6}$$

In Chapter 4, we calculate q_{Y} for a family of defects described by an AdS_3 foliation of AdS_7 [8]. In this codimension-four case, the defect contribution to the universal coefficient is given as

$$q_{\mathrm{Y}} = \frac{1}{3}\left(a_{\mathrm{Y}} - \frac{3}{5}d_2\right) \ , \tag{7}$$

where a_{Y} and d_2 are defect Weyl anomaly coefficients of A- and B-type respectively. We perform a similar computation for codimension-two defects in Chapter 5, where this time [7]

$$q_{\mathrm{Y}} = -4\left(a_{\mathrm{Y}} + \frac{1}{10}d_2\right) \ . \tag{8}$$

Supersymmetric indices

Given a supersymmetric field theory, one defines a set of observables called its supersymmetric indices. These are given as partition functions of the theory on a compact manifolds, $\mathcal{M}$. The standard example is that of the Witten index [52], for which $\mathcal{M} = T^d$, the d-dimensional torus. This index counts the supersymmetric vacua of the theory, and is rigid under renormalisation group (RG) flows. Its trace definition is

$$Z_{T^d} = \mathrm{Tr}_{T^{d-1}} \left((-1)^F e^{-\beta H} \right), \tag{9}$$

where the trace is performed over states on T^{d-1}, F is the fermion number and H is the Hamiltonian. Special cancellations between fermionic and bosonic states above the vacuum energy render the index independent of β.

Another important example is that of the partition function of SCFTs on $S^{d-1} \times S^1$. The index there defined is called the superconformal index (SCI) [28, 29]. It counts supersymmetric states on S^{d-1}, or equivalently the set of local BPS (short) multiplets on $\mathbb{R}^d$ that do not recombine into larger ones. This particular counting renders the partition function independent of exactly marginal couplings, and thus, rigid under RG flows[10]. Let $\mathcal{Q} := \mathcal{Q}_-$ be one of the Poincaré supercharges in flat space, and $\mathcal{Q}^\dagger := \mathcal{S}_+$ the conjugate conformal supercharge. The SCI is equivalently defined as the trace

$$\mathcal{I} = \mathrm{Tr}_{S^{d-1}} \left((-1)^F x^{\frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\}} \right), \tag{10}$$

over the set of states on S^{d-1}.[11] Truthfully, the correspondence between the trace definition above and the partition function on $S^{d-1} \times S^1$ isn't exact. The latter is given by the trace formula, multiplied by an exponential factor that depends on the supersymmetric Casimir energy [53, 54].

The SCI in four dimensions displays numerous properties that render its study of utmost importance. Not only does it contain a vast amount of information about the theory's local BPS states, it is also related to many important indices, such as the Schur and Coulomb-branch indices [5]. In this setting, we understand that the SCI of four-dimensional $\mathcal{N} = 2$ SCFTs possesses information about the Coulomb and Higgs branches of the theory. It is, thus, a shame to realise how difficult it is to evaluate the SCI, generically; doubly so for non-Lagrangian theories for which no Lagrangian fixed point is known.

[10] It is actually invariant under any continuous deformation that preserves a given sub-Lie superalgebra of the full global supersymmetry. Notably, the R-symmetry mustn't be anomalous. We will not detail why this is true, but will instead refer the reader back to the original construction [29].

[11] One can further refine the counting in the trace by introducing fugacities for the various global charges that commute with $\mathcal{Q}$.

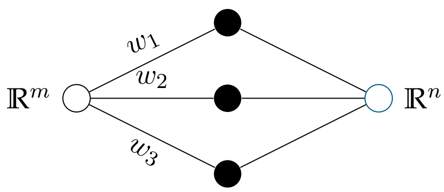
It is the aim of Part III of this thesis to facilitate such constructions. Indeed, we utilise the coupling to background conformal supergravity to construct the partition function of a generic 4d $\mathcal{N}=2$ SCFT on $S^3 \times S^1$, which engineers the superconformal index with a particular set of additional fugacities. We further show, in Chapter 6, that this index is equivalent to the twisted index of the same theory on $S^3 \times S^1$.

In Chapter 7, we search for 4d $\mathcal{N}=4$ conformal supergravity backgrounds that engineer all three twists of those theories, together with their Ω-deformations.

nn-dCFTs

For all intents and purposes, we take a neuron to be a map between two spaces, say $\mathbb{R}^m$ and $\mathbb{R}^n$, $f : \mathbb{R}^m \to \mathbb{R}^n$. Importantly, we allow certain operations between these — composition, addition and scalar multiplication. Then, a *neural network* is a function build from neurons using those compositions. It is standard to represent a network using a diagram where neurons are nodes and the operations are lines.

Take for instance three neurons (i.e. three arbitrary functions) f_i from $\mathbb{R}^m$ to $\mathbb{R}^q$. We decide to add these with weights w_i and compose the sum inside a fourth neuron f_o, which is a function from $\mathbb{R}^q$ to $\mathbb{R}^n$. Mathematically, this is simply the function $f = f_o \circ (\sum_i w_i f_i)$, however it is better represented by the diagram below.



The true power of neural networks is made apparent by the universal approximation theorems (UATs) [55–59]. For a given set of neurons (i.e. basis functions), the UATs state that in a suitable limit where the number of neurons tends to infinity, the neural network can approximate any function in a given space of functions. This could be the set of smooth functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ or another function space, based on the particulars of the network and how the limit is taken.

Thanks to this property, one can envision writing the path integral of a theory in terms of infinitely big neural networks, and integrating over all possible basis neurons. This way of dealing with QFTs is known as the neural network-field theory (nn-FT) correspondence [60].

However, in 2024, Halverson, Naskar and Tian showed that for CFTs this correspondence holds regardless of whether one takes the infinite limit or not [61]. In other words, they were able to construct families of CFTs using only finite-dimensional integrals.

The power behind this formalism comes from the fact the one can easily engineer new CFTs, by starting from a set of nnCFTs and combining them into a larger network. Let us illustrate this with an example. Let $\Phi^i_{\theta_i}(X)$ be a family of neural-network architectures, with parameters θ_i distributed sampled from a distribution $\mathcal{P}_i(\theta)$. Each of these can be thought of as a family of neurons with parameters θ_i. The weighted sum also defines a neural-network CFT,

$$\varphi_{\theta_i, w_i}(X) = \sum_i w_i \Phi^i_{\theta_i}(X) \tag{11}$$

Conformal primary	
$\phi(x) \mapsto \lambda^{-\Delta}\phi(x)$	$\Phi(X) \mapsto \lambda^{-\Delta}\Phi(X)$
Partition function	
$Z[J] = \int \mathcal{D}\phi e^{-S[\phi]} e^{\int d^d x J(x)\phi(x)}$	$Z[J] = \int d\theta \mathcal{P}(\theta) e^{\int d^d x J(x)\Phi(x)}$
Correlators	
$\langle\phi(x)\phi(y)\rangle$	$\mathbb{E}[\Phi(x)\Phi(y)]$

Table 1: A summary of the neural-network-CFT (nnCFT) data, and their standard CFT counterparts. ϕ is a conformal primary, and obeys the conformal primary condition displayed above. In the nnCFT, this translate to requiring a homogenous neuron Φ. The partition function is given as a finite integral over neuron parameters, θ, with distribution $\mathcal{P}(\theta)$. The distribution plays the role of the action in the standard CFT. Correlators are given as standard expectation values of probability variables.

where w_i are distributed according to $\mathcal{P}(w)$. Given that each $\Phi^i_{\theta_i}$ defines a nnCFT, the combined network φ_{θ_i,w_i} also does. The correlation functions of it obey the standard CFT constraints (crossing symmetry, etc).

Given the infinitely many ways of combining neurons into networks, the nnCFT formalism becomes a powerful tool to generate arbitrary CFT data from finite-dimensional integrals.

> We dedicated Part IV of this thesis to the extension of this formalism to the defect case. In our upcoming paper [9], summarised in Chapter 8, we define neural-network-dCFTs and show how their data obey the standard constraints due to defect conformal symmetry and the existence of defect operator product expansions.

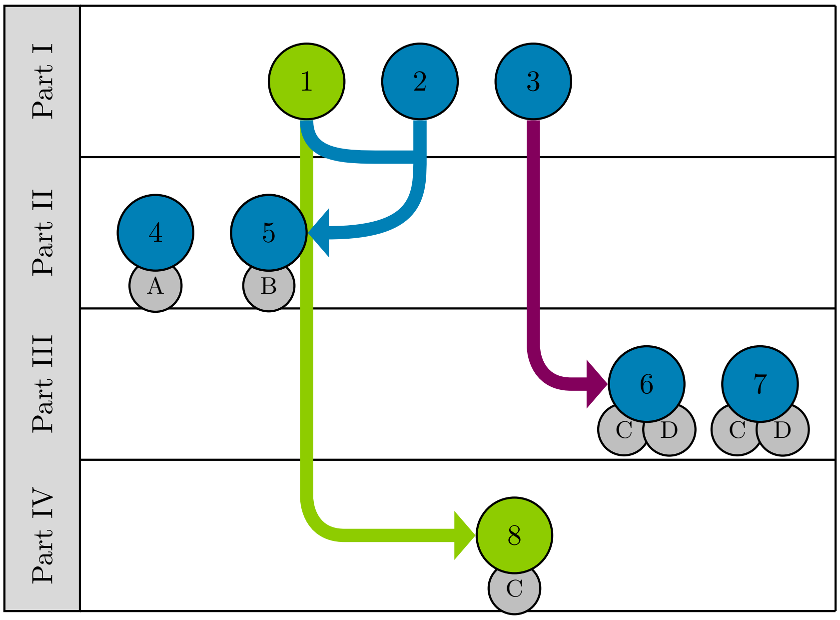


FIGURE 4: A flowchart illustrating the chapter dependencies within this thesis. Chapters marked in blue mainly contain supergravity material. Chapters marked in green are mainly focused on conformal field theory and related topics. Finally, the appendices for each chapter are indicated below them.

Part I

Background Material

Chapter 1

Conformal Field Theory

> Long story made short...
>
> *Andy O'Bannon*

The concept of conformal field theory (CFT) is central to this thesis' content. Not only that, it is also a central concept in quantum field theory (QFT). They appear as the end-point of renormalisation group (RG) flows between quantum field theories. By identifying the set of relevant deformations of a given CFT, one can study the conformal manifold these span. Knowledge of these RG flows turn out to be powerful tools in the study of QFTs. By identifying structures that are independent of RG flows (or monotonic), one can study the conformal limit and infer results on the QFTs that flow to it.

Conformal symmetry is an extension of the Poincaré symmetry, in that it adds invariance under rescaling and so-called special conformal transformations (SCTs). Given the structure of the conformal group, an important question that arises is whether the SCTs come for free. In other words, given a scale-invariant theory, is it also conformally invariant? In general, the answer is no. However, requiring unitarity on top of scale-invariance leaves us with a completely different outcome. In the late 80s, Zamolodchikov and Polchinski [36, 62] proved this to be true in two dimensions. However, asking whether this holds in dimensions greater than two does not give conclusive answers [63]. There has been some recent work towards answering that question, using holography. For instance [64] were able to prove that Scale without Conformal Invariance (SwCI) for certain field theories is only possible if the Null Energy Condition (NEC) is violated. See [65] for an excellent overview of the latest results (as of 2014).

With the avenue of AdS/CFT [10], supersymmetric conformal field theories (SCFTs) have also found a renewed interest. Being the low-energy effective field theories of

D-branes, and M-branes in M-theory, their study opened a surge of results in holography (see the standard reviews [13–25] for more details). In lower dimensions the move away from the large 't Hooft-limit may be permissible in certain settings [66], but is a lot less attainable in higher dimensions. Nevertheless, knowledge of supergravity solutions can inform us about the strong-coupling CFTs, a direction we followed in [7, 8] (reproduced here as Chapters 4 and 5).

In this chapter we will cover some basic concepts of conformal symmetry and CFT. The reader already familiar with those ideas can skip this entirely and move to Part II and beyond. In Section 1.1 we give a brief presentation of the conformal group of $\mathbb{R}^{1,d-1}$. Section 1.2 is dedicated to its algebra, and its representation on the space of fields, after which Section 1.3 lists the standard form for their correlators in a CFT. Section 1.4 is a brief sketch of the important aspects of the operator product expansion in these CFTs. Necessary for an understanding of Chapter 8, Section 1.5 presents the embedding space formalism for Euclidean CFTs, and lists the various form the correlators take therein. Finally, in Section 1.6 we extend all previous concepts to the defect-CFT case, where an extended operator breaks conformal symmetry down to a subgroup.

All the content from this chapter, and more, can be found in the reviews [67–73].

1.1 The Conformal Group

The elementary introduction in this section presents a consistent definition of the conformal group of flat space, with either Euclidean or Minkowskian signature, $\mathbb{R}^{t,s}$, where $(t, s) = (0, d)$ or $(t, s) = (1, d-1)$. Furthermore, we show that on the two-dimensional planes the conformal group structure depends greatly on the spacetime signature — on the Euclidean plane, $\mathbb{R}^{0,2}$, it is finite-dimensional while its sibling, the conformal group of the Minkowski plane, is infinite-dimensional. Most of the material presented here can be found in its original form in [68, 70].

Definition 1.1. Conformal Transformation Let $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ be two (pseudo-) Riemannian (smooth) manifolds of dimension d and let $U \subset \mathcal{M}$, $V \subset \mathcal{N}$, be open subsets of $\mathcal{M}$ and $\mathcal{N}$ respectively. A smooth mapping $\varphi : U \longrightarrow V$ of maximal rank is called a *conformal transformation*, or *conformal map*, if there is a smooth function $\sigma : U \longrightarrow \mathbb{R}$ such that

$$\varphi^* h = e^{2\sigma} g. \tag{1.1}$$

The factor of two in the definition is customary and allows us to simplify various equations down the line. As a first remark, we notice that for any metric g and h, a conformal map only exists if g and h have the same signature, and coincides with a local isometry when $\sigma = 0$. Secondly, we can immediately make connection with our intuitive picture of what a conformal map is supposed to do. Indeed, any such map is allowed to move points of $\mathcal{M}$ around only up to a local rescaling[1], hence the factor of $e^{2\sigma}$. Whenever the manifolds are Riemannian, i.e. g and h are positive definite, this local rescaling property preserves the angle between any two tangent vectors at all points in the manifolds. In more detail, let $X, Y \in T_p\mathcal{M}$ be two vectors on $\mathcal{M}$. The angle between these is defined via the inner-product induced by g,

$$\omega_g(X,Y) = \frac{g(X,Y)}{\sqrt{g(X,X)g(Y,Y)}}. \tag{1.2}$$

Under the action of a conformal map φ, these vectors are mapped to $T\varphi(X), T\varphi(Y) \in T_{\varphi(p)}\mathcal{N}$. The angle between them, however, is identical

$$\omega_h(T\varphi(X), T\varphi(Y)) = \frac{h(T\varphi(X), T\varphi(Y))}{\sqrt{h(T\varphi(X), T\varphi(X))h(T\varphi(Y), T\varphi(Y))}} = \omega_g(X,Y), \tag{1.3}$$

by definition of φ. This angle-preserving property justifies the nomenclature of *con* – the same and *formal* – shape.

Important to our use case are the maps between open subsets of $\mathcal{M}$ alone. The set of such maps for $\sigma = 0$ are precisely the isometries of $\mathcal{M}$, i.e. the set of "symmetries" of the manifold. To all those maps that are continuously connected to the identity there corresponds an infinitesimal map between the tangent spaces, called the algebra of isometries. Furthermore, these are classified by the set of Killing vectors on $\mathcal{M}$. Whenever we allow for non-zero conformal factor σ, we talk about conformal Killing vectors.

In the same way that the set of isomorphisms of $\mathcal{M}$ forms a group, we can define a notion of *conformal group* of $\mathcal{M}$ by specialising to a particular subset of conformal maps. In practice, we are also interested in conformal maps between manifolds with different topologies. For instance, the Euclidean plane $\mathbb{R}^2$ and the two-sphere S^2 are "one conformal transformation away from each other", in the sense that one is the conformal compactification of the other. It turns out that the definition of a conformal group which is the most relevant to Physics is that which places these conformal compactifications are the forefront.

Definition 1.2. Conformal Group The *conformal group* $\mathrm{Conf}(\mathcal{M})$ of a manifold $\mathcal{M}$ is the connected component containing the identity in the group of conformal diffeomorphisms of the conformal compactification of $\mathcal{M}$ [70].

[1]This is to be contrasted with Weyl transformations. A given theory might be invariant under scale transformations but not conformal ones [74].

An important distinction appears when looking at different signatures. For example, the two-planes are either of the Euclidean type of the Minkowskian type. Indeed, these planes behave very differently under conformal compactification. The Euclidean plane $\mathbb{R}^{0,2}$ has a natural conformal compactification to S^2. By our definition above, we find that the set of diffeomorphisms that preserve the metric on the two-sphere up to an overall scale is isomorphic to the Möbius group[2]

$$Conf(\mathbb{R}^{2,0}) \cong SO(3,1) \cong \mathrm{PSL}(2,\mathbb{C}). \tag{1.4}$$

However, this is not the case for the Minkowski plane $\mathbb{R}^{1,1}$. As the two-sphere cannot be given a Lorentzian metric, it cannot be a suitable conformal compactification of $\mathbb{R}^{1,1}$. The natural conformal compactification turns out to be a product of two circles, $S^1 \times S^1$. The corresponding conformal group is then

$$\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1), \tag{1.5}$$

where $\mathrm{Diff}_+(S^1)$ is the group of orientation-preserving diffeomorphisms of the one-sphere. The corresponding infinite-dimensional algebra is known as the Witt algebra and can be engineered as the complexification of the algebra of smooth vector fields on S^1, $\mathrm{Lie}(\mathrm{Diff}_+(S))_{\mathbb{C}} \cong \mathrm{Vect}(S^1)_{\mathbb{C}}$. Upon quantization this algebra gets deformed to its unique central extension, known as the Virasoro algebra.

On the flip side, when considering spacetimes of the form $\mathbb{R}^{t,s}$, with $t+s>2$, the corresponding conformal group is always isomorphic to $SO(t+1,s+1)$. The cases of interest here are $(t,s)=(0,d)$ and $(t,s)=(1,d-1)$, with $d>2$, whose groups are $SO(1,d+1)$ and $SO(2,d)$ respectively.

Having defined a consistent notion of conformal group, and identified its realisation for $\mathbb{R}^{t,s}$, let us now detail its algebraic structure in dimensions larger than two. For those interested in the two-dimensional case, we recommend the standard textbooks [68, 70]. For an approach more focused on vertex operator algebras, we recommend the excellent introduction by Gaberdiel [69].

1.2 The Conformal Algebra

As discussed previously, the set of conformal maps that are connected to the identity form the *conformal algebra* — the Lie algebra associated to the conformal group of the manifold $\mathcal{M}$. Since we will only focus on conformal transformations of $\mathbb{R}^{t,s}$, with $t+s>2$, the conformal algebra will always be given by the standard Lie algebra $\mathfrak{so}(t+1,s+1)$. Furthermore, we will only be interested in the $(t,s)=(0,d)$ and

[2]We ignore here the semi-direct product which generates translations.

$(t,s) = (1, d-1)$ cases; but will keep the notation general to facilitate our presentation. Let us see how to construct this algebra, starting from the set of conformal Killing vectors.

Definition 1.3. Conformal Killing Vector Let $(\mathcal{M}, g)$ be a (pseudo-) Riemannian (smooth) manifold of dimensions d. A vector field $X \in \Gamma(T\mathcal{M})$ is *conformal Killing* if it obeys

$$\mathcal{L}_X g = \frac{2}{d}\operatorname{div}(X) g, \tag{1.6}$$

where $\mathcal{L}_X$ is the Lie derivative along X and $\operatorname{div}(X)$ is its divergence.

If ∇ denotes the Levi-Civita connection associated to g, then (1.6) can be written in terms of it

$$\mathcal{L}_X g = \frac{2}{d}\nabla_\mu X^\mu g. \tag{1.7}$$

It isn't hard to notice that any vector field whose divergence vanishes is also conformal Killing, by definition.

Allow us to apply Definition 1.3 to $\mathbb{R}^{t,s}$, for which the metric is the diagonal metric $\eta = \operatorname{diag}(-1, \ldots, -1, 1, \ldots, 1)$, with t instances of -1 and s of 1. In this simple case the conformal Killing vector equation simplifies to a constraint on the divergence of X (provided $t + s > 2$),

$$\partial_\mu \partial_\nu \operatorname{div}(X) = 0. \tag{1.8}$$

This further implies that each component of X is at most quadratic in the coordinates. Let $a, b \in \mathbb{R}^{t+s}$ be constant vectors, $\lambda \in \mathbb{R}$ a constant scalar and $\omega \in \mathfrak{so}(p,q)$ a global Lorentz transformation. The most general solution to (1.8) may be written as follows

$$X^\mu = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + 2(b \cdot x)x^\mu - (x \cdot x)b^\mu. \tag{1.9}$$

Note that in the expression above, the $\cdot$ notation refers to the inner-product induced by η, i.e. $a \cdot b = \eta(a,b) = a^\mu b^\nu \eta_{\mu\nu}$. The term containing the constant vector a is the familiar Killing vector that generates translations. The first linear term in x generates Lorentz transformations. The corresponding conformal map is $\varphi(x)^\mu = \Lambda^\mu{}_\nu x^\nu$, where $\Lambda \in SO(t,s)$. Together these two form the familiar Poincaré algebra of flat space $ISO(t,s) = SO(t,s) \ltimes \mathbb{R}^{t,s}$. However, we see that conformal transformations allow for more than just the Poincaré algebra. Indeed, the second linear term corresponds to dilatations[3], the conformal map of which is $\varphi(x)^\mu = e^\lambda x^\mu$. The final, quadratic, term

[3] Some people prefer the word *dilation*. The underlying idea is identical, however.

generates so-called *special conformal transformations* (SCTs), whose conformal maps are

$$\varphi(x) = \frac{x - (x \cdot x)b}{1 - 2x \cdot b + (x \cdot x)(b \cdot b)}. \tag{1.10}$$

These SCTs can be constructed via a set of translations sandwiched in between coordinate inversions, $x \mapsto 1/x$[4]. Altogether, the Poincaré subgroup together with dilatations and SCTs forms the conformal group $SO(t+1, s+1) \ltimes \mathbb{R}^{t,s}$ detailed in the previous section. See Table 1.1 for a summary of its generators.

Name	**Conf. Killing vec.**	**Generators**
Translation	a^μ	P_μ
Lorentz Boost	$\omega^\mu{}_\nu x^\nu$	$J_{\mu\nu}$
Dilatation	λx^μ	D
Special Conformal Transformation	$2(b \cdot x)x^\mu - (x \cdot x)b^\mu$	K_μ

TABLE 1.1: A detailed breakdown of the different generators of the conformal group of $\mathbb{R}^{t,s}$. For each generator, the corresponding conformal Killing vector is included.

Action on smooth fields

Of interest to physicists, is the action of such a group on the space of fields (say of smooth functions from $\mathbb{R}^d$ to $\mathbb{R}^J$) and its irreducible representations. We will try and keep the following discussion as general as possible by considering fields $\phi(x)$ valued in some arbitrary representation of the Lorentz group (i.e. with arbitrary spin). By that token, given a field $\phi_a(x)$, where the index a spans the dimensions of said representation, its conformal transformation reads

$$\phi'_a(x) = \sum_b \rho(\mathcal{O})_{ab}\phi_b(x), \qquad \mathcal{O} \in \mathrm{Conf}(\mathbb{R}^{1,d-1}), \tag{1.11}$$

where ρ labels the field representation under which ϕ_a transforms. The infinitesimal version of that transformation above can be written in terms of the generators of $\mathrm{Conf}(\mathbb{R}^{t,s})$, $T^i_{\rho ab}$, in the ρ representation,

$$\begin{aligned} \rho(\mathcal{O})_{ab} &= e^{i\alpha_i T^i_{\rho ab}}, \\ \phi'_a(x) &= \phi_a(x) + i\sum_{i,b} \alpha_i T^i_{\rho ab}\phi_b(x) + o(\alpha). \end{aligned} \tag{1.12}$$

For illustrative purposes, let us consider such a field $\phi(x)$, which sits in the trivial representation of the Lorentz group together with zero scaling dimension. In other

[4]While inversions are not strictly speaking part of the group of conformal transformations, there are still useful for generating these SCTs.

words, we take $\phi(x)$ to be a scalar field. Acting on this field with the different conformal Killing vectors in Table 1.1 allows us to determine a functional form for the generators, in this trivial representation of the conformal group. Starting with translations along a^μ and Lorentz transformations give the standard generators of $ISO(t,s)$,

$$\phi'(x) = \phi(x) + a^\mu \partial_\mu \phi(x) + o(a) \qquad \Rightarrow \qquad P_\mu = -i\partial_\mu, \tag{1.13a}$$

$$\phi'(x) = \phi(x) + \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) + o(\omega) \qquad \Rightarrow \qquad J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu). \tag{1.13b}$$

The dilatation and SCTs, on the other hand, lead to the following expressions for their generators

$$\phi'(x) = \phi(x) + x^\mu \partial_\mu \phi(x) + o(\lambda) \qquad \Rightarrow \qquad D = -i(x\cdot\partial), \tag{1.13c}$$

$$\begin{aligned}\phi'(x) = \phi(x) + 2(b\cdot x)x^\mu \partial_\mu \phi(x) - x^2 b^\mu \partial_\mu \phi(x) + o(b)&\\ \Rightarrow \qquad K_\mu = -i(2x_\mu(x\cdot\partial) - (x\cdot x)\partial_\mu).&\end{aligned} \tag{1.13d}$$

Naturally, this way of representing the generators closes to the conformal algebra on the space of smooth functions (scalar fields). One can verify that their commutation relations are indeed extensions of the Poincaré algebra on $\mathbb{R}^{t,s}$.

$$\left.\begin{aligned}&\left.\begin{aligned}&[P_\rho, J_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)\\ &[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\nu\rho}J_{\mu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma} - \eta_{\nu\sigma}J_{\mu\rho})\end{aligned}\right\}\text{Poincaré}\\ &[D, P_\mu] = iP_\mu\\ &[D, K_\mu] = -iK_\mu\\ &[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - J_{\mu\nu})\\ &[K_\rho, J_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu)\end{aligned}\right\}\text{Conformal} \tag{1.14}$$

All other commutators are trivial.

To make more explicit the $\mathfrak{so}(t+1, s+1)$ structure of the conformal algebra we redefine certain generators as follows:

$$\bar{J}_{\mu\nu} = J_{\mu\nu}, \qquad \bar{J}_{-1,\mu} = \frac{1}{2}(P_\mu - K_\mu), \tag{1.15a}$$

$$\bar{J}_{-1,0} = D, \qquad \bar{J}_{0,\mu} = \frac{1}{2}(P_\mu + K_\mu). \tag{1.15b}$$

Following this redefinition, all the commutation relations which don't involve translations can be summarised in this single commutator

$$\begin{gathered}[\bar{J}_{ab}, \bar{J}_{cd}] = i(\bar{\eta}_{ad}\bar{J}_{bc} + \bar{\eta}_{bc}\bar{J}_{ad} - \bar{\eta}_{ac}\bar{J}_{bd} - \bar{\eta}_{bd}\bar{J}_{ac}),\\ \bar{\eta} = \mathrm{diag}(\underbrace{-1,\dots,-1}_{t+1},\underbrace{1,\dots,1}_{s+1}),\end{gathered} \tag{1.16}$$

which is the standard commutator for the generators of $\mathfrak{so}(t+1, s+1)$. Combining (1.16) with the commutators involving P_μ, we obtain the full conformal algebra $SO(t+1, s+1) \ltimes \mathbb{R}^{t,s}$, as alluded to earlier. This concludes our brief overview of the conformal algebra of $\mathbb{R}^{t,s}$.

1.3 Correlation Functions

Having discussed the conformal group and its algebra in sections 1.1 and 1.2, let us now turn to their application in conformally-invariant quantum field theories, or CFTs. We will assume the reader is familiar enough with the language of QFT and move straight to illustrating the power conformal symmetry has on constraining the form of the CFTs' correlators.

The irreducible representations of the conformal algebra are spanned by so-called *conformal primary operators*. Note that our definition of conformal primaries coincides with the notion of *quasi-primaries* in 2d CFTs, however, we will not need to make that distinction as we work solely in dimensions greater than two. These conformal primaries come with a definite scaling dimension, and commute with the generator of special conformal transformations. Looking back at the conformal algebra $\mathfrak{so}(t+1, s+1)$ in (1.14), we see that acting on a conformal primary with P_μ raises the scaling dimension by one. One can start from a conformal primary and act infinitely many times with P_μ to get a tower of operators, with ever increasing scaling dimensions. These operators are called *descendents* of the conformal primary.

Definition 1.4. Conformal Primary Field A *conformal primary field* is a field $\phi_a(x)$ (in a given representation of the Lorentz group) which commutes with the generators of special conformal transformations, K_μ and is an eigenfunction of the dilatation generator at the origin,

$$[K_\mu, \phi_a(0)] = 0, \tag{1.17a}$$

$$[D, \phi_a(0)] = -i\Delta\phi_a(0). \tag{1.17b}$$

Whenever the above holds, Δ is called the conformal scaling dimension of $\phi_a(x)$, or its scaling dimension for short.

Also note that this definition is not at odds with the Wigner method of classifying irreducible representations according to the eigenvalue of the Casimir operators. The eigenfunctions of these operators, constructed by contracting the generators $\bar{J}_{ab}$ in (1.15), are the conformal primaries and their descendents [75, 76].

To illustrate Definition 1.4 further, let us consider the case of a scalar conformal primary, $\phi(x)$. In general, acting on this field with the dilatation generator gives (obtained by shifting $\phi(0)$ by x, $\phi(x) = e^{ix\cdot P}\phi(0)e^{-ix\cdot P}$)

$$D\phi(x) = (-ix^\mu\partial_\mu - i\Delta)\phi(x), \tag{1.18}$$

where Δ is the scaling dimension of the field. Note that we used the short-hand notation where D is actually the dilatation generator in the corresponding irreducible representation: $\rho_{\text{scalar}}(D)$. Under an arbitrary conformal map $x \mapsto x'$, the field ϕ necessarily transforms following the pattern

$$\phi(x) \mapsto \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d}\phi(x), \tag{1.19}$$

where $\frac{\partial x'}{\partial x}$ is the Jacobian, and $d = t + s$ is the spacetime dimension.

These properties alone are sufficient to constrain the functional form of correlators involving conformal primaries. For correlation functions involving up to two conformal primaries, the symmetries fully determine the function, up to a constant. For correlation functions that involve more than three conformal primaries, we find an arbitrary function of the cross-ratios. Importantly, by studying the operator product expansion between conformal primaries, it is possible to fully determine the CFT data using the four-point functions alone. All higher-order correlators can then be written in terms of this data, known as the conformal block expansion. We give a more detailed overview of this construction and its properties in section 1.4.

Unitary bounds

In any sensible unitary CFT, the Hamiltonian spectrum should be bounded by below. This places a constraint on the conformal primaries and their tower of descendents, by imposing that they all have a non-negative norm. This requirement is translated into a lower bound on the conformal primary scaling dimensions, Δ, which may depend on the spin of the operator [77–81]. This is known as the *unitary bound* on the CFT data. The corresponding notion in Euclidean CFTs is that of reflection positivity. See [82] for an excellent mathematical treatment in general d.

For brevity, we replicate only two of the bounds here (see [83] for detailed references and the fermionic bound)

$$\begin{aligned} &\Delta \geq \frac{d-2}{2}, && \text{scalar,} \\ &\Delta \geq l + d - 2, && \text{symmetric traceless, } l \geq 1. \end{aligned} \tag{1.20}$$

Scalar primaries that saturate the bound are precisely the free bosons.

1.3.1 One-point functions

The simplest kind of correlation function, involving only one insertion of conformal primaries, doesn't require much attention. Indeed, conformal symmetry automatically forces this correlator to vanish, for all conformal primary ϕ. Nevertheless, for completeness let us write this result for our scalar primary,

$$\langle \phi(x) \rangle = 0. \tag{1.21}$$

As we will soon see in section 1.6, this no longer holds when considering defect CFTs, for which the conformal symmetry is broken down to one of its subgroups.

1.3.2 Two-point functions

Moving on to the correlation function between two conformal primaries, with scaling dimensions Δ_1 and Δ_2, we get our first non-trivial result. The functional form of this correlator is fully determined for fields in any representation of the Lorentz group. However, as its expression can be quite complicated in general, we will only detail the spin-less case here. The full spin-full case will be detailed in section 1.5, where the embedding space formalism will allow us to greatly simply notations.

For ϕ_1, ϕ_2, two scalar primaries with scaling dimensions Δ_1 and Δ_2 respectively, we find

$$\langle \phi_1(x_1)\phi_2(x_2) \rangle = \delta_{\Delta_1,\Delta_2} \frac{c_{12}}{x_{12}^{2\Delta_1}}, \tag{1.22}$$

where $x_{12}^2 = (x_1 - x_2) \cdot (x_1 - x_2)$ denotes the squared distance between the two insertions. Note that the factor of Kronecker delta on the right-hand-side constrains this correlator to be non-vanishing if and only if the scaling dimensions of ϕ_1 and ϕ_2 are identical.

1.3.3 Three-point functions

The three-point function follows a similar structure to the two-point function, where the end result depends only on the squared-distance between insertion points. Again, let us illustrate that of the scalar primaries only and come back to the more general spin-full case in later sections.

Let ϕ_1, ϕ_2 and ϕ_3 be scalar primaries with scaling dimensions Δ_1, Δ_2 and Δ_3 respectively. Their correlator is fully determined up to an overall constant,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{1.23}$$

where again $x_{ij}^2 = (x_i - x_j) \cdot (x_i - x_j)$.

1.3.4 Four-point function

The full conformal group leaves the four-point function with more freedom, as it can only fix it up to a function of cross-ratios. Let us now illustrate this with a correlator between four identical scalar primaries, where te more general formula can be found in later sections,

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}. \tag{1.24}$$

In the equation above, $g(u,v)$ is an arbitrary function of the cross-ratios u and v defined by

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad\qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{1.25}$$

Naturally, one can redefine $g(u,v)$ such that the explicit powers that appear in the correlator's expression are those of x_{13} and x_{24}, in agreement with the permutation symmetry exhibited by such correlator.

1.4 The Conformal Blocks and the OPE

A systematic way of obtaining the results from the previous section, regarding the various correlators between conformal primaries, is via the so-called *operator product expansion* (OPE). First introduced in [84] for general QFTs, OPEs are a way of rewriting the product between two operators as a (usually infinite) sum of local operators, where this rewriting is understood as being true when evaluated as an expectation value. This expansion is usually performed in the region where the two operators get infinitely close to one another, as this allows for the infinite sum to be a 'suitable approximation' to the product. While in general such a sum would not converge in the space of operators, it can be shown to converge for CFTs [85, 86]. Furthermore, the convergence holds in a finite region (as opposed to an infinitesimal neighbourhood of the operator insertions). In the 2d CFT context, this can be understood from the fact that the OPE is intricately related to representation theory of the Virasoro algebra [87].

Let us illustrate this phenomenon using two identical scalar primary operators, ϕ. Typically, their OPE will include contributions from spinning fields and other operators which are descendents of conformal primaries. Packaging this neatly, we are able to write

$$\phi(x)\phi(y) \sim \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} P(x-y, \partial_y) \cdot \mathcal{O}(y), \tag{1.26}$$

where the $\sim$ indicates equality as expectation values. Note that the sum is performed over conformal primary operators $\mathcal{O}$ only. The $\cdot$ denotes contractions between all Lorentz indices, when $\mathcal{O}$ is a spinning field. The function P is typically a power series in $x - y$ and ∂_y, which encodes the contributions of $\mathcal{O}$ and its descendents.

Interestingly, one may use this expansion inside the four-point correlation function of scalar primaries, $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$, and rewrite it as a sum of two-point functions. Using (1.22) together with the OPE expansion (1.26), applied once on $\phi(x_1)\phi(x_2)$ and then again $\phi(x_3)\phi(x_4)$, we see that

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \sum_{\mathcal{O}} (f_{\phi\phi\mathcal{O}})^2 \frac{g_{\mathcal{O}}(u,v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}}. \tag{1.27}$$

The overall contribution should match that of (1.24), where the sum of partial functions $g_{\mathcal{O}}(u,v)$ should recover the function of cross-ratios $g(u,v)$. This expansion of the function g is known as the *conformal block* decomposition, or *conformal partial waves* decomposition [76, 88, 89] (see [90] for an introduction). Together with the scaling dimensions of the conformal primaries spectrum, the set of coefficients $(f_{\phi\phi\mathcal{O}})$ form the complete data required to specify the CFT. Notably, however, not every such set of coefficients forms a suitable CFT. Indeed, had we performed the OPE on $\phi(x_1)\phi(x_3)$ first, and then on $\phi(x_2)\phi(x_4)$, we would have ended on a different conformal block decomposition. The coefficients $f_{\phi\phi\mathcal{O}}$ are such that both expansions should match. This constraint on the CFT data is known as crossing symmetry, and is visually depicted in Figure 1.1.

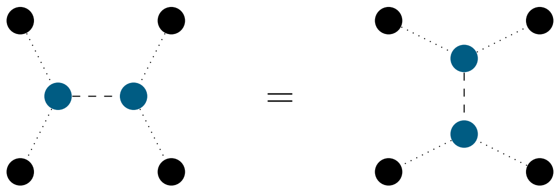


FIGURE 1.1: A visualisation of crossing symmetry in the four-point function. Black circles denote operator insertions, blue circles their decomposition through the OPE.

As these OPEs are generally quite involved, there are ways to simplify the expansion. For instance, in the lightcone limit of Lorentzian CFTs, all operators become lightlike separated and the OPE is dominated by operators with large spin [91–95]. For finite (non-zero) spin the "inversion formula" can be used to reconstruct the CFT data from the correlators [96].

1.5 The Embedding Space Formalism

In preceding sections, we have omitted to display identities relating spinning fields as those are typically quite involved. A simple way of circumventing this problem is to use the *embedding space formalism* [97]. This formalism to prove extremely useful to our presentation of Chapter 8, and as such we will only work with Euclidean CFTs in D dimensions.

First and foremost, recall that the conformal group, $SO(1, D+1)$, acts non-linearly on $\mathbb{R}^D$. This non-linear action is the root cause for complicating calculations with arbitrary spinning fields. The aforementioned formalism aims to solve this by considering a space on which it acts linearly, namely $\mathbb{R}^{1,D+1}$ [98]. Naturally, this embedding only makes sense if there is a well-defined notion extending CFTs to this space, or at the very least a well-defined restriction of QFTs from $\mathbb{R}^{1,D+1}$ to CFTs on $\mathbb{R}^D$. This statement turns out to be true, where the restriction is to a particular subspace of the light-cone, known as the Poincaré section.

We follow the presentation given in the original article [97], and encourage the reader to consult it for further details. Let $X = (X_-, X_+, x)$ be a point on $\mathbb{R}^{1,D+1}$, given in light-cone coordinates, where the metric reads

$$\eta_{AB}X^A X^B = -X_- X_+ + x \cdot x. \tag{1.28}$$

In the above, $\cdot$ denotes the scalar product induced by the Euclidean metric on $\mathbb{R}^D$. The light-cone is the $D+1$-dimensional subspace of $\mathbb{R}^{1,D+1}$ defined by the vanishing of the line element $\eta_{AB}X^A X^B$,

$$\mathrm{LC} = \{X_- X_+ - x \cdot x = 0 \,|\, X \in \mathbb{R}^{1,D+1}\}. \tag{1.29}$$

Furthermore, we can define the projective light-cone as the projective analogue of LC. With the relation $X \sim \lambda X$, for all $X \in \mathrm{LC}\backslash\{0\}$, we define the projective light-cone as

$$\mathbb{P}\mathrm{LC} = \mathrm{LC}\backslash\{0\}/\sim. \tag{1.30}$$

This creates two copies of $\mathbb{R}^D$, both intersecting along a sphere S^{D-1}. A choice of representative for each can be, for instance, $X_- = 1$ and $X_+ = 1$. We call that with $X_+ = 1$ the Poincaré section (PS) of the embedding space. Every point on this section belongs to an $\mathbb{R}^D$ subspace and has the coordinate expression

$$X|_{\mathrm{PS}} = (x^2, 1, x). \tag{1.31}$$

We will often refer to the PS as a being a standard subspace of the embedding space, instead of correctly calling it a ‘projective subspace’. The distinction is made doubly important by the fact that $SO(1, D+1)$ actions do not, in general, preserve the $X_+ = 1$ condition. On the projective light-cone, however, all points related to $X_+ = 1$ by a linear transformations are identified with each other, and we do not need to worry about a potential change in the representative. That being the case, one can treat the PS as a standard subspace of $\mathbb{R}^{1,D+1}$ provided that one rescales X accordingly under conformal transformation, i.e. such that $X_+ = 1$. That is the line of thinking we will follow throughout.

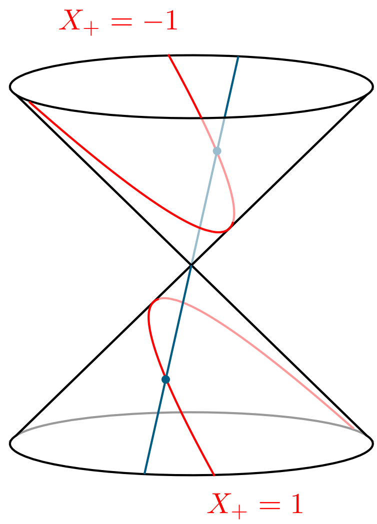


FIGURE 1.2: A visual representation of the null cone $\mathrm{NC} = \{X \in \mathbb{R}^{1,D+1} \,|\, X^2 = 0\}$. The Poincaré section (PS) $X_+ = 1$ and its antipodal locus $X_+ = -1$ are shown in red; they select a unique representative from each projective orbit $X \sim \lambda X$ within each section. An example of such a null ray is shown in blue.

We now move on to describing how to deal with spinning fields. Symmetric traceless tensors (STT)[5] can be encoded as polynomials of an auxiliary complex variable z. In the embedding space, all that needs to be done is to introduce an auxiliary variable Z, and define the polynomial as

$$F(X,Z) = F_{A_1\cdots A_J}(X) Z^{A_1} \cdots Z^{A_J}, \tag{1.32}$$

where $\eta_{AB} Z^A Z^B = 0$. The symmetry under the exchange $Z^{A_i} \leftrightarrow Z^{A_j}$ provides the symmetric constraint on the components of F, while $\eta_{AB} Z^A Z^B = 0$ provides the tracelessness condition. One can move between the polynomial representation and component representation by acting on F with the Todorov differential operator [100]

$$D_A = \left(\frac{D}{2} + Z \cdot \frac{\partial}{\partial Z}\right) \frac{\partial}{\partial Z^A} - \frac{1}{2} Z_A \frac{\partial^2}{\partial Z \cdot \partial Z}. \tag{1.33}$$

Note that in order for F to properly descend to a spinning conformal primary, we further require it to satisfy two additional conditions:

1. (Homogeneity) $F(\lambda X, Z) = \lambda^{-\Delta} F(X,Z)$,

2. (Transversality) $\forall \alpha \in \mathbb{C}, \quad F(X, Z + \alpha Z) = F(X,Z)$.

[5]See [99] for the anti-symmetric case.

Thus, given an $SO(1, D+1)$-invariant QFT, defined on the embedding space, the logical steps towards recovering a CFT in D-dimensions go as follows:

- Define a QFT on the embedding space
- Write down the correlators of homogenous and transverse polynomials
- Restrict them to the PS, for which
  - $X_i \cdot X_j = -\frac{1}{2}x_{ij}^2$ ($X^2 = 0$)
  - $Z \cdot X = 0$
  - $Z_i \cdot Z_j = z_i \cdot z_j$ ($Z^2 = 0$)
  - $X_i \cdot Z_j = x_i \cdot z_j$

With this toolset, we can define a composite tensor C_{AB} built from the coordinate X and auxiliary variable Z,

$$C_{AB} = X_A Z_B - X_B Z_A, \tag{1.34}$$

with which all correlators can be written.

1.5.1 Two-point function

Firstly, we come back to the two-point function of conformal primaries, this time generalising to any spinning fields. Let $F_1(X_1, Z_1)$ and $F_2(X_2, Z_2)$ be two conformal spinning primaries of spin J and conformal scaling dimension Δ_1 and Δ_2, respectively. Their correlator in the embedding space takes the general form

$$\begin{aligned} G(X_1, X_2; Z_1, Z_2) &= \alpha \delta_{\Delta_1, \Delta_2} \frac{\left(\frac{1}{2} C_{1\,AB} C_2^{AB}\right)^J}{(X_1 \cdot X_2)^{\Delta_1 + J}} \\ &= \alpha \delta_{\Delta_1, \Delta_2} \frac{(Z_1 \cdot Z_2 X_1 \cdot X_2 - X_1 \cdot Z_2 X_2 \cdot Z_1)^J}{(X_1 \cdot X_2)^{\Delta_1 + J}}, \end{aligned} \tag{1.35}$$

where α is a constant. One can easily see that in the spinless case, $J = 0$, and after restricting to the PS, we recover the expression given in (1.22).

1.5.2 Three-point function

Let us now look at the three point function of spinning fields. Let $\{F_i(X_i, Z_i)\}_{i \in \{1,2,3\}}$ be three conformal spinning primaries with conformal scaling dimension Δ_i and spin J_i, respectively. The three-point correlator between these has the general expression

$$G(\{X_i, Z_i\}) = \frac{Q(\{X_i, Z_i\})}{(X_1 \cdot X_2)^{\frac{\tau_1+\tau_2-\tau_3}{2}} (X_2 \cdot X_3)^{\frac{\tau_2+\tau_3-\tau_1}{2}} (X_3 \cdot X_1)^{\frac{\tau_3+\tau_1-\tau_2}{2}}}, \tag{1.36}$$

where $\tau_i = \Delta_i + J_i$ and Q is an identically transverse polynomial of degree J_i in each Z_i, with coefficients depending on X_i. By transversality and homogeneity, it must obey

$$Q(\{\lambda_i X_i, \alpha_i Z_i + \beta_i X_i\}) = Q(\{X_i, Z_i\}) \prod_i (\lambda_i \alpha_i)^{J_i}. \tag{1.37}$$

Again, restricting back to the $J_i = 0$ case and imposing the PS constraints, we recover our previous expression (1.23), for the three-point function between scalar conformal primaries.

1.6 Defect CFTs and Boundary CFTs

We are finally ready to move on to the main subject of the thesis, namely defect conformal field theory (dCFT). In this presentation, we define a defect as being any non-local operator that acts on the CFT Hilbert space. Much like how local operators are specified by a position, a defect is specified by a submanifold. We will typical refer to the defects codimensionality instead of its dimension directly. So for instance, if a defect is valued on a p-dimensional surface within a d-dimensional CFT, we will call it a $(d-p)$-codimensional defect.

In general, such a definition is too vague to have any practical use, except in the rare cases where a complete (or fully local) description of the QFT is known [101]. Instead, we will focus on such extended operators that preserve a subgroup of the conformal group. Taking the Euclidean CFT in D dimensions as an example, we will be looking at p-dimensional defects that break $SO(1, D+1)$ into $SO(1, p+1) \times G$, where $G \subseteq SO(d-p)$. We call such defects *conformal defects* by virtue of the fact that they preserve a conformal subgroup along their defining submanifold. Any CFT in the presence of such a defect will be referred to as a defect CFT, or dCFT. In the codimension-1 case, if the CFT is restricted to live only on one side of the defect, we call it a boundary CFT, or bCFT.

We recommend the reviews [102–105] for bCFTs and [106–109] for general dCFTs.

One can engineer conformal defects in many equivalent ways – by assigning singularities to CFT fields on a given submanifold, by coupling the theory to a lower-dimensional one, restricted to a submanifold, following which the lower-dimensional theory is integrated out. Or one could also modify the Ward identities directly[6]. In all these examples, the

[6]Which can be argued to be the most robust way to define conformal defects.

resulting theory is only invariant under the defect conformal subgroup, and their corresponding correlation functions will be adequately modified.

As we will come to see later in this section, preservation of the conformal structure on the defect submanifold implies a CFT-like structure for all fields that are restricted to it. One can see that this holds true in their correlation functions, which follow the form dictated by conformal symmetry, be it in p-dimensions instead of D. What does not necessarily hold true, however, is the existence of a conserved stress-energy tensor. Since the defect itself interacts with the ambient CFT, it may exchange 'energy' between it and the ambient CFT fields. The cases where such a stress-tensor on the defect does exist correspond to decoupled theories and are of no interest to us, amongst other things, due to their triviality.

The specific Ward identity that dictates what the divergence of the stress-energy tensor is, gets modified in the presence of a defect. Since the defect is localised to a submanifold, say Σ, this departure from the divergence-free case should only be true on Σ, and indeed one finds a modification of the form

$$\partial_\mu T^{\mu\nu} = \delta^{(p)}(\Sigma) D^\nu, \tag{1.38}$$

where D^ν is the *displacement operator*. See [35] for an excellent overview of the various results these modified Ward identities can lead to.

Breaking the conformal group offers us a larger set of field representatives of the conformal algebra. In addition to the standard conformal primaries of the *ambient* theory, which one can treat as those that remain when the defect is removed, there are additional *defect* conformal primaries, which are physically restricted to live on the defect submanifold. A direct consequence of this is the opening of a new OPE channel. Indeed, in addition to the standard ambient OPE, for which products of ambient operators can be written as a sum of ambient operators, dCFTs have the defect OPE. There, any ambient field can be written as a sum of defect ones,

$$\phi(x) \sim \sum_{\hat{\mathcal{O}}} f_{\phi\hat{\mathcal{O}}} Q(x_\perp - y, \partial_y) \cdot \hat{\mathcal{O}}(y), \tag{1.39}$$

where ϕ is an ambient conformal primary and $\hat{\mathcal{O}}$ is a defect one. The function Q encapsulates all descendent contributions in this expansion.

One can immediately see that dCFTs then require more data to be fully specified. Indeed, on top of the scaling dimensions of ambient operators (Δ) and their OPE coefficients (λ_{ijk}), we also need to specify the scaling dimensions of defect operators ($\hat{\Delta}$) together with the defect OPE coefficients (λ_{ij}). Naturally, not any set of such coefficients defines a dCFT and one must place constraints on these, similarly to how crossing-symmetry places constraints on the ambient OPE coefficients. In this case, the two OPEs should commute and one is left with a set of relations between the ambient and defect OPE coefficients. Figure 1.3 is a visual representation of this. The search for such dCFT data from the ground up is known as the dCFT bootstrap [107, 110–116].

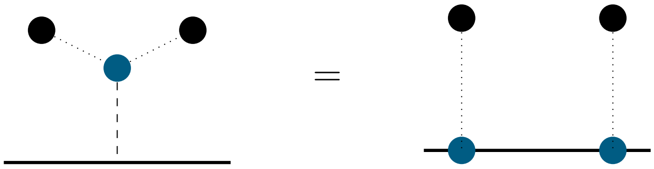


FIGURE 1.3: A visualisation of constraints imposed on the coefficients in the defect OPE. Black circles denote operator insertions, blue circles their decomposition through the OPE and the solid line represents the defect.

Finally, let us close this chapter with a brief presentation of the embedding space formalism for dCFTs, as introduced in [107]. This presentation will be useful in understanding our notation in Chapter 8.

The embedding space has a natural split into defect and orthogonal directions, each being acted on by their respective subgroup of $SO(1, D+1)$. If X denotes the coordinates on $\mathbb{R}^{1,D+1}$, we write X^A, $0 \leq A \leq p+1$, for the defect directions and X^I, $p+2 \leq I \leq p+q+2$, for the orthogonal ones. The dot product, $\cdot$, also sees a natural split into a defect part, $\bullet$, and an orthogonal one, $\circ$,

$$\begin{aligned} X \cdot Y &= X \bullet Y + X \circ Y \\ &= X^A Y^B \eta_{AB} + X^I Y^J \delta_{IJ}. \end{aligned} \tag{1.40}$$

Just as in the non-defect case, ambient spinning conformal primaries which are STTs are encoded by polynomials of an auxiliary coordinate $Z \in \mathbb{R}^{1,D+1}$. Consequently, such insertions obey the same restrictions, when passing down to the Poincaré section, namely

$$X \cdot X = 0 \quad \Leftrightarrow \quad X \bullet X = -X \circ X, \tag{1.41a}$$

$$Z \cdot X = 0 \quad \Leftrightarrow \quad Z \bullet X = -Z \circ X, \tag{1.41b}$$

$$Z \cdot Z = 0 \quad \Leftrightarrow \quad Z \bullet Z = -Z \circ Z. \tag{1.41c}$$

Defect insertions, on the other hand possess an additional $SO(q)$-representation. As such, STTs on the defect require two auxiliary coordinates, Z and W, one for $SO(1, p+1)$ and the other for $SO(q)$. When projecting down to physical space, on top of restricting the coordinates to the Poincaré section of $\mathbb{R}^{1,D+1}$, defect insertions must further be restricted to the 'defect Poincaré section', which a section of the light-cone of the defect subspace $\mathbb{R}^{1,p+1}$. In practice, this means setting

$$X \bullet X = X \circ X = 0, \tag{1.42a}$$
$$Z \bullet X = Z \circ X = 0, \tag{1.42b}$$
$$W \bullet X = W \circ X = 0, \tag{1.42c}$$
$$Z \bullet Z = Z \circ Z = 0, \tag{1.42d}$$
$$W \bullet W = W \circ W = 0. \tag{1.42e}$$

The composite tensor C also breaks down into three components, C_{AB}, C_{AI} and C_{IJ}. For bulk insertions only combinations of C_{AI} appear in correlators, while for defect ones only C_{AB} does.

1.6.1 One-point functions

The existence of the identity operator inside the defect OPE shows that ambient conformal primaries can now pick up a non-trivial one-point function. This can be understood from the perspective that any correlator in the presence of a defect is effectively that of a higher-point function with a defect operator insertion. This introduces a scale, that of the distance to the defect. In our notation, where the defect lies at the origin of the transverse part of the embedding space, this distance is given by $\sqrt{X \circ X}$. Let $\mathcal{O}_{\Delta,J}(X;Z)$ be a spinning conformal primary of spin J and conformal dimension Δ. Then its one-point function, in the presence of a defect, reads

$$\langle \mathcal{O}_{\Delta,J}(X;Z)\rangle = \alpha \frac{Q_J(X;Z)}{(X\circ X)^{\Delta/2}}, \tag{1.43}$$

where the polynomial $Q(X;Z)$ of degree J in Z is given by

$$Q_J(X;Z) = \left(\frac{\frac{1}{2}C_{AI}C^{AI}}{X\circ X}\right)^{J/2} = \left(\frac{(X\circ Z)^2}{X\circ X} - Z\circ Z\right)^{J/2}. \tag{1.44}$$

As expected, this structure mirrors directly that of a two-point function without defects, (1.22).

Defect conformal primaries on the other hand do not develop a one-point function. As alluded to earlier, their correlators are that of a standard CFT, for which operators do not develop non-trivial one-point functions.

1.6.2 Two-point functions

Two-point functions now come in three different classes: defect-defect, ambient-defect and ambient-ambient. In all three cases, the presence of this additional OPE channel grants them a much larger structure than the non-defect two-point functions. In fact, the presence of the defect has the effect of 'doubling' the possible dependencies of this correlator, making it act like a standard four-point function. The only correlator for which this does not apply is that of two defect insertions. As those follow precisely (1.35) (with $\cdot$ replaced by $\bullet$), we refer the reader back to that equation.

Let $\mathcal{O}_{\Delta,J}(X_1;Z_1)$ be an ambient spinning conformal primary of scaling dimension Δ and spin J. We choose $\hat{\mathcal{O}}_{\hat{\Delta},j,s}(X_2;Z_2,W_2)$ to be a defect spinning conformal primary of scaling dimension $\hat{\Delta}$, $SO(1,p+1)$-spin j and $SO(q)$-spin s. Then, their correlation function is a mixed-correlator, or ambient-defect correlator and has the general form

$$\langle \mathcal{O}_{\Delta,J}(X_1;Z_1)\hat{\mathcal{O}}_{\hat{\Delta},j,s}(X_2;Z_2,W_2)\rangle = \left(Q^0_{\text{AD}}\right)^j \sum_{\{n_i\}} b_{n_1\cdots n_4} \frac{\prod_{k=1}^4 \left(Q^k_{\text{AD}}\right)^{n_k}}{(X_1\bullet X_2)^{\hat{\Delta}}(X_1\circ X_1)^{\frac{\Delta-\hat{\Delta}}{2}}}, \tag{1.45}$$

where the sum runs over $n_i \in \mathbb{N}$ that satisfy $n_1+n_3=s$ and $n_2+n_3+2n_4=J-j$. The coefficients $b_{n_1\cdots n_4}$ depend on the specifics of the CFT and can be determined by the OPE expansions. The ambient-defect homogeneous functions Q^i_{AD} are given by

$$Q^0_{\text{AD}} = Z_1\bullet Z_2 - \frac{X_2\bullet Z_1}{X_1\bullet X_2}X_1\bullet Z_2, \tag{1.46a}$$

$$Q^1_{\text{AD}} = \frac{X_1\circ W_2}{(X_1\circ X_1)^{1/2}}, \tag{1.46b}$$

$$Q^2_{\text{AD}} = \frac{X_1\circ Z_1 X_1\circ X_2 - X_2\circ Z_1 X_1\circ X_1}{(X_1\circ X_1)^{1/2}(X_1\bullet X_2)}, \tag{1.46c}$$

$$Q^3_{\text{AD}} = \frac{W_2\circ Z_1 X_1\circ X_1 - X_1\circ W_2 X_1\circ Z_1}{X_1\circ X_1}, \tag{1.46d}$$

$$Q^4_{\text{AD}} = \frac{X_1\circ Z_1}{X_1\circ X_1} - Z_1\circ Z_1. \tag{1.46e}$$

Let us briefly comment on the structure of equation (1.45). One can readily see how the introduction of a defect OPE channel augments the structure of this two-point function, when compared to equation (1.35). The conformally invariant quantities are now the distance between the two insertions on the defect, $X_1\bullet X_2$ and the distance from the defect of the ambient insertion, $X_1\circ X_1$.

Finally, if $\mathcal{O}_{\Delta_1,J_1}(X_1;Z_1)$ and $\mathcal{O}_{\Delta_2,J_2}(X_2;Z_2)$ are two ambient conformal primaries, their correlator takes the general form

$$\langle \mathcal{O}_{\Delta_1,J_1}(X_1;Z_1)\mathcal{O}_{\Delta_2,J_2}(X_2;Z_2)\rangle = \sum_{\{n_i\}} \frac{\prod_{k=1}^8 (Q^k_{AA})^{n_k} f_{n_1\cdots n_8}(\chi,\phi)}{(X_1\circ X_1)^{\Delta_1/2}(X_2\circ X_2)^{\Delta_2/2}}, \tag{1.47}$$

where the sum runs over $n_i \in \mathbb{N}$ that satisfy $n_1+n_2+n_5+n_6+2n_7=J_1$ and $n_3+n_4+n_5+n_6+2n_8=J_2$. This time around, the ambient-ambient homogeneous functions Q^i_{AA} are given by[7]

$$Q^1_{\mathrm{AA}} = \frac{X_1 \bullet X_1 Z_1 \circ X_2 - X_1 \circ X_2 Z_1 \bullet X_1}{X_1 \circ X_1 (X_2 \circ X_2)^{1/2}}, \tag{1.48a}$$

$$Q^2_{\mathrm{AA}} = \frac{X_1 \bullet X_2 Z_1 \circ X_2 - X_1 \circ X_2 Z_1 \bullet X_2}{(X_1 \circ X_1)^{1/2} X_2 \circ X_2}, \tag{1.48b}$$

$$Q^3_{\mathrm{AA}} = \frac{X_1 \bullet X_2 Z_2 \circ X_2 - X_2 \circ X_2 Z_2 \bullet X_1}{(X_1 \circ X_1)^{1/2} X_2 \circ X_2}, \tag{1.48c}$$

$$Q^4_{\mathrm{AA}} = \frac{X_1 \bullet X_2 Z_2 \circ X_1 - X_1 \circ X_2 Z_2 \bullet X_1}{X_1 \circ X_1 (X_2 \circ X_2)^{1/2}}, \tag{1.48d}$$

$$Q^5_{\mathrm{AA}} = \frac{X_1 \bullet X_1 (X_2 \bullet X_2 Z_1 \circ Z_2 - X_2 \bullet Z_2 X_2 \circ Z_1)}{X_1 \circ X_1 X_2 \circ X_2} - \frac{Z_1 \bullet X_1 (X_2 \bullet X_2 X_1 \circ Z_2 - X_2 \bullet Z_2 X_1 \circ X_2)}{X_1 \circ X_1 X_2 \circ X_2}, \tag{1.48e}$$

$$Q^6_{\mathrm{AA}} = \frac{X_1 \circ X_1 (X_2 \circ X_2 Z_1 \bullet Z_2 - X_2 \circ Z_2 X_2 \bullet Z_1)}{(X_1 \circ X_1)^{1/2} (X_2 \circ X_2)^{3/2}} - \frac{Z_1 \circ X_1 (X_2 \circ X_2 X_1 \bullet Z_2 - X_2 \circ Z_2 X_1 \bullet X_2)}{(X_1 \circ X_1)^{1/2} (X_2 \circ X_2)^{3/2}}, \tag{1.48f}$$

$$Q^7_{\mathrm{AA}} = \frac{(X_1 \circ Z_1)^2}{X_1 \circ X_1} - Z_1 \circ Z_1, \tag{1.48g}$$

$$Q^8_{\mathrm{AA}} = \frac{(X_2 \circ Z_2)^2}{X_2 \circ X_2} - Z_2 \circ Z_2. \tag{1.48h}$$

The summands $f_{n_1 \cdots n_8}(\chi, \phi)$ are arbitrary functions of the cross-ratios

$$\xi = -\frac{2 X_1 \cdot X_2}{(X_1 \circ X_1)^{1/2} (X_2 \circ X_2)^{1/2}}, \qquad \cos(\phi) = \frac{X_1 \circ X_2}{(X_1 \circ X_1)^{1/2} (X_2 \circ X_2)^{1/2}}, \tag{1.49}$$

as expected from a correlator that behaves like the four-point function (1.24).

[7] We purposely expanded out every factor of C^{AI}. Please see the original reference for a more compact form [107].

Chapter 2

11D supergravity

En Prouvènço, lou soulèu si lèvo dous còup, lou matin e après lou penequet.

Yvan Audouard

Eleven-dimensional supergravity plays a central role in the landscape of supergravity theories, and it is for that reason that we dedicate an entire chapter to it. Our main motivation originates from the fact that it describes the low-energy limit of M-theory [117]. The extended objects of which (M-branes) are mapped to extended black hole solutions (black branes) in the supergravity theory. As we will survey later in this chapter, the intersections of these extended objects also have a supergravity counterpart. Together with the AdS/CFT correspondence, these allow us to make non-trivial statements about the underlying low-energy theories that live on these M-branes, as well as the defects that they support.

Another point of interest to us, are the various compactifications of 11d supergravity down to lower dimensions. It is well known that maximally-extended supergravity theories in lower dimensions can be obtained via compactification of the 11d theory, i.e. choosing the manifold to be $\mathcal{M}_d \times S^{11-d}$ and letting the radius of S^{11-d} go to zero. Historically, this was what 11d supergravity was originally used for. Just like its non-gravitational counter-parts, $\mathcal{N}$-extended supergravity theories in d dimensions are more simply formulated in terms of $\mathcal{N} = 1$ supergravity in higher dimension.

Thanks to Nahm's classification, we know that eleven is the largest dimension with which one can formulate a consistent supersymmetric theory of gravity [26][1]. The later work by Cremmer, Julia and Scherk [119] showed that the unique supergravity in eleven

[1] Any flat space superalgebra in $d > 11$ will necessarily include fields of spin higher than two, making the supergravity theory inconsistent. This argument, however, does not hold when dealing with an infinite number of fields, curved spaces, or signatures other than $(1, 10)$. For a more recent presentation of Weinberg's original argument, see Schwartz [118], section 9.5.1.

dimensions admits a simple description in terms of three fields: the graviton $g_{\mu\nu}$, the gravitino ψ_μ and a three-form gauge field C_3. These come in the $(2,0,0,0,0)$, $(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $(1,1,1,0,0)$ representations of the Lorentz algebra $\mathfrak{so}(1,10)$, respectively [120].

$\mathfrak{so}(1,10)$ rep.	dim.	name
$(2,0,0,0,0)$	44	graviton
$(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$	128	gravitino
$(1,1,1,0,0)$	84	3-form

FIGURE 2.1: The field content of 11d supergravity and its $\mathfrak{so}(1,10)$ representations.

Let us briefly comment on how the algebra enforces this field content. See [121, 122] for a more detailed exposition. The field content of the theory is given by the massless irreducible representations of the global symmetry algebra, $\mathfrak{so}(1,10)$. Using Wigner's method of induced representations [123], there is a one-to-one correspondence between massless irreducible representations of $\mathfrak{so}(1,10)$ and those of the little group, $\mathfrak{so}(9)$. This follows from the fact that the operator $P_\mu P^\mu$ is Casimir. On this little algebra, only 16 out of the 32 supercharges remain *active*. That is to say, only half of the supercharges act non-trivially. These supercharges turn out to be in the vector representation of Clifford algebra $Cl(16)$, a 256-dimensional representation. Finding the massless field representations then amount to understanding how the vector representation of $Cl(16)$ decomposes into representations of the little algebra $\mathfrak{so}(9)$.

$$\begin{array}{ccccc} Cl(16) & \supset & \mathfrak{so}(16) & \to & \mathfrak{so}(9) \\ \mathbf{256} & \to & \mathbf{128}_s \oplus \mathbf{128}_c & \to & \mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128} \end{array} \tag{2.1}$$

In this decomposition, the irreducible representation of $\mathfrak{so}(9)$ with dimension 44 corresponds to a rank-two symmetric traceless tensor, in other words the graviton field, $g_{\mu\nu}$, which is in the $(2,0,0,0,0)$ representation of $\mathfrak{so}(1,10)$. That with dimension 84 is the three-form C in the $(1,1,1,0,0)$ and that with dimension 128 corresponds to a spin-$\frac{3}{2}$ field, the gravitino, ψ_μ, in the $(\frac{3}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$ representation.

The field content of any maximally-extended supergravities in lower dimensions can then be determined from how these representations decompose into the corresponding global symmetry algebras. For instance, compactifying down to four dimensions yields $\mathcal{N}=8$ supergravity. The field content is determined by the decomposition $\mathfrak{so}(9) \to \mathfrak{so}(2) \oplus \mathfrak{so}(7)$, which gives a graviton, 8 gravitinos, 28 vector fields, 56 spin-$\frac{1}{2}$ fermions, and 70 scalar fields [124].

A standard feature of supergravity theories in higher dimensions seems to be the appearance of higher-form gauge fields. The three-form C_3 being one such example whose dimensional reductions can lead to other higher-form gauge fields in lower dimensions. The subtlety in defining such a theory stems from the fact that the three-form C_3 admits gauge transformations parametrised by a two-form $\lambda^{(2)}$, $C_3 \mapsto C_3 + d\lambda^{(2)}$. Furthermore, this two-form also admits gauge transformations by a one-form, and so on and so forth. While the four-form field strength F_4 is indeed a section of a bundle, $\bigwedge^4 T^*\mathcal{M}$, that clearly isn't the case for the three-form gauge field C_3. It isn't a connection on a principal bundle either, as one would hope. The proper object one needs to consider is a connection on a bundle gerbe. Nevertheless, when describing it locally, one can consider it as a three-form, with the implicit understanding that it admits local gauge transformations and obeys gluing conditions between patches.

This chapter will not serve as a pedagogical introduction to 11d supergravity. Instead, the reader should see it as a useful collection of facts about it and its various black brane solutions. In Section 2.1 we present the 11d supergravity action together with the supersymmetry transformation of its fields and equations of motion. Section 2.2 mentions in passing, the various avenues of compactifying 11d supergravity down to lower dimensions. Section 2.3 will present the five important half-BPS solutions to this supergravity that have relevance within this thesis. Finally, Section 2.4 briefly highlights the key aspects of the asymptotic supersymmetry algebra, namely that of flat eleven-dimensional space.

In attempting to present such a vast topic, we will most likely fail at demonstrating its beauty. For that reason, we recommend reading these excellent reviews instead of this chapter: [125–128]. As a lot of the topics from this section will intersect with concepts in string theory/M-theory, we also recommend these excellent reviews [126, 128–131].

2.1 The action principle

Eleven dimensional supergravity is the unique supersymmetric theory of gravity in signature $(1,10)$ with 32 supercharges. Let $\mathcal{M}$ be an eleven-dimensional smooth, spin manifold equipped with a Lorentzian metric G. Its field content is rather simple as it contains the vielbeins field $e^a{}_\mu$, seen as a section of the bundle $T\mathcal{M} \otimes T^*\mathcal{M}$, a thirty two component spinor field ψ_μ, seen as a section of the bundle $T^*\mathcal{M} \otimes \mathcal{S}$, where $\mathcal{S}$ is the spin bundle. Finally, supersymmetry requires the third field to be a three-form C_3, with four-form field strength $F_4 = dC_3$.

Throughout this chapter, we use greek letters $\mu, \nu, \rho \in \{0, \ldots, 10\}$ to label the coordinates on a patch of $\mathcal{M}$. The non-coordinate basis is labelled by roman letters

$a, b, c \in \{0, \ldots, 10\}$. The Clifford algebra generators $(\Gamma^a)_{a\in\{0,\ldots,10\}}$ are required to obey the following anti-commutator $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}\mathbb{1}_{32}$, where $\eta^{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$ is the Minkowski metric in mostly-plus convention. We also define the antisymmetrised product of these generators as

$$\Gamma^{a_1\cdots a_n} = \frac{1}{n!}\sum_{\sigma\in S^n} \mathrm{sign}(\sigma)\Gamma^{a_{\sigma_1}}\cdots\Gamma^{a_{\sigma_n}}. \tag{2.2}$$

The Clifford algebra generators can then be converted to their "curved-space" analogues using the vielbeins $e^a{}_\mu\Gamma^\mu = \Gamma^a$.

The full supergravity Lagrangian density then reads [119, 132]

$$\begin{aligned}\mathcal{L} = {}&\star R - \frac{1}{2}F\wedge\star F - \frac{1}{6}F\wedge F\wedge C + 2\bar\psi_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho\star 1\\ &+\frac{1}{16}\bar\psi_\mu\Gamma^{\mu\nu\rho}\Gamma^{ab}\psi_\rho\bar\psi_\alpha\Gamma_{\nu ab}{}^{\alpha\beta}\psi_\beta\star 1\\ &-\frac{1}{48}\left(\bar\psi_\mu\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu + 12\bar\psi^\alpha\Gamma^{\gamma\delta}\psi^\beta\right)\left(F_{\alpha\beta\gamma\delta} + 6\bar\psi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\right)\star 1,\end{aligned} \tag{2.3}$$

where $\star 1$ denotes the Hodge dual of unity, i.e. the canonical volume form, and $D_\mu\psi_\nu = \partial_\mu\psi_\nu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab}\psi_\nu$ is the covariant derivative of the gravitino. Do note the presence of a Chern-Simons-like term $F\wedge F\wedge C$, which is be crucial for our understanding of the M-theory description and how its various extended objects can intersect [128]. Additionally, due to the presence of fermions the spin connection will naturally develop a non-zero torsion. If we denote by $\omega^{(0)}$ the Levi-Civita connection, then the torsion-full connection ω takes the form

$$\omega_\mu^{ab} = \omega_\mu^{(0)ab} + K_\mu^{ab}, \tag{2.4a}$$

$$K_\mu^{ab} = \frac{1}{4}(\bar\psi_\alpha\Gamma_\mu{}^{ab\alpha\beta}\psi_\beta - 2(\bar\psi_\mu\Gamma^b\psi^a - \bar\psi_\mu\Gamma^a\psi^b + \bar\psi^b\Gamma_\mu\psi^a)) \quad \text{(contorsion)}. \tag{2.4b}$$

The action built from equation (2.3) is invariant under all superdiffeomorphisms. In particular, given ξ a spinor on $\mathcal{M}$, the odd part of this superdiffeomorphism is called the supersymmetry variation, and for each field of the supergravity multiplet, is given by

$$\delta_\xi e^a{}_\mu = \bar\xi\Gamma^a\psi_\mu, \tag{2.5a}$$

$$\delta_\xi C_{\mu\nu\rho} = 3\bar\xi\Gamma_{[\mu\nu}\psi_{\rho]}, \tag{2.5b}$$

$$\begin{aligned}\delta_\xi\psi_\mu = {}&D_\mu\xi - \frac{1}{2\cdot 144}\left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\Gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)\xi\left(F_{\alpha\beta\gamma\delta} + 6\bar\psi_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\right)\\ &+\frac{1}{16}\bar\psi_\alpha\Gamma_{\mu ab}{}^{\alpha\beta}\psi_\beta\Gamma^{ab}\xi.\end{aligned} \tag{2.5c}$$

Note that any configuration of the fields for which $\delta_\xi = 0$ is known as a supersymmetric configuration, or BPS configuration. The reason for which will be discussed shortly. Nevertheless, such solutions where the fermions are set to zero are known as bosonic fixed points, and will be the central focus of this thesis. Out of the three supersymmetry equations above, only the variation of ψ_μ will impose non-trivial constraints on $\mathcal{M}$ for such a bosonic fixed point. In that context, the variation of the gravitino defines a (generalised) Killing spinor equation

$$D_\mu \xi - \frac{1}{288}\left(\Gamma^{\alpha\beta\gamma\delta}{}_\mu - 8\Gamma^{\beta\gamma\delta}\delta^\alpha_\mu\right) F_{\alpha\beta\gamma\delta}\xi = 0, \tag{2.6}$$

and its solutions, ξ are called (generalised) Killing spinors (see Appendix D for the specific definitions).

Finally, we may also explicitly write down the equations of motions for the graviton, three-form connection and gravitino

$$R_{\alpha\beta} = \frac{1}{12}\left(F_{\alpha\nu\rho\sigma}F_\beta{}^{\nu\rho\sigma} - \frac{1}{12}g_{\alpha\beta}F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}\right), \tag{2.7a}$$

$$d\star F + \frac{1}{2}F\wedge F = 0, \tag{2.7b}$$

$$\begin{aligned}\Gamma^{\mu\nu\rho}\Bigg(D_\nu\psi_\rho - \frac{1}{2\cdot 144}\left(\Gamma^{\alpha\beta\gamma\delta}{}_\nu - 8\Gamma^{\beta\gamma\delta}\delta^\alpha_\nu\right)\psi_\rho\left(F_{\alpha\beta\gamma\delta} + 6\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\right)\\ + \frac{1}{16}\bar{\psi}_\alpha\Gamma_{\nu ab}{}^{\alpha\beta}\psi_\beta\Gamma^{ab}\psi_\rho\Bigg) = 0.\end{aligned} \tag{2.7c}$$

Provided the fields obey suitable reality conditions (obligatory in Lorentzian signature), any solution of the Killing spinor equation is also a solution of the equations of motion[2].

2.2 Its toroidal compactifications

As mentioned in the introduction to this chapter, eleven-dimensional supergravity was original constructed as a tool to facilitate the construction of $\mathcal{N}$-extended supergravity theories in lower dimensions [119]. An important aspect of these lower-dimensional theories that isn't captured by the higher-dimensional parent, however, is the exceptional global symmetry. Indeed, it was shown that 4d $\mathcal{N} = 8$ supergravity exhibits an $E_{7(7)}$-symmetry [124, 133] and in general maximal supergravity in $11 - d$ dimensions exhibits an $E_{d(d)}$ symmetry. This fact was coined the *silver rule of supergravity* [134–136] and these additional symmetries are referred to as *hidden symmetries* by virtue of the fact that they do not appear explicitly in the 11d context.

It turns out that, in order to see the emergence of these hidden symmetries from an 11d perspective, one needs to perform various dualisations of the 11d supergravity fields[3].

[2]With supergravity theories in Euclidean signature we do not need impose any reality conditions on the fields. In that context, solutions to the Killing spinor equation may not be solutions to the equations of motion. However, if one wishes to Wick rotate back to Lorentzian signature, a choice of reality conditions is required. See Chapter 3.

[3]In eleven dimensions, the Hodge dual to the four-form field strength F_4 is a seven-form F_7. The latter can be given the interpretation of the field strength of a 'dual gauge field' C_6.

Toroidal compactifications with the appropriate field redefinitions can then give rise to the E-series of Dynkin labels [137]. We recommend the excellent review by Samtleben [122] for a more recent overview of the emergence of such exceptional global symmetries upon dimensional reduction of the 11d action. The chapter by Dieter Maison [138] in the book [139], is also a great place to learn more about these Kaluza-Klein reductions of Einstein's theory and how the non-abelian symmetries can emerge from them.

The relevance behind these compactifications here won't lie in the hidden symmetries they produce, but rather in their existence in the first place. Indeed, the emergence of M-theory as the large-coupling limit of type IIA string theory [117, 140] suggests a direct relationship from the supergravity limits [125]. The compactification of 11d supergravity on a circle of vanishing radius will indeed lead to the equations of motion of IIA supergravity in ten dimensions [141–143]. Naturally, similar constructions exist in various other dimensions (9d [144], 8d [145], 4d [124], 3d [146–150] and 2d [151–154] just to cite a few). By construction solutions to these lower-dimensional supergravity theories can be 'uplifted' to solutions of the eleven-dimensional theory.

This is the central idea behind the construction of 10d and 11d supergravity solutions in [1] from solutions to certain 7d supergravity. As w will review those uplifts in Chapter 4, Section 4.2, we will not comment further on this.

2.3 BPS solutions

An important subset of solutions to 11d supergravity are given by those which preserve a non-zero number of supercharges. In other words, they are solutions to the equations of motion (2.7) which also offer non-trivial solutions to the generalised Killing spinor equation (2.5c). The supersymmetry thus retained protects them under quantisation and it is expected that a corresponding object exists in the full M-theory [125]. Contrarily to perturbative string theory, whose dynamical objects are strings, those of M-theory are expected to be further extended object known as M-branes. In the supergravity limit, these appear in multiple instances – as supersymmetric black brane solutions (generalising supersymmetric black hole solutions) or directly as central charges in the superalgebra.

Let us focus on the former description within this section. A full classification of all supersymmetric solutions is still out of reach, however, progress was made for solutions with a given amount of supersymmetry. For instance, the maximally supersymmetric solutions, i.e. those that preserve 32 supercharges, are fully known. They break down into either $\mathrm{AdS}_7 \times S^4$, $\mathbb{R}^{1,10}$ or $\mathrm{AdS}_4 \times S^7$, and additional curvature-free solutions [155–158], the Cahen-Wallach pp-wave [159]. We will focus our presentation here on those solutions that preserve half of the maximal supersymmetry, i.e. 1/2-BPS solutions (see for instance [160, 161]). These are essential solutions to 11d supergravity as they describe the supergravity counterparts of the fundamental objects of M-theory, the M-branes. Under the circle compactification that brings us back down to type IIA string theory these objects should have an extended string-theoretic description, as D-branes.

The M-wave (pp-wave)

The M-wave is a solution of 11d supergravity which preserves half of the 32 supercharges and is originally due to [162]. We follow the presentation given in [125]. The M-wave solution only involves a non-trivial profile in the metric and is specified by a harmonic function H on the transverse space $\mathbb{R}^9$,

$$g_{11} = -dt^2 + d\rho^2 + (H-1)(dt+d\rho)^2 + g_{\mathbb{R}^9}, \tag{2.8a}$$

$$H(r) = 1 + \frac{k}{r^7}. \tag{2.8b}$$

However, it can be generalised to include non-trivial 3-form gauge field C [163]

$$g_{11} = -dt^2 + d\rho^2 + (H-1)(dt+d\rho)^2 + \mathcal{A}\otimes(dt+d\rho) + g_{\mathbb{R}^8} + dy^2, \tag{2.9a}$$

$$C = \frac{1}{3}\mathcal{A}\wedge(dt+d\rho)\wedge dy, \tag{2.9b}$$

$$\Delta_{\mathbb{R}^8} H = 0, \qquad \Delta_{\mathbb{R}^8} d\mathcal{A} = 0, \tag{2.9c}$$

where $\mathcal{A}$ is a one-form on $\mathbb{R}^8$.

It was also shown in [164] that this class of solutions can be made to preserve more supersymmetry, from 16 for the original M-wave to 18, 20, 22, 24 and 32 for the Cahen-Wallach pp-wave. Those with 26 supercharges have also been constructed [165]. Note that it is also possible to construct solutions with multiple M-waves, as was shown [166] where the action for such a configuration was constructed.

We finally note that under the S^1 compactification down to type IIA supergravity, the M-wave solution becomes either an M-wave solution of type IIA or a D0-brane.

The M2-brane

The presence of a three-form gauge field C_3 suggests the existence of an object which is electrically charged under it. This is the M2-brane[4], whose supergravity solution is given

[4]We will often use the same terminology to denote M-theory objects and their black brane solutions in supergravity. This might make some readers uneasy but given that one is the low-energy limit of the other, we do not see this as a problem.

by [167]

$$g_{11} = H^{-2/3}(r) g_{\mathbb{R}^{1,2}} \oplus H^{1/3}(r) g_{\mathbb{R}^8}, \tag{2.10a}$$
$$F = vol_{\mathbb{R}^{1,2}} \wedge dH(r)^{-1}, \tag{2.10b}$$
$$H(r) = 1 + \frac{k}{r^6}, \tag{2.10c}$$

where r is the radial direction in $\mathbb{R}^8$.

The presence of the harmonic factor $H(r)$ in front of the three-dimensional subspace $\mathbb{R}^{1,2}$ hints at an extended black hole solution along said subspace. That is indeed the space spanned by the M2-brane as an M-theory object. One can also notice that this solution interpolates between two maximally supersymmetric solutions of 11d supergravity. In the large-r limit, the harmonic factor tends to one and the supergravity solution asymptotes to that of vacuum Minkowski space, $\mathbb{R}^{1,10}$. On the flip side, when taking the small-r limit, the metric becomes that of $\mathrm{AdS}_4 \times S^7$ asymptotically. The latter limit is known as the near-horizon geometry of the M2-brane, and is the typical setting in which a holographic duality between the Aharony Bergman Jafferis Maldacena (ABJM) theory [168] and this geometry can be established.

The fact that this M2-brane interpolates between two maximally supersymmetric solutions is no surprise. Indeed, this is a general feature of solitonic objects. Their presence also affects the supersymmetry algebra via the addition of central terms in the anti-commutator of two supercharges. It is in that way that the M2-brane will appear as a central extension of the 11d supersymmetry algebra.

Under compactification down to type IIA supergravity, the M2-brane solution becomes either a fundamental string (F1), if the M2-brane wraps the compactification circle, or a D2-brane, otherwise.

The M5-brane

Similarly to the M2 case, the existence of a three-form gauge field suggests the existence of an object which is magnetically charged under it. This is the M5-brane, whose supergravity solution is given by [125, 169]

$$g_{11} = H^{-2/3}(r) g_{\mathbb{R}^{1,5}} \oplus H^{1/3}(r) g_{\mathbb{R}^5} \tag{2.11a}$$
$$F = \star_{\mathbb{R}^5} dH(r) \tag{2.11b}$$
$$H(r) = 1 + \frac{k}{r^3} \tag{2.11c}$$

where r is the radial direction in $\mathbb{R}^5$ and $H_{M5} = 1 + \frac{\pi Q_{M5} l_p^3}{r^3} = 1 + \frac{R_{M5}^3}{r^3}$.

We reiterate our observation made with the M2-brane solution. The subspace spanned by the M5-brane corresponds to the singularity subspace of the black brane. In other words, the M5-brane in M-theory spans $\mathbb{R}^{1,5}$. This time, however, it interpolates between $\mathbb{R}^{1,10}$, in the large-r limit, and $\mathrm{AdS}_7 \times S^4$ in the small-r limit. In this near-horizon limit, the $\mathrm{AdS}_7 \times S^4$ geometry should be the holographic dual of a six-dimensional superconformal field theory with $(2,0)$ supersymmetry. Contrarily to the M2-brane, whose low-energy effective field theory is known, the M5-brane does not enjoy such a complete description (see [131] for recent overview). The 6d $(2,0)$ still eludes a Lagrangian description, and the holographic description provides a useful way of calculating quantities that live on it.

Under compactification down to type IIA supergravity, the M5-brane solution becomes either a D4-brane, if the M5 has a leg in the compactification circle, or an NS5-brane, otherwise.

The KK6-monopole

The Kaluza-Klein monopole is another instance of a BPS solution that involves non-trivial configurations of the metric alone. It displays a Taub-Newman-Unti-Tamburino (Taub-NUT) geometry along all points on $\mathbb{R}^{1,6}$, hence the name KK6. The solution may be written as [117]

$$g_{11} = g_{\mathbb{R}^{1,6}} \oplus g_{\mathrm{TN}}, \tag{2.12a}$$

$$g_{\mathrm{TN}} = H(r)dy^i dy^i + H(r)^{-1}(d\psi_{\mathrm{TN}} + V_i(y)dy^i)^2, \tag{2.12b}$$

$$H(r) = 1 + \frac{k}{\|y\|}, \quad \mathrm{curl}(V) = \mathrm{div}(H). \tag{2.12c}$$

Under the circle compactification down to type IIA supergravity, this solution becomes either the KK-monopole of type IIA, or the D6-brane.

The M9-brane

Given the previous results, one can see how the 1/2-BPS solutions of 11d supergravity map to the down extended objects of type IIA supergravity. In other words, we are able to recover the M-wave, F1, NS5 and KK-monopole branes in the Neveu-Schwarz sector. We were also able to recover the D0-, D2-, D4- and D6-branes in the Ramond-Ramond sector (See Figure 1.3 in [170] for a nice pictorial representation of these dualities). Existence of space-filling branes and D8-branes hints at the existence of another type of M-brane. This is known as the M9-brane [171, 172] (see [173] for a recent paper discussing the allowed anomalies in the presence of a chiral boundary in M-theory). As we will see in the following section, the existence of such an object is also hinted at from the algebra.

Quick Remarks

A few remarks are in place. Recall that the M2/M5-brane solutions have a four-form flux turned on in the S^4 or AdS$_4$. The moduli provided by F_4 is such that supersymmetry is maximal at these configurations (i.e. 16 supercharges). Turning the flux on in the other directions can only lead to less or no supersymmetry preserved. This is the case of the Englert [174] and Pope [175] solutions. In fact, it turns out that writing the S^7 as a fibration over S^4 has been useful in finding squashed solutions [176–178] (see [178]).

Additionally, due to the cancellation between forces between such M-branes, it is possible to engineer solutions with multiple parallel M-branes by adding different harmonic forms localised at different positions. Take for instance the harmonic form for the M2-brane solution localised at the origin, $H(r) = 1 + \frac{k}{r^6}$. The multiple-M2-brane solution will then look like

$$H(r)_{\mathrm{M2}} = 1 + \sum_{i=1}^{N} \frac{k}{\|\vec{x} - \vec{x_i}\|^6}, \tag{2.13}$$

where $\vec{x}$ are the $\mathbb{R}^8$ coordinates transverse to the branes, and $\vec{x}_i$ are their respective position in that transverse space.

Another interesting avenue comes from considering multiple M-branes that interest in an orthogonal manner. Those type of configurations might not preserve all 16 supercharges that the parallel solutions do, however, there are ways of constructing them in such a way that at least 8 supercharges are. We will refer to this method of as the *harmonic rule* [179]. It is in that way that solutions of [1] were constructed and later included in the more general set of solutions derived in [2]. From the M-brane theory's perspective, one of these intersection resembles a defect insertion. This is precisely how we are able to study certain supersymmetric conformal defects in the 6d $(2,0)$ theory. We expand upon those statements in our paper [8], or equivalently in Chapter 4.

2.4 Its susy algebra

As promised, let us briefly describe the supersymmetry algebra in eleven dimensions. More specifically, we will focus on the appearance of central charges due to its solitonic objects. This will demonstrate the existence of the five half-BPS configurations mentioned in the previous section. Most of the content here can be found in the following reviews and lecture notes: [125, 127, 129, 130, 180].

From a purely Clifford-algebraic perspective, it can be shown that the standard commutator of supercharges on $\mathbb{R}^{1,10}$, [5]

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta}P_M, \tag{2.14}$$

can be supplemented with various central charges. These come as numbers multiplied to antisymmetrised products of gamma matrices, giving them a representation as differential forms on $\mathbb{R}^{1,10}$. Thus, the most general extension to this supersymmetry algebra takes the form

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^M)_{\alpha\beta}P_M + \frac{1}{2}(C\Gamma_{MN})_{\alpha\beta}Z^{MN} + \frac{1}{5!}(C\Gamma_{MNPQR})_{\alpha\beta}Y^{MNPQR}. \tag{2.15}$$

In both expressions above, P_M is the standard generator of translations in the super-Poincaré algebra. The numbers Z^{MN} and Y^{MNPQR} are the central charges associated to the solitonic objects of M-theory. Do note that, while they are indeed central to the odd part of this Lie superalgebra, they are not central charges of the entire algebra. Given their non-trivial spacetime indices, as one might expect, they transform accordingly under the action of the Poincaré subgroup.

The notion of algebra here is similar to to that of asymptotic symmetry in Einstein gravity, wherein one considers the algebra of generators in the asymptotic limit to flat space. In this case, we are looking at solutions to supergravity which asymptote to $\mathbb{R}^{32|1,10}$, or in other words, which asymptote to $\mathbb{R}^{1,10}$ together with 32 supercharges.

In a similar fashion to how one can couple particle world-line actions to Maxwell-Einstein gravity, it was realised that one can couple 11d supergravity to a closed supermembrane action [181]. That of a supersymmetric membrane that spans two spacial directions and one time direction is none-other than the M2-brane action

$$S = -T_2 \int_{\mathrm{M2}} \mathrm{vol}_g + Q_2 \int_{\mathrm{M2}} C_3. \tag{2.16}$$

Then, given a set of coordinates on the submanifold M2, say X^M, the existence of this coupling naturally induces the aforementioned central charge

$$Z^{MN} = Q_2 \int dX^M \wedge dX^N, \tag{2.17}$$

where the integral is performed along the spacial directions of the M2-brane only. A similar construction can be envisioned for the M5-brane, whose worldvolume spans five spacial directions and one time direction. The central charge this induces is

$$Z^{MNPQR} = Q_5 \int dX^M \wedge dX^N \wedge dX^P \wedge dX^Q \wedge dX^R, \tag{2.18}$$

[5]Recall that we are dealing with $\mathcal{N} = 1$ supersymmetry in eleven dimensions, which has 32 supercharges. They are each labelled by an index $\alpha \in \{1, \ldots, 32\}$.

where, again, the integral is performed on the spacial slice alone.

Both actions described above generate the 2-form central charge Z and five-form central charge Y from an integral over the spacial slice of their worldvolumes. Their Hodge duals, the nine-form $\star Z$ and six-form $\star Y$ should also have a corresponding integral over the worldvolume of a solitonic object. For $\star Z$ this is the M9-brane, whereas for $\star Y$ it is the KK6-monopole [182].

2.5 Further Reading

Given the vastness of the topic, one can never hope to cover all topics within M-theory and 11d supergravity. The sketch given in this thesis only serves the reader to understand the two papers [7, 8] by the author and collaborators, which are transcribed in Chapters 4 and 5.

Nevertheless, it may be important to point out a couple of interesting facts regarding the content discussed thus far.

The concepts are brane worldvolume action and central charges are intimately linked to that of calibration in supergravity (see [183] for a review and [184] for applications to M-branes). Indeed, a calibrated manifold necessarily minimises its volume, in the same way that BPS configurations of M-branes minimise their (generalised) volume [185].

Furthermore, the existence of a 2-form central charge within the supersymmetry algebra of the M5-brane hints at configurations wherein the M2- and M5-branes are bound together [186]. This is a common phenomenon in string theory, wherein configurations of D-branes of different dimensions can be 'bound together'. Alternatively, the smaller D-brane can be seen as 'puffing up' into the larger D-brane when coupled to non-trivial RR-forms of higher degree. This is the celebrated Myers effect [187].

Giving a description of these solitons as an action principle also allows one to study their dynamics, and more specifically their interaction [188–192]. However, proper topological considerations are required if one wishes to discuss anomaly cancellations within this theory [193].

Chapter 3

Supersymmetry on curved spaces

> Passer de la mécanique de Newton à celle d'Esintein doit être un peu, pour le mathématicien, comme de passer du bon vieux dialiecte provençal à l'argot parisien dernier cri. Par contre, passer à la mécanique quantique, j'imagine, c'est passer du français au chinois.
>
> Alexandre Grothendieck, Récoltes et Semailles (1986)

This chapter is mostly concerned with the application of supergravity to supersymmetry on curved spaces in four spacetime dimensions, with a view of using this in Chapters 6 and 7. All the material presented here is mostly self-contained and no prior knowledge of Chapter 2 is assumed. Though, some definitions in Appendix D may help readers who are unfamiliar with Killing spinors. A lot of the standard supergravity material can be found in the books [194, 195]. We also recommend [196], and more specifically [197] for a lightning overview of conformal superspace techniques in $\mathcal{N} = 2$ supergravity.

This chapter is organised as follows. In Section 3.1 we give a brief overview of the seminal work of Festuccia and Seiberg [198] on rigid supergravity and supersymmetry in curved spaces. That presentation is intended as an introductory piece, and as such, will not be thorough. The keen reader might find the lecture notes [199, 200] to be a better introduction to the topic. Sections 3.2.1 and 3.2.2, however, are more technical in that we give a detailed account of the Weyl multiplet fields for 4d $\mathcal{N} = 2$ and 4d $\mathcal{N} = 4$, in view of applying that knowledge in Chapters 6 and 7.

3.1 Rigid Supergravity and Festuccia-Seiberg

Consider a field theory described by a flat-space Lagrangian density $\mathcal{L}_{\mathbb{R}^d}$. Minimally coupling to a background metric g allows us to define a new Lagrangian density $\mathcal{L}_{\mathcal{M}}$ on a Riemannian manifold $(\mathcal{M}, g)$. The procedure of minimal coupling usually entails replacing the Euclidean metric δ with the curved metric g, partial derivatives ∂ with covariant derivatives ∇, and finally adding a volume factor $\sqrt{g}$ to the integration measure. The resulting action functional is suitably invariant under the diffeomorphisms of $\mathcal{M}$. Note that higher order diffeomorphism-invariant terms could be added in; these would, however, constitute a departure from the minimal coupling setup we are considering here.

On the other hand, the minimal coupling procedure can be viewed in a different, yet ultimately equivalent, light. Similarly to how a global $U(1)$ symmetry may be gauged using a background gauge field, minimal coupling may be regarded as a gauging of global symmetries. In this description, one starts from the flat space Lagrangian density $\mathcal{L}_{\mathbb{R}^d}$ and identifies its (generically non-unique) stress-energy tensor $T_{\mu\nu}$, whose existence and conservation follow from locality and translational invariance. The conservation equation, in particular, is given by $\partial_\mu T^{\mu\nu} = 0$, where all indices are raised and lowered using the Euclidean metric δ. The stress-energy tensor can then be coupled to a background rank-2 tensor field, which we will call h. The resulting Lagrangian density

$$\mathcal{L}_{\mathbb{R}^4} - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} \ , \tag{3.1}$$

however, still lacks the proper symmetries that would allow it to be placed on any manifold $\mathcal{M}$, equipped with a curved metric g. If we identify $g = \delta + h$, so that h captures the metric fluctuations about flat space, we can add the correct higher order terms that render the Lagrangian density invariant under general coordinate transformations of g. These are the so-called seagull terms. Putting these together, the action in the presence of the background field h constructed from $\mathcal{L}_{\mathcal{M}}$,

$$S_{\mathcal{M}} = \int_{\mathcal{M}} d^d x \sqrt{g} \mathcal{L}_{\mathcal{M}}, \qquad \mathcal{L}_{\mathcal{M}} = \mathcal{L}_{\mathbb{R}^d} - \frac{1}{2} h_{\mu\nu} T^{\mu\nu} + \mathcal{O}(h^2), \tag{3.2}$$

is now a well defined action functional on $\mathcal{M}$, for any Riemannian metric g. Note that we packed all seagull terms in the term $\mathcal{O}(h^2)$.

Unfortunately, one may face issues when placing a supersymmetric field theory on a curved manifold using the procedure of coupling the conserved stress-energy tensor to a background metric. Indeed, in order to preserve a subset of the flat-space supersymmetry generators, only a select few curved manifolds $\mathcal{M}$ are allowed. Let us denote spinors that generate flat-space supersymmetry transformations as ξ. While on

$\mathbb{R}^d$ any such constant spinor is a suitable supersymmetry generator[1], placing the theory on $\mathcal{M}$ restrict us to those sections of the spin bundle which are parallel with respect to it. This Killing spinor equation $\nabla_\mu \xi = 0$ severely restricts the types of geometries allowed. For instance, only Ricci-flat manifolds can admit parallel Killing spinors. In the case of compact four-dimensional manifolds, only T^4 and K3-surfaces with Ricci-flat Kähler metrics admit non-trivial solutions to the Killing spinor equation.

There exist alternative ways to define a supersymmetric field theory on any curved manifold. If the theory exhibits extended supersymmetry its R-symmetry group can be used in a topological twisting procedure. We remind the reader that in this setup we work with two bundles over the base space $\mathcal{M}$, the spinor bundle, on which the spin connection ω is defined, and the R-symmetry bundle, on which the R-symmetry connection A^R is defined. Introducing this second principal connection amounts to modifying the Killing spinor equation via the addition of a background field as,

$$\left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}A^R_\mu\right)\xi = 0, \tag{3.3}$$

where γ_a are the generators of the Clifford algebra $Cl(d)$. When the R-symmetry is large enough it is possible to identify the the R-symmetry bundle with a subbundle of the spin bundle. In the case of four-dimensional theories with an $SU(2)_R$ R-symmetry, this is done by identifying the R-symmetry connection with the Levi-Civita connection on either $SU(2)$-subbundles of the Spin(4) spin bundle. The resulting Killing spinor equation (3.3) admits solutions for which the spin connection ω and the R-symmetry connection A^R_μ act in a way as to cancel each other out. The solutions are then given by constant spinors, regardless of the metric choice on $\mathcal{M}$, allowing for the resulting theory to exhibit supersymmetry on any curved manifold. This is the topological twisting introduced by Witten [201].

The beauty behind the work of Festuccia and Seiberg [198] is to actually unify the above constructions into a single framework, and allow for more general setups of supersymmetry-preserving theories on curved spaces. To understand how it came about, we must first recall that minimally coupling the stress-energy tensor to a background metric does not, in general, lead to a supersymmetric theory on $\mathcal{M}$, as the Killing spinor equation becomes that of covariantly constant spinors on $\mathcal{M}$. One can look at this from the perspective of the modified Lagrangian $\mathcal{L}_\mathcal{M}$, as a perturbation around flat space $g = \delta + h$, where h is small. In this setup, the higher order seagull terms can be ignored and the linear-order term is the coupling between the background metric h and the stress-energy tensor of the flat-space theory $T^{\mu\nu}$. Since $T^{\mu\nu}$ is not a BPS-operator, this deformation does not generally preserve supersymmetry. On the other hand, this operator always belongs to a supermultiplet, the current supermultiplet, regardless of

[1] Given one chooses the canonical vielbeins, $e^a = dx^a$, formulated in terms of the Cartesian coordinates x^a.

whether the theory admits a Lagrangian formulation or not. It is then possible to introduce other background fields, that each couple to a different member of the supercurrent multiplet in such a way as to counter act the supersymmetry variation of $T^{\mu\nu}$ in the linear coupling, allowing for a supersymmetry-preserving deformation. This was done for four-dimensional $\mathcal{N} = 1$ theories in [202]. The partition function of such theories can then depend on more than just the topology of $\mathcal{M}$ [203, 204].

The systematic way of viewing this coupling of current multiplets is as follows. Consider a dynamical supergravity theory coupled to some matter. Here the specific supergroup we are gauging is not important. As long as the supergravity-matter theory has an off-shell formulation, the following will hold true. If we denote the supergravity fields by Ψ and the matter fields by ϕ, the action takes the schematic form

$$S[\Psi, \phi] = \frac{1}{M_p} \int \text{Kinetic}(\Psi, \partial\Psi, \partial^2\Psi) + \int \text{Coupling}(\Psi, \phi, \partial\phi), \tag{3.4}$$

where M_p is the Planck mass. In the formula above, the action $S[\Psi, \phi]$ is invariant under all superdiffeomorphisms of the underlying supermanifold. In other words, it is invariant under all supersymmetry variations of its fields on top of being invariant under all diffeomorphisms of the base manifold. Taking the *rigid limit* amounts to taking the Planck mass to infinity, $M_p \to \infty$. In doing so, the kinetic term can be discarded and all the supergravity fields, Ψ, freeze to a background configuration, Ψ_B,

$$S[\phi] = \int \text{Coupling}(\Psi_B, \phi, \partial\phi). \tag{3.5}$$

In this rigid limit, the background supergravity fields need not obey any equations of motion and can be frozen to any desired configuration. That being said, the "leftover supersymmetry" will depend on what that background configuration is. Indeed, in choosing a configuration of the supergravity fields we are explicitly breaking the superdiffeomorphism-invariance the original action was gifted with. How can one tell what (if any) amount of supersymmetry is preserved? To answer this question, requires us to look back at the supersymmetry variations of the supergravity fields Ψ. Whatever supergravity theory one starts with, the infinitesimal variations of its fields under any supersymmetry parameter are known[2]. Typically these appear with the structure

$$\delta\Psi = \mathcal{O}(\Psi, \partial\Psi), \tag{3.6a}$$
$$\delta\phi = \mathcal{O}(\Psi, \partial\Psi, \phi, \partial\phi), \tag{3.6b}$$

namely that variations of the supergravity fields only involve supergravity fields, while variations of the matter fields involve both matter and supergravity fields. If the background configuration is chosen in such a way that the supersymmetry variations of

[2] What is often made explicit is the Lie derivative of its superfields with respect to any super-vector field. This contains information about the action of infinitesimal bosonic and fermionic transformations. See [205] for the original construction of such a derivative on spinor fields (*champs de spineurs* therein).

all the supergravity fields vanish, $\delta\Psi = 0$, then the action $S[\phi]$ will be supersymmetric. The amount of supersymmetry is dictated by how many supercharges lead to $\delta\Psi = 0$.

Notably, the multiplet Ψ will always contain the gravitino ψ^i_μ. Its supersymmetry variation is a direct generalisation of the Killing spinor equation, the so-called *generalised Killing spinor equation*[3]. If the background configuration is chosen such that only the metric is set to a non-trivial value[4], then we recover the Killing spinor equations of minimal coupling $\nabla_\mu \xi$. On the other hand, if said background configuration also has an R-symmetry connection set to the spin connection, then we recover the topological twist (3.3).

It turns out that coupling supersymmetric field theories to these rigid supergravities, and further requiring various amount of supersymmetry, places important constraints on the background geometry (for 4 supercharges in 2d see [206], for 4 supercharges in 3d see [203, 207, 208], for four supercharges in 4d see [203, 204, 209–212], for eight supercharges in 4d see [213, 214]). For instance, [203] find that for $\mathcal{M}$ a four-dimensional manifold the existence of one Killing spinor implies that $\mathcal{M}$ is Hermitian. That of two Killing spinors implies that $\mathcal{M}$ is a torus fibration over a Riemann surface and finally that of four Killing spinors implies that $\mathcal{M}$ is either T^4 or $S^3 \times S^1$.[5]

The idea behind Chapters 6 and 7 will be that of finding such background configurations that preserve supersymmetry while also holding other interesting properties. Chapter 6 is concerned with constructing certain supersymmetric configurations of 4d $\mathcal{N} = 2$ conformal supergravity, while Chapter 7 deals with 4d $\mathcal{N} = 4$. For either chapter we need to present a few results on conformal supergravity in four dimensions, and motivate our usage of these.

3.2 Conformal Supergravity

Conformal supergravity is the supergravity theory of local conformal symmetry. In other words, it is the local gauging of a given superconformal algebra. This class of theories was originally introduced as a way of constructing Poincaré supergravities with $\mathcal{N} = 1$ [215–219], and later extended to $\mathcal{N} = 2$ [220]. This construction is now known as superconformal tensor calculus, or sometimes supergravity tensor calculus. A useful representation of this local superconformal algebra is via the *Weyl multiplet*, a

[3] In fact, throughout this thesis, we define the generalised Killing spinor equation to be the set of equations given by the vanishing of the supersymmetry variation of all fermionic fields in te supergravity multiplet. This includes the gravitino, naturally, but may involve other fields too.

[4] For a given choice of multiplet Ψ, minimal coupling might require auxiliary components to have non-trivial values. Here, we are only requiring the other gauge fields to have trivial value.

[5] These results, of course, do not cover all cases and we invite the reader to look up the original article for a more thorough presentation.

supermultiplet which contains the vielbeins[6]. There has been a huge effort by the supergravity community in constructing such multiplets in various dimensions and for various amounts of supersymmetry (2d [221], 3d [222–225], 4d [220, 225–237], 5d [225, 235, 238–242], 6d [225, 235, 241, 243–247]).

The main advantage the superconformal tensor calculus offers is in giving an off-shell formulation of the Poincaré supergravities, obtained from coupling a compensating matter multiplets to the conformal supergravity theory. For example, coupling to a conformal chiral multiplet in 4d $\mathcal{N}=1$ leads to the *old-minimal* Poincaré supergravity [248–250], while coupling to a current multiplet gives the *new-minimal* supergravity [251, 252][7]. This off-shell formulation also lends itself well to applications in localisation [53, 255–257], from which our interest originates (see Chapter 6 for more details).

In this thesis, however, we will only make use of the Weyl multiplets of conformal supergravity in four dimensions. Our interest not lying in Poincaré supergravities, but rather on their conformal parents. The reason being that we wish to study superconformal field theories in curved spaces, using the tools introduced by Festuccia and Seiberg [198], which we described in section 3.1. This coupling to conformal supergravity is natural if one wishes to preserve the conformal structure of the theory [209, 211, 258]. The only notable addition to section 3.1, is the presence of additional fermionic components of the supergravity multiplet[8]. These additional fermionic fields lead to additional constraints on the background configuration, given as a vanishing of their supersymmetric variations (which we will collectively call the generalised Killing spinor equations). We will detail those when appropriate, in the subsections 3.2.1 and 3.2.2.

3.2.1 The Weyl multiplet of 4d $\mathcal{N}=2$ conformal supergravity

Four-dimensional $\mathcal{N}=2$ conformal supergravity is the gauge theory of the superconformal algebra $\mathfrak{su}(2,2|2)$. This algebra contains an eight-dimensional odd part, and as such, describes a conformal supergravity theory with eight supercharges. Its bosonic subalgebra decomposes into the standard conformal algebra in four dimensions, $\mathfrak{su}(2,2)\simeq\mathfrak{so}(2,4)$, and the R-symmetry algebra, $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$[26]. Importantly, however, we will only be interested in its Euclidean counterpart, wherein the conformal algebra is given by $\mathfrak{so}(1,5)$[9]. We will present here its formulation in terms of the Weyl multiplet

[6]In general, this multiplet is not unique and one can trade off various of its field constituents for others. Examples of this are the Dilaton-Weyl multiplet, or Hyperdilaton-Weyl multiplet.

[7]It is also possible to couple to other multiplets of conformal supergravity to obtain non-minimal Poincaré supergravities [253, 254].

[8]The presence of a dilaton field, for instance, will always require a supersymmetric partner, the dilatino.

[9]While the R-symmetry algebra remains identical under Wick rotation, its global structure is no longer that of $SU(2)\times U(1)$ but rather $SU(2)\times SO(1,1)$.

Local symmetry	**Weyl multiplet field**	**$U(1)_r$ weight**
Independent Connections		
Translations	${e^a}_\mu \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$	0
Dilatations	$b_\mu \in \Gamma(T^*\mathcal{M})$	0
$U(1)_r$ R-symmetry	$A_\mu^{U(1)_r} \in \Gamma(T^*\mathcal{M})$	0
$SU(2)_R$ R-symmetry	${A_\mu^{SU(2)_R}}^i{}_k \in \Gamma\Big(T^*\mathcal{M} \otimes \mathrm{Sym}^2\, \mathcal{K}_R\Big)$	0
Q-SUSY	$\psi^{\alpha i}_{+\mu},\ \psi^{\dot\alpha i}_{-\mu} \in \Gamma\Big(T^*\mathcal{M} \otimes S\mathcal{M} \otimes \mathcal{K}_R\Big)$	$\mp\frac{1}{2}$
Composite Connections		
Lorentz	${\omega_\mu}^{ab} \in \Gamma\Big(T^*\mathcal{M} \otimes \Lambda^2 T\mathcal{M}\Big)$	0
S-SUSY	$\phi^{\alpha i}_{+\mu},\ \phi^{\dot\alpha i}_{-\mu} \in \Gamma\Big(T^*\mathcal{M} \otimes S\mathcal{M} \otimes \mathcal{K}_R\Big)$	$\pm\frac{1}{2}$
Special Conformal	${f_\mu}^a \in \Gamma\Big(T^*\mathcal{M} \otimes T\mathcal{M}\Big)$	0
Auxiliary Fields		
—	$D \in \Omega^0(\mathcal{M})$	0
—	$T^\pm_{\mu\nu} \in \Omega^2(\mathcal{M})$	± 1
—	$\chi^{\alpha i}_+,\ \chi^{\dot\alpha i}_- \in \Gamma\big(S\mathcal{M} \otimes \mathcal{K}_R\big)$	$\mp\frac{1}{2}$

TABLE 3.1: Field content of the 4d $\mathcal{N}=2$ Weyl multiplet on an oriented spin Riemannian manifold $\mathcal{M}$. Throughout, $\Gamma(P)$ denotes the space of sections over a bundle $P \longrightarrow \mathcal{M}$, $S\mathcal{M}$ is the spin bundle over $\mathcal{M}$, $\mathcal{K}_R$ is the total space of the associated bundle to the principal $SU(2)_R$ R-symmetry bundle over $\mathcal{M}$, and $\Omega^k(\mathcal{M})$ is the space of k-forms on $\mathcal{M}$. Wherever present, $\pm$ denotes either (self-)anti-self duality or positive/negative chirality. The latter also entails that the spinor is a section of one of the two parts of the spin bundle $\mathcal{SM} \cong \mathcal{S}^+\mathcal{M} \otimes \mathcal{S}^-\mathcal{M}$ only.

constructed in [259] from an off-shell timelike reduction of the Lorentzian 5d Weyl multiplet.

The Weyl multiplet is made up of the connections of the superconformal algebra. Its field content, as well as the various $U(1)_r$ weights, are presented in table 3.1. In addition to the Riemannian metric g, or equivalently the vierbein fields e^a, the multiplet contains the connections for dilatations, $\mathfrak{u}(1)_r$ and $\mathfrak{su}(2)_R$ R-symmetries, and Poincaré supersymmetries, the latter being the gravitino ψ. The remaining fields, namely the spin connection ω, the conformal gravitino ϕ, and the special conformal connection f, are composite, meaning that they can be expressed in terms of the other fields in an algebraic fashion via the so-called conventional constraints of conformal supergravity. For instance, the conformal gravitino with $\mathrm{Spin}(4) \cong SU(2)_\ell \times SU(2)_r$ indices suppressed is expressed as

$$ {\phi_\mu}^i = -\frac{i}{2}\left(\gamma^{\rho\nu}\gamma_\mu - \frac{1}{3}\gamma_\mu\gamma^{\rho\nu}\right)\left(\mathcal{D}_\rho {\psi_\nu}^i + \frac{i}{32}T_{ab}\gamma^{ab}\gamma_\rho {\psi_\nu}^i + \frac{1}{4}\gamma_{\rho\nu}\chi^i\right)\ , \tag{3.7} $$

Local symmetry	Parameter	Weyl weight	$U(1)_r$ weight	χrality
Q-SUSY	$\xi_+^{\alpha i},\ \xi_-^{\dot\alpha i}$	$-\frac{1}{2}$	$\mp\frac{1}{2}$	$\pm$
S-SUSY	$\eta_+^{\alpha i},\ \eta_-^{\dot\alpha i}$	$\frac{1}{2}$	$\pm\frac{1}{2}$	$\pm$

TABLE 3.2: The (Grassmann even) parameters of Q- and S-supersymmetries, and their quantum numbers.

where the 2-form[10] T and the dilatino χ are auxiliary fields whose presence is demanded by off-shell closure of the algebra in field space. For the same reason, one must also introduce an auxiliary scalar field[11] D. This completes the field content of the $\mathcal{N}=2$ Weyl multiplet in four Euclidean dimensions.

The Poincaré (Q-) and conformal (S-) supersymmetries have Dirac spinorial parameters ξ and η, respectively, whose quantum numbers are displayed in table 3.2. Aligning our conventions with those of the supersymmetric localization literature, we take the parameters ξ and η to be Grassmann even, so that their derivations $\delta_{\xi,\eta}$ are Grassmann odd. Following [259], albeit with an opposite sign for the spin connection, the covariant derivatives of ξ and η with respect to Lorentz, dilatation, and R-symmetry transformations are

$$\mathcal{D}_\mu\xi^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}b_\mu + \frac{1}{2}A_\mu^{U(1)_r}\gamma^5\right)\xi^i + \frac{1}{2}A_\mu^{SU(2)_R}{}^i{}_j\xi^j\ , \tag{3.8a}$$

$$\mathcal{D}_\mu\eta^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{1}{2}b_\mu - \frac{1}{2}A_\mu^{U(1)_r}\gamma^5\right)\eta^i + \frac{1}{2}A_\mu^{SU(2)_R}{}^i{}_j\eta^j\ , \tag{3.8b}$$

where γ^5 is the chirality matrix in four Euclidean dimensions (see Appendix D for our detailed conventions). We note in particular our choice of normalization of the R-symmetry gauge fields, for instance the lack of factors of i compared to e.g. [261].

Importantly, what constitutes our generalised Killing spinor equations, namely the supersymmetric variations of the gravitino and dilatino under the parameters (ξ,η), are

$$\delta_{\xi,\eta}\psi_\mu{}^i = 2\mathcal{D}_\mu\xi^i + \frac{i}{16}T_{ab}\gamma^{ab}\gamma_\mu\xi^i - i\gamma_\mu\eta^i\ , \tag{3.9a}$$

$$\delta_{\xi,\eta}\chi^i = \frac{i}{24}\gamma^{ab}\gamma^\mu(\mathcal{D}_\mu T_{ab})\xi^i + \frac{1}{24}T_{ab}\gamma^{ab}\eta^i + \frac{1}{6}R_{\mu\nu}^{SU(2)_R}{}^i{}_j\gamma^{\mu\nu}\xi^j - \frac{1}{3}R_{\mu\nu}^{U(1)_r}\gamma^{\mu\nu}\gamma^5\xi^i + D\xi^i\ , \tag{3.9b}$$

where the supercovariant curvature tensors for the R-symmetry connections are

$$R^{SU(2)_R}{}^i{}_j = dA^{SU(2)_R}{}^i{}_j + \frac{1}{2}A^{SU(2)_R}{}^i{}_k \wedge A^{SU(2)_R}{}^k{}_j + (\text{fermions})\ , \tag{3.10a}$$

$$R^{U(1)_r} = dA^{U(1)_r} + (\text{fermions})\ , \tag{3.10b}$$

[10] The (anti) self-dual components $T^\pm = \frac{1}{2}(T \pm \star T)$ are independent, since $\star^2 = 1$ upon acting on 2-forms on Riemannian manifolds.

[11] Our choice for D coincides with the definition of the field $\tilde{d}$ used in [258], and is related to the field M of [260] by $6D = -3M - R$.

while the covariant derivative of the 2-form T has components

$$\mathcal{D}_\mu T_{ab} = \partial_\mu T_{ab} + \omega_{\mu a}{}^c T_{cb} + \omega_{\mu b}{}^c T_{ac} - A_\mu^{U(1)_r} (\star T)_{ab} \ . \tag{3.11}$$

We also mention in passing the expression for the curvature tensor of the spin connection,[12]

$$R(\omega)^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \ . \tag{3.12}$$

When considering supersymmetric configurations for which the supersymmetric variations of the spinorial components vanish, $\delta_{\xi,\eta}\psi = \delta_{\xi,\eta}\chi = 0$, the S-susy parameter η can be entirely written in terms of the Q-susy parameter

$$\eta^i = -\frac{i}{2}\gamma^\mu \mathcal{D}_\mu \xi^i \ , \tag{3.13}$$

where a term containing the self-dual 2-form T vanishes due to the properties of the Dirac matrices.

For completeness, we also note here the supersymmetric variations of the bosonic fields in the Weyl multiplet (see (3.6) in [259])

$$\delta_\xi e_\mu{}^a = \bar\xi_i \gamma^5 \gamma^a \psi_\mu{}^i \tag{3.14a}$$
$$\delta_\xi b_\mu = \frac{i}{2}\bar\xi_i \gamma^5 \phi_\mu{}^i - \frac{3}{4}\bar\xi_i \gamma^5 \gamma_\mu \chi^i + \frac{i}{2}\bar\eta_i \gamma^5 \psi_\mu{}^i \tag{3.14b}$$
$$\delta_\xi A_\mu^{U(1)_r} = -\frac{i}{2}\bar\xi_i \phi_\mu{}^i - \frac{3}{4}\bar\xi_i \gamma_\mu \chi^i - \frac{i}{2}\bar\eta_i \psi_\mu{}^i \tag{3.14c}$$
$$\delta_\xi A_\mu^{SU(2)_R}{}^i{}_j = 2i\bar\xi_j \gamma^5 \phi_\mu{}^i - 3\bar\xi_j \gamma^5 \gamma_\mu \chi^i - 2i\bar\eta_j \gamma^5 \psi_\mu{}^i - SU(2)_R - (\text{trace}) \tag{3.14d}$$
$$\delta_\xi T_{ab} = -8i\bar\xi_i \gamma^5 R(Q)_{ab}{}^i \tag{3.14e}$$
$$\delta_\xi D = \bar\xi_i \gamma^5 \gamma^\mu \mathcal{D}_\mu \chi^i \tag{3.14f}$$

where the supercovariant curvature of the gravitino is

$$R(Q)_{ab}{}^i = 2\mathcal{D}_{[a}\psi_{b]}{}^i - i\gamma_{[a}\phi_{b]}{}^i + \frac{i}{16}T_{cd}\gamma^{cd}\gamma_{[a}\psi_{b]}{}^i \ . \tag{3.15}$$

Other than our choice of sign convention for the spin connection, all other quantities follow precisely [259]. Consequently, we refer the reader back to them for a detailed expression of all composite fields and supercovariant curvature tensors.

3.2.2 Weyl multiplet of 4d $\mathcal{N} = 4$ conformal supergravity

When considering 4d $\mathcal{N} = 4$, a $\mathfrak{u}(1)$ bosonic factor decouples into a bonus symmetry, so that the simple conformal superalgebra is actually the quotient $\mathfrak{psu}(2,2|4)$ instead of

[12] We wish to emphasise that we adhere to a different sign convention for the spin connection than that used in [259]. In our convention, the sphere enjoys positive scalar curvature.

$\mathfrak{su}(2,2|4)$ [26]. Its bosonic subalgebra splits into the standard conformal algebra in four dimensions, $\mathfrak{su}(2,2) \simeq \mathfrak{so}(2,4)$, and the R-symmetry algebra, $\mathfrak{su}(4)$. Ultimately, we wish to describe the Weyl multiplet in Euclidean signature, just as we did for $\mathcal{N}=2$, however, as the literature lacks such a description, we will instead focus here on the Lorentzian signature, for which the literature is quite rich. Part of our work in Chapter 7, will be to actually come up with a Euclidean formulation by Wick rotating this multiplet, which implies forgoing reality conditions and complexifying its field components.

The original formulation of 4d $\mathcal{N}=4$ conformal supergravity was due to [227, 262–265], even though formulations of $\mathcal{N}=4$ supergravity theories were already known [266–268] without the need of the conformal construction. While the Weyl multiplet and its transformations were known since the 80s [227], a complete action of $\mathcal{N}=4$ conformal supergravity was only found 35 years later [269, 270]. Part of the difficulty lied in the presence of the scalar fields ϕ_α in the Weyl multiplet, which parametrise the coset $SU(1,1)/U(1)$. There is, thus, an entire moduli space of conformal supergravity actions [232]. Our presentation here, however, will follow the one given in [269], whose conventions follow closely [227][13]. A similar presentation can also be found in [271]. As the aim of Chapter 7 is not to construct supergravity actions, but rather to construct supersymmetric configurations of the Weyl multiplet, we will only focus on the latter.

They Weyl multiplet contains the vierbein field $e^a{}_\mu$, the dilatation gauge field b_μ and the $SU(4)$ gauge field $V_\mu{}^i{}_j$ as its independent bosonic connections. The gauge fields associated to special conformal transformations $f^a{}_\mu$, the bonus $U(1)$-gauge field a_μ and the spin connection ω_μ^{ab} are composite. Its fermionic gauge fields are the gravitino $\psi_\mu{}^i$, associated to Q-susy transformations and the composite conformal gravitino $\phi_\mu{}^i$, associated to S-susy transformations. The remainder of this multiplet's bosonic content is made up of the scalar fields ϕ_α, $\alpha \in \{1,2\}$, which parametrise the $SU(1,1)/U(1)$ coset space, the scalar fields E_{ij} and $D^{ij}{}_{kl}$, and the two-form T^{ij}. Finally, the supersymmetric partners of these are the gaugino Λ^i and the dilatino $\chi^{ij}{}_k$. The multiplet and its fields are summarised in Table 3.3, including all constraints obeyed by them and their $SU(4)$ representation.

We also note that, all throughout, we employ the chiral $SU(4)$ notation [227, 262]. This notation assigns conjugate representations of the R-symmetry group to fermions of opposite chirality. For example, the conjugate to the gravitino is a fermionic field in the $\overline{\mathbf{4}}$ representation of $SU(4)$, $(\psi_\mu{}^i) = \psi_{\mu i}$[14]. Since we are working in Lorentzian signature, these fields are not independent and we will not need to consider them separately.

[13] There are certain differences between the two references. We choose to follow the conventions of [269] for which, amongst other choices, the local-frame Levi-Civita symbol obeys $\epsilon^{1234} = -1$. We also decide to part from their spin connection convention and adopt one where the sphere has a positive Ricci scalar curvature.

[14] See the appendix in Bergshoeff's thesis for a detailed description of this procedure [262].

Field	Name	Constraints	SU(4) Rep
${\psi_\mu}^i$	Gravitino	$\gamma^5 {\psi_\mu}^i = {\psi_\mu}^i$	$\mathbf{4} = \ydiagram{1}$
${\phi_\mu}^i$	S-gauge field	(auxiliary)	
Λ_i	Gaugino	$\gamma^5 \Lambda_i = \Lambda_i$	$\overline{\mathbf{4}} = \ydiagram{1,1,1}$
${\chi^{ij}}_k$	Dilatino	${\chi^{ij}}_k = -{\chi^{ji}}_k$, ${\chi^{ij}}_j = 0$, $\gamma^5 {\chi^{ij}}_k = {\chi^{ij}}_k$	$\mathbf{20} = \ydiagram{2,2,1}$
${e^a}_\mu$	Vierbeins		$\mathbf{1}$
ω_μ^{ab}	Spin connection	$\omega_\mu^{ab} = -\omega_\mu^{ba}$	$\mathbf{1}$
b_μ	Dilatation gauge field		$\mathbf{1}$
${V_\mu}^i{}_j$	$SU(4)$-gauge field	$({V_\mu}^i{}_j)^* = -{V_\mu}^j{}_i$, ${V_\mu}^i{}_i = 0$	$\mathbf{15} = \ydiagram{2,1,1}$
${f^a}_\mu$	K-gauge field	(auxiliary)	$\mathbf{1}$
a_μ	$U(1)$-gauge field	(auxiliary)	$\mathbf{1}$
${T_{ab}}^{ij}$		${T_{ab}}^{ij} = -{T_{ab}}^{ji}$, $\frac{1}{2}{\epsilon_{ab}}^{cd} T^{ij}_{cd} = -T^{ij}_{ab}$	$\mathbf{6} = \ydiagram{1,1}$
E_{ij}		$E_{ij} = E_{ji}$	$\overline{\mathbf{10}} = \ydiagram{2,2,2}$
ϕ_α		$(\phi_1)^* = \phi^1$, $(\phi_2)^* = -\phi^2$, $\phi^\alpha \phi_\alpha = 1$	$\mathbf{1}$
${D^{ij}}_{kl}$		${D^{ij}}_{kl} = -{D^{ji}}_{kl} = -{D^{ij}}_{lk}$, ${D^{ij}}_{kl} = \frac{1}{4}\epsilon^{ijmn}\epsilon_{klpq}{D^{pq}}_{mn}$, $({D^{ij}}_{kl})^* = {D^{kl}}_{ij}$, ${D^{ij}}_{kj} = 0$	$\mathbf{20}' = \ydiagram{2,2}$

TABLE 3.3: Field content of the Lorentzian $\mathcal{N} = 4$ Weyl multiplet. The double line separates the four fermionic fields (top) from the bosonic ones (bottom). The third row details all constraints obeyed by the field, including chirality/self-duality (if any). We de not detail those for the auxiliary fields. The final row displays the $SU(4)_R$ representation.

Associated to every Q- and S-supersymmetry transformation of the fields are the parameters ξ^i and η^i, respectively. In our conventions, the Q-susy parameter ξ^i has positive chirality, while that of S-susy η^i has negative chirality, yet both are in the same $\mathbf{4}$ representation of $SU(4)$. These couple to the spin connection, dilaton gauge field, bonus $U(1)$ gauge field and $SU(4)$ gauge field through their covariant derivatives[15],

$$\mathcal{D}_\mu \xi^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}(b_\mu + i a_\mu)\right)\xi^i - V_\mu{}^i{}_j\xi^j, \tag{3.16a}$$

$$\mathcal{D}_\mu \eta^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{1}{2}(b_\mu - i a_\mu)\right)\eta^i - V_\mu{}^i{}_j\eta^j. \tag{3.16b}$$

In parallel to the $\mathcal{N} = 2$ case, the full superconformal derivative of these fields ($D_\mu\xi^i$ and $D_\mu\eta^i$) would also couple them to the fermionic gauge fields. However, as we will never need to consider non-trivial fermionic configurations, we will leave out their full expressions. The $SU(1,1)/U(1)$ coset fields ϕ^α also admit such a superconformal covariant derivative, be it without the spin connection and $SU(4)$ connections. From it, one can define the vectors P_μ, $\bar{P}_\mu$, together with a supercovariant $U(1)$ field strength F_{ab},

$$P_\mu = \epsilon_{\alpha\beta}\phi^\alpha D_\mu\phi^\beta, \tag{3.17a}$$

$$\bar{P}_\mu = -\epsilon^{\alpha\beta}\phi_\alpha D_\mu\phi_\beta, \tag{3.17b}$$

$$F_{ab} = 2i\bar{P}_{[a}P_{b]} - \frac{i}{2}(\bar{\Lambda}^i\gamma_{[a}D_{b]}\Lambda_i - h.c.), \tag{3.17c}$$

where $\epsilon^{12} = \epsilon_{12} = 1$. For completeness, we also note here the covariant derivative acting on the two-form T^{ij} and scalar field E_{ij},

$$\mathcal{D}_\mu T_{ab}{}^{ij} = \partial_\mu T_{ab}{}^{ij} + \omega_{\mu a}{}^c T_{cb}{}^{ij} + \omega_{\mu b}{}^c T_{ac}{}^{ij} - V_\mu{}^i{}_k T_{ab}{}^{kj} - V_\mu{}^j{}_k T_{ab}{}^{ik}, \tag{3.18a}$$

$$\mathcal{D}_\mu E_{ij} = \partial_\mu E_{ij} + V_\mu{}^k{}_i E_{kj} + V_\mu{}^k{}_j E_{ik}. \tag{3.18b}$$

Again, we will not set out to describe the full superconformal derivative of these fields ($D_\mu T_{ab}{}^{ij}$ and $D_\mu E_{ij}$) as we will not need them in the coming work.

Finally, we are ready to present what will become the generalised Killing spinor equations, namely the supersymmetric variations of the spinorial components of the Weyl multiplet. Under a combination of a Q-susy transformation, parametrised by the spinor ξ^i, and an S-susy transformation, parametrised by the spinor η^i, the gravitino,

[15] We emphasise again our differing conventions which spawn a positive sign in front of the spin connection term.

gaugino and dilatino transform following

$$\delta_{\xi,\eta}\psi_\mu{}^i = 2D_\mu\xi^i - \frac{1}{2}\gamma^{ab}T_{ab}{}^{ij}\gamma_\mu\xi_j + \epsilon^{ijkl}\bar{\psi}_{\mu j}\xi_k\Lambda_l - \gamma_\mu\eta^i, \tag{3.19a}$$

$$\delta_{\xi,\eta}\Lambda_i = -2\bar{P}_\mu\gamma^\mu\xi_i + E_{ij}\xi^j + \frac{1}{2}\epsilon_{ijkl}T_{bc}{}^{kl}\gamma^{bc}\xi^j, \tag{3.19b}$$

$$\begin{aligned}\delta_{\xi,\eta}\chi^{ij}{}_k = &-\frac{1}{2}\gamma^{ab}D_\mu T_{ab}{}^{ij}\gamma^\mu\xi_k - \gamma^{ab}R(V)_{ab}{}^{[i}{}_k\xi^{j]} - \frac{1}{2}\epsilon^{ijlm}D_\mu E_{kl}\gamma^\mu\xi_m \\ &+ D^{ij}{}_{kl}\xi^l - \frac{1}{6}\epsilon_{klmn}E^{l[i}\gamma^{|ab|}(T_{ab}{}^{j]n}\xi^m + T_{ab}{}^{|mn|}\xi^{j]}) \\ &+ \frac{1}{2}E_{kl}E^{l[i}\xi^{j]} - \frac{1}{2}\epsilon^{ijlm}\bar{P}_\mu\gamma^\mu\gamma_{ab}T^{ab}{}_{kl}\xi_m - (\text{traces}) \\ &+ \frac{1}{2}T_{ab}{}^{ij}\gamma^{ab}\eta_k + \frac{2}{3}\delta^{[i}_k T_{ab}{}^{j]l}\gamma^{ab}\eta_l - \frac{1}{2}\epsilon^{ijlm}E_{kl}\eta_m + (\text{fermions}).\end{aligned} \tag{3.19c}$$

Since the dilatino $\chi^{ij}{}_k$ obeys a traceless condition in its $SU(4)$ indices, $\chi^{ij}{}_j = 0$, its supersymmetry transformation must too. This explains the presence of the $-$(traces) notation in the equation above. In practice, however, if $\Delta\chi^{ij}{}_k$ denotes the right-hand-side of $\delta_{\xi,\eta}\chi^{ij}{}_k$ without the traces removed, then $\delta_{\xi,\eta}\chi^{ij}{}_k = \Delta\chi^{ij}{}_k + \frac{2}{3}\delta^{[i}{}_k\Delta\chi^{j]l}{}_l$[16].

[16]For completeness, we may also note that a similar expression holds for $D^{ij}{}_{kl}$ and its traceless condition $D^{ij}{}_{kj} = 0$. If $C^{ij}{}_{kl}$ is any $SU(4)$-tensor in the $\mathbf{6}\otimes\mathbf{6}$ representation, then its traceless counterpart can be constructed component-wise following $D^{ij}{}_{kl} = C^{ij}{}_{kl} - (\delta^{[i}{}_k C^{j]m}{}_{lm} - \delta^{[i}{}_l C^{j]m}{}_{km}) + \frac{1}{3}\delta^{[i}{}_k\delta^{j]}{}_l C^{mn}{}_{mn}$.

Part II

Holographic Defects

Chapter 4

Codimension-4 defect in 6d SCFT

Are we meeting in 15 mins?

Pietro Capuozzo

4.1 Introduction

Six dimensional (6d) superconformal field theories (SCFTs) hold a special place among quantum field theories (QFTs). Owing to the classification discovered in the seminal work by Nahm [26], superconformal symmetry is only possible in six and fewer spacetime dimensions. Moreover, $\mathcal{N} = (2,0)$ is the maximal amount of supersymmetry (SUSY) that a 6d theory can have. Combining this amount of SUSY with conformal symmetry constrains a 6d $\mathcal{N} = (2,0)$ SCFT to such a degree that the only additional information that is necessary to completely determine the theory is the choice of a gauge algebra.

The study of the 6d $\mathcal{N} = (2,0)$ theory is thus of fundamental importance in QFT, for many reasons. For example, the 6d $\mathcal{N} = (2,0)$ SCFT determines the physics of many other QFTs in 6d, via SUSY-breaking deformations [272–274] such as orbifolds [275–278]. By suitable (partial) topological twisting, the 6d theory compactified on, e.g., a Riemann surface [279] or a 3-manifold [280] can also determine the physics of infinite families of QFTs in $d < 6$.

The 6d $\mathcal{N} = (2,0)$ SCFT is also of fundamental importance in quantum gravity. Currently, the leading candidate for an ultra-violet (UV)-complete theory of quantum gravity is M-theory. M-theory's fundamental objects are M2-branes [167] and M5-branes [169], and the low-energy worldvolume theory on M coincident M5-branes is the 6d $\mathcal{N} = (2,0)$ SCFT with gauge algebra A_{M-1} [281]. Understanding the 6d $\mathcal{N} = (2,0)$ SCFT is thus essential to understanding M-theory in general. In particular, via the Anti-de Sitter/CFT (AdS/CFT) correspondence, the 6d $\mathcal{N} = (2,0)$ SCFT can provide a fully non-perturbative definition of M-theory on an asymptotically 7d AdS spacetime, AdS_7, times a four-sphere, S^4 [10, 282, 283].

Strongly interacting SCFTs constructed in string- and M-theory, including the non-Abelian 6d $\mathcal{N} = (2, 0)$ SCFT, are prohibitively difficult to study, for many reasons, of which we will mention only three. First, the $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (2, 0)$ SUSY multiplets include a chiral two-form gauge field, and writing a local, gauge-invariant Lagrangian for a non-Abelian higher-form gauge field remains a major open problem. These SCFTs thus have no known Lagrangian descriptions. Second, in the space of renormalization group (RG) flows, these SCFTs are *isolated* fixed points, and in particular they cannot be reached as infra-red (IR) fixed points of RG flows from free ultra-violet (UV) fixed points. Third, these SCFTs are intrinsically strongly interacting. For example, the 6d $\mathcal{N} = (2, 0)$ SCFT has no dimensionless parameter besides M that can be tuned to allow a perturbative expansion.

As a result, practically all of our direct knowledge[1] of interacting 6d SCFTs comes from non-perturbative methods, such as the superconformal bootstrap [285], F-theory [286], and especially AdS/CFT [287], where holographic computations of quantities like Weyl anomalies [288] and entanglement entropy (EE) [289] are used to great effect to characterise 6d SCFTs at large M.

An aspect of 6d SCFTs, and generally of QFTs in three and higher dimensions, that requires particularly careful treatment to characterise is the spectrum of 2d, string-like or surface, defects. In the co-dimension one case, 2d defects in 3d QFTs arise as boundaries or interfaces, and so are more easily studied and, thus, more familiar than their higher co-dimension realizations. Despite being somewhat more exotic in standard treatments of QFTs, 2d defects of co-dimension two and greater show up in a number of settings[2]: from free field theories [291–294] to strongly interacting and non-Lagrangian 4d QFTs, e.g. [295–297], to being fundamental objects in 6d SCFTs [298] and in the study of EE and Renyi entropies [299, 300]. As such, the last few decades have seen tremendous advancements in characterising [107] and constraining [301, 302] the properties of 2d defects, and it is vitally important in the study of QFTs, generally, to continue this effort by finding novel constructions of surface defects and examining their unique physics.

Of interest to us in the current work are the holographic descriptions, afforded by AdS/CFT, of 6d SCFTs and the defects that they support. In particular, we will primarily focus our attention on solutions to 11d supergravity (SUGRA) that are contained in a one-parameter family of solutions with superisometry given by the exceptional Lie superalgebra $\mathfrak{d}(2, 1; \gamma) \oplus \mathfrak{d}(2, 1; \gamma)$ [2]. Crucially, an asymptotically

[1]Indirect methods such as dimensional reduction to 5d $\mathcal{N} \leq 2$ SUSY QFTs have also been used to great effect to study these theories, e.g. by using the resulting lower dimensional Lagrangian description together with supersymmetric localization techniques [284].

[2]This is by no means an exhaustive list of the work done on 2d defects. For a recent review of boundaries and defects in QFTs and further references on the topic, see [290].

$\mathrm{AdS}_7 \times S^4$ solution is possible only at certain values of γ. Indeed, as a historical note, prior to the full classification given in [2], evidence of $\mathfrak{d}(2,1;\gamma) \oplus \mathfrak{d}(2,1;\gamma)$ invariant SUGRA solutions that exist for general γ were found in superconformal Janus solutions in 4d gauged $\mathcal{N} = 8$ SUGRA [303], which extended beyond the known $\gamma = 1$ $\mathrm{AdS}_4 \times S^7$ Janus solution of 11d SUGRA[304].

More pertinent to the cases that we will study below, the most well studied case among the values of γ that admit asymptotically locally $\mathrm{AdS}_7 \times S^4$ solutions is $\gamma = -1/2$, which holographically describes 1/2-BPS Wilson surface operators in the 6d A_{M-1} $\mathcal{N} = (2,0)$ SCFT that preserve a large $\mathcal{N} = (4,4)$ 2d SUSY. These BPS Wilson surface operators have a long history in M-theory descriptions of 6d SCFTs [281, 298, 305–307], and there has been a recent resurgence of interest in these defects where holographic [51, 308, 309] and field theoretic [310–312] computations have characterised these defect CFTs through their EE and Weyl anomalies.

Recently, new solutions in 11d SUGRA have been constructed that are proposed to be holographically dual to 2d BPS surface defects in 6d $\mathcal{N} = (1,0)$ SCFTs preserving "small" $\mathcal{N} = (4,4)$ or $\mathcal{N} = (0,4)$ 2d SUSY [1]. In the sections below, we will clearly demonstrate that these new solutions also fit into the one-parameter family of solutions in [2] in the limit where $\gamma \to -\infty$. It will turn out that the solutions in [1] are, in fact, a particular case within a broader class of solutions that can be realised in the $\gamma \to -\infty$ limit, which we will characterise by computing their contributions to the EE of a spherical region.

A crucial point that we emphasise in our construction of the $\gamma \to -\infty$ solutions is that the superisometry algebra of the near-horizon limit of a stack of M5-branes, $\mathfrak{osp}(8^*|4)$, does not contain $\mathfrak{d}\,(2,1;\gamma) \oplus \mathfrak{d}\,(2,1;\gamma)$ as a sub-superalgebra [313–315]. In the supergravity solution, this is manifested by the appearance of additional terms in the four-form flux as compared to that due to pure M5-branes. Nevertheless, the geometry is still locally asymptotically $\mathrm{AdS}_7 \times S^4$, which is in line with the fact that the bosonic subalgebra of $\mathfrak{d}\,(2,1;\gamma) \oplus \mathfrak{d}\,(2,1;\gamma)$ is a subalgebra of $\mathfrak{osp}(8^*|4)$.

One upshot of our analysis that follows from the identification of the global symmetries in the $\gamma \to -\infty$ limit is that it allows for a suitable regulation scheme, which we will employ when we compute the defect sphere EE. Lacking a generalised program of holographic renormalization for SUGRA solutions dual to defects on the field theory side, we will use a background subtraction scheme in order to remove the ambient degrees of freedom and isolate contributions to physical quantities coming from the defect. The key step in this background subtraction scheme is the correct identification of the vacuum solution, and as will be made clear, the ambient theory with a "trivial defect" is a deformed 6d $\mathcal{N} = (2,0)$ SCFT preserving the bosonic superconformal subalgebra $\mathfrak{so}(2,2) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(5) \subset \mathfrak{osp}(8*|4)$. Since the solutions in [1] belong to the class of solutions that we study below, in finding a vacuum solution that manifests the corresponding isometries and carefully carrying out background subtraction, we will also resolve a puzzle in [1], where physical quantities like the "defect central charge" were divergent.

Ultimately, we will see that that universal part of the defect sphere EE, $S_{\rm EE}^{(univ)}$, i.e. the coefficient of its log-divergent part, is determined in terms of the highest weight vector, ϖ, of an A_{M-1} irreducible representation determined by a Young diagram that encodes the partition of M5-branes that specifies the defect. Explicitly, we will show that

$$S_{\rm EE}^{(univ)} = -\frac{(\varpi,\varpi)}{5}\ , \tag{4.1}$$

where $(\cdot,\cdot)$ is the scalar product on the weight space. This result is similar to the Wilson surface sphere EE [309] in that both are expressible in terms of scalar quantities derived from representation data but differ in that (4.1) is negative definite[3] and is completely determined by the highest weight vector.

In section 4.2, we begin by reviewing the 11d supergravity solutions dual to 2d small $\mathcal{N}=(4,4)$ defects in 6d $\mathcal{N}=(1,0)$ SCFTs found in [1]. In section 4.3 we briefly review the $\mathfrak{d}(2,1;\gamma)\oplus\mathfrak{d}(2,1;\gamma)$-invariant solutions to 11d supergravity found in [2], and we show that by orbifolding the solutions in the $\gamma\to-\infty$ limit, we can recover the solutions in [1]. We then use the $\gamma\to-\infty$ limit to construct new 2d small $\mathcal{N}=(4,4)$ defects with finite Ricci scalar in section 4.4. Further in section 4.4, we demonstrate that the naïve $\mathrm{AdS}_7\times S^4$ vacuum is inappropriate to use in a background subtraction scheme for regulating holographic computations in the $\gamma\to-\infty$ limit, and we identify the correct background to use in this scheme. In section 4.5, we utilise the new $\gamma\to-\infty$ supergravity solutions and correct regulating scheme in a computation of the contribution of a flat 2d small $\mathcal{N}=(4,4)$ superconformal defect to the EE of a spherical region in a 6d SCFT. We then summarise our findings and discuss remaining issues and open questions surrounding these new defect solutions in section 4.6.

In addition, there are two appendices that detail technical aspects of the computations in the main text. First, in Appendix A.1, we analyze the asymptotic expansion of the supergravity data that specify the new solutions in the $\gamma\to-\infty$ limit and construct the map to Fefferman-Graham (FG) gauge. Lastly, in Appendix A.2, we carefully treat the integral in the area functional of the Ryu-Takayanagi (RT) surface in the computation of the holographic EE of the defect in the dual field theory.

	$\mathbb{R}^{1,1}$		r	θ^1	θ^2	χ	z	ρ	φ^1	φ^2	ϕ
KK′	–	–	–	–	–	–	–	·	·	·	ISO
M5′	–	–	–	–	–	–	·	·	·	·	·
M2	–	–	·	·	·	·	–	$\sim$	$\sim$	$\sim$	$\sim$
M5	–	–	·	·	·	·	$\sim$	–	–	–	–
KK	–	–	·	·	·	ISO	–	–	–	–	–

TABLE 4.1: The 1/8-BPS brane setup of [1], with M2-M5 defect branes intersecting orthogonal M5′-branes, and with both stacks of 5-branes probing A-type singularities. In our conventions, –, · and $\sim$ denote directions along which a brane is extended, localised, and smeared, respectively, while ISO(metric) denotes the compact direction of the KK-monopoles.

4.2 Review: small $\mathcal{N} = (4,4)$ surface defects

We begin with a brief summary of the results of [1]. The particular 11d supergravity metric constructed therein is the uplift of a 7d charged $\mathrm{AdS}_3 \times S^3$ domain wall initially found in [316], and is given by

$$\mathrm{d}s^2 = 4kQ_{\mathrm{M5}} H_{\mathrm{M5}'}^{-1/3} \left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{S^3/\mathbb{Z}_k}\right) + H_{\mathrm{M5}'}^{2/3} \left(\mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2 \mathrm{d}s^2_{\tilde{S}^3/\mathbb{Z}_{k'}}\right), \tag{4.2}$$

for some parameter Q_{M5} and a function $H_{\mathrm{M5}'}$ defined over a 4d space parametrised by the coordinates $\{z, \rho, \varphi^1, \varphi^2\}$. The solution above captures the near-horizon geometry of the brane intersection depicted in table 4.1. Namely, a "bound state" (in the sense discussed in footnote 4) of M2- and M5-branes, with charges Q_{M2} and Q_{M5}, intersects an orthogonal stack of M5′-branes, thus forming a 1/4-BPS brane setup. In 2d notation, this corresponds to $\mathcal{N} = (4,4)$ supersymmetry, with the large R-symmetry realised geometrically as the isometry of the two 3-spheres S^3 and $\tilde{S}^3$ with coordinates $\{\chi, \theta^1, \theta^2\}$ and $\{\phi, \varphi^1, \varphi^2\}$, respectively. In addition, the two stacks of 5-branes can be made to probe ALE singularities by introducing two Kaluza-Klein (KK) monopoles, with charges k and k' and Taub-NUT directions ∂_χ and ∂_ϕ, respectively. The inclusion of any one of the two KK monopoles results in a further breaking of the preserved supersymmetries and a degeneration to an 1/8-BPS setup, which in 2d language corresponds to (large) $\mathcal{N} = (0,4)$ supersymmetry. The presence of the second KK monopole does not incur a further loss of supersymmetry, so the final brane configuration is always at least a 1/8-BPS supergravity solution.

Furthermore, the defect M2- and M5-branes are fully localised within a 2d submanifold of the worldvolume of the M5′-branes. This is to be expected from the holographic

[3] Unlike in an ordinary 2d CFT, (4.1) being strictly negative does not necessarily signal that the theory may be non-unitary. Indeed, the 2d defect sphere EE is expressible as a signed linear combination of defect Weyl anomaly coefficients [51], which is not bounded from below.

realization of a surface defect in a 6d SCFT; this interpretation was first attached to the 7d domain wall in [317] and to the full 11d SUGRA background in [1]. On the other hand, the defect branes are smeared in the directions transverse to the worldvolume of the M5′-branes[4], so that their charge is localised within $S^3/\mathbb{Z}_k$, but not $\tilde{S}^3/\mathbb{Z}_{k'}$. Therefore, while the metric in (4.2) manifests the isometry groups of both (orbifolded) 3-spheres, the R-symmetry is partially broken, giving rise to small $\mathcal{N}=(0,4)$ supersymmetry. For $k'=1$, the solution above fits into the classification of $\mathcal{N}=(0,4)$ $\mathrm{AdS}_3\times S^3/\mathbb{Z}_k\times \mathrm{CY}_2$ backgrounds foliated over an interval performed in [319], where $\mathrm{CY}_2=\mathbb{R}^4$ contains the (round) $\tilde{S}^3$. In particular, the solution above corresponds to taking the M5′-branes to be completely localised in their transverse space.

On shell, the defect brane charges are constrained to be equal, $Q_{\mathrm{M2}}=Q_{\mathrm{M5}}$, while the function $H_{\mathrm{M5}'}$ satisfies [1]

$$\nabla^2_{\mathbb{R}^3_{\hat\rho}} H_{\mathrm{M5}'}(z,\hat\rho)+\frac{k'\partial_z^2 H_{\mathrm{M5}'}(z,\hat\rho)}{\hat\rho}=0, \tag{4.3}$$

where we rescaled $\hat\rho=\rho^2/(4k')$, and denoted by $\mathbb{R}^3_{\hat\rho}$ the three-dimensional subspace which is transverse to the M5-branes, parallel to the M5′-branes, and along which the M2-branes are smeared. Following the parametrization adopted in table 4.1, $\mathbb{R}^3_{\hat\rho}$ is then the space spanned by $\{\partial_{\hat\rho},\partial_{\varphi^1},\partial_{\varphi^2}\}$, with $\hat\rho$ being the radial coordinate and $\{\varphi^1,\varphi^2\}$ parametrising a 2-sphere.

In terms of the brane setup described above, then, the spacetime in (4.2) is reached by approaching the brane intersection locus from within the worldvolume of the M5′-branes in a radial fashion, i.e. by taking $r\to 0$. In this limit, the $ISO(1,1)$ isometry group gets promoted to $SO(2,2)$, and the M5′-brane worldvolume becomes $\mathrm{AdS}_3\times S^3/\mathbb{Z}_k$.

A particular solution to (4.3) is, for any $\alpha\in\mathbb{R}$,

$$H_{\mathrm{M5}'}(z,\rho)=\frac{4\sqrt{2}}{g^3}\frac{1}{P_+P_-}\frac{\sqrt{P_+^2+P_-^2-4\alpha^2+2P_+P_-}}{P_+^2+P_-^2+2P_+P_-}, \tag{4.4}$$

where $P_\pm=\sqrt{z^2+(\rho\pm\alpha)^2}$. By redefining

$$\rho=\alpha\frac{\cos\xi}{\sqrt{1-\mu^5}} \tag{4.5a}$$

$$z=\alpha\mu^{\frac{5}{2}}\frac{\sin\xi}{\sqrt{1-\mu^5}} \tag{4.5b}$$

[4]It is this property – that the M2- and M5-branes do not share transverse directions with the M5′-branes, other than those along which the former are smeared – which we are implicitly referring to when we describe the M2- and M5-branes as forming a “bound state”. This aligns with the terminology used in [1], and should not to be confused with the dyonic supermembrane [318], which is a rather different multimembrane solution of 11d supergravity. Indeed, in the setup described by table 4.1, there is no M2-brane charge dissolved within the M5-brane worldvolume, nor do the M2-branes polarise into a fuzzy 3-sphere via the Myers effect.

and setting $\alpha = (2^{7/4} g^{3/2} k Q_{\mathrm{M5}})^{-1}$, the particular solution in (4.4) can be matched to the one found in eq. (2.17) of [1], which we now reproduce for clarity[5]:

$$H_{\mathrm{M5}'}(\mu,\xi) = 2^{27/4} (\sqrt{g} k Q_{\mathrm{M5}})^3 \frac{\mu^{5/2}(1-\mu^5)^{3/2}}{\mu^5 \cos^2\xi + \sin^2\xi} \ . \tag{4.6}$$

Furthermore, in [1] it was argued that, as $\mu \to 1$ (which corresponds to a non-linear limit in the original coordinates $\hat{\rho}$ and z), the near-horizon geometry in (4.2) locally recovers the $\mathrm{AdS}_7/\mathbb{Z}_k \times S^4/\mathbb{Z}_{k'}$ vacuum of M-theory. This was shown by realising the 11d line element as the uplift of the domain wall solution to $\mathcal{N} = 1$, $d = 7$ supergravity (whose gauge coupling constant g appears in (4.6) above) found in [316], which interpolates between AdS_7 (as $\mu \to 1$) and an infrared singularity (at $\mu = 0$).

While the singular nature of the solution is not immediately obvious, it can be made manifest by studying the Ricci scalar, $\mathcal{R}$, in the $z \to 0$ limit. For metrics generally of the form of (4.2), and following [315], we can write the expression for the Ricci scalar for an arbitrary harmonic function $H_{\mathrm{M5}'}$ as

$$\mathcal{R} = H_{\mathrm{M5}'}^{-2/3} \left[\frac{1}{6} \frac{(\partial_z H_{\mathrm{M5}'})^2 + (\partial_\rho H_{\mathrm{M5}'})^2}{H_{\mathrm{M5}'}^2} - \frac{2}{3} \frac{\partial_z^2 H_{\mathrm{M5}'} + \partial_\rho^2 H_{\mathrm{M5}'}}{H_{\mathrm{M5}'}} - 2 \frac{\partial_\rho H_{\mathrm{M5}'}}{\rho H_{\mathrm{M5}'}} \right] . \tag{4.7}$$

The function $H_{\mathrm{M5}'}$ has a branch point located at $z = 0$ and $\rho = \alpha$ and correspondingly admits two different expansions as $z \to 0$, depending on whether ρ is larger or smaller than α. The choice of sign for the branch cut can be determined by the requirement $P_+ P_- \geq 0$. For $\rho > \alpha$, this leads to

$$\mathcal{R} = \frac{g^2 (2\alpha^2 - 3\rho^2)^2}{12 \rho^{2/3} (\rho^2 - \alpha^2)^{5/3}} + \mathcal{O}(z^2) \tag{4.8}$$

which has a pole as $\rho \to \alpha$. This corresponds to setting $\xi = 0$, with the pole appearing as $\mu \to 0$. For $\rho < \alpha$, we find

$$\mathcal{R} = \frac{g^2 \alpha^{2/3} (\alpha^2 - \rho^2)}{12 z^{8/3}} + \mathcal{O}\left(\frac{1}{z^{2/3}} \right) \tag{4.9}$$

which corresponds to taking $\mu \to 0$ with $\xi \neq 0$. Thus for all values of ξ, we find that the Ricci scalar diverges as $\mu \to 0$.

4.3 From large to small $\mathcal{N} = (4,4)$ surface defects

To begin, we will briefly review the classification due to [2, 315] of the $d = 11$ supergravity solutions with superisometry given by two copies of the exceptional Lie

[5] To see this, it is convenient to first rationalise the product $P_+ P_-$ as $P_+ P_- = \alpha^2 \big(1 + \mu^5 - (1 - \mu^5)\cos(2\xi)\big)/2(1-\mu^5)$.

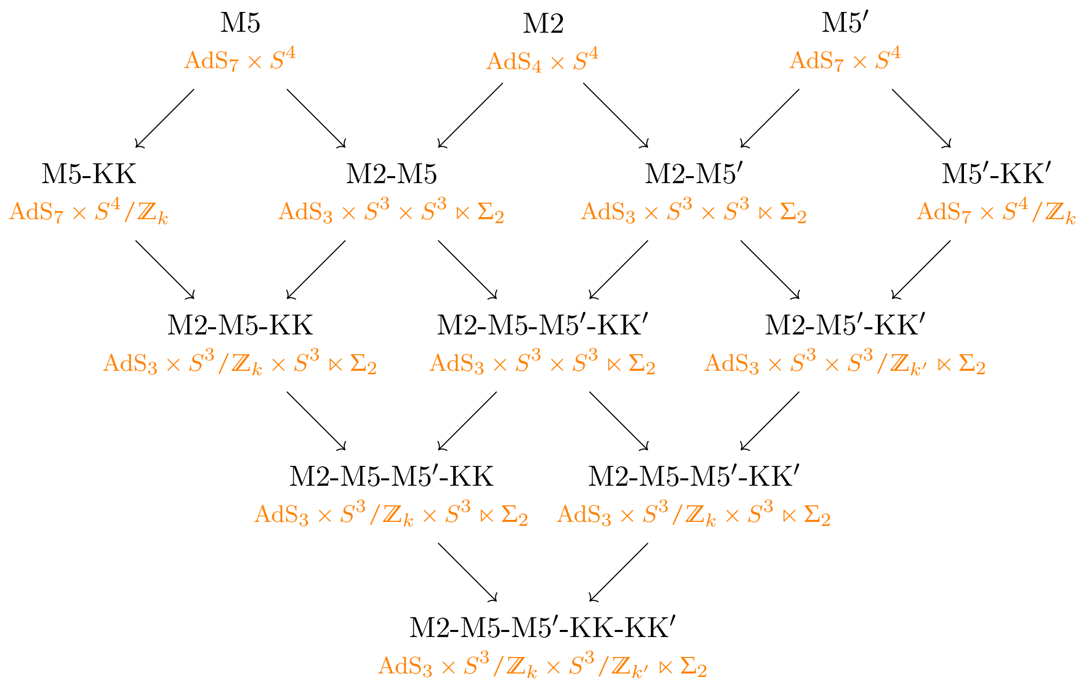


FIGURE 4.1: A diagram representing a constructive approach to building the brane intersection in [1]. One can start with either M2 or M5 branes (top) and consider adding the other branes successively, working downwards. Under each brane setup is the corresponding near horizon geometry. *This content was added for the purpose of this thesis, and does not appear in the original article.*

superalgebra $\mathfrak{d}(2,1;\gamma)$. This classification represents the foundation for the new defect solutions we present below.

In particular, we will be interested in the real form $\mathfrak{d}(2,1;\gamma;0)$ which arises as a real subsuperalgebra of the fixed points of an involutive semimorphism of $\mathfrak{d}(2,1;\gamma)$ [313]. Recall that the 9-dimensional maximal bosonic subalgebra of the real form $\mathfrak{d}(2,1;\gamma;0)$ is $\mathfrak{so}(2,1) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Labelling each factor in this subalgebra by an index $a \in \{1,2,3\}$, the generators $T^{(a)}$ satisfy

$$\forall i,j \in \{1,2,3\}, \qquad [T_i^{(a)}, T_j^{(b)}] = i\delta^{ab}\varepsilon_{ijk}\eta_a^{kl}T_l^{(a)}, \tag{4.10}$$

where ε_{ijk} and η_a^{kl} are the totally anti-symmetric tensor (with $\varepsilon_{123} = 1$) and the canonical metric induced by the Killing form, respectively. In addition to the bosonic sector, $\mathfrak{d}(2,1;\gamma)$ contains an 8-dimensional fermionic generator with components $F_{A_1A_2A_3}$, where the indices $A_a \in \{\pm\}$ transform in the spinorial representation $\mathbf{2}$ of the a^{th} factor in the even subalgebra. Furthermore, the components $F_{A_1A_2A_3}$ obey the following anti-commutation relation

$$\begin{aligned}\{F_{A_1A_2A_3}, F_{B_1B_2B_3}\} = \beta_1 C_{A_2B_2}C_{A_3B_3}(C\sigma^i)_{A_1B_1}T_i^{(1)} \\ + \beta_2 C_{A_1B_1}C_{A_3B_3}(C\sigma^i)_{A_2B_2}T_i^{(2)} \\ + \beta_3 C_{A_1B_1}C_{A_2B_2}(C\sigma^i)_{A_3B_3}T_i^{(3)} \ ,\end{aligned} \tag{4.11}$$

where $C \equiv i\sigma^2$ is the charge conjugation matrix, $\{\sigma^i\}_{i=1}^3$ are the Pauli matrices, and $\{\beta_i\}_{i=1}^3$ are real parameters satisfying $\sum_{a=1}^3 \beta_a = 0$, which follows from the (generalised) Jacobi identity. This last constraint, together with the possibility of absorbing any rescaling $\{\lambda\beta_1, \lambda\beta_2, \lambda\beta_3\}$ for $\lambda \in \mathbb{C}$ into a redefinition of the normalization of the fermionic generator, implies that $\mathfrak{d}(2,1;\gamma)$ is entirely specified by the choice of a ratio of any two of the three β parameters; here, we take $\gamma \equiv \beta_2/\beta_3$. Note that $\mathfrak{d}(2,1;\gamma)$ is the only (finite-dimensional) Lie superalgebra admitting a continuous parametrization.

Amongst the possible values that γ can take, there are clearly three special values corresponding to the vanishing of any one of the β parameters. Specifically, choosing $\beta_1 = 0$ fixes $\gamma = -1$. The more interesting case, and the one relevant to our analysis, is $\beta_3 = 0$, which corresponds to $\gamma \to \pm\infty$. The case $\beta_2 = 0$ corresponds to $\gamma = 0$ and is equivalent under a group involution as discussed at the end of this section. In the limit $\beta_3 = 0$, the anticommutator in (4.11) degenerates to

$$\{F_{A_1A_2A_3}, F_{B_1B_2B_3}\} = \beta_1 \left(C_{A_2B_2}C_{A_3B_3}(C\sigma^i)_{A_1B_1}T_i^{(1)} - C_{A_1B_1}C_{A_3B_3}(C\sigma^i)_{A_2B_2}T_i^{(2)} \right) \ . \tag{4.12}$$

Consequently, the $\mathfrak{so}(4)_R \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ R-symmetry of the large superalgebra is broken into the single $\mathfrak{so}(3)_R$ factor which constitutes the R-symmetry of a small superalgebra. Note that the other $\mathfrak{so}(3)$ factor remains a bosonic symmetry of the supergravity solution; however, it is now realised as a flavour symmetry, rather than as an outer automorphism of the supersymmetry algebra.

In addition, there are isolated points of interest in the γ-parameter space where the real form $\mathfrak{d}(2,1;\gamma;0)$ reduces to classical Lie superalgebras:

$$\mathfrak{d}(2,1;\gamma;0) = \mathfrak{osp}(4^*|2) \qquad \text{for } \gamma \in \{-2, -1/2\} \tag{4.13a}$$

$$\mathfrak{d}(2,1;\gamma;0) = \mathfrak{osp}(4|2;\mathbb{R}) \qquad \text{for } \gamma = 1 \ . \tag{4.13b}$$

In particular, $\gamma = -1/2$ is the only value (up to algebra involution, as we will discuss shortly) for which $\mathfrak{d}(2,1;\gamma;0) \oplus \mathfrak{d}(2,1;\gamma;0)$ admits a canonical inclusion into the superisometry algebra $\mathfrak{osp}(8^*|4)$ of $\mathrm{AdS}_7 \times S^4$. This case was studied extensively in [309]; it is the holographic realization of Wilson surfaces, 1/2-BPS codimension-4 superconformal solitons, within the 6d $\mathcal{N} = (2,0)$ SCFT. In this case, the ambient 6d SCFT is undeformed, in the sense that the supergroup of symmetries preserved by the defect is a subgroup of the 6d superconformal symmetry group. This is a common feature throughout the study of defects embedded in higher-dimensional theories.

For generic values of γ, including the limit $\gamma \to \pm\infty$, the superisometry algebra $\mathfrak{d}(2,1;\gamma;0) \oplus \mathfrak{d}(2,1;\gamma;0)$ is not a subalgebra of the $\mathfrak{osp}(8^*|4)$ superconformal Lie superalgebra [309]. As a result, we expect the 6d ambient theory to be deformed as we tune γ away from the special value $\gamma = -1/2$. In the supergravity solutions we construct below, this deformation will appear at leading order in the asymptotic expansion of the four-form flux. Additionally, the warp factors of the symmetric spaces corresponding to $T^{(1)}$ and $T^{(2)}$ have the correct normalization to create an AdS$_7$ space only when $|\beta_1| = |\beta_2|$, which can happen only when $\gamma = -1/2$ or $\gamma \to \pm\infty$.

Finally, we note that the complex Lie superalgebra $\mathfrak{d}(2,1;\gamma)$ enjoys a triality symmetry generated by $\gamma \mapsto \gamma^{-1}$, $\gamma \mapsto -(\gamma+1)$, and $\gamma \mapsto -\gamma/(\gamma+1)$, any two of which are linearly independent. However, only the $\gamma \mapsto \gamma^{-1}$ involution survives at the level of the real form $\mathfrak{d}(2,1;\gamma;0)$, due to the distinguished nature of the $T^{(1)}$ generator. Therefore, our analysis below can be equivalently carried out in the $\gamma \to \pm\infty$ limits, or even in the $\gamma \to 0$ limit (albeit with a permutation of the two $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ factors contained in two copies of $\mathfrak{d}(2,1;\gamma;0)$)[6]. The physical interpretation of the choice of either limit corresponds to fixing the orientation of the M5-branes that engineer the ambient 6d SCFT.

4.3.1 Supergravity solutions for general γ

In this section, we will review the structure of supergravity solutions with $\mathfrak{d}(2,1;\gamma;0) \oplus \mathfrak{d}(2,1;\gamma;0)$ superisometry algebra, for generic γ, as first described in [2, 315, 320]. The odd subspace of this superalgebra is 16-dimensional for all γ, so that all supergravity backgrounds discussed below are 1/2-BPS. Furthermore, the maximal bosonic subalgebra of $\mathfrak{d}(2,1;\gamma;0) \oplus \mathfrak{d}(2,1;\gamma;0)$ is

$$\mathfrak{so}(2,2) \oplus \mathfrak{so}(4) \oplus \mathfrak{so}(4)\ . \tag{4.14}$$

To realise this superisometry, the supergravity metric must be of the form[7]

$$(\mathrm{AdS}_3 \times S^3 \times \tilde{S}^3) \ltimes \Sigma_2\ , \tag{4.15}$$

for a Riemann surface Σ_2.

The Killing spinor equations on such manifolds can be recast into conditions on two auxiliary functions, h and G, defined over the Riemann surface Σ_2 [2, 315, 320].

[6] In spite of the involution relating the $\gamma \to -\infty$ and $\gamma \to 0$ limits, these two descriptions cannot be smoothly deformed into one another by varying γ. Indeed, the two regimes are separated by the decompactification point $\gamma = -1$.

[7] For the previously mentioned special values $\gamma = -1$ and $\gamma = 0$, AdS$_3$ Wigner-İnönü contracts to $\mathbb{R}^{2,1}$ and one of the 3-spheres decompactifies into $\mathbb{R}^3$, respectively.

Specifically, if we employ local complex[8] coordinates $\{w, \bar{w}\}$ on Σ_2, so that the metric on the Riemann surface is given by $\mathrm{d}s^2_{\Sigma_2} = \mathrm{d}w\mathrm{d}\bar{w}$, the function h is constrained to be $\mathbb{R}$-valued and harmonic

$$\partial_w \partial_{\bar{w}} h = 0 \ , \tag{4.16}$$

while G is $\mathbb{C}$-valued and satisfies the conformally covariant equation

$$\partial_w G = \frac{1}{2}\,(G + \bar{G})\,\partial_w \ln h \ . \tag{4.17}$$

Note that both constraints above hold for generic γ.

Regularity conditions on the auxiliary functions h and G have been discussed in [2]. Here, we will consider Riemann surfaces with a boundary, $\partial\Sigma_2 \neq \emptyset$. In short, regularity of the supergravity solution constrains $G = \pm i$ on $\partial\Sigma_2$, while the (holomorphic part of the) function h must either vanish or have a simple pole at any point of $\partial\Sigma_2$.

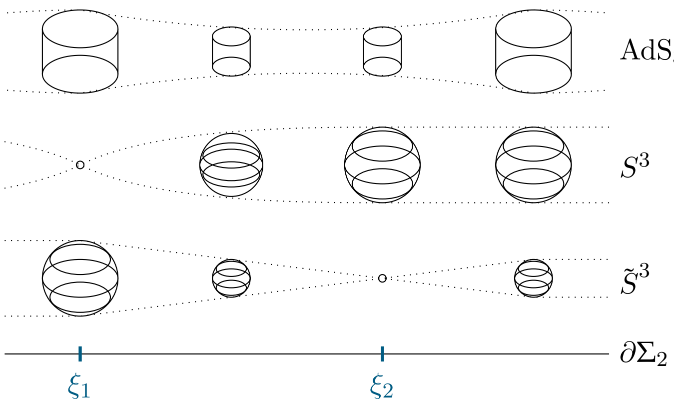


FIGURE 4.2: A visual representation of the solutions in [2] — a foliation of AdS$_3$ and two S^3 over a Riemann surface Σ_2. At the singularities on the boundary of the Riemann surface, ξ_i, either one of the two three-spheres in the foliation collapses.

The triple (h, G, γ) uniquely specifies a bosonic background within the space of 11d supergravity solutions with superisometry algebra $\mathfrak{d}(2,1;\gamma;0) \oplus \mathfrak{d}(2,1;\gamma;0)$. In particular, the metric field in the supergravity solution can be written in terms of this data as

$$\mathrm{d}s^2 = f^2_{\mathrm{AdS}_3}\mathrm{d}s^2_{\mathrm{AdS}_3} + f^2_{S^3}\mathrm{d}s^2_{S^3} + f^2_{\tilde{S}^3}\mathrm{d}s^2_{\tilde{S}^3} + f^2_{\Sigma_2}\mathrm{d}s^2_{\Sigma_2} \ , \tag{4.18}$$

where the metric factors are functions over Σ_2 and can be expressed in terms of the auxiliary functions

$$W_\pm := |G \pm i|^2 + \gamma^{\pm 1}(G\bar{G} - 1) \tag{4.19}$$

[8]The SUGRA orientation and Riemannian metric automatically endow the Riemann surface Σ_2 with a complex structure.

as [2]

$$
\begin{aligned}
f^6_{\mathrm{AdS}_3} &= \frac{h^2 W_+ W_-}{\beta_1^6 (G\bar{G}-1)^2} \ , && (4.20a)\\
f^6_{S^3} &= \frac{h^2 (G\bar{G}-1) W_-}{\beta_2^3 \beta_3^3 W_+^2} \ , && (4.20b)\\
f^6_{\tilde{S}^3} &= \frac{h^2 (G\bar{G}-1) W_+}{\beta_2^3 \beta_3^3 W_-^2} \ , && (4.20c)\\
f^6_{\Sigma_2} &= \frac{|\partial_w h|^6}{\beta_2^3 \beta_3^3 h^4} (G\bar{G}-1) W_+ W_- \ . && (4.20d)
\end{aligned}
$$

The bosonic sector of the 11d supergravity solution is completed by a three-form gauge potential $\mathcal{C}_{(3)}$, which can be written in terms of the (h, G, γ) data as

$$
\mathcal{C}_{(3)} = \sum_{i=1}^{3} b_{\mathcal{M}_i} \, \mathrm{vol}_{\mathcal{M}_i} \tag{4.21}
$$

where $\mathrm{vol}_{\mathcal{M}_i}$ denotes the volume form on the manifold $\mathcal{M}_i = \{\mathrm{AdS}_3, S^3, \tilde{S}^3\}_i$. We also introduced the gauge potentials

$$
\begin{aligned}
b_{\mathrm{AdS}_3} &:= \frac{\tau_1}{\beta_1^3} \left[-\frac{h(G+\bar{G})}{1-G\bar{G}} + (2+\gamma+\gamma^{-1})\Phi - (\gamma-\gamma^{-1})\tilde{h} + b_1^0 \right] \ , && (4.22a)\\
b_{S^3} &:= \frac{\tau_2}{\beta_2^3} \left[-\frac{\gamma h (G+\bar{G})}{W_+} + \gamma(\Phi - \tilde{h}) + b_2^0 \right] \ , && (4.22b)\\
b_{\tilde{S}^3} &:= \frac{\tau_3}{\beta_3^3} \left[\frac{h(G+\bar{G})}{\gamma W_-} - \frac{\Phi + \tilde{h}}{\gamma} + b_3^0 \right] \ , && (4.22c)
\end{aligned}
$$

where $\{b_i^0\}_{i=1}^3$ are integration constants, while the dual harmonic function $\tilde{h}$ and the real auxiliary function Φ are defined in terms of h and G via

$$
\begin{aligned}
\partial_w \tilde{h} &= -i \partial_w h \ , && (4.23a)\\
\partial_w \Phi &= \bar{G} \partial_w h \ . && (4.23b)
\end{aligned}
$$

Finally, in (4.22) we made use of the signs $\tau_i = \pm 1$ for $i \in \{1, 2, 3\}$, which are subject to the constraint

$$
\prod_{i=1}^{3} \beta_i f_{\mathcal{M}_i} + h \prod_{i=1}^{3} \tau_i = 0 \ . \tag{4.24}
$$

As discussed above, regularity conditions force h and G to have prescribed behaviour on $\partial\Sigma_2$. The solutions that we are interested in studying are asymptotically $\mathrm{AdS}_7 \times S^4$, which are indeed described by $G = \pm i$ and h having a single simple pole, which corresponds to having a single asymptotic $\mathrm{AdS}_7 \times S^4$ region. Generically at any point on

$\partial\Sigma_2$, the regularity constraint on h implies that its Laurent expansion reduces to

$$h = -ih_0 w + \text{c. c.} = \frac{2h_0 \sin\vartheta}{\varrho} \ , \tag{4.25}$$

for some real constant h_0. In the second equality, we have introduced new convenient set of coordinates $\{\varrho, \vartheta\}$ on Σ_2 defined by $w = e^{i\vartheta}/\varrho$. Adopting these new polar coordinates and using (4.25) allows us to perturbatively solve (4.17) in small ϱ to find

$$G = -i + a_1 \varrho e^{i\vartheta} \sin\vartheta + \mathcal{O}(\varrho^2) \tag{4.26}$$

for some constant a_1. If we then insert (4.26) into eqs. (4.19) and (4.27), we can perturbatively expand (4.18) in small ϱ to find

$$\mathrm{d}s^2 = L^2 \frac{\mathrm{d}\varrho^2}{\varrho^2} - \frac{2\gamma L^2}{a_1(1+\gamma)^2 \varrho}\left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \frac{(1+\gamma)^2}{\gamma^2}\mathrm{d}s^2_{S^3}\right) + L^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}s^2_{\tilde{S}^3}\right) + \dots \tag{4.27}$$

with

$$L^6 = \frac{a_1^2 h_0^2 (1+\gamma)^6}{\beta_1^6 \gamma^2} \ . \tag{4.28}$$

We can see the $\mathrm{AdS}_7 \times S^4$ asymptotic geometry in (4.27) clearly for certain values of γ. The obvious case is if we choose $\gamma = -1/2$, which was studied extensively in [2, 308, 309]. The other possibility, namely the limits $\gamma \to \pm\infty$, will be discussed in the next section.

For $\gamma < 0$ and for the function h given in (4.25), the general solution to (4.17) for the function G with an even number $2n+2$ of branch points $\{\xi_i\}_{i=1}^{2n+2} \subset \partial\Sigma_2$ was found in [315] to be

$$G = -i\left(1 + \sum_{j=1}^{2n+2} (-1)^j \frac{w - \xi_j}{|w - \xi_j|}\right) \ , \tag{4.29}$$

where G flips sign $\pm i \to \mp i$ upon crossing each branch point ξ_i.

4.3.2 The $\gamma \to -\infty$ limit

Of particular interest to us in the following sections are solutions that are constructed by taking the scaling limit

$$\gamma \to \pm\infty \ , \qquad \gamma a_1 \to \text{constant} \ , \qquad L \to \text{constant} \ . \tag{4.30}$$

From (4.27) and the requirement $\gamma a_1 \to$ constant, we can see that this scaling limit realises an $\mathrm{AdS}_7 \times S^4$ asymptotic geometry. In this and the following subsections, we will consider the $\gamma \to -\infty$ limit in greater detail and show that the solutions engineered in this limit contain the class of solutions found in [1] which we reviewed in section 4.2.

Below, we will expand upon these solutions and construct new surface defects with small $\mathcal{N} = (4,4)$ SUSY.

To begin, we introduce a complex function F via

$$G = -i\left(1 + \gamma^{-1}F\right) \ , \tag{4.31}$$

and we rescale $\beta_1 = \hat{\beta}(-\gamma)^{1/3}$. This ensures that L remains finite as required by the limiting procedure of (4.30). In this limit, the metric in (4.27) becomes

$$\mathrm{d}s^2 = \left[\frac{4h^2}{\hat{\beta}^6(F+\bar{F})}\right]^{\frac{1}{3}}\left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{S^3}\right) + \left[\frac{h^2(F+\bar{F})^2}{16\hat{\beta}^6}\right]^{\frac{1}{3}}\left(\mathrm{d}s^2_{\tilde{S}^3} + \frac{4|\partial_w h|^2}{h^2}\mathrm{d}w\mathrm{d}\bar{w}\right) \ . \tag{4.32}$$

In addition, the gauge potentials are given by

$$b_{\mathrm{AdS}_3} = b_{S^3} = -\frac{2\tilde{h}}{\hat{\beta}^3}, \qquad b_{\tilde{S}^3} = \frac{h}{2\hat{\beta}^3}\frac{-i(F-\bar{F})}{2} - \hat{\Phi} \ , \tag{4.33}$$

where the function $\hat{\Phi}$ is defined via

$$\partial_w\hat{\Phi} = \bar{F}\frac{\partial_w\tilde{h}}{\hat{\beta}^3} \ . \tag{4.34}$$

Let us introduce a new set of coordinates $\{z, \rho\}$ on Σ_2 defined via $w = z + i\rho$, and choose h_0 in (4.25) such that the harmonic function $h = \hat{\beta}^3 h_1\rho$ for some constant h_1. Let us also parametrise the real and imaginary parts of the complex function F such that

$$F = \frac{2\rho^2}{h_1}H + iF_I \ , \tag{4.35}$$

for real functions H and F_I. Note that the latter does not enter the metric field; indeed, we can rewrite the line element in (4.32) entirely in terms of H as

$$\mathrm{d}s^2 = h_1 H^{-\frac{1}{3}}\left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{S^3}\right) + H^{\frac{2}{3}}\left(\mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}s^2_{\tilde{S}^3}\right) \ . \tag{4.36}$$

The condition in (4.17) implies that the complex function F satisfies the first order equation

$$\partial_w F = \frac{1}{2}(F - \bar{F})\partial_w \ln h \ , \tag{4.37}$$

which in turn can be recast into a second order equation for its real part,

$$\partial_\rho^2 H + \frac{3}{\rho}\partial_\rho H + \partial_z^2 H = 0 \ . \tag{4.38}$$

The supergravity solution in the $\{z, \rho\}$ parametrization of Σ_2 is completed by the following expressions for the gauge potentials,

$$b_{\mathrm{AdS}_3} = b_{S^3} = 2h_1 z, \qquad b_{\tilde{S}^3} = \frac{h_1 \rho}{2} F_I + \hat{\Phi} \ . \tag{4.39}$$

In particular, a more explicit form can be given for the derivatives of $b_{\tilde{S}^3}$ as

$$\partial_\rho b_{\tilde{S}^3} = \rho^3 \partial_z H, \qquad \partial_z b_{\tilde{S}^3} = -\rho^3 \partial_\rho H \ . \tag{4.40}$$

These solutions have a scaling symmetry under the transformation

$$z \to \lambda z, \qquad \rho \to \lambda \rho, \qquad H \to \lambda^{-3} H, \qquad h_1 \to \lambda^{-1} h_1, \tag{4.41}$$

for which the metric and flux are invariant. This transformation could be used to fix the value of h_1 so that the solution is uniquely given by choice of function H.

4.3.3 Inclusion of KK-monopoles and recovering known solutions

At the level of the local geometry described by (4.36), KK-monopoles can be included in a straightforward fashion by replacing S^3 and $\tilde{S}^3$ with the lens spaces $S^3/\mathbb{Z}_k$ and $\tilde{S}^3/\mathbb{Z}_{k'}$, respectively, where k and k' are the orbifold charges. The lens spaces can be realised as the total spaces of circle bundles over 2-spheres, where the orbifold acts on the Hopf fibre; this amounts to the substitutions

$$\mathrm{d}s^2_{S^3} \to \mathrm{d}s^2_{S^3/\mathbb{Z}_k} = \frac{1}{4}\left[\left(\frac{\mathrm{d}\chi}{k} + \omega\right)^2 + \mathrm{d}s^2_{S^2}\right] , \tag{4.42a}$$

$$\mathrm{d}s^2_{\tilde{S}^3} \to \mathrm{d}s^2_{\tilde{S}^3/\mathbb{Z}_{k'}} = \frac{1}{4}\left[\left(\frac{\mathrm{d}\phi}{k'} + \eta\right)^2 + \mathrm{d}s^2_{\tilde{S}^2}\right] , \tag{4.42b}$$

where $\mathrm{d}\omega = \mathrm{vol}_{S^2}$ and $\mathrm{d}\eta = \mathrm{vol}_{\tilde{S}^2}$. The inclusion of either KK-monopole incurs the loss of $1/2$ of the existing supersymmetry generators, resulting in a small $\mathcal{N} = (0,4)$ supersymmetry algebra. Indeed, dimensional reduction along the Taub-NUT direction produces a D6-brane which breaks half of the supersymmetries of the massless type IIA AdS_7 vacuum. The second KK-monopole can be added without bringing about any further breaking of the supersymmetries, and provides a second isometric direction upon which the background can be dimensionally reduced to massless type IIA supergravity. Therefore, the final solution is an $1/8$-BPS configuration. At the level of the superconformal symmetry algebra, this corresponds to a reduction from $\mathfrak{d}\,(2,1;\gamma) \oplus \mathfrak{d}\,(2,1;\gamma)$ to $\mathfrak{d}\,(2,1;\gamma)$.

The inclusion of orbifolds allows us to match the $\gamma \to -\infty$ solutions of section 4.3.2 to those found in [1] and reviewed in section 4.2. In particular, we see that the $\gamma \to -\infty$ metric in (4.36), subject to the inclusion of KK-monopoles as in (4.42), matches the

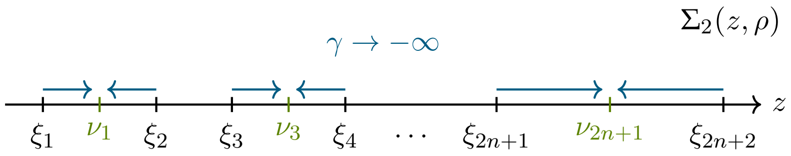


FIGURE 4.3: A particular choice of collapse points $\nu_j \in \partial\Sigma_2$ and the behaviour of the singular points $\xi_i \in \partial\Sigma_2$ as $\gamma \to -\infty$ under pairwise collapse. The collapse points ν_j can be chosen arbitrarily, while maintaining $\nu_j < \nu_{j+1}$, but for clarity in the figure have been depicted at the midpoint in the region $[\xi_j, \xi_{j+1}]$ to demonstrate pairwise collapse.

M2-M5-KK-M5′-KK′ near-horizon in (4.2) under the identifications $h_1 = 4kQ_{\mathrm{M5}}$ and $H = H_{\mathrm{M5}'}$. In particular, the functions H and $H_{\mathrm{M5}'}$ solve the same equation, since (4.38) maps to (4.3) under the rescaling $\hat{\rho} = \rho^2/4$. We have thus managed to fully recover the solutions described in [1] as a limiting case of those in [2].

4.4 New small $\mathcal{N} = (4,4)$ surface defects

In this section, we will construct a new explicit family of solutions $H(z,\rho)$ to (4.38). In particular, without loss of generality we will specialise to the $\gamma \to -\infty$ limit. We recall that for $\gamma < 0$, a general solution with an even number of branch points $\{\xi_i\}_{i=1}^{2n+2} \subset \partial\Sigma_2$ was given by (4.29). We rescale the singular points ξ_j as

$$\xi_j = \nu_j - \gamma^{-1}\hat{\xi}_j \in \partial\Sigma_2 \quad \text{for} \quad j \in \{1,2,\ldots,2n+2\} \,, \tag{4.43}$$

where the collapse points satisfy $\nu_j \leq \nu_{j+1}$ for all j, so as to preserve the total order of the set $\{\xi_j\}$ following the $\gamma \to -\infty$ limit. Furthermore, to ensure finiteness of the complex function F in this limit, we must also demand that all points ν_j, for $1 \leq j \leq 2n+2$, correspond to the collapse of even-dimensional clusters $\{\xi_k, \xi_{k+1}, \ldots, \xi_{k+2m+1}\}$ of singular points, where $k \leq j \leq k+2m+1$. This can be implemented by identifying

$$\nu_k \equiv \nu_{k+1} \equiv \ldots \equiv \nu_{k+2m+1} \tag{4.44}$$

for each cluster. Finally, in order to preserve the ordering of the singular points throughout the $\gamma \to -\infty$ limit, we must also arrange the corresponding collapse parameters within each cluster so that $\hat{\xi}_k < \hat{\xi}_{k+1} < \ldots < \hat{\xi}_{k+2m+1}$. Without loss of generality, then, it suffices to consider a pairwise collapse of neighbouring singular points, which can be realised by identifying

$$\nu_{2i-1} \equiv \nu_{2i} \quad \text{for} \quad i \in \{1,2,\ldots,n+1\} \,. \tag{4.45}$$

However, we will later comment on certain phenomena which appear only when the singular points collapse in clusters of 4 or more.

An illustration of the pairwise collapse described by (4.43) and (4.45) is shown in figure 4.3. Under this collapse dynamic, the complex function F becomes, in the $\gamma \to -\infty$ limit,

$$F(w,\bar{w}) = \sum_{j=1}^{2n+2} (-1)^j \hat{\xi}_j \frac{\bar{w} - w}{2(\bar{w} - \nu_j)|w - \nu_j|} \ . \tag{4.46}$$

The corresponding function H is given by

$$H(z,\rho) = \frac{h_1}{2} \sum_{j=1}^{2n+2} \frac{(-1)^j \hat{\xi}_j}{(\rho^2 + (z - \nu_j)^2)^{3/2}} \ . \tag{4.47}$$

For pure $\mathrm{AdS}_7 \times S^4$, this procedure produces what we refer to as the single-pole vacuum (1PV) solution with

$$F_{1\mathrm{PV}}(w,\bar{w}) = \hat{\xi} \frac{\bar{w} - w}{\bar{w}|w|} \ , \tag{4.48a}$$

$$H_{1\mathrm{PV}}(z,\rho) = \frac{h_1 \hat{\xi}}{(z^2 + \rho^2)^{\frac{3}{2}}} \ , \tag{4.48b}$$

which can be obtained by taking $n = 0$, $\xi_2 = -\xi_1 = \xi$, and $\nu_1 = \nu_2 = 0$. Alternatively, this solution can also be obtained by collapsing all singular points to the origin, i.e. $\nu_j = 0$ for all j. Note that (4.38) is linear in H and admits a translation symmetry under shifts of z. Using these two properties, the general solutions given in (4.47) can be reconstructed from the 1PV solution by taking linear combinations and making use of the translation symmetry.

One may worry that since the solution for the potential H in (4.47) is generically singular at the points $(z,\rho) = (\nu_i, 0)$ for all $i \in \{1, 2, \ldots, 2n+2\}$, the resulting supergravity metric may have a singularity or these points may simply correspond to horizons. In order to investigate the regularity of the spacetime geometry, we compute the scalar curvature at these points. Without loss of generality we can choose to evaluate the Ricci scalar at the point $(z,\rho) = (\nu_{2j}, 0)$. Recall that for metrics generally of the form in (4.36), the scalar curvature is given by (4.7) with $H_{\mathrm{M5}'}$ replaced by H. Note that since all collapse points ν_{2k-1} with odd labels are identified with even-labelled ones ν_{2k} via (4.45), the point $(z,\rho) = (\nu_{2j}, 0)$ is indeed generic within the set of collapsed points. Ultimately, we find

$$\mathcal{R}\Big|_{(z,\rho)=(\nu_{2j},0)} = \frac{3}{2L_{S^4}^2} \left(\frac{\hat{\xi}_{2j} - \hat{\xi}_{2j-1}}{\hat{m}_1} \right)^{-2/3} \ , \tag{4.49}$$

which is indeed finite, where L_{S^4} and $\hat{m}_1$ are strictly positive constants introduced below. This suggests that the geometry is regular at these points.

We also note that the 1PV solution corresponds to a spacetime with a constant scalar curvature given by

$$\mathcal{R} = \frac{3}{2L_{S^4}^2} \ , \tag{4.50}$$

with $L_{S^4} = (h_1\hat{\xi})^{1/3}$.

4.4.1 Asymptotic local behaviour

We will now show explicitly that the solutions described by (4.47) are asymptotically locally $\mathrm{AdS}_7 \times S^4$. As described in greater detail in Appendix A.1, the Riemann surface Σ_2 admits a parametrization by Fefferman-Graham (FG) coordinates $\{v, \phi\}$ in terms of which the line element takes the following asymptotic form for small v,

$$\mathrm{d}s^2 = \frac{4L_{S^4}^2}{v^2}\left[\mathrm{d}v^2 + \alpha_1\left(\mathrm{d}s^2_{\mathrm{AdS}_3} + \mathrm{d}s^2_{S^3}\right)\right] + L_{S^4}^2\left[\alpha_3\mathrm{d}\phi^2 + \alpha_4\sin^2\phi\ \mathrm{d}s^2_{\tilde{S}^3}\right], \tag{4.51}$$

where the metric factors are of the form $\alpha_i = 1 + \mathcal{O}(v^4)$ for $i \in \{1, 3, 4\}$. They are given explicitly, together with the asymptotic mapping to FG coordinates on Σ_2, in Appendix A.1. The asymptotic S^4 radius L_{S^4} can be expressed in terms of the moments

$$\hat{m}_k := \sum_{j=1}^{2n+2}(-1)^j\hat{\xi}_j^k \tag{4.52}$$

as

$$L_{S^4}^3 = \frac{h_1\hat{m}_1}{2} \ . \tag{4.53}$$

As claimed above, we may recognise within (4.51), at leading order in v, the large-x limit of the line element of AdS_7 (with radius $2L_{S^4}$) written in AdS_3 slicing,

$$\mathrm{d}s^2_{\mathrm{AdS}_7} = 4L_{S^4}^2\left[\mathrm{d}x^2 + \cosh^2 x\ \mathrm{d}s^2_{\mathrm{AdS}_3} + \sinh^2 x\ \mathrm{d}s^2_{S^3}\right], \tag{4.54}$$

where the coordinate normal to the AdS_3 foliation is related to the FG coordinate via $x = \log(2/v)$. The large-x limit trivialises the relative warping $\coth^2 x$ between the AdS_3 and the S^3 subspaces, matching the behaviour seen in (4.51).

In the $\gamma \to -\infty$ limit, the auxiliary functions Φ and $\tilde{h}$ admit the following FG expansions,

$$\Phi = -\tilde{h} = \frac{4\hat{m}_1\cos\phi}{v^2} + \frac{2\hat{n}_1}{\hat{m}_1} + \frac{\hat{m}_1\hat{n}_2 - \hat{n}_1^2}{\hat{m}_1^3}\cos\phi\ v^2 + \mathcal{O}(v^4)\ , \tag{4.55}$$

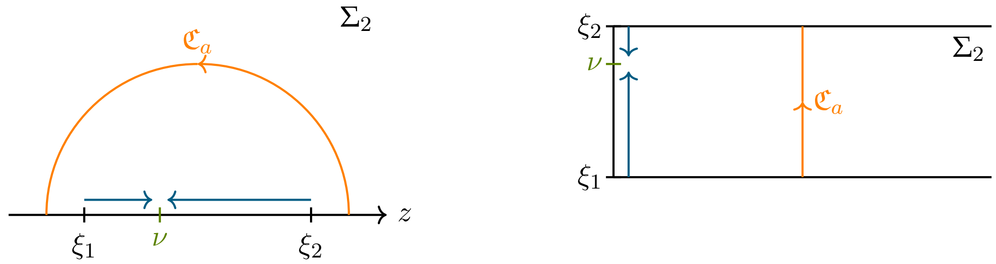


FIGURE 4.4: The 1PV solution described in section 4.4.2 is characterised by two singular points $\xi_{\{1,2\}}$, which collapse to the same $\nu \in \partial\Sigma_2$ in the $\gamma \to -\infty$ limit. As discussed in section 4.4.3, the basis of non-contractible four-cycles for this solution is one-dimensional. The profile of a representative cycle $\mathfrak{C}_a$ along Σ_2 is shown; note that the same 3-sphere collapses at both of its endpoints on $\partial\Sigma_2$. On the right, the 1PV on the upper half plane is mapped to a semi-infinite strip.

where the moments $\hat{n}_i$ are defined in (A.9). Using these expansions, we find that the asymptotic geometry in (4.51) is supported by a four-form flux[9]

$$\begin{aligned}\frac{\mathcal{F}_{(4)}}{L^3_{S^4}} = &-\frac{16\cos\phi}{v^3}\mathrm{d}v\wedge(\mathrm{vol}_{S^3}+\mathrm{vol}_{\mathrm{AdS}_3}) - \frac{8\sin\phi}{v^2}\mathrm{d}\phi\wedge(\mathrm{vol}_{S^3}+\mathrm{vol}_{\mathrm{AdS}_3}) \\ &+ 3\sin^3\phi\;\mathrm{d}\phi\wedge\mathrm{vol}_{\tilde{S}^3} + 4\cos\phi\;\frac{\hat{m}_1\hat{n}_2-\hat{n}_1^2}{\hat{m}_1^4}v\;\mathrm{d}v\wedge(\mathrm{vol}_{S^3}+\mathrm{vol}_{\mathrm{AdS}_3}) + \mathcal{O}(v^2).\end{aligned} \tag{4.56}$$

The first line in the equation above manifests the deformation of the 6d ambient SCFT by the insertion of a source, as already hinted at by the superalgebra structure discussed in section 4.3. This deformation takes the form of an S-wave over the internal $\tilde{S}^3$. It can be compared to the undeformed theory, which corresponds to $\gamma = -1/2$. In that case, the four-form field strength at the conformal boundary $v = 0$ is $\mathcal{F}_{(4)} = 3L^3_{S^4}\,\mathrm{vol}_{S^4}$, where S^4 is the internal 4-sphere spanned by ϕ and $\tilde{S}^3$. We can recognise this term as the first term in the second line of (4.56).

4.4.2 Single-pole vacuum

For later reference, we now identify a vacuum solution within the family of supergravity backgrounds presented above. An appropriate choice of vacuum is necessary for computing properties associated to a defect embedded in a holographic CFT. Indeed, in order to isolate quantities which are intrinsic to a defect, one must ensure that the contributions due to the ambient degrees of freedom are taken into account. The gravitational analogue of this operation, upon recasting a field theory computation into a bulk one, is vacuum subtraction. In particular we must use a vacuum solution which is characterised by the same bulk deformation as the general solutions discussed above. As discussed above, pure $\mathrm{AdS}_7 \times S^4$ with $\gamma = -1/2$ does not satisfy this criteria, as the

[9]In the following, we have made a choice on the signs $\tau_{1,2,3}$ in line with the constraint $\prod_{i=1}^3 \tau_i = 1$ which follows from evaluating (4.24) on the $\gamma \to -\infty$ solutions.

solutions we consider here with $\gamma \to -\infty$ contain a bulk deformation as can be seen in the asymptotic expression for the flux given in (4.56).

We take the vacuum to be the 1PV we identified in (4.48), which is the solution corresponding to the $\gamma \to -\infty$ of pure $\mathrm{AdS}_7 \times S^4$, i.e. two singular points $\xi_{\{1,2\}}$ collapsing to a single point ν, as shown in figure 4.4. More precisely, for general γ, the 1PV metric is given in FG form by

$$\begin{aligned}\frac{\mathrm{d}s^2_{1\mathrm{PV}}(\gamma)}{L^2_{S^4}} = \frac{4}{v^2}\Bigg[\mathrm{d}v^2 &+ \left(1+\frac{2\gamma+3-(2\gamma+1)c_{2\phi}}{16(\gamma+1)^2}v^2\right)\mathrm{d}s^2_{\mathrm{AdS}_3} \\ &+ \left(\frac{(\gamma+1)^2}{\gamma^2}+\frac{2\gamma-1-(2\gamma+1)c_{2\phi}}{16\gamma^2}v^2\right)\mathrm{d}s^2_{S^3}+\mathcal{O}(v^4)\Bigg] \\ &+\left[\left(1+\frac{(2\gamma+1)(2c_{2\phi}+1)}{12(\gamma+1)^2}v^2\right)s^2_\phi \mathrm{d}s^2_{\tilde{S}^3}+\left(1+\frac{(2\gamma+1)c^2_\phi}{4(\gamma+1)^2}v^2\right)\mathrm{d}\phi^2+\mathcal{O}(v^4)\right],\end{aligned} \tag{4.57}$$

where in order to keep the expression manageable, we have adopted the abusive notation

$$\cos x \equiv c_x, \quad \text{and} \quad \sin x \equiv s_x, \tag{4.58}$$

which will be employed from this point forward. For $\gamma = -1/2$, the 1PV recovers the $\mathrm{AdS}_7 \times S^4$ vacuum in FG gauge. In the $\gamma \to -\infty$ limit which is of relevance here, the line element of the 1PV becomes instead

$$\mathrm{d}s^2_{1\mathrm{PV}}(\gamma\to-\infty) = \frac{4L_{S^4}}{v^2}\left[\mathrm{d}v^2+\mathrm{d}s^2_{\mathrm{AdS}_3}+\mathrm{d}s^2_{S^3}+\mathcal{O}(v^4)\right]+L^2_{S^4}\left[s^2_\phi \mathrm{d}s^2_{\tilde{S}^3}+\mathrm{d}\phi^2+\mathcal{O}(v^4)\right]. \tag{4.59}$$

Furthermore, as a quick sanity check, we can take the 1PV limit of (4.49), which recovers $\mathcal{R}|_{1\mathrm{PV}} = 3/(2L^2_{S^4})$ as expected.

As remarked above, the 1PV contains no explicit defect data – as we will see later, no Young Tableau can be associated to it. However, it does fully capture the bulk S-wave deformation discussed previously. This follows from the fact that all terms in (4.56) which do not vanish at the conformal boundary $v = 0$ are independent of the moments $\{\hat{m}_i, \hat{n}_j\}$. Therefore, the 6d ambient theory dual to the 1PV is deformed by the same sources as the theories dual to completely generic $\gamma \to -\infty$ bulk solutions, and so, in this limit, there is no smooth deformation of the field theory parameters that restores the ambient conformal symmetry in full. This is to be contrasted with the global $\mathrm{AdS}_7 \times S^4$ solution, which enjoys the full $SO(6,2)$ conformal symmetry. Lastly, in taking the 1PV limit, the solution exhibits a flavour symmetry enhancement $SO(4) \to SO(5)$.

In the construction of [1] reviewed in section 4.2, the 1PV solution can be obtained from (4.3) by taking $\alpha \to 0$ and setting $g^3 = 2\sqrt{2}/h_1\hat{\xi}$. Alternatively, it can also be obtained in the limit $g \to 0$. To see this, first make a scale transformation using the scaling symmetry given by (4.41), take g to scale with λ as $g^3 = 2\sqrt{2}/h_1\hat{\xi}\lambda^3$, and then take $\lambda \to \infty$.

4.4.3 Partition data

In this subsection, our aim is to identify within the $\gamma \to -\infty$ solutions of (4.47) a basis of independent, non-contractible cycles threaded by four-form flux. Integrating the flux along these cycles will allow us to compute the integral M-brane charges that label a supergravity solution. In turn, this characterization will enable us to recast the specification of a supergravity solution in the form of a partition containing the representation data associated to the defect string in the dual gauge theory, in analogy with the Wilson surfaces of the $\gamma = -1/2$ solutions [309]. We begin by searching for independent, non-contractible four-cycles through the 11d geometry of the pairwise-collapsed $\gamma \to -\infty$ solutions. Later, we will comment on how the cycles are modified if less generic collapse dynamics are considered.

It is straightforward to see that, in the $\gamma \to -\infty$ limit, the volume of S^3 vanishes on $\partial\Sigma_2 - \{\nu_j\}_{j=1}^{2n+2}$, i.e. all along the boundary of the Riemann surface, except at the locations of the collapse points ν_j. In turn, any open curve on Σ_2 with end points on $\partial\Sigma_2 - \{\nu_j\}_{j=1}^{2n+2}$ will be a closed curved in the full 11d geometry. Therefore, any curve encircling at least one of the distinct collapse midpoints ν_j will not be contractible to a point. This observation allows us to build a basis for non-contractible four-cycles $\mathfrak{C}_a$, by taking the product of such curves on Σ_2 with $\tilde{S}^3$, i.e.

$$\mathfrak{C}_a \equiv \{Re^{i\theta} - \nu_{2a-1} \,|\, 0 \leq \theta \leq \pi\} \times \tilde{S}^3 \,, \tag{4.60}$$

for $1 \leq a \leq n+1$. Note that any curve enveloping multiple ν_a's can be decomposed into a linear combination of curves enclosing a single collapse point; hence, to build an irreducible basis of cycles, we take the radius R above such that $\mathfrak{C}_a$ encircles only a single collapse point ν_a.

The $\mathfrak{C}_a$ cycles alone do not exhaust the set of all non-contractible four-cycles. Indeed, we can consider a four-cycle constructed from a curve on Σ_2 connecting singular points ν_a. For irreducibility, we only consider curves connecting neighbouring collapse points. While the singular nature of their endpoints might make such curves appear problematic at first glance, building a regular four-cycle from such a curve is possible as the volume of S^3 vanishes at both endpoints, while the volume of $\tilde{S}^3$ remains finite. Indeed, the metric factor of S^3 contains $h_1 H^{-1/3}$, while the metric factor of $\tilde{S}^3$ has $\rho^2 H^{2/3}$, and from (4.47), the function H goes as ρ^{-3} near any collapse point ν_a. Thus, the cycle

$$\mathfrak{C}'_a \equiv \left\{ \frac{1}{2}(\nu_{2a+1} - \nu_{2a-1})e^{i\theta} + \frac{1}{2}(\nu_{2a+1} + \nu_{2a-1}) \,\middle|\, 0 \leq \theta \leq \pi \right\} \times S^3 \,, \tag{4.61}$$

has the desired behaviour of a non-contractible four-cycle. The distinction between the $\mathfrak{C}_a$ and $\mathfrak{C}'_a$ cycles is illustrated in figure 4.5. Given that, by construction, they connect neighbouring collapse points, there are n distinct such cycles $\mathfrak{C}'_a$. Combining them with

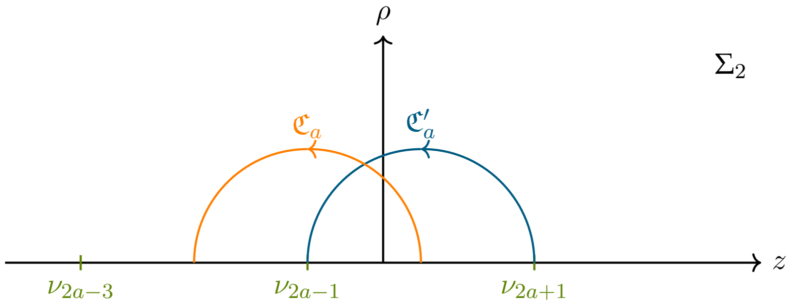


FIGURE 4.5: The profile of the non-contractible four-cycles $\mathfrak{C}_a$ and $\mathfrak{C}'_a$ in Σ_2. Every point on the red and blue curves is a 3-sphere.

the $n+1$ cycles $\mathfrak{C}_a$, we can thus build a basis consisting of $2n+1$ non-contractible four-cycles in the solutions defined by (4.47).

Having identified a basis for non-contractible four-cycles, we are now in a position to derive the M-brane charges by integrating the four-form flux over the $\mathfrak{C}_a$. To ease the computation, we abstractly write the flux as [1]

$$\begin{aligned}\mathcal{F}_{(4)} &= 2h_1 \,\mathrm{vol}_{\mathrm{AdS}_3} \wedge \mathrm{d}z + 2h_1 \,\mathrm{vol}_{S^3} \wedge \mathrm{d}z \\ &\quad + \partial_z H \rho^3 \mathrm{d}\rho \wedge \mathrm{vol}_{\tilde{S}^3} - \partial_\rho H \rho^3 \mathrm{d}z \wedge \mathrm{vol}_{\tilde{S}^3} \ .\end{aligned} \tag{4.62}$$

The pull-back of the four-form field strength $\mathcal{F}_{(4)}$ onto any of the $\mathfrak{C}_a$ cycles eliminates the first two terms in (4.62). Integrating along a given $\mathfrak{C}_a$ then yields

$$\int_{\mathfrak{C}_a} P_{\mathfrak{C}_a}[\mathcal{F}_{(4)}] = 2h_1 \,\mathrm{Vol}(\tilde{S}^3)(\hat{\xi}_{2a} - \hat{\xi}_{2a-1}) \,. \tag{4.63}$$

The choice of orientation we made while defining the cycles $\mathfrak{C}_a$ ensures that the integral above is positive. This enables us to define the following charge

$$M_a = \frac{1}{2(4\pi^2 G_N)^{1/3}} \int_{\mathfrak{C}_a} P_{\mathfrak{C}_a}[\mathcal{F}_{(4)}] \,, \tag{4.64}$$

which can be interpreted as the number of M5-branes in the a^{th} stack [2], which obey $\sum_a M_a = M$ with M being the total number of M5-branes.

It is also possible to generalise the construction above. As we mentioned previously, we can in fact build four-cycles surrounding more than one collapse point ν_a. The four-cycle defined by $\mathfrak{C}_{bc} \equiv \sum_{a=b}^{c} \mathfrak{C}_a$, with $1 \leq b \leq c \leq n+1$, is also non-contractible by construction, and is characterised by a charge

$$\int_{\mathfrak{C}_{bc}} P_{\mathfrak{C}_{bc}}[\mathcal{F}_{(4)}] = 2h_1 \,\mathrm{Vol}(\tilde{S}^3) \sum_{a=b}^{c} (\hat{\xi}_{2a} - \hat{\xi}_{2a-1}) \,, \tag{4.65}$$

under the four-form field strength.

We can follow a similar analysis for the flux threading the other set of non-contractible four-cycles, which we labelled $\mathfrak{C}'_a$ earlier. In this case, only the second term in (4.62) provides a non-vanishing contribution after pulling back the four-form field strength $\mathcal{F}_{(4)}$ to the cycle $\mathfrak{C}'_a$. Integrating the flux through $\mathfrak{C}'_a$ gives

$$\int_{\mathfrak{C}'_a} P_{\mathfrak{C}'_a}[\mathcal{F}_{(4)}] = 2h_1 \, \mathrm{Vol}(S^3)(\nu_{2a+1} - \nu_{2a-1}) \; . \tag{4.66}$$

Similar to the integrated fluxes through the $\mathfrak{C}_a$ cycles, we define the charge

$$M'_a = \frac{1}{2(4\pi^2 G_N)^{1/3}} \int_{\mathfrak{C}'_a} P_{\mathfrak{C}'_a}[\mathcal{F}_{(4)}] \, , \tag{4.67}$$

which is read as the number of M5′-branes in the a^{th} stack (see table 4.1).

Again, we can easily generalise this analysis to four-cycles that connect non-neighbouring collapse points ν_a. Denoting the sum of four-cycles as $\mathfrak{C}'_{bc} \equiv \sum_{a=b}^{c} \mathfrak{C}'_a$, where given the construction of the $\mathfrak{C}'_a$ in (4.66) $1 \leq b \leq c \leq n$, the integral of the flux through this cycle is simply

$$\int_{\mathfrak{C}'} P_{\mathfrak{C}'}[\mathcal{F}_{(4)}] = 2h_1 \, \mathrm{Vol}(S^3) \sum_a (\nu_{2a+1} - \nu_{2a-1}) \, . \tag{4.68}$$

In addition, following [2] we can deduce the number of M2-branes ending on the a^{th} stack of M5-branes, which we denote

$$N_a = \sum_{b=a}^{n} M'_b = \frac{h_1 \, \mathrm{Vol}(S^3)}{(4\pi^2 G_N)^{1/3}} \sum_{b=a}^{n} (\nu_{2b+1} - \nu_{2b-1}) \, . \tag{4.69}$$

One may notice how the definition of the collapse points ν_a influences the properties of N_a. Indeed, since $\{\nu_a\}$ is an ordered set, we see that $N_a \geq N_b$ for $a \leq b$. In other words, the set $\{N_a\}$ forms a partition of the total number of M2-branes $N = \sum_a N_a$, and so it is possible to define a Young diagram to encode the brane charges as illustrated in figure 4.6a.

We can now recast the various moments defined in eqs. (4.52) and (A.9) in terms of this partition data. Since only $\hat{m}_1$ appears in the asymptotic expansions in Appendix A.1, and so also in the physical quantities computed using those expansions, we will not treat the higher $\hat{m}_j$ moments and simply write

$$\hat{m}_1 = \sum_{j=1}^{2n+2} (-1)^j \hat{\xi}_j = \frac{(4\pi^2 G_N)^{1/3}}{h_1 \, \mathrm{Vol}(\tilde{S}^3)} M \, , \tag{4.70}$$

which gives back the usual relation between M and the length scale on the S^4,

$$L^3_{S^4} = \frac{h_1 \hat{m}_1}{2} = \frac{(G_N)^{1/3}}{(2\pi)^{4/3}} M \, . \tag{4.71}$$

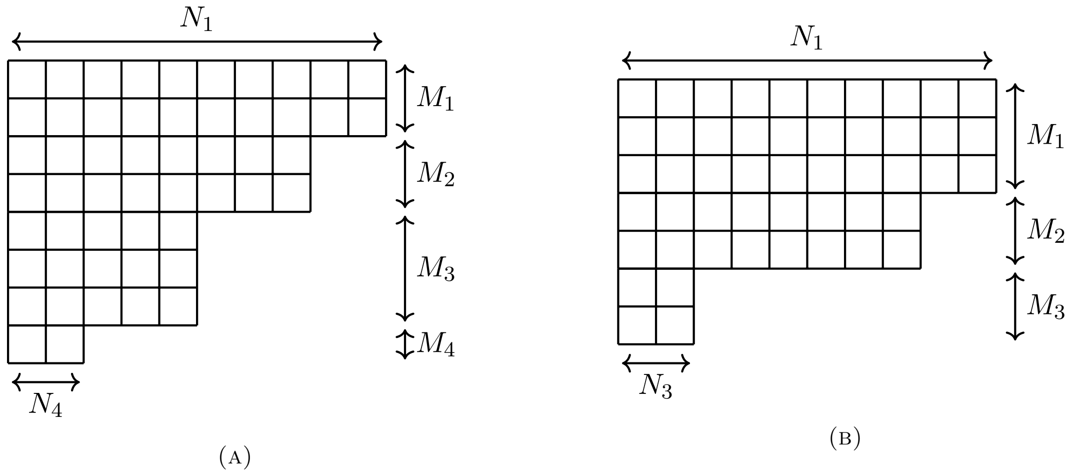


FIGURE 4.6: (a) Young diagram corresponding to the partition specifying a $\gamma \to -\infty$ solution with 5 distinct ν_a constructed by pairwise collapse of 10 different ξ_j. (b) Young diagram obtained by "multiwise" collapse of 10 different ξ_j to 4 distinct ν_a. Another way to realise the construction of (b) starts from the partition in (a) and collapses $\nu_2 = \nu_3$.

The $\hat{n}_j$ moments are a bit trickier to re-express in terms of M_a and N_a, but one can show that the following relation holds for any j

$$\sum_{k=0}^{j} (-1)^{j+k} \frac{j!}{k!(j-k)!} \nu_{2n+1}^k \, \hat{n}_{j-k} = \left(\frac{G_N^{1/3}}{h_1 \pi (2\pi)^{1/3}} \right)^{j+1} \sum_{a=1}^{n+1} M_a N_a^j \,, \tag{4.72}$$

where it is understood that $\hat{n}_0 = \hat{m}_1$. However, we will only require relations up to $j = 2$ moving forward. Using the expressions above, we can write

$$\hat{n}_1^2 - \hat{m}_1 \hat{n}_2 = \frac{G_N^{4/3}}{h_1^4 \pi^4 (2\pi)^{4/3}} \left[\left(\sum_{a=1}^{n+1} M_a N_a \right)^2 - M \sum_{a=1}^{n+1} M_a N_a^2 \right] \,, \tag{4.73}$$

which will be useful in the following section.

Before moving on, we recall from our previous discussion that it is also possible to consider solutions where more than two branch points ξ_j collapse to a single point ν_a and, of course, build non-contractible four-cycles around or connecting them. Let us again index the p distinct loci of collapse as ν_a, where now $1 \leq a \leq p+1$ with $p \leq n$. The limiting case $p = n$ recovers the pairwise collapse described above. It will be useful to label I_a and K_a respectively as the smallest and largest j-indices of the ξ_j branch points which collapse to a given ν_a, i.e. $\nu_a := \nu_{I_a} = \nu_{I_a+1} = \cdots = \nu_{K_a}$. The ordering of the collapse points ν_a and of the branch points ξ_j was discussed previously in section 4.4; in particular, we recall that the parameters $\hat{\xi}_j$'s associated to the branch points collapsing to the same ν_a are ordered amongst themselves. In the end, the analysis for these "multiwise" collapse solutions is identical to the one presented above for the pairwise collapse scenario, up to the replacements

$$\hat{\xi}_{2a} - \hat{\xi}_{2a-1} \quad \longrightarrow \quad \sum_{j=I_a}^{K_a} (-1)^j \hat{\xi}_j \tag{4.74}$$

and $n \to p$ throughout the expressions above. The net effect is that the partition data yields a differently shaped Young diagram, illustrated in figure 4.6b. In particular, since the multiwise collapse can greatly reduce the number of singular points on the boundary of the Riemann surface, we see that the Young tableaux specifying the $\gamma \to -\infty$ solutions have at most p rows. This is a striking difference compared to the Young tableau construction of [309] for the Wilson surface solutions with $\gamma = -1/2$, whose associated Young tableaux always have n rows. This difference is maximal in the 1PV solution described in section 4.4.2, to which one cannot attach any Young tableau interpretation at all. This is due to the fact that the 1PV solution features a single collapse point ν, so that the construction of the $\mathfrak{C}'_a$ cycle fails, thus preventing the definition of the M'_a and N_a charges.

4.5 Entanglement entropy of small $\mathcal{N} = (4,4)$ surface defects

The entanglement entropy S_{EE} of a spatial subregion $\mathcal{B}$ within a QFT is defined as the von Neumann entropy of the reduced density matrix obtained by tracing out the states in the complementary region $\overline{\mathcal{B}}$ of the QFT. For CFTs with weakly coupled gravity duals, the RT prescription [48, 49, 321] holographically recasts the computation of S_{EE} into the following Plateau problem in the asymptotically AdS bulk,

$$S_{\text{EE}} = \min_{\zeta} \frac{\mathcal{A}[\zeta]}{4G_{\text{N}}} , \tag{4.75}$$

where $\mathcal{A}[\zeta]$ is the area functional evaluated on a bulk hypersurface ζ which is homologous to the chosen spatial subregion in the dual CFT: $\zeta \cup \mathcal{B} = \partial b$ for some static bulk subregion b. We will denote this extremal bulk hypersurface as ζ_{RT}. In the computations below, we choose the spatial subregion to be an Euclidean 5-ball, $\mathcal{B} = \mathbb{B}^5_R \hookrightarrow \mathbb{R}^5$, with radius R. We take $\mathcal{B}$ to be centred on the spatial extent of the surface defect, which has a Lorentzian worldvolume $\text{Y}_2 = \mathbb{R}^{1,1}$.

In an ordinary QFT, the presence of highly entangled UV degrees of freedom induces short-distance divergences near the surface $\partial\mathcal{B} = S^4$. Most of the divergences in the EE of a general QFT are non-universal and shape-dependent. However, in even dimensional theories, there are universal log-divergent contributions to the EE, which are generically related at conformal fixed points to the Weyl anomalies of the CFT.

This is true in the presence of a defect as well, but we see additional divergences that arise from the defect degrees of freedom near $\mathrm{Y}_2 \cap \partial\mathcal{B}$. In order to isolate these defect-localised contributions, we will adopt a scheme where we subtract off the EE of the deformed, vacuum ambient CFT, $S_{\mathrm{EE}}[\emptyset]$, from the EE computed in the presence of the defect $S_{\mathrm{EE}}[\mathrm{Y}_2]$. We can then extract the coefficient of the universal, log-divergent part of the defect contribution to the sphere EE by

$$S^{(univ)}_{\mathrm{EE}} = R\frac{\mathrm{d}}{\mathrm{d}R}\left(S_{\mathrm{EE}}[\mathrm{Y}_2] - S_{\mathrm{EE}}[\emptyset]\right)\Big|_{R\to 0} . \tag{4.76}$$

In [51], it was shown that contribution to the log-divergent part of the EE of a spherical region coming from a flat, 2d conformal defect embedded in a d-dimensional flat-space ambient CFT takes the form of a linear combination of defect localised Weyl anomalies. Explicitly, for a 6d ambient CFT

$$S^{(univ)}_{\mathrm{EE}} = \frac{1}{3}\left(a_{\mathrm{Y}} - \frac{3}{5}d_2\right) . \tag{4.77}$$

where a_{Y} is the A-type defect Weyl anomaly coefficient–i.e. appearing with the intrinsic Euler density–and d_2 is the B-type anomaly that enters with the trace of the pullback of the ambient Weyl tensor. Importantly, while [51] demonstrated that $d_2 \geq 0$ based on energy conditions, a_{Y}, though obeying a defect "c-theorem", has no positivity constraints. This means that as opposed to an ordinary, unitary 2d CFT where the universal part of the EE is is proportional to the central charge [322] and, hence, is non-negative, it is clear from (4.77) that $S^{(univ)}_{\mathrm{EE}}$ is not similarly bounded nor is it RG monotonic.

In completing the holographic computation, we also need to contend with the fact that the FG expansion is not globally defined as it typically breaks down in a region near the AdS submanifold that is dual to the insertion of the defect in the field theory. However, we have a full analysis of the asymptotic expansions of the data specifying the $\gamma \to -\infty$ solutions in Appendix A.1, and we have the general prescription for the FG transformation suitable for defect EE in [323]. Together, we will be able to unambiguously holographically compute $S^{(univ)}_{\mathrm{EE}}$.

To begin the holographic computation of the defect EE, we choose the following parametrization for the AdS_3 subspace in (4.51),

$$\mathrm{d}s^2_{\mathrm{AdS}_3} = \frac{1}{u^2}\left(\mathrm{d}u^2 - \mathrm{d}t^2 + \mathrm{d}x_{\|}^2\right). \tag{4.78}$$

Furthermore, we take the RT hypersurface ζ to wrap both S^3 and $\tilde{S}^3$, and its profile in the remaining subspace to be described by $x_{\|}(u,\rho,z)$. The area of ζ as measured against the metric in (4.51) is then

$$A[\zeta] = \mathrm{Vol}(S^3)\,\mathrm{Vol}(\tilde{S}^3)\int \mathrm{d}u \int_{\Sigma_2} \mathrm{d}\rho\mathrm{d}z\ \mathcal{L} , \tag{4.79}$$

where the Lagrangian is

$$\mathcal{L} = \frac{h_1^2 \rho^3}{u^2} \left[h_1 \left((\partial_\rho x_\|)^2 + (\partial_z x_\|)^2 \right) H(z,\rho) + u^2 \left(1 + (\partial_u x_\|)^2 \right) H(z,\rho)^2 \right]^{1/2}. \tag{4.80}$$

As shown in [324], the minimal area surface wraps the Riemann surface Σ_2 too, so that $\partial_\rho x_\| = \partial_z x_\| = 0$. The Lagrangian is thus minimised by

$$x_\|^2 + u^2 = R^2, \tag{4.81}$$

for a constant R, so that the area of the extremal hypersurface $\zeta_{\rm RT}$ is

$$A[\zeta_{\rm RT}] = h_1^3 \,\mathrm{Vol}(S^3)\,\mathrm{Vol}(\tilde{S}^3) \log\left(\frac{2R}{\epsilon_u}\right) \int_{\Sigma_2} \mathrm{d}\rho \mathrm{d}z \; \rho^3 \sum_{j=1}^{2n+2} \frac{(-1)^j \hat{\xi}_j}{(\rho^2 + (z-\nu_j)^2)^{3/2}} + \mathcal{O}(\epsilon_u^2), \tag{4.82}$$

where we introduced a small-u cutoff $\epsilon_u > 0$.

The evaluation of the integral above is performed in detail in Appendix A.2. The resulting holographic entanglement entropy is

$$S_{\rm EE}[\mathrm{Y}_2] = \frac{\pi^4 L_{S^4}^9}{G_{\rm N}} \log\left(\frac{2R}{\epsilon_u}\right) \left[\frac{64}{3}\frac{1}{\epsilon_v^4} + \frac{16}{5}\frac{\hat{n}_1^2}{\hat{m}_1^4} - \frac{16}{5}\frac{\hat{n}_2}{\hat{m}_1^3} + \mathcal{O}(\epsilon_v^2) \right] + \mathcal{O}(\epsilon_u^2), \tag{4.83}$$

where $\epsilon_v > 0$ is a small-v cutoff in the FG parametrization.

Subtracting off the 1PV contribution to the entanglement entropy, $S_{\rm EE}^{\rm 1PV}$, precisely removes the ϵ_v^{-4} divergence from (4.83), and leaves the $\mathcal{O}(\epsilon_v^0)$ term unchanged. To see this, we recall that the 1PV limit takes $\hat{n}_k \to 0$, which in (4.83) gives

$$S_{\rm EE}^{\rm 1PV} = \frac{\pi^4 L_{S^4}^9}{G_{\rm N}} \log\left(\frac{2R}{\epsilon_u}\right) \left[\frac{64}{3}\frac{1}{\epsilon_v^4} + \mathcal{O}(\epsilon_v^2) \right] + \mathcal{O}(\epsilon_u^2). \tag{4.84}$$

Therefore, we can at once compute the coefficient of the universal part of the defect sphere EE using (4.76) and plugging in eqs. (4.83) and (4.84) with $S_{\rm EE}[\emptyset] = S_{\rm EE}^{\rm 1PV}$ to find

$$S_{\rm EE}^{(univ)} = \frac{16}{5}\frac{\pi^4 L_{S^4}^9}{G_{\rm N}} \frac{\hat{n}_1^2 - \hat{m}_1 \hat{n}_2}{\hat{m}_1^4} \tag{4.85a}$$

$$= \frac{1}{5M} \left[\left(\sum_{a=1}^{n+1} M_a N_a \right)^2 - M \sum_{a=1}^{n+1} M_a N_a^2 \right], \tag{4.85b}$$

where we have mapped to field theory quantities using $L_{S^4}^3 = \frac{G_N^{1/3}}{(2\pi)^{4/3}} M$ and used the definitions of moments $\hat{n}_j$, $\hat{m}_j$ in terms of the numbers of branes in eqs. (4.70) and (4.73). In terms of the highest weight ϖ of the A_{M-1} irreducible representation encoded in the Young diagrams that specify the defect discussed in the previous section, we can re-express the defect sphere EE as [309]

$$S_{\rm EE}^{(univ)} = -\frac{(\varpi,\varpi)}{5} \ , \tag{4.86}$$

where $(\cdot,\cdot)$ is the scalar product on the weight space induced by the Killing form.

We also note that the contribution of the aforementioned bulk deformation to the coefficient of the universal, log-divergent component of the vacuum-subtracted entanglement entropy is

$$S_{\rm EE}^{(univ,bulk-def.)} = \frac{8}{3}\frac{\pi^4 L_{S^4}^9}{G_{\rm N}} = \frac{M^3}{6} \ . \tag{4.87}$$

This is independent of the moments $(\hat{m}_i, \hat{n}_j)$, in line with the lack of Young Tableau data associated to the bulk deformation. Had we subtracted in (4.76) the $\mathrm{AdS}_7 \times S^4$ vacuum, rather than the 1PV, the resulting entanglement entropy would have received both the defect and bulk deformation contributions above.

Finally, we note that the same quantity can be trivially computed in the orbifolded theory described in section 4.3.3 simply by rescaling

$$S_{\rm EE}^{(univ)} \longrightarrow \frac{\mathrm{Vol}(S^3/\mathbb{Z}_k)\,\mathrm{Vol}(\tilde{S}^3/\mathbb{Z}_{k'})}{\mathrm{Vol}(S^3)\,\mathrm{Vol}(\tilde{S}^3)} S_{\rm EE}^{(univ)} = \frac{S_{\rm EE}^{(univ)}}{kk'} \ . \tag{4.88}$$

4.6 Summary and Outlook

In this work, we have constructed a novel class of solutions in 11d SUGRA that are holographically dual to 2d superconformal defects preserving small $\mathcal{N}=(4,4)$ and $\mathcal{N}=(0,4)$ SUSY in 6d SCFTs at large M. These solutions fit into the one-parameter family organised in a general classification scheme of 11d SUGRA solutions with superisometry $\mathfrak{d}(2,1;\gamma)\oplus\mathfrak{d}(2,1;\gamma)$ [2]; specifically, they are obtained in the $\gamma\to-\infty$ limit. There are several features of these new solutions that separate them from the more familiar $\gamma=-1/2$ case that holographically corresponds to 1/2-BPS Wilson surface type defects in the 6d $\mathcal{N}=(2,0)$ A_{M-1} SCFT.

Within the one-parameter family of solutions labelled by γ, the $\gamma\to-\infty$ limit is slightly unusual from the superalgebra perspective. Despite producing an $\mathrm{AdS}_7\times S^4$ asymptotic geometry as shown in section 4.3, taking the $\gamma\to-\infty$ limit means that $\mathfrak{d}(2,1;\gamma)\oplus\mathfrak{d}(2,1;\gamma)$ is not a subalgebra of the $\mathfrak{osp}(8^*|4)$ superisometry of $\mathrm{AdS}_7\times S^4$. On the field theory side of the holographic duality, this means that the ambient theory into which the defects are inserted is some deformation of the 6d $\mathcal{N}=(2,0)$ SCFT.

We have seen that choosing all of the singular loci in the internal space to collapse to a single point – a configuration which we label 1PV – destroys the data that specifies the defect, i.e. the Young diagram corresponding to the arrangement of M5-branes. However, as is clear from the discussion in section 4.4.2, the vacuum that we arrive at has an isometry group of $SO(2,2) \times SO(3) \times SO(5)$, as opposed to the vacuum solution at $\gamma = -1/2$, which instead enjoys the full $SO(6,2)$. In the latter case, the trivial defect corresponds to a Wilson surface transforming in the $\mathbf{1}$ of A_{M-1}, and the conformal symmetry of the ambient 6d $\mathcal{N} = (2,0)$ theory is restored. For $\gamma \to -\infty$, the trivial defect dual to the 1PV still possesses what looks like the 'defect' conformal symmetry despite being the vacuum solution, which renders giving a precise definition for and interpretation of the defect CFT difficult. The 1PV does, however, enable us to correctly employ a background subtraction scheme[10] and arrive at a finite result for $S_{\text{EE}}^{(univ)}$ and, we believe, resolves the puzzling appearance of divergences in the "defect central charge" computed in [1].

On a more fundamental level, the small $\mathcal{N} = (4,4)$ defects at $\gamma \to -\infty$ cannot be viewed as a smooth deformation of the Wilson surface defects at $\gamma = -1/2$. Indeed, the $\gamma \to -\infty$ solutions cannot even be smoothly deformed into the solution at $\gamma = 0$, to which they are related by the involution $\gamma \mapsto 1/\gamma$ with an exchange of the $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ factors in $\mathfrak{d}(2,1;\gamma;0) \oplus \mathfrak{d}(2,1;\gamma;0)$. The reason is that there is a special point at $\gamma = -1$ where the real form $\mathfrak{d}(2,1;\gamma;0)$ becomes $\mathfrak{osp}(4|2;\mathbb{R})$. At this value of γ, $SO(2,2)$ Wigner-İnönü contracts to $ISO(1,2)$, and AdS$_3$ becomes $\mathbb{R}^{2,1}$. Therefore, the $\gamma \to -\infty$ solutions are isolated from the other class of asymptotically AdS$_7 \times S^4$ geometries.

In light of the new small $\mathcal{N} = (4,4)$ solutions that we have constructed and holographically studied, there are a number of open questions that remain to be answered.

Firstly, as we discussed at the start of section 4.5 , the contribution from a flat 2d conformal defect to the log-divergent, universal part of the EE of a spherical region in a $d \geq 4$ ambient CFT is built from a linear combination of two defect Weyl anomaly coefficients, a_Υ and d_2 that characterise the defect theory. In order to disentangle these two fundamental defect quantities, we would compute a second holographic quantity that contains either a_Υ or d_2. For instance, d_2 controls the normalization of the one-point function $\langle T_{\mu\nu} \rangle$ of the stress-energy tensor, which can be readily computed for most 10d or 11d supergravity solutions by dimensional reduction on the internal space [325, 326]. Therefore, it is natural to try to compute d_2 and $S_{\text{EE}}^{(univ)}$ in order to isolate the independent defect Weyl anomalies[11]. This was successfully done for the Wilson surfaces at $\gamma = -1/2$ in [309] and for codimension-2 defects in [7]. However, the $\gamma \to -\infty$ solutions are more subtle, and a naïve application of dimensional reduction

[10]Here, "correctly" refers to a background subtraction which also removes any contributions from the trivial defect. As we have demonstrated, the same cannot be said of a subtraction scheme which utilises vacuum AdS$_7 \times S^4$.

[11]In fact, a_Υ and d_2 are the only independent defect Weyl anomaly coefficients for superconformal defects preserving at least 2d $\mathcal{N} = (0,2)$ supersymmetry. This was shown for co-dimension four defects in 6d SCFTs in [311] and co-dimension two defects in 4d SCFT in [327].

techniques would be inappropriate. Namely, in reducing the 11d solutions to 7d, the presence of the non-trivial four-form flux modifies the gravitational equations of motion at the conformal boundary of AdS_7, which violates the assumptions in [325]. Therefore, computing d_2 holographically from $\langle T_{\mu\nu} \rangle$ requires a generalization to account for flux contributions, which is the subject of ongoing work.

Furthermore, it is natural to look for physical observables which can be reliably computed on the field theory side and employed to test the holographic predictions made above. For 2d BPS conformal defects in 6d A_{M-1} and D_M $\mathcal{N} = (2,0)$ SCFTs at large M, recent developments in analytic bootstrap methods have enabled the computation of correlations functions in the presence of 2d defects that are controlled by anomalies [311, 328]. Further, despite the lack of a Lagrangian description and of supersymmetric localization methods for 6d SCFTs at large M, chiral algebra methods have also been shown to give exact results for defect correlators [328] and the defect SUSY Casimir energy [284, 310]. Currently, only Wilson surface type defects, i.e. the holographically dual theories to the $\gamma = -1/2$ solutions, have been studied using these field theory techniques. It is reasonable to wonder whether any of these methods are applicable to the types of defects in the deformed 6d theory that we have constructed in this work.

The biggest hurdle to clear in trying to generalise bootstrap or chiral algebra methods for use in the dual to the 1PV of the $\gamma \to -\infty$ solutions is clarifying the precise role of the deformation parameter γ. As we have explained in the $\gamma \to -\infty$ limit, the ambient 6d theory has reduced global and conformal symmetries, and there is no smooth path in field theory space as γ is varied through $\gamma = -1$ to get from the 6d A_{M-1} $\mathcal{N} = (2,0)$ theory to the dual of the 1PV. It is unclear at the moment precisely what symmetry breaking operators are sourced on the field theory side in the $\gamma \to -\infty$ limit.

Chapter 5

Codimension-2 defect in 6d SCFT

5.1 Introduction

Knowing the spectrum of local operators in a given quantum field theory (QFT) is insufficient to uniquely specify it in field theory space [329], and so operators with non-trivial extension along submanifolds embedded in the background spacetime ('defects') play an important role in classifying QFTs [32]. However, the way that the presence of these defects affects, say, correlation functions of local operators depends on the dimension d and geometry of the background manifold $\mathcal{M}_d$, the co-dimension $d-\mathfrak{d}$ and embedding of the $\mathfrak{d}$-dimensional defect submanifold $\Sigma_{\mathfrak{d}}$, and the couplings between ambient and defect degrees of freedom[1]. Thus, it is crucial to characterise allowable defects in a given theory and precisely determine how ambient physical observables change under the deformation by defect operators.

In this effort, some of the most powerful tools that we have come from imposing symmetries on both the ambient and defect theories. The ambient field theories we consider are 6d, supersymmetric, and invariant under 6d flat-space conformal symmetry $SO(6,2)$; superconformal field theories (SCFTs). The defects that we study in this work are supported on embedded co-dimension 2 submanifolds, $\Sigma_4 \hookrightarrow \mathcal{M}_6$, that will preserve at least 1/4 of the total supersymmetries, i.e. $\mathcal{N} \geq 1$ 4d supersymmetry, as well as an $SO(4,2) \times U(1)_N \subset SO(6,2)$ global symmetry representing the defect conformal symmetry and $U(1)_N$ rotations in $\mathcal{M}_6/\Sigma_4$. We will refer to these theories as defect [super]conformal field theories (D[S]CFTs).

In the following, we will focus on ambient theories that are maximally superconformal $\mathcal{N} = (2,0)$ SCFTs with gauge algebra A_{N-1} in the large N limit and the 1/4- and 1/2-BPS co-dimension 2 defects that they support. Despite the highly restrictive symmetries imposed, 6d $\mathcal{N} = (2,0)$ SCFTs and their defect operators pose a challenge

[1] See [290] for a recent review of defects of various (co-)dimension in QFTs.

to direct study. We know from the worldvolume theory of a stack of coincident M5-branes [281] or M5-branes probing ADE singularities [330] that 6d $\mathcal{N} \geq (1,0)$ SCFTs exist, but generally they have no known Lagrangian description. We also know that 6d SCFTs constructed from M-theory support 4d BPS defect operators engineered at the intersection of orthogonal stacks of M5 branes. Since we often lack a Lagrangian description, our efforts to characterise these $\mathfrak{d} = 4$ dSCFTs are limited to analysing their global properties using techniques such as anomaly inflow (e.g. [331]) and chiral algebra methods [332]. That said, there is a tremendous amount that we can learn about the defect theory by studying its conformal anomalies.

As with any systems preserving an $SO(d,2)$ global conformal symmetry, putting the ambient theory on a curved $\mathcal{M}_d$ results in a non-trivial Weyl anomaly. Crucial to our understanding of dCFTs, the theory supported on $\Sigma_{\mathfrak{d}} \hookrightarrow \mathcal{M}_d$ has its own defect-localised contributions to the total Weyl anomaly that are sensitive to both the intrinsic submanifold geometry and its embedding in the ambient space. The resulting defect Weyl anomaly can be far more complicated than that of an ordinary $\mathfrak{d}$-dimensional theory. For example, it is common knowledge that the Weyl anomaly in $d = 4$ is a combination of an 'A-type' anomaly $\sim aE_4$, where E_4 is the 4d Euler density, and a 'B-type' anomaly $\sim c|W|^2$ with $W_{\mu\nu\rho\sigma}$ denoting the Weyl-tensor [333]. On the other hand, it was recently discovered in [34] that the Weyl anomaly of a $\mathfrak{d} = 4$ defect in an ambient theory with $d \geq 6$ has a total of 29 terms[2].

The challenge thus far has been finding tractable, non-trivial $\mathfrak{d} = 4$ defect systems beyond free theories (e.g. [294]) in which any of the 29 available defect Weyl anomalies can be computed[3]. In light of recently discovered 11d supergravity (SUGRA) solutions that holographically describe certain $\mathfrak{d} = 4$ BPS defects in 6d SCFTs [3, 4], we have a window on strongly coupled, non-Lagrangian defect systems that can be approached with standard tools in holography to compute quantities known to be controlled by defect anomalies.

In this work, we study both the 1/4-BPS 'two-charge' solutions in 11d constructed as the uplift of domain wall solutions in 7d gauged SUGRA and the 1/2-BPS 'electrostatic' solutions for bubbling geometries [336, 337] built along the lines of those in [338, 339] but with non-compact internal spaces. By holographically computing the one-point function of the stress-energy tensor and the flat defect contribution to the entanglement entropy (EE) of a spherical region co-original with the defect, we will be able to extract two of the 29 possible defect Weyl anomaly coefficients. In doing so, we find two independent pieces of data that characterise these defect systems.

[2]These 29 terms include 6 terms that break parity on the defect submanifold. The limit case of a co-dimension 1 defect in 5d has 12 (including 3 parity odd) terms in the Weyl anomaly [34, 334].

[3]For probe branes wrapping $\mathrm{AdS}_5 \subset \mathrm{AdS}_{d+1}$, all of 23 of the parity even anomalies can be holographically computed [34] using the work of Graham and Reichert in [335].

The results that we obtain for these $\mathfrak{d} = 4$ defect anomalies have implications and open up questions beyond their roles in 6d SCFTs. In particular, the 'electrostatic' solutions holographically describe defects in the 6d A_{N-1} $\mathcal{N} = (2,0)$ SCFT labelled equivalently by a Lie algebra homomorphism $\vartheta : \mathfrak{sl}(2) \to \mathfrak{su}(N)$, the choice of Levi subalgebra $\mathfrak{l} \subset \mathfrak{su}(N)$ associated with the Levi subgroup $L = S(U(N_1) \times \ldots U(N_n))$, or the Young diagram corresponding to the partition of $N = \sum_{a=1}^{n} N_a$. Under the (partially) twisted dimensional reduction on genus-$\mathbf{g}$ Riemann surface $\mathcal{C}_{\mathbf{g}}$ either with Σ_4 wrapping two legs along $\mathcal{C}_{\mathbf{g}}$ or with $\mathcal{C}_{\mathbf{g}}$ orthogonal to Σ_4 in $\mathcal{M}_6$, the theory descends to a 4d class$-\mathcal{S}$ $\mathcal{N} = 2$ SCFT [279, 340] deformed by a $\mathfrak{d} = 2$ surface defect [297, 341] or by (possibly irregular [342–344]) punctures on its UV curve. For example in the case of Σ_4 wrapping $\mathcal{C}_{\mathbf{g}} = \mathbb{T}^2$, the 4d description is of Gukov-Witten defects in $\mathcal{N} = 4$ $SU(N)$ super-Yang Mills theory [295, 296], whose defect Weyl anomalies are known [51, 345, 346]. While the precise map between the 29 defect anomalies and the two independent Weyl anomalies of a surface operator in 4d [327] or the central charges of the 4d SCFT itself is unknown at present, our results provide some insight into how some of the defect data in 6d is reorganised into 4d (defect) Weyl anomalies.

The following work is structured as follows: In section 5.2, we first review the pertinent aspects of Weyl anomalies for 4d defects and highlight their connection to physical quantities that we will compute in later sections. We will also briefly review the solutions in 11d SUGRA that holographically describe 1/4-BPS and 1/2-BPS co-dimension 2 defects in 6d SCFTs. In section 5.3, we compute the holographic stress-energy tensor one-point function for both the 1/4-BPS, two-charge solution and a generic 1/2-BPS electrostatic solution, which we use to find the defect B-type Weyl anomaly that we call d_2. In section 5.4, we holographically compute the defect contribution to the EE of a spherical region, which we use to determine defect A-type Weyl anomaly, a_Σ. In section 5.5, we discuss comparisons to field theory results and future directions.

In addition, a number of useful intermediate results are contained in appendices. In Appendix B.1 we detail the asymptotic maps of the metrics for the solutions we consider into Fefferman-Graham form. In appendix B.2 we compute the on-shell action for the 11d uplift of the two-charge solutions and highlight a discrepancy with the same computation done in the domain wall description in 7d $\mathcal{N} = 4$ gauged SUGRA. Finally, in appendix B.3, we discuss the details of the regulating scheme for the on-shell action including the vacuum solution that we use in background subtraction as well as the renormalised volume of the AdS$_5$ geometry.

5.2 Review

In this section, we will very briefly review some key background material in order to orient the subsequent computations. In the first subsection, we will introduce the two defect Weyl anomalies and discuss the physical quantities that they control, which will be the focus of the computations to follow. In the second subsection, we will give a short overview of the two solutions to 11d SUGRA that will be the focus of our holographic study.

5.2.1 Defect Weyl anomalies

Up to a total derivative, the Weyl anomaly of an ordinary 4d CFT has two independent contributions[4],

$$T^\mu{}_\mu = \frac{1}{4\pi^2}(-a_{4d}E_4 + c|W|^2). \tag{5.1}$$

The first term proportional to the Euler density E_4 is the so-called "A-type" anomaly in the classification of [333], which exists in all even-dimensional CFTs and is unique in that it transforms as a total derivative under Weyl transformations. The second term given by the square of the Weyl tensor is a "B-type" anomaly. In arbitrary even-dimensional CFTs, there is generally a tower of B-type anomalies each of which is exactly Weyl invariant and built out of non-topological, rank-$\frac{d}{2}$ monomials in curvatures. The Weyl anomaly coefficients of a 4d CFT control correlation functions of the stress-energy tensor [348], and have strong upper and lower bounds on their ratio [349]; a_{4d} also appears in the EE [350], and obeys an 'a'-theorem under renormalization group (RG) flows [38, 40]. For 4d SCFTs with an R-symmetry, a_{4d} and c are both related to the cubic and mixed R-anomalies through non-perturbative formulae [351].

The Weyl anomaly of a conformal defect supported on $\Sigma_\mathfrak{d} \hookrightarrow \mathcal{M}_d$ is much richer due to the additional freedom of building submanifold conformal invariants out of not only the intrinsic curvature but also the normal bundle curvature, the pullback of curvature tensors from the ambient space, and the second fundamental form for the embedding. For conformal defects on $\Sigma_4 \hookrightarrow \mathcal{M}_d$ of co-dimension 2 or greater[5], there are a total of 23 anomalies respecting submanifold parity [34][6]. The complete form of the 4d defect Weyl anomaly is cumbersome, and so we will only display the parts relevant to the computations in the following sections (see eq. 3.1 of [34] for the full expression):

$$T^\mu{}_\mu|_{\Sigma_4} \supset \frac{1}{(4\pi)^2}\Big(-a_\Sigma \overline{E}_4 + d_2\mathcal{J}_2 + \dots\Big). \tag{5.2}$$

The first term is recognizable as the defect A-type anomaly proportional to the *intrinsic* Euler density, $\overline{E}_4$, of Σ_4. The second term $\mathcal{J}_2$ is a B-type anomaly built out of a

[4]This basis is not unique, and one can exchange either E_4 or W^2 for Branson's Q-curvature [347] and a total derivative, which gives a basis for the 4d Weyl anomaly that is particularly convenient for holography.

[5]The limit case of co-dimension one is far more restricted and only leads to 9 parity even anomalies [34, 334].

[6]There are an additional 6 parity odd defect Weyl anomalies, but as of yet, there are neither any known physical quantities in which they appear nor any no-go theorem to forbid them.

complicated linear combination of the submanifold pullback of the ambient curvatures, connection on the normal bundle, normal bundle curvature, and the second fundamental form for the embedding (see eq. 3.2 of [34] for the full expression). Importantly, $\mathcal{J}_2$ does not contain a term like the pullback of $|W|^2$ or the square of the intrinsic Weyl tensor, and so is not analogous to the B-type anomaly of a standalone 4d CFT above.

While it is unclear what physics the vast majority of terms in the full expression of the defect Weyl anomaly control, the two anomalies displayed above appear in two physical quantities that will be the primary focus of the following work.

The first quantity we will analyze is the one-point function of the stress-energy tensor. For a $\mathfrak{d}$-dimensional conformal defect embedded in a d-dimensional CFT, conformal symmetry preserved by the defect constrains the form of the one-point function of the stress-energy tensor a distance $x_\perp$ away from the defect to be of the form [107, 352]

$$\langle T^{ab}\rangle = -h_T\frac{(d-\mathfrak{d}-1)\delta^{ab}}{|x_\perp|^d}\,, \qquad \langle T^{ij}\rangle = h_T\frac{(\mathfrak{d}+1)\delta^{ij}-d\frac{x_\perp^i x_\perp^j}{|x_\perp|^2}}{|x_\perp|^d}\,, \tag{5.3}$$

where a, b index directions parallel to the defect and i, j label directions normal to the defect. By starting from the defect geometry $\Sigma_4 = \mathbb{R}^4 \hookrightarrow \mathbb{R}^d$ and then finding the totally transverse log divergent parts of the effective action in the presence of a linear ambient metric perturbation [34, 300], it can be shown that the normalization of the stress-energy tensor one-point function is determined by

$$h_T = -\frac{\Gamma\left(\frac{d}{2}-1\right)}{\pi^{\frac{d}{2}}\,(d-1)}d_2\,. \tag{5.4}$$

In the case that we are particularly interested in for the following work, i.e. $d=6$,

$$h_T = -\frac{1}{5\pi^3}d_2\,. \tag{5.5}$$

There is a constraint on the sign of d_2 that follows from the assumption that the average null energy condition (ANEC) holds in the presence of a defect. That is, the statement of the ANEC is that for any state $|\Psi\rangle$ of a QFT, the expectation value of the stress-energy tensor projected along a null direction v^μ in that state satisfies

$$\int_{-\infty}^{\infty} d\lambda\ \langle\Psi|T_{\mu\nu}|\Psi\rangle\, v^\mu v^\nu \geq 0, \tag{5.6}$$

where λ parametrises the null geodesic. From (5.4), we see that by taking the ambient theory to be a CFT and $|\Psi\rangle$ to be the vacuum state of the theory deformed by a defect and orienting the null ray v^μ to be parallel to the defect and separated by a distance $x_\perp$ in the normal direction, $h \geq 0$, which implies $d_2 \leq 0$ [34, 51][7].

[7] In fact, it has recently been argued that the quantum null energy condition (QNEC), which is a stronger energy condition valid in any ambient QFT and reduces to ANEC in a certain limit (see e.g.

The other physical quantity controlled by defect Weyl anomalies that we will study below is the contribution to the EE of a spherical region of size R centred on $\Sigma_4 = \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}^{1,d-1}$. Following the same logic that formed the basis of the proof for 2d defects [51, 302], it was shown in [34] that for a 4d defect of co-dimension $d-4$, the coefficient of the universal, i.e. the log divergent, part of the defect EE is

$$S_{\mathrm{EE}}[\Sigma]\Big|_{\log} = -4\left[a_\Sigma + \frac{1}{4}\frac{(d-4)(d-5)}{d-1}\,d_2\right]\log\left(\frac{R}{\epsilon}\right), \tag{5.7}$$

where $\epsilon \ll R$ is a UV cutoff scale and $\Big|_{\log}$ denotes dropping the leading non-universal divergences as well as the trailing scheme dependent terms.

For a conformal defect on Σ_4, we will use a background subtraction scheme to isolate the defect contribution to the EE. That is, our computations below will use

$$4a_\Sigma + \frac{2}{5}d_2 = -R\partial_R\left(S_{\mathrm{EE}}[\Sigma] - S_{\mathrm{EE}}[\emptyset]\right)|_{R\to 0}, \tag{5.8}$$

where $S_{\mathrm{EE}}[\emptyset]$ is the EE computed without the defect, i.e. the EE of the vacuum of the 6d ambient theory. Thus, combining the computation of d_2 from $\Delta\langle T_{ij}\rangle$ with the result of (5.8), we can compute the defect A-type anomaly unambiguously.

Unlike d_2, however, there is no constraint on the sign of a_Σ. Indeed, in the simple case of a free scalar on a 5d manifold with a boundary, $a_\Sigma > 0$ for Neumann (Robin) boundary conditions, while $a_\Sigma < 0$ for Dirichlet [334][8].

5.2.2 11d SUGRA solutions

Two-charge solutions

We now briefly review the domain wall solutions in 7d $\mathcal{N}=4$ gauged SUGRA found in [3] and uplifted to 11d in [4]. The bosonic 7d gauged SUGRA action built from the metric g, two scalars $\Phi_{1,2}$ and two $U(1)$ gauge fields $A_{1,2}$ takes the following form:

$$S = -\frac{1}{16\pi G_N^{(7)}}\int d^7x\sqrt{|g|}\left(\mathcal{R} - \frac{1}{2}|\partial_\mu\Phi_I|^2 - \hat{g}^2 V(\Phi) - \frac{1}{4}\sum_{I=1}^{2} e^{\vec{a}_I\vec{\Phi}}F_I^2\right). \tag{5.9}$$

Using $\vec{a}_1 = (\sqrt{2}, \sqrt{2/5})$, $\vec{a}_2 = (-\sqrt{2}, \sqrt{2/5})$, the potential is given by

$$V = -4e^{-\frac{1}{2}(\vec{a}_1+\vec{a}_2)\vec{\Phi}} - 2\left(e^{\frac{1}{2}(\vec{a}_1+2\vec{a}_2)\vec{\Phi}} + e^{\frac{1}{2}(2\vec{a}_1+\vec{a}_2)\vec{\Phi}}\right) + \frac{1}{2}e^{2(\vec{a}_1+\vec{a}_2)\vec{\Phi}}. \tag{5.10}$$

[353]), holds in the presence of a defect [354]; putting $d_2 \le 0$ and any other sign constraint derived from such energy conditions on even firmer ground.

[8]Note we are using the conventions for the definition of the 4d defect A-type anomaly a_Σ as in [34], which differs from the defect A-type anomaly, a, in [334] by $a_\Sigma \leftrightarrow -a/5760$.

The domain wall solution to (5.9) describing the double analytic continuation of a charged black hole is given by

$$ds_7^2 = (yP(y))^{\frac{1}{5}} ds^2_{\text{AdS}_5} + \frac{y(yP(y))^{\frac{1}{5}}}{4Q(y)} dy^2 + \frac{yQ(y)}{(yP(y))^{\frac{4}{5}}} dz^2, \tag{5.11}$$

where the polynomials P, Q are given by

$$P(y) = H_1(y)H_2(y), \tag{5.12a}$$

$$Q(y) = -y^3 + \mu y^2 + \frac{\hat{g}^2}{4} P(y), \tag{5.12b}$$

where $H_I(y) = y^2 + q_I$, $I \in \{1, 2\}$. The gauge fields in this solution[9] are given by

$$A_I = \left(\sqrt{1 - \frac{\mu}{q_I}} \frac{q_I}{H_I(y)} + a_I \right) dz \ . \tag{5.13}$$

In order to find BPS solutions, SUSY forces $\mu = 0$. For both $q_I \neq 0$, the solutions are 1/4-BPS, while setting one charge, say q_2, to zero allows for 1/2-BPS solutions. In the following, we will refer to the former 1/4-BPS cases as 'two-charge solutions' and the latter 1/2-BPS cases as 'one-charge solutions'. The coordinate y ranges from y_+, the largest root of $Q(y)$, to infinity. To have a smooth geometry one can choose the gauge so that $A_I(y_+) = 0$ by appropriate choice of the a_I. Setting $\hat{g} = 2$, the $\text{AdS}_5 \times S^1$ geometry does not have a conical deficit provided $z \in [0, 2\pi)$ (this will be assumed in the uplift to 11d). At $y = y_+$, the geometry either has a smooth cap or a conical deficit $2\pi \frac{\hat{n}-1}{\hat{n}}$ with $\hat{n}$ related to y_+ by the constraint $\hat{n}\, Q'(y_+) = y_+^2$.

The conditions $Q(y_+) = 0$ and $\hat{n}\, Q'(y_+) = y_+^2$ can be solved to determine q_1 and q_2 in terms of $\hat{n}$ and y_+ as follows

$$q_I = y_+ \left(\frac{3\hat{n} + 1}{\hat{n}\hat{g}^2} - y_+ \pm \frac{2}{\hat{g}} \sqrt{\frac{(1 + 3\hat{n})^2}{4\hat{g}^2\hat{n}^2} - y_+} \right), \tag{5.14}$$

where q_1 and q_2 are chosen with opposite signs for the square root. This has real solutions provided $0 \leq y_+ \leq y_{+,\text{max}}$ with $y_{+,\text{max}} = (1 + 3\hat{n})^2/4\hat{g}^2\hat{n}^2$. It will be useful later to notice that the sum $q_1 + q_2$ is always non-negative as is evident from

$$\frac{q_1 + q_2}{2y_+} = \left(\frac{3\hat{n} + 1}{\hat{n}\hat{g}^2} - y_+ \right) \geq \left(\frac{3\hat{n} + 1}{\hat{n}\hat{g}^2} - y_{+,\text{max}} \right) = \frac{(\hat{n} - 1)(3\hat{n} + 1)}{4\hat{g}^2\hat{n}^2} \geq 0. \tag{5.15}$$

[9] Note that, in general, the action in (5.9) does not qualify as a consistent truncation of 11d supergravity. The 7d solutions considered here, however, are characterised by $F_1 \wedge F_2 = 0$; this guarantees that their uplift produces consistent solutions of the 11d theory [355].

Uplifting to 11d, the metric for the two-charge 1/4-BPS solutions can be written schematically as

$$ds_{11}^2 = \hat{f}_{\rm AdS}^2 ds_{\rm AdS_5}^2 + \hat{f}_y^2 dy^2 + \hat{f}_z^2 dz^2 + \hat{f}_{\phi_i}^2 d\phi_i^2 + \hat{f}_{z\phi_i}^2 dz d\phi_i + \hat{f}_\psi^2 d\psi^2 + \hat{f}_\zeta^2 d\zeta^2 + \hat{f}_{\psi\zeta} d\psi d\zeta, \tag{5.16}$$

where each of the $\hat{f}$'s displayed in (B.1) is a function of the y, ψ, and ζ coordinates and also depends on the q_I's and a_I's. Note that in (B.1), we have introduced the slightly abusive shorthand

$$\sin x \equiv s_x, \qquad \cos x \equiv c_x, \tag{5.17}$$

in order to compactly express some of the more cumbersome expressions, and we will adopt this notation throughout the following sections. Continuing on, the uplifted four-form field strength can be inferred from

$$\begin{aligned}\frac{\star_{11} F_4}{\kappa^2} =& -2(\hat{H}(X_0 + 2(X_1+X_2)) - 2X_0^2 + 2(X_0^2 - X_1^2)s_\zeta^2 + 2(X_0^2 - X_2^2)c_\psi^2 c_\zeta^2)\Upsilon_7 \\ &+ \frac{c_\zeta^2 c_\psi s_\psi}{2X_0X_2}(X_2 \star_7 dX_0 - X_0 \star_7 dX_2)\wedge d\psi + \frac{c_\zeta s_\zeta}{2X_1}\star_7 dX_1 \wedge d\zeta \\ &- \frac{c_\zeta s_\zeta}{2X_0X_2}(X_2 s_\psi^2 \star_7 dX_0 + X_0 c_\psi^2 \star_7 dX_2)\wedge d\zeta + \frac{c_\zeta s_\zeta}{4X_1^2} d\zeta \wedge (d\phi_1 + 2A_1)\wedge \star_7 dA_1 \\ &- \frac{c_\zeta c_\psi}{4X_2^2}(c_\zeta s_\psi d\psi + s_\zeta c_\psi d\zeta)\wedge(d\phi_2 + 2A_2)\wedge \star_7 dA_2\end{aligned} \tag{5.18}$$

where we have set $\hat{g} = 2$, and where Υ_7 is the 7d volume form. We also defined

$$X_1 = \frac{(yH_2(y))^{\frac{2}{5}}}{H_1(y)^{\frac{3}{5}}}, \quad X_2 = \frac{(yH_1(y))^{\frac{2}{5}}}{H_2(y)^{\frac{3}{5}}}, \quad X_0 = (X_1X_2)^{-2} \tag{5.19}$$

as well as

$$\hat{H} = \frac{X_2(H_2 - q_2 c_\psi^2)c_\zeta^2}{y^2} + X_1 s_\zeta^2 \ . \tag{5.20}$$

Electrostatic solutions

In this subsection, we review the construction of an infinite class of 'bubbling' solutions to 11d SUGRA with $\mathrm{AdS}_5 \times S^1$ boundary geometries that holographically describe 1/2-BPS co-dimension 2 defects in 6d SCFTs [4]. There is a long history of AdS_5 compactifications in 11d SUGRA and M-theory holographically dual to 4d $\mathcal{N} = 2$ SCFTs, e.g. [336, 337, 356, 357]. The class into which the solutions of [3, 4] are embedded are a particular type of Lin-Lunin-Maldacena (LLM) 'bubbling' geometries [336, 337].

Recall that the general LLM solution consists of an 11d geometry with a warped product $\mathrm{AdS}_5 \times S^2$ over $\mathcal{M}_4$ realised as a $U(1)_\chi$-fibration over a 3d base space $\mathcal{B}_3$ supported by four-form flux. The data that specifies the solution is encoded in a function that satisfies a non-linear Toda equation on $\mathcal{B}_3$, which is generically difficult to solve. However, by imposing that $\mathcal{B}_3$ has an additional $U(1)_\beta$ isometry, the Toda equation can be cast in an axi-symmetric form that can be solved more easily. Further facilitating finding general solutions to the axi-symmetric Toda equation, one can perform a Bäcklund transformation to map to a Laplace-type equation on $\mathbb{R}^3$, and so the problem is turned into an 'electrostatic' one [4, 331, 339, 358–360]. Hence, the class of bubbling geometries reviewed below will be referred to as 'electrostatic solutions' in the following sections.

In the formulation as a Laplace-type equation, finding a solution to the SUGRA equations of motion amounts to specifying a linear charge density ϖ which determines the electrostatic potential V. Exploiting the axial symmetry of the problem on $\mathcal{B}_3$, we take $\varpi = \varpi(\eta)$ to be aligned along the η-axis, i.e. the fixed point of the $U(1)_\beta$ rotations. The bosonic sector of these solutions takes the form

$$
\begin{aligned}
ds^2_{11} &= \kappa_{11}^{\frac{2}{3}} \left(\frac{\dot{V}\sigma}{2V''}\right)^{\frac{1}{3}} \left(4ds^2_{\mathrm{AdS}_5} + \frac{2V''\dot{V}}{\sigma} d\Omega_2^2 + \frac{2(2\dot{V}-\ddot{V})}{\dot{V}\sigma}\left(d\beta + \frac{2\dot{V}\dot{V}'}{2\dot{V}-\ddot{V}} d\chi\right)^2\right. \qquad (5.21a)\\
&\qquad \left. + \frac{2V''}{\dot{V}}\left(dr^2 + \frac{2\dot{V}}{2\dot{V}-\ddot{V}} r^2 d\chi^2 + d\eta^2\right)\right)\\
&\equiv f^2_{\mathrm{AdS}} ds^2_{\mathrm{AdS}_5} + f_{S^2} d\Omega_2^2 + f^2_\beta d\beta^2 + f^2_\chi d\chi^2 + f^2_{\beta\chi} d\beta d\chi + f_3^2 (dr^2 + d\eta^2)\ ,\\
C_3 &= \frac{2\kappa_{11}}{\sigma}\left(\left(\dot{V}\dot{V}' - \sigma\eta\right) d\beta - 2\dot{V}^2 V'' d\chi\right) \wedge \Upsilon_{S^2}\ , \qquad (5.21b)
\end{aligned}
$$

where we have adopted the notation where $\Upsilon_{\mathcal{M}} := \sqrt{|g_{\mathcal{M}}|} dx^1 \wedge \ldots \wedge dx^d$ is the volume form on a d-dimensional manifold $\mathcal{M}$. In this notation, the coordinates $\{r, \eta, \beta\}$ span $\mathcal{B}_3$, $\kappa_{11} = \pi \ell_P^3/2$, and

$$
V' \equiv \partial_\eta V, \qquad \dot{V} \equiv r\partial_r(V), \qquad \sigma \equiv V''(2\dot{V}-\ddot{V}) + (\dot{V}')^2. \qquad (5.22)
$$

In this background, away from sources, the electrostatic potential $V(r,\eta)$ satisfies

$$
\ddot{V}(r,\eta) + r^2 V''(r,\eta) = 0, \qquad (5.23)
$$

subject to the boundary condition $\partial_r V|_{\eta=0} = 0$. Exploiting the $U(1)_\beta$ isometry imposed on $\mathcal{B}_3$, the line charge distribution $\varpi(\eta)$ specifying the solution is related to the Laplace potential V by

$$
\varpi(\eta) = \lim_{r\to 0^+} \dot{V}(r,\eta). \qquad (5.24)
$$

Given an appropriate $\varpi(\eta)$, the solution to (5.23) can be expressed in terms of a Green's function, $G(r,\eta,\eta')$, as

$$
V(r,\eta) = -\frac{1}{2}\int d\eta' G(r,\eta,\eta')\varpi(\eta'). \qquad (5.25)
$$

By the symmetry of the problem, the Green's function can be written simply using the method of images as [4, 339]

$$G(r,\eta,\eta') = \frac{1}{\sqrt{r^2+(\eta-\eta')^2}} - \frac{1}{\sqrt{r^2+(\eta+\eta')^2}}. \tag{5.26}$$

The complete description of the solution to the 11d SUGRA field equations is thus given by finding a $\varpi(\eta)$ that obeys a set of necessary conditions.

For a generic $\varpi(\eta)$, the constraints that follow from charge conservation and regularity (modulo A_k singularities on $\mathcal{M}_4$) of the full 11d geometry were given in [357]. Satisfying these constraints determines the profile of $\varpi(\eta)$ to be a continuous, convex piecewise linear function of η with integer slope, whose slope decreases by integer values at discrete η_a. In general, the boundary conditions and symmetry imposed on V in solving (5.23) require $\varpi(0) = 0$. However, there are generally two cases for the behaviour of ϖ as η increases.

In the first case, apart from the zero at the origin, ϖ has a zero at some value $\eta = \eta_c > 0$ where the internal space closes off. The geometry of the 11d SUGRA solution is then a warped product of AdS$_5$ over the compact internal space $\mathcal{M}_6 = \mathcal{C}_{\mathbf{g}} \times \mathcal{M}_4$, and holographically describes a 4d theory that descends from the compactification of a 6d SCFT on a Riemann surface $\mathcal{C}_{\mathbf{g}}$. The generic charge distribution is decomposed into $n+1$ 'regular' intervals with positive slope and an 'irregular' interval $[\eta_n, \eta_c]$ with negative slope fixed by ratios of four-form flux. The data associated with the kinks between the regular parts of the charge distribution, namely a partition of N, label a regular puncture on $\mathcal{C}_{\mathbf{g}}$, while the data specifying the slope of the irregular interval is mapped to an irregular puncture [339]. This construction – reminiscent of other spindle compactifications engineering 4d SCFTs [338, 361–368] – was argued in [339] to be the SUGRA dual to class-$\mathcal{S}$ constructions [279] of certain classes of Argyres-Douglas theories [369] by analysing anomalies and counting of Coulomb and Higgs branch operators in the field theory. While we will not study these types of solutions further here, we will mention some of their properties as they pertain to the results of holographic calculations of defect anomalies.

The second case, relevant for our study, is where $\varpi(\eta)$ has non-trivial support over the whole range $\eta \in [0,\infty)$ [4]. Since $\varpi(\eta)$ never turns around to hit the η-axis, the geometry $\mathcal{M}_6$ in the 11d SUGRA solution is non-compact, and the 11d geometry can be engineered to be asymptotically locally AdS$_7 \times S^4$ where the geometry of the conformal boundary of the AdS$_7$ factor is AdS$_5 \times S^1$. These solutions are, thus, interpreted as holographically describing co-dimension 2 defect operators in 6d SCFTs, where the defect operator 'lives' at the conformal boundary of AdS$_5$.

As a simple example of a line charge density that gives rise to a non-compact geometry, it was shown in [4] that the one-charge solution reviewed in the previous subsection can be recast in the language of the electrostatic solutions as a $\varpi(\eta)$ with two segments:

$$\varpi(\eta) = \begin{cases} \left(1 + \frac{1}{\sqrt{1-4q_1}}\right)\eta, & \eta \in \left[0, \frac{N}{2}\sqrt{1-4q_1}\right] \\ \eta + N/2, & \eta \in \left[\frac{N}{2}\sqrt{1-4q_1}, \infty\right). \end{cases} \tag{5.27}$$

Due to ϖ being continuous and piecewise linear, we will refer to the solution engineered by (5.27) as a 'single kink solution'. This relation between the $q_2 \to 0$ limit of the two-charge solutions and the simple single kink line charge distribution for the electrostatic solutions will be useful in later sections as a consistency check for our computations. We should also note that the constraint that the change in slope of $\varpi(\eta)$ is integral forces $q_1 = \frac{j^2-1}{4j^2}$ for $j \in \mathbb{N}$.

Generalizing beyond the single kink solutions, the constraints on $\varpi(\eta)$ realizing a defect solution allow for a generic n-kink charge profile. Since $\varpi(\eta)$ is piecewise linear, its behavior on the a^{th} interval, where $\eta \in [\eta_a, \eta_{a+1}]$ and $a \in \{0, 1, \ldots, n\}$, can be written as [4, 331]

$$\begin{aligned} \varpi_a(\eta) &= \left(1 + \sum_{b=a+1}^{n} k_b\right)\eta + \sum_{b=1}^{a} \eta_b k_b \\ &\equiv p_{a+1}\eta + \delta_{a+1}, \end{aligned} \tag{5.28}$$

where in the second line we have introduced a convenient short hand for the slope p_{a+1} and intercept δ_{a+1} of the line continued from the a^{th} segment. From the boundary condition $\varpi(0) = 0$ it is understood that $\eta_0 = 0$, and due to the semi-infinite domain of support we take $\eta_{n+1} \to \infty$.

As a simple visualization of an arbitrary distribution, see the left side of figure 5.1. Note that from the constraint following from the quantization of four-form flux $N = 2\sum_{a=1}^{n} \eta_a k_a$ along with the quantization of the η_a and their ordering along the η-axis $(0 < \ldots < \eta_a < \eta_{a+1} < \ldots < \eta_n)$, there is a natural interpretation of the data (η_a, k_a) specifying the charge distribution as a Young diagram, which is displayed on the right side of figure 5.1.

In the language of the field theory description, the Young diagram corresponding to the specific $\varpi(\eta)$ is in correspondence to both the Lie algebra homomorphism $\vartheta : \mathfrak{sl}(2) \to \mathfrak{g}$ and to the choice of Levi subalgebra $\mathfrak{l}$ of the A_{N-1} gauge algebra. Furthermore, the slope change $k_a \in \mathbb{Z}$ between the $(a-1)^{\text{th}}$ and a^{th} intervals corresponds to the monopole charge at the $\mathbb{R}^4/\mathbb{Z}_{k_a}$ orbifold point located at $(r, \eta) = (0, \eta_a)$ in the internal manifold. These points are the holographic realization of the non-Abelian summands $\mathfrak{su}(k_a)$ of the global symmetry algebra [357].

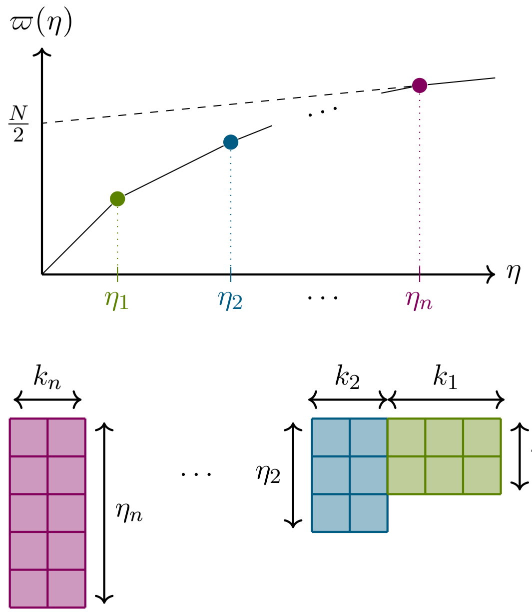


FIGURE 5.1: **(Top)** A generic line charge distribution $\varpi(\eta)$, with n kinks at positions η_a along the axis of cylindrical symmetry, specifying a solution to the axially symmetric Laplace equation in $\mathbb{R}^3$. The $\mathrm{AdS}_7 \times S^4$ vacuum corresponds to the single-kink ($n = 1$) charge distribution with $k_1 = 1$; the location of the kink is then given by $\eta_1 = N/2$. **(Bottom)** The Young Tableau corresponding to the partition $N = 2\sum_{a=1}^{n} k_a \eta_a = \sum_{a=1}^{n} N_a$. The height and width of the a-th block are given by the location $\eta_a \in \mathbb{Z}$ and slope change $k_a \in \mathbb{Z}$ of the a-th kink in $\varpi(\eta)$, respectively. The $\mathrm{AdS}_7 \times S^4$ vacuum is associated to the $\mathbf{1}$ of $\mathfrak{su}(N)$ determined by $n = k_1 = 1$ and $\eta_1 = N/2$.

Lastly, for use in future computations, it will be convenient to define the 'moments' of the potential as in [4]

$$m_j = \sum_{a=1}^{n} (p_a - p_{a+1}) \eta_a^j = \sum_{a=1}^{n} k_a \eta_a^j . \tag{5.29}$$

For most of the following, we will only need the first and third moments

$$m_1 = \frac{N}{2} \qquad \text{and} \qquad m_3 = \sum_a \frac{N_a^3}{8k_a^2} \tag{5.30}$$

respectively.

5.3 Holographic stress-energy tensor one-point function

In this section, we will compute the contribution of a co-dimension 2 defect to the one-point function of the stress-energy tensor of the ambient 6d SCFT. To do so, we will reduce the 11d SUGRA backgrounds described in the previous section on the internal S^4 and employ the holographic renormalization methods of [325]. In their original formulation, these methods are meant to apply to asymptotically AdS solutions of pure Einstein-Hilbert gravity; therefore, we must ensure that the presence of the four-form flux in the dimensionally reduced M-theory solutions does not necessitate a modification of those methods. In both two-charge and electrostatic solutions, we will show that the field strength decays sufficiently fast as the conformal boundary of AdS_7 is approached so that it produces a vanishing contribution to the field equations on the boundary.

Following the general procedure in [325], we begin by recasting the 11d metric as a perturbation h_{11} about the $\mathrm{AdS}_7 \times S^4$ vacuum:

$$ds_{11}^2 = g_{\mathrm{AdS}_7 \times S^4} + h_{11}. \tag{5.31}$$

Dimensionally reducing on the internal S^4 then leads to the 7d line element

$$ds_7^2 = \left(1 + \frac{\bar{\varsigma}}{5}\right) g_{\mathrm{AdS}_7} + \bar{h}_7, \tag{5.32}$$

where g_{AdS_7} is the metric on AdS_7, the 7d field h_7 captures the fluctuations about the AdS_7 geometry, and ς is the trace of the fluctuations in the internal manifold. Bars indicate zero modes on the internal space; for instance[10],

$$\bar{\varsigma} = \frac{3}{4} \int_{S^4} \sqrt{g_{S^4}}\; h^{ab} g^{(0)}_{ab}. \tag{5.33}$$

Mapping the 7d line element into Fefferman-Graham (FG) gauge,

$$ds_7^2 = \frac{L^2}{u^2}\left(du^2 + g\right), \tag{5.34}$$

where the 6d metric g admits the power series expansion

$$g = g_{(0)} + g_{(2)} u^2 + g_{(4)} u^4 + g_{(6)} u^6 + h_{(6)} u^6 \log u^2 + \dots, \tag{5.35}$$

the 6d stress-energy tensor one-point function can be computed

$$\langle T_{ij}\rangle\, dx^i dx^j = \frac{3L^5}{8\pi G_N^{(7)}}\left(g_{(6)} - A_{(6)} + \frac{S}{24}\right) \tag{5.36a}$$

$$= \frac{N^3}{4\pi^3}\left(g_{(6)} - A_{(6)} + \frac{S}{24}\right), \tag{5.36b}$$

where $A_{(6)}$ and S are rank-2 tensors built out of $g_{(0)}$, and in the second line we have used the holographic map to field theory quantities

$$\frac{1}{G_N^{(7)}} = \frac{\mathrm{vol}(S^4)}{G_N^{(11)}}, \quad G_N^{(11)} = 2^4 \pi^7 \ell_P^9, \quad L^3 = \pi N \ell_P^3, \quad \mathrm{vol}(S^4) = \frac{L^4 \pi^2}{6}. \tag{5.37}$$

[10] Our index conventions in this section are that $\mu, \nu, \dots$ are AdS_7 indices, $a, b, \dots$ are S^4 indices, and $i, j, \dots$ are 6d indices on the conformal boundary of AdS_7.

Note that in our conventions the internal S^4 has curvature scale $L^2/4$. Explicit expressions for $A_{(6)}$ and S are provided in [325][11]. Once the appropriate vacuum subtraction is performed, the defect contribution to h_T, and therefore to d_2, can be extracted via (5.3) and (5.5).

5.3.1 Two-charge solutions

In this subsection, we will focus on the 11d uplift of the two-charge solutions described in section 5.2.2 and compute $\langle T_{ij}\rangle$ with the methods described above. In order to isolate the contributions from the holographic dual to the defect, we will employ a background subtraction scheme where we remove the contributions from vacuum $\mathrm{AdS}_7 \times S^4$.

Before jumping in to the computation of $\langle T_{ij}\rangle$, we need to carefully check that we can properly utilise our chosen holographic renormalization scheme. One of the crucial assumptions in the construction of (5.36a) is that Einstein's equations near the boundary of the dimensionally reduced AdS_7 geometry are not modified by contributions coming from non-trivial fluxes, such as the four-form curvature F_4. So, we must be careful to make sure that in the asymptotic small u region, the components of the variation of the $F_{MNPQ}F^{MNPQ}$ part of the 11d SUGRA action involving AdS_7 directions fall off sufficiently fast so as to not modify the boundary equations of motion.

For the solutions in eqs. (5.16) and (5.19), it suffices to show the fall-off conditions for the single charge case. Setting $q_2 \to 0$ and $a_2 \to 0$, transforming $\phi_I \to \varphi_I - 2a_I z$, and mapping to FG gauge as in appendix B.1.1, a quick computation shows the small u behaviour to be (up to overall numerical prefactors)

$$\begin{aligned}
F_a^{MNP}F_{bMNP} &\sim c_\theta^2 g_{ab} + \dots\ , \\
F_{\varphi_1}^{MNP}F_{\varphi_1 MNP} &\sim s_\theta^2 + \dots\ , \\
F_\theta^{MNP}F_{\theta MNP} &\sim 1 + \dots\ , \\
F_z^{MNP}F_{zMNP} &\sim q_1^2(13 - 5c_{2\theta})u^8 + \dots, \\
F_z^{MNP}F_{\varphi_1 MNP} &\sim q_1 s_\theta^2 u^4 + \dots\ , \\
F_y^{MNP}F_{yMNP} &\sim q_1^2 s_{2\theta}^2 u^{12} + \dots\ ,
\end{aligned} \tag{5.38}$$

where g_{ab} are components along the $S^2 \subset S^4$ and the AdS_5 components of the variation vanish. From the zz- and $z\varphi_1$-components of the variation of F_4^2, we can see that the contributions to the boundary equations of motion dies at worst as u^4 as $u \to 0$. The analysis of the two-charge solution follows similarly, and so we can proceed using (5.36a) without modification. Allowing for $q_2 \neq 0$ modifies the variation of $F_{MNPQ}F^{MNPQ}$ but crucially does not introduce any leading terms in the small u expansion.

[11] Note that the differences in sign are due the fact that we are using the convention that, in units of L^{2d}, the scalar curvature $\mathcal{R} < 0$ for a space of constant "negative curvature"; whereas the authors of [325] use the opposite convention, $\mathcal{R} > 0$.

Now that we have established that the variation of F_4^2 decays sufficiently fast near the AdS_7 boundary, we can proceed using the logic of [325] recapped above to compute $\langle T_{ij}\rangle$. To do so, we first map (5.16) to FG gauge as in (B.4), which we reproduce here for clarity

$$\begin{aligned} ds^2_{\mathrm{FG}} =& \frac{L^2}{u^2}(du^2 + \hat{\alpha}_{\mathrm{AdS}} ds^2_{\mathrm{AdS}_5} + \hat{\alpha}_z dz^2) + L^2 s_\theta^2 \hat{\alpha}_{z\varphi_1} dz d\varphi_1 + L^2 c_\aleph^2 c_\theta^2 \hat{\alpha}_{z\varphi_2} dz d\varphi_2 \\ &+ \frac{L^2}{4}(\hat{\alpha}_\theta d\theta^2 + s_\theta^2 \hat{\alpha}_{\varphi_1} d\varphi_1^2 + c_\theta^2(\hat{\alpha}_\aleph d\aleph^2 + c_\aleph^2 \hat{\alpha}_{\varphi_2} d\varphi_2^2) + \hat{\alpha}_{\theta\aleph} d\theta d\aleph). \end{aligned}$$

The $\hat{\alpha}$ metric functions are given in (B.6). In order to put the dimensionally reduced metric in the form of (5.32), we then write ds^2_{FG} as a fluctuation around $\mathrm{AdS}_7 \times S^4$

$$ds^2 = (g^{(0)}_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu \tag{5.39}$$

where

$$g^{(0)}_{\mu\nu} dx^\mu dx^\nu = \frac{L^2 du^2}{u^2} + \frac{L^2}{u^2}\left(\left(1 + \frac{u^2}{2} + \frac{u^4}{16}\right) ds^2_{\mathrm{AdS}_5} + \left(1 - \frac{u^2}{2} + \frac{u^4}{16}\right) dz^2\right) + \frac{L^2}{4} d\Omega_4^2. \tag{5.40}$$

Using the expressions in (B.6), we can compute the zero modes of the fluctuations around the AdS_7 directions

$$\bar{h}_7 = -\frac{2L^2(q_1+q_2)}{15} u^4 (ds^2_{\mathrm{AdS}_5} - 5dz^2). \tag{5.41}$$

Similarly, the trace fluctuations on the S^4 are found to be

$$\varsigma = \frac{10 q_2 c_{2\aleph} c_\theta^2 + 5(q_2 - 2q_1) c_{2\aleph} + 2q_1 - 3q_2}{8} u^4 + \ldots. \tag{5.42}$$

Integrating the internal space fluctuations over the S^4 gives $\bar{\varsigma} = 0$. The vanishing of the zero modes of the trace fluctuations means that the dimensionally reduced metric is already in FG form. The resulting stress-energy tensor one-point function is

$$\langle T_{ij}\rangle \; dx^i dx^j = \frac{N^3}{192\pi^3}\left[1 - \frac{32}{5}(q_1+q_2)\right]\left(ds^2_{\mathrm{AdS}_5} - 5dz^2\right). \tag{5.43}$$

In order to isolate the holographic quantities associated with the defect, we will subtract off the value of $\langle T^{(vac)}_{ij}\rangle$ computed using vacuum $\mathrm{AdS}_7 \times S^4$. Note that, taking $q_I \to 0$ in (5.41) kills the fluctuations and gives the exact AdS_7 metric upon dimensional reduction, as expected. So, taking $q_I \to 0$ in (5.43) yields the vacuum 1-pt function

$$\langle T^{(vac)}_{ij}\rangle \; dx^i dx^j = \frac{N^3}{192\pi^3}\left(ds^2_{\mathrm{AdS}_5} - 5dz^2\right). \tag{5.44}$$

Subtracting this vacuum contribution from (5.43) computes the change in the

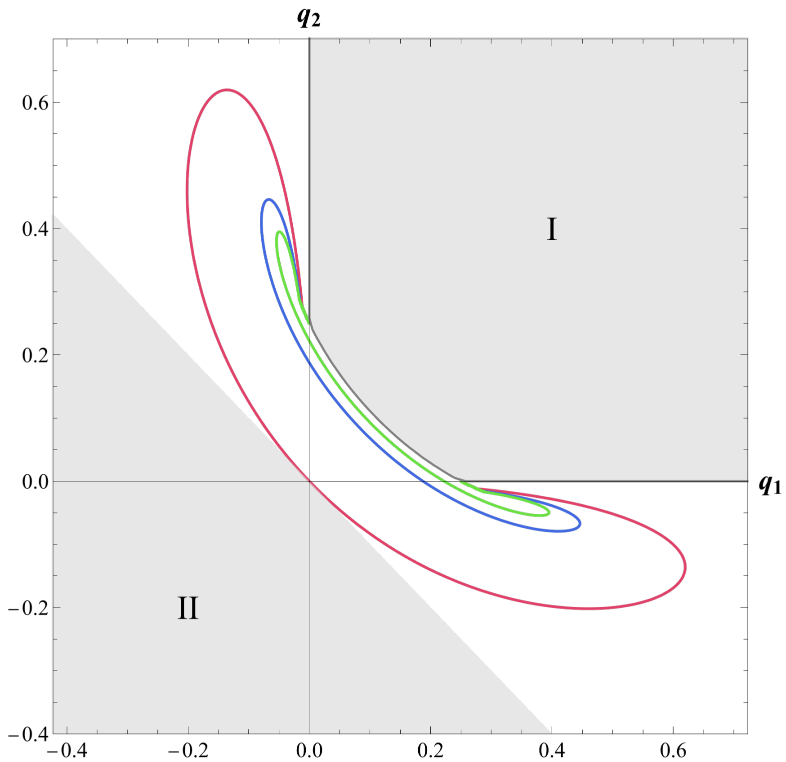


FIGURE 5.2: The solutions to the constraint in (5.15) for $\hat{n} = 1$ (red), $\hat{n} = 2$ (blue), and $\hat{n} = 3$ (green) on the (q_1, q_2) plane, reproduced from [3, 4]. The shaded regions correspond to the two-charge configurations for which $Q(y) = 0$ admits no real solutions (region I) or which violate the defect ANEC (region II).

stress-energy tensor one-point function due to the introduction of the holographic dual to the field theory defect:

$$\Delta \langle T_{ij} \rangle \, dx^i dx^j = -\frac{N^3(q_1 + q_2)}{30\pi^3}(ds^2_{\mathrm{AdS}_5} - 5dz^2), \tag{5.45}$$

which recovers the results in [3] up to subtraction of the contribution from the $\mathrm{AdS}_7 \times S^4$ vacuum. Using (5.3) we arrive at

$$h_T = \frac{N^3(q_1 + q_2)}{30\pi^3} \tag{5.46}$$

Thus, one of the B-type anomaly coefficients for 1/4-BPS co-dimension 2 operators in a 6d $\mathcal{N} = (2,0)$ A_{N-1} SCFT holographically described by the two-charge solutions is found to be

$$d_2 = -\frac{1}{6}N^3(q_1 + q_2). \tag{5.47}$$

Recall that in (5.15), we found that the linear combination $q_1 + q_2 \geq 0$ for all $\hat{n}$. Further, we know that (5.6) implies $d_2 \leq 0$, and so all of the two-charge solutions studied in [3, 4] are consistent with the defect ANEC. In figure 5.2, we reproduce the curves for solutions obeying (5.15) as appears in [3, 4] together with the region excluded by consistency with defect ANEC. We see that, indeed, all of the $\hat{n} = 1, 2, 3$ solutions lie above the line $q_1 + q_2 \geq 0$ with only $\hat{n} = 1$ saturating the bound at $q_1 = q_2 = 0$.

5.3.2 Electrostatic solutions

Prior to approaching the holographic computation of $\Delta \langle T_{ij} \rangle$ for the electrostatic solutions using the methods outlined above, we again must verify that the boundary equations of motion in the dimensionally reduced geometry are unmodified by the four-form flux. From (5.21b), we can compute F_4. For brevity, we will immediately define $r = \varrho c_\omega$ and $\eta = \varrho s_\omega$ to map (5.21b) into (ϱ, ω) coordinates on the internal space and adopt (z, φ) using (B.14) as our angular coordinates and compute the large ϱ expansion to leading order in each component

$$\begin{aligned}\frac{F_4}{2\kappa_{11}} = \Bigg[& c_\omega^2 s_\omega^3 \frac{5m_3 - 2m_1^3}{\varrho^3}\, d\varrho \wedge dz + 3c_\omega s_\omega^2 m_1 d\omega \wedge dz \\ &+ s_\omega^3 \frac{4(m_3 - m_1^3)}{\varrho^3} d\varrho \wedge d\varphi + c_\omega s_\omega^2 \frac{6(m_1^3 - m_3)}{\varrho^2} d\omega \wedge d\varphi \Bigg] \wedge \ \mathrm{vol}(S^2) + \ldots\end{aligned} \tag{5.48}$$

where we have fixed $\mathcal{C}_z = -2$ following the discussion in appendix B.1.2.

Now, we can check the fall off of the contribution of the variation of F_4^2 to the equations of motion. Keeping $\mathcal{C}_z = -2$ fixed and transforming into FG gauge, we find the leading behaviour in the small-u expansion (up to numerical factors)

$$\begin{aligned} F_{uMNP} F_u{}^{MNP} &\sim s_{2\theta}^2 (m_1^3 - m_3)^2 u^6 + \ldots \ , \\ F_{zMNP} F_z{}^{MNP} &\sim (13 - 5c_{2\theta})(m_1^3 - m_3)^2 u^8 + \ldots \ , \\ F_{\varphi MNP} F_z{}^{MNP} &\sim (m_1^3 - m_3) s_\theta^2 u^4 + \ldots \ , \\ F_{aMNP} F_b{}^{MNP} &\sim g_{S^4} + \ldots \ , \end{aligned} \tag{5.49}$$

where a, b are indices for S^4 coordinates $\{\theta, \varphi, S^2\}$, g_{S^4} is the metric on the unit S^4 in $S^1 \times S^2$ fibration. Note that the variations in the AdS_5 directions vanish identically. So, in the $u \to 0$ limit, there are no surviving contributions to the equations of motion in the dimensionally reduced geometry coming from the variation of the F_4^2 term.

We can now proceed with [325]. First, we rewrite the metric in (B.16) as fluctuations around $\mathrm{AdS}_7 \times S^4$. The perturbation away from $\mathrm{AdS}_7 \times S^4$ takes the form

$$\begin{aligned} h_{11} = \frac{L^2}{u^2}\left(\alpha_{\mathrm{AdS}} - 1 - \frac{u^2}{2} - \frac{u^4}{16}\right) ds^2_{\mathrm{AdS}_5} + \frac{L^2}{u^2}\left(\alpha_z - 1 + \frac{u^2}{2} - \frac{u^4}{16}\right) dz^2 \\ + \frac{L^2}{4}(\alpha_\theta - 1) d\theta^2 + \frac{L^2 s_\theta^2}{4}(\alpha_\varphi - 1) d\varphi^2 + \frac{L^2 c_\theta^2}{4}(\alpha_{S^2} - 1) d\Omega_2^2 + L^2 s_\theta^2 \alpha_{z\varphi} dz d\varphi. \end{aligned} \tag{5.50}$$

Fixing $\chi = -z - \varphi$ and $\beta = 2z + \varphi$, we can compute the zero modes for the AdS$_7$ part of the fluctuations,

$$\bar{h}_7 = L^2 \frac{m_3 - m_1^3}{30 m_1^3} u^4 ds^2_{\mathrm{AdS}_5} + L^2 \frac{m_1^3 - m_3}{6 m_1^3} u^4 dz^2 + \dots \,. \tag{5.51}$$

The trace S^4 fluctuations are found to be

$$\varsigma = (1 - 5c_{2\theta}) \frac{m_1^3 - m_3}{16 m_1^3} u^4 - 11(1 - 5c_{2\theta}) \frac{m_1^3 - m_3}{216 m_1^3} u^6 + \dots \,. \tag{5.52}$$

Integrating ς over the S^4, we find the zero modes $\bar{\varsigma} = 0$. The reduced geometry

$$g_7 = \left(1 + \frac{\bar{\varsigma}}{5}\right) g^{(0)} + \bar{h}_7 \tag{5.53}$$

is thus already in FG form. So, the dimensionally reduced metric is

$$\begin{aligned} ds_7^2 =& \frac{L^2}{u^2} \left[du^2 + \left(1 + \frac{u^2}{2} + \frac{u^4}{16} + \frac{(m_3 - m_1^3) u^6}{30 m_1^3}\right) ds^2_{\mathrm{AdS}_5} \right. \\ & \left. + \left(1 - \frac{u^2}{2} + \frac{u^4}{16} + \frac{(m_1^3 - m_3) u^6}{6 m_1^3}\right) dz^2 \right], \end{aligned} \tag{5.54}$$

where we have suppressed higher powers of u. From this expression for ds_7^2, we can easily read off $g_{(0)}$, $g_{(2)}$, $g_{(4)}$, and $g_{(6)}$. Note if we take $n = 1$ and $k_1 = 1$, then $m_3 = m_1^3 = N^3/8$, and so in this limit, (5.54) reduces to the exact AdS$_7$ metric, which is expected from eqs. (5.21a), (5.23), and (5.28).

Proceeding with the computation in the same way as the previous subsection, we find that the holographic stress-energy tensor one-point-function takes the form

$$\langle T_{ij} \rangle \, dx^i dx^j = -\frac{N^3 (3 m_1^3 - 8 m_3)}{960 \pi^3 m_1^3} \left(ds^2_{\mathrm{AdS}_5} - 5 dz^2 \right). \tag{5.55}$$

Regulating this result by subtracting the AdS$_7 \times S^4$ vacuum contribution $\langle T_{ij}^{(vac)} \rangle$ in (5.44) produces

$$\Delta \langle T_{ij} \rangle \, dx^i dx^j = -\frac{N^3 (m_1^3 - m_3)}{120 \pi^3 m_1^3} \left(ds^2_{\mathrm{AdS}_5} - 5 dz^2 \right). \tag{5.56}$$

As a quick check, computing the trace of (5.56) gives $\Delta \langle T^i{}_i \rangle = 0$ as expected due to defect conformal symmetry. Comparing (5.56) to (5.3), we find

$$h_T = \frac{m_1^3 - m_3}{15 \pi^3}. \tag{5.57}$$

We can thus read off the defect Weyl anomaly coefficient d_2 from (5.5):

$$d_2 = -\frac{m_1^3 - m_3}{3} \tag{5.58a}$$
$$= -\frac{1}{24}\left(N^3 - \sum_a \frac{N_a^3}{k_a^2}\right), \tag{5.58b}$$

where in the second line we have rewritten d_2 in terms of the parameters N_a and k_a which are more suitable for comparison to field theory. For any partition of $N = \sum_a N_a$, it is clear that $d_2 \leq 0$. The upper bound $d_2 = 0$ is only saturated in the vacuum case $n = 1$, $k_1 = 1$ where there is no defect.

There is a non-trivial consistency check on the value of d_2 in the $n = 1$ case. As mentioned above, the 11d uplift of the 1/2-BPS one-charge solutions is related to the single-kink electrostatic solutions by setting $n = 1$ and $k_1 = 1/\sqrt{1-4q_1}$. Plugging these values into in to (5.58b) results in $d_2 = -N^3 q_1/6$. Checking this against the one-charge solutions found by taking $q_2 \to 0$ in (5.47), we also find $d_2 = -N^3 q_1/6$. Thus, the values of d_2 computed in the two-charge and n-kink electrostatic solutions are consistent in this limit.

5.4 Defect sphere EE and the defect A-type anomaly

In the following subsections we will use the techniques developed in [323, 324] to holographically compute the defect contribution to the EE of a spherical region in the dual 6d A_{N-1} $\mathcal{N} = (2,0)$ SCFT at large N for both the 1/4-BPS two-charge and 1/2-BPS electrostatic co-dimension 2 defects. Leveraging the results of the previous section and eqs. (5.7) and (5.8), we will be able to compute the defect A-type anomaly a_Σ.

To facilitate the discussion below, let us briefly review some of the relevant background concepts for defect EE. We will restrict our discussion here to the holographic duals to 6d (D)SCFTs.

To start, we will need the Ryu-Takayanagi (RT) formula for holographic EE [48–50], which we write agnostic to the presence of a defect as

$$S_{\text{EE}} = \frac{\mathcal{A}_{\min}}{4G_N}. \tag{5.59}$$

The quantity $\mathcal{A}_{\min}$ is the area of the extremal surface that minimises the bulk area functional subject to the condition that the surface anchored at the conformal boundary of AdS_7 is homologous to the entangling region in the dual theory. For our computations below, we take the entangling region in the 6d SCFT at a fixed time slice to be a Euclidean 5-ball $\mathcal{B} = \mathbb{B}^5 \hookrightarrow \mathbb{R}^5$ of radius R. When we consider the theory deformed by a flat embedding of a Lorentzian defect on $\Sigma = \mathbb{R}^{1,3}$, we will take the defect to be co-original with the entangling surface such that $\partial\mathcal{B} \cap \Sigma = S^2$ sitting along the equator of $\partial\mathcal{B}$.

By including a defect in the field theory, there are subtleties that arise in directly applying (5.7). On the field theory side, S_{EE} will now have short-distance divergences near $\partial\mathcal{B}$ due to highly entangled UV modes in both ambient and defect localised theories. In the holographic description, one needs to adopt a suitable regularization scheme that isolates the defect contribution to S_{EE}; we will use a background subtraction scheme ($S_{\text{EE}}[\Sigma] - S_{\text{EE}}[\emptyset]$) akin to the one used in computing the holographic stress-energy tensor one-point function. One further complication in the holographic computation is the fact that the FG expansion is generally not globally defined, and so one must be careful to find the asymptotic form of the map to FG gauge in order to define the UV cutoff slice at fixed AdS$_7$ radius $\Lambda \gg L$. A general formula for finding the asymptotic form of the FG transformation and cutoff slice was found in [323], which we will use in the computations below.

Since we are considering a spherical entangling region, the solution for $\mathcal{A}_{\text{min}}$ takes a particularly simple form; even in the presence of a defect. It was shown in [324] that for a bulk geometry realizing the defect symmetry group $SO(2, d - \mathfrak{d}) \times SO(\mathfrak{d})$, the relative warp factors of the AdS$_{\mathfrak{d}+1}$ and $S^{\mathfrak{d}-1}$ spaces are largely immaterial, and the logic of [370] can be generalised to prove (5.7) for these backgrounds. In the process, the authors of [324] proved that for the holographic defect spherical EE the surface $\mathcal{A}_{\text{min}}$ is simply a hemispherical region extending into the bulk anchored at $\mathcal{B}$. For the 11d backgrounds corresponding to both the two-charge and electrostatic solutions that we consider, if we write the line element on the AdS$_5$ in the form

$$ds^2_{\text{AdS}_5} = \frac{1}{w^2}(dw^2 - dt^2 + dr_\parallel^2 + r_\parallel^2 d\Omega_2^2) \ , \tag{5.60}$$

then $\mathcal{A}_{\text{min}}$ is the surface $w^2 + r_\parallel^2 = R^2$. We will exploit the simplicity of the minimal surface to great effect in the subsequent computations.

5.4.1 Two-charge solutions

To begin computing the defect spherical EE for the two-charge solutions, we need to express the area functional $\mathcal{A}$ in terms of the metric functions, $\hat{f}$ in (5.16) with the AdS$_5$ factor written as in (5.60). Evaluating on the extremal surface $r_\parallel^2 + w^2 = R^2$, we regularise the w integration by introducing a UV cutoff $\epsilon_w \ll 1$ and performing the integral over the angular coordinates ϕ_1, ϕ_2 and z to obtain

$$\mathcal{A}_{\text{min}}[\Sigma] = 8\pi^4 L^9 R \int_{\epsilon_w}^{\infty} dw \frac{\sqrt{R^2 - w^2}}{w^3} \mathcal{I} = 4\pi^4 \left(\frac{R^2}{\epsilon_w^2} - \log \frac{2R}{\epsilon_w} + \ldots \right) \mathcal{I} \ , \tag{5.61}$$

where we have defined the remaining integral

$$\mathcal{I} \equiv \int d\psi\, d\zeta \int_{y_+}^{\Lambda_y(\epsilon_u,\psi,\zeta)} dy\, \hat{f}_{\rm AdS}^3 f_y \sqrt{(4\hat{f}_\psi^2\hat{f}_\zeta^2 - \hat{f}_{\psi\zeta}^4)(\hat{f}_{\phi_1}^2\hat{f}_{z\phi_2}^4 + \hat{f}_{\phi_2}^2\hat{f}_{z\phi_1}^4 - 4\hat{f}_{\phi_1}^2\hat{f}_{\phi_2}^2\hat{f}_z^2)}\,. \tag{5.62}$$

Despite the initially complicated appearance of the integrand upon substituting the form of the metric functions in (B.1), we find after a bit of algebra that the remaining integral drastically simplifies to

$$\mathcal{I} = \frac{1}{8}\int d\psi\ d\zeta\ c_\psi c_\zeta^2 s_\zeta \int_{y_+}^{\Lambda_y(\epsilon_u,\psi,\zeta)} dy\ y. \tag{5.63}$$

Using the double-cutoff prescription to compute $\mathcal{I}$ as in [308, 323], we first map the radial coordinate y to the FG coordinate u leaving the remaining angular coordinates ψ and ζ in their original frame. We then impose a cutoff $\epsilon_u \ll 1$, which induces a cutoff in large y, $\Lambda_y(\epsilon_u,\psi,\zeta)$. Recalling the asymptotic FG map in appendix B.1.1 used in the previous section and recasting the FG angular coordinates $\aleph,\theta$ in terms of ψ,ζ, we find that

$$\Lambda_y(\epsilon_u,\psi,\zeta) = \frac{1}{\epsilon_u^2} + \frac{1}{2} + \frac{3 - 10q_1 - 9q_2 - 2q_2c_{2\psi}c_\zeta^2 + (2q_1 - q_2)c_{2\zeta}}{48}\epsilon_u^2 + \dots\,. \tag{5.64}$$

Evaluating the integral $\mathcal{I}$ with this cutoff is straightforward, yielding

$$\mathcal{I} = \frac{1}{24\epsilon_u^4} + \frac{1}{24\epsilon_u^2} + \frac{1}{960}(15 - 16(q_1+q_2) - 40y_+^2) + \dots \tag{5.65}$$

In order to find the contributions coming from the defect, we must regulate the ϵ_u divergences present in $\mathcal{A}_{\rm min}$. In order to do so, we employ the same vacuum subtraction scheme as was used in computing $\Delta\langle T_{ij}\rangle$ above. For the two-charge solution, the vacuum is obtained by setting $q_1 = q_2 = 0$ and $a_1 = a_2 = 0$, which sets $y_+^{(\rm vac)} = 1$. Recomputing $\mathcal{A}_{\rm min}[\emptyset]$ for the vacuum solution and subtracting it from $\mathcal{A}_{\rm min}[\Sigma]$, the regulated area functional gives

$$\mathcal{A}_{\rm min}[\Sigma] - \mathcal{A}_{\rm min}[\emptyset] = -\frac{\pi^4 L^9}{30}(2q_1 + 2q_2 + 5(y_+^2 - 1))\left(\frac{R^2}{\epsilon_w^2} - \log\frac{2R}{\epsilon_w} + \dots\right)\,, \tag{5.66}$$

free from ϵ_u divergences.

In order to compute a_Σ for the defect theory, we insert (5.66) in (5.59). Mapping to field theory quantities by $L^3 = 4\pi N\ell_P^3$ and $G_N = 2^4\pi^7\ell_P^9$, we can read off the coefficient of the universal part of the defect sphere EE from (5.59)

$$-R\partial_R(S_{\rm EE}[\Sigma] - S_{\rm EE}[\emptyset])|_{R\to 0} = -\frac{N^3}{30}(2(q_1+q_2) + 5(y_+^2 - 1)). \tag{5.67}$$

Hence, using $d_2 = -\frac{N^3}{6}(q_1 + q_2)$ derived above in (5.8) we find

$$a_\Sigma = \frac{N^3}{24}(1 - y_+^2). \tag{5.68}$$

One interesting consequence of this computation is that one can show that A-type anomaly of the general two-charge solution must satisfy $a_\Sigma \geq 0$. To see this more clearly, recall from (5.15) that

$$y_+ \leq \frac{3\hat{n} + 1}{4\hat{n}} \leq 1. \tag{5.69}$$

The second inequality follows from $\hat{n} \in \mathbb{N}$, and so the upper bound is saturated only for $\hat{n} = 1$. Thus, for all consistent two-charge solutions, $a_\Sigma \geq 0$.

5.4.2 Electrostatic solutions

Continuing with the logic used in the previous subsection, we now turn our attention to the electrostatic solutions. Our starting point for the computation is in transforming the metric in (5.21a) using (B.14) and reading off the metric functions. Since only $\mathcal{C}_z = -2$ gives an asymptotic form for the metric suitable for mapping into FG gauge, we fix the transformation $\chi = -z - \varphi$ and $\beta = 2z + \varphi$ and arrive at

$$ds_{11}^2 = f_{\mathrm{AdS}}^2 ds_{\mathrm{AdS}_5}^2 + f_{S^2} d\Omega_2^2 + f_z^2 dz^2 + f_\varphi^2 d\varphi^2 + f_{z\varphi}^2 dz d\varphi + f_\varrho^2 d\varrho^2 + f_\omega^2 d\omega^2. \tag{5.70}$$

We will also write the AdS_5 line element as in (5.60).

Plugging in the expression for the minimal surface, $r_\parallel^2 + w^2 = R^2$, into the area functional, we first integrate over the two S^2 factors as well as the angular coordinates $z \in [0, 2\pi]$ and $\varphi \in [0, 2\pi]$, which yields

$$\mathcal{A}_{\min}[\Sigma] = 32\pi^4 R \int dw \frac{\sqrt{R^2 - w^2}}{w^3} \mathcal{I}[\Sigma] \ . \tag{5.71}$$

where

$$\mathcal{I}[\Sigma] \equiv \int_0^{\pi/2} d\omega \int_0^{\Lambda_\varrho(\epsilon_u, \omega)} f_{\mathrm{AdS}}^3 f_{S^2}^2 f_\omega f_\varrho \sqrt{4 f_z^2 f_\varphi^2 - f_{z\varphi}^4} \ . \tag{5.72}$$

Note that we have introduced the large ϱ cutoff, Λ_ϱ, that was induced by the small u cutoff in FG gauge ϵ_u:

$$\Lambda_\varrho(\epsilon_u, \omega) = \frac{2m_1}{\epsilon_u^2} + \frac{2m_1^3 s_\omega^2 - (1 + 5c_{2\omega})m_3}{48 m_1^2}\epsilon_u^2 + s_\omega^2 \frac{m_3 - m_1^3}{36 m_1^2}\epsilon_u^4 + \dots \ . \tag{5.73}$$

Since the metric functions f are independent of w, the w integral can be performed over $[\epsilon_w, \infty)$, where $\epsilon_w \ll 1$,

$$\mathcal{A}_{\text{min}}[\Sigma] = 16\pi^4 \left(\frac{R^2}{\epsilon_w^2} - \log \frac{2R}{\epsilon_w} + O(\epsilon_w^0) \right) \mathcal{I}[\Sigma] \ . \tag{5.74}$$

Using the expressions for the metric functions in (5.21a) in terms of the potential, we find that $\mathcal{I}$ can be expressed as a total derivative. To see this more clearly, we note that in (ϱ, ω) coordinates

$$\mathcal{I}[\Sigma] = 64\kappa_{11}^3 \int_0^{\pi/2} d\omega \int_0^{\Lambda_\varrho(\epsilon_u,\omega)} d\varrho\, \varrho^2 c_\omega \dot{V} V'' \ . \tag{5.75}$$

Switching to (r, η) coordinates and using the Laplace equation $\ddot{V} = -r^2 V''$, we arrive at

$$\mathcal{I}[\Sigma] = -32\kappa_{11}^3 \int_0^{\Lambda_\eta} d\eta \int_0^{\Lambda_r} dr\, \partial_r \dot{V}^2 \ , \tag{5.76}$$

where we have mapped the asymptotic cutoff in ϱ back to the (r, η) frame,

$$\Lambda_r = \Lambda_\varrho(\epsilon_u, \omega) c_\omega \ , \qquad \Lambda_\eta = \Lambda_\varrho(\epsilon_u, \omega) s_\omega \ . \tag{5.77}$$

The remaining integral in $\mathcal{I}$ is identical to the one found in computing the central charge for the compact electrostatic solutions in [339] and again in [4]. For clarity, let us analyze $\mathcal{I}$ in detail here. We can integrate the total derivative in (5.76) and find that the surviving contributions come from the boundary of the region in the $\varrho - \omega$ quarter-plane spanned by the η-axis at $\omega = \pi/2$ and the contour at fixed $\varrho = \Lambda_\varrho$ between $\omega = 0$ and $\omega = \pi/2$. The integral along the η-axis can be decomposed into the regions of $\eta \in [0, \eta_n]$ and $\eta \in [\eta_n, \Lambda_\varrho(\epsilon_u, \pi/2)]$; in the latter region, the line charge density takes the form $\lambda(\eta) = \eta + m_1$. In all,

$$\frac{\mathcal{I}[\Sigma]}{32\kappa_{11}^3} = \underbrace{\int_0^{\eta_n} d\eta \varpi(\eta)^2}_{I_1} + \underbrace{\int_{\eta_n}^{\Lambda_\varrho(\epsilon_u,\pi/2)} d\eta (\eta + m_1)^2}_{I_2} - \underbrace{\int_{\omega=0}^{\omega=\pi/2} \dot{V}^2\Big|_{\Lambda_r} d(\Lambda_\rho(\epsilon_u, \omega))}_{I_3}, \tag{5.78}$$

where $\dot{V}^2\Big|_{\Lambda_r}$ in I_3 is held at fixed $r = \Lambda_r$ in the integration over ω.

Let's take each of the I's individually, starting with I_2. Performing the integral is trivial and leads to the small ϵ_u expansion

$$I_2 = \frac{8m_1^2}{3\epsilon_u^6} + \frac{4m_1^3}{\epsilon_u^4} + \frac{13m_1^3 + 2m_3}{6\epsilon_u^2} + \frac{8m_3 + m_1^3 - 18m_1^2\eta_n - 18m_1\eta_n^2 - 6\eta_n^3}{18} + \dots \ . \tag{5.79}$$

The integral I_3 can also be easily taken. First, we expand the integrand using the large ϱ expansions of the potential in (B.10). Then after computing $d\Lambda_r(\epsilon_u, \omega)$, we expand in small ϵ_u and integrate term-by-term in $\omega \in [0, \pi/2]$, which gives

$$I_3 = \frac{8m_1^3}{3\epsilon_u^6} + \frac{8m_1^3}{3\epsilon_u^4} + \frac{5m_1^3 + 2m_3}{6\epsilon_u^2} + \frac{m_1^3 + 14m_3}{45} + \dots \,. \tag{5.80}$$

Combining I_2 and I_3, we see

$$I_2 - I_3 = \frac{4m_1^3}{3\epsilon_u^4} + \frac{4m_1^3}{3\epsilon_u^2} + \frac{4m_3 + m_1^3 - 10\eta_n(\eta_n^2 + 3\eta_n m_1 + 3m_1^2)}{30} + \dots \,. \tag{5.81}$$

Lastly, we need to take care of the integral I_1. To do so, we break up the the integral over $\eta \in [0, \eta_n]$ into a sum over the intervals $[\eta_a, \eta_{a+1}]$ for $a = 0, \dots, n-1$ with $\eta_0 = 0$. Then, using $\varpi_a = p_{a+1}\eta + \delta_{a+1}$ over each interval we find

$$I_1 = \frac{1}{3}\sum_{a=0}^{n-1}\Big(p_{a+1}^2(\eta_{a+1}^3 - \eta_a^3) + 3\delta_{a+1}p_{a+1}(\eta_{a+1}^2 - \eta_a^2) + 3\delta_{a+1}^2(\eta_{a+1} - \eta_a)\Big)\,. \tag{5.82}$$

Combining everything we get

$$\begin{aligned}\frac{\mathcal{I}[\Sigma]}{32\kappa_{11}^3} = \frac{4m_1^3}{3\epsilon_u^4} + \frac{4m_1^3}{3\epsilon_u^2} + \frac{4m_3 + m_1^3}{30} + \frac{1}{3}\sum_{a=0}^{n}(p_{a+1}^2\eta_{a+1}^3 - \eta_a^3) \\ + \sum_{a=0}^{n}\delta_{a+1}p_{a+1}(\eta_{a+1}^2 - \eta_a^2) + \sum_{a=0}^{n}\delta_{a+1}^2(\eta_{a+1} - \eta_a),\end{aligned} \tag{5.83}$$

where we slightly abuse the notation by setting $\eta_{n+1} = 0$ in this sum to make the expressions a bit more compact.

The ϵ_u divergences in $\mathcal{I}[\Sigma]$ need to be regulated. We again adopt the background subtraction scheme as before, where the background vacuum $\mathrm{AdS}_7 \times S^4$ solution is obtained by taking $n = 1$ and $k_1 = 1$. Taking this limit in (5.83) yields

$$\frac{\mathcal{I}[\emptyset]}{32\kappa_{11}^3} = \frac{4m_1^3}{3\epsilon_u^4} + \frac{4m_1^3}{3\epsilon_u^2} - \frac{5m_1^3}{6} + \dots \,. \tag{5.84}$$

We then arrive at the expression for the regulated $\mathcal{I}$:

$$\begin{aligned}\frac{\mathcal{I}[\Sigma] - \mathcal{I}[\emptyset]}{32\kappa_{11}^3} =& \frac{2m_3 + 13m_1^3}{15} + \frac{1}{3}\sum_{a=0}^{n}p_{a+1}^2(\eta_{a+1}^3 - \eta_a^3) + \sum_{a=0}^{n}\delta_{a+1}p_{a+1}(\eta_{a+1}^2 - \eta_a^2) \\ &+ \sum_{a=0}^{n}\delta_{a+1}^2(\eta_{a+1} - \eta_a)),\end{aligned} \tag{5.85}$$

which recovers the result of the integral for the non-compact electrostatic solutions in [4]. Thus, the regulated minimal area is given by

$$\mathcal{A}_{\min}[\Sigma] - \mathcal{A}_{\min}[\emptyset] = 2^9\pi^4\kappa_{11}^3\left(\frac{R^2}{\epsilon_w^2} - \log\frac{2R}{\epsilon_w} + O(1)\right)(\mathcal{I}[\Sigma] - \mathcal{I}[\emptyset]). \tag{5.86}$$

Proceeding with the computation of a_Σ, we feed (5.86) in (5.59) to get S_{EE}. Computing the log derivative with respect to R of the regularised minimal area functional at $R = 0$ gives the universal part of defect entanglement entropy

$$R\partial_R(S_{\text{EE}}[\Sigma] - S_{\text{EE}}[\emptyset]) = -\left(\mathcal{I}[\Sigma] - \mathcal{I}[\emptyset]\right), \tag{5.87}$$

where we mapped to the field theory variables using $G_N^{(11)} = 2^{13}\pi^4\kappa_{11}^3$ and $\kappa_{11} = L^3/8N$. Using (5.8) we can read off the A-type anomaly coefficient using $d_2 = -\frac{1}{3}(m_1^3 - m_3)$

$$a_\Sigma = \frac{\left(\sum_{a=1}^n k_a\eta_a\right)^3}{4} + \frac{1}{12}\sum_{a=0}^{n}(p_{a+1}^2(\eta_{a+1}^3 - \eta_a^3) + 3\delta_{a+1}p_{a+1}(\eta_{a+1}^2 - \eta_a^2) + 3\delta_{a+1}^2(\eta_{a+1} - \eta_a)). \tag{5.88}$$

Recall that the η_a are ordered by $0 = \eta_0 < \eta_1 < \ldots < \eta_n$, and so $(\eta_{a+1}^j - \eta_a^j) > 0$ for any $j \in \mathbb{N}$ and for all a. Further, the orbifold parameters are non-negative $k_a \in \mathbb{N}$, and so by definition are the p_a, and in addition $2\delta_a \in \mathbb{N}$. Hence, we see that $a_\Sigma \geq 0$. Note that the inequality is saturated at $n = k_1 = 1$ i.e. $a_\Sigma = 0$, which is expected since this line charge density configuration corresponds to having no defect.

For completeness, we can rewrite a_Σ in terms of the ranks, N_a, of the factors in the Levi subalgebra $\mathfrak{l} \subset A_{N-1}$ and their associated monopole charges, k_a,

$$a_\Sigma = \frac{N^3}{32} - \frac{1}{96}\sum_{a=1}^{n}\left(\frac{1+2k_a}{k_a^2}N_a^3 + \sum_{b=a+1}^{n}N_ak_b\left(\frac{N_a^2}{k_a^2} + 3\frac{N_b^2}{k_b^2}\right)\right). \tag{5.89}$$

While the definite sign of a_Σ is a bit less clear in terms of the gauge algebra data, it is nonetheless non-negative following from (5.88).

As we mentioned toward the end of section 5.3.2, there is a non-trivial consistency check of our results in (5.88) from the comparison to the one-charge ($q_2 \to 0$) solutions. Setting $n \to 1$ and $k_1 \to 1/\sqrt{1-4q_1}$ in (5.88) results in

$$a_\Sigma\Big|_{n=1} = \frac{N^3}{48}\left(1 + 2q_1 - \sqrt{1-4q_1}\right). \tag{5.90}$$

Looking back to the computation of a_Σ for the two-charge solutions, we need the largest root of $Q(y)$ with $q_2 \to 0$, which is simply $y_+(q_1) = \frac{1}{2}(1 + \sqrt{1-4q_1})$. Plugging $y_+(q_1)$ into (5.68) exactly matches (5.90).

We now compare a_Σ to the computations of the 'defect central charge' for these solutions. The 'defect central charge' was computed in [4] using the standard formula for the central charge c of *standalone* 4d $\mathcal{N} = 2$ SCFTs at large N holographically dual to

AdS_5 solutions in M-theory [371]

$$c = \frac{2^5\pi^3\kappa_{11}^3}{(2\pi\ell_P)^9}\int_{\mathcal{M}_6}\left(\frac{\dot{V}\sigma}{2V''}\right)^{\frac{3}{2}}, \tag{5.91}$$

which applies to 11d metrics of the form

$$ds_{11}^2 = \left(\frac{\kappa_{11}^2\dot{V}\sigma}{2V''}\right)^{\frac{1}{3}}(ds^2_{\mathrm{AdS}_5} + ds^2_{\mathcal{M}_6}). \tag{5.92}$$

This formula had been used to find the holographic central charge dual to electrostatic solutions with compact internal space engineering irregular punctures [339, 372]. Despite the integrals in eqs. (5.85) and (5.91) having the same form, the crucial difference is in the interpretation of the result: the relative difference between a_Σ and c is a factor of $-2d_2/5$.

Lastly, while monotonicity of the universal part of the defect sphere EE has yet to be tested for 4d dCFTs, in the case of a co-dimension 4 Wilson surface in 6d SCFTs the universal defect contribution to the sphere EE does not behave monotonically under defect RG flows (see e.g. [373]). Due to the relative sign in ΔS_{EE} and the fact that only a_Σ is known to obey a weak defect a-theorem[12], it is expected that (5.87) is not a monotone along defect RG flows.

5.5 Discussion

In this work, we have analysed solutions in 11d SUGRA that holographically describe $1/4-$ and $1/2-$BPS co-dimension 2 defects in the 6d A_{N-1} $\mathcal{N}=(2,0)$ SCFT at large N.

Our holographic computations of the defect contribution to the one-point function of the stress energy tensor have revealed simple expressions for the defect Weyl anomaly coefficient d_2 in section 5.3. For the 1/4-BPS two-charge solutions specified by charges q_1, q_2, we have found that $d_2 \propto N^3(q_1+q_2)$. For the 1/2-BPS electrostatic solutions determined by a potential solving a Laplace-type equation with moments m_j, $d_2 \propto (m_1^3 - m_3) \sim N^3 - \sum_a N_a^3$ where $N = \sum_a N_a$. Using the 4d form of the defect ANEC, which states $d_2 \le 0$, we have demonstrated that all of the allowed two-charge solutions found in [3] and the electrostatic solutions in [4] obey the bound and are thus consistent with this known defect energy condition [354]. We were also able to compare against a similar computation for the two-charge solutions done in 7d $\mathcal{N}=4$ gauged SUGRA, and found an agreement with $\langle T_{ij}\rangle$ in [3].

[12] The recent entropic proof in [354] of the irreversibility of defect RG flows in addition to the dilaton effective action methods (à la [40]) in [374] have firmly established the existence of at least a weak defect a-theorem.

In section 5.4, we used the tools developed in [323, 324] to holographically compute the contribution of flat, co-dimension 2 defects to the EE of a spherical region in the dual field theory. By isolating the universal, log-divergent part of the defect sphere EE, we were able to find closed form expressions for the A-type anomaly a_Σ for both defect systems considered. Since we know that the universal part of the defect sphere EE, is a linear combination of a_Σ and d_2 as in (5.7), by combining $\Delta \langle T_{ij} \rangle$ and ΔS_{EE}, we have a direct computation of a_Σ: for the two-charge solutions we found $a_\Sigma \propto N^3(1-y_+^2)$ where y_+ is the largest root of the quartic polynomial in (5.12b), while a_Σ for the electrostatic solutions in (5.88) is a complicated function of the data of the line charge distribution that specifies the solution. For the electrostatic solutions, we have shown that the computation of the holographic 'central charge' in [4] is proportional to the universal part of the defect sphere EE. Further, we were able to show that the complicated sum over line charge density data that appears in a_Σ is the same sum that determines the large N 'central charge' $c(=a)$ for the *compact* electrostatic solutions describing 4d $\mathcal{N}=2$ SCFTs; the important difference is that the defect a_Σ has an additional contribution of $N^3/32$. In both classes of defects, we have also shown that $a_\Sigma \geq 0$, where the inequality is only saturated for a trivial defect.

Curiously, in appendix B.2, we showed that the holographically renormalised on-shell action for the 11d uplift of the two-charge solutions using the full form of the radial cutoff in FG gauge and found that the log divergent part of the action cannot be written in terms of either a_Σ, as was expected from the same computation done in 7d gauged SUGRA description of the two-charge defects [3], d_2, or a linear combination of them. The reason for this discrepancy is unclear at this time, but may be related to the insufficiency of the background subtraction scheme for the on-shell action, which highlights a need for a full covariant holographic renormalization scheme for 'defects' in 11d SUGRA.

With the holographic predictions for a_Σ and d_2 in hand, let us compare to results in the field theory at large N. We will focus entirely on the 1/2-BPS electrostatic solutions in the following comparisons.

Defect supersymmetric Casimir energy

In ordinary 4d SCFTs with R-symmetry placed on $S^1_\beta \times S^3$, the supersymmetric localised partition function can be decomposed as a product of an exponential prefactor multiplying the superconformal index

$$\mathcal{Z}_{S^1_\beta \times S^3} = e^{-\beta E_C} \mathcal{I}. \tag{5.93}$$

The supersymmetric Casimir energy (SCE), E_C, can be expressed in terms of the conformal anomalies a and c [53, 375] of the theory, the equivariant integral of the anomaly polynomial [376], or 't Hooft anomalies [377]. Given the results in [284] for the localised partition functions a 1/2-BPS co-dimension 2 defect in a 6d $\mathcal{N} = (2,0)$ A_{N-1} SCFT labelled by ϑ wrapping $\Sigma = S^1_\beta \times S^3 \subset S^1_\beta \times S^5$, it was conjectured in [310] that the change in the exponential prefactor due to the introduction of the defect was in fact the defect SCE and could be related to defect conformal anomalies[13]. Now that we have holographic predictions for two defect anomalies, we can look for a superficial match to this field theory quantity.

As a very brief overview, we start the comparison by putting the ambient theory on the squashed $S^1_\beta \times S^5_\mathfrak{b}$ and reducing along the S^1 factor. The localised partition function of the 6d $\mathcal{N} = (2,0)$ A_{N-1} SCFT in the unrefined limit becomes the partition function of 5d $\mathcal{N} = 2$ $U(N)$ super-Yang-Mills theory on $S^5_\mathfrak{b}$, which determines the ambient SCE

$$E_C[\emptyset] \equiv \frac{\mathfrak{c}}{24}\,, \qquad \text{where} \qquad \mathfrak{c} = N(N^2-1)(\mathfrak{b}+\mathfrak{b}^{-1})^2 + N - 1. \tag{5.94}$$

The quantity $\mathfrak{c}$ in this picture is the central charge of the 2d W_N-algebra on the plane orthogonal to the directions that defect will eventually wrap [284, 332]. The introduction of a co-dimension 2 defect breaks the gauge algebra to the Levi subalgebra $\mathfrak{l} = \mathfrak{s}\left[\bigoplus_{a=1}^n \mathfrak{u}(N_a)\right]$. The most general 1/2-BPS defect configuration allows for monodromy parameters $\vec{\mathfrak{w}} = (\mathfrak{w}_1, \ldots, \mathfrak{w}_n)$ for the Levi factors. The change in the SCE due to introducing the defect along Σ labelled by $\vartheta : \mathfrak{sl}(2) \to \mathfrak{g}$ with monodromy parameters $\vec{\mathfrak{w}}$ was found to be given by [284, 310]

$$\begin{aligned} E_C[\Sigma]_{\vartheta,\vec{\mathfrak{w}}} - E_C[\emptyset] &= \frac{1}{2}(\mathfrak{b}+\mathfrak{b}^{-1})^2[(\hat{\varrho}_\mathfrak{l}, \hat{\varrho}_\mathfrak{l}) - (\hat{\varrho}_\mathfrak{g}, \hat{\varrho}_\mathfrak{g})] + \frac{1}{2}(\vec{\mathfrak{w}}, \vec{\mathfrak{w}}), \\ &= -\frac{1}{6}\left(N^3 - \sum_{a=1}^n N_a^3 - 3(\vec{\mathfrak{w}}, \vec{\mathfrak{w}})\right). \end{aligned} \tag{5.95}$$

In the second line we took the limit $\mathfrak{b} \to 1$, and replaced the scalar product of the Weyl vectors – denoted $\hat{\varrho}_\mathfrak{l}$ and $\hat{\varrho}_\mathfrak{g}$ for $\mathfrak{l}$ and $\mathfrak{g} = \mathfrak{su}(N)$, respectively – with

$$(\hat{\varrho}_\mathfrak{l}, \hat{\varrho}_\mathfrak{l}) = \frac{1}{12}\sum_{a=1}^n (N_a^3 - N_a), \qquad (\hat{\varrho}_\mathfrak{g}, \hat{\varrho}_\mathfrak{g}) = \frac{1}{12}(N^3 - N). \tag{5.96}$$

Turning off the monodromy parameters[14] ($\mathfrak{w}_a = 0$) in (5.95) we see the superficial relation

$$E_C[\Sigma]_{\vartheta,\vec{0}} - E_C[\emptyset] = 4 d_2|_{k_a \to 1}\,, \tag{5.97}$$

[13] Evidence for a version of this conjecture for $\mathfrak{d} = 2$ defects gathered from studying various examples appeared to support the claim, and in [328], a relation between the SCE and h_T was established using the chiral algebra description of the defect insertion, which gives a much stronger argument for E_C being controlled by d_2. We thank Maxime Trépanier for pointing out the chiral algebra proof in [328] to us.

[14] In light of the compact LLM-type solutions found recently in [378] where the additional internal $U(1)$ symmetry is broken by the presence of scalar fields, which are interpreted as monodromy parameters, it may be possible to pin down a more precise relation between E_C and defect anomalies by computing $\langle T_{\mu\nu}\rangle$ if similar non-compact solutions allowing for $\mathfrak{w}_a \neq 0$ can be constructed.

where on the right hand side we take all orbifold parameters $k_a \to 1$ in (5.58b).

Since the expression for the defect SCE in terms of explicit defect Weyl anomalies is still unknown and 4d dCFTs have 23 possible parity even anomalies, we cannot definitively state that the defect SCE is determined solely by d_2. We note, though, that a similar relation was found for co-dimension 4 Wilson surface defects: the defect SCE in that case was also related to the 2d dCFT equivalent of d_2. Since 2d dSCFT preserving at least $\mathcal{N} = (2,0)$ supersymmetry have only two independent Weyl anomalies[15], which for the Wilson surface defect can be clearly distinguished from one another [51], it was conjectured that d_2 alone fixed the defect SCE [310]. So, while it is not inconceivable that d_2 could appear in the defect SCE for co-dimension 2 defects, we leave establishing the precise relation for future work.

R-anomalies

Ordinarily in 4d SCFTs, there are non-perturbative formulae that relate the A-type and B-type Weyl anomalies to 't Hooft anomalies for the superconformal R symmetry [351]. In [374], it was conjectured that a_Σ obeys the same relation to defect R-anomalies as a standalone theory[16]:

$$a_\Sigma = \frac{9k_{rrr} - 3k_r}{32}, \tag{5.99}$$

where k_{rrr} and k_r are the cubic and mixed $U(1)_r$ R-anomalies. Importantly for the defect theory written in 4d $\mathcal{N} = 1$ language, the superconformal r_Σ symmetry is a linear combination of the Cartan generator of the ambient $SU(2)_R$ R-symmetry and the generator of normal bundle rotation M_φ [374]

$$r_\Sigma = \frac{2}{3}(2r_{6d} - M_\varphi). \tag{5.100}$$

It was further stated in [374] that precisely for the types of defects holographically described by the electrostatic solutions considered above, in order to determine the R and mixed anomaly we should use the counting formulae [331]

$$k_{rrr} = \frac{2}{27}(\mathfrak{n}_v - \mathfrak{n}_h) + \frac{8}{9}\mathfrak{n}_v, \qquad k_r = \frac{2}{3}(\mathfrak{n}_v - \mathfrak{n}_h), \tag{5.101}$$

[15] This was first proven for superconformal surface defects in 4d $\mathcal{N} = 2$ SCFTs in [327], and later, it was proven for 2d defects in the 6d $\mathcal{N} = (2,0)$ theory in [311].

[16] It was also conjectured that a B-type defect anomaly built out of the square of intrinsic Weyl tensor ($c_\Sigma|\bar{W}|^2$) obeys the usual relation [351]

$$c_\Sigma = \frac{9k_{rrr} - 5k_r}{32}. \tag{5.98}$$

However, the basis used in [34] did not include $|\bar{W}|^2$. From the Gauss-Codazzi and Ricci relations, $|\bar{W}|^2$ is related to several anomalies in the original basis (none of which include d_2). So it is unclear at the this time, what observables can be used to compute c_Σ. Though it is reasonable to expect that the defect limit of $\langle T_{\mu\nu} T_{\rho\sigma}\rangle$ may be the appropriate correlator to compute c_Σ, proving this is the subject of future work.

where $\mathfrak{n}_v$ is the number of 4d vector multiplets and $\mathfrak{n}_h$ is the number hypermultiplets. In turn, both $\mathfrak{n}_h$ and $\mathfrak{n}_v$ are determined by the Young diagram data.

As we have pointed out around (5.88), the defect A-type anomaly contains a contribution that is precisely of the form of the central charge c of 4d SCFTs engineered from irregularly punctured Riemann surface compactifications of 6d $\mathcal{N} = (2, 0)$ A_{N-1} series SCFTs dual to electrostatic solutions of the type studied above. Further, in [331, 339], a match was found between the holographic computation of c of the dual 4d SCFT and the large N behaviour of the central charge computed in the field theory using the R-anomalies and (5.101). However since we have found $a_\Sigma \sim c + N^3/32$, it is clear that the naïve application of (5.99) and (5.101) do not directly match.

5.5.1 Future directions and open questions

The work that we have presented in this paper is only scratching the surface of 4d defects. While a full accounting of all of the defect Weyl anomalies of these systems through computing entropies, correlation functions, or other physical quantities is not currently possible, there are a number of questions opened up by our analysis that we will leave for future work.

Probe branes

Even though we have access to the full 11d SUGRA bubbling geometry solution, it is useful to consider limit cases where we can instead appeal to a probe brane construction. By finding κ-symmetric embeddings of probe M5-branes in an $\mathrm{AdS}_7 \times S^4$ background wrapping $\mathrm{AdS}_5 \subset \mathrm{AdS}_7$ and an S^1 living either in the internal S^4 or in the AdS_7, we expect to be able to holographically study defects engineered by Young diagrams associated to totally symmetric or totally antisymmetric representations of $\mathfrak{su}(N)$ similar to the co-dimension 4 Wilson surface defects from M2 and M5 probe branes [373, 379, 380]. One advantage of studying these defect systems using probe brane holography is that we will have clearer access to the study of defect RG flows, which will provide holographic tests of the defect a_Σ-theorem in a strongly coupled theory, a means to study defect phase transitions, and a setting to test the monotonicity of the defect sphere EE along an RG flow [373]. Further taking inspiration from AdS_5 holography [381–383], if one was able to construct a κ-symmetric probe M5 brane embedding in global AdS_7, say with an $S^1 \times S^5$ boundary, one could try to compare to recent results in type IIB probe brane holography and supersymmetric localization in $3d/5d$ systems on a sphere [384, 385]. These questions are currently being investigated in work currently in progress.

Dimensional reduction

By (partial) topologically-twisted dimensional-reduction on a Riemann surface or a 3-manifold, 6d SCFTs can be used to engineer large classes of 4d [341, 386] and 3d [280, 387] theories. Further, we can enrich the algorithm to determine the lower dimensional theory by starting from a 6d theory deformed by their natural co-dimension 2 and 4 defects to end up with a dimensionally reduced theory possibly with defects [297, 388, 389]. As we have seen in the computation of the A-type anomaly for co-dimension 2 defects in the 6d $\mathcal{N}=(2,0)$ A_{N-1} series SCFTs, there is a connection to the central charge of a 4d SCFT engineered on a Riemann surface with regular punctures, at least in the large N limit. It is natural, then, to wonder how the rest of the data contained in the other 22 parity even defect Weyl anomalies can be used to characterise the lower dimensional theory, or whether the remaining unknown defect Weyl anomalies are vanishing or fixed by a_Σ and d_2. For BPS Wilson surfaces in 6d preserving at least 2d $\mathcal{N}=(2,0)$ supersymmetry, the defect supersymmetry imposes non-trivial relations among the B-type defect Weyl anomalies [311], but as of yet, there is no known relation imposed by 4d $\mathcal{N}=2$ defect supersymmetry.

A special case of dimensional reduction of the 6d $\mathcal{N}=(2,0)$ A_{N-1} theory is taking the Riemann surface to be $\mathbb{T}^2$, which reduces to 4d $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory. The co-dimension 2 defects labelled by $\vartheta:\mathfrak{sl}(2)\to\mathfrak{su}(N)$ in the parent theory that we have holographically studied above wrapped on $\mathbb{T}^2$ reduce to Gukov-Witten type defects. In the absence of complex structure deformations on $\mathbb{T}^2$, all of the defect Weyl anomalies are equal to one another and are $\propto N^2-\sum_a N_a^2$ [51, 345, 346, 390], which is closer in appearance to d_2 in (5.58b) than a_Σ in (5.88). However, an exact relation to determine the anomalies of the Gukov-Witten defect from the higher dimensional defect anomalies is as of yet unknown.

Defect Weyl anomalies and ‘t Hooft anomalies

As we saw in the attempt to match the any of the holographic results for a_Σ or d_2 to large N field theory computations, there are points of tension that should be resolved. One of the biggest issues, though, is that the putative relation between defect ‘t Hooft anomalies and defect Weyl anomalies seemed to disagree with the holographic results. While it remains a possibility that the issue stems from the holographic side of the story, there is an open question on the field theory side that must be addressed as well. Namely, the formulae conjectured in [374] only relate two of the twenty-three parity even defect Weyl anomalies to the defect R-anomalies for co-dimension ≥ 2 4d defects. That is, only the $\overline{E}_4$ and $|\overline{W}|^2$ structures in the defect anomaly have been supersymmetrised. A similar supersymmetrisation of the defect Weyl anomaly for 2d defects limited to be sensitive only to the intrinsic geometry of the defect submanifold was carried out in [391]. This naturally leads one to wonder if it is possible to supersymmetrise the full defect Weyl anomaly including the anomalies containing the second fundamental form and normal bundle curvature in order to arrive at a complete set of non-perturbative formulae for defect Weyl anomalies.

Part III

Supersymmetry and Indices

Chapter 6

Interpolating Index

It ain't over 'til it's over.

Yogi Berra

This chapter presents some novel results in the construction of supersymmetric backgrounds of 4d $\mathcal{N}=2$ conformal supergravity. By coupling general 4d SCFTs with eight Poincaré supercharges to conformal supergravity, it is possible to study these on various geometric backgrounds, while retaining some amount of supersymmetry. The background of interest to us here is the product manifold $S^3 \times S^1$. We are interested in constructing a background supergravity configuration whereby evaluating the partition function on it will lead to either the superconformal index or the twisted index — two quantities that count local operators within the SCFT. The superconformal index does also admit a definition as a trace over a Hilbert space of operators on S^3, and the relation between the partition function and trace definitions is known [53].

Both indices admit a dual representation as AdS black hole configurations, for holographic theories [30]. Our results show that the twisted index is equal to the value of the superconformal index at a special point within its moduli space, inherently relating their holographic descriptions. We do not further comment on this relation between black hole microstate countings and simply detail our proof on the CFT side, in a theory-agnostic fashion.

6.1 Introduction

Four-dimensional superconformal field theories with eight supercharges are uniquely positioned within the space of supersymmetric field theories and possess many interesting structures. Firstly, the amount of supersymmetry is sufficient that one can derive exact results, yet not too constraining that they still display rich dynamics. Many of these theories do not admit weak-coupling descriptions and one must rely on a class-$\mathcal{S}$ construction [279] to build them. Thanks to the Alday-Gaiotto-Tachikawa (AGT) correspondence [340], they are intimately tied to two-dimensional conformal field theories. This amount of symmetry also gifts these theories with observables that are rigid under certain renormalisation group (RG) flows.

The superconformal index [28, 29] is an example of such a quantity. It performs a graded counting of local BPS operators within the theory, up to those short multiplets that recombine into larger ones under certain RG flows. It therefore captures fundamental data about the SCFT at hand. Invariance under RG flows[1] also means that the index can be computed in the free-theory limit and be related to other conformal field theories at various fixed points.

This type of index is defined as a Witten index with respect to a given supercharge of the theory (see Definition 6.1). For every subalgebra that commutes with this supercharge, one can introduce a fugacity that further refines the index by counting separately operators that fall under different representations of said algebra. In four dimensions, all theories with sixteen Poincaré supercharges ($\mathcal{N} = 4$) admit a refinement with four parameters. Theories with eight supercharges generically have three such parameters. See Figure 6.1 or [392] for further details. Furthermore, one can consider constructing limits of these parameters to construct new sub-indices, that count differently the various operators in the theory. For instance, the Schur index, which counts operators in the Schur subsector, is given as a one-dimensional sub-index[2]. The Coulomb-branch index [5] can also be obtained, as a two-dimensional sub-index (see Figure 6.2). The latter will take the center stage in this chapter.

The R-symmetry group of these four-dimensional theories with eight supercharges is also large enough that one can consider identifying a subgroup of the Lorentz group with it. This is the twisting procedure introduced by Witten [201]. Such a construction will always lead to the conservation of at least one supercharge, on any (smooth) orientable manifold, known as the Donaldson-Witten supercharge. The cohomological theory thus defined is topological and its partition function, known as the twisted index, counts operators in the cohomology of that supercharge.

Summary of new results

[1] More specifically, it is invariant under any continuous transformation that preserves its defining supercharge.

[2] It recently came to my attention that Deb and Razamat constructed a new limit, dubbed the 'generalised Schur index', which exhibits fascinating properties relating it to the Schur index of different SCFTs [393].

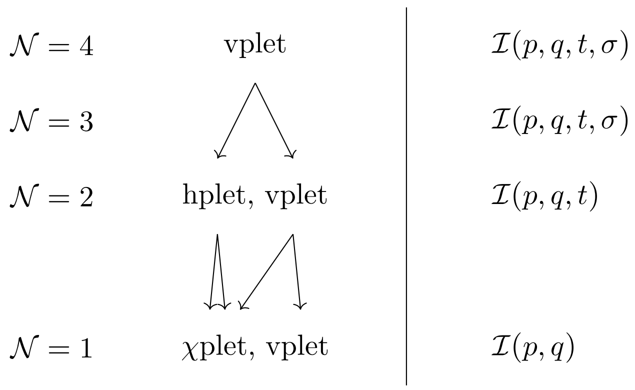


FIGURE 6.1: Decomposition of four-dimensional superconformal multiplets for various amounts of supersymmetry. On the right-hand-side, we denote the corresponding index, illustrating how many parameters it depends on.

It is the aim of this chapter to relate the twisted index and the Coulomb-branch index, further than a simple identification of their defining supercharges. Indeed, both indices fall under the unified framework of Festuccia and Seiberg [198] (see Chapter 3 for an overview), whereby the trace over operators is equivalently given as the partition function of the original theory on $\mathcal{M}_3 \times S^1$, where $\mathcal{M}_3$ is a three-dimensional manifold. In this context, the various fugacities which the indices may depend on are encoded as values of background gauge fields. All throughout, we will consider geometries of the form $S^3 \times S^1$, which correspond to removing any fugacities associated to the two $\mathfrak{su}(2)$ subalgebras of the global $\mathfrak{so}(4)$. The consequence of this is two-fold. Firstly, the Coulomb-branch index there created is restricted to live at a given point in its parameter space. In other words, if $\mathcal{I}_{\text{CB}}(\rho,\sigma)$ denotes the general Coulomb-branch index, our background can only engineer $\mathcal{I}_{\text{CB}}(\rho_0,\sigma_0)$, for a given ρ_0 and σ_0.[3] Secondly, it is the only geometrical configuration which admits an 'exact interpolation', in a sense which is defined clearly in later sections.

We use the coupling to background $\mathcal{N}=2$ conformal supergravity, reviewed in Chapter 3, to find supersymmetric background configurations that give either the twisted index or the Coulomb-branch index when evaluating the partition function of the coupled theory. Furthermore, we also construct a background which interpolates between the two. Together with a novel formula, relating the supersymmetric variation of current multiplets to that of the background supergravity fields, we show that this interpolation is exact. One can then conclude that indices on the real line connecting the twisted index and the Coulomb-branch index are all equal. We further emphasise that this result holds for any 4d $\mathcal{N}=2$ SCFT, regardless of whether a Lagrangian description exists or not. Given that the supercharge that defines the Coulomb-branch index is the Donaldson-Witten supercharge, it may come to no surprise that these turn

[3]This result cannot be otherwise. Since we claim that the twisted index, which doesn't depend on any moduli, is equal to the Coulomb-branch index, which depends on two fugacities, this can only happen at a particular point in that moduli.

out to be equal. However, our background supergravity construction gives a novel way of viewing this result, and allows for a more precise identification of the fugacities.

6.2 The Coulomb-branch index

The superconformal index of a d-dimensional theory captures non-trivial information about the spectrum of protected operators. It coincides, up to a factor related to the supersymmetric Casimir energy [53], with the Euclidean partition function of the radially quantised theory on S^3. We now introduce the basic partition function, and then explain its 3-parameter refinement and the resulting Coulomb-branch index.

Definition 6.1. SCI The superconformal index of a superconformal field theory in four dimensions is defined as the Witten index of the theory in radial quantisation. Let $\mathcal{Q} := \mathcal{Q}_-$ be one of the Poincaré supercharges, and $\mathcal{Q}^\dagger := \mathcal{S}_+$ the conjugate conformal supercharge. It is defined as a trace over the Hilbert space of operators on S^3,

$$\mathcal{I}(\mu_i) = \mathrm{Tr}_{S^3}\left((-1)^F x^{\frac{1}{2}\{\mathcal{Q},\mathcal{Q}^\dagger\}} \prod_{i=1}^{N} \mu_i^{\mathcal{M}_i}\right). \tag{6.1}$$

In the above, $\delta = 2\{\mathcal{Q},\mathcal{Q}^\dagger\}$ is a linear combination of the Cartan generators of the 4d $\mathcal{N}=2$ superconformal algebra $\mathfrak{su}^*(4|2)$, with coefficients determined by the quantum numbers of the chosen supercharge $\mathcal{Q}$, while $\{\mu_i\}$ is a complete set of fugacities associated to symmetry generators Q_i which commute amongst themselves, as well as with $\mathcal{Q}$. The centraliser of any one supercharge in $\mathfrak{su}^*(4|2)$ is $\mathfrak{su}^*(2|2)$, which has rank 3; therefore, the superconformal index generically depends on three superconformal fugacities, as well as any flavour fugacities which may be present in the theory.. On the other hand, due to Bose-Fermi cancellations amongst the $\delta \neq 0$ states, the superconformal index only enumerates the harmonic representatives of $\mathcal{Q}$-cohomology classes, and so it is independent of β.[4]

The definition given above is an application of linear algebra — a simple counting of operators that act on a Hilbert space. Our interest rather lies in a more geometric construction of it, namely through its definition as a partition function on $\mathcal{M}_3 \times S^1$, up to factors of the supersymmetric Casimir energy. We emphasise once again that the trace definition follows from radial quantisation of the theory on flat space, whereby the radial direction plays the role of Euclidean time. It may be instructive to understand how placing the theory on $S^3 \times S^1$ can describe that same algebra. Furthermore, a precise identification of this procedure is required if one wishes to translate between the

[4]The independence of the superconformal index on β is reminiscent of the stability of the index of a Fredholm operator under additive perturbations by (relatively) compact operators [394].

two pictures, for instance, when deciding which supercharge is used to define the index (see Appendix D.2).

As reviewed in Chapter 3, four-dimensional $\mathcal{N} = 2$ superconformal field theories on $\mathbb{R}^4$ are gifted with an $\mathfrak{su}^*(4|2)$ Lie superalgebra. By applying a conformal transformation to $\mathbb{R}^4 \setminus \{0\}$, the full superconformal algebra can be preserved on the round $S^3 \times \mathbb{R}_t$, where t parametrises Euclidean time. The conformal Killing spinor equation on $S^3 \times \mathbb{R}_t$ allows for the presence of an arbitrary complex Wilson line for the $U(1)_r$ gauge field along $\mathbb{R}_t$. While generic configurations of $A^{U(1)_r}$ lead to Killing spinors with exponential temporal profiles, preventing the compactification of the $\mathbb{R}_t$ factor into a circle S^1_β of radius β, it is nevertheless possible to render the supercharges independent of Euclidean time by selecting the following gauge field configuration [198, 210],

$$A^{U(1)_r} = -\beta dt \ . \tag{6.2}$$

Consequently, in this R-symmetry background, we can compactify $S^3 \times \mathbb{R}_t$ into $S^3 \times S^1_\beta$. In this context, we consider a Riemannian metric g,

$$g = d\theta^2 + \sin^2(\theta) d\varphi^2 + \cos^2(\theta) d\tau^2 + \beta^2 dt^2 \ , \tag{6.3}$$

where $\{\theta, \varphi, \tau\}$ are toroidal coordinates on the round 3-sphere. A more detailed discussion of this geometry, and our choice of conventions, can be found in Appendix C.2.

The resulting rigid supersymmetries on $S^3 \times S^1_\beta$ are a (non-invariant) mixture of Poincaré (or Q-) and conformal (or S-type) supersymmetries; equivalently, in terms of the Euclidean Weyl multiplet of conformal supergravity described in Chapter 3, the conformal supersymmetry parameter η is generically non-vanishing. The generalised Killing spinors ξ on this background satisfy

$$\left(\nabla_\mu - \frac{1}{2\beta}\delta_\mu{}^t\gamma^5\right)\xi^i = \frac{1}{4}\gamma_\mu\gamma^\nu\left(\nabla_\nu - \frac{1}{2\beta}\delta_\nu{}^t\gamma^5\right)\xi^i \ . \tag{6.4}$$

This generalised Killing spinor equation is a specialisation of equation (3.9a) to the background presently under consideration, and is solved by any Dirac spinor which is constant, but otherwise generic, when measured against the spacetime and Clifford algebra bases summarised in Appendix D.2.

Let us now consider a deformation of the background discussed above. The partition function evaluated on this background will compute, up to a multiplicative factor determined by the supersymmetric Casimir energy [53], the superconformal index of the theory. In particular, we introduce a geometric deformation [395–397] with parameters

$\epsilon_{1,2}$ by modifying the vierbein according to

$$e^a \longrightarrow e^a + e^a{}_\mu v^\mu dt \ , \tag{6.5}$$

where $v = \epsilon_1 \partial_\varphi + \epsilon_2 \partial_\tau$ parametrises the Cartan subalgebra of the $\mathfrak{so}(4)$ isometry of S^3. This deformation metrically fibres S^3 over S^1_β; the resulting Riemannian metric g_Ω reads

$$g_\Omega = d\theta^2 + \sin^2(\theta)(d\varphi + \epsilon_1 dt)^2 + \cos^2(\theta)(d\tau + \epsilon_2 dt)^2 + \beta^2 dt^2 \ . \tag{6.6}$$

To preserve the reality of the metric g_Ω, we will take the equivariant parameters $\epsilon_{1,2}$ to be real. Their complexification would allow us to make contact with the abundant literature concerning partition functions on squashed spherical backgrounds (see Appendix C.3).

Furthermore, we include an $SU(2)_R$ deformation with parameter $m_R \in \mathbb{C}$ by introducing a background $SU(2)_R$ gauge field

$$A^{SU(2)_R} = -i m_R \sigma_3 dt \ . \tag{6.7}$$

Preservation of (at least) two constant supercharges on $S^3 \times_\Omega S^1_\beta$ can be achieved by performing a concurrent and further deformation of the $U(1)_r$ gauge field to

$$A^{U(1)_r} = (-\beta + i(m_R - \epsilon_1 - \epsilon_2))dt \ . \tag{6.8}$$

The fact that the R-symmetry holonomy is complexified is not problematic, since throughout this work we do not insist on attaching any reality conditions to the background fields. In the Clifford algebra basis summarised in Appendix D.2, the 3-parameter background thus described possesses two generalised Killing spinors

$$\xi_\pm = c_\pm(\sigma_1 \mp i\sigma_2) \ , \tag{6.9}$$

where $c_\pm$ are constants. Their corresponding conformal supersymmetry parameters, as determined by equation (3.13), are

$$\eta_\pm = c_\mp(\sigma_1 \pm i\sigma_2) \ . \tag{6.10}$$

Additional conserved Killing spinors can be found at specific points (or submanifolds) in the space of geometric and $SU(2)_R$ deformation parameters. For instance, and anticipating the relation between the Coulomb-branch and topologically twisted indices, the specialization to the $m_R = \epsilon_1 + \epsilon_2$ locus results in a supersymmetry enhancement.

In particular, the Killing spinors satisfy

$$\left(\nabla_\mu - \frac{1}{2\beta}\delta_\mu{}^t\gamma^5\right)\xi^i - \frac{i}{2}(\epsilon_1+\epsilon_2)\delta_\mu{}^t{\sigma_3}^i{}_j\xi^j = \tag{6.11}$$
$$= \frac{1}{4}\gamma_\mu\gamma^\nu\left[\left(\nabla_\nu - \frac{1}{2\beta}\delta_\mu{}^t\gamma^5\right)\xi^i - \frac{i}{2}(\epsilon_1+\epsilon_2)\delta_\mu{}^t{\sigma_3}^i{}_j\xi^j\right]$$

and can be written in the chiral basis as follows,

$$\xi_\pm = c^1_\pm\sigma_1 + c^2_\pm\sigma_2\ , \tag{6.12}$$

for constants $c^{1,2}_\pm$. Quantisation of the two negative- and two positive-chirality constant Killing spinors preserved along the $m_R = \epsilon_1+\epsilon_2$ subspace leads to two negative-chirality supercharges $\mathcal{Q}$, and their adjoints. The superconformal index on this fugacity subspace therefore computes the cohomology of the commuting pair of supercharges.

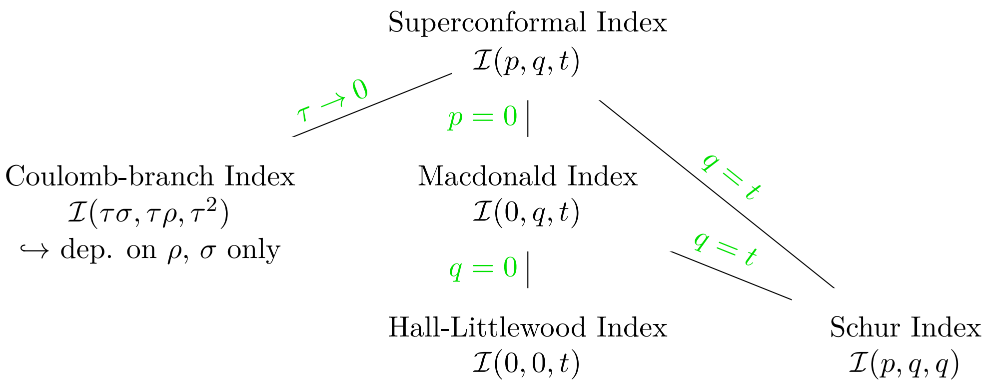


FIGURE 6.2: The various limits of the sci considered in [5]. Note that the Schur index, while explicitly dependent on two parameters, p and q, only actually dependents on the latter.

The index thus defined (in particular, specialised to $m_R = \epsilon_1+\epsilon_2$) coincides with the Coulomb-branch index introduced in [5], which is obtained in the limit wherein the three original fugacities vanish, while two ratios are kept fixed (see Figure 6.2). In [5], it was argued that this index only counts the short multiplets of type $\overline{\mathcal{E}}$, in the notation of [398]. These supermultiplets are shortened by a vanishing charge under the Cartan of $SU(2)_R$, and arise by acting with the superconformal generators on a lowest dimension operator which is spinless under one Cartan of the spin group, and in an arbitrary representation of the other [398]. The lowest components of these short multiplets are the gauge-invariant operators spanning the Coulomb-branch of the theory, such as $\{\mathrm{Tr}\,\phi^i\}_{i=2}^{\mathrm{rank}\,\mathcal{G}}$ for a theory with special unitary gauge group $\mathcal{G}$ [5].

How does one identify this background as engineering the Coulomb-branch index, as opposed to any other specialisation of the superconformal index? One way of answering this question is as follows. Take the four spinors that are Killing with respect to this background configuration, equation (6.12). These are all spinors of $S^3\times_\Omega S^1_\beta$. It is, nevertheless, possible to map them back to flat space spinors using, for instance, our results from Appendix D.2. Doing so one finds an exact match with the spinor used to defined the Coulomb-branch index, in the notation of [5]. The partition function of a theory on this background can then only describe the Coulomb-branch index, as stated previously.

Having described the background that engineers the full Coulomb-branch index, from this point on, we will restrict back to backgrounds with $\epsilon_{1,2} = 0$, for which the metric diagonalises to $S^3 \times S^1_\beta$. The Coulomb-branch index therefore sees its fugacities fixed, and the partition function on this background evaluates to $\mathcal{I}_{\rm CB}(\rho_0, \sigma_0)$. The precise values of these coordinates on the moduli depend on the functional dependence of ρ and σ on the deformation parameters $\epsilon_{1,2}$. For instance, identifying our metric with that of a primary Hopf surface, we see that the parameters p and q, appearing in the $\mathcal{N} = 1$ superconformal index, are related to $\epsilon_{1,2}$ via $\arg(p) = \epsilon_1$ and $\arg(q) = \epsilon_2$ [203]. Translating this back to the Coulomb-branch index parameters, the $\epsilon_{1,2} \to 0$ limit would correspond to the $\rho, \sigma \to \infty$ limit. Despite this, we will continue to refer to $\mathcal{I}_{\rm CB}(\rho_0, \sigma_0)$ as the Coulomb-branch index, with the understanding that the limit is implicit.

6.3 The twisted index

As reviewed in Chapter 3, topological twisting allows an $\mathcal{N} = 2$ theory to be defined consistently on any oriented Riemannian 4-manifold; it does so by leveraging the theory's $SU(2)_R$ symmetry [201]. The procedure can be phrased in terms of the introduction of a principal bundle with $SU(2)_R$ structure group, chosen so that it, and its connection $A^{SU(2)_R}$, are locally isomorphic to the $SU(2)_r$ spinor bundle and its Levi-Civita spin connection, respectively. Twisting the theory in this fashion leads to the existence of spinors on which the action of the $SU(2)_R$ background gauge field exactly cancels that of the spin connection. Any such spinors which are constant are therefore also covariantly constant and Killing in a generalised sense. By viewing the metric and the $SU(2)_R$ gauge field as being embedded in an off-shell formulation of (conformal) supergravity [198], which enjoys an $SU(2)_R$ local symmetry, it is then natural to consider deforming the theory further by the inclusion of non-trivial backgrounds for the fields in the supersymmetric completion of the multiplet.

In particular, on the product manifold $S^3 \times S^1_\beta$, rigid supersymmetry can be preserved by taking $A^{U(1)_r}$ and T to vanish, while tuning the $SU(2)_R$ background to the following configuration,

$$A^{SU(2)_R}_\mu = \frac{1}{2}\omega_\mu{}^{ab}\sigma_{ab} \ , \tag{6.13}$$

where σ_{ab} is the anti-symmetrised product of Pauli matrices, as defined in equation (D.16). In the presence of this $SU(2)_R$ background, the generalised Killing spinor equation, equation (3.9a), reads

$$\nabla_\mu \xi^i + \frac{1}{4}\omega_\mu{}^{ab}\sigma_{ab}{}^i{}_j\xi^j = 0 \tag{6.14}$$

and admits, on any oriented Riemannian manifold, regardless of the latter's holonomy, a solution of definite chirality, which in our conventions is $\xi_- = c_-\sigma_2$, for constant c_-. On $S^3 \times S^1_\beta$, the self-duality of the spin connection 1-form actually engenders a second, positive-chirality Killing spinor $\xi_+ = c_+\sigma_2$, for constant c_+. For both $\xi_\pm$, the associated conformal supersymmetry parameters vanish, $\eta_\pm = 0$.

The topological supercharge associated to either generalised Killing spinor is nilpotent on gauge-invariant operators, and is of the type used by Witten to define the cohomological field theory which arises from the Donaldson-Witten topological twist of $SU(2)$ $\mathcal{N}=2$ super Yang-Mills (SYM) [201]. The path integral of topologically twisted SYM theories localises onto a subspace of $\mathcal{A}/\mathcal{G}$, where $\mathcal{G} = \mathrm{Aut}(P) = \Gamma(\mathrm{Ad}\,P)$ is the gauge group associated to the principal G-bundle with total space P, and $\mathcal{A}$ is the (affine) space of connections. This subspace coincides with the moduli space of vacua protected by supersymmetry (or, equivalently, which are cohomologically trivial); for the Donaldson-Witten theory, this is famously the zero locus of the section $F^+ = \frac{1}{2}(1+\star)F$, i.e. it is the (finite-dimensional) moduli space of instantons, which are anti-self-dual (ASD) connections on the gauge bundle. Hence, the topologically twisted theory provides a field theoretic description of Donaldson invariants [399]. The finite dimensionality of the moduli space of instantons is related to the ellipticity of the problem $F^+ = 0$.

We remind the reader that the topological twist rearranges the flat-space supersymmetries into representations of the twisted symmetry group. Under this effect, the Donaldson-Witten supercharge becomes scalar and the twist can subsequently be defined on any smooth manifold, whether a spin structure exists or not. As such, one is free to evaluate the partition function of the twisted theory on any background $\mathcal{M}_4$, however, in anticipation of our interpolation with the Coulomb-branch index, we wish to consider its partition function on $S^3 \times S^1_\beta$. By definition, this partition function will perform a counting of twisted operators, and as such, will return a number. It is worthwhile to note that this twisted partition function is equivalent to the original theory's partition function on that same geometric background, in the presence of the $SU(2)_R$ background gauge field. To illustrate this further, we write

$$\mathcal{I}_{\text{twisted}} = Z[\Psi_{\text{twist}}] := \int \mathcal{D}\phi\, e^{-S[\Psi_{\text{twist}},\phi]}, \tag{6.15}$$

where Ψ_{twist} denotes the set of background supergravity fields that engender the twist. Amongst these is the previously-mentioned $SU(2)_R$ gauge field, tuned to its configuration in equation (6.13). The value of all other auxiliary fields of the $\mathcal{N}=2$

Weyl multiplet are then determined from the generalised Killing spinor equations; the evaluation of which we move on to now.

Consistency of the bosonic nature of the fixed point given by the R-symmetry background in equation (6.13) entails the vanishing of the supersymmetric variation of the dilatino in equation (3.9b); evaluated on the topologically twisted background, this reads

$$\delta_\xi \chi^i = \frac{1}{6} R_{\mu\nu}^{SU(2)_R}{}^i{}_j \gamma^{\mu\nu} \xi^j + D\xi^i \ . \tag{6.16}$$

The identification in equation (6.13) of the $SU(2)_R$ R-symmetry vector bundle and the negative chirality spinor bundle, and their corresponding connections, implies that

$$R_{\mu\nu}^{SU(2)_R}{}^i{}_j = \frac{1}{2} R(\omega)_{\mu\nu}{}^{ab} \sigma_{ab}{}^i{}_j \tag{6.17}$$

where the curvature of the spin connection satisfies

$$R(\omega)_{ab}{}^{ab} = R \ , \tag{6.18}$$

where R is the Ricci scalar. Note that, in our conventions, spaces of constant positive curvature, such as the round 3-sphere, have positive Ricci scalar. The vanishing of $\delta_\xi \chi^i$ in equation (6.16) against either of the generalised Killing spinors $\xi_\pm$ therefore necessitates the following background for the auxiliary scalar D [400],

$$D = -\frac{R}{6} \ . \tag{6.19}$$

6.4 Current multiplet transformations

In this section we will give an important result in conformal supergravity coupled to any 4d $\mathcal{N} = 2$ SCFT. In general, the supersymmetric variation of the current multiplet is theory dependent and may be rather complicated. Here, we propose a formula that relates it to the supersymmetric variation of the background supergravity fields instead, allowing for a theory-agnostic formulation of these variations.

Let us denote by $S[\Psi, \phi]$ the action of a 4d $\mathcal{N} = 2$ SCFT coupled to conformal supergravity. We choose to collectively denote all the Weyl multiplet fields by $\Psi = \{e_\mu{}^a, \psi_\mu{}^i, b_\mu, A_\mu^{U(1)_R}, A_\mu^{SU(2)_R i}{}_j, T_{ab}, \chi^i, D\}$, and all the SCFT field by ϕ. Note that, by consequence of this coupling, the action $S[\Psi, \phi]$ is invariant under all super-diffeomorphisms and super-conformal transformations; said otherwise, it is invariant under all supersymmetry transformations with arbitrary (Q- and S-)spinor parameters.

Of importance here is the existence of a supersymmetric fixed point of the background, denoted Ψ_0. By definition, if Ψ_0 exists there also exists a subset of all possible supersymmetry transformations which preserves this background. If we denote by ξ and η, respectively, the Q- and S-supersymmetry parameters which generate this subset, then the fixed point satisfies $\delta_{\xi,\eta}\Psi_0 = 0$. Given a particular choice of fixed point Ψ_0, let us consider an expansion of the supergravity fields around it. In other words, we write $\Psi = \Psi_0 + t\Delta\Psi$. The parameter t is a dimensionless quantity which we will use to quantify deviations away from the fixed point. Assuming sufficient smoothness of the action, it may also be expanded in powers of t,

$$\begin{aligned} S[\Psi,\phi] &= S[\Psi_0,\phi] + t \sum_{\Psi\in\text{Weyl}} \left\langle \Delta\Psi, \left.\frac{\delta S[\Psi,\phi]}{\delta\Psi}\right|_{\Psi=\Psi_0} \right\rangle + \mathcal{O}(t^2) \\ &= S[\Psi_0,\phi] + t \sum_{\Psi\in\text{Weyl}} \langle \Delta\Psi, J_\Psi[\Psi_0,\phi]\rangle + \mathcal{O}(t^2), \end{aligned} \tag{6.20}$$

where $\langle\cdot,\cdot\rangle$ denotes a suitable contraction of indices followed by integration, and where we defined the current multiplet

$$J_\Psi = \left.\frac{\delta S[\Psi,\phi]}{\delta\Psi}\right|_{t=0} . \tag{6.21}$$

We recognise here the action coupled to background rigid supergravity at the zero$^{\text{th}}$ order and the linear coupling between the (shifted) Weyl multiplet fields, $\Delta\Psi$, and the current multiplet fields, J_Ψ, at the first order in t. Importantly, our definition of the currents deviates from the standard definition by also including the volume factor. We further emphasise that these currents are functions of the fixed point, Ψ_0, and the SCFT fields, ϕ, only.

As equation (6.20) is a simple rewriting of the full action $S[\Psi,\phi]$, it is still invariant under all supersymmetry transformations. In particular, it is invariant under the infinitesimal transformations which leave the fixed point, Ψ_0, invariant. Again, we use $\delta_{\xi,\eta}$ to denote this action on the fields. With our particular choice of conformal supergravity, $\delta_{\xi,\eta}$ splits into two actions on the supergravity fields — one which preserves the powers of t and one which increases the powers of t by one.[5] With that in mind we can now take the supersymmetric variation of equation (6.20) and collect terms by powers of t. Since this supersymmetric action is a symmetry of both the full action $S[\Psi,\phi]$ and the action at the fixed point $S[\Psi_0,\phi]$, the resulting equation has no zero-order term. For our current investigation, it will be sufficient to look at the linear

[5]This split of $\delta_{\xi,\eta}$ can simply be derived from the Q- and S-susy transformations of the Weyl multiplet. The right hand side being at most linear in the multiplet fields, any t-expansion will result in a t-independent term and a linear term in t.

term only,

$$\sum_{\Psi\in\text{Weyl}} \left(\langle\delta_{\xi,\eta}\Delta\Psi, J_\Psi\rangle + \langle\Delta\Psi, \delta_{\xi,\eta}J_\Psi\rangle\right)\Big|_{t=0} = 0 \ . \tag{6.22}$$

From that expression, taking a function derivative gives us a formula for the supersymmetric variation of the current multiplet in terms of that of the Weyl multiplet, at the bosonic fixed point,

$$\delta_{\xi,\eta}J_\Phi = -\frac{\delta}{\delta\Phi}\sum_{\Psi\in\text{Weyl}} \langle\delta_{\xi,\eta}\Psi, J_\Psi\rangle\Big|_{t=0} \ , \tag{6.23}$$

which in our case expands to

Theorem 6.2.

$$\delta_\xi J_\Phi = -\sum_{\Psi\in\,Weyl}\left(\frac{\partial(\delta_\xi\Psi)}{\partial\Phi}J_\Psi - \partial_\mu\left(\frac{\partial(\delta_\xi\Psi)}{\partial(\partial_\mu\Phi)}J_\Psi\right)\right) \ . \tag{6.24}$$

Notice how the right-hand-side only involves supersymmetric variations of the background Weyl multiplet. It is thanks to that fact that we are able to prove exactness of our interpolation, regardless of the SCFT chosen.

6.5 Interpolating index

Having introduced both the Coulomb-branch index and the twisted index in the previous sections, we now construct a novel one-parameter bosonic background of 4d $\mathcal{N}=2$ Euclidean conformal supergravity which smoothly interpolates between the two, and which is supersymmetric at all points in the interpolation. Furthermore, we show that, given a generic 4d $\mathcal{N}=2$ superconformal field theory coupled to this background, the dependence of its action on the interpolation parameter is exact in a supersymmetric sense. Therefore, all of the *a priori* distinct partition functions and indices obtained at each point in the interpolation are actually equivalent. In particular, the Coulomb-branch index coincides with the twisted index, as expected by their common defining supercharge.

6.5.1 Interpolating background

We briefly recall that the rigid supergravity backgrounds introduced in sections 6.2 and 6.3 allowed us to identify the corresponding Euclidean partition functions with a particular limit of the superconformal index in the former case, and with the topologically twisted index in the latter. The former is in part introduced via the geometric background $S^3 \times S^1_\beta$. These supergravity configurations correspond to particular supersymmetric fixed points of Euclidean 4d $\mathcal{N} = 2$ conformal supergravity. As reviewed above, a necessary condition for supersymmetry to be preserved by the 4d theory coupled to the off-shell Weyl multiplet is for a subset of the spinors, namely those generating Q- and S-type supersymmetries, to solve the generalised Killing spinor equations, (3.9a) and (3.9b).

In particular, we constructed a particular point in the Coulomb-branch index moduli, which corresponds to the following bosonic background of conformal supergravity,

$$\begin{array}{lll} g = g_{S^3\times S^1_\beta} & D = 0 & A^{U(1)_r} = -\beta dt \\ T^- = 0 & T^+ = 0 & A^{SU(2)_R} = 0 \ , \end{array} \tag{6.25}$$

where we recall that the metric $g_{S^3\times S^1_\beta}$ is defined in equation (6.3).

While it may seem unnecessary to specify the fields T and D at this stage, their inclusion will ease the comparison with the field content of the twisted index background. Additionally, it will become useful to reiterate now the particular form of the Killing spinors in equation (6.12), which are given by $\xi_\pm = c^1_\pm \sigma_1 + c^2_\pm \sigma_2$. In particular, the Killing spinor $(\xi_+, \xi_-) = (0, \sigma_2)$ given by $c^{1,2}_+ = c^1_- = 0$ and $c^2_- = 1$ will play a prominent role in the ensuing discussion.

The twisted index is associated with non-trivial backgrounds for the auxiliary scalar field D and the $SU(2)_R$ gauge field required by the topological twisting procedure, notably. Summarising the results from that analysis here provides us with the following bosonic fixed point,

$$\begin{array}{lll} g = g_{S^3\times S^1_\beta} & D = -\frac{1}{6}R & A^{U(1)_r} = 0 \\ T^- = 0 & T^+ = 0 & A^{SU(2)_R} = \frac{1}{2}\omega^{ab}\sigma_{ab} \ . \end{array} \tag{6.26}$$

The corresponding constant Killing spinors are the two chiral spinors $\xi_\pm = c_\pm \sigma_2$.

We can observe that both supergravity backgrounds above preserve two common Killing spinors, namely, $\xi_\pm = \sigma_2$. It is then interesting to ask whether there exists a conformal supergravity background which interpolates between the Coulomb-branch index and twisted indices, while preserving those same Killing spinors throughout the interpolation. We now show that this is indeed the case, by constructing the explicit interpolation. Indeed, the configuration

$$\begin{array}{lll} g = g_\Omega & D = \dfrac{u(u-2)}{6}R & A^{U(1)_r} = \beta(u-1)dt \\ T^- = 0 & T^+ = 0 & A^{SU(2)_R} = \dfrac{u}{2}\omega^{ab}\sigma_{ab} \ . \end{array} \tag{6.27}$$

with interpolating parameter u, connects the superconformal point (at $u = 0$) and the twisted one (at $u = 1$) in a smooth fashion. One can verify that the chiral spinors

$$\xi_{\pm} = c_{\pm}\sigma_2 \tag{6.28}$$

are indeed (generalised) Killing for all $u \in [0, 1]$. Their corresponding conformal supersymmetry parameters η are

$$\eta_{\pm} = c_{\mp}(1-u)\sigma_2 \ . \tag{6.29}$$

One can also check that, against the background of equation (6.27), the supersymmetry variation of the dilatino in equation (3.9b) vanishes. From this point onward, we will refer to the background configuration in equation (6.27), together with the supersymmetry parameters ξ and η in equations (6.28) and (6.29), as the interpolating background, and to the partition function evaluated on it,

$$\mathcal{I}(u) := \int \mathcal{D}\phi e^{-S[\Psi(u),\phi]} \ , \tag{6.30}$$

as the interpolating index, where ϕ denotes the field content of a generic 4d $\mathcal{N} = 2$ superconformal field theory.

6.5.2 Supersymmetric exactness of the interpolation

In the previous subsection, we showed how the interpolating background gives rise to a continuous line of indices $\mathcal{I}(u)$, which *a priori* are distinct for all values of the interpolating parameter u. We now argue that $\mathcal{I}(u)$ not only provides a smooth interpolation between the Coulomb-branch index at $u = 0$ and the topologically twisted index at $u = 1$, but in fact manifests their equivalence.

In particular, we contend that the dependence of the coupled action $S[\Psi(u), \phi]$ on the interpolating parameter u is exact in a supersymmetric sense:

$$\frac{dS}{du} = \delta_{\xi}\mathcal{O} \ , \tag{6.31}$$

where ξ and $\mathcal{O}$ are a Killing spinor and an otherwise generic gauge-invariant operator, respectively. We emphasise that their explicit expressions are unimportant. Indeed, the bare existence of data $\{\xi, \mathcal{O}\}$ satisfying equation (6.31) suffices to imply the

Weyl multiplet field Ψ	$\frac{\delta\delta_\xi\Psi}{\delta\chi^i}$	$\frac{\delta\delta_\xi\Psi}{\delta\partial_\mu\chi^i}$
$e_\mu{}^a$	0	0
b_μ	$-\frac{1}{2}\bar{\xi}_i\gamma^5\gamma_\mu$	0
$A_\mu^{U(1)_r}$	$-\bar{\xi}_i\gamma_\mu$	0
$A_\mu^{SU(2)_R}{}^j{}_k$	$-2\bar{\xi}_k\gamma^5\gamma_\mu\delta^j{}_i + \delta^j{}_k\bar{\xi}_i\gamma^5\gamma_\mu$	0
T_{ab}	$4i\bar{\xi}_i\gamma^5\gamma_{ab}$	0
D	$\bar{\xi}_i\gamma^5\gamma^\mu\left(\frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}A_\mu^{U(1)_r}\gamma^5\right) + \frac{1}{2}\bar{\xi}_j\gamma^5\gamma^\mu\ A_\mu^{SU(2)_R}{}^j{}_i$	$\bar{\xi}_i\gamma^5\gamma^\mu$

TABLE 6.1: The functional dependence of the supersymmetric variations of the fields in the Weyl multiplet on the dilatino χ and $\partial_\mu\chi$.

independence of the interpolating index $\mathcal{I}(u)$ on the parameter u:

$$\begin{aligned}\frac{d\mathcal{I}}{du} &= \frac{d}{du}\int \mathcal{D}\phi\ e^{-S[\Psi(u),\phi]} \\ &= -\int \mathcal{D}\phi\ e^{-S[\Psi(u),\phi]}\delta_\xi\mathcal{O} \\ &= -\int \mathcal{D}\phi\ \delta_\xi\left(e^{-S[\Psi(u),\phi]}\mathcal{O}\right) \\ &= 0\ ,\end{aligned} \tag{6.32}$$

where in the second line we employed equation (6.31), while the third line follows from the assumption that ξ is a Killing spinor for the coupled action $S[\Psi(u), \phi]$. Therefore, if equation (6.31) holds, the indices $\mathcal{I}(u)$ coincide for all u, including in particular the Coulomb-branch index and the topologically twisted index.

Our precise claim, whose proof will preoccupy us for the remainder of this section, is that equation (6.31) is satisfied by the spinor in equation (6.28),

$$\begin{pmatrix}\xi_+ \\ \xi_-\end{pmatrix} = \begin{pmatrix}c_+\sigma_2 \\ c_-\sigma_2\end{pmatrix}\ , \tag{6.33}$$

which we recall is Killing for all $u \in [0, 1]$, and by the following gauge-invariant operator,

$$\mathcal{O} = \int_{S^3\times S^1_\beta}\left(\bar{J}_{\chi i}\zeta^i + \bar{J}^\mu_{\psi i}\tilde{\zeta}^i_\mu\right)\ , \tag{6.34}$$

where J_χ and J_ψ are the supercurrent multiplet fields corresponding to the dilatino χ and the gravitino ψ, respectively, while ζ and $\tilde{\zeta}^\mu$ are, for now, arbitrary spinors. The

Weyl multiplet field Ψ	$\frac{\delta\delta_\xi\Psi}{\delta\psi_\nu{}^i}$	$\frac{\delta\delta_\xi\Psi}{\delta\partial_\tau\psi_\nu{}^i}$
$e_\mu{}^a$	$\bar\xi_i\gamma^5\gamma^a\delta^\nu_\mu$	0
b_μ	$\frac{i}{2}\bar\xi_j\gamma^5\frac{\delta\phi_\mu{}^j}{\delta\psi_\nu{}^i}+\frac{i}{2}\bar\eta_i\gamma^5\delta^\nu_\mu$	$\frac{i}{2}\bar\xi_j\gamma^5\frac{\delta\phi_\mu{}^j}{\delta\partial_\tau\psi_\nu{}^i}$
$A^{U(1)_r}_\mu$	$-\frac{i}{2}\bar\xi_j\frac{\delta\phi_\mu{}^j}{\delta\psi_\nu{}^i}-\frac{i}{2}\bar\eta_i\delta^\nu_\mu$	$-\frac{i}{2}\bar\xi_j\frac{\delta\phi_\mu{}^j}{\delta\partial_\tau\psi_\nu{}^i}$
$A^{SU(2)_R j}_\mu{}_k$	$2i\bar\xi_k\gamma^5\frac{\delta\phi_\mu{}^j}{\delta\psi_\nu{}^i}-2i\bar\eta_k\gamma^5\delta^\nu_\mu\delta^j_i-SU(2)_R$ trace	$2i\bar\xi_k\gamma^5\frac{\delta\phi_\mu{}^j}{\delta\partial_\tau\psi_\nu{}^i}-SU(2)_R$ trace
T_{ab}	$-8i\bar\xi_j\gamma^5\frac{\delta R(Q)_{ab}{}^j}{\delta\psi_\nu{}^i}$	$-8i\bar\xi_j\gamma^5\frac{\delta R(Q)_{ab}{}^j}{\delta\partial_\tau\psi_\nu{}^i}$
D	0	0
where		
$\frac{\delta\phi_\mu{}^j}{\delta\psi_\nu{}^i}=-\frac{i}{4}\left(\gamma^{\rho\nu}\gamma_\mu-\frac{1}{3}\gamma_\mu\gamma^{\rho\nu}\right)\left[\left(\frac{1}{2}\omega_\rho{}^{ab}\gamma_{ab}+b_\rho+A^{U(1)_r}_\rho\gamma^5+\frac{i}{16}T_{ab}\gamma^{ab}\gamma_\rho\right)\delta^j_i+A^{SU(2)_R j}_\rho{}_i\right]$		
$\frac{\delta\phi_\mu{}^j}{\delta\partial_\tau\psi_\nu{}^i}=-\frac{i}{2}\left(\gamma^{\tau\nu}\gamma_\mu-\frac{1}{3}\gamma_\mu\gamma^{\tau\nu}\right)\delta^j_i$		
$\frac{\delta R(Q)_{\mu\rho}{}^j}{\delta\psi_\nu{}^i}=\left[\left(\frac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}+b_{[\mu}+A^{U(1)_r}_{[\mu}\gamma^5\right)\delta^j_i+A^{SU(2)_R j}_{[\mu}{}_i\right]\delta^\nu_{\rho]}-i\gamma_{[\mu}\frac{\delta\phi_{\rho]}{}^j}{\delta\psi_\nu{}^i}+\frac{i}{16}T_{ab}\gamma^{ab}\gamma_{[\mu}\delta^\nu_{\rho]}\delta^j_i$		
$\frac{\delta R(Q)_{\mu\rho}{}^j}{\delta\partial_\tau\psi_\nu{}^i}=2\delta^\tau_{[\mu}\delta^\nu_{\rho]}\delta^j_i-i\gamma_{[\mu}\frac{\delta\phi_{\rho]}{}^j}{\delta\partial_\tau\psi_\nu{}^i}$		

TABLE 6.2: The functional dependence of the supersymmetric variations of the fields in the Weyl multiplet on the gravitino ψ and $\partial_\mu\psi$.

claim is then that the supersymmetric variation $\delta_\xi\mathcal{O}$ of the operator above coincides with

$$\begin{aligned}\frac{dS}{du}&=\sum_{\Psi\in\text{Weyl}}\left\langle\frac{d\Psi}{du},\hat\Psi\right\rangle\\&=\left\langle\frac{de}{du},J_e\right\rangle+\left\langle\frac{db}{du},J_b\right\rangle+\left\langle\frac{dA^{U(1)_R}}{du},J^{U(1)_R}\right\rangle+\left\langle\frac{dA^{SU(2)_R}}{du},J^{SU(2)_R}\right\rangle+\\&\quad+\left\langle\frac{dT}{du},J_T\right\rangle+\left\langle\frac{dD}{du},J_D\right\rangle\,.\end{aligned}\tag{6.35a}$$

The computation of $\delta_\xi\mathcal{O}$ amounts to determining the supersymmetric variations of $\bar J_\chi$ and $\bar J_\psi$. Using the Q- and S-supersymmetry transformations of the 4d $\mathcal{N}=2$ Euclidean Weyl multiplet presented in equation (3.14), we find the results collated in Tables 6.1 and 6.2. Leveraging the argument in equation (6.24), the supersymmetric variation of

$\bar{J}_\chi$ is given by

$$\begin{aligned}\delta_\xi \bar{J}_{\chi i} = & \frac{1}{2}\bar{\xi}_i\gamma^5\gamma_\mu J_b^\mu + \bar{\xi}_i\gamma^\mu J_\mu^{U(1)_r} + 2\bar{\xi}_k\gamma^5\gamma^\mu J_\mu^{SU(2)_R}{}^k{}_i - \bar{\xi}_i\gamma^5\gamma^\mu J_\mu^{SU(2)_R}{}^k{}_k - 4i\bar{\xi}_i\gamma^5\gamma_{\mu\nu}J_T^{\mu\nu} \\ & + \Bigg[\underbrace{\bar{\xi}_i\gamma^5\gamma^\mu\left(\overleftarrow{\partial}_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{1}{2}A_\mu^{U(1)_r}\gamma^5\right) - \frac{1}{2}\bar{\xi}_j\gamma^5\gamma^\mu A_\mu^{SU(2)_R}{}^j{}_i}_{=\,2i\bar{\eta}_i\gamma^5} \Bigg] J_D \\ & + \bar{\xi}_i\gamma^5\gamma^\mu\partial_\mu J_D \ , \end{aligned} \tag{6.36}$$

where we exploited the (Dirac conjugate of the) Killing spinor equation to exchange differential terms in $\bar{\xi}$ with an algebraic one in $\bar{\eta}$. Equation (6.24) similarly determines the supersymmetric variation of the source field $\bar{J}_\psi$ to be

$$\begin{aligned}\delta_\xi \bar{J}_{\psi\mu i} = & -\bar{\xi}_i\gamma^5\gamma_a J_{e\mu}{}^a - \frac{i}{2}\bar{\xi}_j\gamma^5\frac{\delta\phi^{\nu j}}{\delta\psi^{\mu i}}J_{b\nu} - \frac{i}{2}\bar{\eta}_i\gamma^5 J_{b\mu} + \frac{i}{2}\bar{\xi}_j\frac{\delta\phi^{\nu j}}{\delta\psi^{\mu i}}J_\nu^{U(1)_r} + \frac{i}{2}\bar{\eta}_i J_\mu^{U(1)_r} + \\ & - 2i\bar{\xi}_k\gamma^5\frac{\delta\phi^{\nu j}}{\delta\psi^{\mu i}}J_\nu^{SU(2)_R k}{}_j + i\bar{\xi}_k\gamma^5\frac{\delta\phi^{\nu k}}{\delta\psi^{\mu i}}J_\nu^{SU(2)_R j}{}_j + 2i\bar{\eta}_k\gamma^5 J_\mu^{SU(2)_R k}{}_i - i\bar{\eta}_i\gamma^5 J_\mu^{SU(2)_R k}{}_k \\ & + 8i\bar{\xi}_j\gamma^5\frac{\delta R(Q)_{ab}{}^j}{\delta\psi^{\mu i}}J_T^{ab} + \frac{i}{2}\partial_\nu\left(\bar{\xi}_j\gamma^5\frac{\partial\phi^{\rho j}}{\partial\partial_\nu\psi^{\mu i}}J_{b\rho}\right) - \frac{i}{2}\partial_\nu\left(\bar{\xi}_j\frac{\partial\phi^{\rho j}}{\partial\partial_\nu\psi^{\mu i}}J^{U(1)_r}{}_\rho\right) \\ & + 2i\partial_\nu\left(\bar{\xi}_k\gamma^5\frac{\partial\phi^{\rho j}}{\partial\partial_\nu\psi^{\mu i}}J_\rho^{SU(2)_R k}{}_j\right) - i\partial_\nu\left(\bar{\xi}_j\gamma^5\frac{\partial\phi^{\rho j}}{\partial\partial_\nu\psi^{\mu i}}J_\rho^{SU(2)_R k}{}_k\right) \\ & - 8i\partial_\nu\left(\bar{\xi}_j\gamma^5\frac{\partial R(Q)_{ab}{}^j}{\partial\partial_\nu\psi^{\mu i}}J_T^{ab}\right) \end{aligned} \tag{6.37}$$

The proof is then complete by finding spinors ζ and $\tilde{\zeta}_\mu$ such that the supersymmetric variation of $\mathcal{O}$ with parameter ξ recovers the u-dependence of the coupled action; a sufficient set of conditions which can be imposed upon the spinors ζ and $\tilde{\zeta}_\mu$ for this to occur is

$$\frac{d}{du}e_\mu{}^a = -\bar{\xi}_i\gamma^5\gamma^a\tilde{\zeta}_\mu{}^i \ , \tag{6.38a}$$

$$\frac{d}{du}b_\mu = \frac{1}{2}\bar{\xi}_i\gamma^5\gamma_\mu\zeta^i - \frac{i}{2}\bar{\xi}_j\gamma^5\frac{\partial\phi_\mu{}^j}{\partial\psi_\nu{}^i}\tilde{\zeta}_\nu{}^i - \frac{i}{2}\bar{\eta}_i\gamma^5\tilde{\zeta}_\mu{}^i - \frac{i}{2}\bar{\xi}_j\gamma^5\frac{\partial\phi_\mu{}^j}{\partial\partial_\nu\psi_\rho{}^i}\partial_\nu\tilde{\zeta}_\rho{}^i \ , \tag{6.38b}$$

$$\frac{d}{du}A_\mu^{U(1)_r} = \bar{\xi}_i\gamma_\mu\zeta^i + \frac{i}{2}\bar{\xi}_j\frac{\partial\phi_\mu{}^j}{\partial\psi_\nu{}^i}\tilde{\zeta}_\nu{}^i + \frac{i}{2}\bar{\eta}_i\tilde{\zeta}_\mu{}^i + \frac{i}{2}\bar{\xi}_j\frac{\partial\phi_\mu{}^j}{\partial\partial_\nu\psi_\rho{}^i}\partial_\nu\tilde{\zeta}_\rho{}^i \ , \tag{6.38c}$$

$$\begin{aligned}\frac{d}{du}A_\mu^{SU(2)_R i}{}_j = & 2\bar{\xi}_j\gamma^5\gamma_\mu\zeta^i - 2i\bar{\xi}_j\gamma^5\frac{\partial\phi_\mu{}^i}{\partial\psi_\nu{}^k}\tilde{\zeta}_\nu{}^k + 2i\bar{\eta}_j\gamma^5\tilde{\zeta}_\mu{}^i + \\ & - 2i\bar{\xi}_j\gamma^5\frac{\partial\phi_\mu{}^i}{\partial\partial_\nu\psi_\rho{}^k}\partial_\nu\tilde{\zeta}_\rho{}^k - (SU(2)_R \text{ trace}), \end{aligned} \tag{6.38d}$$

$$\frac{d}{du}T_{ab} = -4i\bar{\xi}_i\gamma^5\gamma_{ab}\zeta^i + 8i\bar{\xi}_j\gamma^5\frac{\partial R(Q)_{ab}{}^j}{\partial\psi_\mu{}^i}\tilde{\zeta}_\mu{}^i + 8i\bar{\xi}_j\gamma^5\frac{\partial R(Q)_{ab}{}^j}{\partial\partial_\mu\psi_\nu{}^i}\partial_\mu\tilde{\zeta}_\nu{}^i \tag{6.38e}$$

$$\frac{d}{du}D = 2i\bar{\eta}_i\gamma^5\zeta^i, \tag{6.38f}$$

all of which are to be evaluated against the interpolating background of equation (6.27).

While seemingly overconstrained, the conditions above in fact admit a simultaneous solution

$$\zeta = \begin{pmatrix} -\frac{i}{4c_-}\sigma_2 \\ \frac{i}{4c_+}\sigma_2 \end{pmatrix}, \qquad \tilde{\zeta}_\mu = c_\mu \begin{pmatrix} -\sigma_2 \\ \frac{c_-}{c_+}\sigma_2 \end{pmatrix}. \tag{6.39}$$

The existence of these spinors satisfying equation (6.38) completes the proof of our claim in equation (6.31). We conclude that the Coulomb-branch index (at a point of its moduli) and the topologically twisted index, as well as the indices at all other values of the interpolating parameter u, coincide.

6.6 Conclusion/Remarks

Let us take this opportunity to pause and look back at what was achieved within this chapter, and postulate on the possible extensions and implications our work brings forth.

Under the formalism of Festuccia and Seiberg, we considered arbitrary 4d $\mathcal{N} = 2$ SCFTs coupled to Euclidean conformal supergravity. Via the construction of explicit bosonic fixed points of the background supersymmetry, we recovered the well-established configurations that engineer the Coulomb-branch and twisted indices. These are both given as partition functions of the theory on $S^3 \times S^1$, with a given subset of background supergravity fields turned on.[6]

We then constructed a supersymmetric configuration that interpolates between the twisted index, $\mathcal{I}_{\text{twisted}}$, and a point on the Coulomb-branch index, $\mathcal{I}_{\text{CB}}(\rho_0, \sigma_0)$. The interpolation relies on a dimensionless parameter u, and so does the partition function of a theory on said background, which we coined the *interpolating index*, $\mathcal{I}(u)$. Having shown that the background preserves two chiral supercharges, we moved on to proving the main result, namely that the interpolation is exact. This exactness translates into the fact that the u-variation of the action is a supersymmetry-exact quantity, written as $\delta_{\xi,\eta}\mathcal{O}$, for some operator $\mathcal{O}$; and consequently the interpolation is shown to be independent of u. Proving this for arbitrary 4d $\mathcal{N} = 2$ SCFT is reliant on our intermediary result in equation (6.24), thanks to which we are able to recast the supersymmetry variation of arbitrary current multiplets into that of background supergravity fields alone. These results, in turn, assure us that for any 4d $\mathcal{N} = 2$ SCFT, regardless of whether it admits a Lagrangian description or not, the twisted index is equivalent to the Coulomb-branch index at a given point in its moduli space. Furthermore, the whole interval of indices, $\{\mathcal{I}(u)\}_{u\in\mathbb{R}}$, is also equal to the twisted index.

[6] Truthfully, the former partition function equates to the Coulomb-branch index, multiplied by a supersymmetric Casimir energy [53].

One may point to the fact that both the Coulomb-branch index and the twisted index are defined from the same supercharge — the Donaldson-Witten supercharge — and, as such, a relation between the two is bound to appear. However, our presentation shows this explicitly, even allowing one to find the precise point on the moduli of the Coulomb-branch for which this is true.

It would be interesting to understand this relation further; demonstrably, a precise understanding of the specific point in the Coulomb-branch index moduli, (ρ_0, σ_0), is still lacking. Our work here only presents this relation from the background supergravity perspective; it would be instructive to understand it from the field theory side. Nevertheless, our work unearthed a relation between the counting of operators in a given theory — on one side, operators on the Coulomb-branch are counted with $U(1)_R$ and $SU(2)_R$ charge-dependent weight; on the other side, the twisted index counts operators in the cohomology of the Donaldson-Witten supercharge — where both countings are equal. Given the theory-agnostic derivation, we are convinced that this demonstrates a deeper property of 4d $\mathcal{N}=2$ theories.

Why restrict to one point on the space of CB indices?

It is known that the twisted index of 4d $\mathcal{N}=2$ admits a refinement, whereby operators in the cohomology of the Donaldson-Witten supercharge are graded with respect to the two $\mathfrak{su}(2)$ subalgebras of the flat space isometry $\mathfrak{so}(4)$. This refined index is known as the equivariant Donaldson-Witten index, and depends on the two Ω-deformation parameters $\epsilon_{1,2}$ (which are valued in the Cartan subalgebras of each $\mathfrak{su}(2)$ [397, 401]). Following our presentation in this chapter, one can easily show that most results hold when replacing the product manifold $S^3 \times S^1$ by the metric fibration $S^3 \times_\Omega S^1$, whose line element reads[7]

$$g_\Omega = d\theta^2 + \sin^2(\theta)(d\varphi + \epsilon_1 dt)^2 + \cos^2(\theta)(d\tau + \epsilon_2 dt)^2 + \beta^2 dt^2 \ . \tag{6.40}$$

On the Coulomb-branch side of the interpolation, this engineers the full moduli of indices, where the standard parameters ρ and σ are recast into $\epsilon_{1,2}$.[8] As stated above, one can show that this background is supersymmetric — it preserves one supercharge. However, we were unable to find a pair of spinors ζ^i, $\tilde{\zeta}^i_\mu$ such that the u-variation of the action can be written in terms of these (see equation (6.31)). In other words, we were unable to show that the interpolation is exact.

[7]Further adjustments must be made to the background supergravity fields on the twisted side to accommodate for the equivariance (any continuous geometric deformation cannot affect a topological index). Notably, one must turn on the background two-form as follows $T^- = 4u(dv^\#)^-$, where $v = \epsilon_1\partial_\varphi + \epsilon_2\partial_\tau$.

[8]Truthfully, this only engineers a subspace of the moduli where $\epsilon_{1,2}$ are real. A proper treatment of complex $\epsilon_{1,2}$ would require squashing the spheres.

One might wonder why bother bringing this up, if the interpolation may be doomed to fail. The answer to this query lies in two key observations:

- The interpolation would have been exact, had the b-field constraint in equation (6.38b) been ignored.
- Both indices now exhibit a two-dimensional moduli. Given that they match at one point, one could expect them to match at every point therein.

The first observation above is crucial. Indeed, most applications of 4d conformal supergravity utilise K-gauge, thanks to which the b-field can be completely ignored (i.e. set to zero). If we naively apply this gauge before establishing our interpolating conditions, both indices are found to be equal; a strong showing for the subtlety this gauge choice brings forth.

Our failure in finding spinors that solve the interpolating equations (6.38), naturally, doesn't prevent a solution from existing. It would be interesting to see if such a solution can be unearthed. The most interesting reason to the author being the holographic interpretation it would lead to. Indeed, for holographic theories, both indices have known AdS completions: a Kerr-Newman AdS_5 black hole in the superconformal index case [402, 403] and a magnetically charged one in the twisted case [30]. An equality of these indices at the CFT level would imply a relation between their black holes microstate counting.

Chapter 7

The $\mathcal{N} = 4$ Ω-deformations

K'i sayè métre o aprenti, on crwé ouvri a crwé outi.
Whether master or an apprentice, a bad worker has bad tools.

Savoie proverb

The present chapter constitutes a departure from the discussions made in Chapter 6, and can be read independently of it.

After looking at supergravity backgrounds that engineer the superconformal and twisted indices in 4d $\mathcal{N} = 2$ SCFTs, we are naturally drawn to doing the same for 4d $\mathcal{N} = 4$ SCFTs. In fact, we focus solely on constructing backgrounds that give the three main twists of $\mathcal{N} = 4$, the half-twist, the Kapustin-Witten twist and the Vafa-Witten twists. These constitute three different ways of getting a topological field theory from an $\mathcal{N} = 4$ SCFT. As a reminder, the twist is built by identifying a subgroup of the R-symmetry with a Lorentz symmetry. With a large-enough R-symmetry group, such as that of 4d $\mathcal{N} = 4$, there are three inequivalent ways of doing this, hence the three twists. This identification can be done by choosing a value for the background R-symmetry gauge field that cancels with the spin connection. Our results are novel in that we are the first to construct these twists from an $\mathcal{N} = 4$ background conformal supergravity perspective, which adheres to the Festuccia-Seiberg paradigm [198] (see Chapter 3 for an overview).

Additionally to constructing the regular twists, we attempt to expand these results by including possible Ω-deformations. This deforms the topological field theory in a way which refines the counting of operators in the partition function. In the supergravity construction, it appears via the inclusion of non-trivial configurations of the two-form T which depend on a conformal Killing vector field v. Subsequently, the Killing spinors will also gain a linear v-dependence, rendering the BPS equations generically non-linear.

We successfully find Ω-deformations for the half- and Kapustin-Witten twists, but are unable to do so for the Vafa-Witten one.

7.1 Introduction

Four dimensional supersymmetric field theories exist with 4 ($\mathcal{N} = 1$), 8 ($\mathcal{N} = 2$), 12 ($\mathcal{N} = 3$) and 16 ($\mathcal{N} = 4$) Poincaré supercharges. Any larger amount of supersymmetry will necessarily lead to the inclusion of fields with spin greater than $3/2$, i.e. gravitational theories. Those with $\mathcal{N} = 4$ supersymmetry are the most restrictive – the only multiplet there is the vector multiplet. The large amount of R-symmetry does offer many avenues in twisting these theories, however. Those with $\mathcal{N} = 2$ have a much richer structure. While a full classification is unknown, many exact results have been found, see for instance the solutions of Seiberg and Witten [404, 405]. Many of these theories do not admit a weak-coupling coupling limit, yet, we are able to construct some of them implicitly from twist-compactifications of the 6d $(2, 0)$ theory. Theories constructed in that way are called class-$\mathcal{S}$ [279]. They also admit a subset of operators, known as the Schur subsector, which are constrained by 'infinite' 2d-symmetries [406]. One would expect that $\mathcal{N} = 3$ theories lie somewhere in the middle, however, they do not admit a Lagrangian description and neither do they admit full topological twists [407].

The topological twist, introduced by Witten [201], is a powerful tool in studying supersymmetric field theories. When the R-symmetry is large enough, such as in the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases, one can identify it with the spin group. This effectively 'twists' the spin representations of the theory, allowing for supersymmetric theories to be defined on any smooth manifold. Furthermore, the set of operators in the cohomology of the supercharge define a field theory that is independent of the metric. Theories with eight supercharges in four dimensions only have one inequivalent topological twist[1], the Donaldson-Witten twist. The construction by Witten identified the partition function of the pure $\mathcal{N} = 2$ vector multiplet theory, topologically twisted, with the generator of Donaldson invariants of the underlying manifold [408]. We now refer to any twisted $\mathcal{N} = 2$ theory as a Donaldson-Witten theory. The partition function of which computes various topological invariants depending on the original field content, see Table 7.1 for a summary.

Theories with sixteen supercharges, on the other hand, have a larger R-symmetry group, $SO(6)$. As a consequence, there are three inequivalent topological twists one can perform. One of them is the $\mathcal{N} = 4$ uplift of the one available in the $\mathcal{N} = 2$ setting, known here as the half-twists [409]. The other two are the Vafa-Witten twist [409, 410] and Kapustin-Witten (or Geometric Langlands) twist [411, 412]. The former is characterised by the presence of two supercharges with identical chirality, while the

[1] We are not counting here the half-twists, for which there are more possibilities.

latter has two of the opposite chirality. We recommend the review [413] for a complete list of available twists in various dimensions.

Donaldson-Witten twist of:	Invariant of $\mathcal{M}_{\mathbf{k}}$
$\mathcal{N}=2$ vector multiplet	Donaldson invariant
$\mathcal{N}=2$ vector multiplet with adjoint hypermultiplets	Euler characteristic
$\mathcal{N}=2$ vector multiplet with fundamental hypermultiplets	Segre numbers
$\mathcal{N}=2$ vector multiplet coupled to gravity	Family Donaldson invariant
$\mathcal{N}=1$ 5d vector multiplet on $X\times S^1$	Holomorphic Euler characteristic
$\mathcal{N}=1$ 6d vector multiplet on $X\times T^2$	Elliptic genus

TABLE 7.1: List of invariants computed by the Donaldson-Witten twist for various $\mathcal{N}=2$ gauge theories. Taken from [6].

In practice, these cohomological partition functions can still be complicated objects to manipulate, as one still needs to perform an infinite dimensional integral over a field space. In the context of four dimensional $\mathcal{N}=2$ theories, Nekrasov showed how one can break this path integral down into integrals over finite-dimensional moduli of instantons by deforming the original theory [397, 401]. This is the celebrated Ω-deformation, which is now used in various dimensions with various amount of supersymmetry. There are many ways of constructing such a deformation. Typically, one starts from an action functional in higher dimensions, say six. There, one places the theory on the product $\mathcal{M}_4\times T^2$, and introduces a geometric twist between the torus and the four-manifold. Compactifying back down to four dimensions gives the Ω-deformed theory of Nekrasov. Another completely geometric construction is that of the flux-trap background in string/M-theory (see for instance [414]). Theories constructed on such backgrounds are known as equivariant versions of the standard twists, or Ω-deformed twists. In the context of Donaldson-Witten theory, we talk about equivariant Donaldson-Witten twist [395, 396, 415, 416]. Using this method [417, 418] constructed an Ω-deformation for all three types of twists of $\mathcal{N}=4$, by starting from a ten-dimensional theory and twist compactifying down to four.

An equivalent way of defining the Ω-background is through the use of background supergravity. There, the higher-dimensional Ω-deformation parameters are remapped to an auxiliary bosonic field of the chosen background supergravity. For theories with eight supercharges, we already constructed such a background in Chapter 6. Here we will aim to do so for the theories with sixteen supercharges. As this work is still ongoing, we will only present the preliminary results gathered so far.

In Section 7.2 we propose a Euclidean formulation of the Weyl multiplet of $\mathcal{N}=4$ conformal supergravity [227] (see Chapter 3). In Section 7.3 we give the corresponding supergravity backgrounds that engineer all three twists of these theories, detailing every time the corresponding supercharges. Our results are summarised in Table 7.4. Finally, in Section 7.4 we attempt to construct those backgrounds that produce an Ω-deformation of each twist. We are successful in doing so for the Kapustin-Witten and half-twists but fail for the Vafa-Witten twist.

7.2 Euclidean Weyl multiplet

In Chapter 3, we gave an account of the Weyl multiplet of $\mathcal{N}=4$ conformal supergravity in Lorentzian signature. Given that we wish to construct background configurations that engineer the various twists of $\mathcal{N}=4$, together with their Ω-deformation, we will need to Euclidean formulation of it. In principle, one could construct it directly with known supergravity methods, however, we will use a simpler trick. As Wick rotations have the effect of complexifying fields and disentangling spinors from their conjugates, one can start from a Lorentzian supergravity and forgo any reality constraints on its fields. This has the desired effect of making spinors and their conjugates independent as well as complexifying its bosonic fields. If the Lorentzian picture contains a complex field, say m, then the Euclidean formulation will spawn two independent complex fields, m and $\bar{m}$.

The Lorentzian Weyl multiplet fields, together with their reality conditions and constraints, are listed in Table 3.3. The main constraint we will be lifting is that given implicitly by the chiral-$SU(N)$ notation. Therein, all fields in a given $SU(4)$ representation have a 'conjugate' field that lives in the conjugate representation. This includes bosonic fields, for which the 'conjugate field' is the complex conjugate. For instance, the auxiliary field E_{ij} has, as its complex conjugate, E^{ij}. Being a complex field from the start (see Table 2 in [271]), we now wish to make E_{ij} and E^{ij} independent.

Let us, thus, proceed with building such a Euclidean Weyl multiplet using the rules specified above. We do point out, however, that we will not seek to complexify the metric or the $SU(4)$ connection. The reason being that we still wish to consider real geometries, as well as their topological twists. Any complexification of those fields would be a departure from that setup. The field content being identical to the Lorentzian case, we will not repeat our presentation of it and instead refer the reader back to Chapter 3 for that. For better readability, we have separated our summary of these Euclidean fields into two tables, one for the fermions (see Table 7.2) and one for the bosons (see Table 7.3). In each, we break down the chiral-$SU(N)$ notation so as to list all independent such fields.

Additionally, under this 'Euclideanisation' procedure the Q- and S-supersymmetry spinors ξ^i and η^i also see themselves disentangled from their chiral-$SU(4)$ counterparts. As such, any Q-supersymmetry transformation will be specified by the spinor ξ^i (resp. ξ_i) for its positive (resp. negative) chirality and any S-supersymmetry transformation

Field	**Constraints**	**Chirality**	**SU(4) Rep**
${\psi_\mu}^i$		$\gamma^5 {\psi_\mu}^i = {\psi_\mu}^i$	$\mathbf{4} = \square$
${\phi_\mu}^i$ Λ^i		$\gamma^5 {\phi_\mu}^i = -{\phi_\mu}^i$ $\gamma^5 \Lambda^i = -\Lambda^i$	
$\psi_{\mu i}$		$\gamma^5 \psi_{\mu i} = -\psi_{\mu i}$	$\bar{\mathbf{4}} = \begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\phi_{\mu i}$ Λ_i		$\gamma^5 \phi_{\mu i} = \phi_{\mu i}$ $\gamma^5 \Lambda_i = \Lambda_i$	
${\chi^{ij}}_k$	${\chi_{ij}}^k = -{\chi_{ji}}^k$, ${\chi_{ij}}^j = 0$	$\gamma^5 {\chi^{ij}}_k = {\chi^{ij}}_k$	$\mathbf{20} = \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$
${\chi_{ij}}^k$	${\chi_{ij}}^k = -{\chi_{ji}}^k$, ${\chi^{ij}}_j = 0$	$\gamma^5 {\chi_{ij}}^k = -{\chi_{ij}}^k$	$\overline{\mathbf{20}} = \begin{smallmatrix}\square\square\\\square\end{smallmatrix}$

TABLE 7.2: List of the fermionic components of the Euclidean $\mathcal{N} = 4$ Weyl multiplet. In the second column, any remaining constraints are detailed. The chirality is given in the third row, while the final row contains their $SU(4)_R$ representation.

will be specified by the spinor η^i (resp. η_i) for its negative (resp. positive) chirality. As there are now twice as many independent spinorial components in the Weyl multiplet, we also see a doubling of the generalised Killing spinor equations. Those of the two chiral parts of the gaugino become

$$\delta_{\xi,\eta} {\psi_\mu}^i = 2D_\mu \xi^i - \frac{1}{2}\gamma^{ab} {T_{ab}}^{ij} \gamma_\mu \xi_j + \epsilon^{ijkl} \bar{\psi}_{\mu j} \xi_k \Lambda_l - \gamma_\mu \eta^i, \tag{7.1a}$$

$$\delta_{\xi,\eta} \psi_{\mu i} = 2D_\mu \xi_i - \frac{1}{2}\gamma^{ab} T_{ab\, ij} \gamma_\mu \xi^j + \epsilon_{ijkl} {\bar{\psi}_\mu}{}^j \xi^k \Lambda^l - \gamma_\mu \eta_i. \tag{7.1b}$$

Those of the gaugino are

$$\delta_{\xi,\eta} \Lambda_i = -2\bar{P}_\mu \gamma^\mu \xi_i + E_{ij} \xi^j + \frac{1}{2}\epsilon_{ijkl} {T_{bc}}^{kl} \gamma^{bc} \xi^j, \tag{7.1c}$$

$$\delta_{\xi,\eta} \Lambda^i = -2P_\mu \gamma^\mu \xi^i + E^{ij} \xi_j + \frac{1}{2}\epsilon^{ijkl} T_{bc\, kl} \gamma^{bc} \xi_j, \tag{7.1d}$$

Field	**Constraints**	**Self-duality**	**SU(4) Rep**
$e^a{}_\mu$			**1**
ω_μ^{ab}	$\omega_\mu^{ab} = -\omega_\mu^{ba}$		**1**
b_μ			**1**
$V_\mu{}^i{}_j$	$(V_\mu{}^i{}_j)^* = -V_\mu{}^j{}_i$, $V_\mu{}^i{}_i = 0$		**15** =
$f^a{}_\mu$	(auxiliary)		**1**
a_μ	(auxiliary)		**1**
T_{abij} $T_{ab}{}^{ij}$	$T_{abij} = -T_{abji}$ $T_{ab}{}^{ij} = -T_{ab}{}^{ji}$	$\star T_{ij} = T_{ij}$ $\star T^{ij} = -T^{ij}$	**6** =
E^{ij}	$E^{ij} = E^{ji}$		**10** =
E_{ij}	$E_{ij} = E_{ji}$		$\overline{\mathbf{10}}$ =
ϕ_α	$(\phi_1)^* = \phi^1$, $(\phi_2)^* = -\phi^2$ $\phi^\alpha\phi_\alpha = 1$		**1**
$D^{ij}{}_{kl}$	$D^{ij}{}_{kl} = -D^{ji}{}_{kl} =$ $-D^{ij}{}_{lk}$		**20′** =

TABLE 7.3: List of the bosonic components of the Euclidean $\mathcal{N}=4$ Weyl multiplet. In the second column, any remaining constraints are detailed. The self-duality is given in the third row, while the final row contains their $SU(4)_R$ representation. Note that the vierbeins e^a, spin connection ω^{ab} and $SU(4)$ connection $V^i{}_j$ can be taken as complex through this procedure. However, we will only focus on backgrounds where they are real.

and finally those due to the dilatino are

$$\begin{aligned}
\delta_{\xi,\eta}\chi^{ij}{}_k = & -\frac{1}{2}\gamma^{ab}D_\mu T_{ab}{}^{ij}\gamma^\mu\xi_k - \gamma^{ab}R(V)_{ab}{}^{[i}{}_k\xi^{j]} - \frac{1}{2}\epsilon^{ijlm}D_\mu E_{kl}\gamma^\mu\xi_m \\
& + D^{ij}{}_{kl}\xi^l - \frac{1}{6}\epsilon_{klmn}E^{l[i}\gamma^{|ab|}(T_{ab}{}^{j]n}\xi^m + T_{ab}{}^{|mn|}\xi^{j]}) \\
& + \frac{1}{2}E_{kl}E^{l[i}\xi^{j]} - \frac{1}{2}\epsilon^{ijlm}\bar{P}_\mu\gamma^\mu\gamma_{ab}T^{ab}{}_{kl}\xi_m - (\text{traces}) \\
& + \frac{1}{2}T_{ab}{}^{ij}\gamma^{ab}\eta_k + \frac{2}{3}\delta^{[i}_k T_{ab}{}^{j]l}\gamma^{ab}\eta_l - \frac{1}{2}\epsilon^{ijlm}E_{kl}\eta_m + (\text{fermions}), \qquad (7.1\text{e}) \\
\delta_{\xi,\eta}\chi_{ij}{}^k = & -\frac{1}{2}\gamma^{ab}D_\mu T_{ab\,ij}\gamma^\mu\xi^k + \gamma^{ab}R(V)_{ab}{}^k{}_{[i}\xi_{j]} - \frac{1}{2}\epsilon_{ijlm}D_\mu E^{kl}\gamma^\mu\xi^m \\
& + D^{kl}{}_{ij}\xi_l - \frac{1}{6}\epsilon^{klmn}E_{l[i}\gamma^{ab}(T_{|ab|\,j]n}\xi_m + T_{|ab\,mn|}\xi_{j]}) \\
& + \frac{1}{2}E^{kl}E_{l[i}\xi_{j]} - \frac{1}{2}\epsilon_{ijlm}P_\mu\gamma^\mu\gamma_{ab}T^{ab\,kl}\xi^m - (\text{traces}) \\
& + \frac{1}{2}T_{ab\,ij}\gamma^{ab}\eta^k + \frac{2}{3}\delta^k_{[i}T_{|ab|\,j]l}\gamma^{ab}\eta^l - \frac{1}{2}\epsilon_{ijlm}E^{kl}\eta^m + (\text{fermions}). \qquad (7.1\text{f})
\end{aligned}$$

For completeness, let us also rewrite the covariant derivates of our spinor parameters, where we only focus on the coupling between them and the bosonic supergravity fields,

$$D_\mu \xi^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}(b_\mu + ia_\mu)\right)\xi^i - V_\mu{}^i{}_j\xi^j, \tag{7.2a}$$

$$D_\mu \xi_i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{1}{2}(b_\mu - ia_\mu)\right)\xi_i + V_\mu{}^j{}_i\xi_j, \tag{7.2b}$$

$$D_\mu \eta^i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{1}{2}(b_\mu - ia_\mu)\right)\eta^i - V_\mu{}^i{}_j\eta^j, \tag{7.2c}$$

$$D_\mu \eta_i = \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{1}{2}(b_\mu + ia_\mu)\right)\eta_i + V_\mu{}^j{}_i\eta_j. \tag{7.2d}$$

Another important set of covariant derivatives, is that of the conjugate fields E^{ij} and T_{ij}. Being in conjugate representations of $SU(4)$, it is straightforward to determine what their covariant derivates should be. Nevertheless, let us write them down explicitly,

$$D_\mu T_{ab}{}^{ij} = \partial_\mu T_{ab}{}^{ij} + \omega_{\mu a}{}^c T_{cb}{}^{ij} + \omega_{\mu b}{}^c T_{ac}{}^{ij} - V_\mu{}^i{}_k T_{ab}{}^{kj} - V_\mu{}^j{}_k T_{ab}{}^{ik}, \tag{7.3a}$$

$$D_\mu T_{ab\,ij} = \partial_\mu T_{ab\,ij} + \omega_{\mu a}{}^c T_{cb\,ij} + \omega_{\mu b}{}^c T_{ac\,ij} + V_\mu{}^k{}_i T_{ab\,kj} + V_\mu{}^k{}_j T_{ab\,ik}, \tag{7.3b}$$

$$D_\mu E_{ij} = \partial_\mu E_{ij} + V_\mu{}^k{}_i E_{kj} + V_\mu{}^k{}_j E_{ik}, \tag{7.4a}$$

$$D_\mu E^{ij} = \partial_\mu E_{ij} - V_\mu{}^i{}_k E^{kj} - V_\mu{}^j{}_k E^{ik}. \tag{7.4b}$$

We conclude this section by point out that on a given supersymmetric configuration of the Weyl multiplet, one where $\delta_{\xi,\eta}\psi_\mu{}^i = \delta_{\xi,\eta}\psi_{\mu\,i} = 0$, the S-supersymmetry parameter is completely determined from the Q-supersymmetry one,

$$\eta^i = \frac{1}{2}\gamma^\mu D_\mu \xi^i \qquad\qquad \eta_i = \frac{1}{2}\gamma^\mu D_\mu \xi_i. \tag{7.5}$$

7.3 The twists of $\mathcal{N}=4$

In this section we will use the machinery built in the previous section to construct topological twists of $\mathcal{N}=4$ gauge theories. Put differently, we will search for supersymmetric configurations of the background Euclidean Weyl multiplet for which the $SU(4)$ connection is proportional to the spin connection. We will come to see that this successfully reproduces the Vafa-Witten twist [409, 410], Kapustin-Witten twist [411, 412] and half-twists [409] of $\mathcal{N}=4$. Our results are summarised in Table 7.4.

Firstly, let us set the stage. We wish to consider topological twists of arbitrary smooth, orientable, Riemannian manifolds, $\mathcal{M}$. However, let us start by considering those that are also spin, as required by the initial presence of spinors on $\mathcal{M}$. In turn, through the twisting procedure, one can relax that condition. Nevertheless, let g be the Riemannian metric on $\mathcal{M}$. Through the Levi-Civita connection, associated to g, we see that $\mathcal{M}$ comes equipped with a spin connection, ω.

Secondly, the twisting procedure requires an identification of the spin bundle with the R-symmetry bundle. Equivalently, one can specify an injective homomorphism from the spin group to the R-symmetry group, $\varphi : SU(2)_\ell \times SU(2)_r \to SU(4)_R$. This homomorphism takes a simple form if we write elements of $SU(4)$ as $SU(2)$ block-diagonal matrices. In this way, the notation will completely mirror that of Chapter 6, whereby a basis of $SU(2)$ is given in terms of the Pauli matrices (see Appendix D for our conventions).

Let us proceed as follows. We take the $SU(4)$ connection to have the following ansatz,

$$V_\mu{}^i{}_j = -\frac{1}{4}\omega_\mu^{ab}\Sigma_{ab}{}^i{}_j, \tag{7.6}$$

where Σ_{ab} is a traceless 4×4 matrix. We also set all other, non-geometric, Weyl multiplet fields to zero, except for the auxiliary field $D^{ij}{}_{kl}$, which we will come back to shortly. Now, different choices of Σ will lead to different topological twists.

The half-twists

We can start with the simplest example, that which recovers the Donaldson-Witten twist, available to us in $\mathcal{N}=2$, and discussed at length in Chapter 6. In that case, we beak down $SU(4)$ to a single $SU(2)$ R-symmetry group and consider the homomorphism $SU(2)_l \times SU(2)_r \to SU(2)_R$, which is given by either

$$\Sigma_{ab}{}^i{}_j = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & 0 \end{pmatrix}^i{}_j, \qquad \text{or} \qquad \Sigma_{ab}{}^i{}_j = \begin{pmatrix} \bar\sigma_{ab} & 0 \\ 0 & 0 \end{pmatrix}^i{}_j, \tag{7.7}$$

depending on the choice whether we are twisting the left $SU(2)$ (twist) or the right one (anti-twist). One can verify that both these ansätze do indeed lead to non-trivial (and constant) solutions to the generalised Killing spinor equations (7.1a) and (7.1b). The two constant solutions to these equations are given in Table 7.4. Note that, to make our presentation more palatable, we use a matrix notation for our supersymmetry parameters where the first index labels the (chiral) spin index, and the second one the $SU(4)$ index. As such, a chiral Q-susy parameter ξ^i will be represented by a 2×4 matrix, sometimes written as two 2×2 matrices. The two equations derived from the variation of the gaugino, (7.1c) and (7.1d), are trivially satisfied on this background. The final two equations, (7.1e) and (7.1f), are solved by those same two spinors if we set

$\varphi: \; SU(2)_\ell \times SU(2)_r \to SU(4)_R$, $(A,B) \mapsto \varphi(A,B)$	${\Sigma_{ab}}^i{}_j$	Killing spinors ξ^i	Killing spinors ξ_i
Donaldson-Witten twists			
$\varphi(A,B) := \mathrm{diag}(A, I_2)$	$\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & 0 \end{pmatrix}$	0	$\begin{pmatrix} c_1 I_2 & 0 \end{pmatrix}$
$\varphi(A,B) := \mathrm{diag}(B, I_2)$	$\begin{pmatrix} \bar\sigma_{ab} & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} c_1 \sigma_2 & 0 \end{pmatrix}$	0
$\varphi(A,B) := \mathrm{diag}(I_2, A)$	$\begin{pmatrix} 0 & 0 \\ 0 & \sigma_{ab} \end{pmatrix}$	0	$\begin{pmatrix} 0 & c_1 I_2 \end{pmatrix}$
$\varphi(A,B) := \mathrm{diag}(I_2, B)$	$\begin{pmatrix} 0 & 0 \\ 0 & \bar\sigma_{ab} \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 \sigma_2 \end{pmatrix}$	0
Kapustin-Witten twists			
$\varphi(A,B) := \mathrm{diag}(A, B)$	$\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar\sigma_{ab} \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 \sigma_2 \end{pmatrix}$	$\begin{pmatrix} c_2 I_2 & 0 \end{pmatrix}$
$\varphi(A,B) := \mathrm{diag}(B, A)$	$\begin{pmatrix} \bar\sigma_{ab} & 0 \\ 0 & \sigma_{ab} \end{pmatrix}$	$\begin{pmatrix} c_1 \sigma_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_2 I_2 \end{pmatrix}$
Vafa-Witten twists			
$\varphi(A,B) := \mathrm{diag}(A, A)$	$\begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \sigma_{ab} \end{pmatrix}$	0	$\begin{pmatrix} c_1 I_2 & c_2 I_2 \end{pmatrix}$
$\varphi(A,B) := \mathrm{diag}(B, B)$	$\begin{pmatrix} \bar\sigma_{ab} & 0 \\ 0 & \bar\sigma_{ab} \end{pmatrix}$	$\begin{pmatrix} c_1 \sigma_2 & c_2 \sigma_2 \end{pmatrix}$	0

TABLE 7.4: The topological (anti-)twists of $\mathcal{N}=4$ theories on oriented Riemannian (but otherwise arbitrary) manifolds, as defined by their respective homomorphic injection φ of the Euclidean rotations into the $SU(4)_R$ R-symmetries. For each twist, we provide the data Σ defining the $SU(4)_R$ connection via equation (7.6) and the positive- (negative-)chirality Killing spinor(s) ξ^i (ξ_i). I_2 is the unit 2×2 matrix.

the auxiliary field ${D^{ij}}_{kl}$ to the following background value

$$ {D^{ij}}_{kl} = -\frac{R}{4}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}^{ij}\begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}_{kl} - (\text{trace}), \tag{7.8} $$

where the specific meaning of $-(\text{trace})$ is given in footnote 16 of Chapter 3.

A similar formulation exists for those two twists where a different $SU(2)$ subgroup is chosen within $SU(4)$. In that case, the $SU(4)$ connection is set using the matrices

$$ {\Sigma_{ab}}^i{}_j = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{ab} \end{pmatrix}^i{}_j, \qquad \text{or} \qquad {\Sigma_{ab}}^i{}_j = \begin{pmatrix} 0 & 0 \\ 0 & \bar\sigma_{ab} \end{pmatrix}^i{}_j. \tag{7.9} $$

The corresponding two Killing spinors are also given in Table 7.4. This time, however, the background value of $D^{ij}{}_{kl}$ takes the following form

$$D^{ij}{}_{kl} = -\frac{R}{4}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}^{ij}\begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}_{kl} - (\text{trace}), \tag{7.10}$$

due to the change in $SU(2)$ subgroup.

The Kapustin-Witten twist

Arguably more interesting are the twists that are not available in $\mathcal{N}=2$. The Kapustin-Witten is one such twist that has preserves two constant spinors of opposite chirality. Its defining injective homomorphism is that which identifies $SU(2)$-left and $SU(2)$-right with different $SU(2)$ subgroups of the $SU(4)$ R-symmetry. In the supergravity background, this can be engineered in two different ways, using two different Σ_{ab} matrices

$$\Sigma_{ab}{}^{i}{}_{j} = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix}^{i}{}_{j}, \qquad \text{or} \qquad \Sigma_{ab}{}^{i}{}_{j} = \begin{pmatrix} \bar{\sigma}_{ab} & 0 \\ 0 & \sigma_{ab} \end{pmatrix}^{i}{}_{j}. \tag{7.11}$$

Both engender two constant solutions to the first two generalised Killing spinor equations, (7.1a) and (7.1b), with opposite chirality. The two twists are mapped to each other under chiral flips, as expected from the fact that one is just the 'anti-' twists of the other. As usual, we reported those solutions in Table 7.4.

The two Killing spinor equations given by the supersymmetry variation of the gaugino are still trivially satisfied on our background. The final two equations, given as the variation of the dilatino, (7.1e) and (7.1f), are solved by those two constant spinors provided we set

$$\begin{aligned} D^{ij}{}_{kl} = -\frac{R}{6}\Big(&(I_2\otimes\sigma_2)^{ij}(I_2\otimes\sigma_2)_{kl} + (\sigma_1\otimes\sigma_2)^{ij}(\sigma_1\otimes\sigma_2)_{kl} \\ &+ (\sigma_3\otimes\sigma_2)^{ij}(\sigma_3\otimes\sigma_2)_{kl}\Big) - (\text{trace}). \end{aligned} \tag{7.12}$$

In the above I_2 denotes the 2×2 identity matrix, and $\otimes$ is the tensor product, whose components are defined such that

$$(I_2\otimes\sigma_2)^{ij} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}^{ij}. \tag{7.13}$$

The Vafa-Witten twist

Finally, we can present the supergravity background that engineers the Vafa-Witten twist of $\mathcal{N}=4$ gauge theories. This twist is characterised by the fact that it preserves two constant spinors of identical chirality, distinguishing it from the Kapustin-Witten twist which has two spinors of opposite chirality. Its defining injective homomorphism maps the same $SU(2)$ subgroup of the spin group to the R-symmetry group. As such, the correct background supergravity configuration is that specified by

$$\Sigma_{ab}{}^{i}{}_{j} = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \sigma_{ab} \end{pmatrix}^{i}{}_{j}, \qquad \text{or} \qquad \Sigma_{ab}{}^{i}{}_{j} = \begin{pmatrix} \bar{\sigma}_{ab} & 0 \\ 0 & \bar{\sigma}_{ab} \end{pmatrix}^{i}{}_{j}. \tag{7.14}$$

The first configuration leads to two constant Killing spinors, both of negative chirality. We will label that configuration the 'twist'. The second configuration leads to two constant positive chirality Killing spinors. We will label that one the 'anti-twist'.

As usual, we reported these solutions in Table 7.4. They properly solve all generalised Killing spinor equations, provided the auxiliary field $D^{ij}{}_{kl}$ takes the following form

$$\begin{aligned} D^{ij}{}_{kl} = -\frac{R}{6}\Big(&(I_2\otimes\sigma_2)^{ij}(I_2\otimes\sigma_2)_{kl} + (\sigma_1\otimes\sigma_2)^{ij}(\sigma_1\otimes\sigma_2)_{kl} \\ &+ (\sigma_3\otimes\sigma_2)^{ij}(\sigma_3\otimes\sigma_2)_{kl}\Big) - (\text{trace}), \end{aligned} \tag{7.15}$$

identically to the Kapustin-Witten case.

Whenever a given manifold $\mathcal{M}$ exhibits more symmetry, additional non-trivial solutions to the generalised Killing spinor equations may emerge. For instance, on any product manifold $\mathcal{M}_3 \times S^1$, where the canonical choice of vierbeins is taken, the spin connection satisfies $\omega^{4a} = 0$. From our choice of $SU(2)$ basis, this further shows that the $SU(4)$ connections for the Vafa-Witten and Kapustin-Witten twists coincide. The manifold then admits four constant Killing spinor.

7.4 Ω-deformed twists

Recall from our exposition in Chapter 6 that the Donaldson-Witten twist admits a $U(1)$ deformation. This is often referred to as the equivariant Donaldson-Witten twist, or Ω-deformed Donaldson-Witten twist. In our background supergravity setup, we constructed such a twist by additionally turning on the two-form Weyl multiplet field T. Its values being dictated by a $U(1)$-vector of $\mathcal{M}$.

Let us present here the various generalisations of this construction to the $\mathcal{N} = 4$ conformal supergravity background. This time, however, the two-form T is $SU(4)_R$-valued and must somehow be related to the auxiliary fields E^{ij} and E_{ij} through the gaugino variation. These facts, generally, complicated the search for supersymmetric configurations quite a bit. Let us nevertheless, give an account for the ones we were able to construct explicitly.

The half-twists

In this first setup, we will mirror directly the configuration given in $\mathcal{N}=2$ conformal supergravity for the Ω-deformation of the Donaldson-Witten twist. Here, we start from the background configuration of the half-twist and turn on one of the two-forms T^{ij} or T_{ij}, depending on whether we are looking at the twist or anti-twist. In our conventions, the former has

$$\Sigma_{ab}{}^{i}{}_{j} = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & 0 \end{pmatrix}^{i}{}_{j}. \tag{7.16}$$

Consequently, we wish to turn on the anti-self-dual two-form T_ij. Let v be a vector field on $\mathcal{M}$, then the ansatz

$$T^{ij} = \frac{1}{2}(dv)^{-} \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}^{ij}, \tag{7.17}$$

leads to a non-trivial solution to the equations of motion, provided further constraints on v. Indeed, one can show that the spinors

$$(\xi^{\alpha\, i}) = v^{\mu}\bar{\sigma}_{\mu}\left(c_1\sigma_2 \; 0\right), \qquad\qquad (\xi^{\alpha}_{i}) = \left(c_1 I_2 \; 0\right), \tag{7.18}$$

are Killing on this background provided that v is a conformal Killing vector of $\mathcal{M}$. This result mirrors our finding in Chapter 6.

The Kapustin-Witten twist

Let us now start from the supergravity background that manufactures the Kapustin-Witten twist, as discussed in the previous section. For simplicity, we will focus on that given by

$$\Sigma_{ab}{}^{i}{}_{j} = \begin{pmatrix} \bar{\sigma}_{ab} & 0 \\ 0 & \sigma_{ab} \end{pmatrix}^{i}{}_{j}, \tag{7.19}$$

in other words, what we call the anti-twist. Similar result can be found for the twist, with a flip of chirality/self-duality. In the present setting, the $SU(4)$ connection forces us to turn on both self-dual and anti-self-dual two-form fields T^{ij} and T_{ij}.

Let v be a vector field on $\mathcal{M}$. Then, the ansatz

$$T_{ij} = \frac{1}{2}(dv)^{+} \begin{pmatrix} \sigma_2 & 0 \\ 0 & 0 \end{pmatrix}_{ij}, \qquad\qquad T^{ij} = \frac{1}{2}(dv)^{-} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \end{pmatrix}^{ij}, \tag{7.20}$$

leads to the following Killing spinors

$$(\xi^{\alpha\, i}) = (c_1\sigma_2 \; c_2 v^\mu \bar{\sigma}_\mu \sigma_2)\,, \qquad\qquad (\xi_i^\alpha) = (c_1 v^\mu \bar{\sigma}_\mu \; c_2 I_2)\,, \tag{7.21}$$

provided v is a conformal Killing vector field of $\mathcal{M}$. Additionally to the two-forms, one must also introduce a non-trivial configuration for the bosonic fields E_{ij} and E^{ij}, in order to solve the second sets of Killing spinor equations, (7.1c) and (7.1d). The simplest form these take can be written as

$$E_{ij} = \frac{1}{2}(dv^-)_{ab} \begin{pmatrix} \sigma_2 \bar{\sigma}_{ab} & 0 \\ 0 & 0 \end{pmatrix}_{ij}\,, \qquad E^{ij} = \frac{1}{2}(dv^+)_{ab} \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2 \sigma_{ab} \end{pmatrix}^{ij}\,. \tag{7.22}$$

With those fields specified, the final two Killing spinor equations, (7.1e) and (7.1f), are automatically solved. This confirms the supersymmetric nature of the proposed background configuration.

The Vafa-Witten twist

Surprisingly enough, a similar construction for the Vafa-Witten twist doesn't seem to allow for the introduction of an Ω-deformation. At the very least, no simple construction like the ones introduced for the half-twists and Kapustin-Witten twist solve all Killing spinor equations.

Let us, nevertheless, presented here our attempt at constructing such a background and pin-point where this attempt fails. Starting from the $SU(4)$ connection, let us consider the configuration that gives the 'twist'. A similar ansatz exists for the anti-twist. Together with the matrix configuration

$$\Sigma_{ab}{}^i{}_j = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \sigma_{ab} \end{pmatrix}^i{}_j\,, \tag{7.23}$$

we give the two-form T^{ij} a non-trivial configuration that depends on a vector field v,

$$T^{ij} = \frac{1}{2}(dv)^- \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}^{ij}\,. \tag{7.24}$$

The spinors

$$(\xi^{\alpha\, i}) = v^\mu \bar{\sigma}_\mu \,(c_1\sigma_2 \; c_2\sigma_2)\,, \qquad\qquad (\xi_i^\alpha) = (c_1 I_2 \; c_2 I_2)\,, \tag{7.25}$$

are indeed solutions of the first two Killing spinor equations, (7.1a) and (7.1b), provided v is conformal Killing. Solving the variation of the gaugino, (7.1c) and (7.1d), further requires us to set

$$E_{ij} = \frac{1}{2}(dv^-)_{ab}\frac{v^c v^d}{\|v\|^2}\left(I_2 \otimes \sigma_2\sigma_c\bar{\sigma}^{ab}\bar{\sigma}_d\right)_{ij}. \tag{7.26}$$

Unfortunately, this configuration doesn't solve the variation of the dilatino. The failure of which originates from the $D_\mu E_{kl}$ term in (7.1e). The equation would be solved by our ansatz, if its sign were inverted, possibly hinting at an error in our construction.

7.5 Conclusion/Remarks

In this chapter we proposed a Euclidean formulation of the Weyl multiplet of $\mathcal{N}=4$ conformal supergravity, starting from the known literature (detailed in Chapter 3). Using this, we were able to construct BPS configurations that engineer the three twists of 4d $\mathcal{N}=4$, the half-, Kapustin-Witten and Vafa-Witten twists. Each case is distinguished by its Lie group homomorphism from the Lorentz group to the R-symmetry group, φ. In each case, we solved the variations of the gravitino, gaugino and dilatino; reporting our results in Table 7.4.

These twists admit a further refinement, whereby the two-form fields T^{ij} and T_{ij} are turned on, and depend on a conformal Killing vector v. Mirroring the construction done for $\mathcal{N}=2$ theories, we believe these engineer the Ω-deformation of their corresponding twists. A more thorough identification with the alternative definition [418] has yet to be done, however. More specifically, it would be interesting to see if the algebra generated by our supercharges match that found in that article.

The lack of apparent Ω-deformed solution for the Vafa-Witten twist is troubling. As pointed out below equation (7.26), our ansatz fails only due to a singular sign within the dilatino variation. This could possibly hint at a mistake made when constructing the Euclidean formula of the Weyl multiplet.

Where to go from there?

Those used to performing Ω-deformations via twist-compactifications might find the background supergravity formulation too abstract. However, it does seem to provide a simpler, group-theoretic, way of constructing these deformations — by choosing how to decompose the anti-symmetric representation of T^{ij} into $SU(2)$ representations, with the usual vector field dependence. Notably, this allows us to construct general ansätze for these fields, by giving them an arbitrary block decomposition in terms of $SU(2)$ generators.

For instance, let us consider a deformation of the Vafa-Witten twist,

$$\Sigma_{ab}{}^{i}{}_{j} = \begin{pmatrix} \sigma_{ab} & 0 \\ 0 & \sigma_{ab} \end{pmatrix}^{i}{}_{j},$$

where we give the two-form T^{ij} a configuration that now depends on three vector fields v_1, v_2 and v_3,

$$T^{ij} = \frac{1}{2} \begin{pmatrix} \sigma_2 (dv_1)^- & \sigma_2 (dv_3)^- \\ \sigma_2 (dv_3)^- & \sigma_2 (dv_2)^- \end{pmatrix}^{ij} . \tag{7.27}$$

One can show that the gravitino variation, (7.1a) and (7.1b), is solved by the spinors

$$(\xi^{\alpha\, i}) = \bar{\sigma}_\mu \left((v_1^\mu c_1 + v_3^\mu c_2)\sigma_2 \;\; (v_3^\mu c_1 + v_2^\mu c_2)\sigma_2 \right), \qquad (\xi_i^\alpha) = \left(c_1 I_2 \;\; c_2 I_2 \right), \tag{7.28}$$

provided the vector fields $V_1 = v_1^\mu c_1 + v_3^\mu c_2$ and $V_2 = v_3^\mu c_1 + v_2^\mu c_2$ are conformal Killing. This is a promising start as it seems to engineer a background with a *non-abelian* Ω-deformation, where the algebra generated by these spinors mixes the vectors v_i. However, we were unable to find solutions to the other four generalised Killing spinor equations, (7.1c), (7.1d), (7.1e) and (7.1f); and cannot conclude whether such a background configuration is indeed supersymmetric. Further look into this is expected.

Part IV

Neural networks

Chapter 8

nn-dCFT

Patti chjari amici cari.
Clear contracts makes dear friends.

Corsican proverb

The central idea behind this chapter is the so-called *neural-network-field-theory* (nn-FT) correspondence. Without aiming for a complete description, we will at least outline the main points, by focusing on its realisation in correlation functions. For a more comprehensive overview, and a good point to start reading on this, we recommend the introductory sections of [419] or the TASI lecture notes [60].

In 2024, Halverson, Naskar and Tian successfully specialised this correspondence to conformal field theories (CFTs) [61]. By leveraging the constructive power of neural networks, they developed a formalism that assigns a CFT to every neural network configuration of a given class. This comes in opposition to the standard nn-FT correspondence, for which one gets a QFT only in a suitable limit.

In this chapter, we will expand on this formalism by extending the construction to defect conformal field theories (dCFTs). In a manner similar to the CFT case, one can assign a dCFT to every neural network of a given class. Basic operations between neural-networks also allows one to iteratively build new dCFTs from given dCFT data. This construction is highly desirable as it allows us to engineer an infinite class of (potentially new) dCFTs using a finite function space. Additionally, while the original formalism was only explicitly built for scalar conformal primaries, we will suggest a potential extension to any spinning conformal primary that can be packaged into symmetric-traceless tensors (STTs)[1].

[1]We will not present a proof that our extension works in every case. This will be ironed out in a follow-up work to [9].

8.1 The nn-FT correspondence

Let us restrict our discussion to Euclidean field theories in d-dimensions, and recall some basic facts about those QFTs. In the Lagrangian approach, the generating functional is called the partition function $Z[\{J_i\}]$. For every field ϕ^i in the theory we can associate a current J_i and define $Z[\{J_i\}]$ to be the functional integral of the exponentiated action $S[\{\phi^i\}]$ of the fields multiplied by the exponential of the current couplings.
Schematically, we write this as an integral with measure $\mathcal{D}\phi$ (even if no such measure is known formally),

$$Z[\{J_i\}] = \int \mathcal{D}\phi e^{-S[\{\phi^i\}]} \prod_i e^{\int d^dx J_i(x)\phi^i(x)}. \tag{8.1}$$

The expression above must then be understood as a formal expression, used to illustrated the properties of this generating functional [420]. We also note that ϕ^i is a placeholder for fields in any representation of Spin(d). Any spacetime indices those would contain are understood as being contracted with those of J_i, resulting in a spacetime scalar.

All correlators of the fields ϕ^i can be retrieved from $Z[\{J_i\}]$ by performing multiple functional derivatives with respect to the currents J_i. For example, the n-point correlation function $G^{(n)}(x_1,\ldots,x_n)$ between the fields $\phi^1,\ldots,\phi^n$ is constructed as

$$G^{(n)}(x_1,\ldots,x_n) = \langle \phi^1(x_1)\cdots\phi^n(x_n)\rangle = \frac{\delta^n Z[\{J_i\}]}{\delta J_1(x_1)\cdots\delta J_n(x_n)}. \tag{8.2}$$

By symmetry arguments, it is then possible to restrict the functional forms of these correlators (see Chapter 1). They can even be given a more formal definition, and be exactly determined in some special cases[2].

Let us now see how neural-networks can be used to engineer these QFTs. The Universal Approximation Theorems (UATs) are a category of theorems that describe under which conditions neural networks are dense in a given function space [55–59][3]. Placing ourselves in such conditions satisfied by the UATs, we can quickly understand how neural networks can be used to approximate a field theory. Let us illustrate this constructively.

Firstly, recall that a neural network is a map $\Phi_\theta : \mathbb{R}^d \to \mathbb{R}^q$, the parameters of which can take any value in a measurable set $\theta \in E$. We can then define a partition function, $Z[\{J_i\}]$, for this family of q neural networks. Since the networks are labelled by a measurable set, E, we can define any functional integral involving these fields as a standard integral over E, with a given choice of measure $\mu_{\mathcal{E}}$[4]. An important step in this

[2]The examples that come to mind are integrable theories like TQFTs, and certain supersymmetric QFTs.

[3]We refer the reader to the classic surveys [421–423] for more information.

[4]Here $\mathcal{E}$ denotes a σ-algebra on E. In practice, we will always take $\mathcal{E} = \mathcal{P}(E)$, the set of all subsets of E, and choose E to be $\mathbb{R}^d$ or some subset thereof.

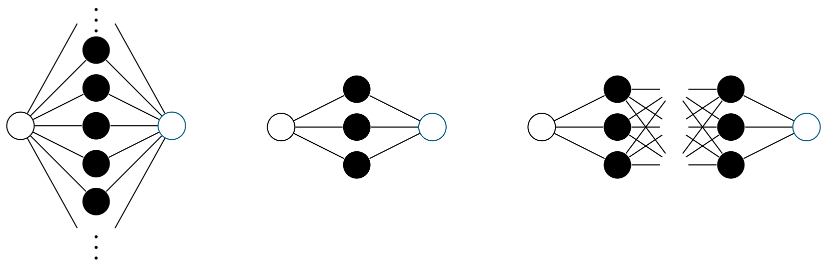


FIGURE 8.1: Illustration of the large-width limit (left) and large-depth limit (right) of the neural network (middle).

construction lies in the fact that we wish to give all correlators a statistical interpretation. This is possible when the measure $\mu_{\mathcal{E}}$ is a probability density. In other words, it must obey the axioms of probability:

- $\forall U \in \mathcal{E}$, $\mu_{\mathcal{E}}(U) \in [0, 1]$,
- $\mu_{\mathcal{E}}(E) = 1$.

In that instance, the partition function $Z[\{J_i\}]$, written explicitly as

$$Z[\{J_i\}] = \int_E \mu_{\mathcal{E}} \prod_{i=1}^{q} e^{\int d^d x J_i(x) \Phi^i_\theta(x)}, \tag{8.3}$$

describes a type of field theory known as a *neural-network quantum field theory* (nn-QFT). The explicit choice of Φ^i_θ, as a function of θ, is known as a *choice of architecture*.

Allow us to briefly point out the similarities between the standard partition function (8.1) and that of the nn-QFT (8.3). In the Feynman interpretation of the partition function, every field configuration $\phi^i_\star$ is weighted by its action, $S[\{\phi^i_\star\}]$, through the exponential term. Those with the smallest action contribute the most to the integral. In other words, $\mathcal{D}\phi e^{-S[\{\phi^i\}]}$ acts as a measure on the space of fields, just like $\mu_{\mathcal{E}}$ does on E. From the nn-QFT side of things, a choice of measure amounts to a choice of action $S[\{\phi^i\}]$ in the QFT language. The UATs tell us that in a suitable limit of the number of parameters θ, the networks Φ_θ will be dense in the set of functions from $\mathbb{R}^d$ to $\mathbb{R}^q$. In other words, the integration in (8.3) will coincide with the functional integral in (8.1). The neural-network-field-theory correspondence has emerged.

Such a correspondence should not appear as a surprise. For most standard neural-networks with a parameter N, say the number of neurons in the layer, taking N to infinity amounts to drawing the network from a Gaussian process. This is the *neural-network-Gaussian-Process* (nn-GP) correspondence [424]. Using this correspondence, [425, 426] showed that the infinite-width limit of such nn-QFTs leads to a free field theory. The interactions emerge as finite-width corrections, i.e. deviations away from the Gaussian limit. [427] extend this construction to the non-perturbative renormalisation group.

8.2 Neural-network-CFTs

Constructing neural-network-CFTs (nn-CFTs) is now a simple extension of that of nn-QFTs described above. This section follows the work of Halverson, Naskar and Tian [61] and uses the embedding space formalism of a D-dimensional CFT in flat space, detailed in Chapter 1, Section 1.5. We also restrict the current discussion to scalar conformal primary fields, $\phi(x)$. We will propose a similar formalism for spinning fields at the end of Section 8.4.

Firstly, recall that under any special conformal transformations (SCTs) of the coordinates a conformal primary field must transform by a power of the scaling factor. This power is called the scaling dimension of ϕ, denoted Δ. If b_μ are the parameters of this transformation, then $x^\mu \mapsto x^\mu - 2b\cdot x x^\mu + x^2 b^\mu$ and the scalar conformal primary obeys

$$\phi(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta}. \tag{8.4}$$

In general, the action of K_μ, the generators of SCTs, is non-linear on $\mathbb{R}^D$ and imposing such a constraint on a neural-network architecture might be highly non-trivial. Thankfully, when utilising the embedding space formalism, the action of $SO(1, D+1)$ is linear and such a constraint can be imposed on neural-network architectures. The idea is then as follows. Define a neural-network-CFT (nn-CFT) as a neural-network-QFT on the embedding space $\mathbb{R}^{1,D+1}$ such that the architecture Φ_θ is both homogeneous and obeys the conformal primary constraint. In practice, however, we will often work with a Euclidean version of the embedding space, $\mathbb{R}^{D+2}$.

While we can always Wick rotate the embedding space back to $\mathbb{R}^{1,D+1}$, in doing so, however, we are spoiling the $SO(D+2)$-symmetry that was built into the measure $\mu_{\mathcal{E}}$. It is, therefore, also necessary to “Wick rotate” the measure, making it $SO(1, D+1)$-invariant instead. The downside to this approach is that Wick rotation is not well-defined on the space of measures. For instance, the Cauchy distribution defined using the $SO(D+2)$-invariant dot product on the embedding space

$$\mu_{\mathcal{P}(\mathbb{R}^{D+2})} = \frac{d^{D+2}\theta}{\theta\cdot\theta + 1}, \tag{8.5}$$

can be Wick rotated by simply changing the dot product to the Lorentzian one. Unfortunately, the object this defines violates the first axiom of probability – the measure is no longer positive and bounded.

There are two conceptual ways around this hurdle. The first is to consider the theory and its correlators built from the Euclidean embedding space, as the fundamental objects. All integrals are performed in that setting, and only the final output is Wick rotated to the standard theory. This way of thinking completely sidesteps the measure problem by doing everything on $\mathbb{R}^{D+2}$. The second way is to relax the conditions on $\mu_{\mathcal{E}}$. Instead of strictly requiring a probability measure, we allow for non-positive (and potentially non-bounded) measures as long as the integrals $\int_E \mu_{\mathcal{E}} \Phi_\theta^n$ can be regulated. This approach is similar to the standard QFT approach, where all but a few integrals must be regulated and renormalised. In essence, the specific interpretation will not be relevant to us, as we will sometime perform the integrals in Euclidean signature, and some other times in Lorentzian signature.

To conclude this section, we collect here the various ingredients needed to define a nn-CFT. Restricting this discussion to nn-CFT built from scalar conformal primaries only, those are

1. a $SO(1, D+1)$-invariant measure (or $SO(D+2)$-invariant probability density),
2. a neural-network architecture $\Phi_\theta(X)$ on $\mathbb{R}^{1,D+1}$ (or $\mathbb{R}^{D+2}$) which is
   (a) Homogeneous: $\Phi_\theta(\lambda X) = \lambda^{-\Delta}\Phi_\theta(X)$,
   (b) Conformal Primary: $\Phi_\theta(K(b)\cdot X) = (1 + 2b\cdot X + b^2X^2)^\Delta \Phi(X)$,

where $K(b)\cdot X$ denotes the action of a SCT with parameter b on the coordinate X. With these ingredients set, the final requirement, which is also the hardest to ensure, is

3 well defined (finite) correlators.

[61] illustrate this construction with a couple of examples, all based on the same architecture, dubbed the *standard architecture*. It is labelled so because it is the simplest such architecture that obeys the homogeneity and conformal primary conditions set above.

Definition 8.1. The spinless nn-CFT architecture The *standard nn-CFT architecture* with scaling dimension Δ is defined as [61]

$$\Phi_\theta(X) = (X\cdot\theta)^{-\Delta}, \tag{8.6}$$

where the dot product is performed either on the standard embedding space, $\mathbb{R}^{1,D+1}$, or on its Euclidean counterpart, $\mathbb{R}^{D+2}$.

8.3 Neural-network-defect-CFTs

We now wish to extend the previously-described formalism to accommodate defect-Conformal Field Theories (dCFTs) [9]. As before, we wish to consider a CFT in D-dimensions. We will take the defect to be p-dimensional and flat. In that setting, the embedding space has a natural splitting in terms of $p+2$ defect directions X^A and $q = D - p$ transverse direction X^I. A more thorough treatment can be found in Chapter 1, Section 1.6 or in the original article [107].

Since introducing a defect breaks down the conformal group to its defect subgroup, we must allow for measures $\mu_{\mathcal{E}}$ that preserve this subgroup. A general p-dimensional conformal defect may break down $SO(1, D+1)$ fully to $SO(1, p+1)$, but can also preserve a subgroup of $SO(q)$ on top of that. For this flat-defect case that we are interested in, the full transverse subgroup is preserved, which implies that we must choose a measure that is $SO(1, p+1) \times SO(q)$-invariant. Following this choice, we must also find a nn-CFT analogue of the defect and ambient conformal primaries. We label these the *defect architecture* and *ambient architecture*. The defect architecture should obey all the conditions placed on a standard nn-CFT architecture, be it restricted to the $p+2$ defect directions of the embedding space. The ambient architecture on the other hand, should obey those of a nn-CFT architecture that lives on the full $D+2$-dimensional embedding space.

These choices can be justified as follows. The scalar defect insertions in any dCFT still form a well-defined CFT on their own. Thus, their nn-dCFT counterparts should also behave as such, on the $p+2$-dimensional subspace of the embedding space. The scalar ambient insertions, on the other hand, behave like $SO(1, D+1)$ scalar conformal primaries whose correlators are supplemented by a defect operator. This defect operator insertion, is what the $SO(1, p+1) \times SO(q)$-invariant measure engineers for us.

Extending the construction in [61], we define the simplest scalar neural-network architectures for defect and ambient insertions in Definition 8.2.

Definition 8.2. The spinless nn-dCFT architecture In the language of nn-QFTs, defect insertions are represented by the *defect architecture* $\varphi(X)$, the standard form of which we define as follows

$$\varphi(X) = (X \bullet \theta)^{-\hat{\Delta}}. \tag{8.7}$$

Any such insertion is restricted to the defect, and thus obeys the constraints $X^I = 0$.

The *ambient architecture*, representing ambient operator insertions, follows the standard nn-CFT architecture described in the previous section

$$\phi(X) = (X \cdot \theta)^{-\Delta}. \tag{8.8}$$

Here, the coordinate X has no further restrictions, until brought down to the Poincaré section.

8.3.1 Example: monomial nn-dCFTs

In this first example, we wish to illustrate the effectiveness of our formalism at describing dCFT correlation functions for non-unitary theories. Indeed, we will focus on architectures with negative scaling dimensions, $\Delta, \hat{\Delta} < 0$. To facilitate things, we will take the measure $\mu_{\mathcal{E}}$ to be a product of two centred multivariate normal distributions, one along the defect directions with variance matrix $\hat{\sigma}_{AB} = \hat{\mu}_2 \delta_{AB}$, and the other in the transverse directions with variance matrix $\tilde{\sigma}_{IJ} = \tilde{\mu}_2 \delta_{IJ}$,

$$\mu_{\mathcal{E}} = \mathcal{N}(0, \hat{\sigma}) \mathcal{N}(0, \tilde{\sigma}) d^{D+2}\theta. \tag{8.9}$$

This choice preserves the full defect symmetry $SO(p+2) \times SO(q)$, which will be reflected in the correlators we will compute. One can also choose a measure that breaks all the transverse symmetry, or only part of it, by either changing $\tilde{\sigma}$ or by choosing a different distribution altogether, however, that would be a departure from the flat defect description we wish to stick to.

The correlation function between n-defect insertions φ, and m-ambient insertions ϕ is given by the expectation value

$$\mathbb{E}\left[\prod_{i=1}^{n} \varphi_i(X_i) \prod_{j=1}^{m} \phi_j(Y_j)\right] = \int_{\mathbb{R}^{p+2}} d^{p+2}\theta \mathcal{N}(0, \hat{\sigma}) \prod_{i=1}^{n} \varphi_i(X_i) \left(\int_{\mathbb{R}^q} d^q\theta \mathcal{N}(0, \tilde{\sigma}) \prod_{j=1}^{m} \phi_j(Y_j)\right). \tag{8.10}$$

Since all scaling dimensions considered in this example are negative, we wil adopt the notation $\Delta_i = -n_i$, where n_i is positive. All correlators below will reference this “inverse” scaling dimension for better readability.

Before proceeding, we also note the binomial expansion of the ambient architecture ϕ, which will be useful in our computations

$$\phi(X) = (X \cdot \theta)^n = \sum_{d=0}^{n} \binom{n}{d} (X \bullet \theta)^d (X \circ \theta)^{n-d}. \tag{8.11}$$

Note that the expansion above, while obvious for negative scaling dimensions, is a prototypical example of the defect operator product expansion (dOPE) of ambient architectures. This OPE not only is finite, but also only involves scalar conformal primaries. None of the spinning fields, or descendents are required for this expansion to work.[5]

One-point functions

There are two types of one-point functions – the defect and the ambient ones. In both cases, the embedding space formalism requires us to restrict them to the Poincaré section (P.S.) in order to get a physical-space correlator. The defect one-point function should always vanish, when restricted to the defect P.S., and indeed, our explicit calculations show that these expectation values obey this and are exactly of the form in equation (1.43).

$$\mathbb{E}[\varphi(X)] = \mathbb{E}[(X \bullet \theta)^n] = \begin{cases} 2^{n/2}\pi^{-1/2}\Gamma\left(\frac{n+1}{2}\right)(\hat{\mu}_2 X \bullet X)^{\frac{n}{2}} & n \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$
$$\overset{P.S.}{=} 0 \tag{8.12a}$$

$$\mathbb{E}[\phi(X)] = \mathbb{E}[(X \cdot \theta)^n] = \begin{cases} \frac{\Gamma(n+1)}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}(\hat{\mu}_2 X \bullet X + \tilde{\mu}_2 X \circ X)^{\frac{n}{2}} & n \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$
$$\overset{P.S.}{=} \begin{cases} \frac{\Gamma(n+1)}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}+1\right)}(\tilde{\mu}_2 - \hat{\mu}_2)^{\frac{n}{2}}(X \circ X)^{\frac{n}{2}} & n \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{8.12b}$$

A detailed proof of these results can be found below Lemma C.1.

Two-point functions

The two-point functions, on the other hand, come in three varieties. The defect-defect, mixed and ambient-ambient correlators. As discussed in Section 1.6, both the mixed and ambient-ambient correlators gain more structure thanks to the additional OPE channels that are available to them. Nevertheless, the structure of these correlators in known [107] and our computations show perfect agreement with those.

Starting with the defect-defect correlator,

$$\mathbb{E}[\varphi_1(X_1)\varphi_2(X_2)] \overset{P.S.}{=} \delta_{n_1,n_2}\Gamma(n_1+1)\tilde{\mu}_2^{n_1}(X_1 \bullet X_2)^{n_1}. \tag{8.13a}$$

[5]The fact that no descendents appear in this expansion might not be as surprising as one would think. Indeed, given the simplicity of the network architecture, any descendent is directly proportional to a scalar primary with neighbouring scaling dimension

$$\partial_A (X \bullet \theta)^{\hat{n}} = \hat{n}\theta_A (X \bullet \theta)^{\hat{n}-1}.$$

The mixed defect-ambient correlator takes the form

$$\mathbb{E}[\varphi_1(X_1)\phi_2(X_2)] \overset{P.S.}{=} \frac{\Gamma(n_2+1)}{\Gamma\left(\frac{n_2-n_1}{2}+1\right)} 2^{\frac{n_1-n_2}{2}} \tilde{\mu}_2^{n_1} (\hat{\mu}_2 - \tilde{\mu}_2)^{\frac{n_2-n_1}{2}} (X_2 \circ X_2)^{\frac{n_2-n_1}{2}} (X_1 \bullet X_2)^{n_1}, \tag{8.13b}$$

when $n_2 \geq n_1$ and $n_1 + n_2 \in 2\mathbb{Z}$, and vanishes otherwise.

The two-point correlator between two ambient insertions requires the result from Theorem C.3. After resumming that result, we find

$$\begin{aligned}\mathbb{E}[\phi_1(X_1)\phi_2(X_2)] =& \frac{\Gamma(n_2+1)}{\Gamma(n_1+1)\Gamma\left(\frac{n_2-n_1}{2}+1\right)} \left(\frac{\alpha_{22}}{2}\right)^{\frac{n_2-n_1}{2}} \alpha_{12}^{n_1} \\ & {}_2F_1\left(\frac{1-n_1}{2}, -\frac{n_1}{2}; \frac{n_2-n_1}{2}+1; \frac{\alpha_{11}\alpha_{22}}{\alpha_{12}^2}\right)\end{aligned} \tag{8.13c}$$

when $n1 \leq n_2$ and $n_1 + n_2 \in 2\mathbb{Z}$,

$$\begin{aligned}\mathbb{E}[\phi_1(X_1)\phi_2(X_2)] =& \frac{\Gamma(n_1+1)}{\Gamma(n_2+1)\Gamma\left(\frac{n_1-n_2}{2}+1\right)} \left(\frac{\alpha_{11}}{2}\right)^{\frac{n_1-n_2}{2}} \alpha_{12}^{n_2} \\ & {}_2F_1\left(\frac{1-n_2}{2}, -\frac{n_2}{2}; \frac{n_1-n_2}{2}+1; \frac{\alpha_{11}\alpha_{22}}{\alpha_{12}^2}\right)\end{aligned} \tag{8.13d}$$

when $n_1 \geq n_2$ and $n_1 + n_2 \in 2\mathbb{Z}$, and zero otherwise. In both expressions above, we also defined $\alpha_{ij} = \tilde{\mu}_2 X_i \bullet X_j + \hat{\mu}_2 X_i \circ X_j$. In the special cases $i = j$, these simplify to $\alpha_{ii} \overset{P.S.}{=} (\hat{\mu}_2 - \tilde{\mu}_2) X_i \circ X_i$ on the Poincaré section.

Higher-point functions

Given the simplicity of these architectures and distributions, it is a surprise to no one that the non-unitary dCFT this constructs is fully solvable. Indeed, while we gave closed-form expressions for the one- and two-point functions, all higher-point functions can simply be determined by performing various Wick contractions between the insertion vectors X_i. The elementary example of such a contraction is presented in Lemma C.2. Let us illustrate how one would construct a correlator between arbitrary number of insertions.

For every insertion X_i with weight n_i, one can perform $n_i/2$ contractions $X_i \bullet X_i$ iff n_i is even. Every self contraction gives a factor of $Q(n)$ (see C.45). We must then also consider all possible contractions between X_i and X_j, summing over these possibilities with appropriate weights. All-in-all, the correlator $\mathbb{E}[(X_1 \bullet \theta)^{n_1} \cdots (X_k \bullet \theta)^{n_k}]$ will contain many constrained sums based on the evenness of the inverse scaling dimensions n_i.

In this construction, the difficulty lies in resumming these integrals into closed-form expressions, such as into the hypergeometric functions given previously.

$$\mathbb{E}[(X_1 \bullet \theta)^{n_1} \cdots (X_k \bullet \theta)^{n_k}] =$$

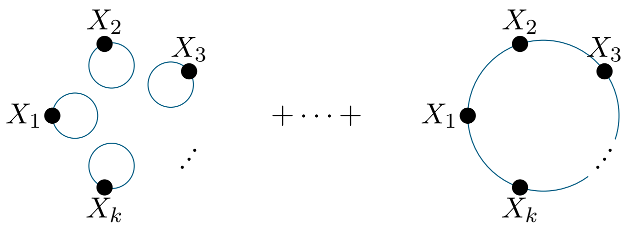


FIGURE 8.2: A pictorial representation of all possible Wick contractions that build the k-point expectation value $\mathbb{E}[(X_1 \bullet \theta)^{n_1} \cdots (X_k \bullet \theta)^{n_k}]$.

The correlators above contain the necessary information to fully describe the defect structure of this theory. Indeed, abstracted behind the hypergeometric functions is an OPE expansion in the defect conformal primaries. For every such hypergeometric function, we are in possession of the full finite expansion, and thus, know everything about the defect theory.

8.3.2 Second example: reciprocal nn-dCFT

We now wish to consider defect and ambient insertions, $\varphi(X) = (X \bullet \theta)^{-\hat{\Delta}}$ and $\phi(X) = (X \cdot \theta)^{-\Delta}$, with positive scaling dimensions $\hat{\Delta}, \Delta > 0$. It is possible to analytically continue the Gaussian integrals with those positive scaling dimensions. Without going into details, let us outline the main formulae needed to calculate those partition functions.

Firstly, the moments of a single θ-parameter vanish for odd values and equals

$$\int_{\mathbb{R}} \mathcal{N}(0, \mu_2) \theta^s d\theta = \frac{1}{\sqrt{\pi}} \left(\frac{\mu_2}{2} \right)^{-\frac{s}{2}} \Gamma \left(\frac{1+s}{2} \right), \tag{8.14}$$

for even values. We can further extend the validity of the above formula to any s even, positive or negative.

Secondly, the various correlation functions can be written in a manifestly hypergeometric form, using the Feynman reparametrisation trick

$$\frac{1}{A^{n_1}} \frac{1}{B^{n_2}} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 du \int_0^1 dv \delta(1-u-v) \frac{u^{n_1-1} v^{n_2-1}}{(uA+vB)^{n_1+n_2}}. \tag{8.15}$$

This holds whenever 0 is not included in the segment from A to B. We can make a choice of regularisation, wherein the segment is either above or below the real line. We argue that any of these choices leads to the correct dCFT correlator structure.

One-point function

The one-point function of ambient insertions $\phi(X)$ obey the desired structure given in equation (**??**eq:dCFT-1pt-ES)). They are

$$\mathbb{E}[\phi(X)] \overset{\text{P.S.}}{=} (-1)^{\frac{\Delta}{2}} 2^{-\frac{\Delta}{2}} \frac{\pi^{1/2}}{\Gamma\left(\frac{\Delta+1}{2}\right)} \frac{(\hat{\mu}_2 - \tilde{\mu}_2)^{\frac{\Delta}{2}}}{(X \circ X)^{\Delta/2}}, \tag{8.16}$$

when Δ is even and zero otherwise.

Two-point functions

The two-point functions also obey the desired dCFT correlator structure, as in the polynomial nn-dCFT example. For conciseness, we only report here the ambient-ambient correlator, and urge the reader to refer to our publication [9] for more details. We find

$$\mathbb{E}[\phi(X_1)\phi(X_2)] \propto (1-\beta)^{-\Delta_1} \alpha_{22}^{-\frac{\Delta_1+\Delta_2}{2}} {}_2F_1\left(\Delta_1, \frac{\Delta_1+\Delta_2}{2}; \Delta_1+\Delta_2; \frac{\alpha-\beta}{1-\beta}\right), \tag{8.17}$$

where $\alpha_{ij} = \frac{X_i \bullet X_j}{\hat{\mu}_2} + \frac{X_i \circ X_j}{\tilde{\mu}_2}$ and $\alpha = \alpha_{22}^{-1}(-\alpha_{12} + \alpha_{22} - \sqrt{\alpha_{12} - \alpha_{11}\alpha_{22}})$ and $\beta = \alpha_{22}^{-1}(-\alpha_{12} + \alpha_{22} + \sqrt{\alpha_{12} - \alpha_{11}\alpha_{22}})$. The proportionality indicates the additional presence of combinatorial factors of Δ_1 and Δ_2 which we do not specify here.

8.4 Spinning nn-dCFT

Before concluding this chapter, we wish to write down our proposal for describing nn-CFTs and nn-dCFTs with spinning fields. This proposal has only been tested on a handful of examples and lacks a proper proof. However, we still wish to commit it to writing, as it may prove useful, were it to work out in general[6].

Spinning nn-CFTs

Since the embedding space formalism uses an auxiliary variable Z, we wish to do the same with our parameters. We take $\theta \sim P(\theta)$ and $\eta \sim P(\eta)$ to be two sets of $D+2$ random variables, distributed according to centred multivariate distributions with $\sigma_{ij} = \delta\mu_2$. We propose the following nn-CFT with scaling dimension Δ and spin J,

$$\begin{aligned}\Phi_{\Delta,J}(X,Z) &= (\theta \cdot X)^{-\Delta-J}(C_{MN}\theta^M\eta^N)^J \\ &= (\theta \cdot X)^{-\Delta-J}(Z \cdot \theta P \cdot \eta - Z \cdot \eta P \cdot \theta)^J.\end{aligned} \tag{8.18}$$

Critically, this ansatz obeys the conditions required by a STT conformal primary

[6] I would like to take accountability for any errors in the proposals below. As these were solely proposed by myself, my collaborators are in no way responsible for their failure, were that to happen.

1. Homogeneity: $\Phi_{\Delta,J}(\lambda X, Z) = \Phi_{\Delta,J}(\lambda X, Z)$,
2. Spin-J: $\Phi_{\Delta,J}(X, \lambda Z) = \lambda^J \Phi_{\Delta,J}(X, Z)$,
3. Transversality: $\Phi_{\Delta,J}(X, Z + \lambda X) = \Phi_{\Delta,J}(X, Z)$.

One can evaluate the various correlators by performing an integration over $P(\theta)$ and $P(\eta)$,

$$\begin{aligned} G(X_1, X_2; Z_1, Z_2) &= \langle \Phi_{\Delta_1,J_1}(X_1, Z_1)\Phi_{\Delta_2,J_2}(X_2, Z_2)\rangle \\ &= \int d^{D+2}\theta d^{D+2}\eta P(\theta)P(\eta)\Phi_{\Delta_1,J_1}(X_1, Z_1)\Phi_{\Delta_2,J_2}(X_2, Z_2). \end{aligned} \tag{8.19}$$

Using a few explicit values for the spin and scaling dimensions, the above ansatz produces the expected form for the two-point function of spinning fields, equation (1.35), and three-point function of spinning fields, equation (1.36).

Spinning nn-dCFTs

When inserting a defect, the dot product in the embedding space $\cdot$ splits into its defect part $\bullet$ and its transverse part $\circ$. We will also split the distributions for each random variables into a defect part and a transverse part. To accommodate defect spinning fields we also introduce a second auxiliary random variable, ζ, which will encode the $SO(q)$-indices. The defect neural network CFT with scaling dimension $\hat{\Delta}$, defect spin j and transverse spin s is postulated to be

$$\varphi_{\hat{\Delta},j,s} = (\theta \bullet X)^{-\hat{\Delta}-j-s}(Z \bullet \theta X \bullet \eta - Z \bullet \eta X \bullet \theta)^j (W \bullet \theta X \bullet \eta - W \bullet \eta X \bullet \theta)^s. \tag{8.20}$$

The ambient insertions keep the same functional form as in the non-defect case.

Again, the ansatz there given passes multiple checks for the one-point and two-point functions of defect-defect and ambient-defect correlators.

8.5 Conclusion/Remarks

In this chapter, we successfully extended the formalism of [61] to the defect-CFT case. We proposed two neural-network architectures — one for ambient insertions and the other for defect insertions. Through the explicit calculation of correlation functions, we demonstrated how our non-unitary data satisfies the properties of a defect-CFT. The power behind this formalism lies in the ability to generate families of CFTs (or dCFTs) from a finite-dimensional integral. While we only demonstrated this for single neurons of simple scalar primaries, one can combine multiple of these neurons to create new dCFTs. One then has access to an infinite family of dCFTs, built from recurring, finite integrals.

A proposal was also given to extend both construction to include STT conformal primaries. If the proposals hold true, this opens up an even larger class of dCFTs to be constructed from this neuron construction.

Given how recent this field is, relatively few results can be found. Nevertheless, it would be interesting to see if, through the nn-CFT formalism, one can engineer known CFT data. Naturally, any spinorial data is out of reach for the moment as the formalism doesn't yet accommodate spinors. A generalisation in that direction would also be interesting.

The formalism also allows for the construction of CFT data in any number of dimensions, whose conformal block decomposition is generically not that of a free CFT. One might wonder whether the simple architecture in equation (8.6) applied to the $D = 6$ case can engineer non-trivial, unitary, interacting CFTs; given that those have yet to be found.[7]

Recently, the fuzzy-sphere regularisation has emerged as a powerful tool for studying CFTs in 3d. By placing them on a product manifold $\mathbb{R} \times S^2$, where the S^2 promoted to the fuzzy-sphere, the space of functions gets truncated (as expected from non-commutative field theory) providing a suitable regularisation scheme [428]. For instance, He was able to place the free boson CFT on this fuzzy geometry and ended up with a theory which approximates the 3d Ising CFT [429]. This way of regularising the CFT partition function shares some similarities with our construction of neural-network CFTs, wherein the space of functions is also truncated in a way which still preserves conformal invariance. It would be interesting to see if those two topics are somehow related.

[7] I thank N. Benjamin for bringing that observation to light.

Chapter 9

Conclusion

In this thesis we presented three main avenues along which one can study conformal field theories — the holographic picture, the exact supersymmetric picture, and the neural-network picture. Respectively, these make up Part II, III and IV of the present work. Our contributions in each domain, while small, were meaningful. We detail below what those were and what their possible extensions are.

In Part II, we reproduced our publications [7, 8] which aimed at computing defect Weyl anomaly coefficients for codimension-4 and codimension-2 superconformal defects in the 6d $\mathcal{N} = (2,0)$ SCFTs at large-N. More specifically, in [8] (Chapter 4) studied a particular limit of the 11d supergravity solutions in [2], wherein the superisometry parameter γ was taken to infinity. This degenerate limit of the superisometry $\mathfrak{d}(2,1;\gamma) \oplus \mathfrak{d}(2,1;\gamma)$ required a careful rescaling of the singularities on the Riemann surface found in the foliation. The supergravity solutions one ended up with were of the type constructed in [1], be it with a finite Ricci scalar, and describe certain codimension-4 defects in the 6d theory. What separates this solution from other constructions is the apparent inability to "turn off the defect". In other words, the field theory in which the defect is present is a deformation of the 6d theory. This, in turn, rendered the study of Weyl anomaly coefficients more complex. Indeed, the holographic entanglement entropy allowed us to extract a linear combination of A-type and B-type defect Weyl anomaly coefficients, however, separating them out requires the use of another holographic observable. One such observable is the stress-tensor one point function, which dictates the B-type coefficient, and can be computed using standard holographic renormalisation in seven dimensions. The problem lies in that our supergravity solution doesn't exhibit the correct asymptotics to allow for such an analysis — the four-form field strength contains leading-order terms which, upon dimensional reduction to 7d, don't give Einstein gravity. It would be interesting to know whether one can devise such a holographic renormalisation scheme, in the presence of flux.

In [7] (Chapter 5) we performed a similar analysis for solutions that describe 1/4- and 1/2-BPS codimension-2 defects. These are holographically described by two supergravity solutions — the two-charge and electrostatic solutions. In both cases, we computed the stress-energy tensor one-point function and holographic entanglement entropy, allowing us to extract the A-type and one of the B-type anomaly coefficients.

For both types of defects described above, we computed the holographic entanglement entropy for a spherical region around the defect; leading to a quantity which is a linear combination of the A-type and one of the B-type anomaly coefficients. Our work, therefore, was only able to probe (at most) two of the many coefficients these defects come equipped with. A natural extension to this would be to probe for other coefficients by, for instance, deforming the entangling region. It would be interesting to see if one can perform such a computation, and see which coefficients are computationally attainable that way. Another way would be via the computation of other observables, such as the on-shell action or the partition function on S^4.

In Part III, we studied various supersymmetric configurations of background conformal supergravity in four dimensions. We introduced, in Chapter 6, a supergravity background which interpolates between two known configurations — the twisted index and the Coulomb-branch index. Any 4d $\mathcal{N}=2$ SCFT coupled to that background can see its partition function evaluated at any point along the interpolation. This defines an index which we coined *the interpolating index.* At one end of the interpolation, the partition function of the coupled theory evaluates to the twisted index, which computes Donaldson-Witten invariants of the manifold; while on the other end it evaluates to a point on the moduli of Coulomb-branch indices. Not only is the interpolation supersymmetric, it is also exact in the sense that both indices are rendered equal to one another. Furthermore, we derived a formula that recasts the supersymmetry variation of a current multiplet into that of the background supergravity fields; allowing us to derive our exactness statement in a theory-agnostic way. Thus, we proved that the twisted index and a point on the moduli of Coulomb-branch indices are equal for all 4d $\mathcal{N}=2$ SCFTs. An interesting question arises when looking at potential generalisations of this equality. Indeed, the twisted index is known to admit a refinement, called equivariant Donaldson-Witten, or Ω-deformed twisted index. Most of the relations derived in this chapter hold true in this more general case; where, instead of interpolating to a point on the moduli of Coulomb-branch indices, we can interpolate to the full Coulomb-branch index moduli. The background there-defined is supersymmetric, however, it does not seem to be exact. A more thorough treatment of our exactness equations would be required to definitely conclude whether that is the case or not. If it were true, this could spell out an interesting duality between these two types of indices. Indeed, their known holographic completions are very different AdS black hole configurations, and equating the indices would indicate a relation between the black hole microstate countings. Naturally, a field theory understanding would also be warranted.

In Chapter 7, on the other hand, we strayed away from the 4d $\mathcal{N} = 2$ constructions described above and focused our attention on the $\mathcal{N} = 4$ case. We proposed a formulation of 4d $\mathcal{N} = 4$ conformal supergravity by extending the know Weyl multiplet to the Euclidean case. This description of the Weyl multiplet is a novel finding, and allows us to look for supersymmetric configurations that engineer the various twists of $\mathcal{N} = 4$. We enumerated configurations of the background fields that describe the half-, Kapustin-Witten and Vafa-Witten twists of these 4d theories. We further searched for Ω-deformations of these twists, which are typically found by turning on the bosonic two-form field in the Weyl multiplet. We do so for all three twists, and find satisfactory results for the half- and Kapustin-Witten twists; but fail to do so for the Vafa-Witten twist. It would be interesting to see whether such a configuration can be found using our Euclidean formulation. Naturally, given the preliminary nature of our results, some errors might be present within the Euclidean formulation itself. Furthermore, a thorough analysis of these Ω-deformations is warranted. Notably, one should look at the algebra generated by their Killing spinors and compare it to the known literature. Our results are, nevertheless, interesting to those who wish to study the twisted partition functions of 4d $\mathcal{N} = 4$ theories, as we provide the suitable conformal supergravity background to do so.

Finally, in Part IV (Chapter 8), we presented a new formalism that allows one to construct dCFT data using finite-dimensional integrals. Most of the work there will appear in an upcoming paper [9]. The non-defect case is a known formalism [61], wherein a conformal primary is represented by a node within a neural-network, and is given a functional form on the embedding space. By integrating over all parameters in this function, one arrives at a partition function for a CFT with said conformal primary. We introduced two new “nodes” — one for the defect conformal primaries and one for the bulk ones. By integrating over all their parameters, whose distribution is only invariant under the defect subgroup, we arrived at a partition function for a dCFT. We showed, through explicit examples, that the various correlation functions do obey the functional form for dCFT correlators. The power behind this formalism is two-fold. Firstly, it allows one to generate (d)CFT data using a finite number of parameters — this data always obeys crossing symmetry, defect OPE constraints, etc. Secondly, one can take any such (d)CFT and combine its nodes into a diagram, called neural-network; leading to an infinite family of (d)CFT data — one for each neural-network. One of the main hurdles this formalism currently faces is the lack of a description for conformal primaries other than the scalar ones. We, nevertheless, proposed a functional form for a node that encodes symmetric traceless tensors, and leave its study to a future publication. It would also be interesting to see whether fermionic CFTs can be incorporated in this framework. On a similar note, it would be interesting to see whether these neural-network-CFTs can be used to generate CFT data of known models, such as strongly interacting SCFTs in six dimensions or otherwise. As the topic is quite recent, we have high hopes for the number of results one can get out of it.

Part V

Appendices

Appendix A

Appendix to Chapter 4

A.1 Fefferman-Graham parametrization

In this Appendix, we will be concerned with finding a reparametrization $\{w, \bar{w}\} \to \{v, \phi\}$ of the Riemann surface Σ_2 under which the 11d SUGRA solution in (4.18) can be asymptotically (in particular, for small v) recast into the following form,

$$\mathrm{d}s^2 = \frac{4L_{S^4}^2}{v^2}\left[\mathrm{d}v^2 + \alpha_1 \mathrm{d}s^2_{\mathrm{AdS}_3} + \alpha_2 \mathrm{d}s^2_{S^3}\right] + L^2_{S^4}\left[\alpha_3 \mathrm{d}\phi^2 + \alpha_4 s_\phi^2 \mathrm{d}s^2_{\tilde{S}^3}\right] , \tag{A.1}$$

for some $\mathcal{O}(v^0)$ metric factors $\{\alpha_i\}_{i=1}^4$. By imposing that the metric factor in the v direction be exact in v, as above, we can reconstruct the asymptotic reparametrization order by order in v and in terms of polar coordinates

$$r = \sqrt{z^2 + \rho^2} \quad \text{and} \quad \theta = \arctan(\rho/z) \tag{A.2}$$

on Σ_2 to be the following,

$$\begin{aligned} r(v,\phi) = &-\frac{2(\gamma+1)^2 m_1}{\gamma v^2} + \frac{(2\gamma+1)m_1\left(c_{2\phi}-3\right)}{24\gamma} + \frac{m_2 c_\phi}{2m_1} + \left[\frac{3\gamma^2 m_2^2\left(7c_{2\phi}+1\right)}{4m_1^3}\right. \\ &-\frac{\gamma^2 m_3\left(5c_{2\phi}+3\right)}{m_1^2} - \frac{2\gamma(2\gamma+1)m_2 c_\phi s_\phi^2}{m_1} - \frac{1}{16}(8\gamma(\gamma+1)+3)m_1 c_{2\phi} \\ &\left.+\frac{37}{48}\gamma(\gamma+1)m_1 + \frac{73 m_1}{192} + \frac{19}{192}(2\gamma+1)^2 m_1 c_{4\phi}\right]\frac{v^2}{48\gamma(\gamma+1)^2} + \mathcal{O}(v^4), \end{aligned} \tag{A.3a}$$

$$\begin{aligned} \theta(v,\phi) = \phi &+ \frac{(2\gamma+1)m_1^2 c_\phi + 3\gamma m_2}{12(\gamma+1)^2 m_1^2} s_\phi v^2 + \left[\frac{9\gamma^2 m_2^2 s_{2\phi}}{8m_1^4} - \frac{5\gamma^2 m_3 s_{2\phi}}{6m_1^3}\right. \\ &+\frac{c_\phi s_\phi\left(5(2\gamma+1)^2 c_{2\phi} - 24\gamma(\gamma+1) - 7\right)}{48} \end{aligned} \tag{A.3b}$$

$$+\frac{\gamma(2\gamma+1)m_2\,(3c_{2\phi}-1)\,s_\phi}{12m_1^2}\Bigg]\frac{v^4}{16(\gamma+1)^4}+\mathcal{O}(v^6)\ ,$$

where we introduced the moments

$$m_k := \sum_{j=1}^{2n+2}(-1)^j\xi_j^k\ . \tag{A.4}$$

Under this mapping, the metric takes the desired form of (A.1) with the following metric factors

$$\begin{aligned}
\alpha_1(\gamma) &= 1+\frac{2\gamma+3-(1+2\gamma)c_{2\phi}}{16(\gamma+1)^2}v^2+\Big[9\Big(4(3\gamma-13)\gamma+16\gamma^2\kappa-17\Big) && \text{(A.5a)}\\
&\quad -67(2\gamma+1)^2c_{4\phi}+12\Big(8(3\gamma+1)\gamma+20\gamma^2\kappa+3\Big)c_{2\phi}\Big]\frac{v^4}{18432(\gamma+1)^4}+\mathcal{O}(v^6)\ ,\\
\alpha_2(\gamma) &= \frac{(\gamma+1)^2}{\gamma^2}+\frac{2\gamma-1-(1+2\gamma)c_{2\phi}}{16\gamma^2}v^2+\Big[9\Big(4(3\gamma+19)\gamma+16\gamma^2\kappa+47\Big) && \text{(A.5b)}\\
&\quad -67(2\gamma+1)^2c_{4\phi}+12\Big(8(3\gamma+5)\gamma+20\gamma^2\kappa+19\Big)c_{2\phi}\Big]\frac{v^4}{18432\gamma^2(\gamma+1)^2}+\mathcal{O}(v^6)\ ,\\
\alpha_3(\gamma) &= 1+\frac{(2\gamma+1)c_\phi^2}{4(\gamma+1)^2}v^2+\Big[-12(\gamma+1)\gamma-24\gamma^2\kappa-9+6(2\gamma+1)^2c_{2\phi} && \text{(A.5c)}\\
&\quad +7(2\gamma+1)^2c_{4\phi}\Big]\frac{v^4}{768(\gamma+1)^4}+\mathcal{O}(v^6)\ ,\\
\alpha_4(\gamma) &= 1+\frac{(2\gamma+1)(2c_{2\phi}+1)}{12(\gamma+1)^2}v^2+\Big[-52(\gamma+1)\gamma-24\gamma^2\kappa-19 && \text{(A.5d)}\\
&\quad +79(2\gamma+1)^2c_{4\phi}+12\Big(-2(\gamma+1)\gamma-10\gamma^2\kappa-3\Big)c_{2\phi}\Big]\frac{v^4}{4608(\gamma+1)^4}+\mathcal{O}(v^6)\ ,
\end{aligned}$$

where, to the order shown, the moments $\{m_i\}$ enter the metric factors only in the combination

$$\kappa \equiv \frac{3m_2^2-4m_1m_3}{m_1^4}\ . \tag{A.6}$$

Note that κ is invariant under coordinate transformations $\{\xi_i \mapsto \xi_i+\lambda\}$, even though m_2 and m_3 are not individually. From this mapping, we also deduce the curvature scale of the asymptotic local S^4 in FG gauge,

$$L_{S^4}^3 = \frac{|1+\gamma|^3}{\gamma^2}\frac{h_1m_1}{2}\ . \tag{A.7}$$

We can now take the $\gamma\to-\infty$ limit, accompanied by appropriate rescalings as described in 4.4, to determine the asymptotic local behaviour of the metric in (4.36). In this limit, we find that

$$\frac{\kappa}{\gamma^2}\to\frac{12(\hat{n}_1^2-\hat{m}_1\hat{n}_2)}{\hat{m}_1^4}\ , \tag{A.8}$$

where we introduced the additional moments

$$\hat{n}_k := \sum_{j=1}^{2n+2} (-1)^j \nu_j^k \hat{\xi}_j \ . \tag{A.9}$$

Furthermore, we find that this limiting procedure trivialises the relative warping between AdS_3 and S^3; that is, $\alpha_1(\gamma \to -\infty) = \alpha_2(\gamma \to -\infty)$. Overall, the limit $\gamma \to -\infty$ produces the FG line element advertised in (4.51), with the following metric factors,

$$\begin{aligned}
\alpha_1(\gamma \to -\infty) = 1 &+ \frac{(3+5c_{2\phi})(\hat{n}_1^2 - \hat{m}_1\hat{n}_2)}{32\hat{m}_1^4} v^4 \qquad \text{(A.10a)}\\
&- \frac{5(1+7c_{2\phi})(2\hat{n}_1^3 - 3\hat{m}_1\hat{n}_1\hat{n}_2 + \hat{m}_1^2\hat{n}_3)c_\phi}{144\hat{m}_1^6} v^6 + \mathcal{O}(v^8),\\
\alpha_3(\gamma \to -\infty) = 1 &+ \frac{3(\hat{m}_1\hat{n}_2 - \hat{n}_1^2)}{8\hat{m}_1^4} v^4 + \frac{5(2\hat{n}_1^3 - 3\hat{m}_1\hat{n}_1\hat{n}_2 + \hat{m}_1^2\hat{n}_3)c_\phi}{12\hat{m}_1^6} v^6 + \mathcal{O}(v^8), \qquad \text{(A.10b)}\\
\alpha_4(\gamma \to -\infty) = 1 &+ \frac{(1+5c_{2\phi})(\hat{m}_1\hat{n}_2 - \hat{n}_1^2)}{16\hat{m}_1^4} v^4 \qquad \text{(A.10c)}\\
&+ \frac{5(5c_\phi + 7c_{3\phi})(2\hat{n}_1^3 - 3\hat{m}_1\hat{n}_1\hat{n}_2 + \hat{m}_1^2\hat{n}_3)}{144\hat{m}_1^6} v^6 + \mathcal{O}(v^8) \ ,
\end{aligned}$$

and with S^4 radius given by (4.53).

Finally, we note that, if all points are taken to collapse to the origin, $\nu_j = 0$ for $j \in \{1, 2, \ldots, 2n+2\}$, the asymptotic reparametrization in (A.3) becomes exact,

$$r(v,\phi) = \frac{2\hat{m}_1}{v^2} \tag{A.11a}$$

$$\theta(v,\phi) = \phi \ . \tag{A.11b}$$

A.2 Area of the Ryu-Takayanagi hypersurface

In this appendix, we fill in the technical details of the computation of the holographic entanglement entropy in (4.83).

We begin by treating the integral in (4.82). In order to exploit the geometry of the internal space and facilitate an easier path to evaluating the remaining integral, we first transform to polar coordinates $\{r, \theta\}$, introduced in (A.2), with $\theta \in [0, \pi]$. We can then expand the summands in (4.82) on the complete basis of $L^2(0, \pi)$ functions spanned by the Legendre polynomials, $P_k(c_\theta)$. The harmonic function H can then be written in two equivalent representations

$$H(r,\theta)=\begin{cases}\frac{h_1}{2}\sum_{j=1}^{2n+2}(-1)^j\hat{\xi}_j\,|\nu_j|^{-3}\left(\sum_{k=0}^{\infty}r^k\nu_j^{-k}P_k(c_\theta)\right)^3, & r\in[0,|\nu_j|),\\ \frac{h_1}{2}\sum_{j=1}^{2n+2}(-1)^j\hat{\xi}_j r^{-3}\left(\sum_{k=0}^{\infty}r^{-k}\nu_j^{k}P_k(c_\theta)\right)^3, & r\in(|\nu_j|,\Lambda_r(\epsilon_v,0)],\end{cases} \tag{A.12}$$

which converge when integrated in r over their respective domains. We have also introduced in the second line of (A.12) a cutoff scale Λ_r at large r, which from the transformation to FG gauge in (A.3) can be expressed in a small ϵ_v expansion as

$$\Lambda_r(\epsilon_v,\theta)=\frac{2\hat{m}_1}{\epsilon_v^2}+\frac{\hat{n}_1c_\theta}{\hat{m}_1}+\frac{(3+5c_{2\theta})\hat{m}_1\hat{n}_2-(5+3c_{2\theta})\hat{n}_1^2}{16\hat{m}_1^3}\,\epsilon_v^2+\mathcal{O}(\epsilon_v^4). \tag{A.13}$$

Thus, the remaining integral in the area functional can be partitioned into two separate contributions

$$A[\zeta_{\rm RT}]=\frac{32\pi^4L_{S^4}^9}{\hat{m}_1^3}\log\left(\frac{2R}{\epsilon_u}\right)\sum_{j=1}^{2n+2}(-1)^j\hat{\xi}_j\left(\mathcal{I}_j^{(1)}+\mathcal{I}_j^{(2)}\right)+\mathcal{O}(\epsilon_u^2), \tag{A.14}$$

where both integrals

$$\mathcal{I}_j^{(1)}=\int_0^\pi d\theta\int_0^{|\nu_j|}dr\;r^4s_\theta^3\left(\frac{1}{|\nu_j|}\sum_{k=0}^{\infty}\left(\frac{r}{\nu_j}\right)^kP_k(c_\theta)\right)^3, \tag{A.15a}$$

$$\mathcal{I}_j^{(2)}=\int_0^\pi d\theta\int_{|\nu_j|}^{\Lambda_r(\epsilon_v,\theta)}dr\;r\;s_\theta^3\left(\sum_{k=0}^{\infty}\left(\frac{\nu_j}{r}\right)^kP_k(c_\theta)\right)^3, \tag{A.15b}$$

are convergent.

At first glance, (A.15) may not seem to put us in a better position to evaluate the integral, but we can exploit the properties of $P_k(c_\theta)$ over the interval $\theta\in[0,\pi]$. In particular, we can make use of the orthogonality relation of the triple-product of Legendre polynomials

$$\int_0^\pi d\theta\;s_\theta\;P_{\ell_1}(c_\theta)P_{\ell_2}(c_\theta)P_{\ell_3}(c_\theta)=2\begin{pmatrix}\ell_1&\ell_2&\ell_3\\0&0&0\end{pmatrix}^2, \tag{A.16}$$

where the right-hand side is the Wigner $3j$-symbol. Since this particular $3j$-symbol has vanishing magnetic quantum numbers, if $\ell\equiv\ell_1+\ell_2+\ell_3$ is even and together the ℓ_i satisfy the triangle inequality, then it can neatly be expressed as

$$\begin{pmatrix}\ell_1&\ell_2&\ell_3\\0&0&0\end{pmatrix}=\frac{(-1)^{\ell/2}(\ell/2)!}{\sqrt{(\ell+1)!}}\prod_{i=1}^{3}\frac{\sqrt{(\ell-2\ell_i)!}}{(\ell/2-\ell_i)!}\;. \tag{A.17}$$

Otherwise, if ℓ is not even or the triangle inequality is violated, the $3j$-symbol vanishes. Additionally, the following identity

$$P_{\ell_1}P_{\ell_2} = \sum_{\ell_3=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 (2\ell_3+1)P_{\ell_3} \ , \tag{A.18}$$

is particularly useful in the evaluation of the θ-integrals in (A.15), where for brevity we denoted $P_\ell \equiv P_\ell(c_\theta)$. Eventually, after applying both eqs. (A.16) and (A.18), we find that the θ-integrals in $\mathcal{I}_j^{(1)}$ can be handled with the help of

$$\int_0^\pi \mathrm{d}\theta \ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3} = \frac{4}{3}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 - \frac{4}{3}\sum_{k=|2-\ell_1|}^{2+\ell_1}(2k+1)\begin{pmatrix} 2 & \ell_1 & k \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} k & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}^2 . \tag{A.19}$$

The integrals $\mathcal{I}_j^{(1)}$ and $\mathcal{I}_j^{(2)}$ may be further simplified by considering the convergence properties of the sum, which allow us to integrate each term in r separately. Doing so, we rapidly see the usefulness of the previous θ-integral formulae, which appear explicitly as

$$\mathcal{I}_j^{(1)} = \sum_{\ell_1,\ell_2,\ell_3} \frac{1}{5+\ell}\frac{\nu_j^{\ell+2}}{|\nu_j|^\ell}\int_0^\pi \mathrm{d}\theta \ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3}, \tag{A.20a}$$

$$\begin{aligned}\mathcal{I}_j^{(2)} &= \sum_{\substack{\ell_1,\ell_2,\ell_3 \\ \ell\neq 2}} \frac{\nu_j^\ell}{2-\ell}\int_0^\pi \mathrm{d}\theta \ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3}\left[\Lambda_r^{2-\ell}(\epsilon_v,\theta) - \frac{\nu_j^2}{|\nu_j|^\ell}\right] \\ &\quad + \sum_{\substack{\ell_1,\ell_2,\ell_3 \\ \ell=2}} \nu_j^2 \int_0^\pi \mathrm{d}\theta \ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3} \ln\left(\frac{\Lambda_r(\epsilon_v,\theta)}{|\nu_j|}\right) ,\end{aligned} \tag{A.20b}$$

where we have isolated the $\ell = 2$ mode in $\mathcal{I}_j^{(2)}$ in order to handle the potential log divergence as a separate case.

Starting with the sum on the second line of (A.20b), we first expand the integral in small ϵ_v using (A.13). Using the integral formulae for the Legendre polynomials above, we find that the leading $\ln\left(2\hat{m}_1/\epsilon_v^2\right)$ divergence contains no additional θ-dependence, and so due to the constraint that $\ell = 2$, is weighted with $P_1^2 + P_2$, and vanishes upon integration. Thus, we find that

$$\sum_{j=1}^{2n+2}(-1)^j\hat{\xi}_j \sum_{\substack{\ell_1,\ell_2,\ell_3 \\ \ell=2}} \nu_j^2 \int_0^\pi \mathrm{d}\theta \ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3} \ln\left(\frac{\Lambda_r(\epsilon_v,\theta)}{|\nu_j|}\right) = \mathcal{O}(\epsilon_v^4) \ . \tag{A.21}$$

Hence, the only meaningful contributions to $A[\zeta_{\mathrm{RT}}]$ from $\mathcal{I}_j^{(2)}$ come from the first line in (A.20b).

Moving on to the cutoff-dependent integrand on the first line of (A.20b), we can again utilise the small ϵ_v expansion in (A.13). Since the leading divergence in $\Lambda_r(\epsilon_v,\theta)$ is

$\mathcal{O}(1/\epsilon_v^2)$, we can neglect any integral for $\ell > 2$ as it will vanish as $\mathcal{O}(\epsilon_v^2)$. Truncating to the sum to $\ell < 2$, expanding in small ϵ_v, and evaluating the sum over j, we find that the total contribution to $A[\zeta_{\mathrm{RT}}]$ from the first sum in (A.20b) is

$$\sum_{j=1}^{2n+2}(-1)^j\hat{\xi}_j \sum_{\substack{\ell_1,\ell_2,\ell_3\\ \ell\neq 2}} \frac{\nu_j^\ell}{2-\ell}\int_0^\pi \mathrm{d}\theta\ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3}\Lambda_r^{2-\ell}(\epsilon_v,\theta) = \frac{8\hat{m}_1^3}{3\epsilon_v^4}+\frac{2\hat{n}_1^2}{5\hat{m}_1}+\mathcal{O}(\epsilon_v^2). \quad \text{(A.22)}$$

Finally, we treat the remaining sums in (A.20a) and the second term on the first line of (A.20b) together. Firstly, we note that the integral at $\ell = 2$ in $\mathcal{I}_j^{(1)}$ vanishes due the integrand being of the form $P_1^2 + P_2$. Secondly, we make use of (A.19) explicitly and observe that only the $\ell = 0$ term contributes. That is, if we decompose the sum as partial sums in ℓ,

$$\begin{aligned}\sum_{\substack{\ell_1,\ell_2,\ell_3\\ \ell\neq 2}}\left(\frac{1}{5+\ell}-\frac{1}{2-\ell}\right)\int_0^\pi \mathrm{d}\theta\ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3} &= \sum_{\substack{a=0\\ a\neq 2}}^{\infty}\left(\frac{1}{5+a}-\frac{1}{2-a}\right)\sum_{\substack{\ell_1,\ell_2,\ell_3\\ \ell=a}}\int_0^\pi \mathrm{d}\theta\ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3} \\ &= -\frac{2}{5}-\frac{5}{6}\sum_{\substack{\ell_1,\ell_2,\ell_3\\ \ell=1}}\int_0^\pi \mathrm{d}\theta\ s_\theta^3 P_{\ell_1}P_{\ell_2}P_{\ell_3}+\ldots,\end{aligned} \quad \text{(A.23)}$$

we find that all the partial sums with $\ell > 0$ vanish and $-2/5$ is the exact result.

Putting all of the results above together and taking the sum over j, we find

$$\sum_{j=1}^{2n+2}(-1)^j\hat{\xi}_j(\mathcal{I}_j^{(1)}+\mathcal{I}_j^{(2)}) = \frac{8\hat{m}_1^3}{3\epsilon_v^4}+\frac{2}{5}\frac{\hat{n}_1^2-\hat{n}_2\hat{m}_1}{\hat{m}_1}+\mathcal{O}(\epsilon_v^2). \quad \text{(A.24)}$$

Plugging in to (A.14), the unregulated area of the RT hypersurface is

$$A[\zeta_{\mathrm{RT}}] = \pi^4 L_{\mathrm{S}^4}^9\log\left(\frac{2R}{\epsilon_u}\right)\left[\frac{256}{3}\frac{1}{\epsilon_v^4}+\frac{64}{5}\frac{\hat{n}_1^2}{\hat{m}_1^4}-\frac{64}{5}\frac{\hat{n}_2}{\hat{m}_1^3}+\mathcal{O}(\epsilon_v^2)\right]+\mathcal{O}(\epsilon_u^2). \quad \text{(A.25)}$$

It is then straightforward to see that S_{EE} is given by (4.83).

Appendix B

Appendix to Chapter 5

B.1 Fefferman-Graham coordinates

The starting point for computing holographic quantities associated with the two-charge solutions and electrostatic solutions is finding the asymptotic transformation which maps their respective metrics into FG gauge. In this appendix, we will first derive the transformations of (5.16) and find the asymptotic expressions for the metric functions in FG gauge. We will also derive the transformation of (5.21a) into FG gauge. In this process, we will find necessary conditions on the mixing of two of the angular coordinates that allow for the metrics to be put into FG form. The interpretation of this mixing of angular coordinates is interpreted in the field theory language as an identification of the defect superconformal R-symmetry.

B.1.1 Two-charge solutions

In this subsection, we will focus on putting the two-charge solutions in FG gauge. The explicit forms of the metric functions in (5.16) are as follows:

$$\hat{f}_{\mathrm{AdS}}^2 = \kappa^{2/3}\left[\frac{c_\zeta^2\left(q_1+y^2\right)\left(q_2-q_2c_{2\psi}+2y^2\right)}{2y}+y\left(q_2+y^2\right)s_\zeta^2\right]^{1/3}, \tag{B.1a}$$

$$\hat{f}_y^2 = \kappa^{2/3}\frac{\hat{f}_{\mathrm{AdS}}^2 y}{4\left(q_1+y^2\right)\left(q_2+y^2\right)-4y^3}, \tag{B.1b}$$

$$\begin{aligned}\hat{f}_z^2 = \kappa^{2/3}\Bigg[&\frac{c_\zeta^2\left(c_{2\psi}\left(\left(a_2+1\right)^2q_2y+a_2^2y^3-q_2\left(q_1+y^2\right)\right)+\left(a_2+1\right)^2q_2y+\left(a_2^2-2\right)y^3\right)}{2y\hat{f}_{\mathrm{AdS}}^4}\\ &+\frac{s_\zeta^2\left(y\left(\left(a_1^2-1\right)y+\left(q_2+y^2\right)\right)+\left(a_1+1\right)^2q_1\right)}{\hat{f}_{\mathrm{AdS}}^4}+\frac{c_\zeta^2\left(q_1+y^2\right)\left(q_2+2y^2\right)}{2\hat{f}_{\mathrm{AdS}}^4y}\Bigg],\end{aligned} \tag{B.1c}$$

$$\hat{f}_{\phi_1}^2 = \kappa^{2/3} \frac{(q_1 + y^2)\, s_\zeta^2}{4\hat{f}_{\mathrm{AdS}}^4}\,, \tag{B.1d}$$

$$\hat{f}_{\phi_2}^2 = \kappa^{2/3} \frac{c_\psi^2 c_\zeta^2\,(q_2 + y^2)}{4\hat{f}_{\mathrm{AdS}}^4}\,, \tag{B.1e}$$

$$\hat{f}_{z\phi_1}^2 = \kappa^{2/3} \frac{s_\zeta^2\,(a_1 q_1 + a_1 y^2 + q_1)}{\hat{f}_{\mathrm{AdS}}^4}\,, \tag{B.1f}$$

$$\hat{f}_{z\phi_2}^2 = \kappa^{2/3} \frac{c_\psi^2 c_\zeta^2\,(a_2 q_2 + a_2 y^2 + q_2)}{\hat{f}_{\mathrm{AdS}}^4}\,, \tag{B.1g}$$

$$\hat{f}_{\psi}^2 = \kappa^{2/3} \frac{c_\zeta^2\,(q_2 - q_2 c_{2\psi} + 2y^2)}{8\hat{f}_{\mathrm{AdS}}^4}\,, \tag{B.1h}$$

$$\hat{f}_{\zeta}^2 = \kappa^{2/3} \frac{q_1 c_{2\zeta} + 2q_2 c_\psi^2 s_\zeta^2 + q_1 + 2y^2}{8\hat{f}_{\mathrm{AdS}}^4}\,, \tag{B.1i}$$

$$\hat{f}_{\psi\zeta}^2 = \kappa^{2/3} \frac{q_2 c_\psi c_\zeta s_\psi s_\zeta}{2\hat{f}_{\mathrm{AdS}}^4}\,, \tag{B.1j}$$

where we denote $\kappa = \hat{g}^3 N \ell_{\mathrm{P}}^3/2$.

We seek an asymptotic map from $\{y,\, \psi,\, \zeta\}$ to the FG coordinates $\{u, \aleph, \theta\}$ in the large-y/small-u regime. By solving

$$\hat{f}_y^2 dy^2 + \hat{f}_\psi^2 d\psi^2 + \hat{f}_\zeta^2 d\zeta^2 + \hat{f}_{\psi\zeta}^2 d\psi d\zeta = \frac{L^2}{u^2} du^2 + \frac{L^2}{4}\left(c_\theta^2 \hat{\alpha}_\aleph d\aleph^2 + \hat{\alpha}_\theta d\theta^2 + \hat{\alpha}_{\theta\aleph} d\theta d\aleph\right) \tag{B.2}$$

order by order in u, we find that the appropriate asymptotic map is

$$\begin{aligned} y &= \frac{1}{u^2} + \frac{1}{2} + \frac{(2q_1 - q_2)\, c_{2\theta} - 2q_2 c_{2\aleph} c_\theta^2 - 10q_1 - 9q_2 + 3}{48} u^2 + \ldots, \\ \psi &= \aleph + \frac{q_2 s_{2\aleph}}{24} u^4 + \ldots, \\ \zeta &= \theta - \frac{s_{2\theta}\,(q_1 - q_2 c_\aleph^2)}{24} u^4 + \ldots\ , \end{aligned} \tag{B.3}$$

where we have suppressed higher orders in u due to their cumbersome expressions. To complete this map, we need to identify $\kappa = L^3$, where L denotes the radius of the asymptotic AdS$_7$ spacetime.

Mapping all of the other metric functions in (5.16), we find the FG form of the metric to be

$$\begin{aligned} ds_{\mathrm{FG}}^2 = &\frac{L^2}{u^2}(du^2 + \hat{\alpha}_{\mathrm{AdS}} ds_{\mathrm{AdS}_5}^2 + \hat{\alpha}_z dz^2) + L^2 s_\theta^2 \hat{\alpha}_{z\varphi_1} dz d\varphi_1 + L^2 c_\aleph^2 c_\theta^2 \hat{\alpha}_{z\varphi_2} dz d\varphi_2 \\ &+ \frac{L^2}{4}(\hat{\alpha}_\theta d\theta^2 + s_\theta^2 \hat{\alpha}_{\varphi_1} d\varphi_1^2 + c_\theta^2(\hat{\alpha}_\aleph d\aleph^2 + c_\aleph^2 \hat{\alpha}_{\varphi_2} d\varphi_2^2) + \hat{\alpha}_{\theta\aleph} d\theta d\aleph), \end{aligned} \tag{B.4}$$

where we have transformed the angular coordinates using

$$\phi_I = \varphi_I - 2a_I z. \tag{B.5}$$

Note that since ϕ_I and z are all 2π-periodic and $a_I \in \mathbb{Z}/2$, the new angular coordinates φ_I are also 2π-periodic. The metric functions have the asymptotic behaviour

$$\hat{\alpha}_{\mathrm{AdS}} = 1 + \frac{u^2}{2} + \frac{3 - 2q_1 + 3q_2 - 10q_2 c_{2\aleph} c_\theta^2 + 5(2q_1 - q_2)c_{2\theta}}{48} u^4 + \ldots, \tag{B.6a}$$

$$\hat{\alpha}_z = 1 - \frac{u^2}{2} + \frac{3 - 2q_1 + 3q_2 - 10q_2 c_{2\aleph} c_\theta^2 + 5(2q_1 - q_2)c_{2\theta}}{48} u^4 + \ldots, \tag{B.6b}$$

$$\hat{\alpha}_{\varphi_1} = 1 + \frac{10q_2 c_{2\aleph} c_\theta^2 + 5(q_2 - 2q_1)c_{2\theta} + 14q_1 - 11q_2}{24} u^4 + \ldots, \tag{B.6c}$$

$$\hat{\alpha}_{\varphi_2} = 1 + \frac{10q_2 c_{2\aleph} c_\theta^2 + 5(q_2 - 2q_1)c_{2\theta} - 6q_1 + 9q_2}{24} u^4 + \ldots, \tag{B.6d}$$

$$\hat{\alpha}_{z\varphi_1} = q_1 u^4 - q_1 u^6 + \ldots, \tag{B.6e}$$

$$\hat{\alpha}_{z\varphi_2} = q_2 u^4 - q_2 u^6 + \ldots, \tag{B.6f}$$

$$\hat{\alpha}_\theta = 1 + \frac{5q_2 c_{2\aleph} + 2q_1 - 3q_2}{12} u^4 + \ldots, \tag{B.6g}$$

$$\hat{\alpha}_\aleph = 1 + \frac{5(q_2 - 2q_1)c_{2\theta} - 10q_2 c_{2\aleph} s_\theta^2 - 6q_1 - q_2}{24} u^4 + \ldots, \tag{B.6h}$$

$$\hat{\alpha}_{\theta\aleph} = \frac{5q_2 s_{2\theta} s_{2\aleph}}{12} u^4 + \ldots\ . \tag{B.6i}$$

If we had not transformed to φ_I, we would not have been able to put the metric in FG form. We can see this in the original ϕ_I coordinates, where $\hat{\alpha}_{z\phi_I}$ has an $O(1)$ term which is proportional to a_I. FG gauge requires $\hat{\alpha}_{z\phi_I} \sim u^4$, which would mean setting $a_I = 0$. However, the values of the a_I's are set by regularity, i.e.

$$a_I = -\frac{q_I}{q_I + y_+^2}\ , \tag{B.7}$$

and so, we cannot simply tune them to zero without also setting the corresponding $q_I = 0$, which lands us on the pure $\mathrm{AdS}_7 \times S^4$ solution.

B.1.2 Electrostatic solutions

We now turn to deriving the FG form of the metric for the electrostatic solutions. Finding the asymptotic expansions of the metric factors in (5.21a) requires explicit expressions for $\dot{V}$, $\ddot{V}$, $\dot{V}'$, V'', and σ. We can compute the indefinite integral in V for a trial line charge distribution $\varpi_a(\eta) = p_{1+a}\eta + \delta_{1+a}$,

$$-\frac{1}{2}\int d\eta' G(r,\eta,\eta')\varpi_a(\eta') = \frac{p_{1+a}}{2}\Bigg(\sqrt{r^2+(\eta+\eta')^2} - \sqrt{r^2+(\eta-\eta')^2} \\ - \eta\tanh^{-1}\left(\frac{\eta+\eta'}{\sqrt{r^2+(\eta+\eta')^2}}\right) + \eta\tanh^{-1}\left(\frac{\eta-\eta'}{\sqrt{r^2+(\eta-\eta')^2}}\right)\Bigg) \\ + \frac{\delta_{1+a}}{2}\left(\tanh^{-1}\left(\frac{\eta+\eta'}{\sqrt{r^2+(\eta+\eta')^2}}\right) + \tanh^{-1}\left(\frac{\eta-\eta'}{\sqrt{r^2+(\eta-\eta')^2}}\right)\right), \tag{B.8}$$

and then build up the full potential by summing over the intervals. Clearly, evaluating the result above in the $\eta' \to \infty$ region leads to linear and logarithmic divergences. However, when evaluating derivatives of the right-hand side above, these divergences are eliminated, and only derivatives of V appear in all of the computations carried out below and in the main body of the text.

The asymptotically $\mathrm{AdS}_7 \times S^4$ region corresponds to the limits r, $\eta \to \infty$. In order to facilitate the expansion of the derivatives of the electrostatic potential in this region, we redefine $r = \varrho c_\omega$ and $\eta = \varrho s_\omega$, with $\omega \in [0, \pi/2]$, so that

$$f_3^2(dr^2 + d\eta^2) \to f_\varrho^2 d\varrho^2 + f_\omega^2 d\omega^2, \tag{B.9}$$

with $f_\varrho^2 = f_3^2$ and $f_\omega^2 = f_3^2\varrho^2$. The $\mathrm{AdS}_7 \times S^4$ region now lies in the $\varrho \to \infty$ limit. We can compute the asymptotic expansions of the derivatives of the electrostatic potential in this region in terms of its moments as follows,

$$\dot{V} = \varrho s_\omega + m_1 s_\omega - \frac{m_3 c_\omega^2 s_\omega}{2\varrho^2} + \frac{m_5\left(7c_{2\omega}-1\right)c_\omega^2 s_\omega}{16\varrho^4} + \dots\,, \tag{B.10a}$$

$$\ddot{V} = -m_1 c_\omega^2 s_\omega + \frac{m_3\left(5c_{2\omega}+1\right)c_\omega^2 s_\omega}{4\varrho^2} - \frac{m_5\left(28c_{2\omega}+63c_{4\omega}+29\right)c_\omega^2 s_\omega}{64\varrho^4} + \dots\,, \tag{B.10b}$$

$$\dot{V}' = 1 + \frac{m_1 c_\omega^2}{\varrho} + \frac{m_3\left(3-5c_{2\omega}\right)c_\omega^2}{4\varrho^3} + \frac{3m_5\left(21c_{4\omega}-28c_{2\omega}+15\right)c_\omega^2}{64\varrho^5} + \dots\,, \tag{B.10c}$$

$$V'' = \frac{m_1 s_\omega}{\varrho^2} - \frac{m_3\left(5c_{2\omega}+1\right)s_\omega}{4\varrho^4} + \frac{m_5\left(28c_{2\omega}+63c_{4\omega}+29\right)s_\omega}{64\varrho^6} + \dots\,. \tag{B.10d}$$

From these expressions, we can also find the asymptotic behaviour of σ in terms of the moments of the electrostatic potential to be

$$\sigma = 1 + \frac{2m_1}{\varrho} - \frac{m_1^2\left(c_{2\omega}-3\right)}{2\varrho^2} + \frac{m_3\left(1-3c_{2\omega}\right)}{2\varrho^3} + \frac{m_3 m_1\left(1-12c_{2\omega}+3c_{4\omega}\right)}{8\varrho^4} + \dots\,. \tag{B.10e}$$

Together, these expansions can be inserted into the definitions of the metric functions in (5.21a) to give

$$\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}} f_{\mathrm{AdS}}^2 = 4\varrho + 4m_1 + \frac{5m_3 c_{2\omega} + 4m_1^3 s_\omega^2 + m_3}{3m_1\varrho} + \frac{4\left(m_3 - m_1^3\right) s_\omega^2}{3\varrho^2} + \dots\,, \tag{B.11a}$$

$$\frac{(2m_1)^{1/3}}{s_\omega^2 \kappa_{11}^{2/3}} f_{S^2}^2 = 2m_1 - \frac{\left(1 + 5c_{2\omega}\right) m_3 + 4s_\omega^2 m_1^3}{3\varrho^2} + \frac{8m_1\left(m_1^3 - m_3\right) s_\omega^2}{3\varrho^3} + \dots\,, \tag{B.11b}$$

$$\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}} f_\varrho^2 = \frac{2m_1}{\varrho^2} - \frac{(1 + 5c_{2\omega})m_3 - 2s_\omega^2 m_1^3}{3\varrho^4} + \frac{4m_1\left(m_3 - m_1^3\right) s_\omega^2}{3\varrho^5} + \dots\,, \tag{B.11c}$$

$$\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}} f_\beta^2 = 4\varrho + m_1\left(c_{2\omega} - 3\right) + \frac{(1 + 5c_{2\omega})m_3 + 4m_1^3 s_\omega^2}{3m_1\varrho} + \dots\,, \tag{B.11d}$$

$$\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}} f_\chi^2 = 4\varrho + 4m_1 c_{2\omega} + \frac{5m_3 c_{2\omega} + 4m_1^3 s_\omega^2 + m_3}{3m_1\varrho} + \dots\,, \tag{B.11e}$$

$$\frac{(2m_1)^{1/3}}{\kappa_{11}^{2/3}} f_{\beta\chi}^2 = 8\varrho - 8m_1 s_\omega^2 + \frac{2\left(5m_3 c_{2\omega} + 4m_1^3 s_\omega^2 + m_3\right)}{3m_1\varrho} + \dots\,. \tag{B.11f}$$

We again look for an asymptotic map to a set of coordinates $\{u, \theta\}$ in terms of which the metric is in FG form. By taking $\varrho = \varrho(u, \theta)$ and $\omega = \omega(u, \theta)$, and expanding in small u to solve

$$f_\varrho^2 d\varrho^2 + f_\omega^2 d\omega^2 = \frac{L^2}{u^2} du^2 + \frac{L^2}{4}\alpha_\theta d\theta^2 \tag{B.12}$$

order by order, we find

$$\rho = \frac{2m_1}{u^2} + \frac{2m_1^3 c_\theta^2 + m_3\left(5c_{2\theta} - 1\right)}{48m_1^2} u^2 + \frac{\left(m_3 - m_1^3\right) c_\theta^2}{36m_1^2} u^4 + \dots\,, \tag{B.13a}$$

$$\omega = \theta + \frac{\pi}{2} - \frac{\left(m_1^3 + 5m_3\right) s_{2\theta}}{96m_1^3} u^4 + \frac{\left(m_1^3 - m_3\right) s_{2\theta}}{216m_1^3} u^6 + \dots\,. \tag{B.13b}$$

The asymptotic expansions of f_χ^2, f_β^2, and $f_{\beta\chi}^2$ under the above transformation reveal an ambiguity as to which of the angular coordinates should be identified as parametrizing the external $S^1 \subset \mathrm{AdS}_7$ and which as parametrizing the internal $S^1 \subset S^4$ upon mapping to FG gauge. That is, both are characterised by $1/u^2$ divergences at small-u, so that the resulting asymptotic metric is not in FG gauge. To resolve this issue, we introduce

$$\chi = (1 + \mathcal{C}_z)z + a_\varphi\varphi, \qquad \beta = -\mathcal{C}_z z + b_\varphi\varphi, \tag{B.14}$$

where $\mathcal{C}_z \in \mathbb{Z}$ and a_φ and b_φ are arbitrary constants. Note that this transformation parallels the one taken in [339], where $\mathcal{C}_z = 1/\mathcal{C}$ is fixed by the ratio of four-form flux through two 4-cycles, which in turn fixes the mixing parameter between the $U(1)$ symmetries leading to $U(1)_r$ symmetry $\partial_\chi = \partial_z + \frac{1}{\mathcal{C}}\partial_\varphi$ in the field theory. Here we are following the conventions of [4] where the corresponding $\mathcal{C}$ is negative. We then find that the metric functions for the transformed coordinates display the following asymptotic behaviour,

$$\frac{f_\varphi^2}{L^2} = \frac{(a_\varphi + b_\varphi)^2}{u^2} - \frac{1}{8}\Big((2a_\varphi + b_\varphi)^2 c_{2\theta} + b_\varphi(4a_\varphi + 3b_\varphi)\Big) + \dots\ , \tag{B.15a}$$

$$\frac{f_{z\varphi}^2}{L^2} = \frac{2(a_\varphi + b_\varphi)}{u^2} + \frac{1}{4}(2a_\varphi\mathcal{C}_z + b_\varphi(\mathcal{C}_z - 2) - (2a_\varphi + b_\varphi)(\mathcal{C}_z + 2)c_{2\theta}) + \dots\ , \tag{B.15b}$$

$$\frac{f_z^2}{L^2} = \frac{1}{u^2} + \frac{1}{8}(\mathcal{C}_z(\mathcal{C}_z + 4) - (\mathcal{C}_z + 2)^2 c_{2\theta}) + \dots\ , \tag{B.15c}$$

where we introduced the AdS$_7$ radius $L = (16m_1\kappa_{11})^{1/3}$. Setting $a_\varphi = -b_\varphi = -1$ removes the $1/u^2$ divergences in the asymptotic expansions of f_φ^2 and $f_{z\varphi}^2$. In particular, $f_\varphi^2 = L^2 s_\theta^2/4 + \dots$. This identifies the φ-circle as the internal $S^1 \subset S^4$. Furthermore, $f_z^2 = L^2/u^2 + \dots$, as required for the external $S^1 \subset \mathrm{AdS}_7$. The final requirement to achieve an FG parametrization is that $f_{z\varphi}^2 \sim O(u^2)$. Eliminating the u^0 behaviour of $f_{z\varphi}^2$ fixes $\mathcal{C}_z \equiv -2$. Recalling the role of $\mathcal{C}_z$, we see that the defect superconformal R-symmetry is $\partial_\chi = \partial_z - 2\partial_\varphi$.

Having identified the correct combination of angular variables, we can at once express the metric in FG gauge as

$$\begin{aligned} ds^2_{\mathrm{FG}} = &\frac{L^2}{u^2}(du^2 + \alpha_{\mathrm{AdS}} ds^2_{\mathrm{AdS}_5} + \alpha_z dz^2) + L^2 s_\theta^2 \alpha_{z\varphi} dz d\varphi \\ &+ \frac{L^2}{4}\Big(s_\theta^2\alpha_\varphi d\varphi^2 + c_\theta^2 \alpha_{S^2} d\Omega_2^2 + \alpha_\theta d\theta^2\Big)\ , \end{aligned} \tag{B.16}$$

where the metric functions have asymptotic behaviour

$$\alpha_{\mathrm{AdS}} = 1 + \frac{u^2}{2} + \frac{1}{96}\left(10c_\theta^2 + \frac{m_3\,(1-5c_{2\theta})}{m_1^3}\right)u^4 + \frac{(m_3 - m_1^3)\,c_\theta^2}{18m_1^3}u^6 \dots\ , \tag{B.17a}$$

$$\alpha_z = 1 - \frac{u^2}{2} + \frac{1}{96}\left(10c_\theta^2 + \frac{m_3\,(1-5c_{2\theta})}{m_1^3}\right)u^4 + \frac{(m_3 - m_1^3)\,(5c_{2\theta} - 13)}{72m_1^3}u^6 + \dots\ , \tag{B.17b}$$

$$\alpha_\varphi = 1 + \frac{(m_3 - m_1^3)\,(5c_{2\theta} - 7)}{48m_1^3}u^4 + \frac{(m_1^3 - m_3)\,(10c_{2\theta} - 17)}{108m_1^3}u^6 + \dots\ , \tag{B.17c}$$

$$\alpha_{S^2} = 1 + \frac{(m_3 - m_1^3)\,(5c_{2\theta} + 3)}{48m_1^3}u^4 + \frac{(m_1^3 - m_3)\,(5c_{2\theta} + 4)}{54m_1^3}u^6 + \dots\ , \tag{B.17d}$$

$$\alpha_{z\varphi} = \frac{m_1^3 - m_3}{4m_1^3}u^4 - \frac{m_1^3 - m_3}{4m_1^3}u^6 + \dots\ , \tag{B.17e}$$

$$\alpha_\theta = 1 + \frac{m_1^3 - m_3}{24m_1^3}u^4 + \frac{(m_3 - m_1^3)\,(5c_{2\theta} + 9)}{216m_1^3}u^6 + \dots\ . \tag{B.17f}$$

Note that, upon being evaluated on the single kink electrostatic profile in (5.27), the asymptotic metric above recovers the $q_2 = 0$ instance of (B.4); in particular, the coordinate φ maps over to φ_1, while $\aleph$ and φ_2 correspond to, respectively, the polar and azimuthal angles on the asymptotic internal $S^2 \subset S^4$ in the electrostatic description.

B.2 On-shell action

Given a solution to SUGRA equations of motion, one of the most basic quantities that one can compute is the on-shell action. Holographically, the on-shell action is mapped to the free energy of the theory, and so with an even dimensional spherical boundary, has universal divergences that are related to anomalies. In this section, we will compute the on-shell action for the $11d$ uplift of the two-charge solutions and compare the results of the log-divergent parts to the holographic defect anomalies computed in the preceding sections.

B.2.1 Two-charge solutions

In this subsection, we consider the on-shell action for the two-charge solutions. The computation of the on-shell action for the two-charge solutions was originally carried out in their realization as $7d$ $\mathcal{N}=4$ gauged SUGRA domain wall solutions [3]. Here, we will work with the $11d$ uplift in 5.2.2 using the regulating scheme where we subtract off the on-shell action for the $\mathrm{AdS}_7 \times S^4$ vacuum computed in B.3.1.

The starting point for computing the on-shell action for the electrostatic solution is the bosonic part of the $11d$ SUGRA action

$$S = \frac{1}{16\pi G_N^{(11)}} \int_{\mathcal{M}} d^{11}x \, \sqrt{-g_{11}} \left(\mathcal{R} - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right) + \frac{1}{8\pi G_N^{(11)}} \int_{\partial\mathcal{M}} K \Upsilon_{\partial\mathcal{M}} + S_{\mathrm{CS}}, \tag{B.18}$$

where $\Upsilon_{\partial\mathcal{M}}$ is the natural volume form associated to the metric induced on the boundary $\partial\mathcal{M}$, while K is the trace of the boundary extrinsic curvature $K_{MN} = -\frac{1}{2}(\nabla_M \nu_N + \nabla_N \nu_M)$ with ν_M denoting the components of the outward-pointing normal vector to $\partial\mathcal{M}$ and where capital Latin indices M, $N \in \{0, \ldots, 10\}$. Using the equations of motion for the $11d$ metric we can write the bulk term as

$$\sqrt{-g_{11}} \left(\mathcal{R} - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right) d^{11}x = -\frac{1}{3} F_4 \wedge \star F_4. \tag{B.19}$$

Note that for this particular solution, the four-form flux obeys the equation

$$d \star F_4 = 0, \tag{B.20}$$

and consequently the Chern-Simons term S_{CS} vanishes. As a further consequence of the equations of motion for the four-form flux, we can freely exchange $\star F_4$ for dC_6, which due to C_6 being better behaved will make the following computation a bit easier. Using this fact and the bulk equations of motion, the bulk integrand can be expressed as a total derivative. Thus, the on-shell action can be written as a boundary integral

$$S_{\mathrm{OS}} = \frac{1}{16\pi G_N^{(11)}} \int_{\partial\mathcal{M}} \left(2K \Upsilon_{\partial\mathcal{M}} - \frac{1}{3} F_4 \wedge C_6 \right) =: S_{\mathrm{OS,GHY}} + S_{\mathrm{OS,bulk}}. \tag{B.21}$$

The particular solutions we are interested in are asymptotically locally $\mathrm{AdS}_7 \times S^4$. So, in order to regularise the boundary integral, we first map the metric into FG form as in (B.4) using the explicit asymptotic coordinate transformation derived in (B.3). That is, we will define a regulating hypersurface at $u = \epsilon_u$ that will become $\partial\mathcal{M}$ as we take $\epsilon_u \to 0$. Note that due to the presence of an AdS_5 factor, an additional regularization procedure will have to be applied, which we will address later.

Before beginning the computation in earnest, we will need the asymptotic $u \ll 1$ expansions of F_4 and C_6. First, we compute C_6 from (5.19), which yields

$$\begin{aligned} C_6 = L^6 \Bigg\{ & \frac{1}{2} q_2 c_\zeta^2 c_\psi^2 d\phi_2 - \frac{1}{2} q_1 c_\zeta^2 d\phi_1 + \Bigg[y(y^2+q_2) - \frac{c_\zeta^2}{2y} \left(q_2 c_{2\psi} \left(y\left(y - a_2 - 1\right) + q_1 \right) \right. \\ & \left. + 2q_1 y \left(a_1 - y + 1\right) + q_2 y \left(y - a_2 - 1\right) - q_2 q_1 \right) \Bigg] dz \Bigg\} \wedge \Upsilon_{\mathrm{AdS}_5}. \end{aligned} \tag{B.22}$$

We can then use the residual gauge freedom to shift $C_6 \mapsto C_6 + d\Lambda_5 =: \tilde{C}_6$ such that $\tilde{C}_6$ is regular at $y = y_+$. At $y = y_+$, we can use the values for a_I determined from $A_I(y_+) = 0$ to show

$$C_6(y_+) = L^6 \left\{ \frac{1}{2} q_2 c_\zeta^2 c_\psi^2 d\phi_2 - \frac{1}{2} q_1 c_\zeta^2 d\phi_1 + y_+ H_2(y_+) dz \right\} \wedge \Upsilon_{\mathrm{AdS}_5}, \tag{B.23}$$

where the terms in $\Upsilon_{\mathrm{AdS}_5} \wedge dz$ depending on the angular coordinates vanish due to a common factor of $Q(y_+)$ appearing in their coefficients. By demanding that the $\Upsilon_{\mathrm{AdS}_5} \wedge dz$ part of C_6 vanishes at $y = y_+$, we find the appropriate gauge transformation to be

$$\Lambda_5 = -\, z L^6 y_+ H_2(y_+) \Upsilon_{\mathrm{AdS}_5}. \tag{B.24}$$

Using the gauge transformation by Λ_5, we map $\phi_I \to \varphi_I$ and find the asymptotic expansion of $\tilde{C}_6$ to be

$$\tilde{C}_6 = L^6 \left[\left(\frac{1}{u^6} + \frac{3}{2u^4} - \frac{1}{16u^2} (2q_1 - 3(5+q_2) + 10 q_2 c_{2\aleph} c_\theta^2 + 5(q_2 - 2q_1) c_{2\theta}) \right) dz \right] \wedge \Upsilon_{\mathrm{AdS}_5} + \dots\,. \tag{B.25}$$

Next, we need to find F_4, which we can easily compute from (5.19). We then map into FG coordinates, fix $\hat{g} = 2$, and expand in small u. Keeping the most relevant singular terms, we find

$$F_4 = \frac{L^3}{8}\left\{\left[3c_\theta^2 s_\theta d\varphi_1\wedge d\theta + \frac{c_\theta^3}{2}\left(5s_\theta^2\left(2q_1 - q_2 c_{2\aleph} - q_2\right)du\wedge d\varphi_1 + 16q_1 du\wedge dz\right)u^3\right]\wedge \Upsilon_{S^2}\right.$$
$$\left. + \frac{s_\theta|c_\aleph|}{2}du\wedge d\varphi_1\wedge\left(5q_2c_\theta^2 s_{2\aleph}d\theta\wedge d\varphi_2 + 8q_2 dz\wedge\left(\frac{2s_\aleph}{c_\aleph}d\theta - s_{2\theta}d\aleph\right)\right)u^3\right\} + \dots \,. \tag{B.26}$$

Now that we have the asymptotics of the metric, $\tilde{C}_6$, and F_4, we are in position to compute the on-shell action for the two-charge solutions. To begin, we first examine the Gibbons-Hawking-York (GHY) term. We note that after mapping to FG coordinates as in (B.4), the volume form on the regulating cutoff slice at $u = \epsilon_u$ can be easily seen to have small ϵ_u expansion

$$\Upsilon_{\partial\mathcal{M}} = \frac{L^{10}}{16}\left(\frac{1}{\epsilon_u^6} + \frac{1}{\epsilon_u^4} + \frac{5}{16\epsilon_u^2} + \frac{5\left(5c_{2\theta}(q_2 - 2q_1) + 2q_1 + q_2(10c_\theta^2 c_{2\aleph} - 3)\right)}{432}\right)\Upsilon_{\mathrm{AdS}_5}\wedge dz\wedge\Upsilon_{S^4} + \dots \,. \tag{B.27}$$

where we denote $\Upsilon_{S^4} := |c_\aleph| c_\theta^2 s_\theta d\phi_1\wedge d\phi_2\wedge d\theta\wedge d\aleph$. A quick calculation also shows the trace of the extrinsic curvature on the cutoff slice to be given by

$$K = -\frac{6}{L} + \frac{2\epsilon_u^2}{L} - \frac{3\epsilon_u^4}{4L} + \frac{\left(25c_{2\theta}(q_2 - 2q_1) + 10q_1 + 50q_2c_\theta^2 c_{2\aleph} - 15q_2 + 9\right)\epsilon_u^6}{72L} + \dots, \tag{B.28}$$

where we have dropped terms at $O(\epsilon_u)^8$ that depend on the charges but do not contribute to the final result as $\epsilon_u \to 0$. Thus, we find

$$S_{\mathrm{OS,GHY}} = -\mathrm{vol}(\mathrm{AdS}_5)\frac{\pi^2 L^9}{8G_N^{(11)}}\left(\frac{2}{\epsilon_u^6} + \frac{4}{3\epsilon_u^4} + \frac{5}{24\epsilon_u^2}\right) + \dots \,. \tag{B.29}$$

Note that despite K and $\Upsilon_{\partial\mathcal{M}}$ containing non-trivial dependence on the charges, the end result in (B.29) is independent of the charges to $O(\epsilon_u)^0$, and the ϵ_u^0 part of the GHY term explicitly vanishes.

Moving on to find $S_{\mathrm{OS,\ bulk}}$, combining eqs. (B.26) and (B.25) and pulling back on to the $u = \epsilon_u$ hypersurface, we arrive at

$$S_{\mathrm{OS,\ bulk}} = -\,\mathrm{vol}(\mathrm{AdS}_5)\frac{\pi^2 L^9}{16G_N^{(11)}}\left(\frac{2}{3\epsilon_u^6} + \frac{1}{\epsilon_u^4} + \frac{5}{8\epsilon_u^2} - \frac{2q_1(q_2 + y_+(2 + 3y_+))}{15y_+}\right. \tag{B.30}$$
$$\left. - \frac{32q_2 + 48q_2y_+ + 80y_+^3 - 25}{120}\right) + \dots \,.$$

Thus, combining eqs. (B.29) and (B.30) and subtracting of the on-shell action for the $\mathrm{AdS}_7 \times S^4$ vacuum in (B.41), which is recovered by setting $q_I = a_I = 0$ and $y_+ = 1$, the full regulated on-shell action is

$$S_{\rm OS} - S_{\rm OS}^{(\rm vac)} = \frac{\mathrm{vol}(\mathrm{AdS}_5)\pi^2 L^9}{120 y_+ G_N^{(11)}} \Big(q_1 q_2 + (q_1+q_2) y_+ (2+3y_+) + 5y_+ (y_+^3 - 1) \Big) . \tag{B.31}$$

Note that choosing a different Λ_5 while maintaining regularity at $y = y_+$ does not change the final result. Further, using the form of the regulated AdS_5 volume in B.3.2, the log divergent part of the on-shell action for the two-charge solution is given by

$$\begin{aligned} S_{\rm OS}^{(ren)}\Big|_{\log} &= -\frac{N^3}{1920 y_+} \Big(q_1 q_2 + (q_1+q_2) y_+ (2+3y_+) + 5y_+ (y_+^3 - 1) \Big) \\ &= \frac{N^3 (4 q_1 q_2 - 2(q_1+q_2) y_+ (1-y_+) + 5 y_+ (1 - y_+^2))}{1920 y_+} , \end{aligned} \tag{B.32}$$

where in the second line we used $Q(y_+) = 0$. Taking the one-charge $(q_2 \to 0)$ limit, we find

$$\begin{aligned} S_{\rm OS}^{(ren)}\Big|_{\log, 1\text{-charge}} &= \frac{N^3}{1920} (1-y_+)(5 - 2q_1 + 5y_+) \\ &= \frac{N^3}{7680} (1 - \sqrt{1-4q_1})(5(3+\sqrt{1-4q_1}) - 4q_1) . \end{aligned} \tag{B.33}$$

Finally, let us compare (B.32) to the on-shell action of the domain wall solution in $7d$ gauged SUGRA. In [3], the authors, using a similar background subtraction regulating scheme as above, found that the coefficient of the log divergent part of regulated on-shell action for the domain wall solution to be given by

$$S_{\rm OS}^{(ren)}\Big|_{\log} = -\frac{\pi L^5}{8 G_N^{(7)}} (1 - y_+^2) = -\frac{N^3}{768\pi} (1 - y_+^2) , \tag{B.34}$$

where in the last equality we mapped to field theory variables using $G_N^{(7)} = G_N^{(11)}/\mathrm{vol}(S^4)$. We can immediately see a discrepancy with the on-shell action computed in the $11d$ uplift owing to the different dependence on the q_I and y_+. An explanation for the mismatch is not entirely obvious, but it could be rooted in the background subtraction scheme in some way being inadequate for the purpose of this computation. This potential failure mode for such a simple regulating scheme could be interrogated if we had access to a full holographic renormalization scheme for defects in 11d SUGRA.

B.3 Regulating the on-shell action

In this appendix, we collect some of the details of the regulating scheme for the computation of the on-shell action for both the two-charge and electrostatic solutions. Below we compute the vacuum $\mathrm{AdS}_7 \times S^4$ on-shell action, which we will use in the background subtraction scheme. This value will also give a good diagnostic for the known limit case, $q_I = a_I = 0$ for the two charge solutions, that recovers the vacuum geometry. We also briefly discuss computing the renormalised volume of the AdS_5 part of the geometry.

B.3.1 $\mathbf{AdS_7} \times S^4$

In this subsection, we compute the on-shell action for the vacuum $\mathrm{AdS}_7 \times S^4$ geometry that we use in our background subtraction scheme. The data relevant to specify this solution to bosonic theory in (B.18) is the metric

$$ds_{11}^2 = L^2 \left(dx^2 + \cosh^2(x) ds_{\mathrm{AdS}_5}^2 + \sinh^2(x) dz^2\right) + \frac{L^2}{4} d\Omega_4^2, \tag{B.35}$$

with $0 \le x \le \infty$, and the four-form flux and its Hodge dual

$$F_4 = -\frac{3L^3}{8} \mathrm{Y}_{S^4}, \tag{B.36a}$$

$$\star_{11} F_4 = 6L^6 \cosh^5(x) \sinh(x) dx \wedge dz \wedge \mathrm{Y}_{\mathrm{AdS}_5}. \tag{B.36b}$$

Since we are working with vacuum $\mathrm{AdS}_7 \times S^4$, the transformation to FG gauge is simply

$$x = -\ln(u/2), \tag{B.37}$$

where the FG radial coordinate is valued $0 \le u \le 2$. In FG gauge, the metric takes the form

$$ds_{11}^2 = \frac{L^2}{u^2} \left(du^2 + \left(1 + \frac{u^2}{2} + \frac{u^4}{16}\right) ds_{\mathrm{AdS}_5}^2 + \left(1 - \frac{u^2}{2} + \frac{u^4}{16}\right) dz^2\right) + \frac{L^2}{4} d\Omega_4^2. \tag{B.38}$$

The four-form flux has no functional dependence on x and is unchanged in transforming to FG gauge, while the seven-form flux becomes

$$\star_{11} F_4 = 6L^6 \left(\frac{1}{u^7} + \frac{1}{u^5} + \frac{5}{16u^3} - \frac{5u}{256} - \frac{u^3}{256} - \frac{u^5}{4096}\right) du \wedge dz \wedge \mathrm{Y}_{\mathrm{AdS}_5} + \dots\,. \tag{B.39}$$

With the asymptotics of the metric and fluxes in hand, we can easily compute the on-shell action. Note that the GHY term for the vacuum $\mathrm{AdS}_7 \times S^4$ solution is trivially identical to the expression found in (B.29), and so we will not reproduce it here. The bulk action is then computed from the $F_4 \wedge \star F_4$ term, which after inserting eqs. (B.36a) and (B.39), introducing a radial cutoff at $u = \epsilon_u \ll 1$, and integrating over the

$\mathrm{AdS}_7 \times S^4$ geometry gives

$$S^{(\mathrm{vac})}_{\mathrm{OS,bulk}} = -\frac{L^9\pi^2}{8G_N^{(11)}}\,\mathrm{vol}(\mathrm{AdS}_5)\left(\frac{1}{3\epsilon_u^6}+\frac{1}{2\epsilon_u^4}+\frac{5}{16\epsilon_u^2}-\frac{11}{48}\right)+\dots\,. \tag{B.40}$$

Combining with the GHY term, we find

$$S^{(\mathrm{vac})}_{\mathrm{OS}} = -\frac{\pi^2L^9}{8G_N^{(11)}}\,\mathrm{vol}(\mathrm{AdS}_5)\left(\frac{1}{3\epsilon_u^6}+\frac{5}{3\epsilon_u^5}+\frac{1}{2\epsilon_u^4}+\frac{1}{\epsilon_u^3}+\frac{5}{16\epsilon_u^2}+\frac{5}{48\epsilon_u}-\frac{11}{48}\right)+\dots\,. \tag{B.41}$$

Finally, we note that since $d\star F_4 = 0$ we can introduce C_6 so that $dC_6 = \star_{11}F_4$. We can then perform the bulk integral of $F_4 \wedge C_6$ over the radial cutoff slice at ϵ_u with the pullback of the six-form potential being given by

$$C_6 = 3L^6\left(\frac{1}{3\epsilon_u^6}+\frac{1}{2\epsilon_u^4}+\frac{5}{16\epsilon_u^2}-\frac{11}{48}+\frac{5\epsilon_u^2}{256}+\frac{\epsilon_u^4}{512}+\frac{\epsilon_u^6}{12288}\right)dz\wedge \Upsilon_{\mathrm{AdS}_5}\,. \tag{B.42}$$

Crucially, we have used the residual gauge freedom to fix the six-form potential to be regular at the origin of AdS_7, i.e. we pick a gauge such that $C_6\Big|_{u=2} = 0$. The computation using C_6 then gives the same result as above.

B.3.2 Renormalised AdS_5 volume

Even accounting for the removal of divergences coming from the asymptotically AdS_7 part of the geometry via background subtraction, we are still left to deal with the volume of the AdS_5 factor in the on-shell action. In order to regularise the remaining polynomial divergences and read off the universal log-divergent part of the on-shell action, we will simply treat the intrinsic parts of the AdS_5 geometry using standard counterterms in holographic renormalization and neglecting any divergences associated with the embedding. This renormalization scheme is admittedly simplistic as it only treats the set of counterterms associated with the intrinsic geometry of the AdS_5 submanifold. However, since the background subtraction scheme leaves behind only divergences from the volume of the AdS_5 and we choose the boundary geometry to be $S^4 \hookrightarrow \mathbb{R}^6$, only defect Weyl anomalies constructed purely from the intrinsic geometry should contribute, which will be accounted for in the scheme we have chosen. The caveat is that there may be structures for which we have not accounted in the full set of $11d$ counterterms, which is difficult to construct, whose pullback to the AdS_5 submanifold contains terms that contribute to the log divergence in a similar way. Absent a full holographic renormalization scheme for solutions to SUGRA dual to defects, which would replace background subtraction scheme as well, this scheme choice constructing counterterms only for the intrinsic geometry of the AdS_5 submanifold is the best tool available.

Moving on, the volume of AdS_5 has well known divergences. In order to systematically remove them and reveal any universal log-divergent terms, we consider AdS_5 in global coordinates with an S^4 boundary:

$$ds^2_{\mathrm{AdS}_5} = dx^2 + \sinh^2(x)\ d\Omega_4^2. \tag{B.43}$$

For simplicity, we consider the round metric on S^4. Computing the AdS_5 volume requires regulating the large x behaviour, and so we introduce a radial cutoff $\Lambda_x \equiv -\log\frac{\epsilon_x}{2}$ for $\epsilon_x \ll 1$. Then, expanding in small ϵ_x

$$\mathrm{vol}(\mathrm{AdS}_5) = \frac{8\pi^2}{3}\int_0^{\Lambda_x} dx \sinh^4(x) = \frac{2\pi^2}{3\epsilon_x^4} - \frac{4\pi^2}{3\epsilon_x^2} - \pi^2\log\frac{\epsilon_x}{2} + \dots\ . \tag{B.44}$$

We regulate the volume using covariant counterterms[1] added on the radial cutoff slice that are standard in AdS_5 holographic renormalization [288, 325]

$$S_{\mathrm{CT},1} = -\frac{1}{4}\int d\Omega_4\sqrt{|g_{\epsilon_x}|} = -\frac{2\pi^2}{3\epsilon_x^4} + \frac{2\pi^2}{3\epsilon_x^2} - \frac{\pi^2}{4} + \dots, \tag{B.45a}$$

$$S_{\mathrm{CT},2} = \frac{1}{48}\int d\Omega_4\sqrt{|g_{\epsilon_x}|}\mathcal{R}_{\epsilon_x} = \frac{2\pi^2}{3\epsilon_x^2} - \frac{\pi^2}{3} + \dots, \tag{B.45b}$$

where $\sqrt{|g_{\epsilon_x}|} = (1-\epsilon_x)^4\sqrt{|g_{S^4}|}/16\epsilon_x^2$ and $\mathcal{R}_{\epsilon_x} = 12\,\mathrm{csch}^2(\epsilon_x)$ are the volume form and the intrinsic Ricci scalar on the cutoff slice, respectively, built from the induced AdS_5 metric. Adding these counterterms to the bulk action, we see that the holographically renormalised volume of the unit AdS_5 takes the well-known form

$$\mathrm{vol}(\mathrm{AdS}_5) = -\pi^2\log\frac{\epsilon_x}{2} + \dots\ . \tag{B.46}$$

To complete the regularization of the on-shell actions for the vacuum $\mathrm{AdS}_7\times S^4$ and two-charge solutions and extract the universal contributions to the defect free energy, we replace $\mathrm{vol}(\mathrm{AdS}_5) = -\pi^2\log(\epsilon_x/2)$ wherever it appears.

[1] To be complete, we should also fix finite counterterms to ensure that we are in a supersymmetry preserving scheme, but we will forego addressing this here as it is not germane to the problem at hand.

Appendix C

Useful identities

猿も木から落ちる
(Even monkeys fall from trees)

Japanese idiom

In this appendix, we enumerate various useful identities and conventions that we used throughout the thesis. We have organised them as follows. Section C.1 is dedicated to differential geometry conventions and identities. Most conventions follow [194]. For completeness, we also include basic identities which are not explicitly used within the thesis, but could prove useful to the reader. Section C.2 is entirely dedicated to the geometric conventions of Chapter 6. Section C.4 will present some basic definitions and useful formulae for Young tableaux. They can prove useful in understanding our Young tableaux description of the $SU(4)$ representations of Weyl multiplet field content in Chapter 7. Finally, Section C.5 presents basic definitions and identities of the hypergeometric functions. These are used to resum the various correlation functions found in Chapter 8.

C.1 Differential Geometry Identities

Let $\mathcal{M}$ be a (smooth) differential manifold of dimension d. Throughout this thesis, we only consider such manifolds which are orientable and equipped with a metric g. When the metric is positive-definite, we say that $\mathcal{M}$ is a Riemannian manifold and its metric has signature $(0, d)$[1]. We are also interested in the setup where the signature is so-called Lorentzian, $(1, d-1)$, for which the metric is no-longer positive. In both cases, the metric g is an example of a structure on $\mathcal{M}$ which reduces its structure group from

[1]Importantly, we always use the notation where we indicate the number of negative eigenvalues first. This extends to our notation for groups whose elements preserve such metrics. eg. $SO(1, d-1)$.

$GL(d,\mathbb{R})$ to one of its subgroups. We give a (non-exhaustive) list of these additional structures and their structure group in Table C.1.

Structure on $\mathcal{M}$	G-structure
Orientation	$SL(d,\mathbb{R})$
Metric	$O(p,q)$, $p+q=d$
Complex	$GL(d/2,\mathbb{C})$
Hermitian	$U(d/2,\mathbb{C})$
Kähler	$U(d/2,\mathbb{C})$
Calabi-Yau	$SU(d/2,\mathbb{C})$
Symplectic	$Sp(d,\mathbb{R})$

TABLE C.1: List of common structures on a differential manifold $\mathcal{M}$, together with their corresponding G-structure group. The map is not one-to-one as many structures can require additional integrability conditions which are not made explicit from the G-structures. Almost complex and complex structures for instance share the same G-structures but the former requires the vanishing of the Nijenhuis tensor.

Not every smooth differential manifold can admit a given structure, and determining which manifold $\mathcal{M}$ can admit which structure has been long standing topic of research. Certain obstructions to endowing $\mathcal{M}$ with a given structure can be measured through a set of differential invariants known as *characteristic classes.* These are cohomology classes of the manifold's vector bundles or principal bundles. The Stiefel-Whitney classes for instance are used to determine whether the manifold is orientable w_0, admits a spin structure w_1, or a spin-c structure w_2, etc. Unless stated otherwise, we only consider oriented, Riemannian/Lorentzian, spin manifolds in this thesis.

Summary of conventions:

1. In Lorentzian signature, we used the mostly plus convention $\eta = \text{diag}(-1,1,\dots,1)$. We denote the signature by the number of negative eigenvalues followed by the number of positive ones, eg $(1,d-1)$ for the Lorentzian one.

2. The spin connection is so that the spheres has positive Ricci scalar curvature, ${\omega_\mu}^{ab} = {e^a}_\nu(\partial_\mu e^{b\nu} + {\Gamma^\nu}_{\mu\rho}e^{b\rho})$.

3. In Lorentzian signature, the Levi-Civita symbol obeys $\epsilon^{01\cdots d} = -1$ and $\epsilon_{01\cdots d} = 1$.

C.1.1 The ϵ-δ identities (3d Cartesian)

Let us now consider a Riemannian manifold of dimension 3 with the standard Euclidean metric, δ. Its local patches U_i admits a set of coordinates x^i, $i \in \{1, 2, 3\}$ for which δ_{ij} is the Kronecker delta symbol. The Levi-Civita tensor on this manifold is the totally anti-symmetric tensor of rank 3 for which $\epsilon_{123} = 1$. The indices are raised and lowered using the Euclidean metric, $\epsilon^i{}_{kl} = \delta^{ij}\epsilon_{jkl}$. The following identities then follow

$$\epsilon^{ijk}\epsilon_{ijk} = 3! \tag{C.1a}$$
$$\epsilon^{ijk}\epsilon_{ijl} = 2\delta^k{}_l \tag{C.1b}$$
$$\epsilon^{ijk}\epsilon_{ilm} = \delta^j{}_l\delta^k{}_m - \delta^j{}_m\delta^k{}_l := \delta^{jk}_{lm} \tag{C.1c}$$
$$\epsilon^{ijk}\epsilon_{lmn} = \delta^{ijk}_{lmn} \tag{C.1d}$$

C.1.2 The ϵ-δ identities (4d Lorentzian)

Here we consider a Lorentzian manifold of dimension 4 with the standard Minkowski metric, η. Its local patches U_i admits a set of coordinates x^μ, $\mu \in \{0, 1, 2, 3\}$ for which $\eta_{\mu\nu}$ is represented by the matrix $\mathrm{diag}(-1, 1, 1, 1)$[2]. The Levi-Civita tensor on this manifold is the totally anti-symmetric tensor of rank 4 for which $\epsilon_{0123} = 1$. The indices are raised and lowered using the Minkowski metric, $\epsilon^\mu{}_{\alpha\beta\gamma} = \eta^{\mu\nu}\epsilon_{\nu\alpha\beta\gamma}$. A direct consequence of this is that the tensor with all indices raised obeys $\epsilon^{0123} = -1$. The following identities then follow

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta} = -4! \tag{C.2a}$$
$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\nu\rho\sigma} = -3!\delta^\mu{}_\alpha \tag{C.2b}$$
$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\rho\sigma} = -2\delta^{\mu\nu}_{\alpha\beta} \tag{C.2c}$$
$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\sigma} = -\delta^{\mu\nu\rho}_{\alpha\beta\gamma} \tag{C.2d}$$
$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = -\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \tag{C.2e}$$

C.1.3 General identities

Let $\mathcal{M}$ be a pseudo-Riemannian (smooth) manifold of dimension d. Pick a patch U_i and a set of local coordinates on U_i, say x^μ. If g denotes the metric on $\mathcal{M}$, we can be slightly abusive with the notation and define $g_{\mu\nu}$ as the local representation of it on this patch. The Levi-Civita tensor is an element of $\Gamma(\bigwedge^d T^*\mathcal{M})$, in other words it is a top form on $\mathcal{M}$. Its components on U_i, $\epsilon_{\mu_1\cdots\mu_d}$, can be defined using a local parallelisation of the frame bundle, i.e. the vielbeins $e^a{}_\mu$

$$\epsilon_{\mu_1\cdots\mu_d} = e^{a_1}{}_{\mu_1}\cdots e^{a_1}{}_{\mu_d}\epsilon_{a_1\cdots a_d}, \tag{C.3}$$

[2]We follow our conventions and use the mostly plus metric.

where $\epsilon_{a_1\cdots a_d}$ is the Levi-Civita symbol in d dimensions. We decide on the convention where $\epsilon_{12\cdots m} = 1$. Its indices can be lowered and raised using the metric and its inverse, as usual for tensorial quantities.

We are now ready to formulate the following identities involving the Levi-Civita tensor and the metric components

$$\epsilon^{\mu_1\cdots\mu_d} = \det(g)^{-1}\epsilon_{\mu_1\cdots\mu_d}, \tag{C.4a}$$

$$\epsilon^{\mu_1\cdots\mu_d}\epsilon_{\mu_1\cdots\mu_d} = d!\det(g), \tag{C.4b}$$

$$\epsilon^{\mu_1\cdots\mu_r\mu_{r+1}\cdots\mu_d}\epsilon_{\mu_1\cdots\mu_r\nu_{r+1}\cdots\nu_d} = \frac{r!(d-r)!}{\det(g)}\delta^{[\mu_{r+1}}{}_{\nu_{r+1}}\cdots\delta^{\mu_d]}{}_{\nu_d}. \tag{C.4c}$$

Furthermore, if $a_\mu{}^\nu$ are the components of an element of $\Gamma(T\mathcal{M}\oplus T^*\mathcal{M})$, the following holds true on U_i

$$a_{\mu_1}{}^{\nu_1}\cdots a_{\mu_d}{}^{\nu_d}\epsilon_{\nu_1\cdots\nu_d} = \det(a)\epsilon_{\mu_1\cdots\mu_d}. \tag{C.5}$$

The Hodge star operator maps p-forms to $(d-p)$-forms,
$\star:\Gamma(\bigwedge^p T^*\mathcal{M}) \longrightarrow \Gamma(\bigwedge^{d-p} T^*\mathcal{M})$ bijectively. This is possible since the vector space of p-forms at a point in $\mathcal{M}$ has the same dimension as that of $(d-p)$-forms, namely $\binom{d}{p}$. On the patch U_i, we can use the Levi-Civita tensor to define the star of any p-form ω,

$$\omega = \frac{1}{p!}\omega_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \tag{C.6a}$$

$$\star\omega = \frac{\sqrt{|g|}}{p!(d-p)!}\omega_{\mu_1\cdots\mu_p}\epsilon^{\mu_1\cdots\mu_p}{}_{\mu_{p+1}\cdots\mu_m}dx^{\mu_{p+1}}\wedge\cdots\wedge dx^{\mu_d}. \tag{C.6b}$$

Taking two instances of the Hodge dual gives a automorphism of the space of p-forms. Depending on the dimension d, rank of the forms p and signature of the manifold, this maps a p-form to plus or minus itself,

$$\star\star\omega = (-1)^{p(d-p)}\omega \quad \text{Riemannian}, \tag{C.7a}$$

$$\star\star\omega = -(-1)^{p(d-p)}\omega \quad \text{Lorentzian}. \tag{C.7b}$$

Let α and β be two p-forms on $\mathcal{M}$, i.e. $\alpha,\beta\in\Gamma(\bigwedge^p T^*\mathcal{M})$. The Hodge dual can be used to define a natural inner product on p-forms given as the integral of $\alpha\wedge\star\beta$. In terms of components, this is written as

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \tag{C.8a}$$

$$\beta = \frac{1}{p!}\beta\mu_1\cdots\mu_p dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}, \tag{C.8b}$$

$$\alpha\wedge\star\beta = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\beta^{\mu_1\cdots\mu_p}\star 1 \tag{C.8c}$$

$$= \frac{\sqrt{|g|}}{p!}\alpha_{\mu_1\cdots\mu_p}\beta^{\mu_1\cdots\mu_p}dx^1\wedge\cdots dx^d. \tag{C.8d}$$

In other words $\alpha\wedge\star\beta$ is proportional to the canonical volume form $\sqrt{|g|}dx^1\wedge\cdots\wedge dx^d$. To derive the above formulae we used the following way of rewriting a generic top-form basis, in therms of the Levi-Civita tensor,

$$dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_d} = -\epsilon^{\mu_1\cdots\mu_d}dx^1\wedge\cdots\wedge dx^d. \tag{C.9}$$

C.1.3.1 Cartan structure equations

Our conventions of the spin connection will always follow that of [194], namely that for which the sphere has positive Ricci curvature. Given a set of vielbeins for $\mathcal{M}$, the spin connection $\omega\in\Gamma(\mathfrak{so}(p,q)\otimes T^*\mathcal{M})$ (given as an $\mathfrak{so}(p,q)$-valued one form) obeys the Cartan structure equations

$$T^a = de^a + \omega^a{}_b\wedge e^b, \tag{C.10a}$$

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b. \tag{C.10b}$$

Where T^a are the torsion one-forms and $R^a{}_b$ the curvature two-forms. Requiring a torsion-free connection, one can solve the above equation for ω either implicitly or using the explicit formulation in terms of its components on a local patch,

$$\omega_\mu{}^{ab} = e^a{}_\nu\nabla_\mu e^{b\nu}, \tag{C.11a}$$

$$\nabla_\mu e^{b\nu} = \partial_\mu e^{b\nu} + \Gamma^\nu{}_{\mu\rho}e^{b\rho}, \tag{C.11b}$$

where $\Gamma^\mu{}_{\nu\rho}$ are the Christoffel symbols associated to the Levi-Civita connection ∇.

C.2 Geometric conventions of Chapter 6

We will now describe the various geometric conventions used throughout Chapter 6. Importantly, we will describe the metric fibration of $S^3 \times S^1$ in terms of two parameters, ϵ_1 and ϵ_2, and detail our choice of vielbeins for the resulting Riemannian manifold $S^3 \times_\Omega S^1$. Finally, we will display the corresponding spin connection one-form.

However, before introducing the metric fibration, allow us to consider a geometry which is that of the Cartesian product between a unit three-sphere, S^3 and a one-sphere, S^1_β, of radius β. We will use toroidal coordinates on the former and write the metric as

$$\begin{aligned} &ds^2 = d\theta^2 + \sin^2(\theta)d\varphi^2 + \cos^2(\theta)d\tau^2 + \beta^2 dt^2, \\ &\theta \in [0, \frac{\pi}{2}[\;, \quad \varphi \in [0, 2\pi[\;, \quad \tau \in [0, 2\pi[\;, \quad t \in [0, 2\pi[\;. \end{aligned} \tag{C.12}$$

The vielbeins allow us to identify the tangent bundle of $S^3 \times S^1_\beta$ with a vector bundle of structure group $SO(4)$. In other words, in any patch on the base manifold, we can express the metric in terms of the vielbeins via $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \delta_{ab}$. Then, $e^a{}_\mu$ are the local component expressions of the vielbein one-form $e^a = e^a{}_\mu dx^\mu$. While it is possible to construct diagonal vielbeins for this metric, such a choice would lead to a rather verbose expression for the spin connection one-form. Instead, we will consider the following vielbeins,

$$e^1 = \sin(\varphi+\tau)d\theta + \cos(\theta)\sin(\theta)\cos(\varphi+\tau)d\varphi - \cos(\theta)\sin(\theta)\cos(\varphi+\tau)d\tau \;, \tag{C.13a}$$

$$e^2 = -\cos(\varphi+\tau)d\theta + \cos(\theta)\sin(\theta)\sin(\varphi+\tau)d\varphi - \cos(\theta)\sin(\theta)\sin(\varphi+\tau)d\tau \;, \tag{C.13b}$$

$$e^3 = \sin^2(\theta)d\varphi + \cos^2(\theta)d\tau \;, \tag{C.13c}$$

$$e^4 = \beta dt \;. \tag{C.13d}$$

The spin connection, a connection on the tangent bundle, is locally a 1-form ω valued in the special orthogonal Lie algebra $\mathfrak{so}(d)$, where d is the dimension of the base manifold.

By this fact, the matrix representation of $\mathfrak{so}(d)$ allows us to identify ω_{ab} as the components of a skew-symmetric, traceless matrix. After identifying a set of vielbeins, the one-form ω is completely determined by solving the Cartan structure equations with vanishing torsion, $de^a + \omega^a{}_b \wedge e^b = 0$. This can either be done via ansatz methods or by using the coordinate expression $\omega_\mu{}^{ab} = e^a{}_\nu \nabla_\mu e^{b\nu}$, where ∇_μ is the Levi-Civita connection along the unit vector ∂_μ. Its action on the vielbein component $e^{b\nu}$ takes the form $\nabla_\mu e^{b\nu} = \partial_\mu e^{b\nu} + \Gamma^\nu{}_{\mu\rho} e^{b\rho}$, where $\Gamma^\nu{}_{\mu\rho}$ are the Christoffel symbols associated to the metric. One can easily check that the one-form built from the components defined above automatically satisfies the torsion-less Cartan structure equations. Using either of the two methods, and with the choice of vielbeins made above, the spin connection 1-form takes the simple form

$$\omega_{ab} = \begin{cases} \displaystyle\sum_{c=1}^{3} \varepsilon_{abc} e^c & \text{if } a, b \in \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases} \tag{C.14}$$

Our sign convention for the curvature of the spin connection is given in equation (3.12).

We now wish to introduce a geometric deformation of $S^3 \times S^1_\beta$ in the form of a metric fibration of the S^3 along the S^1_β. This is similar to the Ω-deformation of $\mathbb{R}^4$ first introduced in [395–397] in the context of Seiberg-Witten theory. However, the deformation here will remain purely geometric and will not further grade the twisted theory. The constants ϵ_1 and ϵ_2 parameterise this fibration, as written in equation (6.5),

$$e^a \to e^a + e^a{}_\mu v^\mu dt, \qquad\qquad \text{where} \quad v = \epsilon_1 \partial_\varphi + \epsilon_2 \partial_\tau . \tag{C.15}$$

More explicitly, the deformed vielbeins take the following form,

$$\begin{aligned} e^1 &= \sin(\varphi+\tau) d\theta + \cos\theta \sin\theta \ \cos(\varphi+\tau) d\varphi - \cos\theta\sin\theta \ \cos(\varphi+\tau) d\tau \\ &\quad + \cos\theta\sin\theta \ \cos(\varphi+\tau)(\epsilon_1 - \epsilon_2) dt \ , & \text{(C.16a)} \\ e^2 &= -\cos(\varphi+\tau) d\theta + \cos\theta\sin\theta \ \sin(\varphi+\tau) d\varphi - \cos\theta\sin\theta \ \sin(\varphi+\tau) d\tau \\ &\quad + \cos\theta\sin\theta \ \sin(\varphi+\tau)(\epsilon_1-\epsilon_2) dt \ , & \text{(C.16b)} \\ e^3 &= \sin^2\theta \ d\varphi + \cos^2\theta \ d\tau + (\epsilon_1 \sin^2\theta + \epsilon_2\cos^2\theta) dt \ , & \text{(C.16c)} \\ e^4 &= \beta dt \ . & \text{(C.16d)} \end{aligned}$$

Following this deformation, the metric is given by equation (6.6). Naturally, in the limit of vanishing $\epsilon_{1,2}$, it recovers equation (C.12), as expected.

Finally, we may note that the deformation changes the spin connection only by a single factor along the fourth vielbein,

$$\omega_{ab} = \begin{cases} \displaystyle\sum_{c=1}^{3} \varepsilon_{abc} e^c - \beta(\epsilon_1+\epsilon_2)\varepsilon_{ab3} e^4 & \text{if } a, b \in \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases} \tag{C.17}$$

C.3 Complexification of the equivariant parameters

In this section, we briefly review how the complexification of the equivariant parameters $\epsilon_{1,2} = \mathrm{Re}(\epsilon_{1,2}) + i\mathrm{Im}(\epsilon_{1,2})$ which specify (together with β) a primary Hopf surface $\mathcal{M}_4^{p,q}$ leads to a violation of the local roundness condition $|p| = |q|$. As mentioned before, the local holomorphicity of the partition function in the complex structure moduli p and q

guarantees that any such squashing would be invisible in our analysis, which is why we could retain full generality while restricting to the $|p| = |q|$ fugacity subspace.

Consider the following Hermitian metric on a primary Hopf surface $\mathcal{M}_4^{p,q}$,

$$g_{\mathcal{M}_4^{p,q}} = \sqrt{\frac{\beta + \mathrm{Im}(\epsilon_1)}{\beta + \mathrm{Im}(\epsilon_2)}} e^{2(\beta+\mathrm{Im}(\epsilon_2))t} dw d\bar{w} + \sqrt{\frac{\beta + \mathrm{Im}(\epsilon_2)}{\beta + \mathrm{Im}(\epsilon_1)}} e^{2(\beta+\mathrm{Im}(\epsilon_1))t} dz d\bar{z} \ . \tag{C.18}$$

Via the transformation

$$w = p^t e^{i\tau} \cos\theta \ , \qquad\qquad z = q^t e^{i\varphi} \sin\theta \ , \tag{C.19}$$

we find the following metric

$$\begin{aligned} g_{\mathcal{M}_4^{p,q}} = ds^2_{S^3_\ell} &+ 2\ell^{-1}\mathrm{Re}(\epsilon_1)\sin^2\theta d\varphi dt + 2\ell\mathrm{Re}(\epsilon_2)\cos^2\theta d\tau dt \\ &+ \frac{1}{\beta^{-2}}\Big[\ell^{-1}(1 + 2\beta^{-1}\mathrm{Im}(\epsilon_1) + \beta^{-2}|\epsilon_1|^2)\sin^2\theta \\ &\qquad + \ell(1 + 2\beta^{-1}\mathrm{Im}(\epsilon_2) + \beta^{-2}|\epsilon_2|^2)\cos^2\theta\Big] dt^2 \end{aligned} \tag{C.20}$$

where the squashed S^3 metric is

$$g_{S^3_\ell} = (\ell^{-1}\cos^2\theta + \ell\sin^2\theta)d\theta^2 + \ell^{-1}\sin^2\theta d\varphi^2 + \ell\cos^2\theta d\tau^2 \tag{C.21}$$

with squashing parameter

$$\ell = \sqrt{\frac{1 + \beta^{-1}\mathrm{Im}(\epsilon_1)}{1 + \beta^{-1}\mathrm{Im}(\epsilon_2)}} \ . \tag{C.22}$$

Note that, if $\epsilon_{1,2}$ are taken to be real again, then $\ell = 1$: the S^3 unsquashes, and the metric in equation (C.20) recovers precisely the deformed metric on $S^3 \times_\Omega S^1$.

C.4 Young Tableau

To every irreducible representation of $SU(N)$ there is a corresponding Young tableau with at most N rows.[3]

It is customary to denote a representation using its dimension alone. Whenever two irreps have the same dimension, we add a prime. If two irreps are conjugate to each other, we prefer the overline notation. For instance, the fundamental representation of $SU(N)$ is N-dimensional and its Young tableau is given by the one box $\square$. We denote it by $\mathbf{N}$. Its conjugate representation, on the other hand, has the same dimension but is described by $N-1$-stacked boxes $\begin{array}{c}\square\\ \vdots\\ \square\end{array}$. We denote it by $\overline{\mathbf{N}}$. While we will try our best, no guaranteed is made that we will stick to this refined notation. More often than not, we might omit primes or overlines, and only specify the dimension.

C.4.1 Calculating the dimension

Let λ be a Young tableau for a given irrep of $SU(N)$. We wish to determine the dimension of said irrep from the tableau alone. This can be done by counting the total number of *standard Young tableau* λ can accommodate. One can alleviate the need to explicitly count those via the following algorithm.

First, in each box, count the total number of boxes below b and to the right r plus one. Each box within the Young tableau will then contain the numbers $b_i + r_i + 1$. The *hook length* h_λ is defined as the product of all these numbers for all the cells in λ

$$h_\lambda = \prod_{i\in\lambda}(b_i + r_i + 1). \tag{C.23}$$

Example C.1. *For the Young tableau* $\ydiagram{4,3,1}$*, the labelling described above gives*

6	4	3	1
4	2	1	
1			

From this, we find that the hook length is $h_\lambda = 6\cdot 4\cdot 3\cdot 1\cdot 4\cdot 2\cdot 1\cdot 1 = 576$.

[3]Young tableaux are also useful in classifying irreducible representations of $GL(N,\mathbb{C})$, $SL(N,\mathbb{C})$, $U(N)$, $O(N)$, $SO(N)$ and $Sp(2N)$. However, we will only need this machinery for $SU(N)$ representations.

The number of standard Young tableaux is then given as the ratio between two quantities. The numerator is calculated as follows. Label the top left cell with N. For every cell to the right, increase the number by one. Repeat this construction on the second row, starting from $N-1$ and on the third row and so on until all cells are assigned a number. Finally, take the product of all these numbers to get the numerator. The denominator is the hook length described previously. If we give each cell in the tableau a coordinate (i,j), where $(1,1)$ labels the top left cell, and i increases downwards and j rightwards, the dimension formula is more easily given as the product

$$\dim(\lambda) = \prod_{(i,j)\in\lambda} \frac{N+j-i}{h_\lambda}. \tag{C.24}$$

Example C.2. *For example, consider the Young tableau used in example C.1 for the group $SU(4)$. We apply the construction described above to the numerator and denominator, which gives the following ratio.*

$$\dim\left(\ydiagram{4,3,1}\right) = \frac{\begin{ytableau} 4 & 5 & 6 & 7 \\ 3 & 4 & 5 \\ 2 \end{ytableau}}{\begin{ytableau} 6 & 4 & 3 & 1 \\ 4 & 2 & 1 \\ 1 \end{ytableau}} = \frac{4\cdot5\cdot6\cdot7\cdot3\cdot4\cdot5\cdot2}{6\cdot4\cdot3\cdot1\cdot4\cdot2\cdot1\cdot1} = 174 \tag{C.25}$$

Relevant to this thesis, are the Young tableaux for $SU(4)$ representations. The ones that appear in the 4d $\mathcal{N}=4$ conformal supergravity multiplet from Chapter 7 are $\mathbf{4} = \ydiagram{1}$, $\bar{\mathbf{4}} = \ydiagram{1,1,1}$, $\mathbf{20} = \ydiagram{2,2,1}$, $\overline{\mathbf{20}} = \ydiagram{2,1}$, $\mathbf{6} = \ydiagram{1,1}$, $\mathbf{10} = \ydiagram{2}$, $\overline{\mathbf{10}} = \ydiagram{2,2,2}$, $\mathbf{20'} = \ydiagram{2,2}$.

C.4.2 Tensor product of Young tableaux

Very often, one would like to consider the tensor product of irreducible representations of $SU(N)$. These are, in general, decomposable into a direct sum of irreducible representations. One way of finding such decomposition is by working directly with the Young tableau representations.[4] Taking the tensor product of two Young tableaux, λ_A and λ_B, amounts to following the algorithm described below.

1. Label the cells in the second Young tableau, λ_B with a's in the first row, b's in the second etc..

2. Add all the a cells to the first Young tableau in all possible ways that still result in a Young tableau and remove them from the second tableau. The only exception being that a cells cannot appear in an antisymmetrised way (i.e. two a in the same column). If identical tableaux appear in that construction, only keep one.

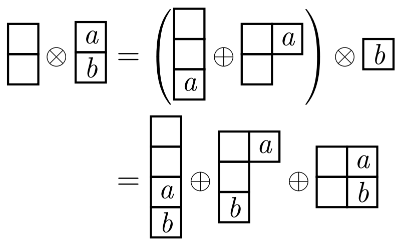
[4]Another popular way is to deal with the highest weight representations directly.

3. Repeat that step with all the b cells, then the c cells, etc. The only difference now being that tableaux which are identical up to an exchange of a, b,.. labels count as distinct ones.

4. Counting from left-to-right and top-to-bottom the number of a cells, b cells etc, if at any point the number of b cells exceeds that of a cells, or the number of c cells that of b cells,... that tableau must be removed. One direct consequence of this rule is that, from right-to-left and top-to-bottom, the first a cell must appear before the first b cell which must appear before the first c cell etc.

5. Finally, unlabel the remaining tableaux and add them using a direct sum, $\oplus$.

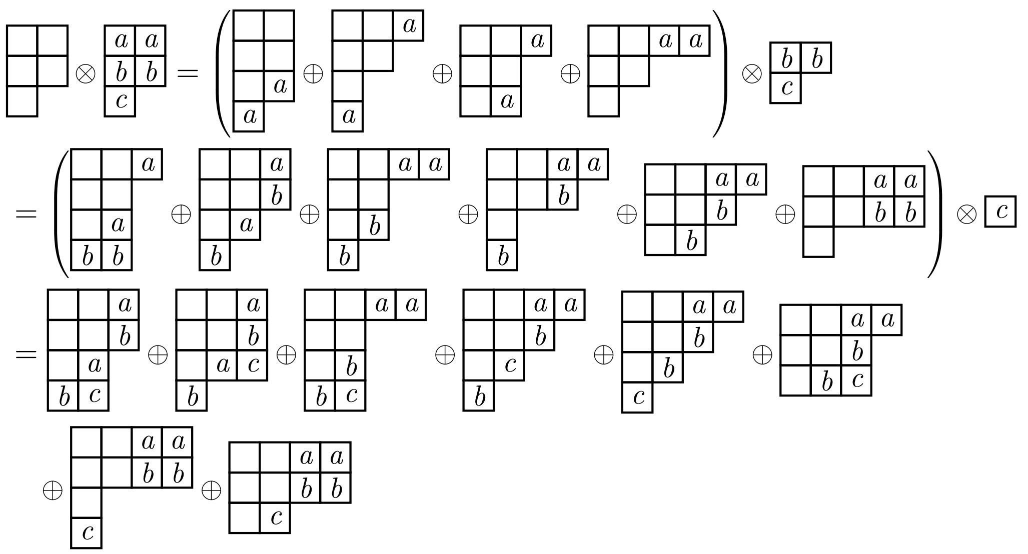
Example C.3. $\mathbf{6} \otimes \mathbf{6}$ *As an illustrative example, let us consider the tensor product between two copies of the* **6** *representation of* $SU(4)$*. In terms of Young tableaux, we add the labels a and b to the second copy and proceed with the algorithm detailed above.*



$$= \mathbf{1} \oplus \mathbf{15} \oplus \mathbf{20'}$$

As a sanity check, we see that dimensions match on both sides of the equality.

Example C.4. $\mathbf{20} \otimes \mathbf{20}$ *To make sure we understand the algorithm, let us consider the tensor product between two copies of the* **20** *representation of* $SU(4)$*. In terms of Young tableaux, we add the labels a and b an c to the second copy and proceed.*



$$= \mathbf{6} \oplus \mathbf{10} \oplus \mathbf{10} \oplus \mathbf{64} \oplus \mathbf{64} \oplus \mathbf{70} \oplus \mathbf{50} \oplus \mathbf{126}$$

Again, we can see that on both sides of the equality the dimensions match. In other words, they are both 400*.*

C.5 Hypergeometric functions

Hypergeometric series often pop up in conformal field theory. In this instance, they appear as closed-form solutions for the correlators of our nn-dCFTs in Chapter 8. Let us outline a few definition and properties of these series that are useful for our applications.

A generalised hypergeometric function of one variable takes the form

$$ {}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{C.28} $$

where $(a)_k$ is the Pochhammer symbol, defined as $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$, for $k \geq 0$ and a not a negative integer. We can extend the symbol's definition to negative integers as follows. If $a < 0$ then $(a)_k$ is defined for all $k \leq -a$ as

$$ (a)_k = (-1)^k \frac{\Gamma(-a+1)}{\Gamma(-a-k+1)}. \tag{C.29} $$

Consequently, whenever one of the top arguments a_i is negative, the infinite sum terminates and the function is well defined for all values of z.

If one wishes to derive many results from this section, they will find useful these following identities that involve the gamma function. There is the Euler reflection formula

$$ \Gamma(z-n) = (-1)^{n-1}\frac{\Gamma(-z)\Gamma(1+z)}{\Gamma(1-z+n)} \qquad n \in \mathbb{Z},\ z \notin \mathbb{Z}, \tag{C.30a} $$

and the Legendre duplication formula

$$ \Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{C.30b} $$

Out of the family of functions in (C.28), the Gauß hypergeometric function is the most common one. We denote it by ${}_2F_1(a,b;c;z)$, the semi-colon indicating the switch to the bottom set of Pochhammer symbols. Its definition in terms of the infinite sum converges for $|z| < 1$

$$ {}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{C.31} $$

but as pointed out above, for negative a or b, this sum terminates and we find

$$ {}_2F_1(-m,b;c;z) = \sum_{k=0}^{m}\binom{m}{k}\frac{(b)_k}{(c)_k}(-z)^k, \tag{C.32a} $$

$$ {}_2F_1(-m,-n;c;z) = \sum_{k=0}^{\min(m,n)}\binom{m}{k}\binom{n}{k}\frac{k!}{(c)_k}z^k. \tag{C.32b} $$

Generalising the previous construction to hypergeometric series in two-variables gives the so-called Kampé de Fériet hypergeometric function

$$F^{p,r,t}_{q,s,u}\left(\begin{matrix}a_1,\dots a_p\\ b_1,\dots b_q\end{matrix};\begin{matrix}c_1,\dots c_r\\ d_1,\dots d_s\end{matrix};\begin{matrix}e_1,\dots e_t\\ f_1,\dots f_u\end{matrix};x,y\right)=\sum_{n,m=0}^{\infty}\frac{(\vec{a})_{n+m}(\vec{c})_m(\vec{e})_n}{(\vec{b})_{n+m}(\vec{d})_m(\vec{f})_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{C.33}$$

where $(\vec{a})_m=(a_1)_m\cdots(a_p)_m$ and similarly for the other Pochhammer symbols. There are multiple ways of recovering the generalised hypergeometric in one variable, of which we list a few

$$F^{0,p,0}_{0,q,0}\left(;\begin{matrix}c_1,\dots c_p\\ d_1,\dots d_q\end{matrix};;x,0\right)={}_pF_q\left(\begin{matrix}c_1,\dots,c_p\\ d_1,\dots,d_q\end{matrix};x\right), \tag{C.34a}$$

$$F^{0,0,p}_{0,0,q}\left(;;\begin{matrix}e_1,\dots e_p\\ f_1,\dots f_q\end{matrix};0,y\right)={}_pF_q\left(\begin{matrix}e_1,\dots,e_p\\ f_1,\dots,f_q\end{matrix};y\right), \tag{C.34b}$$

$$F^{p,0,0}_{q,0,0}\left(\begin{matrix}a_1,\dots a_p\\ b_1,\dots b_q\end{matrix};\frac{x}{2},\frac{x}{2}\right)={}_pF_q\left(\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};y\right). \tag{C.34c}$$

An important subset of these are the four Appell functions,

$$F_1(a,b_1,b_2;c;x,y)=\sum_{n,m=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}x^my^n, \tag{C.35a}$$

$$F_2(a,b_1,b_2;c_1,c_2;x,y)=\sum_{n,m=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_nm!n!}x^my^n, \tag{C.35b}$$

$$F_3(a_1,a_2,b_1,b_2;c;x,y)=\sum_{n,m=0}^{\infty}\frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n}m!n!}x^my^n, \tag{C.35c}$$

$$F_4(a,b;c_1,c_2;x,y)=\sum_{n,m=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_nm!n!}x^my^n. \tag{C.35d}$$

C.5.1 Special values

We list here a few examples where these hypergeometric functions reduce to well-known functions.

When multiple parameters are identical, the series expansion coincide precisely with the generalised binomial and trinomial expansions,

$${}_2F_1(a,b;b;z)=(1-z)^{-a}, \tag{C.36}$$

$$F_2(a,b,b;b,b;x,y) = (1-x-y)^{-a}. \tag{C.37}$$

C.5.2 Integral formulae

See [430] Section 7.5 for a more complete list of integral formulae.

First, we have the beta function integral formula, valid for $\mathrm{Re}(a), \mathrm{Re}(b) > 0$,

$$\int_0^1 u^{a-1}(1-u)^{b-1}du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{C.38}$$

Second, we have that of the Gauß hypergeometric, valid for $\mathrm{Re}(b), \mathrm{Re}(c) > 0$,

$$\int_0^1 u^{a-1}(1-u)^{c-b-1}(1-xu)^{-a}du = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a,b;c;x). \tag{C.39}$$

Finally, we have that of the Appell function, valid for $\mathrm{Re}(a), \mathrm{Re}(c) > 0$,

$$\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b_1}(1-yu)^{-b_2}du = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F_1(a,b_1,b_2;c;x,y). \tag{C.40}$$

An analytic continuation of $\mathbb{E}[(\theta \cdot X)^{-n}]$

In general the Gaussian integral $\int_{\mathbb{R}} \mathcal{N}(0,\mu_2)\theta^{-s}d\theta$ diverges when $s \geq 1$[5]. However, we can analytically continue it to any complex s as follows. Using Cauchy's principal value method, we see that the integral is trivial for all $s \in 2\mathbb{Z}+1$,

$$\int_{\mathbb{R}} \mathcal{N}(0,\mu_2)\theta^{-s}d\theta = \lim_{\theta_0 \to 0} \sqrt{\frac{\mu_2}{2\pi}}\,(1+(-1)^s)\int_{\theta_0}^{\infty} e^{-\theta^2\frac{\mu_2}{2}}\theta^{-s}d\theta = 0. \tag{C.41}$$

For $s \in 2\mathbb{Z}$, it evaluates to

$$\int_{\mathbb{R}} \mathcal{N}(0,\mu_2)\theta^{-s}d\theta = \frac{1}{\sqrt{\pi}} 2^{-s/2}\mu_2^{s/2}\Gamma\left(\frac{1-s}{2}\right). \tag{C.42}$$

All other values of s are irrelevant to our examples.

Finding the value of $\mathbb{E}[(\theta \cdot X)^{-n}]$ comes down to the following simple steps.

Firstly, we use the Feynman reparametrisation trick, which says that for $\mathrm{Re}(n_i) > 0$ [431],

$$A_1^{-n_1}\cdots A_k^{-n_k} = \frac{\Gamma(n_1+\cdots+n_k)}{\Gamma(n_1)\cdots\Gamma(n_k)}\int_{[0,1]^k} d^k u \frac{\delta(1-\sum_{i=1}^k u_i)u_1^{n_1-1}\cdots u_k^{n_k-1}}{(\vec{n}\cdot\vec{A})^{\sum_{i=1}^k n_i}}. \tag{C.43}$$

[5] We remind the reader that $\mathcal{N}(0,\mu_2) = \sqrt{\frac{\mu_2}{2\pi}}e^{-\theta^2\frac{\mu_2}{2}}$ is the Gaussian distribution for one random variable, centred at the origin.

This, however, is only valid as long as the convex hull created by the complex numbers A_i does not contain the origin.

C.6 Lemmas of Chapter 8

Lemma C.1. *Let $\mathcal{N}(0,\sigma)$ be multivariate Gaussian distribution centred at the origin with variance matrix $\sigma_{AB} = \tilde{\mu_2}\delta_{AB}$. The expectation value of the defect variables $\tilde{\theta}$ then obeys*

$$\mathbb{E}[(X \bullet \theta)^{2\lambda}] = Q(2\lambda)(X \bullet X)^{\lambda}\tilde{\mu}_2^{\lambda} \overset{P.S.}{=} 0 \tag{C.44}$$

for all $\lambda \in \mathbb{N}^*$.

The function $Q(n)$ is defined as counting all possible pairings between n (even) objects

$$Q(n) = 2^{n/2}\pi^{-1/2}\Gamma\left(\frac{n+1}{2}\right) \tag{C.45}$$

Proof. Let us prove this statement recursively.

When $\lambda = 1$, we explicitly find $\mathbb{E}[(X \bullet \theta)^2] = X \bullet X\,\tilde{\mu}_2$.

Let us assume that $\mathbb{E}[(X \bullet \theta)^{2\lambda}] = \frac{(2\lambda-1)!}{2^{\lambda-1}(\lambda-1)!}(X \bullet X)^{\lambda}\tilde{\mu}_2^{\lambda}$ for some $\lambda \in \mathbb{N}^*$. Then, using Stein's lemma (with $\tilde{\mu}_1 = 0$ and $\sigma_{AB} = \tilde{\mu}_2\delta_{AB}$)

$$\mathbb{E}[g(\theta)\theta_A] = \tilde{\mu}_2 \sum_B \delta_{AB}\mathbb{E}[\partial_B g(\theta)], \tag{C.46}$$

we find the intermediate result

$$\mathbb{E}[\theta_{A_1}\cdots\theta_{A_{2\lambda}}\theta_{B_1}\theta_{B_2}] = \tilde{\mu}_2 \sum_{i=1}^{2\lambda} \delta_{B_2 A_i}\mathbb{E}[\theta_{A_1}\cdots\widehat{\theta_{A_i}}\cdots\theta_{A_{2\lambda}}\theta_{B_1}] + \tilde{\mu}_2\delta_{B_1B_2}\mathbb{E}[\theta_{A_1}\cdots\theta_{A_{2\lambda}}]. \tag{C.47}$$

Contracting this with the vector components of X, we are able to evaluate the expression at $\lambda + 1$,

$$\begin{aligned}\mathbb{E}[(X \bullet \theta)^{2\lambda+2}] &= X^{A_1}\cdots X^{A_{2\lambda}}X^{B_1}X^{B_2}\mathbb{E}[\theta_{A_1}\cdots\theta_{A_{2\lambda}}\theta_{B_1}\theta_{B_2}] \\ &= (2\lambda+1)(X \bullet X)\tilde{\mu}_2\mathbb{E}[(X \bullet X)^{2\lambda}] \\ &= \frac{(2\lambda+1)!}{2^{\lambda}\lambda!}(X \bullet X)^{\lambda+1}\tilde{\mu}_2^{\lambda+1}. \end{aligned} \tag{C.48}$$

□

Lemma C.2. *Let $\mathcal{N}(0,\sigma)$ be multivariate Gaussian distribution centred at the origin with variance matrix $\sigma_{AB} = \mu_2\delta_{AB}$. Let X_1, X_2 be two distinct $D+2$-dimensional* *vectors. Then the expectation value of the random variables* $\tilde{\theta}$ *obeys*

$$\mathbb{E}[(X_1 \bullet \theta)^{n_1}(X_2 \bullet \theta)^{n_2}] = \sum_{\substack{n=0 \\ n_1-n=0\,[2] \\ n_2-n=0\,[2]}}^{\min(n_1,n_2)} (X_1 \bullet X_2)^n (X_1 \bullet X_1)^{\frac{n_1-n}{2}} (X_2 \bullet X_2)^{\frac{n_2-n}{2}} \tilde{\mu}_2^{\frac{n_1+n_2}{2}} \frac{2^{\frac{2n-n_1-n_2}{2}} \Gamma(n_1+1)\Gamma(n_2+1)}{\Gamma(n+1)\Gamma(\frac{n_1-n+2}{2})\Gamma(\frac{n_2-n+2}{2})} \tag{C.49}$$

for all $n_1, n_2 \in \mathbb{N}$*. A similar expression holds for the tangent space random variables* $X_1 \circ \theta$*, where* $\bullet$ *should be replaced by* $\circ$ *and* $\tilde{\mu}_2$ *by* $\hat{\mu}_2$*.*

For defect variables, all terms proportional to $X_1 \bullet X_1$ *and* $X_2 \bullet X_2$ *vanish on the defect Poincaré section, and so the expectation value simplifies to*

$$\mathbb{E}[(X_1 \bullet \theta)^{n_1}(X_2 \bullet \theta)^{n_2}] \overset{P.S.}{=} \delta_{n_1 n_2} \Gamma(n_1+1)(X_1 \bullet X_2)^{n_1} \tilde{\mu}_2^{n_1}. \tag{C.50}$$

Proof. The expectation value is given by all possible contractions between n_1 copies of the vector X_1 and n_2 copies of the vector X_2. We can classify them according to the number of contractions $X_1 \bullet X_2$, call it n.

For a given n, there are $\frac{1}{n!}P(n_1,n)P(n_2,n)$ ways of constructing these n contractions of X_1 and X_2. $P(m,n)$ denotes the number of permutations of n objects within a set of m objects, and is defined as $P(m,n) = \frac{m!}{(m-n)!}$. The remaining $n_1 - n$ copies of X_1 must then be contracted within themselves. Similarly for the $n_2 - n$ copies of X_2. Of course, this is only possible if both $n_1 - n$ and $n_2 - n$ are even. Whenever that is the case, there are $Q(n_1 - n)Q(n_2 - n)$ ways of contracting these vectors among themselves, as shown in lemma C.1. □

One can show that the right-hand-side of Lemma C.2 is resummable into hypergeometric functions. This, however, must be done separately for even and odd $n_1 + n_2$, leading to the following result.

Theorem C.3. *Let* $\mathcal{N}(0,\sigma)$ *be multivariate Gaussian distribution centred at the origin with variance matrix* $\sigma_{AB} = \mu_2 \delta_{AB}$*. Let* X_1*,* X_2 *be two distinct* $D+2$*-dimensional vectors. Then the expectation value of the random variables* $\tilde{\theta}$ *obeys*

$$\mathbb{E}[(X_1 \bullet \theta)^{n_1}(X_2 \bullet \theta)^{n_2}] = \left(\frac{\tilde{\mu}_2}{2} X_1 \bullet X_1\right)^{\frac{n_1}{2}} \left(\frac{\tilde{\mu}_2}{2} X_2 \bullet X_2\right)^{\frac{n_2}{2}} \frac{\Gamma(n_1+1)\Gamma(n_2+1)}{\Gamma\left(\frac{n_1}{2}+1\right)\Gamma\left(\frac{n_2}{2}+1\right)} {}_2F_1\left(-\frac{n_1}{2}, -\frac{n_2}{2}; \frac{1}{2}; \frac{(X_1 \bullet X_2)^2}{X_1 \bullet X_1 X_2 \bullet X_2}\right) \tag{C.51}$$

when $n_1 + n_2$ *is even,*

$$\mathbb{E}[(X_1 \bullet \theta)^{n_1}(X_2 \bullet \theta)^{n_2}] = 2\left(\frac{\tilde{\mu}_2}{2} X_1 \bullet X_2\right)\left(\frac{\tilde{\mu}_2}{2} X_1 \bullet X_1\right)^{\frac{n_1-1}{2}}\left(\frac{\tilde{\mu}_2}{2} X_2 \bullet X_2\right)^{\frac{n_2-1}{2}}$$
$$\frac{\Gamma(n_1+1)\Gamma(n_2+1)}{\Gamma\left(\frac{n_1-1}{2}+1\right)\Gamma\left(\frac{n_2-1}{2}+1\right)} {}_2F_1\left(-\frac{n_1-1}{2}, -\frac{n_2-1}{2}; \frac{3}{2}; \frac{(X_1 \bullet X_2)^2}{X_1 \bullet X_1 X_2 \bullet X_2}\right) \tag{C.52}$$

when $n_1 + n_2$ *is odd, and vanishes otherwise.*

Appendix D

Spinor Conventions

Avec tout ce que je sais, on pourrait faire un livre… il est vrai qu'avec tout ce que je ne sais pas, on pourrait faire une bibliothèque

Sacha Guitry

In this appendix we will present the various spinor conventions used throughout Chapters 5,4 and Chapters 6,7. We will also detail some basic definitions and formulae that are useful in this context. As such, you can see this chapter as a hybrid between a background material chapter from Part I and an appendix.

We start by outlining the main definitions of a Clifford algebra in Section D.1. Most of the definitions and theorems there can be found in reviews [121, 432–434].

D.1 Clifford Algebras

Clifford algebras are a type of algebra associated to a quadratic form Q on a vector space V. Our main motivation for introducing these objects lies in the fact that they are the central objects used to describe the action of the spin group on spinors. Understanding how to construct such algebras, one can understand the corresponding spinor representations. Unless stated otherwise, in this section $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

First and foremost, allow us to to introduce the first ingredient in constructing Clifford algebras, namely a quadratic vector space.

Definition D.1. Quadratic Vector Space Let V be a finite-dimensional $\mathbb{K}$-vector space, let $B : V \times V \to \mathbb{K}$ be a symmetric bilinear form and let $Q : V \to \mathbb{K}$ denote the corresponding quadratic form, defined by $Q(x) = B(x,x)$. Note that one can recover B from Q by *polarisation*, namely

$$B(x,y) = \frac{1}{2}\left(Q(x+y) - Q(x) - Q(y)\right) \tag{D.1}$$

The pair (V,Q) is called a *quadratic* $\mathbb{K}$-vector space.

The symmetric bilinear form B plays the role of a metric on V, and indeed, in most cases we will consider $B = \eta$ or $B = \delta$ (the Minkowski metric or the Euclidean metric).

Example D.1. *real pre-Hilbert space Recall that a real pre-Hilbert space E is an $\mathbb{R}$-vector space equipped with an inner product $\langle \cdot, \cdot \rangle : E \times E :\to \mathbb{R}$. In other words, the map $\langle \cdot, \cdot \rangle$ must obey the following properties. It must be*

1. *bilinear:* $\forall x, y, z \in E, \forall \lambda \in \mathbb{R}$

$$\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$$
$$\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$$

2. *symmetric:* $\forall x, y \in E$, $\langle x, y \rangle = \langle y, x \rangle$

3. *positive definite:* $\forall x \in E$, $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0_E$

Notice how the inner product on E is simply a special case of the symmetric bilinear form B in Definition. D.1. We may also recall that from an inner product, we can immediately define a norm by

$$\|x\| = \sqrt{\langle x, x \rangle} \tag{D.2}$$

Using the properties of the inner product we can see that $\|\cdot\|$ does indeed obey the required properties of a norm (positive definite, homogeneity, triangular inequality). However, one may also notice that $\|\cdot\|^2$ obeys the axioms of a quadratic form on E. In that way we see that both finite dimensional pre-Hilbert spaces and finite-dimensional normed $\mathbb{K}$-vector spaces can be extended into quadratic vector spaces.

The following two definitions are the main ones we wish to focus on. The Clifford map will also make an appearance in our definitions for the Killing spinor equations in Section D.3. There,

Definition D.2. Clifford Map Let (V,Q) be a quadratic vector space and let A be an associative, unital associative $\mathbb{K}$-algebra. We say that a $\mathbb{K}$-linear map $\phi : V \to A$ is *Clifford* if for all $x \in V$

$$\phi(x) \cdot \phi(x) = Q(x) 1_A \tag{D.3}$$

where $\cdot$ is the product on A and 1_A is its associated unit.

Definition D.3. Clifford Algebra Let (V, Q) be a quadratic vector space. The Clifford algebra, if it exists, is given by an associative algebra $Cl(V, Q)$ together with a Clifford map $i : V \to Cl(V, Q)$ such that for every Clifford map $\phi : V \to A$ there is a unique algebra morphism $\Phi : Cl(V, Q) \to A$ making the following triangle commute

$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\ Cl(V,Q) \arrow[rr, "\Phi"] & & A \end{tikzcd} \tag{D.4}$$

D.1.1 Relation to the Physics language

Let us choose a basis on V, given by $(e_i)_{1 \leq i \leq \dim(V)}$. Relative to this basis, the bilinear form $B(\cdot, \cdot)$ may be decomposed into its components $B(e_i, e_j) = B_{ij} = B_{ji}$[1]. Let Γ_i denote the image of e_i under $i : V \to Cl(V, Q)$. We recall from Definition D.3 that since i is Clifford, equation (D.3) will hold in $Cl(V, Q)$. Hence, we may proceed by defining $x_{jk} = e_j + e_k \in V$ for any $j, k \in \{1, \dots, \dim(V)\}$. By linearity of the Clifford map the left-hand-side of equation (D.3) becomes

$$\begin{aligned} i(x_{jk})i(x_{jk}) &= (i(e_j) + i(e_k))\,(i(e_j) + i(e_k)) = \Gamma_j^2 + \Gamma_j\Gamma_k + \Gamma_k\Gamma_j + \Gamma_k^2 \\ &= B_{jj}\mathbf{1} + \Gamma_j\Gamma_k + \Gamma_k\Gamma_j B_{kk}\mathbf{1} \end{aligned}$$

On the other hand, by bilinearity of B, the right-hand-side becomes

$$B(x_{jk}, x_{jk})\mathbf{1} = B(e_j + e_k, e_j + e_k)\mathbf{1} = B_{jj}\mathbf{1} + 2B_{jk}\mathbf{1} + B_{kk}\mathbf{1}$$

Comparing both sides, it is now straightforward to see that the Γ_i satisfy

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 2B_{ij}\mathbf{1} \tag{D.5}$$

where $\mathbf{1}$ is the unit in $Cl(V, Q)$.

The expression above is the familiar way in which Clifford algebras are defined in Physics, where the Γ_j are called Gamma matrices and the symmetric product of those may be packaged into the anti-commutator $\{\Gamma_i, \Gamma_j\}$.

The Clifford algebra is consequently the associative algebra generated by the Γ_i subject to the above relation. In other words, any element X of $Cl(V, Q)$ may be written as a linear combination of these elements.

$$X = X^0\mathbf{1} + \sum_{i=1}^{\dim(V)} X^i\Gamma_i + \sum_{i,j=1}^{\dim(V)} X^{i,j}\Gamma_i\Gamma_j + \dots \tag{D.6}$$

[1]Again, think of this as a basis of $\mathbb{R}^d$ with which we decompose a metric into its matrix components.

Since the product of gamma matrices are generally not independent, we can simplify these using the previously-described identities. Using eq. D.5 we find

$$\Gamma_i\Gamma_j = \frac{1}{2}(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i) + \frac{1}{2}(\Gamma_i\Gamma_j + \Gamma_j\Gamma_i) = \Gamma_{ij} - B_{ij}\mathbf{1} \tag{D.7}$$

$$\Gamma_i\Gamma_j\Gamma_k = \Gamma_i\Gamma_{jk} - B_{jk}\Gamma_i = \Gamma_{ijk} - B_{ij}\Gamma_k + B_{ik}\Gamma_j - B_{jk}\Gamma_i \tag{D.8}$$

and so one and so forth. One can then prove that any product of these matrices may be decomposed into the antisymmetrised products $\Gamma_{i_1\cdots i_n}$ and lower order products of generators. In that way $Cl(V,Q)$ is the linear span of $\mathbf{1}, \Gamma_i, \Gamma_{ij}, \ldots, \Gamma_{i_1\cdots i_{\dim(V)}}$. Its dimension is thus

$$\dim(Cl(V,Q)) = 1 + \dim(V) + \begin{pmatrix} 2 \\ \dim(V) \end{pmatrix} + \ldots + \begin{pmatrix} \dim(V) \\ \dim(V) \end{pmatrix} = 2^{\dim(V)} \tag{D.9}$$

Whenever one looks at the vector space $V = \mathbb{R}^d$ together with the quadratic form generated by the Euclidean metric δ, the shorthand notation $Cl(V,Q) = Cl(d)$ is used. When describing such an algebras built from metrics with different signature, say (p,q), where $p+q=d$, then we used the notation $Cl(p,q)$ instead. As such, the dimensions of $Cl(d)$ and $Cl(p,q)$ are both 2^d.

D.1.2 A basis in any dimension and any signature

The constructive approach described in the previous subsection is only useful if one can find a systematic way of generating a representation of $Cl(p,q)$ in any dimension. Let us briefly describe here one way of doing so.

Starting in three-dimensions, recall that the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{D.10}$$

obey the following condition on their anticommutator[2]

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}. \tag{D.11}$$

Consequently, the vector space of matrices spanned by the Pauli matrices is a representation of the Clifford algebra $Cl(3)$. To make the identification more clear, we will write $\Gamma_i = \sigma_i$ (abusively). The other two signatures in three dimension, can be had by setting $\Gamma_1 \to i\Gamma_1$ and/or $\Gamma_2 \to i\Gamma_2$. This idea will extend to any dimension, for which the substitutions $\Gamma_i \to i\Gamma_i$ will lead to a representation of the algebra in all other

[2]This can easily be derived if one starts with the formula $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ and takes the symmetrised product.

signatures. Without loss of generality, let us illustrate how to construct a basis of $Cl(d)$. The other signature then follow from that.

In d dimensions, the Clifford algebra has dimension 2^d and the gamma matrices are represented by $2^{\lfloor d/2 \rfloor} \times 2^{\lfloor d/2 \rfloor}$ matrices. These can be built as follows

$$\begin{aligned}
\Gamma_1 &= \sigma_1 \otimes I_2 \otimes I_2 \otimes \cdots \otimes I_2, \\
\Gamma_2 &= \sigma_2 \otimes I_2 \otimes I_2 \otimes \cdots \otimes I_2, \\
\Gamma_3 &= \sigma_3 \otimes \sigma_1 \otimes I_2 \otimes \cdots \otimes I_2, \\
\Gamma_4 &= \sigma_3 \otimes \sigma_2 \otimes I_2 \otimes \cdots \otimes I_2, \\
&\vdots \\
\Gamma_d &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \otimes \sigma_2,
\end{aligned} \tag{D.12}$$

where the tensor product $\otimes$ is performed $2^{\lfloor d/2 \rfloor}$ times and I_2 is the 2×2 identity matrix. The algebra obeyed by the Pauli matrices automatically ensures these Γ_i matrices generate the Clifford algebra $Cl(d)$.

D.2 Spin(4) conventions

In four-dimensional Euclidean field theory, every field falls under projective representations of the isometry group $SO(4)$. These representations can equivalently be identified with ordinary representations of the double cover of $SO(4)$, namely the group $\mathrm{Spin}(4)$. This spin group exhibits an accidental isomorphism

$$\mathrm{Spin}(4) \cong SU(2)_\ell \times SU(2)_r \; . \tag{D.13}$$

Every representation of the spin group can consequently be written in terms of representations of the left- and right-handed $SU(2)$ factors. This implies that every Dirac spinor can then be decomposed into two Weyl spinors, each transforming with respect to one of the above $SU(2)$ factors, and inert with respect to the other. The Weyl representations $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ are pseudoreal, or quaternionic, meaning that they are related to their respective complex conjugate representations by a similarity transformation.

Since each symmetry group will require its own matrix representation, including the R-symmetry group, we need to make a clear distinction between their respective indices. Our choices are summarised in table D.1.

Indices	**Description**	**Range**
$\alpha, \beta, \ldots$	left-handed spinor indices	$\{1,2\}$
$\dot\alpha, \dot\beta, \ldots$	right-handed spinor indices	$\{1,2\}$
$\mu, \nu, \ldots$	coordinate indices for local chart	$\{1,2,3,4\}$
$a, b, \ldots$	non-coordinate indices for local frame	$\{1,2,3,4\}$
$i, j, \ldots$	$\mathfrak{su}(\mathcal{N})$ R-symmetry indices	$\{1,\ldots,\mathcal{N}\}$

TABLE D.1: Description of the various index labels used in Chapters 6 and 7.

First and foremost, we will describe our choice of conventions for the Clifford algebra generators used throughout Chapters 6 and 7. This choice does differ from the basis described in equation D.12. To make the distinction clearer, we will denote the gamma matrices with the lowercase gamma letter. The Clifford algebra $Cl(4)$, associated to the Euclidean metric on $\mathbb{R}^4$, is an associative algebra generated by elements γ_a subject to the condition $\{\gamma_a, \gamma_b\} = 2\delta_{ab}\mathbf{1}$, where $\mathbf{1}$ denotes the unit element. The group $\mathrm{Spin}(4)$ is then seen as a certain subgroup of units of $Cl(4)$. By a suitable choice of representation of the latter algebra, we can make explicit the accidental isomorphism of $\mathrm{Spin}(4)$, as will been seen by the diagonal nature of the chirality matrix γ_5.

Let us start by writing down the Pauli matrices, which will serve as the starting point for building a representation of $C\ell(4)$,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{D.14}$$

We then define

$$\bar{\sigma}_{1,2,3} = \sigma_{1,2,3} \ , \qquad \sigma_4 = -\bar{\sigma}_4 = -iI_2 \ , \tag{D.15}$$

where I_k is the $k \times k$ identity matrix, and note that these exhibit the index structure $(\sigma_a)^{\dot\alpha}{}_{\beta}$ and $(\bar\sigma_a)^{\alpha}{}_{\dot\beta}$. As is standard, those indices may be raised and lowered using the $\mathfrak{su}(2)_\ell$-invariant (resp. $\mathfrak{su}(2)_r$-invariant) tensor, namely the Levi-Civita symbol $\varepsilon_{\alpha\beta}$ (resp. $\varepsilon_{\dot\alpha\dot\beta}$). In our conventions, $\varepsilon_{12} = \varepsilon_{\dot1\dot2} = -\varepsilon^{12} = -\varepsilon^{\dot1\dot2} = 1$. Introducing the anti-symmetric product of these Pauli matrices as

$$\sigma_{ab} = \frac{1}{2}(\sigma_a\bar\sigma_b - \sigma_b\bar\sigma_a) \ , \tag{D.16}$$

$$\bar\sigma_{ab} = \frac{1}{2}(\bar\sigma_a\sigma_b - \bar\sigma_b\sigma_a) \ , \tag{D.17}$$

we can define the gamma matrices in block form,

$$\gamma_a = \begin{pmatrix} 0 & \bar\sigma_a \\ \sigma_a & 0 \end{pmatrix}. \tag{D.18}$$

One can readily check that they obey the anti-commutator relation $\{\gamma_a, \gamma_b\} = 2\delta_{ab}I_4$. The Clifford algebra can be generated by considering the antisymmetric products of the γ-matrices, $\gamma_{ab} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$, $\gamma_{abc} = \ldots$ and so on. See Section D.1 for a more details on this. The Euclidean chirality matrix, written in this basis, takes the conventional form

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{D.19}$$

exhibiting the decomposition of Spin(4) representations as two $SU(2)$ representations. The chirality matrix γ_5 can be used to project spinors onto those chiral components, via $P_\ell = \frac{1}{2}(I_4 + \gamma_5)$ and $P_r = \frac{1}{2}(I_4 - \gamma_5)$. With this choice, we have implicitly defined the Weyl spinors of $SU(2)_\ell$ (resp. $SU(2)_r$) as having positive (resp. negative) chirality.

This, in turn, means that the four-component vector representation of the Dirac spinors admits an orthogonal split in terms of two-component vector representations of the chiral Weyl spinors, $\psi_\pm$,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}. \tag{D.20}$$

The charge conjugation matrix C acts on our γ-matrices and is defined to satisfy

$$C^{-1}\gamma_a C = -\gamma_a^T\,, \tag{D.21}$$
$$C^{-1}\gamma_5 C = \gamma_5^T\,. \tag{D.22}$$

In particular, it takes the explicit form

$$C = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix}, \tag{D.23}$$

where $(C_-)_{\dot\alpha\dot\beta} = \varepsilon_{\dot\alpha\dot\beta}$ and $(C_+)_{\alpha\beta} = \varepsilon_{\alpha\beta}$. The Euclidean analogue of the Dirac conjugate of the spinor ψ is then defined as

$$\overline{\psi} = (C\psi)^T. \tag{D.24}$$

Throughout Chapter 6, however, we work with spinors that form a representation of the $\mathcal{N}=2$ supersymmetric algebra in four dimensions, while for Chapter 7 we do so with $\mathcal{N}=4$ spinors. In both cases, we utilise the so-called *chiral* $SU(\mathcal{N})$ notation, whereby spinors with opposing chirality are assigned a conjugate representation of the $SU(\mathcal{N})$ R-symmetry algebra. To avoid any repetition, please see Chapter 3 for a more detailed explanation, together with references. This chiral notation, however, is anecdotal in $\mathcal{N}=2$ as all representations are isomorphic to their conjugate. In that case, we will simply denote spinors with opposing chirality with the same R-symmetry index.

$\mathcal{N}=2$ spinors

In four dimensions, the R-symmetry algebra of $\mathcal{N}=2$ supersymmetry[3] is $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$. It is then in our best interest to define $\mathcal{N}=2$ spinors as $SU(2)_R$ doublets, with $i \in \{1,2\}$,

$$\psi^i = \begin{pmatrix} \psi^i_+ \\ \psi^i_- \end{pmatrix}. \tag{D.25}$$

The notion of Dirac conjugation extends onto these doublets via an additional action on the R-symmetry components as

$$\overline{\psi}_i = \varepsilon_{ij}(C\psi^j)^T. \tag{D.26}$$

We will frequently write the chiral components of these spinor doublets as 2×2 matrices with components $\psi_+^{\alpha i}$ or $\psi_-^{\dot\alpha i}$. For completeness, allow us to rewrite the Dirac conjugates

[3]This corresponds to eight supercharges, which we will describe explicitly in flat space later.

of these chiral spinors in terms of their matrix representation,

$$\overline{\psi}_{\pm} = -\varepsilon \psi_{\pm} \varepsilon. \tag{D.27}$$

D.3 Killing Spinors

The Killing spinor equation plays a central role in this thesis. Its role is to determine the number of allowed supersymmetries in a given theory and classifies the BPS configurations therein. As this construct is referenced in almost every chapter, let us give a brief overview of the main definitions and their extensions, namely that of Killing spinor equations and generalised Killing spinor equations. All definitions and theorems can be found in [432, 434].

Killing Spinor Equation

Definition D.4. Killing spinor Let ϵ be a section of the Spin bundle of a smooth manifold, $\mathcal{M}$. The spinor ϵ is said to be Killing if it satisfies the Killing spinor equation

$$\forall X \in \Gamma(T\mathcal{M}), \quad \nabla_X \epsilon = \lambda X \cdot \epsilon, \tag{D.28}$$

for some $\lambda \in \mathbb{C}$. Here $\cdot$ denoted the Clifford product and ∇ the Levi-Civita connection on $\mathcal{M}$.

We will note that λ is strongly constrained by the geometry of the manifold. Indeed, acting on the definition with ∇_Y and taking the antisymmetric product, we recover the relation $R = 4\lambda^2 d(d-1)$, where R is the Ricci scalar and d is the dimension of $\mathcal{M}$.

Note that the existence of such a spin bundle on $\mathcal{M}$ is not guaranteed. Following our notation from Section D.1, we need a vector bundle $\mathcal{S}$ with some action of the spin group which carries a globally well-defined Clifford product. If such a structure exists, $\mathcal{S}$ becomes a bundle of modules over the fibres of the Clifford bundle $Cl(T\mathcal{M}, g)$. We then say that $\mathcal{M}$ admits a spin structure. However, some supergravity theories might exist without requiring that $\mathcal{M}$ is spin and indeed one can construct vector bundles with a globally well-defined Clifford products without it. In this case, one needs a Lipschitz structure on $\mathcal{M}$ [435].

In Definition D.4, reality conditions on the metric constrains λ^2 to be real, and as such there are three cases to consider. When λ is pure imaginary (real), the Killing spinor is said to be imaginary (real). The particular case when $\lambda = 0$ corresponds to parallel Killing spinors. We immediately see that if our manifold admits Killing spinors, it is automatically Einstein and if it admits parallel Killing spinors it is Ricci flat.

Example D.2. *Killing spinors on* $\mathbb{R}^3$ *Consider the flat metric on* $\mathbb{R}^3$ *written in Cartesian coordinates*

$$g = dx^2 + dy^2 + dz^2. \tag{D.29}$$

We choose the vielbeins to be $e^1 = dx$, $e^2 = dy$ *and* $e^3 = dz$. *It is straightforward to see that the Cartan structure equations are solved by the trivial spin connection* $\omega^a{}_b = 0$. *The Ricci scalar being null, we look for parallel Killing spinors on that manifold. The Killing spinor equation simply reads*

$$d\epsilon = 0. \tag{D.30}$$

Its solutions are simply given by constant spinors $\epsilon = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix}$.

When dealing with more complicated manifold, such as those constructed in [2] (see Chapter 4), one might wish to solve for the Killing spinors without explicitly evaluating every spinor component. A firs path towards that is to find a smart way of solving for the spin connection on such fibred manifolds.

Example D.3. *Spin-connection on any fibered space This example will show how to compute the spin-connection of any product manifold or manifold fibred over an interval (can probably be extended to higher fibre). Let our metric be given by*

$$g = \sum_i f_i^2(x) g_{\mathcal{M}_i} + g^2(x) dx^2, \tag{D.31}$$

where $\mathcal{M}_i$ *are the manifolds fibred on the* x*-interval.*
We introduce the vielbeins

$$e^{a_i} = f_i(x) \hat{e}_i^{a_i}, \qquad e^{d-1} = g(x) dx, \tag{D.32}$$

where $\hat{e}_i$ *are the vielbeins on* $\mathcal{M}_i$ *and* $a_i \in \{\sum_{j=1}^{i-1} \dim(\mathcal{M}_j), \ldots, \sum_{j=1}^{i} \dim(\mathcal{M}_j)\}$.
If we define $\hat{\omega}_i^{a_i}{}_{b_i}$ *to be the spin connection on* $\mathcal{M}_i$, *then it must obey the Cartan structure equation*

$$d\hat{e}_i^{a_i} + \hat{\omega}_i^{a_i}{}_{b_i} \wedge \hat{e}_i^{b_i} = 0. \tag{D.33}$$

Consequently, one can easily check that the spin connection on the fibred space has non-vanishing components

$$\omega^{a_i}{}_{b_i} = \hat{\omega}_i^{a_i}{}_{b_i} \qquad \omega^{a_i}{}_x = \frac{f_i'(x)}{g(x)} \hat{e}_i^{a_i} \tag{D.34}$$

As an example, we may take the $AdS_5 \times S^1$ *fibration of* AdS_7 *and look at the metric of* $AdS_7 \times S^4$

$$g = L^2(\cosh^2(x) g_{\mathrm{AdS}_5} + \sinh^2(x) dz^2 + dx^2) + \frac{L^2}{4} g_{S^4}. \tag{D.35}$$

We introduce the vielbeins,

$$e^\mu = L\cosh(x)\hat{e}^\mu, \qquad e^z = L\sinh(x)dz, \qquad e^x = Ldx, \qquad e^\alpha = \frac{L}{2}\tilde{e}^\alpha, \tag{D.36}$$

where $\hat{e}^\mu$ *are those of* AdS_5 *and* $\tilde{e}^\alpha$*, those of* S^4*. The spin connection of* $AdS_7 \times S^4$ *written is these coordinates is given by*

$$\omega^\mu{}_\nu = \hat{\omega}^\mu{}_\nu, \qquad \omega^\alpha{}_\beta = \tilde{\omega}^\alpha{}_\beta, \qquad \omega^\mu{}_6 = \sinh(x)\hat{e}^\mu, \qquad \omega^\alpha{}_6 = \cosh(x)\tilde{e}^\alpha.$$

A similar computation for $AdS_7 \times S^4$ *written in FG form*

$$g = \frac{L^2}{z^2}(g_{\mathbb{R}^{1,5}} + dz^2) + \frac{L^2}{4}g_{S^4}, \tag{D.37}$$

yields the spin connection

$$\omega^\mu{}_\nu = \hat{\omega}^\mu{}_\nu, \qquad \omega^\alpha{}_\beta = \tilde{\omega}^\alpha{}_\beta, \qquad \omega^\mu{}_6 = -\frac{1}{z}\hat{e}^\mu, \tag{D.38}$$

where $\hat{}$ *refers to quantities in* $\mathbb{R}^{1,5}$ *and* $\tilde{}$ *in* S^4*.*

Generalised Killing Spinor Equation

As shown in Chapters 2 and 3, the Killing spinor equation gets supplemented with fields other than the Levi-Civita connection when dealing with supergravity theories. This calls for a generalisation of Definition D.4 to more general connections[4].

Definition D.5. Generalised Killing Spinor [436] Let $(\mathcal{M}, g)$ be a pseudo-Riemannian manifold of signature (p,q), equipped with a bundle of irreducible real Clifford modules $\mathcal{S}$. Let $\mathcal{D} : \Gamma(S) \to \Gamma(T^\star\mathcal{M} \otimes S)$ be a connection on $\mathcal{S}$, which may depend on various geometric structures on $(\mathcal{M}, g)$. A section of $\mathcal{S}$, $\xi \in \Gamma(\mathcal{S})$, is called *generalised Killing spinor* if it obeys the *generalised Killing spinor equation*

$$\mathcal{D}\xi = 0. \tag{D.39}$$

[4]It is the author's belief that when taking $\mathcal{M}$ to be the total space of all the vector bundles of a given theory, the Koszul connection on $\mathcal{M}$ should decompose in such a way that the 'standard' Killing spinor equation in Definition D.4 becomes the generalised Killing spinor equation in Definition D.5. However, lacking a proper proof of that statement in the general case, the author will not state it as fact.

D.4 Mapping $S^3 \times S^1$ KS to $\mathbb{R}^4$ KS

In Chapter 6 we constructed a supergravity background which engineers a particular limit of the superconformal index of 4d $\mathcal{N} = 2$ theories [28, 29]. In attempting to relate our background supergravity notation from de Wit [259] to that of Gadde, Rastelli, Razamat and Yan [5], we are forced to consider how to map $S^3 \times S^1$ spinors to $\mathbb{R}^4$ spinors. Since the superconformal index language developed in [5] is based on the *canonical spinors* of $\mathbb{R}^4$, we must suitably adapt our notation to theirs. As we will see below, given that $S^3 \times \mathbb{R}$ and $\mathbb{R}^4$ are one conformal compactification away from each other, translating between their spinors amounts to a change of basis of the local frame.

Let us write the metric of $\mathbb{R}^4$ in its Cartesian form,

$$g_{\mathbb{R}^4} = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \tag{D.40}$$

and choose the canonical vierbeins $e^a_{\mathbb{R}^4} = dx^a$. Then, a basis for the conformal Killing spinors on this space can be constructed by embedding the following spinors

$$\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \xi^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \xi^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{D.41}$$

$$\xi^{(5)} = -x^a\gamma_a\xi^{(1)}, \quad \xi^{(6)} = -x^a\gamma_a\xi^{(2)}, \quad \xi^{(7)} = -x^a\gamma_a\xi^{(3)}, \quad \xi^{(8)} = -x^a\gamma_a\xi^{(4)}, \tag{D.42}$$

as left- and right-handed spinors $\xi^{(A)}_{\mathbb{R}^4}$ satisfying the conformal Killing spinor equation

$$\partial_a \xi^{(A)}_{\mathbb{R}^4} = \frac{1}{4}\gamma_a\gamma^b\partial_b\xi^{(A)}_{\mathbb{R}^4} \ , \tag{D.43}$$

where $A \in \{1, 2, \dots, 8\}$.

We can now perform a change of variables to translate the metric into a form more reminiscent of our parametrisation of S^3. We perform the change of coordinates

$$\begin{aligned} x_1 &= e^{-t\beta}\sin\theta\cos\varphi, \\ x_2 &= e^{-t\beta}\sin\theta\sin\varphi, \\ x_3 &= e^{-t\beta}\cos\theta\cos\tau, \\ x_4 &= e^{-t\beta}\cos\theta\sin\tau, \end{aligned} \tag{D.44}$$

where $t \in [0, \infty[$, $\theta \in [0, \pi/2]$, $\varphi \in [0, 2\pi]$ and $\tau \in [0, 2\pi]$. Under this new parametrisation of $\mathbb{R}^4$ the metric reads

$$g_{\mathbb{R}^4} = e^{-2t\beta}\left(\beta^2 dt^2 + d\theta^2 + \sin^2(\theta)d\varphi^2 + \cos^2(\theta)d\tau^2\right). \tag{D.45}$$

From (D.45), one can readily see that the conformal transformation that takes $g_{\mathbb{R}^4}$ to $e^{2t\beta}g_{\mathbb{R}^4}$ is the one which maps the Euclidean space to $S^3 \times \mathbb{R}$.

The change of coordinates described above will affect our expression for the canonical vierbeins $e^a_{\mathbb{R}^4}$, naturally. However, after performing a Weyl rescaling of these by a factor of $e^{-t\beta}$ it is possible to bring them into the form in equation (C.12) via an $SO(4)$ rotation. In other words, our vierbeins for $S^3 \times S^1$ are related to those of $\mathbb{R}^4$ as follows

$$e^a_{S^3\times S^1} = e^{-t\beta} R^a{}_b e^b_{\mathbb{R}^4}, \tag{D.46}$$

where R is the $SO(4)$ matrix

$$R = \begin{pmatrix} c_\theta s_\tau & c_\theta c_\tau & -s_\theta s_\varphi & -c_\varphi s_\theta \\ -c_\theta c_\tau & c_\theta s_\tau & c_\varphi s_\theta & -s_\theta s_\varphi \\ -s_\theta s_\varphi & c_\varphi s_\theta & -c_\theta s_\tau & c_\theta c_\tau \\ -c_\varphi s_\theta & -s_\theta s_\varphi & -c_\theta c_\tau & -c_\theta s_\tau \end{pmatrix}. \tag{D.47}$$

The shorthand notation $c_\theta = \cos(\theta)$ and $s_\theta = \sin(\theta)$ was used to simplify the expressions above. Any global $SO(4)$ rotation of the coordinates in (D.44) will lead to the same conformal metric to that of $S^3 \times \mathbb{R}$, and as such will require its own $SO(4)$ rotation of the vierbeins. The one given above follows from our choice of canonical vierbeins of $\mathbb{R}^4$.

Now, consider a rotation of the frame bundle, described as an $SO(4)$ rotation of the vierbeins, just like that described previously,

$$e^a_{\mathbb{R}^4} \mapsto R^a{}_b e^b_{\mathbb{R}^4}, \qquad R \in SO(4). \tag{D.48}$$

This will induce a rotation of the spin bundle, given by the uplift of the $SO(4)$ action to a Spin(4) action. One way of finding this uplift is by considering the following surjective homomorphism from Spin(4) to $SO(4)$, whose matrix components are given by

$$R^a{}_b = \frac{1}{4}\,\mathrm{tr}\left(\gamma^a \mathcal{R} \gamma_b \mathcal{R}\right). \tag{D.49}$$

In the above, γ_i are the gamma matrices that form a basis of $Cl(4)$, and $\mathcal{R}$ is the (non-unique) uplift of the $SO(4)$ rotation R to a Spin(4) action. With a given $SO(4)$ matrix R, one can determine a suitable uplift by solving for $\mathcal{R}$ in the above.

Using the $SO(4)$ frame rotation in equation (D.47), and together with the uplift formula (D.49), we can solve for one (of the two) uplifts $\mathcal{R}$,

$$\mathcal{R} = \begin{pmatrix} -e^{-i\tau} c_\theta & -e^{-i\varphi} s_\theta & 0 & 0 \\ e^{i\varphi} s_\theta & -e^{i\tau} c_\theta & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} . \tag{D.50}$$

This, in turn, implies that for any Killing spinor on $S^3 \times S^1$, $\xi_{S^3 \times S^1}$, the corresponding Killing spinor on $\mathbb{R}^4$ is given by[5]

$$\xi_{\mathbb{R}^4} = \mathcal{R}^{-1} \xi_{S^3 \times S^1} . \tag{D.51}$$

For instance, consider the following two Killing spinors preserved by the supergravity background (**??**)

$$\xi = \begin{pmatrix} 0 & 0 \\ a & 0 \\ 0 & b \\ 0 & 0 \end{pmatrix} , \tag{D.52}$$

where the second matrix direction counts the $SU(2)$ R-symmetry components. These are mapped to the $\mathbb{R}^4$ conformal Killing spinors $\xi^{(3)}$ and $\xi^{(7)}$. In the notation used in [5], these parametrise the supercharges $Q^{2\dot{+}}$ and $S^{1\dot{-}}$[6].

[5] The Spin(4) action is sufficient as the Killing spinor equation is Weyl covariant.

[6] The eight constant conformal Killing spinors on $\mathbb{R}^4$ parametrised the Q-supersymmetries labelled by their R-symmetry index and non-vanishing spinor component, $Q^{i\alpha}$. The other eight conformal Killing spinors, have non-vanishing S-supersymmetry parameter $\eta^{i\alpha}$ and such parametrise the eight S-susy parameters $S^{i\alpha}$.

D.5 Supergroups and Lie superalgebras

The aim of this section will be to give a brief introduction to Lie superalgebras and their supergroups. This section might seem rather tangential at first. However, we will see that the various supergravity solutions in Chapter 4 necessarily exhibit supergroup symmetry. That is, they are invariant under a given supergroup which describes the usual "bosonic" symmetry of the spacetime as well the "fermionic" symmetry, generated by the Killing spinors. Furthermore, we also wish to elaborate on the construction of the exceptional groups $D(2,1;\gamma)$ that appears in the supergravity solutions of [2].

Most of the elementary definitions presented here are a watered-down version of the original presentation in [437]. We encourage the reader to consult this reference for a more thorough presentation of the topic.

Starting from the original building block, let us give a brief description of what a Lie superalgebra is, from first principles.

Definition D.6. Superalgebra Let $(A,\cdot)$ be a $\mathbb{K}$-algebra. We call *superalgebra* the $\mathbb{Z}_2$-grading of A. In other words, for A to be a *superalgebra* it must admit a decomposition into a direct sum of subspaces

$$A = A_0 \oplus A_1 \tag{D.53}$$

such that

$$\forall \alpha, \beta \in \mathbb{Z}_2, \quad A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta} \tag{D.54}$$

The elements of A_0 are called *even*, those of A_1 *odd*. If $a \in A_\alpha$, we say that a is *homogenous* of degree α and we write $\deg(a) = \alpha$.

The tensor product of two superalgebras $A \oplus B$ can be defined though the induced $\mathbb{Z}_2$-grading and the operation

$$\forall a_1, a_2 \in A, \forall b_1, b_2 \in B, \quad (a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = (-1)^{\deg(a_2)\deg(b_1)} a_1 a_2 \otimes b_1 b_2 \tag{D.55}$$

Definition D.7. Lie superalgebra A *Lie superalgebra* is a superalgebra L endowed with a bilinear map $[\,,\,] : L \times L \mapsto L$, satisfying for all homogenous $a, b, c \in L$,

$$\text{anticommutativity}: \quad [a,b] = -(-1)^{\deg(a)\deg(b)}[b,a] \tag{D.56}$$

$$\text{Jacobi identity}: \quad [a,[b,c]] = [[a,b],c] + (-1)^{\deg(a)\deg(b)}[b,[a,c]] \tag{D.57}$$

Let us briefly comment on the naming choice for the above algebra. Many authors choose *super Lie algebras* or *graded Lie algebras*, however, we will follow Victor Kac in calling them *Lie superalgebras* [437]. While all Lie algebras are indeed Lie superalgebras, the converse isn't true. Lie superalgebras are, in general, not Lie algebras.

Example D.4. *If A is an associative superalgebra, then the bracket*

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba \tag{D.58}$$

turns A into a Lie superalgebra.

Given a graded vector space V, one can consider the set of endomorphisms of V, $\mathrm{End}(V)$. Together with the composition operation $\circ$, $\mathrm{End}(V)$ can be turned into a superalgebra.

Definition D.8. superalgebra $l(V)$ Let $V = V_0 \oplus V_1$ be a $\mathbb{Z}_2$-graded $\mathbb{K}$-space. The algebra $\mathrm{End}(V)$ is endowed with a $\mathbb{Z}_2$-grading and so becomes a superalgebra, denoted $l(V)$ or $l(m|n)$, where $m = \dim(L_0)$ and $n = \dim(L_1)$.

Following Example D.4, we can introduce a bracket on $\mathrm{End}(V)$ turning it into a Lie superalgebra. For completeness we also note here the following definition.

Definition D.9. Let $(e_i)_{1 \geq i \geq m}$ be a basis of V_0 and $(e_j)_{m+1 \geq j \geq m+n}$ a basis of V_1. Together these form a so-called homogeneous basis of V. In this basis, a matrix representation of elements of $l(V)$ is given by

$$l(V) \ni a \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{D.59}$$

where α is an $(m \times m)-$, δ an $(n \times n)-$, β an $(m \times n)-$, and γ an $(n \times m)-$matrix. The *supertrace* of such an element is then defined as

$$\mathrm{str}(a) = \mathrm{tr}(\alpha) - \mathrm{tr}(\delta) \tag{D.60}$$

D.5.1 Classification of classical Lie superalgebras

A finite-dimensional Lie superalgebra $L = L_0 \oplus L_1$ is called *classical* if it is simple and the representation of L_0 on L_1 is completely reducible.

We may now follow a similar construction to the classical Lie algebras, namely, we look at the ideals of $l(m|n)$. One such ideal is the subspace defined by the vanishing of the supertrace

$$sl(m|n) = \{a \in l(m|n) \,|\, \mathrm{str}(a) = 0\}. \tag{D.61}$$

This subspace defines the first set of classical Lie superalgebras, those of type A,

$$A(m, n) = sl(m + 1|n + 1) \quad m, n \geq 0, m \neq n, \tag{D.62a}$$

$$A(n, n) = sl(n + 1|n + 1)/\langle 1_{2n+2} \rangle \quad n > 0. \tag{D.62b}$$

Other other three, non-exceptional, Lie superalgebras are constructed as follows. Let F be a non-degenerate consistent supersymmetric bilinear form on V, so that V_0 and V_1 are orthogonal and the restriction of F to V_0 is symmetric and to V_1 skew-symmetric. We define in $l(m|n)$, the orthogonal-symplectic superalgebra $osp(m|n)$ by

$$osp(m|n)_\alpha = \{a \in l(m|n)_\alpha \,|\, F(a(x), y) = -(-1)^{\alpha\,\deg(x)} F(x, a(y))\}. \tag{D.63}$$

The classical Lie superalgebras of type B, D and C are then defined as

$$B(m,n) = osp(2m+1|2n) \quad m \geq 0, n > 0, \tag{D.64}$$

$$D(m,n) = osp(2m|2n) \quad m \geq 2, n > 0, \tag{D.65}$$

$$C(n) = osp(2|2n-2) \quad n \geq 2. \tag{D.66}$$

By virtue of their definitions, the Lie superalgebras of type A, B, C, and D mirror the classical Cartan series A_n, B_n, C_n and D_n, explaining the naming scheme. To this classification, however, one must supplement the 40-dimensional $F(4)$ and 31-dimensional $G(3)$ exceptional Lie superalgebras, as well a family of deformations of the 17-dimensional $D(2,1)$, denoted by $D(2,1;\gamma)$. This latter algebra is unique in that is depends on a continuous parameter γ. As pointed out in Chapter 4, for certain values of γ this degenerates to other known Lie superalgebras. Surprisingly, the classification doesn't stop there as there are also two so-called 'strange' Lie superalgebras: $P(n)$ and $Q(n)$.

This classification can be package into one theorem, curtsey of Victor Kac [437].

Theorem D.10. *Kac A simple finite-dimensional Lie superalgebra over an algebraically closed field* $\mathbb{K}$ *of characteristic* 0 *is isomorphic either to one of the simple Lie algebras or to one of the Lie superalgebras* $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, $D(2,1;\gamma)$, $F(4)$, $G(3)$, $P(n)$, $Q(n)$, $W(n)$, $S(n)$, $\tilde{S}(n)$ *or* $H(n)$.

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