Quantum-corrected Floquet dynamics in the Rabi model

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Semiclassical descriptions of the dynamics of a few-level system coupled to a mode of the electromagnetic field which effectively reduce the contribution of the field to a time-dependent term in the Hamiltonian of the few-level system are widely used. For example, such an approach is typically taken in quantum control applications. However, the underlying quantum character of the field will lead to corrections to the semiclassical dynamics which, given sufficient time, can lead to significant changes. Here we develop an approach for calculating these quantum corrections systematically, building on the time-dependent Floquet dynamics that emerges in the semiclassical limit. Using the Rabi model of a spin–field system as an illustrative example, we obtain approximate analytic expressions for the first-order quantum corrections to the semiclassical dynamics of the spin for a range of initial field states. These expressions describe the initial stages of the full quantum dynamics accurately, though they eventually fail for sufficiently long times. Our work has relevance both for understanding the fundamental properties of emergent semiclassical behavior and as a potential tool for assessing corrections to semiclassical control techniques.

The interaction of electromagnetic fields with two- or few-level quantum systems — spins, atoms, superconducting qubits, quantum dots — underpins many emerging quantum technologies [1–3]. Increasingly sophisticated control techniques have been developed under the assumption that the fields may be treated classically [4–6]. Nevertheless, such fields are quantized and the dynamics they give rise to will never be exactly that of the corresponding semiclassical system. Indeed, even when the field is prepared in the most classical of quantum states, corrections to the semiclassical dynamics can lead over time to dramatically different behavior [7]. Understanding the nature and magnitude of the quantum corrections to the semiclassical dynamics is of fundamental interest, but also has practical significance as they represent a source of error that will increase with time in quantum control algorithms [8]. Furthermore, unlike other forms of noise and decoherence that can be mitigated through improved engineering, quantization of the field is an irreducible source of error in classical control protocols.

Intensive efforts stretching over several decades have sought effective means of calculating the fully quantum dynamics of few-level systems coupled to quantum fields. Many studies have relied on numerical computation, as analytical approximation techniques for calculating dynamical variables are complicated and provide limited physical insight [9–13]. Although this work has uncovered the rich phenomenology present in even the simplest models, it has not managed to construct a complete, coherent picture of how quantum field dynamics can give rise to semiclassical behavior. The series of studies on quasiclassical trajectories in the Jaynes-Cummings and Rabi models by Gea-Banacloche [14–16], together with Finney [17], comes the closest. While their work bears some similarities to that presented here, it is rooted in phenomenological observations rather than the more general first-principles approach that we take.

Recent work has established that the quantum-to-semiclassical transition can be expressed at the Hamiltonian level as a well-defined mathematical limit, resolving the long-standing correspondence problem by taking the quantum coupling to zero alongside the field amplitude going to infinity [18]. This provides both the physical insight and the analytical framework necessary for understanding how corrections to the semiclassical dynamics will emerge. With a rigorous mathematical formulation of the semiclassical limit in hand, it becomes possible to systematically study exactly how the limit is approached and to characterize the quantum corrections.

In this Letter we outline a method for calculating the time evolution of the joint state of a few-level quantum system interacting with a quantized field in terms of the Floquet solutions to the corresponding semiclassical Hamiltonian, an approach we call 'quantum-corrected Floquet dynamics' (QCFD). We further introduce an exactly solvable first-order approximation, the Floquet-basis rotating-wave approximation (FBRWA). Focusing on the case of a spin coupled to a field mode, the celebrated Rabi model, we obtain closed-form analytical expressions for the dynamics of the system with arbitrary initial states of the quantum field. As illustrations, we derive expressions for the dynamics of the spin population in both the Rabi model and the simpler Jaynes-

Cummings model for a variety of different field states. We demonstrate that our approximate analytic expressions for the lowest-order quantum corrections to the semiclassical dynamics accurately describe the collapse of the Rabi oscillations, but not their later revival.

The conceptual basis of QCFD is that the dynamics induced by the field can be separated into semiclassical and quantum components. This goes beyond simply providing a powerful technique for calculating dynamical quantities. It enables us to cleanly delineate which features in the evolution are semiclassical in nature and which arise from the underlying quantum character of the field, a question of both foundational and practical importance. Moreover, the same approach is likely to apply to a wider class of settings involving interactions between fields and few-level systems.

To introduce the principles of QCFD, we employ the paradigmatic Rabi model: a two-level quantum system coupled to a single mode of the electromagnetic field [18–20]. For simplicity, we use the term 'spin' for the two-level system. We write the Hamiltonian in the form

$$H = \omega_0 \hat{a}^\dagger \hat{a} + \tfrac{1}{2}\Omega \hat{\sigma}_z + \lambda[\hat{f}(\hat{a},\hat{a}^\dagger)\hat{\sigma}_+ + \hat{f}^\dagger(\hat{a},\hat{a}^\dagger)\hat{\sigma}_-], \quad (1)$$

where $\hat{\sigma}_\pm = |\pm z\rangle\langle\mp z|$ are the raising and lowering operators for the spin states $|\pm z\rangle$, which represent the energy eigenstates of the bare spin Hamiltonian. The energy eigenstates of the field are the Fock states $|n\rangle$. With the choice $\hat{f}(\hat{a},\hat{a}^\dagger) = (\hat{a}+\hat{a}^\dagger)$, Eq. (1) corresponds to the quantum Rabi model (QRM). If the standard rotating-wave approximation (RWA) is applied to the Rabi model, $\hat{f}(\hat{a},\hat{a}^\dagger) = \hat{a}$ and the resulting Hamiltonian is commonly known as the Jaynes-Cummings model (JCM).

Central to the development of QCFD is the separation of the Hamiltonian into semiclassical and quantum terms. Applying the first two steps of the semiclassical limiting procedure outlined in [18] — transforming to the interaction picture with respect to the field, followed by a displacement transformation $\hat{D}(\alpha)$ on the field operators — allows the Hamiltonian (1) to be written in the form

$$\tilde{H}^I(t) = H_{sc}(t) \otimes \hat{I}_f + H_q^I(t), \quad (2)$$

where $H_{sc}(t)$ is the corresponding semiclassical Hamiltonian, $\hat{I}_f$ is the identity for the field, and $H_q^I(t)$ describes the interaction with the quantized field. (See Appendix for details.) In the semiclassical limit $\lambda/\omega_0 \to 0$ and $|\alpha| \to \infty$, $H_q^I(t)$ is eliminated. Here, however, we are concerned with calculating corrections to the semiclassical dynamics induced by the quantum interaction term.

From Floquet theory, the time-dependent semiclassical Hamiltonian admits solutions

$$|\Psi_\pm(t)\rangle = e^{-iq_\pm t}|\psi_\pm(t)\rangle \quad (3)$$

that satisfy

$$\left[H_{sc}(t) - i\tfrac{\partial}{\partial t}\right]|\psi_\pm(t)\rangle = q_\pm|\psi_\pm(t)\rangle, \quad (4)$$

where the Floquet quasienergies $q_\pm$ are defined up to an integer multiple of ω_0 [17, 21]. These Floquet states form a complete orthonormal basis at any time t, so an arbitrary state of the joint quantum system may be written as

$$|\Phi(t)\rangle = |\Psi_+(t)\rangle \otimes |\tilde{\phi}_+^I(t)\rangle + |\Psi_-(t)\rangle \otimes |\tilde{\phi}_-^I(t)\rangle. \quad (5)$$

The field states are given in the transformed basis as $|\tilde{\phi}_\pm^I\rangle = \sum_{n=0}^{+\infty} c_\pm^n(t)\hat{D}^\dagger(\alpha)e^{i\omega_0 t\hat{a}^\dagger\hat{a}}|n\rangle$, with probability amplitudes $c_\pm^n(t)$ [22]. Inserting this form of the wavefunction into the Schrödinger equation and projecting onto the Floquet states yields a pair of coupled differential equations for the field associated with each Floquet state of the spin:

$$\begin{aligned} i\frac{\partial}{\partial t}|\tilde{\phi}_\pm^I(t)\rangle = \langle\Psi_\pm(t)|H_q^I(t)|\Psi_\pm(t)\rangle|\tilde{\phi}_\pm^I(t)\rangle \\ + \langle\Psi_\pm(t)|H_q^I(t)|\Psi_\mp(t)\rangle|\tilde{\phi}_\mp^I(t)\rangle. \end{aligned} \quad (6)$$

This is the key conceptual result of the QCFD approach. Working in the Floquet basis effectively diagonalizes the semiclassical term in the Hamiltonian, allowing us to isolate the corrections due to the quantum nature of the field.

This separation, however, comes at the expense of explicit time dependence in both the spin basis states and the quantum Hamiltonian. With the choice $q_- = -q_+$, the Floquet states have a Fourier series representation [17, 23][24]

$$|\Psi_\pm(t)\rangle = e^{-iq_\pm t}(\pm A_\pm(t)|\pm z\rangle + B_\pm(t)|\mp z\rangle), \quad (7)$$

where $A_\pm(t) = \sum_{k=-\infty}^{\infty} A_{2k}e^{\pm 2ki\omega_0 t}$ and $B_\pm = \sum_{l=-\infty}^{\infty} B_{2l+1}(t)e^{\pm(2l+1)i\omega_0 t}$ with A_{2k}, B_{2l+1} real. When $H_q^I(t)$ is expressed in this basis (see Appendix for detailed expressions), terms that are diagonal in the Floquet states $|\Psi_\pm(t)\rangle$ evolve at integer multiples of ω_0. By contrast, the off-diagonal terms contain the additional factor $e^{\pm i(q_+ - q_-)t}$. We see, then, that the two terms on the lhs of Eq. (6) are distinct in physical character and act on different timescales. The first leaves the Floquet states of the spin unchanged but generates quantum evolution of the associated field components; the second creates transitions between the Floquet states on a timescale that depends on the difference between the Floquet quasienergies.

Up to this point, the treatment has been exact. The quantum interaction terms in the Floquet basis are cumbersome [see Eqs. (20)-(21)]. However, their form — especially the separation of timescales between the diagonal and off-diagonal terms — invites an approximation strategy in the spirit of the standard RWA. To lowest order, we retain only the time-independent terms in $H_q^I(t)$, an approximation that we term the FBRWA.

The resulting Hamiltonian for both the JCM and the full Rabi model takes the remarkably simple form

$$H^I_{\mathrm{FBRWA}}(t) = \lambda^{\mathrm{eff}}(\hat{a}^\dagger+\hat{a})(|\Psi_+(t)\rangle\langle\Psi_+(t)|-|\Psi_-(t)\rangle\langle\Psi_-(t)|), \tag{8}$$

where [25]

$$\lambda^{\mathrm{eff}} = \lambda \sum_{k=-\infty}^{\infty} A_{2k}(B_{2k+1} + B_{2k-1}). \tag{9}$$

The effective equations of motion governing the field evolution decouple and are readily solved. Transforming back to the original frame, the solutions for the field states are given by

$$|\phi_\pm(t)\rangle = e^{-i\omega_0 t\hat{a}^\dagger\hat{a}} e^{\pm 2i\lambda^{\mathrm{eff}}|\alpha|t}\hat{D}(\eta_\pm(t))|\phi_\pm(0)\rangle, \tag{10}$$

where the time-dependent displacement of the field in phase space is $\eta_\pm(t) = \mp ie^{-i\phi}\lambda^{\mathrm{eff}}t$. This evolution of the coupled system may be interpreted in terms of a dynamical polaron transformation. As in the usual polaron transformation, the field is displaced in a direction that depends on the state of the spin [26–28]; here, the displacement amplitude is a function of time as well as coupling strength.

Together with the Floquet solutions for the spin, Eq. (10) constitutes a closed-form analytical approximation for the time evolution of the joint spin–field state vector. Any dynamical quantity may then be calculated from the state vector. In the following discussion we focus on the excited-state probability $P(+z) = |\langle +z|\Phi(t)\rangle|^2$ of the spin, a quantity with a clear connection to well-known results in quantum optics and significant relevance to quantum control. Selected examples illustrate the simplicity and power of this new approach together with the intuitive physical interpretation of complex dynamics that it offers.

Consider the initial state $|\Phi(0)\rangle = |+z\rangle \otimes |\phi_0\rangle$, with $\alpha = \langle\phi_0|\hat{a}|\phi_0\rangle$ taken to be real. The FBRWA provides a general expression for the excited-state probability of the spin at later times:

$$\begin{aligned} P(+z) = {} & |A_+(0)|^2|A_+(t)|^2 + |B_-(0)|^2|B_-(t)|^2 \\ & + \big[A^*_+(0)B_-(0)A_+(t)B^*_-(t)e^{-i(q_+-q_-)t} \\ & \quad \times \langle\phi_-(t)|\phi_+(t)\rangle + c.c.\big]. \end{aligned} \tag{11}$$

The dynamics is determined by the Floquet coefficients and quasienergies, together with the inner product between the field states associated with different Floquet states.

Let us first look at the Jaynes-Cummings model. The corresponding semiclassical model is exactly solvable. On resonance ($\Omega = \omega_0$), the quasienergies are $q_\pm = \pm(\frac{1}{2}\omega_0 + \lambda|\alpha|)$ and the only non-zero coefficients of the Floquet modes are $A_0 = B_1 = 1/\sqrt{2}$. Hence $\lambda^{\mathrm{eff}} = \lambda/2$. Taking the initial state of the field to be a coherent state $|\alpha\rangle$, the excited-state probability is readily worked out to be

$$P(+z) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\lambda^2t^2/2}\cos(2\lambda|\alpha|t). \tag{12}$$

This, of course, is the famous expression for sinusoidal Rabi oscillations with a Gaussian collapse envelope first obtained by Cummings [29]. In Cummings' approach, the time evolution of the joint state is given as a sum over the sinusoidal Rabi oscillations induced by each Fock state $|n\rangle$ contained in the initial coherent state. For large values of $|\alpha| = \sqrt{\bar{n}}$, the weighting factor becomes sharply peaked around $\sqrt{\bar{n}}$ and the sum may be approximately evaluated, leading to the above expression. The collapse results from destructive interference of oscillations at incommensurate frequencies.

Here we see a different physical interpretation of the collapse. The initial spin state is a superposition of the two semiclassical Floquet states. Within the FBRWA, the quantum interaction term does not cause transitions between the Floquet states, but the evolution of the field depends on which Floquet state it is associated with. As the two field components become displaced in opposite directions, their inner product decreases, causing the collapse of the semiclassical Rabi oscillations. Gea-Banacloche showed similarly that the collapse occurs when the field states associated with two spin states become macroscopically distinguishable [14, 15].

Next, consider an initial displaced Fock state $|\alpha, n\rangle = \hat{D}(\alpha)|n\rangle$. Numerical studies of the dynamics have been carried out [30, 31], but no analytical results appear to have been previously reported. Here the generalization from the coherent-state case is practically trivial. The only change is in the field overlap, giving

$$P(+z) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\lambda^2t^2/2}L_n(\lambda^2t^2)\cos(2\lambda|\alpha|t). \tag{13}$$

The Gaussian collapse envelope characteristic of a coherent field is modulated by the Laguerre polynomial $L_n(\lambda^2t^2)$. Figure 1 illustrates this behavior. Rabi oscillations persist over longer times because the displaced Fock states $|\alpha, n\rangle$ are not minimum-uncertainty states, but have a width in phase space that increases with n. A larger relative displacement is required for the overlap between the two field states to become negligible; or, in other words, for the states to become macroscopically distinguishable. Nodes in the envelope result from destructive interference between the two displaced Fock states.

Another interesting example is a superposition of two displaced Fock states:

$$|\phi_0\rangle = (|\beta, 0\rangle + e^{-i\xi}|\beta, 1\rangle)/\sqrt{2}. \tag{14}$$

For $\beta = |\beta|e^{-i\phi}$ and $\xi = \phi$, the excited-state probability of the spin becomes

$$\begin{aligned} P(+z) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\lambda^2t^2/2}\big[&(1 - \tfrac{1}{2}\lambda^2t^2)\cos(2\lambda|\beta|t) \\ &-\lambda t\sin(2\lambda|\beta|t)\big]. \end{aligned} \tag{15}$$

In this case the result cannot be simply expressed as a sinusoidal oscillation modified by a collapse envelope. Nevertheless, the analytical expression captures the dynamical behavior quite well (see Fig. 3 in the Appendix).

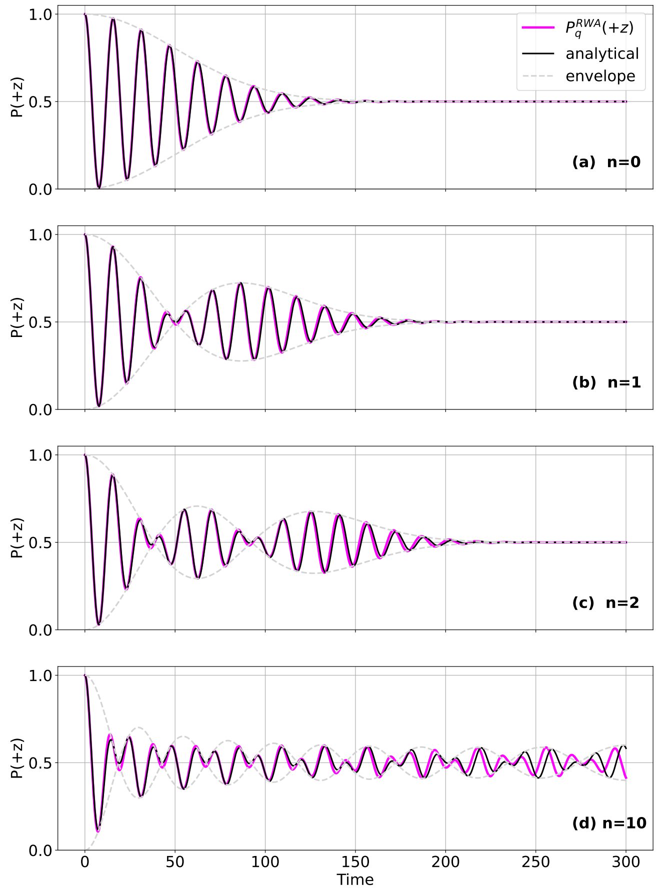

FIG. 1. Collapse of Rabi oscillations for different initial field states. The excited-state probability of the spin is plotted against time. Parameter values are $\alpha = 10$, $\lambda = 0.05$, and $\Omega = \omega_0 = 1$. Numerical solution of the full JCM Hamiltonian (magenta) is compared with Eq. (13) (black), dashed grey lines indicate the collapse envelope from Eq. (13). The initial state of the system is $|\Phi(0)\rangle = |+z\rangle \otimes |\alpha, n\rangle$ with $n =$ (a) 0, (b) 1 (c) 2, (d) 10.

It is worth noting that all the above results can be trivially extended to the off-resonance case. The only change required is to replace the Floquet quasienergies and states by the equivalent solutions for $\Omega \neq \omega_0$.

We now turn to the full Rabi model. The FBRWA solutions still have the same form in the Floquet basis, differing only in the value of λ^{eff}. Finding the Floquet states of the Rabi model, however, is a challenging problem in itself. To illustrate the properties of the FBRWA solution with the Rabi interaction, we employ the approximate Floquet solutions for $\Omega = \omega_0$ given in [17], which are valid up to third order in $\epsilon \equiv \lambda|\alpha|/(2\Omega)$.

Figure 2 compares the population dynamics of the spin given by the analytical FBRWA expression in the Rabi model with a numerical solution of the full Hamiltonian. The field is initialized in the displaced Fock state $|\alpha, 1\rangle$. As the field components $|\phi_\pm(t)\rangle$ become displaced, the growing entanglement between the spin and field suppresses the coherence between the Floquet states that generates oscillations at the Rabi frequency. As in the JCM, the collapse is governed by the product of a Gaussian and a Laguerre polynomial.

However, the collapse here is not complete; oscillations persist in the 'quiescent' region, at a frequency near $2\omega_0$ [32, 33]. The origin of this phenomenon is readily understood from Eq. (11). In the quiescent region, $\langle\phi_-(t)|\phi_+(t)\rangle \approx 0$. The spin dynamics is effectively determined by $|A_+(t)|^2$ and $|B_-(t)|^2$, which represent the probability of $|+z\rangle$ in the Floquet states $|\Psi_+(t)\rangle$ and $|\Psi_-(t)\rangle$, respectively. Within the RWA, the Floquet coefficients each have a single frequency component and the probabilities are constant. When counter-rotating terms are included, the Floquet coefficients develop components at higher frequencies. Since $A_+(t)$ ($B_-(t)$) contains only even (odd) multiples of ω_0, the dominant correction involves terms with a frequency difference of $2\omega_0$. Consequently, the residual oscillations are unrelated to the quantum nature of the field. They arise because the Floquet states are time-dependent coherent superpositions of the bare spin eigenstates, a purely semiclassical effect. Interestingly, a similar effect can be seen around the node of the $n = 1$ Laguerre polynomial in the collapse envelope, for the same reason.

The analytical curves plotted in Fig. 2 involve two approximations: the FBRWA itself, and the approximate Floquet solutions. The validity of each approximation depends on the parameters in different ways. In (a), the Floquet approximation is indistinguishable from the numerical solution of the semiclassical model, so differences between the two curves indicate limitations of the FBRWA. In the lower two panels, the larger value of ϵ creates discrepancies in the Floquet approximation that are visible at earlier times, especially in the central region of the bottom plot. By contrast, as α increases from top to bottom, the FBRWA improves, particularly for the residual oscillations in the collapse region.

As with any rotating-wave-type approximation, the validity of the FBRWA is predicated on a separation of timescales: the frequency of the rotating phase should be large compared to the associated transition matrix element. Put differently, transitions between Floquet states will be less likely if they are well separated in (quasi)energy. For the on-resonance JCM, this gives the criterion $q_+ - q_- - \omega_0 \gg \lambda/2$, which reduces to $|\alpha| \gg 1/4$. Hence the approximation improves as the phase-space displacement $\langle\phi_0|\hat{a}|\phi_0\rangle$ of the initial field state is increased. The details differ for the off-resonant case and for the Rabi interaction (and in the latter case, time-dependent terms that are diagonal in the Floquet basis have also been neglected), but the essential scaling with $|\alpha|$ holds, as seen from Fig. 2. In general, the error in rotating-wave approximations also increases as time goes on [34–37], which is true here as well.

It is clear that the FBRWA can only predict the col-

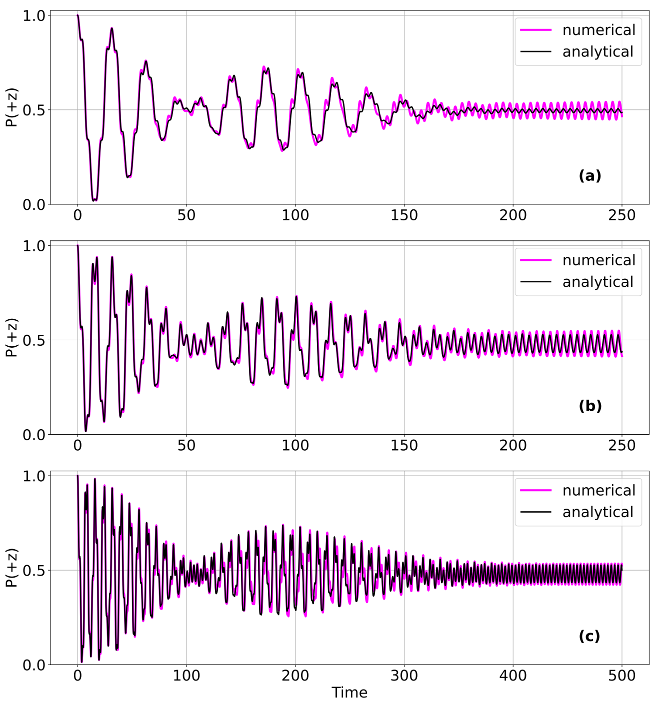

FIG. 2. Collapse of Rabi oscillations with the Rabi interaction Hamiltonian in the on-resonance case, $\Omega = \omega_0 = 1$. The initial state is $|+z\rangle \otimes |\alpha, 1\rangle$. Numerical solutions of the full Rabi Hamiltonian (magenta) are compared with . the analytical solution (black) from the FBRWA together with an analytical approximation for the Floquet states from [17]. The parameters are (a) $\lambda = 0.02$, $\alpha = 10$, (b) $\lambda = 0.02$, $\alpha = 20$ and (c) $\lambda = 0.01$, $\alpha = 40$.

lapse of the Rabi oscillations, not the revivals. The interaction-induced displacement of the field components increases linearly with time, so their overlap will asymptotically vanish for long times. Revivals, then, must be created by the terms in $H_q^I(t)$ that are off-diagonal in the Floquet states. This leads to the intriguing conclusion that the collapse and the revivals may be attributed to distinct physical processes in the Floquet basis. The collapse results from interaction terms that leave the semiclassical Floquet states unchanged but cause a state-dependent displacement of the field, generating entanglement between spin and field and thereby destroying the coherent oscillations between Floquet states. Revivals, on the other hand, are linked to interaction terms that drive transitions between the Floquet states.

While we have considered only the JCM and the standard Rabi model here, the QCFD approach and the FBRWA are likely to prove useful for a range of related models. Extension to an asymmetric [38–40] or anisotropic [41] Rabi model is simple: adding a bias term to the spin will change the Floquet solutions, while allowing different coupling constants for the co-rotating and counter-rotating terms will change λ^{eff}. Multiphoton interactions will be more complicated to treat but intriguing to explore from this perspective. Of particular interest is the polaron-transformed form of the Rabi Hamiltonian, which is widely employed to study the regimes of very strong coupling to a high-frequency field and whose semiclassical limit can also be obtained through the limiting procedure described above [18, 27, 28].

Quantum-corrected Floquet dynamics is a generalizable technique that opens up broad prospects for future work, both within and beyond the FBRWA. Closed-form analytic expressions are readily derived from the FBRWA yet accurately capture even quite complicated short-time dynamics. Such computational and conceptual simplicity lends itself to assessing quantum field effects on control sequences deployed in quantum technologies. More fundamentally, this approach provides the physical intuition that draws various aspects of dynamics in the Jaynes-Cummings and Rabi models into a cohesive picture. Both practical applications and foundational implications for how classical and semiclassical behavior emerge from quantized fields offer tantalizing avenues for exploration.

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Appendix

QCFD for the Rabi model

Following the 'recipe' of [18] for obtaining the semiclassical limit requires a series of transformations. First, the Hamiltonian is written in the interaction picture with respect to the field, using the operator $\hat{U}(t) = e^{-i\omega_0 t \hat{a}^\dagger \hat{a}}$:

$$H^I(t) = \tfrac{1}{2}\Omega\hat{\sigma}_z + \lambda\left[\hat{f}(e^{-i\omega_0 t}\hat{a}, e^{i\omega_0 t}\hat{a}^\dagger)\hat{\sigma}_+ + \hat{f}^\dagger(e^{-i\omega_0 t}\hat{a}, e^{i\omega_0 t}\hat{a}^\dagger)\hat{\sigma}_-\right]. \quad (16)$$

Next, a displacement transformation $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ with $\alpha \equiv |\alpha|e^{i\phi}$ is applied, giving

$$\tilde{H}^I(t) = H_{sc}(t) \otimes \hat{I}_f + H^I_q(t), \quad (17)$$

where

$$H_{sc}(t) = \tfrac{1}{2}\Omega\hat{\sigma}_z + \lambda\left[f(|\alpha|e^{-i(\omega_0 t+\phi)}, |\alpha|e^{i(\omega_0 t+\phi)})\hat{\sigma}_+ + f^*(|\alpha|e^{-i(\omega_0 t+\phi)}, |\alpha|e^{i(\omega_0 t+\phi)})\hat{\sigma}_-\right] \quad (18)$$

is the semiclassical Hamiltonian corresponding to Eq. (1), $\hat{I}_f$ is the identity for the field, and

$$H^I_q(t) \equiv \lambda[\hat{f}(e^{-i\omega_0 t}\hat{a}, e^{i\omega_0 t}\hat{a}^\dagger)\hat{\sigma}_+ + \hat{f}^\dagger(e^{-i\omega_0 t}\hat{a}, e^{i\omega_0 t}\hat{a}^\dagger)\hat{\sigma}_-] \quad (19)$$

is the quantum interaction term. Equivalently, this may be thought of as writing the Hamiltonian $H^I(t)$ in the basis of the displaced Fock states $|\alpha, n\rangle \equiv \hat{D}(\alpha)|n\rangle$.

$H^I_q(t)$ in the Floquet basis

Using the $|\Psi_\pm\rangle$ basis, the interaction term $\lambda e^{-i\omega_0 t}\hat{a}\hat{\sigma}_+$ takes the form

$$\begin{aligned}\lambda e^{-i\omega_0 t}\hat{a}\hat{\sigma}_+ = \lambda\hat{a}\sum_{k,l=-\infty}^{\infty} \big[&A_{2k}B_{2l+1}e^{2(l-k)i\omega_0 t}(|\Psi_+(t)\rangle\langle\Psi_+(t)| - |\Psi_-(t)\rangle\langle\Psi_-(t)|) \\ &+ e^{-i(q_+-q_-)t}B_{2k+1}B_{2l+1}e^{(2k+2l+1)i\omega_0 t}|\Psi_-(t)\rangle\langle\Psi_+(t)| \\ &- e^{i(q_+-q_-)t}A_{2k}A_{2l}e^{-(2k+2l+1)i\omega_0 t}|\Psi_+(t)\rangle\langle\Psi_-(t)|\big].\end{aligned} \quad (20)$$

Similarly, the counter-rotating term appearing in the quantum Rabi interaction becomes

$$\begin{aligned}\lambda e^{i\omega_0 t}\hat{a}^\dagger\hat{\sigma}_+ = \lambda\hat{a}^\dagger\sum_{k,l=-\infty}^{\infty} \big[&A_{2k}B_{2l+1}e^{2(l-k+1)i\omega_0 t}(|\Psi_+(t)\rangle\langle\Psi_+(t)| - |\Psi_-(t)\rangle\langle\Psi_-(t)|) \\ &+ e^{-i(q_+-q_-)t}B_{2k+1}B_{2l+1}e^{(2k+2l+3)i\omega_0 t}|\Psi_-(t)\rangle\langle\Psi_+(t)| \\ &- e^{i(q_+-q_-)t}A_{2k}A_{2l}e^{-(2k+2l+3)i\omega_0 t}|\Psi_+(t)\rangle\langle\Psi_-(t)|\big].\end{aligned} \quad (21)$$

Expressed in this basis, $H^I_q(t)$ appears as a complicated function of time with both diagonal and off-diagonal terms in the Floquet states, making the coupled differential equations (6) for the field difficult to solve.

A very attractive simplification is obtained if the terms which are off-diagonal in the Floquet basis can be dropped. This is the Floquet-basis rotating-wave approximation. However, care is needed to identify the different frequency-components involved and hence to understand when the approximation is justified. The first line of Eq. (20), which is diagonal in the Floquet states, has zero-frequency components for all $l = k$. The off-diagonal terms, however, have an additional time dependence arising from the quasienergy difference $q_+ - q_-$. This means they only possess zero-frequency components at isolated resonances satisfying the condition $q_+ - q_- = (2k+2l+1)\omega_0$. Away from any such resonances, the interaction terms that drive transitions between the Floquet states are suppressed relative to the time-independent diagonal terms and may be neglected to lowest order.

Rabi oscillation collapse for superposition states

Figure 3 shows that the FBRWA expression for the excited-state spin population, Eq. (15)], arising from an initial superposition state [Eq. (14)] agrees extremely well with a full numerical integration of the JCM, capturing all of the Rabi oscillation collapse.

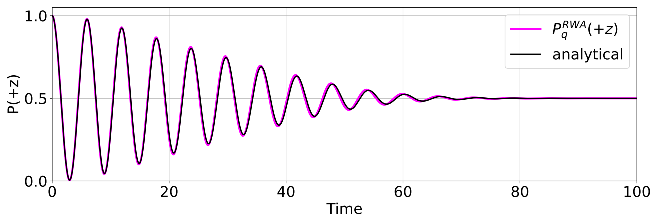

FIG. 3. Collapse of Rabi oscillations for an initial superposition field state Eq. (14). The excited-state probability of the spin is plotted as a function of time. Parameter values are $\beta = 10$, $\lambda = 0.05$, and $\Omega = \omega_0 = 1$. Numerical solution of the full RWA Hamiltonian (magenta) is compared with Eq. (15) (black).