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Equitability and explosive synchronisation in multiplex and higher-order networks

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Cluster synchronisation is a key phenomenon observed in networks of coupled dynamical units. Its presence has been linked to symmetry and, more generally, to equability of the underlying network pattern of interactions between identical dynamical units. In this article, we clarify once and for all the relation between equitability and cluster synchronisation on a very general dynamical system which allows multi-layer and higher-order interactions. Namely, we show that equitability is a necessary, and sufficient, condition for the existence of independent cluster synchronised solutions. As an important consequence, our results explain the ubiquity of explosive synchronisation, as opposed to cluster synchronisation, in multi-layer and higher-order networks: equitability imposes additional constraints that must be simultaneously satisfied on the same set of nodes. Our results have important implications for the design of complex dynamical systems of coupled dynamical units with arbitrary cluster synchronisation patterns and coupling functions.

INTRODUCTION

The study of synchronization in coupled networks of oscillators [1] serves as a fundamental pillar to understand the dynamics of interconnected systems, holding profound implications across numerous disciplines [2–4]. At its core, the existence of a synchronized state represent how individual elements within a network coordinate their behaviours, fostering coherence or collective dynamics. In recent years, traditional results of global synchronizability in standard networks [5] have been extended to both systems with multiple layers of interactions [4, 6–9] as well as systems with many-body interactions [10, 11], constituting a new paradigm that allows researchers to better model the complex processes of real-world systems [12].

A synchronized state may not involve the whole network. In fact, under specific conditions, one can observe one or more subsets of nodes that synchronize their motion independently of the other groups of nodes and of the rest of the network, giving rise to the phenomenon known as *cluster synchronization*. Notably, even though the nodes within a specific cluster evolve in a coordinated fashion, displaying similar patterns of activity or dynamics, nodes in other clusters may have different synchronization patterns or even remain desynchronized from each other. Understanding how clusters within a network synchronize their activities impacts multiple fields, as it offers insights into the general processes of organization and dynamics in complex systems [13–20], including brain networks, social networks, power grids, and biological systems.

Recently, the study of synchronisation has been considered in generalisations of network models, such as multilayer and hypergraph models [8, 11]. In contrast to network models, explosive synchronisation has been commonly observed in higher-order network dynamics [21, 22]. In this article, we formally explain the role of equitability in cluster synchronisation for a very general system of coupled identical dynamical units in multilayer and hypergraph networks.

In particular, we show that the occurrence of equitability, and thus (generic, or independent) cluster synchronisation, is much more rare in multilayer and higher-order networks, as it must be simultaneously and satisfied on each layer, or at each order of multi-body interaction, independently. This explains why cluster synchronisation is uncommon in such systems, which, instead, almost universally feature explosive synchronisation.

More specifically, we show that equitability (of the partition of the underlying interaction multiplex, or hypergraph, into clusters) is a necessary condition for the existence of *independent* cluster-synchronised solutions, that is, synchronised solutions for which the coupling functions between clusters are linearly independent over trajectories. Conversely, equitability is also a sufficient condition for synchronisation: we can construct cluster synchronised solutions with respect to any given partition, as long as it is equitable.

The results are independent of the specific form of the coupling functions or of the coupling strength parameters, and depend only of the structural properties (equitability) of the underlying interaction patterns (a network, multiplex, or hypergraph), directly relating structure to function. The caveat is that we only concern ourselves with the existence and not the stability of the synchronised solutions, which, indeed, typically depends on the specific form of the coupling functions and, for fixed coupling functions, on the values of the coupling strength parameters. Nevertheless, our results show how equitability gives strong restrictions on the existence of synchronised solutions, and, in particular, how the restrictions are much harder to achieve for higher-order dynamical interactions.

The existence of cluster synchronised solutions has been previously related to the presence of network symmetries and, more generally, of equitable partitions [14] and balanced equivalent relations [23]. Our structural equitability results can be seen as an instantiation of the formalism in [23] to multilayer networks, and an extension to many-body interactions (hypergraphs). In addition, we characterise non-equitable solutions in terms of the linear independence of the coupling functions over trajectories, study difference notions of dynamical and structural equitability (fully in the SI), and explain the role of higher-order equitability in explosive synchronisation.

DYNAMICAL SYSTEM AND CLUSTER SYNCHRONISATION

The behaviour of a networked system of N identical D-dimensional units is described by a system of dynamical equations of the form (see SI for full details)

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \mathbf{h}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N) \qquad i = 1, \ldots, N \tag{1}$$

where $\mathbf{x}_i(t)$ is a D-dimensional vector describing the state of the ith unit at time $t \in T \subseteq \mathbb{R}$, $\mathbf{f}$ is a smooth map that encodes the internal dynamics of the units, and $\mathbf{h}_i(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ is a coupling term, representing the total dynamical input of all other nodes to node i. In a multilayer network (strictly speaking, a multiplex network [6], i.e. same N nodes on each layer) with M layers, the coupling is of the general form

$$\mathbf{h}_i^{\text{multi}} = \sum_{m=1}^{M} \sigma_m \sum_{j=1}^{N} A_{i,j}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) \, , \tag{2}$$

whereas in a higher-order network (hypergraph) with highest multi-body interactions order M, it is [11]

$$\mathbf{h}_i^{\text{hyper}} = \sum_{m=1}^{M} \sigma_m \sum_{j_1,\ldots,j_m=1}^{N} A_{i,j_1,\ldots,j_m}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_m}) \, . \tag{3}$$

In the equations above, $\sigma_m > 0$ is the coupling strength for layer m or order m, $A^{(m)}$ is the adjacency matrix for layer m, or the hypergraph adjacency tensor of order $m+1$, and the coupling functions $\mathbf{g}$ are assumed to be synchronization non-invasive, that is, $\mathbf{g}^{(m)}(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) = 0$ for all m and any $\mathbf{x}$. For $M = 1$, both the multilayer and the hypergraph case reduce to the network (or monolayer) case,

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} a_{ij} \, \mathbf{g}(\mathbf{x}_i, \mathbf{x}_j), \tag{4}$$

where a_{ij} are the entries of the network adjacency matrix $A = (a_{ij})$. An important example in this case is the graph Laplacian coupling, $\mathbf{g}(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_j - \mathbf{x}_i$.

A solution $\vec{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ to the dynamical system Eq. (1) is *globally synchronised* if $\mathbf{x}_1 = \ldots = \mathbf{x}_N$ (identical synchronisation). Since $\mathbf{g}$ is non-invasive, this is equivalent to the synchronised solution $\mathbf{x}_s = \mathbf{x}_i$ being a solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Many systems, however, exhibit *partial synchronisation*, that is, identical synchronisation in a subset, or several subsets, of nodes or dynamical units [3, 4, 13]. Formally, given a solution $\vec{\mathbf{x}}$, we say that a subset of nodes $C \subseteq V$ is *synchronised* if $\mathbf{x}_i = \mathbf{x}_j$ for all $i, j \in C$. If $\mathcal{P}$ is a partition of the node set V (a collection of pairwise disjoint subsets of V, called *clusters*, whose union is V), we say that a solution $\mathbf{x}$ is $\mathcal{P}$*-synchronised* if every $C \in \mathcal{P}$ is synchronised, that is,

$$\mathbf{x}_i = \mathbf{x}_j \text{ for all } i, j \in C, \tag{5}$$

for each $C \in \mathcal{P}$. This includes any form of partial synchronisation, as we can put single nodes on (trivially) synchronised clusters on their own — we call such cluster a *singleton*.

In this article, we are concerned with very general necessary and sufficient conditions for the *existence* of cluster synchronised solutions, based on the equitability properties of the underlying pattern of interactions. We will not make any assumptions or claims on the *stability* of the solutions, which typically depends on the actual form of the dynamical system, such as the coupling functions $\mathbf{g}^{(m)}$, or, for given coupling functions, the values of the coupling strength parameters σ_m (cf. Fig. 2).

For simplicity, we focus on the multilayer case first, and discuss the hypergraph case later, in the Section called 'Results for hypergraph dynamics'.

DYNAMICAL AND STRUCTURAL EQUITABILITY

We want to relate cluster (or partial) synchronisation to equitability, that is, the fact that every node receives the same input from nodes in other clusters. We first introduce a notion of dynamical equitability, then relate this to a more standard notion of structural equitability, normally referred to as 'external equitability' in the cluster synchronisation literature [14, 24].

Given a subset of nodes $C \subseteq V$ and a solution $\vec{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ of the dynamical system, we write

$$\mathbf{h}_i^C = \sum_{m=1}^{M} \sigma_m \sum_{j \in C}^{M} A_{i,j}^{(m)} \mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) \tag{6}$$

the total dynamical input to node i from nodes in cluster $C \subseteq V$. We can see $\mathbf{h}_i^C$ as a function of $\mathbf{x}_1, \ldots, \mathbf{x}_N$, or as a function of t as $\mathbf{x}_j = \mathbf{x}_j(t)$ for all j. Note that $\mathbf{h}_i^C$ is a non-invasive function of $\mathbf{x}_1, \ldots, \mathbf{x}_N$, since all the coupling functions $\mathbf{g}^{(m)}$ are non-invasive, and additive, in the sense that $\mathbf{h}_i^{A \cup B} = \mathbf{h}_i^A + \mathbf{h}_i^B$ whenever A and B are disjoint clusters.

We can now rewrite Eq. (1) in the multilayer case in terms of the internal and external dynamical inputs with respect to a partition $\mathcal{P}$,

$$\dot{\mathbf{x}}_i = f(\mathbf{x}_i) + \mathbf{h}_i^{\text{int}} + \mathbf{h}_i^{\text{ext}}, \tag{7}$$

where $i \in C$, $C \in \mathcal{P}$, $\mathbf{h}_i^{\text{int}} = \mathbf{h}_i^C$, and $\mathbf{h}_i^{\text{ext}} = \mathbf{h}_i^{V \setminus C}$. In particular, if $\vec{\mathbf{x}}$ is a $\mathcal{P}$-synchronised solution, the internal dynamical input to each node is zero ($\mathbf{h}_i^{\text{int}}(t) = 0$ all $t \in T$, all i) since the coupling functions are non-invasive, and the external contribution to two nodes in the same cluster is equal, that is,

$$\mathbf{h}_i^{\text{ext}} = \mathbf{h}_j^{\text{ext}} \tag{8}$$

whenever $i, j \in C$, $C \in \mathcal{P}$. This motivates the following definition of dynamical equitability. We call a solution $\vec{\mathbf{x}}$ *dynamically $\mathcal{P}$-equitable* if Eq. (8) is satisfied for all $i, j \in C$, for all $C \in \mathcal{P}$. Thus we have showed that dynamical equitability is a necessary condition for cluster synchronisation.

We would like to relate this notion to a more standard notion of structural equitability, that is, equitability with respect to the interaction pattern of the nodes in the underlying multilayer network. If $C \subseteq V$, we write

$$h_i^{C,m} = \sum_{k \in C} A_{ik}^{(m)}. \tag{9}$$

for the structural input to node i from all nodes in C on layer m. This is a constant that depends only on the partition $\mathcal{P}$ and the underlying interaction multilayer network, but not on the coupling functions $\mathbf{g}^{(m)}$ nor on a solution $\mathbf{x}$. We say that $\mathcal{P}$ is *structurally equitable* if, for all $C, C' \in \mathcal{P}$, $C \neq C'$, and $1 \leq m \leq M$,

$$h_i^{C',m} = h_j^{C',m} \tag{10}$$

for all $i, j \in C$. The condition $C \neq C'$ results in 'external' equitability, and is included since the internal connectivity in each cluster does not play a role in cluster synchronisation, as shown above. A partition $\mathcal{P}$ satisfying Eq. (10) is often called *external equitable partition* in the literature, and has been extensively related to cluster synchronisation [14, 15, 23].

It is easy to show that structural equitability implies dynamical equitability for any cluster synchronised solution (see SI). We will show that, under a natural cluster linear independence condition (next section), dynamical and structural equitability are in fact equivalent for cluster-synchronised solutions, and relate them more generally to cluster synchronisation.

In the SI, we study several definitions of dynamical and structural equitability and their relationships in more detail, and justify our choices above.

INDEPENDENT CLUSTER SYNCHRONISATION

We call a cluster *independent* if the coupling functions (across clusters, and layers) with all other clusters are linearly independent over trajectories. Formally, given $\mathcal{P}$ a partition and $\vec{\mathbf{x}}$ a $\mathcal{P}$-synchronised solution, we call the solution $\vec{\mathbf{x}}$ *$\mathcal{P}$-independent* if, for each $C \in \mathcal{P}$, the $M \cdot (|\mathcal{P}| - 1)$ functions

$$\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C'}),\ m = 1, \ldots, M,\ C' \in \mathcal{P},\ C' \neq C, \tag{11}$$

are linearly independent over the time domain T, where we write $\mathbf{x}_C(t)$ for the synchronised solution on C (i.e. $\mathbf{x}_C(t) = \mathbf{x}_i(t)$ for all $i \in C$), and similarly for $\mathbf{x}_{C'}$ (see the SI for full details). We call this *(linearly) independent cluster synchronisation*: cluster synchronisation (Eq. (5)) with linearly independent coupling functions over trajectories (Eq. (11)).

In some sense, independent cluster synchronisation is 'generic': clusters are independent unless they are synchronised ($\mathbf{x}_C = \mathbf{x}_{C'}$, which we can resolve by merging synchronised clusters together), or there is a non-trivial linear relation between them, as we illustrate next.

There are several ways in which a cluster may fail to be independent:

(1) there are *synchronised clusters*, that is, $\mathbf{x}_C = \mathbf{x}_{C'}$ for some $C \neq C'$, which immediately implies $\mathbf{g}^{(m)}(x_C, x_{C'}) = \mathbf{0}$ identically 0;

(2) the *coupling* between two clusters is *zero* despite not being (identically) synchronised, that is, $\mathbf{g}^{(m)}(x_C, x_{C'}) = \mathbf{0}$ even though $\mathbf{x}_C \neq \mathbf{x}_{C'}$;

(As an example, consider a sinusoidal Kuramoto coupling function

$$\mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_j) = \sin(\mathbf{x}_i - \mathbf{x}_j) \tag{12}$$

and a solution with $\mathbf{x}_C(t) = \mathbf{x}_{C'}(t) + k\pi$ for an integer k.)

(3) there are *linearly dependent layers*, for example $\mathbf{g}^{(1)} = \mathbf{g}^{(2)} + \mathbf{g}^{(3)}$ on some trajectory $(\mathbf{x}_C(t), \mathbf{x}_{C'}(t))$, even if $\mathbf{g}^{(1)} \neq \mathbf{g}^{(2)} + \mathbf{g}^{(3)}$ in general;

(For an example, consider the coupling functions

$$\begin{cases} \mathbf{g}^{(1)}(x_1, x_2) &= 4x_1 - 1, \\ \mathbf{g}^{(2)}(x_1, x_2) &= x_1 + x_2, \\ \mathbf{g}^{(3)}(x_1, x_2) &= x_1 + x_2 + 1, \end{cases} \tag{13}$$

with the trajectories given by

$$\begin{cases} x_C(t) = t, \\ x_{C'}(t) = t - 1, \end{cases} \tag{14}$$

then $\mathbf{g}^{(1)} \neq \mathbf{g}^{(2)} + \mathbf{g}^{(3)}$ in general but $\mathbf{g}^{(1)}(t) = \mathbf{g}^{(2)}(t) + \mathbf{g}^{(3)}(t)$ over the trajectory $(x_C(t), x_{C'}(t))$.)

(4) there are *linearly dependent clusters within a layer*, for example $\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}) = \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_2}) + \mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_3})$;

(As an example, consider the coupling function $\mathbf{g}^{(m)}(x_1, x_2) = x_1 + x_2$ and cluster solutions $\mathbf{x}_C(t) = t$, $\mathbf{x}_{C_1}(t) = 6t$, $\mathbf{x}_{C_2}(t) = 2t$, and $\mathbf{x}_{C_3}(t) = 3t$.)

and any combination of the above.

All but case (1), which forces synchronised nodes to be in the same cluster, depend on the specific form of the M coupling functions $\mathbf{g}^{(m)}$, and in particular on their algebraic properties, and thus we cannot say anything more in general. From now on, therefore, we will focus on independent cluster synchronisation.

INDEPENDENT CLUSTER SYNCHRONISATION IMPLIES EQUITABILITY

Our first result is that, under independent synchronisation, dynamical and structural equitability are equivalent. That is, if $\vec{\mathbf{x}}$ is a $\mathcal{P}$-synchronised (Eq. (5)), $\mathcal{P}$-independent (Eq. (11)) solution, then the following are equivalent:

(i) $\vec{\mathbf{x}}$ is dynamically $\mathcal{P}$-equitable (Eq. (8));

(ii) $\mathcal{P}$ is structurally equitable (Eq. (10)).

(See the SI for a full derivation.) Since we have shown that dynamically equitability is a necessary condition for (linearly independent) cluster synchronisation, so is structural equitability. This restricts the groups of nodes in a network that can support cluster synchronisation to those forming equitable partitions simultaneously on each layer.

Note that less restrictive notions of structural equitability are possible, for instance equitability with respect to all layers instead of per layer, or with respect to all external nodes instead of per cluster, but we can show that they are *not* necessary for cluster synchronisation (see SI for a full study on difference notions of equitability). Therefore, in the context of independent cluster synchronisation, a partition must be equitable *on each layer simultaneously*. This explains why cluster synchronisation is much harder to achieve in multi-layer networks: the equitability condition must be simultaneously satisfied on each layer, for the same nodes.

Let us illustrate this simultaneous structural equitability condition with the two multilayer networks shown in Fig. 1. These are biplex networks, that is, multiplex networks (same nodes on all layers) with two layers. The first biplex network $G_A = (G_A^{(1)}, G_A^{(2)})$ has a partition which is structurally (or externally) equitable in both layers simultaneously: the one with nodes $C = \{7, 8, 9, 10\}$ as one cluster and all other nodes as singletons. Later, we show that cluster synchronisation does occur on C, on a numerical example (Fig. 2). On the other hand, the second biplex network $G_B = (G_B^{(1)}, G_B^{(2)})$ violates the simultaneous equitable condition: even though there are almost identical external equitable partitions for every layer individually, $\{6, 7, 8, 9, 10, 11\}$ on layer 1 and $\{7, 8, 9, 10, 11\}$ on layer 2, there are no partitions in which they are equitable for both layers (node 6 is essential to achieve equitability on layer 1, but prevents equitability on layer 2). Again, we show later on a numerical example (Fig.2) that only explosive synchronisation is observed in this case. This example illustrates how much harder it is to achieve equitability (and thus cluster synchronisation) in the multilayer setting.

Altogether, we have shown that cluster synchronisation can only occur in one of these cases:

- there is a linear dependence relation between the coupling functions on trajectories (as illustrated in the previous section); or, if not,
- the partition of the underlying interaction network must be layer-by-layer equitable (Eq. (10)).

This summarises the key insight of our article: unless there are non-trivial linear dependencies between coupling functions (on trajectories), cluster synchronisation can only occur if the underlying partition is equitable.

In the next section, we show the converse statement, that is, that structural equitability (layer by layer) is not

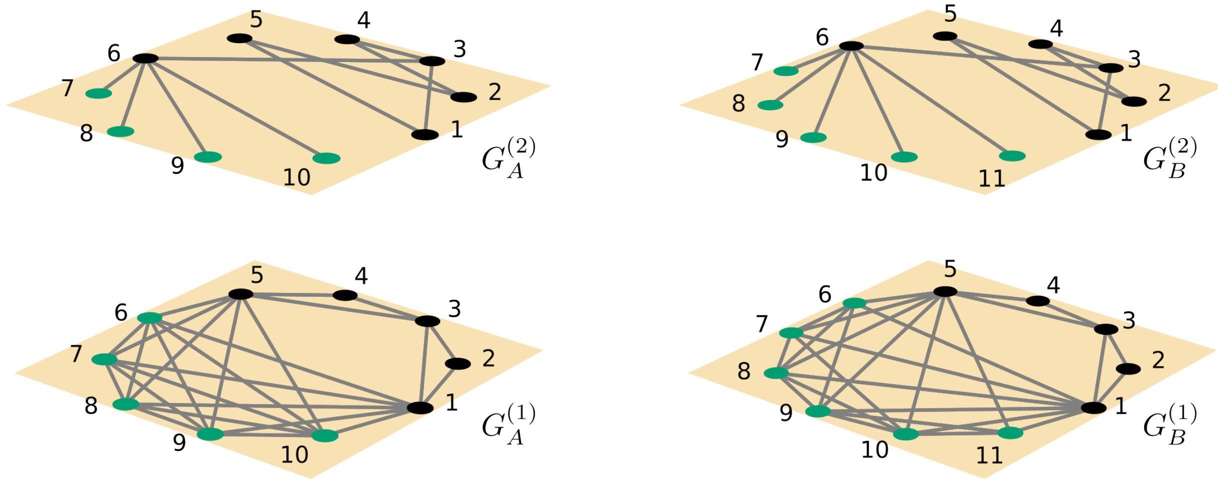

FIG. 1. Two biplex networks $G_A = (G_A^{(1)}, G_A^{(2)})$ on the left and $G_B = (G_B^{(1)}, G_B^{(2)})$ on the right that exhibit drastically different dynamical behaviour: cluster synchronization on G_A, and explosive synchronization on G_B (see Fig. 2 for the accompanying numerical results). This can be easily explained in terms of equitability: the green nodes form an equitable cluster in their respective layers, but only the common nodes $\{7, 8, 9, 10\}$ are equitable overall on G_A, with no similar equitable partition on G_B.

only necessary, but sufficient to guarantee the existence of a synchronised solution, with the help of the quotient network formalism.

Before moving on, we briefly remark on structural equitability and its relation to symmetry. Network symmetries (formally, permutation of the nodes preserving (weighted) edges) are an important source of equitable partitions, namely partitions into orbits of structurally equivalent nodes [17]. For multilayer (multiplex) networks, and also for hypergraphs, we can extend the definition to permutations of the nodes preserving edges on each layer, respectively hyperedges of the same order. As an example, the equitable partition in the biplex G_A in Fig. 1 is a symmetry partition into orbits.

QUOTIENT DYNAMICS

If A is an $N \times N$ real matrix, say the adjacency matrix of a weighted, directed network with N nodes, and $\mathcal{P} = \{C_1, \dots, C_K\}$ is a partition of the node set $\{1, \dots, N\}$ into K clusters, the *quotient* of A with respect to $\mathcal{P}$ is the $K \times K$ matrix B with (k, l)-entry

$$B_{k,l} = \frac{1}{|C_k|} \sum_{i \in C_k, j \in C_l} A_{i,j}, \tag{15}$$

the average connectivity from all nodes in C_l to a node in C_k.

Given the dynamical system (Eqs. (1) and (2)) and a partition $\mathcal{P}$ of the dynamical units (equivalently, of the set $\{1, \dots, N\}$) into K clusters, we define its *quotient dynamical system* with respect to $\mathcal{P}$ as the same dynamical system (same internal dynamics $\mathbf{f}$, coupling functions $\mathbf{g}^{(m)}$, and coupling strengths σ_m) on the quotient multilayer (multiplex) network, that is, the multiplex network with quotient adjacency matrices $B^{(m)}$, $m = 1, \dots, M$. Explicitly, we have dynamical units $\mathbf{y}_1, \dots, \mathbf{y}_K$ and K differential equations

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i) + \mathbf{h}_i(\mathbf{y}_1, \dots, \mathbf{y}_K), \tag{16}$$

where

$$\mathbf{h}_i(\mathbf{y}_1, \dots, \mathbf{y}_K) = \sum_{m=1}^{M} \sigma_m \sum_{j=1}^{K} B_{i,j}^{(m)} \, \mathbf{g}^{(m)}(\mathbf{y}_i, \mathbf{y}_j). \tag{17}$$

We refer to this dynamical system as the *quotient* dynamics, and to the original one as the *parent* dynamics.

In the SI, we show a correspondence between $\mathcal{P}$-synchronised $\mathcal{P}$-equitable solutions $\vec{\mathbf{x}}$ of the parent dynamical system, and (global) solutions $\vec{\mathbf{y}}$ of the quotient dynamical system, by setting $\mathbf{y}_k = \mathbf{x}_i$ for any $i \in C_k$. Formally, $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ is a $\mathcal{P}$-synchronised, $\mathcal{P}$-equitable solution of Eq. (1) if and only if $\vec{\mathbf{y}} = (\mathbf{y}_1, \dots, \mathbf{y}_M)$ is a solution of Eq. (16), where $\mathbf{y}_k = \mathbf{x}_i$ for any $i \in C_k$.

This means that we can construct equitable, cluster synchronised solutions on an arbitrary multilayer network with respect to an arbitrary partition, simply by 'lifting' (repeating on each node in a cluster) a solution of the quotient multilayer network. Again, we are not making any claims on the *stability* of the synchronised solutions. Indeed, even if the quotient solution is stable, the parent solution will be stable under uniform perturbations of all the nodes in the cluster, but not necessarily under arbitrary, non-uniform perturbations.

As an example, the biplex network G_A in Fig. 1 was constructed in this way: starting with an arbitrary (quotient) network on 7 nodes, substitute node 7 by an arbitrary graph (on four nodes 7, 8, 9 and 10 in this case) and connect the new nodes to the other nodes in an equitable way (each node 1 to 6 connects to either all, or none, of the nodes 7 to 10 in the cluster) to create the parent graph. (Of course, other, more complex, equitable connectivity patterns are possible.)

NUMERICAL RESULTS

We demonstrate the relation between equitability and cluster synchronisation with numerical simulations of coupled Lorenz oscillators on the two biplex network shown in Fig. 1 as we vary the coupling strength parameters σ_1 and σ_2. Full details of the simulations, including the dynamical functions involved, which belong to Class II [1] on each layer (global synchronisation guaranteed for high enough values of σ_i), and the definition of the synchronisation error, can be found in the SI.

The results of the simulations are shown in Fig. 2. Panels A1 to A3 (respectively B1 to B3) show the total synchronisation errors on the biplex network G_A (respectively G_B) as well as the cluster synchronisation error for the nodes $\{7, 8, 9, 10\}$, which form a layer-by-layer equitable cluster in G_A (but not in G_B). Panels A1, A2, B1 and B2 show that cluster synchronisation occurs on each layer separately (the cluster synchronisation error drops before the global one does) for both G_A and G_B. However, only G_A exhibits multilayer cluster synchronisation: panel A3 shows a significant region of coupling values (σ_1, σ_2) (red area) for which the cluster $\{7, 8, 9, 10\}$ synchronises while the network is not globally synchronised; this is however numerically negligible for G_B, as shown in panel B3, which suggests that the system undergoes a critical transition from no synchronisation to global synchronisation (explosive synchronisation).

The general results in our paper fully explain the numerically observed behaviour: the nodes $\{7, 8, 9, 10\}$ form a layer-by-layer equitable partition on G_A, but no such partition exists on G_B (Fig. 1), exemplifying the prevalence of explosive synchronisation on multilayer (and higher-order) networks compared to standard (monolayer) networks.

RESULTS FOR HYPERGRAPH DYNAMICS

Now we turn our attention to higher-order networks (hypergraphs), with dynamical interactions described by Eqs. (1) and (3), and show that all our previous results hold in this setting. In this section, we state the key definitions and main results, with full details deferred to the SI.

We first introduce the following notation, where $\mathcal{P}$ is a partition of the node set, $C, C_1, \ldots, C_m \in \mathcal{P}$, $i \in C$, and $1 \le m \le M$,

$$h_i^{C_1,\ldots,C_m} = \sum_{j_1 \in C_1} \cdots \sum_{j_1 \in C_m} A^{(m)}_{i,j_1,\ldots,j_m}, \tag{18}$$

$$\tag{19}$$

We also define corresponding dynamical quantities $\mathbf{h}_i^{C_1,\ldots,C_m}$, given a solution $\vec{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, by adding $\mathbf{g}^{(m)}(\mathbf{x}_i, \mathbf{x}_{j_1}, \ldots, \mathbf{x}_{j_m})$ to Eq. (18) above. These account for the total structural, respectively dynamical, input to node i from all nodes in $C_1, \ldots, C_m$.

Unlike the 2-body (pairwise) case, we allow in general the 'input' clusters $C_1, \ldots, C_m$ to be equal to C or to each other since interactions are now not only internal $(i, j_1, \ldots, j_m \in C)$ or external $(j_1, \ldots, j_m \notin C)$ but 'mixed' in general. Dynamical equitability is still independent of the purely internal connectivity (due to the non-invasiveness of the coupling functions $\mathbf{g}^{(m)}$) so we remove the case $C_1 = \ldots = C_m = C$ in the definitions of equitability, below.

We call a $\mathcal{P}$-synchronised solution $\vec{\mathbf{x}}$ *dynamically $\mathcal{P}$-equitable* if, for all $C \in \mathcal{P}$, $i, j \in C$, we have

$$\mathbf{h}_i^{\text{ext}} = \mathbf{h}_j^{\text{ext}}, \tag{20}$$

where $\mathbf{h}_i^{\text{ext}} = \mathbf{h}_i - \sum_{m=1}^{M} \sigma_m \, \mathbf{h}_i^{C,\overset{m}{\ldots},C}$ with $C,\overset{m}{\ldots},C$ representing C repeated m times. We call the partition $\mathcal{P}$ *structurally equitable* if, for all $C \in \mathcal{P}$, $i, j \in C$, we have

$$h_i^{C_1,\ldots,C_m} = h_j^{C_1,\ldots,C_m}, \tag{21}$$

for all $C_1, \ldots, C_m \in \mathcal{P}$ not all equal to C, and all $1 \le m \le M$.

Similarly, we extend the definition of independent cluster synchronisation to multi-body interactions. We call a $\mathcal{P}$-synchronised solution $\vec{\mathbf{x}}$ *$\mathcal{P}$-independent* if, for every $C \in \mathcal{P}$, the family of functions

$$\mathbf{g}^{(m)}(\mathbf{x}_C, \mathbf{x}_{C_1}, \ldots, \mathbf{x}_{C_m}) \tag{22}$$

where $1 \le m \le M$, and $C_1, \ldots, C_m \in \mathcal{P}$ not all equal to C, is linearly independent over the time domain T.

As in the multilayer case, we can show that cluster synchronisation implies dynamical equitability, and an equivalence between dynamical and structural equitability for independent cluster synchronisation (see the SI).

This shows why we expect explosive synchronisation to be prevalent in generic higher-order dynamics as well: for a partition to be structurally equitable (a necessary condition for independent cluster synchronisation), the equitability condition must be satisfied simultaneously for each subset of m clusters for all m (Eq. (21)).

To demonstrate our theoretical results numerically, we simulated Lorenz oscillators in a hypergraph with 20 nodes, four clusters, and both 2- and 3-body interactions (see the SI for a full description of the system).

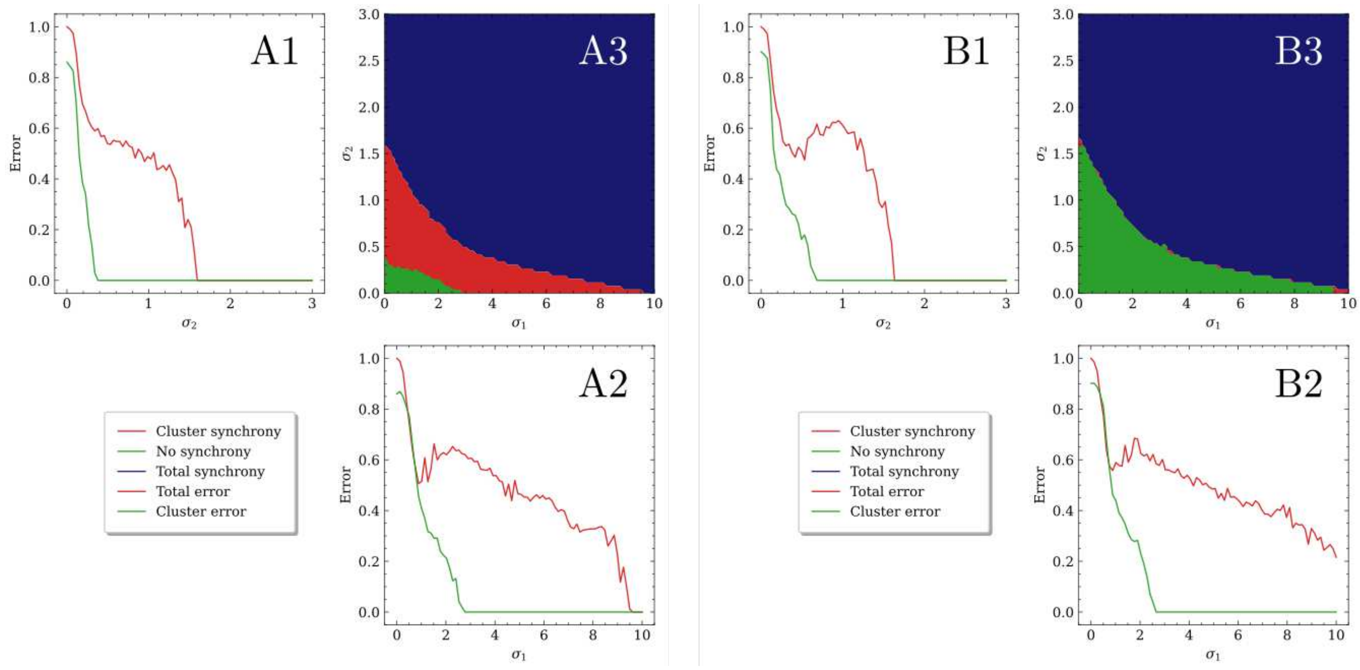

FIG. 2. Synchronization error of coupled Lorenz oscillators in the two biplex networks shown in Fig. 1 with respect to the cluster $\{7, 8, 9, 10\}$. Panels A1 and A2 (respectively B1 and B2) show the cluster synchronization in G_A (respectively G_B) when only layer 2, or layer 1, is active. Panels A3 and B3 show the 10^{-5} contours of the synchronization errors for $(\sigma_1, \sigma_2) \in [0, 10] \times [0, 10]$ for networks G_A and G_B, respectively. See the SI for full details.

The clusters were chosen such that C_1 is (externally) equitable with respect to pairwise but not triadic interactions; cluster C_2 is equitable for triadic but not pairwise interactions; cluster C_3 satisfy neither pairwise nor triadic equitability; and cluster C_4 is the only one that simultaneously satisfy pairwise and triadic equitability. The results are shown on Fig. 3, which shows the mean square error of the four clusters for a fixed and small value of the triadic interaction coupling ($\sigma_2 = 0.2$), and varying pairwise interaction coupling (σ_1). We can see that only cluster C_4 synchronises before the whole system does. We also note that cluster C_1 becomes almost synchronous due to triadic interactions becoming small with respect to the pairwise interactions for which C_1 does have structural equitability.

Finally, we can also prove the reciprocal statement, that is, that the structural equitability of a given partition implies the existence of cluster synchronised solutions, by 'lifting' (repeating on each cluster) a solution of the quotient dynamics to the parent hypergraph. For completeness, we give below the definitions of quotient dynamics for multi-body interactions, with full details included in the SI.

If A is a square tensor (that is, all indices have the same dimension N, for example an adjacency tensor for a hypergraph with N nodes) and $\mathcal{P}$ is a partition of the set $\{1, \ldots, N\}$ into K clusters, the *quotient tensor* of A with respect to $\mathcal{P}$ is the square tensor B defined by

$$B_{k_1,\ldots,k_m} = \frac{1}{|C_{k_1}|} \sum_{i_1 \in C_{k_1}, \ldots, i_m \in C_{k_m}} A_{i_1,\ldots,i_m}, \qquad (23)$$

the average connectivity of nodes in $C_{k_2}, \ldots, C_{k_m}$ to a node in C_{k_1}. The *quotient dynamics* of the dynamical system Eqs. (1), (3) with respect to a partition $\mathcal{P}$ is the system with dynamical units $\mathbf{y}_1, \ldots, \mathbf{y}_K$, where $K = |\mathcal{P}|$, and differential equations

$$\dot{\mathbf{y}}_i = \mathbf{f}(\mathbf{y}_i) + \mathbf{h}_i(\mathbf{y}_1, \ldots, \mathbf{y}_K), \qquad (24)$$

where

$$\mathbf{h}_i(\mathbf{y}_1, \ldots, \mathbf{y}_K) = \sum_{m=1}^{M} \sigma_m \sum_{j_1,\ldots,j_m=1}^{K} B^{(m)}_{i,j_1,\ldots,j_m} \mathbf{g}^{(m)}(\mathbf{y}_i, \mathbf{y}_{j_1}, \ldots, \mathbf{y}_{j_m}), \qquad (25)$$

and $B^{(m)}$ is the quotient tensor of $A^{(m)}$ with respect to $\mathcal{P}$, for each m.

CONCLUSIONS

In this article, we give a full characterisation of all possible cluster synchronised solutions in terms of cluster

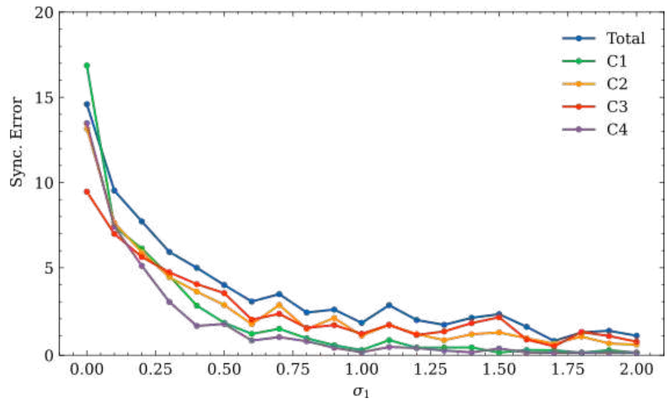

FIG. 3. Synchronization error of coupled Lorenz oscillators for four cluster on a hypergraph (fully described in the SI) with 20 nodes and both pairwise and triadic interactions. The clusters satisfy only pairwise respectively triadic structural equitability (clusters C_1 respectively C_2), none (cluster C_3) or simultaneously both (cluster C_4). The coupling parameter σ_2 is fixed at 0.2 while σ_1 is increased. Only cluster C_4 synchronises before the whole system does, as shown by the synchronisation error, which agrees with our prediction, as the only simultaneously equitable cluster for all types of interactions. Cluster C_1 becomes almost synchronous due to the triadic interactions becoming small with respect to the pairwise interactions for which C_1 does have structural equitability.

independence, and equitability, for two very general systems of coupled dynamics (multiplex, and hypergraph). Namely, we show that cluster synchronisation, with respect to a partition $\mathcal{P}$, can only occur in either of these two cases:

- there is a non-trivial linear dependence relation between the coupling functions on trajectories (see Eqs.(11) and (22); or, if not,
- $\mathcal{P}$ is an equitable partition of the underlying interaction multiplex or hypergraph (Eq. (10) respectively (21)).

The first case depends on the particular algebraic properties of the coupling functions, while the second one is a structural property of the underlying pattern of interactions, and thus independent of the particular form of the coupling function.

Moreover, in the cluster independent, or equitable, case, we show that the correct notion of structural equitability is simultaneous equitability per layer (multiplex case, Eq. (10)) or per many-body interaction (hypergraph case, Eq. (21)), explaining why cluster synchronisation is much harder to achieve in general in multi-layer and higher-order coupled dynamical systems. Our theoretical findings are illustrated with numerical simulations for both multiplex networks and hypergraphs.

Although we cannot make any general claims about the stability of the cluster synchronised solutions, which will depend on the specific form of the coupling functions, and values of the coupling strength parameters, our results provide strong restrictions about the groupings of nodes that can independently synchronise for any choice of coupling function, based solely on the equitability of the underlying interaction network, multiplex, or hypergraph.

Finally, using the quotient dynamics, we also show that equitable solutions exists for any pre-determined clustering structure (even though we cannot say anything in general about their stability), paving the way for control applications in clustering engineering and control on arbitrary systems.

Our work focuses on two general systems of dynamical interactions, each generalising standard pairwise (network) dynamics: multiplex (different 'types' or layers of edges or pairwise interactions) and hypergraph (multibody interactions). Of course, a even more general system would combine several layers of multi-body interactions — the interested reader can easily adapt the results in our paper to such systems as needed.

Our results formalise and clarify the relationship between cluster synchronisation and equitability, including a new concept of dynamical stability, on networks, and higher-order networks, but several important open questions remain. These include fast and exhaustive algorithms to find equitable partitions in arbitrary multiplexes and hypergraphs; the realisation and ordering problem, that is, which equitable partitions and in which order they synchronise as we increase the coupling strength parameters (see [25] for the network case); the stability question, that is, finding general conditions that guarantee the stability and a synchronised solution, for instance from a quotient to a parent solution; and an extension to non-identical dynamical units such as general multi-layer networks and to other synchronisation types beyond identical synchronisation.

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Author contribution. R.J.S.-G., S.B. and C.I.d.G. conceived the project. The theoretical framework was developed by R.J.S.-G. and K.K. The numerical simulations were done by G.C.-A., K.K. and C.I.d.G. All authors wrote and reviewed the final manuscript.

Data availability. All data needed to replicate the findings is synthetic and can be reconstructed from the description in the Supplementary Information.

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