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Gravitational waves from quasielliptic compact binaries in scalar-tensor theory to one-and-a-half post-Newtonian order

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Abstract

The orbit-averaged fluxes of energy and angular momentum generated by a compact binary system of nonspinning particles are obtained in a popular class of massless scalar-tensor theories with first-and-a-half post-Newtonian (1.5PN) accuracy, i.e. 2.5PN accuracy beyond the leading -1 PN dipolar radiation. The secular evolution of the orbital elements (frequency and eccentricity) are then also obtained with 1.5PN accuracy. Three technical advances were necessary to obtain these results: (i) the decomposition of the scalar dipole moment as a Fourier series at 1PN order, along with the other moments at Newtonian order, (ii) the derivation of the formula for the passage to the center-of-mass frame at 2.5PN, including a novel hereditary term arising from the contribution of the radiation, and (iii) the complete treatment of the memory-like term appearing in the angular momentum flux. Moreover, as a byproduct of this work, I revisit in the appendix the treatment *in general relativity* of the memory integral appearing at 2.5PN order in the angular momentum flux (Arun *et al* 2009 *Phys. Rev. D* **80** 124018), and find that the spurious divergence in the initial eccentricity e_1 found there is cured by a careful split between AC and DC terms.



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1. Introduction

General relativity (GR) still passes with flying colors all precision astrophysical tests so far, such as Solar System tests [1–5], binary pulsar tests [6–10] and gravitational wave constraints [11]. However, it is expected that GR is only the low-energy limit of some UV complete theory of quantum gravity, and deviations to GR would first manifest themselves as some extra degrees of freedom, such as a scalar field. This could be useful in explaining phenomena such as the accelerating Universe, inflation, the Hubble tension [12], etc.

Our understanding of GR itself has also dramatically improved. Focusing on gravitational waveforms, we now have post-Newtonian waveform templates at very high 4.5PN order [13, 14], numerical relativity templates are now full blown [15–17], gravitational self-force techniques have yielded generic waveform template at adiabatic order [18, 19] and waveforms for circular orbits at second post-adiabatic order [20], and methods such as effective-one-body (EOB) [21–23] or phenomenological models [24, 25] have been very successful at unifying these different waveforms.

In parallel, there has been a recent effort towards developing a complete template bank in alternative theories of gravity. To date, the most studied alternative theory is a class of scalar-tensor theories introduced by Damour and Esposito-Farèse [26] (see [27] for an equivalent formulation, which I will be using), which generalizes Brans–Dicke gravity [28, 29]. This theory has first been studied within the post-Newtonian framework, focusing on the equations of motion [27, 30, 31] and waveform templates for nonspinning circular orbits [32–35]. There has also been recent interest in studying the post-Newtonian dynamics within the EFT framework [36, 37], in which the spinning case [38] and tidal effects [39] were recently studied. These PN results have also been used to inform EOB models [40–44]. Finally, this class of theories has also recently been studied in the context of numerical relativity [45, 46], which found good agreement with post-Newtonian templates in the region where they overlap.

The present work aims at generalizing some of the results obtained for nonspinning *quasi-circular* orbits in [35], henceforth referred to as [paper I](#), to the case of nonspinning *quasielliptic* orbits. It crucially relies on the recent quasi-Keplerian (QK) parametrization at 2PN order for ST theories obtained in [47], henceforth referred to as [paper II](#). The main result I will be deriving is the orbit-averaged fluxes of energy and angular momentum at 1.5PN (i.e. 2.5PN beyond the leading –1PN dipolar radiation), along with the secular evolution of the orbital parameters (e.g. the frequency and eccentricity) at 2.5PN. The latter result generalizes the ‘chirp’ to the case of quasielliptic orbits, and is thus the main GW observable. To this end, three technical advances were needed. First, the expansion of scalar dipole moment as a Fourier series in time is computed for the first time with 1PN accuracy, along with the other required moments at Newtonian order. Secondly, the formula for the passage to the center-of-mass (CM) frame is obtained for the first time at 2.5PN order, including a novel hereditary term that arises from the contribution of the radiation to the definition of the CM frame, through the fluxes of linear momentum and CM position. This extra contribution was discovered only very recently in the context of GR [48]. Thirdly, the tail sector of the flux is computed for the first time in terms of enhancement functions, whose small-eccentricity expansions are also presented. Finally, the memory piece of the angular momentum flux is treated consistently, where the AC part

is computed using the Fourier expansion of the scalar dipole moment, while the DC part is treated thanks to the secular evolution equations of the orbital parameters derived in [47].

Despite the stringent constraints on this class of theories, stemming mostly from the constraints on the existence of dipolar radiation [49], the motivation for this work is two-fold. First, future ground-based detectors such as Einstein Telescope [50] or Cosmic Explorer [51] will have drastically higher sensitivities than current ones, and will be able to place much more stringent constraints on deviations from GR. In this context, it is interesting to have alternative template banks for popular theories that cover the entire parameter space (masses, spins, eccentricity, tidal effects, ...), e.g. in the event that we detect an alternative signal with very small coupling constants. Moreover, it is not excluded that a subpopulation of signals exhibit GW signals that differ greatly from GR, and have thus been missed by matched filtering [52] — alternative templates would be very useful in this regard. The second motivation is that this class of theories is simple enough to explore the phenomenology of extra degrees of freedom at high post-Newtonian order, and serve as a testbed for more complicated theories. For example, there has been recent interest for scalar-Gauss-Bonnet theories, which build on results in ST theories to obtain their waveforms [42, 46, 53–55]. Moreover, more general theories, e.g. in the Horndeski class, can exhibit screening mechanisms which cannot yet be correctly treating within the post-Newtonian framework [56, 57]. Thus, I view this program as an incremental step towards the ultimate goal of obtaining waveforms for some robust future extension of GR.

The plan of the paper is as follows. After notational reminders, I will introduce in section 2 the ST theory under scrutiny, and review the QK parametrization at 2PN for ST theories in section 3. In section 4, I will go over the crucial step of decomposing the multipole moments into Fourier series. In section 5, I will discuss the passage to the CM frame, with the appearance of non-local terms first discovered in [48]. The fluxes of energy and angular momentum for eccentric orbits will be derived in section 6 at 1.5PN order beyond quadrupolar radiation (i.e. 2.5PN beyond the leading -1 PN dipolar radiation), and will be used in section 7 to obtain the secular evolution of the orbital elements through 2.5PN order. After a short conclusion in section 8, many lengthy coefficients are relegated in the Appendices. Most results are also given in machine-readable form in the supplemental material [58]. Finally, I complete *in GR* the memory piece of the angular momentum flux in appendix A, curing the spurious divergence in the initial eccentricity appearing in the previous treatment of [59].

1.1. Main notations and summary of parameters

The convention in this work is that all stated PN orders are, by default, relative to the the Newtonian dynamics and to the standard quadrupole radiation in GR. Thus, the dominant dipole radiation enters at -0.5 PN order in the waveform and at -1 PN order in the energy flux. The QK parametrization at second post-Newtonian order computed in [paper II](#) is therefore next-to-next-to-leading-order, while the Newtonian fluxes and waveforms are next-to-leading order.

After a $3+1$ decomposition of spacetime, and using the convention that boldface letters represent three-dimensional Euclidean vectors, the field-point spatial vector in radiative coordinates is denoted by $\mathbf{X} = R\mathbf{N}$, where $\mathbf{N}$ has unit norm. In spherical coordinates, the coordinates of the field point are denoted by (R, Θ, Φ) . Time is denoted by t , and retarded time is defined as $t_{\text{ret}} = t - R/c$, so as to not confuse it with the eccentric anomaly of the QK motion, u .

The positions of particles 1 and 2 are denoted by $\mathbf{y}_1$ and $\mathbf{y}_2$, and $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ is the separation vector. The orbital radius is given by $r = |\mathbf{x}|$, and the unit vector $\mathbf{n} = \mathbf{x}/r$ is introduced.

The relative velocity is given by $\mathbf{v} = d\mathbf{x}/dt$. One can then introduce the rotating orthonormal triad $(\mathbf{n}, \boldsymbol{\lambda}, \boldsymbol{\ell})$ and the nonrotating reference triad $(\mathbf{n}_0, \boldsymbol{\lambda}_0, \boldsymbol{\ell}_0)$, which are defined rigorously in section 4. For nonspinning binaries, there is no precession, so $\boldsymbol{\ell} = \boldsymbol{\ell}_0$ and one can describe the orbital motion with polar coordinates (r, ϕ) in the orbital plane¹ such that $\mathbf{n} = \cos \phi \mathbf{n}_0 + \sin \phi \boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda} = -\sin(\phi) \mathbf{n}_0 + \cos(\phi) \boldsymbol{\lambda}_0$. Notice that the relative velocity is given by $\mathbf{v} = \dot{r} \mathbf{n} + r \dot{\phi} \boldsymbol{\lambda}$. The following relations then hold: $\mathbf{n} \cdot \mathbf{v} = \dot{r}$, $v^2 = \dot{r}^2 + r^2 \dot{\phi}^2$, and $\mathbf{n} \times \mathbf{v} = r \dot{\phi} \boldsymbol{\ell}$.

The notation $L = i_1 \cdots i_\ell$ stands for multi-index with ℓ spatial indices (and K would stand for a multi-index with k spatial indices). One can then write $\partial_L = \partial_{i_1} \cdots \partial_{i_\ell}$, $\partial_{aL-1} = \partial_a \partial_{i_1} \cdots \partial_{i_{\ell-1}}$ and so on; similarly, $n_L = n_{i_1} \cdots n_{i_\ell}$, $n_{aL-1} = n_a n_{i_1} \cdots n_{i_{\ell-1}}$. The symmetric trace-free (STF) part is indicated using a hat or angled brackets: for instance, $\text{STF}_L[\partial_L] = \hat{\partial}_L = \partial_{\langle i_1} \partial_{i_2} \cdots \partial_{i_\ell \rangle}$, $\text{STF}_L[n_L] = \hat{n}_L = n_{\langle i_1} n_{i_2} \cdots n_{i_\ell \rangle}$, and $\text{STF}_L[x_L] = \hat{x}_L = r^\ell n_{\langle i_1} n_{i_2} \cdots n_{i_\ell \rangle}$. The n th time derivative of a function $F(t)$ is denoted $F^{(n)}(t) = d^n F / dt^n$.

The constant asymptotic value of the scalar field at spatial infinity is denoted ϕ_0 , and the normalized scalar field is defined by $\varphi \equiv \phi / \phi_0$. The Brans–Dicke-like scalar function $\omega(\phi)$ is expanded around the asymptotic value $\omega_0 \equiv \omega(\phi_0)$ and the mass functions $m_A(\phi)$ (see section 2 for a definition) are expanded around the asymptotic values $m_A \equiv m_A(\phi_0)$, where $A \in \{1, 2\}$. In the CM frame, one then defines the asymptotic total mass $m = m_1 + m_2$, reduced mass $\mu = m_1 m_2 / m$, symmetric mass ratio $\nu = \mu / m \in]0, 1/4]$, and relative mass difference $\delta = (m_1 - m_2) / m \in]1, 1[$. Note that the asymptotic symmetric mass ratio and the relative mass difference are linked by the relation $\delta^2 = 1 - 4\nu$.

Following [30, 35], a number of parameters describing these expansions are introduced in table 1. The ST parameters are defined directly from the expansions of the Brans–Dicke-like scalar function $\omega(\phi)$ and of the mass functions $m_A(\phi)$. The PN parameters are combinations of the ST parameters that naturally extend and generalize the usual PPN parameters to the case of a general ST theory [60, 61]. Moreover, these parameters can be mapped back to those of the formulation of Damour and Esposito–Far  se [26], see e.g. table I of [62] for the complete correspondence.

2. Massless scalar-tensor theories

As in paper I, I will consider a generic class of ST theories in which a single massless scalar field ϕ minimally couples to the metric $g_{\mu\nu}$, described by the action:

$$S_{\text{ST}} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right] + S_{\text{m}}(\mathbf{m}, g_{\alpha\beta}), \quad (2.1)$$

where R and g are, respectively, the Ricci scalar and the determinant of the metric, ω is a function of the scalar field and $\mathbf{m}$ stands generically for the matter fields. Since these theories break the strong equivalence principle, I will follow the standard prescription by Eardley [63] (see also [64]), which consists in modeling the inertial masses of each particle as a function of the scalar field, namely $m_A(\phi)$, where $A \in \{1, 2\}$ for a binary system. The effective matter action for point particles thus reads

$$S_{\text{m}} = -c \sum_A \int m_A(\phi) \sqrt{-(g_{\alpha\beta})_A} dy_A^\alpha dy_A^\beta, \quad (2.2)$$

¹ The notation ϕ is used both for the scalar field and the polar angle, but its meaning will always be clear in context. Moreover, r will be used at one occasion to denote the radial coordinate of the field point in harmonic coordinates, at this will be stated clearly.

Table 1. Summary of parameters for the general ST theory and notations for PN parameters.

ST parameters	
general	$\omega_0 = \omega(\phi_0), \quad \omega'_0 = \left. \frac{d\omega}{d\phi} \right _{\phi=\phi_0}, \quad \omega''_0 = \left. \frac{d^2\omega}{d\phi^2} \right _{\phi=\phi_0}, \quad \varphi = \frac{\phi}{\phi_0}, \quad \tilde{g}_{\mu\nu} = \varphi g_{\mu\nu},$ $\tilde{G} = \frac{G(4+2\omega_0)}{\phi_0(3+2\omega_0)}, \quad \zeta = \frac{1}{4+2\omega_0},$ $\lambda_1 = \frac{\zeta^2}{(1-\zeta)} \left. \frac{d\omega}{d\varphi} \right _{\varphi=1}, \quad \lambda_2 = \frac{\zeta^3}{(1-\zeta)} \left. \frac{d^2\omega}{d\varphi^2} \right _{\varphi=1}, \quad \lambda_3 = \frac{\zeta^4}{(1-\zeta)} \left. \frac{d^3\omega}{d\varphi^3} \right _{\varphi=1}.$
sensitivities	$s_A = \left. \frac{d \ln m_A(\phi)}{d \ln \phi} \right _{\phi=\phi_0}, \quad s_A^{(k)} = \left. \frac{d^{k+1} \ln m_A(\phi)}{d(\ln \phi)^{k+1}} \right _{\phi=\phi_0}, \quad (A = 1, 2)$ $s'_A = s_A^{(1)}, \quad s''_A = s_A^{(2)}, \quad s'''_A = s_A^{(3)},$ $\mathcal{S}_+ = \frac{1-s_1-s_2}{\sqrt{\alpha}}, \quad \mathcal{S}_- = \frac{s_2-s_1}{\sqrt{\alpha}}.$
PN parameters	
N	$\alpha = 1 - \zeta + \zeta(1 - 2s_1)(1 - 2s_2)$
1PN	$\bar{\gamma} = -\frac{2\zeta}{\alpha}(1 - 2s_1)(1 - 2s_2),$ $\bar{\beta}_1 = \frac{\zeta}{\alpha^2}(1 - 2s_2)^2(\lambda_1(1 - 2s_1) + 2\zeta s'_1),$ $\bar{\beta}_2 = \frac{\zeta}{\alpha^2}(1 - 2s_1)^2(\lambda_1(1 - 2s_2) + 2\zeta s'_2),$ $\bar{\beta}_+ = \frac{\bar{\beta}_1 + \bar{\beta}_2}{2}, \quad \bar{\beta}_- = \frac{\bar{\beta}_1 - \bar{\beta}_2}{2}.$ Degeneracy $\alpha(2 + \bar{\gamma}) = 2(1 - \zeta)$
2PN	$\bar{\delta}_1 = \frac{\zeta(1-\zeta)}{\alpha^2}(1 - 2s_1)^2, \quad \bar{\delta}_2 = \frac{\zeta(1-\zeta)}{\alpha^2}(1 - 2s_2)^2,$ $\bar{\delta}_+ = \frac{\bar{\delta}_1 + \bar{\delta}_2}{2}, \quad \bar{\delta}_- = \frac{\bar{\delta}_1 - \bar{\delta}_2}{2},$ $\bar{\chi}_1 = \frac{\zeta}{\alpha^3}(1 - 2s_2)^3[(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1)(1 - 2s_1) - 6\zeta\lambda_1 s'_1 + 2\zeta^2 s'^2_1],$ $\bar{\chi}_2 = \frac{\zeta}{\alpha^3}(1 - 2s_1)^3[(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1)(1 - 2s_2) - 6\zeta\lambda_1 s'_2 + 2\zeta^2 s'^2_2],$ $\bar{\chi}_+ = \frac{\bar{\chi}_1 + \bar{\chi}_2}{2}, \quad \bar{\chi}_- = \frac{\bar{\chi}_1 - \bar{\chi}_2}{2}.$ Degeneracy $16\bar{\delta}_1\bar{\delta}_2 = \bar{\gamma}^2(2 + \bar{\gamma})^2$

where y_A^α denotes the space-time positions of the particles, and $(g_{\alpha\beta})_A$ is the metric evaluated² at the position of particle A . Despite minimal coupling, the point particle effective model introduces a dependence in the scalar field. The action of equation (2.1) is usually called the Jordan-frame action, as the matter only couples to the Jordan or ‘physical’ metric $g_{\alpha\beta}$. It is then very useful to introduce a rescaled scalar field and a conformally related metric,

$$\varphi \equiv \frac{\phi}{\phi_0} \quad \text{and} \quad \tilde{g}_{\alpha\beta} \equiv \varphi g_{\alpha\beta}, \quad (2.3)$$

where ϕ_0 is the constant cosmological background value of the scalar field, such that the physical and conformal metrics have the same asymptotic behavior at spatial infinity. Quantities expressed in terms of the pair $(\varphi, \tilde{g}_{\alpha\beta})$ are said to be in the ‘Einstein frame.’ From these, one defines the scalar and metric perturbation variables $\psi \equiv \varphi - 1$ and $h^{\mu\nu} \equiv \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu} - \eta^{\mu\nu}$,

² Divergences are treated with Hadamard regularization [65], which is equivalent, at this order, to dimensional regularization [66, 67].

where $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Within this perturbative setup, the arbitrary Brans-Dicke coupling function will be Taylor-expanded around the cosmological background value ϕ_0 , namely $\omega(\phi) = \omega_0 + (\phi - \phi_0)\omega'_0 + \mathcal{O}((\phi - \phi_0)^2)$. Moreover, the mass functions $m_A(\phi)$ will be described perturbatively in terms of their sensitivities, defined as

$$s_A^{(k)} \equiv \frac{d^{k+1} \ln m_A(\phi)}{d(\ln \phi)^{k+1}} \Big|_{\phi=\phi_0}, \quad (2.4)$$

where $s_A \equiv s_A^{(1)}$. In these theories, no hair theorems [68, 69] impose that isolated black holes carry no scalar charge. This strongly suggests that black holes in a binary system also have vanishing scalar charge, and this is partially confirmed by numerical relativity computations [70]. Since the scalar charge α_A of the particle A is related to its sensitivity by the relation [34]

$$\alpha_A = \frac{1 - 2s_A}{\sqrt{3 + 2\omega_0}}, \quad (2.5)$$

this means that black holes in a binary system have sensitivity $s_A = 1/2$. In all post-Newtonian computations, it was indeed found that the results of GR are recovered by setting $s_1 = s_2 = 1/2$.

With all these tools in hand, the field equations in the Landau-Lifschitz formulation [71] can now be rewritten [27]

$$\square_\eta h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}, \quad (2.6a)$$

$$\square_\eta \psi = -\frac{8\pi G}{c^4} \tau_s, \quad (2.6b)$$

where $\square_\eta$ denotes the ordinary flat space-time d'Alembertian operator, and where the source terms read

$$\tau^{\mu\nu} = \frac{\varphi}{\phi_0} (-g) T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu} [h, \psi], \quad (2.7a)$$

$$\tau_s = -\frac{\varphi}{\phi_0 (3 + 2\omega(\phi))} \sqrt{-g} \left(T - 2\varphi \frac{\partial T}{\partial \varphi} \right) - \frac{c^4}{8\pi G} \Lambda_s [h, \psi]. \quad (2.7b)$$

Here $T^{\mu\nu} = 2(-g)^{-1/2} \delta S_m / \delta g_{\mu\nu}$ is the matter stress-energy tensor, $T \equiv g_{\mu\nu} T^{\mu\nu}$ is its trace and $\partial T / \partial \varphi$ is defined as the partial derivative of $T(g_{\mu\nu}, \varphi)$ with respect to φ holding $g_{\mu\nu}$ constant. Moreover, $\Lambda^{\mu\nu}$ and Λ_s are given explicitly in (2.8) and (2.9) of [paper I](#) as functionals of the $h^{\mu\nu}$ and ψ that are at least quadratic in the fields.

The field equations of (2.6) are solved using the PN-MPM construction [72, 73]. The reader is invited to refer to [paper I](#) for a detailed description of this construction in the case of ST theories. The main point is that the fields in vacuum zone exterior to the source are written as a formal expansion in the bookkeeping parameter G , namely $h^{\mu\nu} = Gh_1^{\mu\nu} + G^2 h_2^{\mu\nu} + \mathcal{O}(G^3)$ and $\psi = G\psi_1 + G^2 \psi_2 + \mathcal{O}(G^3)$. The linearized piece is parametrized by the so-called source moments I_L , J_L and I_L^s , along with some gauge moments W_L , X_L , Y_L , and Z_L , which are pure functions of retarded time and are STF in their indices. These moments uniquely parametrize the linearized metric $h_1^{\mu\nu}$ and the linearized scalar field ψ_1 . For example, the linearized scalar field reads explicitly

$$\psi_1 = -\frac{2}{c^2} \sum_{\ell=0}^{+\infty} \frac{(-)^\ell}{\ell!} \partial_L \left[\frac{I_L^s(t-r/c)}{r} \right], \quad (2.8)$$

where in this particular case, r denotes the radial coordinate of the field point in harmonic coordinates (rather than the radial separation of the binary). For a given source, these moments can be computed explicitly as functions of the orbital variables, see e.g. appendix (B.1) of [paper I](#). A feature of ST theory is that the scalar monopole Γ^s is not constant but its time-variation will be a small PN effect, i.e. $d\Gamma^s/dt = \mathcal{O}(c^{-2})$. This is why the leading-order radiation is dipolar, and not monopolar. It is thus practical to define

$$\Gamma^s = \frac{1}{\phi_0} \left[m^s + \frac{E^s}{c^2} \right], \quad (2.9a)$$

where

$$m^s = -\frac{1}{3 + 2\omega_0} \sum_A m_A (1 - 2s_A) \quad (2.9b)$$

is constant and E^s is the time-varying, PN correction.

The parametrized linearized piece is reinjected iteratively into the field equations (2.6), where $T^{\mu\nu} = 0$, since only the vacuum zone is considered here. This yields field equations for $h_n^{\mu\nu}$ and ψ_n (for $n \geq 2$) which can be iteratively solved at quadratic [74, 75], cubic [76–78] and quartic order [79]. Once this MPM metric is constructed at desired order, one can extract its leading order expression in the large radii expansion (this is the only thing an asymptotic observer will see). This asymptotic metric, written in terms of some radiative coordinates $(t, R, \mathcal{N})$ [or alternatively (t, R, Θ, Φ)] which describe the position of the field point, admits a multipolar decomposition as well, which is uniquely parametrized, after TT projection, by three families of *radiative* moments [35, 73], denoted $\mathcal{U}_L$, $\mathcal{V}_L$ and $\mathcal{U}_L^s$. Introducing $t_{\text{ret}} = t - R/c$, namely retarded time in the radiative coordinate system [77, 80], one find that that the asymptotic metric reads in a transverse-traceless gauge

$$h_{ij}^{\text{TT}} = -\frac{4G}{c^2 R} \perp_{ijab}^{\text{TT}} \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!} \left[N_{L-2} \mathcal{U}_{abL-2}(t_{\text{ret}}) - \frac{2\ell}{c(\ell+1)} N_{cL-2} \epsilon_{cd(a} \mathcal{V}_{b)dL-2}(t_{\text{ret}}) \right] + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (2.10a)$$

$$\psi = -\frac{2G}{c^2 R} \sum_{\ell=0}^{+\infty} \frac{1}{c^\ell \ell!} N_L \mathcal{U}_L^s(t_{\text{ret}}) + \mathcal{O}\left(\frac{1}{R^2}\right), \quad (2.10b)$$

where $\perp_{ijab}^{\text{TT}} = \perp_{a(i} \perp_{j)b} - \frac{1}{2} \perp_{ij} \perp_{ab}$ with $\perp_{ij} = \delta_{ij} - N_i N_j$. The radiative moments are generically related to the source moments by the relations

$$\mathcal{U}_L = \overset{(\ell)}{\mathbf{I}}_L + \mathcal{O}\left(\frac{1}{c^3}\right), \quad \mathcal{V}_L = \overset{(\ell)}{\mathbf{J}}_L + \mathcal{O}\left(\frac{1}{c^3}\right), \quad \mathcal{U}_L^s = \overset{(\ell)}{\mathbf{I}}_L^s + \mathcal{O}\left(\frac{1}{c^3}\right), \quad (2.11)$$

where the small $\mathcal{O}(1/c^3)$ corrections correspond to the nonlinear corrections of the MPM construction. The full expressions needed for the 1.5PN fluxes were obtained in [paper I](#), and read

$$\begin{aligned} \mathcal{U}_{ij} = & \mathbf{I}_{ij}^{(2)} + \frac{2GM}{\phi_0 c^3} \int_0^\infty d\tau \mathbf{I}_{ij}^{(4)}(t_{\text{ret}} - \tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12} \right] \\ & + \frac{G(3+2\omega_0)}{3c^3} \left(-\mathbf{I}_{\langle i}^{(3)} \mathbf{I}_{j\rangle}^{(3)} - \mathbf{I}_{\langle i}^{(1)} \mathbf{I}_{j\rangle}^{(2)} - \frac{1}{2} \mathbf{I}^{(3)} \mathbf{I}_{ij}^{(3)} + \int_0^\infty d\tau \left[\mathbf{I}_{\langle i}^{(2)} \mathbf{I}_{j\rangle}^{(2)} \right] (t_{\text{ret}} - \tau) \right) + \mathcal{O}\left(\frac{1}{c^5}\right). \end{aligned} \quad (2.12a)$$

$$\begin{aligned} \mathcal{U}^s = & \mathbf{I}^s + \frac{2GM}{\phi_0^2 c^5} \int_0^\infty d\tau E^s(t_{\text{ret}} - \tau) \ln\left(\frac{c\tau}{2b_0}\right) + \frac{G}{c^5} \left(1 - \frac{\phi_0 \omega_0'}{3+2\omega_0} \right) \\ & \times \left[\frac{2}{9} \mathbf{I}_k^{(3)} \mathbf{I}_k^s - 2 \mathbf{I}^s \frac{E^s}{\phi_0} \right] + \mathcal{O}\left(\frac{1}{c^7}\right), \end{aligned} \quad (2.12b)$$

$$\begin{aligned} \mathcal{U}_i^s = & \mathbf{I}_i^s + \frac{2GM}{\phi_0 c^3} \int_0^\infty d\tau \mathbf{I}_i^{(3)}(\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1 \right] + \frac{G}{c^5} \left[-\frac{1}{5} \mathbf{I}_k^{(1)} \mathbf{I}_{ik}^{(3)} - \frac{1}{5} \mathbf{I}_k^{(2)} \mathbf{I}_{ik}^{(2)} \right. \\ & + \frac{3}{5} \mathbf{I}_k^{(3)} \mathbf{I}_{ik}^{(1)} + \frac{3}{5} \mathbf{I}_k^{(4)} \mathbf{I}_{ik}^{(0)} - \epsilon_{iab} J_a \mathbf{I}_b^{(3)} - 4 \mathbf{W} \mathbf{I}_i^{(1)(2)} - 4 \mathbf{W} \mathbf{I}_i^{(2)(1)} + 4 \mathbf{I}^s \mathbf{Y}_i \\ & \left. + \left(1 - \frac{\phi_0 \omega_0'}{3+2\omega_0} \right) \left(-2 \frac{E^s}{\phi_0} \mathbf{I}_i^{(1)} - 2 \frac{E^s}{\phi_0} \mathbf{I}_i^{(2)} + \frac{2}{5} \mathbf{I}_k^{(3)} \mathbf{I}_{ik}^{(1)} + \frac{2}{5} \mathbf{I}_k^{(4)} \mathbf{I}_{ik}^{(0)} \right) \right] + \mathcal{O}\left(\frac{1}{c^7}\right), \end{aligned} \quad (2.12c)$$

$$\mathcal{U}_{ij}^s = \mathbf{I}_{ij}^{(2)} + \frac{2GM}{\phi_0 c^3} \int_0^\infty d\tau \mathbf{I}_{ij}^{(4)}(t_{\text{ret}} - \tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{3}{2} \right] - \frac{G}{c^3} \mathbf{I}^s \mathbf{I}_{ij}^{(3)} + \mathcal{O}\left(\frac{1}{c^5}\right), \quad (2.12d)$$

where the instantaneous moments are implicitly evaluated at t_{ret} . Note the presence of hereditary tails (associated as usual to the arbitrary length scale b_0) as well and memory integrals in the latter expressions.

3. The QK parametrization

In previous works [32, 33, 35, 62], the fluxes of energy and angular momentum were computed in terms of the orbital variables of the binary in the CM frame, namely the relative position $\mathbf{x} = \mathbf{r}$ and the relative velocity $\mathbf{v}$. However, in order to provide a ready-to-use post-Newtonian waveform template, one still has to solve equations of motions such as to obtain a parametrization of the orbital motion, which is then explicitly injected into these fluxes. This parametrization is very easy if one assumes that the orbit is bound and quasicircular [35, 81], which is relevant astrophysically, since compact binaries are expected to have circularized by the time they enter the frequency band of ground based GW detectors. However, other mechanisms, such as von Zeipel–Lidov–Kozai resonances [82] or dynamical captures, can lead to non-negligible eccentricities even within the detector's frequency band, and there are hints that eccentric binaries have in fact already been detected [83]. For an quasielliptic orbit, one of the standard prescriptions in GR is the QK parametrization [84], which have been fully extended up to 3PN order [85–87], and partially up to 4PN order [88] (see also [89] for an equivalent osculating-orbit prescription). In [paper II](#), I have extended the QK parametrization

to the case of scalar-tensor theories at 2PN order, which is exactly what will be needed for this work. At 2PN order, the parametrization reads

$$r = a_r (1 - e_r \cos(u)) , \quad (3.1a)$$

$$\ell = n(t - t_0) = u - e_t \sin(u) + f_t \sin(v) + g_t(v - u) , \quad (3.1b)$$

$$\phi - \phi_0 = K[v + f_\phi \sin(2v) + g_\phi \sin(3v)] , \quad (3.1c)$$

where t is time, (r, ϕ) are the polar coordinates in the orbital plane, u is an affine parameter called the *eccentric anomaly*, and the *true anomaly* is defined as

$$\begin{aligned} v &= 2 \arctan \left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\left(\frac{u}{2}\right) \right] + 2\pi \left\lfloor \frac{u+\pi}{2\pi} \right\rfloor \\ &= u + 2 \arctan \left[\frac{\beta(e_\phi) \sin(u)}{1 - \beta(e_\phi) \cos(u)} \right] , \end{aligned} \quad (3.2)$$

where

$$\beta(e) \equiv \frac{e}{1 + \sqrt{1 - e^2}} . \quad (3.3)$$

The QK parameters are the mean motion n , the rate of pericenter advance K , the semi major axis a_r , the time eccentricity e_t , the radial eccentricity e_r , the angular eccentricity e_ϕ , the rate of pericenter advance K , and the higher-order parameters f_ϕ, f_t, g_ϕ , and g_t . The motion is characterized by two generically incommensurable periods, namely the radial period P (i.e. the time between two pericenter passages), which is related to the mean motion by $n = 2\pi/P$; and the angular period (i.e. the time needed for ϕ to go from 0 to 2π), related to the angular frequency $\omega = Kn$. The mean anomaly is then defined as $\ell = n(t - t_0)$, and the advance in the pericenter is typically decomposed as $K = 1 + k$, where k is a small PN quantity. Moreover, the three eccentricities e_r, e_ϕ and e_t differ by small PN corrections, and are thus equal in the Newtonian limit. It will also be extremely useful to introduce the small frequency-related PN parameter

$$x = \left(\frac{Gm\omega}{c^3} \right)^{2/3} = \left(\frac{GmnK}{c^3} \right)^{2/3} . \quad (3.4)$$

Crucially, these various parameters are explicitly expressed in terms of a set of two parameters that entirely characterize the orbits, e.g. the dimensionless energy and angular momentum (ε, j) , which are defined from the mechanical energy and angular momentum (E, J) as

$$\varepsilon = -\frac{2E}{\mu c^2} \quad \text{and} \quad j = -\frac{2J^2 E}{\mu^3 (\tilde{G}\alpha m)^2} . \quad (3.5)$$

This is the main result of [47], and the interested reader is encouraged to refer to it for more details. Another useful pair of gauge invariant variables is (x, ι) where $\iota \equiv x\mathcal{B}/k$ and

$$\mathcal{B} = 3 + 2\bar{\gamma} - \bar{\beta}_+ + \bar{\beta}_- \delta . \quad (3.6)$$

The variable ι is defined such that $\iota \rightarrow j$ when $x \rightarrow 0$. However, the limiting case of circular orbit limit is not immediate to obtain with this choice. Thus, from now on, I will work with the

pair of variables (x, e_t) , which has the good property of yielding the results for circular orbits by taking the limit $e_t \rightarrow 0$, but has the unfortunate property of not being gauge invariant.

Moreover, for the purpose of the following derivations, it will be very practical to define

$$w = v(u) - u = 2 \arctan \left[\frac{\beta(e_\phi) \sin(u)}{1 - \beta(e_\phi) \cos(u)} \right], \quad (3.7)$$

$$\tilde{w} = 2 \arctan \left[\frac{\beta(e_t) \sin(u)}{1 - \beta(e_t) \cos(u)} \right], \quad (3.8)$$

where β is defined in equation (3.3). We then denote $\tilde{v} = u + \tilde{w}$, such that w and $\tilde{w}$ represent the periodically oscillating pieces of v and $\tilde{v}$. Finally, I will often need to apply orbit averaging. For any time-depending quantity f , I define its orbit average over a radial period as

$$\langle f \rangle = \frac{1}{P} \int_0^P dt f(t) = \frac{1}{2\pi} \int_0^{2\pi} d\ell f(\ell) = \frac{1}{2\pi} \int_0^{2\pi} du \frac{d\ell}{du} f(u), \quad (3.9)$$

where the 2PN-accurate expression for $d\ell/du$ can be derived from the results presented in [paper II](#), and reads

$$\begin{aligned} \frac{d\ell}{du} = & 1 - e_t \cos u + \frac{x^2}{8} \left\{ \frac{(1 - e_t^2)(15 + 8\bar{\gamma} - \nu)\nu}{[1 - e_t \cos u]^2} + \frac{1}{1 - e_t \cos u} [60 + 60\bar{\gamma} + 15\bar{\gamma}^2 \right. \\ & - 4\bar{\delta}_+ - 4\bar{\delta}_- \delta + \nu(-39 - 24\bar{\gamma} + 8\bar{\beta}_+ - 8\bar{\beta}_- \delta) + \nu^2] \\ & \left. - \frac{1}{\sqrt{1 - e_t^2}} [60 + 60\bar{\gamma} + 15\bar{\gamma}^2 - 4\bar{\delta}_+ - 4\bar{\delta}_- \delta + \nu(-24 - 16\bar{\gamma} + 8\bar{\beta}_+ - 8\bar{\beta}_- \delta)] \right\} \\ & + \mathcal{O}(x^3). \end{aligned} \quad (3.10)$$

4. The Fourier decomposition of the multipolar moments

One of the challenges when computing high-order PN waveforms is the treatment of hereditary integrals, such as the tails or the memory. These integrals span over the whole history of the binary, and require a model for its past evolution. Moreover, these integrals can be easily computed analytically in the case of quasicircular orbits, but the treatment of elliptic orbits is more complicated. The standard strategy is to use the periodicity (or double-periodicity) of the multipolar moments in order to express their time-evolution as a Fourier series. In GR, the Fourier coefficients required for the 3PN flux were computed in [90]. These results were then improved by Loutrel and Yunes [91], and I will follow the latter reference, adapting it to the case of ST theories.

4.1. Moments at Newtonian order

At Newtonian order, the three eccentricities (e_t, e_r, e_ϕ) are interchangeable and denoted e . Moments at Newtonian order are periodic in $\ell = n(t - t_0)$, so they can be decomposed as a Fourier series as

$$\mathbf{K}_L(t) = \sum_{p \in \mathbb{Z}} {}_p \tilde{\mathbf{K}}_L e^{ip\ell}, \quad (4.1)$$

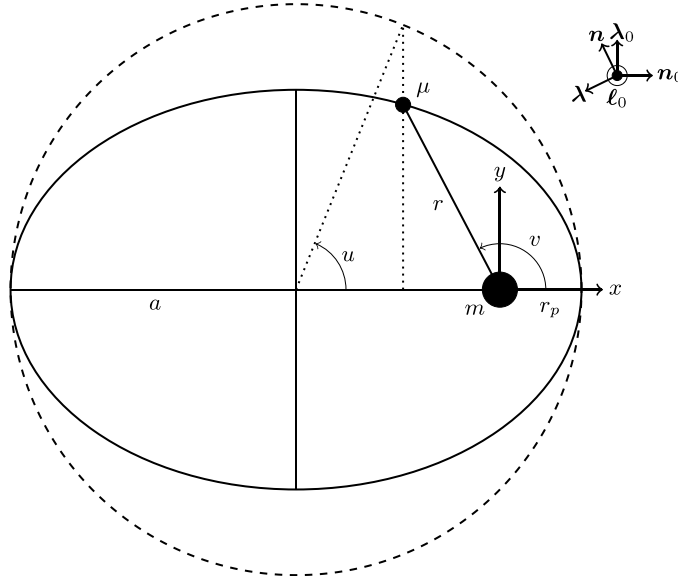


Figure 1. Elliptical motion.

where K_L stands either for I_L , J_L , I_L^s , or E^s . The Fourier coefficients are given by

$${}_p\tilde{K}_L = \frac{1}{2\pi} \int_0^{2\pi} d\ell K_L(t) e^{-ip\ell}. \quad (4.2)$$

In practice, the multipolar moments, which are known in terms of r , $\dot{r}$, ϕ , and $\dot{\phi}$ (see (4.6) and (4.7) of [paper I](#)), are first expressed in terms of the eccentric anomaly u using the Keplerian representation, and projected onto the time-independent orthonormal triad $(\mathbf{n}_0, \boldsymbol{\lambda}_0, \boldsymbol{\ell}_0)$, which corresponds, respectively, to the components (x, y, z) . As illustrated in figure 1: $\mathbf{n}_0$ is parallel to the major axis and points from the center of the ellipse towards the periastron; $\boldsymbol{\lambda}_0$ is parallel to the minor axis, and oriented such that $v^i \lambda_0^i > 0$ at each periastron passage; $\boldsymbol{\ell}_0$ is orthogonal to the orbital plane, and oriented such that $(\mathbf{n}_0, \boldsymbol{\lambda}_0, \boldsymbol{\ell}_0)$ is a positively-oriented orthonormal basis. I will also use the corotating orthonormal basis $(\mathbf{n}, \boldsymbol{\lambda}, \boldsymbol{\ell})$ defined by $\mathbf{n} = \cos(\phi)\mathbf{n}_0 + \sin(\phi)\boldsymbol{\lambda}_0$, $\boldsymbol{\lambda} = -\sin(\phi)\mathbf{n}_0 + \cos(\phi)\boldsymbol{\lambda}_0$ and $\boldsymbol{\ell} = \boldsymbol{\ell}_0$. Since the Kepler equation reads at Newtonian order $\ell = 1 - e \cos u$, after a change of variables, the only integrals we will have to evaluate can in fact be reduced to Bessel functions, namely [\[91\]](#)

$$J_p(x) = \frac{1}{2\pi} \int_0^{2\pi} du e^{i(pu - x \sin u)}. \quad (4.3)$$

Moreover, following Loutrel and Yunes [\[91\]](#), the expressions of the Fourier coefficients are simplified by exploiting identities between the Bessel functions and their derivatives (see their equations (46)–(48)). Thus, the following rules were recursively applied, valid for $x \neq 0$:

$$J_{p+n}(x) = \frac{2(p+n-1)}{x} J_{p+n-1}(x) - J_{p+n-2}(x) \quad \text{if } n \geq 2, \quad (4.4a)$$

$$J_{p-n}(x) = \frac{2(p-n+1)}{x} J_{p-n+1}(x) - J_{p-n+2}(x) \quad \text{if } n \leq 1, \quad (4.4b)$$

$$J_{p+1}(x) = \frac{p}{x} J_p(x) - J'_p(x). \quad (4.4c)$$

To simplify the presentation, the Fourier coefficients are normalized as

$${}_p\tilde{K}_L = \mathcal{K}_\ell \times {}_p\hat{K}_L, \quad (4.5)$$

where the generic normalization $\mathcal{K}_\ell$ is explicitly given for each moment by

$$\mathcal{E}_0^s = \frac{8\sqrt{\alpha}\zeta m\nu c^2 x}{3(1-\zeta)} \left[\mathcal{S}_+ - \frac{1}{4}\mathcal{S}_-\delta - 3\bar{\gamma}^{-1}(\mathcal{S}_+\bar{\beta}_+ + \mathcal{S}_-\bar{\beta}_-) \right], \quad (4.6a)$$

$$\mathcal{I}_1^s = -\frac{2\mathcal{S}_-\zeta\tilde{G}\alpha^{3/2}m^2\nu}{c^2(1-\zeta)\phi_0 x}, \quad (4.6b)$$

$$\mathcal{I}_2^s = -\frac{(\mathcal{S}_+ - \mathcal{S}_-\delta)\zeta\tilde{G}^2\alpha^{5/2}m^3\nu}{c^4(1-\zeta)\phi_0 x^2}, \quad (4.6c)$$

$$\mathcal{I}_2 = \frac{\tilde{G}^2\alpha^2 m^3\nu}{c^4\phi_0 x^2}. \quad (4.6d)$$

Omitting the terms that vanish or that can immediately be deduced from the fact that the moments are symmetric in their indices, I find, for $p \neq 0$ (the $p = 0$ terms will never be needed):

$${}_p\hat{E}^s = J_p(ep), \quad (4.7a)$$

$${}_p\hat{I}_x^s = \frac{1}{p}J'_p(ep), \quad (4.7b)$$

$${}_p\hat{I}_y^s = -\frac{i\sqrt{1-e^2}}{ep}J_p(ep), \quad (4.7c)$$

$${}_p\hat{I}_{xx}^s = {}_p\hat{I}_{xx} = \frac{2}{ep} \left[(1-e^2)J'_p(ep) + \frac{(e^2-3)}{3ep}J_p(ep) \right], \quad (4.7d)$$

$${}_p\hat{I}_{xy}^s = {}_p\hat{I}_{xy} = \frac{2i\sqrt{1-e^2}}{ep} \left[\frac{1}{p}J'_p(ep) - \frac{1-e^2}{e}J_p(ep) \right], \quad (4.7e)$$

$${}_p\hat{I}_{yy}^s = {}_p\hat{I}_{yy} = -\frac{2}{ep} \left[(1-e^2)J'_p(ep) + \frac{(2e^2-3)}{3ep}J_p(ep) \right], \quad (4.7f)$$

$${}_p\hat{I}_{zz}^s = {}_p\hat{I}_{zz} = \frac{2}{3p^2}J_p(ep), \quad (4.7g)$$

where it is immediate to verify that ${}_p\hat{I}_{xx} + {}_p\hat{I}_{yy} + {}_p\hat{I}_{zz} = 0$ and ${}_p\hat{I}_{xx}^s + {}_p\hat{I}_{yy}^s + {}_p\hat{I}_{zz}^s = 0$.

4.2. The scalar dipole moment at 1PN

The Fourier decomposition of I_i^s will also be required at 1PN order. For this, the decomposition will be slightly different from the Newtonian case, because the relativistic precession of the periastris requires that the doubly periodic structure of the motion be correctly account for.

Since the ellipse is precessing, the $(\mathbf{n}_0, \boldsymbol{\lambda}_0, \ell_0)$ triad is now defined based on the osculating ellipse at current time t , and will remain constant throughout the past motion. Thus, at times $t' < t$, the major and minor axes are not necessarily aligned with $(\mathbf{n}_0, \boldsymbol{\lambda}_0)$.

Following Arun *et al* [90], I first express I_i^s in terms of r and ϕ , and use the decomposition

$$\phi = K\ell + W(\ell) = k\ell + \ell + W(\ell), \quad (4.8)$$

where, at this order, $W(\ell) = K(v - \ell)$ is a 2π -periodic function of ℓ . The explicit expression of I_i^s is solely composed of terms proportional to $e^{\pm i\phi}$, so it is immediate to factor out the dependence in $k\ell$, namely

$$I_i^s = \sum_{m \in \{-1, 1\}} \tilde{m} \tilde{I}_i^s e^{imk\ell} = \mathcal{I}_1 \sum_{m \in \{-1, 1\}} \widehat{m} \widehat{I}_i^s e^{imk\ell}, \quad (4.9)$$

where the normalization $\mathcal{I}_1$ is the same as for the Newtonian case. The residual $\widehat{m} \widehat{I}_i^s$ terms can then be written as a sum of terms proportional to $e^{\pm i[\ell + W(\ell)]}$, which are all 2π -periodic in ℓ because $W(\ell)$ is. Thus, it can be decomposed into a Fourier series. The final expansion reflects the double periodicity of the motion, and reads

$$I_i^s = \mathcal{I}_1 \sum_{m \in \{-1, 1\}} \sum_{p \in \mathbb{Z}} \widehat{m} \widehat{I}_i^s e^{i(mk+p)\ell}, \quad (4.10)$$

where

$$\widehat{m} \widehat{I}_i^s = \frac{1}{2\pi} \int_0^{2\pi} d\ell \, \tilde{m} \tilde{I}_i^s e^{-ip\ell} = \frac{1}{2\pi} \int_0^{2\pi} du \, \frac{d\ell}{du} \tilde{m} \tilde{I}_i^s e^{-ip(u - e_t \sin u)}, \quad (4.11)$$

and where we recall from (3.10) that $d\ell/du = 1 - e_t \cos u + \mathcal{O}(c^{-4})$. In order to explicitly compute $\widehat{m} \widehat{I}_i^s$, recall that $\tilde{m} \tilde{I}_i^s(t)$ only contains terms proportional to $e^{\pm i[\ell + W(\ell)]}$, and use the 1PN-accurate relation

$$\ell + W(\ell) = v + k(w + e_t \sin u) + \mathcal{O}(c^{-4}). \quad (4.12)$$

The exponential is PN-expanded to find

$$e^{\pm i[\ell + W(\ell)]} = e^{\pm iv} (1 \pm ik[\tilde{w} + e_t \sin u] + \mathcal{O}(c^{-4})), \quad (4.13)$$

where I have replaced at this order w by $\tilde{w}$, and where k can be replaced by its leading order expression in terms of x and e_t . The exponential term is easily computed using trigonometric functions, namely

$$e^{\pm iv} = \frac{\cos u - e_\phi \pm i\sqrt{1 - e_\phi^2} \sin u}{1 - e_\phi \cos u}, \quad (4.14)$$

where e_ϕ should then be replaced in terms of its 1PN expression in terms of x and e_t . After all replacements are made, I obtain a 1PN accurate expression in terms only of x , e_t and u . From

the structure of this expression, it appears that the Fourier coefficients can only be computed by introducing, on top of the Bessel functions, the real-valued³ special function

$$\Theta_p^q(e_t) \equiv \frac{i}{2\pi} \int_0^{2\pi} du e^{-i[(p+q)u - pe_t \sin u]} \tilde{w}(u, e_t), \quad (4.15)$$

which is not entirely novel since it appears in a different form in (68) of [91]. Finally, following [90, 91], it is useful for presentation purposes to expand the Fourier coefficients in the variables x and ν , so as to present coefficients that depend only on the eccentricities. The decomposition is denoted:

$${}_p\hat{\Gamma}_i^s = {}_p\hat{\Gamma}_i^{s,00} + x \left[{}_p\hat{\Gamma}_i^{s,01} + \nu {}_p\hat{\Gamma}_i^{s,11} \right] + \mathcal{O}(x^2). \quad (4.16)$$

First, I found that $\forall p \in \mathbb{Z}$, ${}^{\pm 1}\hat{\Gamma}_z^s = 0$, and that the following relation holds

$${}^{\pm 1}\hat{\Gamma}_y^s = (\mp i) {}^{\pm 1}\hat{\Gamma}_x^s. \quad (4.17)$$

Thus, recalling the definition (3.6) for the recurrent parameter $\mathcal{B}$, all the information is contained in the following expressions for the x -component:

$${}^{\pm 1}\hat{\Gamma}_x^{s,00} = \pm \frac{\sqrt{1-e_t^2}}{2pe_t} J_p(pe_t) + \frac{1}{2p} J'_p(pe_t), \quad (4.18a)$$

$${}^{\pm 1}\hat{\Gamma}_x^{s,11} = \pm \frac{14 + e_t^2}{30pe_t\sqrt{1-e_t^2}} J_p(pe_t) + \frac{7}{15p} J'_p(pe_t), \quad (4.18b)$$

$$\begin{aligned} {}^{\pm 1}\hat{\Gamma}_x^{s,01} = J_p(pe_t) & \left[\mp \frac{i\mathcal{B}}{pe_t} + (1 - e_t^2) \frac{\bar{\beta}_+ \mathcal{S}_- + \bar{\beta}_- \mathcal{S}_+ - \delta(\bar{\beta}_+ \mathcal{S}_+ + \bar{\beta}_- \mathcal{S}_-) - \frac{1}{4}\bar{\gamma}(\mathcal{S}_- - 2\mathcal{S}_+ \delta)}{e_t \bar{\gamma} \mathcal{S}_-} \right. \\ & + \frac{i\mathcal{B}}{p^2 e_t \sqrt{1-e_t^2}} \mp \frac{\sqrt{1-e_t^2} \mathcal{S}_+ \delta}{10p \mathcal{S}_- e_t} \mp \frac{39 + 10\bar{\gamma} + 20(\bar{\beta}_+ - \bar{\beta}_- \delta) + e_t^2(21 + 10\bar{\gamma})}{60pe_t \sqrt{1-e_t^2}} \Big] \\ & + J'_p(pe_t) \left[\pm \frac{i\mathcal{B}}{p^2(1-e_t^2)} - \frac{\mathcal{S}_+ \delta}{10p \mathcal{S}_-} - \frac{39 + 10\bar{\gamma} + 20(\bar{\beta}_+ - \bar{\beta}_- \delta) - e_t^2(99 + 50\bar{\gamma})}{60p(1-e_t^2)} \right. \\ & - \frac{i\mathcal{B}}{p(1-e_t)^2} \pm \sqrt{1-e_t^2} \frac{\bar{\beta}_+ \mathcal{S}_- + \bar{\beta}_- \mathcal{S}_+ - \delta(\bar{\beta}_+ \mathcal{S}_+ + \bar{\beta}_- \mathcal{S}_-) - \frac{1}{4}\bar{\gamma}(\mathcal{S}_- - 2\mathcal{S}_+ \delta)}{\bar{\gamma} \mathcal{S}_-} \Big] \\ & \mp \frac{\mathcal{B}}{8(1-e_t^2)} \left[e_t(1 \mp \sqrt{1-e_t^2}) \Theta_p^{+2}(e_t) + e_t(1 \pm \sqrt{1-e_t^2}) \Theta_p^{-2}(e_t) \right. \\ & \left. - 2(1 + e_t^2 \mp \sqrt{1-e_t^2}) \Theta_p^{+1}(e_t) - 2(1 + e_t^2 \pm \sqrt{1-e_t^2}) \Theta_p^{-1}(e_t) + 6e_t \Theta_p^0(e_t) \right]. \end{aligned} \quad (4.18c)$$

It is easy to verify that,

$${}_p\hat{\Gamma}_x^s \equiv e^{+ik\ell} \left({}_p^{+1}\hat{\Gamma}_x^s \right) + e^{-ik\ell} \left({}_p^{-1}\hat{\Gamma}_x^s \right) \xrightarrow{c \rightarrow +\infty} {}_p^{+1}\hat{\Gamma}_x^{s,00} + {}_p^{-1}\hat{\Gamma}_x^{s,00} = \frac{1}{p} J'_p(pe_t),$$

³ By a $u' = -u$ variable change, it is clear that $\Theta_{-p}^{-q}(e_t) = -\Theta_p^q(e_t)$, by oddness of arctan. This also means that $\Theta_p^q(e_t) = \Theta_p^q(e_t)^*$, so $\Theta_p^q(e_t) \in \mathbb{R}$.

which is indeed the expression I found in the Newtonian case—the same holds for the y -component. Finally, the following identities, valid for any $(p, q) \in \mathbb{Z}^2$, will prove very useful⁴ in section 6:

$$\sum_{\substack{m \in \{-1, 1\} \\ s \in \{-1, 1\}}} (m + s) \begin{pmatrix} m \hat{\Gamma}_i^s \\ p \hat{\Gamma}_i \end{pmatrix} \begin{pmatrix} m \hat{\Gamma}_i^s \\ q \hat{\Gamma}_i \end{pmatrix} = 0, \quad (4.19a)$$

$$\sum_{\substack{m \in \{-1, 1\} \\ s \in \{-1, 1\}}} (m + s) \epsilon_{ijk} \begin{pmatrix} m \hat{\Gamma}_j^s \\ p \hat{\Gamma}_j \end{pmatrix} \begin{pmatrix} m \hat{\Gamma}_k^s \\ q \hat{\Gamma}_k \end{pmatrix} = 0. \quad (4.19b)$$

4.3. Special functions

The Fourier coefficients computed previously feature some special functions. The most common ones are the Bessel functions, which are well known, but the new special function $\Theta_p^q(e_t)$ deserves some attention. For $p + q \neq 0$, it behaves as $\Theta_p^q(e_t) = \mathcal{O}(e_t^{|p+q|})$ when $e_t \rightarrow 0$, whereas $\Theta_p^{-p}(e_t) = \mathcal{O}(e_t^2)$. It is therefore useful to define the renormalized function

$$\tilde{\Theta}_p^q(e_t) = \begin{cases} e_t^{-2} \Theta_p^q(e_t) & \text{if } p + q = 0 \\ e_t^{-|p+q|} \Theta_p^q(e_t) & \text{otherwise} \end{cases}, \quad (4.20)$$

such that $\tilde{\Theta}_p^q(0)$ is always finite and nonzero (except for $p = q = 0$, since $\Theta_0^0(e_t)$ is identically zero). These special functions only appear in the unnormalized enhancement functions $\rho_1^{s,01}(e_t)$ and $\tilde{\rho}_1^{s,01}(e_t)$ (see equations (6.23)), which means that they only appear in the normalized enhancement functions $\alpha_1^s(e_t)$ and $\tilde{\alpha}_1^s(e_t)$ (see equations (6.29)). In figure 2, I have plotted the exact values of all the special functions which are needed to evaluate these enhancement functions, along with the case $p = 0$, which was never needed but is nonetheless instructive.

In view of obtaining fully analytical results, it is also useful to control the small eccentricity expansion of these special functions. First, recall that the Bessel function and its derivative can be expanded for $p \neq 0$ as follows:

$$\mathcal{J}_p(pe_t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n + |p|)!} \left(\frac{|p|e_t}{2} \right)^{2n+|p|}, \quad (4.21)$$

$$\mathcal{J}'_p(pe_t) = \text{sgn}(p) \sum_{n=0}^{\infty} \frac{(-1)^n (2n + |p|)}{2 (n!) (n + |p|)!} \left(\frac{|p|e_t}{2} \right)^{2n+|p|-1}, \quad (4.22)$$

where I have introduced the sign function $\text{sgn}(p) = 1$ if $p > 0$, $\text{sgn}(p) = -1$ if $p < 0$.

The Taylor expansion of $\Theta_p^q(e_t)$ as $e_t \rightarrow 0$ can be easily obtained at any order for given values of p and q (e.g. with *Mathematica*), but its general term is very cumbersome. Here,

⁴ The proof goes as follows. Performing an explicit summation over the contracted indices, I find that $\forall (m, s) \in \{-1, 1\}^2$,

$$\begin{aligned} (m + s) \begin{pmatrix} m \hat{\Gamma}_i^s \\ p \hat{\Gamma}_i \end{pmatrix} \begin{pmatrix} m \hat{\Gamma}_i^s \\ q \hat{\Gamma}_i \end{pmatrix} &= (m + s) (1 - ms) \begin{pmatrix} m \hat{\Gamma}_x^s \\ p \hat{\Gamma}_x \end{pmatrix} \begin{pmatrix} m \hat{\Gamma}_x^s \\ q \hat{\Gamma}_x \end{pmatrix}, \\ (m + s) \epsilon_{ijk} \begin{pmatrix} m \hat{\Gamma}_j^s \\ p \hat{\Gamma}_j \end{pmatrix} \begin{pmatrix} m \hat{\Gamma}_k^s \\ q \hat{\Gamma}_k \end{pmatrix} &= i (m + s) (m - s) \epsilon_{ixy} \begin{pmatrix} m \hat{\Gamma}_x^s \\ p \hat{\Gamma}_x \end{pmatrix} \begin{pmatrix} m \hat{\Gamma}_y^s \\ q \hat{\Gamma}_y \end{pmatrix}, \end{aligned}$$

where I have used (4.17) and the fact that the z component of the dipole vanishes. But the right hand sides of the previous equations vanish identically for any allowed values of m and s , which concludes the proof.

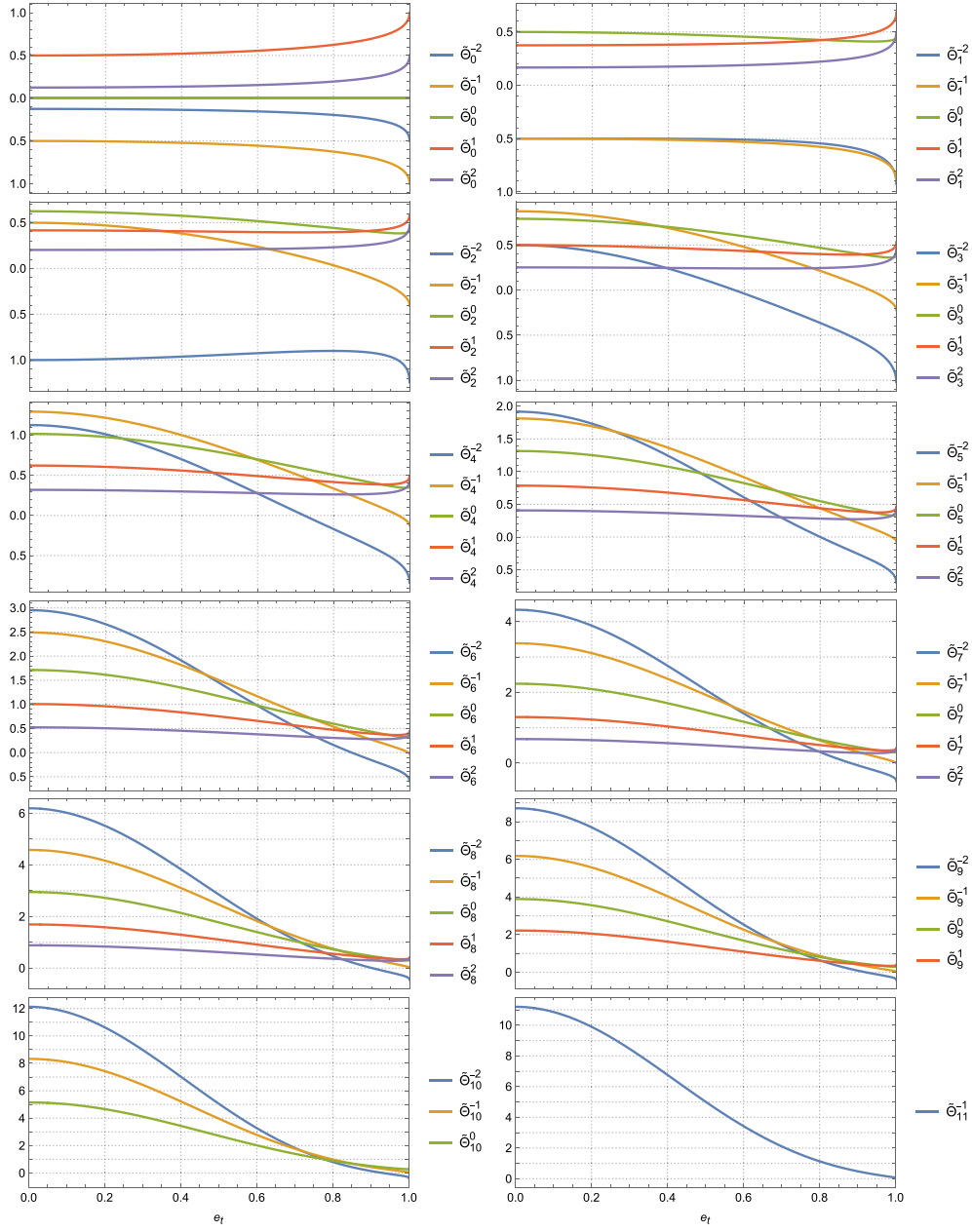


Figure 2. Numerical plots of the $\tilde{\Theta}_p^q(e_t)$ required for this computation, as well as the special case $p = 0$.

I will only present the results which are required to obtain the expansions of the normalized enhancement functions $\alpha_1^s(e_t)$ and $\tilde{\alpha}_1^s(e_t)$ at tenth order in the eccentricity (see equations (6.30) and appendix C). The corresponding expansions are:

$$\Theta_1^{-2}(e_t) = -\frac{1}{2}e_t - \frac{23}{384}e_t^5 - \frac{181}{4608}e_t^7 - \frac{43093}{1474560}e_t^9 + \mathcal{O}(e_t^{11}) \quad (4.23a)$$

$$\Theta_1^{-1}(e_t) = -\frac{1}{2}e_t^2 - \frac{1}{16}e_t^4 - \frac{59}{1152}e_t^6 - \frac{203}{6144}e_t^8 - \frac{58183}{2457600}e_t^{10} + \mathcal{O}(e_t^{12}) \quad (4.23b)$$

$$\Theta_1^0(e_t) = \frac{1}{2}e_t - \frac{1}{8}e_t^3 + \frac{1}{128}e_t^5 + \frac{41}{9216}e_t^7 + \frac{2143}{491520}e_t^9 + \mathcal{O}(e_t^{11}) \quad (4.23c)$$

$$\Theta_1^1(e_t) = \frac{3}{8}e_t^2 + \frac{1}{32}e_t^4 + \frac{113}{3072}e_t^6 + \frac{4589}{184320}e_t^8 + \mathcal{O}(e_t^{10}) \quad (4.23d)$$

$$\Theta_1^2(e_t) = \frac{1}{6}e_t^3 + \frac{35}{768}e_t^5 + \frac{25}{278}e_t^7 + \mathcal{O}(e_t^9) \quad (4.23e)$$

$$\Theta_2^{-2}(e_t) = -e_t^2 + \frac{1}{4}e_t^4 - \frac{7}{72}e_t^6 - \frac{7}{192}e_t^8 + \mathcal{O}(e_t^{10}) \quad (4.23f)$$

$$\Theta_2^{-1}(e_t) = \frac{1}{2}e_t - \frac{3}{4}e_t^3 + \frac{1}{12}e_t^5 - \frac{53}{1152}e_t^7 - \frac{25}{1152}e_t^9 + \mathcal{O}(e_t^{11}) \quad (4.23g)$$

$$\Theta_2^0(e_t) = \frac{5}{8}e_t^2 - \frac{5}{16}e_t^4 + \frac{1}{24}e_t^6 - \frac{13}{11520}e_t^8 + \mathcal{O}(e_t^{10}) \quad (4.23h)$$

$$\Theta_2^1(e_t) = \frac{5}{12}e_t^3 - \frac{13}{192}e_t^5 + \frac{67}{1920}e_t^7 + \mathcal{O}(e_t^9) \quad (4.23i)$$

$$\Theta_2^2(e_t) = \frac{13}{64}e_t^4 + \frac{1}{96}e_t^6 + \mathcal{O}(e_t^8) \quad (4.23j)$$

$$\Theta_3^{-2}(e_t) = \frac{1}{2}e_t - \frac{7}{4}e_t^3 + \frac{101}{128}e_t^5 - \frac{59}{256}e_t^7 + \mathcal{O}(e_t^9) \quad (4.23k)$$

$$\Theta_3^{-1}(e_t) = \frac{7}{8}e_t^2 - \frac{39}{32}e_t^4 + \frac{367}{1024}e_t^6 - \frac{1867}{20480}e_t^8 + \mathcal{O}(e_t^{10}) \quad (4.23l)$$

$$\Theta_3^0(e_t) = \frac{19}{24}e_t^3 - \frac{151}{256}e_t^5 + \frac{741}{5120}e_t^7 + \mathcal{O}(e_t^9) \quad (4.23m)$$

$$\Theta_3^1(e_t) = \frac{1}{2}e_t^4 - \frac{53}{256}e_t^6 + \mathcal{O}(e_t^8) \quad (4.23n)$$

$$\Theta_3^2(e_t) = \frac{323}{1280}e_t^5 + \mathcal{O}(e_t^7) \quad (4.23o)$$

$$\Theta_4^{-2}(e_t) = \frac{9}{8}e_t^2 - \frac{47}{16}e_t^4 + \frac{691}{384}e_t^6 + \mathcal{O}(e_t^8) \quad (4.23p)$$

$$\Theta_4^{-1}(e_t) = \frac{31}{24}e_t^3 - \frac{47}{24}e_t^5 + \frac{1687}{1920}e_t^7 + \mathcal{O}(e_t^9) \quad (4.23q)$$

$$\Theta_4^0(e_t) = \frac{65}{64}e_t^4 - \frac{193}{192}e_t^6 + \mathcal{O}(e_t^8) \quad (4.23r)$$

$$\Theta_4^1(e_t) = \frac{149}{240}e_t^5 + \mathcal{O}(e_t^7) \quad (4.23s)$$

$$\Theta_5^{-2}(e_t) = \frac{23}{12}e_t^3 - \frac{3661}{768}e_t^5 + \mathcal{O}(e_t^7) \quad (4.23t)$$

$$\Theta_5^{-1}(e_t) = \frac{29}{16}e_t^4 - \frac{2365}{768}e_t^6 + \mathcal{O}(e_t^8) \quad (4.23u)$$

$$\Theta_5^0(e_t) = \frac{1683}{1280}e_t^5 + \mathcal{O}(e_t^7) \quad (4.23v)$$

$$\Theta_6^{-2}(e_t) = \frac{189}{64}e_t^4 + \mathcal{O}(e_t^6) \quad (4.23w)$$

$$\Theta_6^{-1}(e_t) = \frac{797}{320}e_t^5 + \mathcal{O}(e_t^5) \quad (4.23x)$$

5. Passage to the CM frame at 2.5PN order

In order to use the QK parametrization, but also to yield simple and unambiguous results, the fluxes will be presented in the CM frame. However, the transformations from a general frame to the CM frame (including both conservative and dissipative contributions) are currently known only up to 2PN in scalar tensor theory [35], but the computation of the angular momentum flux requires these laws at 2.5PN. This is the object the present section. Importantly, it was recently rediscovered (in GR) that it is vital to define the CM frame as the barycenter of both the matter and radiation [48], the latter entering at high PN order as nonlocal integrals over the radiation escaping at infinity. Previous results [92, 93] that ignored the radiation contribution were therefore erroneous. In this section, I will therefore be careful to include these contributions.

5.1. Flux balance laws for the linear momentum and position of the center of mass

Due the the symmetries of Minkowski spacetime, the ten Poincaré invariants of the binary system (its energy E , angular momentum J^i , linear momentum P^i and CM position G^i) are conserved in the absence of gravitational radiation. When including the radiation, these quantities are only approximately conserved, and are described by flux-balance laws. For the energy and angular momentum, these are described in [paper II](#) for the case of ST theory, whereas the linear momentum, P^i , and the position of the CM, G^i , satisfy

$$\frac{dP^i}{dt} = -\mathcal{F}_P^i - \mathcal{F}_{s,P}^i, \quad (5.1a)$$

$$\frac{dG^i}{dt} = P^i - \mathcal{F}_G^i - \mathcal{F}_{s,G}^i. \quad (5.1b)$$

The tensorial fluxes, $\mathcal{F}_P^i$ and $\mathcal{F}_G^i$, are given (up to global factor ϕ_0) in terms of the radiative moments by (4.12b)–(4.15b) of [94], and the scalar fluxes are defined by

$$\mathcal{F}_{s,P}^i = \frac{c^4 \phi_0}{16\pi G} \int d^2S_j \Lambda_\phi^{ij}, \quad (5.2a)$$

$$\mathcal{F}_{s,G}^i = \frac{c^3 \phi_0}{16\pi G} \int d^2S_j \left[x^i \Lambda_\phi^{0j} - r \Lambda_\phi^{ij} \right], \quad (5.2b)$$

where $\int d^2S_j$ denotes integration over the sphere. When taking the $r \rightarrow 0$ limit, only the asymptotic expression of Λ_ϕ are needed, namely

$$\Lambda_\phi^{\mu\nu} = (3 + \omega_0) \left(\eta^{\mu\alpha} \eta^{\nu\beta} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta} \right) \partial_\alpha \partial_\beta \psi + \mathcal{O}\left(\frac{1}{r^3}\right). \quad (5.3)$$

Discarding the $\mathcal{O}(1/r^3)$ term which contains at least cubic terms in the multipolar moments,⁵ and plugging this expression into (5.2), I find

$$\mathcal{F}_{s,P}^i = \frac{c^2 (3 + 2\omega_0) \phi_0 r^2}{16\pi G} \int d\Omega n_i \dot{\psi}^2, \quad (5.4a)$$

$$\begin{aligned} \mathcal{F}_{s,G}^i = & -\frac{c^3 (3 + 2\omega_0) \phi_0 r^3}{16\pi G} \int d\Omega \left[n_j \partial_i \psi \partial_j \psi + \frac{n_i n_j}{c} \dot{\psi} \partial_j \psi \right. \\ & \left. - \frac{1}{2} n^i \left(\partial_k \psi \partial_k \psi - \frac{1}{c^2} \dot{\psi}^2 \right) \right], \end{aligned} \quad (5.4b)$$

where $\int d\Omega$ denotes integration over angles. Finally, replacing ψ by its multipolar expansion (2.10b) yields the final expressions⁵

$$\mathcal{F}_{s,P}^i = G\phi_0 (3 + 2\omega_0) \sum_{\ell=0}^{\infty} \frac{2}{c^{2\ell+3} \ell! (2\ell+3)!!} \mathcal{U}_L^{(1)s} \mathcal{U}_{iL}^{(1)s}, \quad (5.5a)$$

$$\mathcal{F}_{s,G}^i = G\phi_0 (3 + 2\omega_0) \sum_{\ell=0}^{\infty} \frac{\ell+2}{c^{2\ell+3} \ell! (2\ell+3)!!} \left(\mathcal{U}_L^{(1)s} \mathcal{U}_{iL}^{(1)s} - \frac{\ell}{\ell+2} \mathcal{U}_{iL}^{(1)s} \mathcal{U}_L^{(1)s} \right). \quad (5.5b)$$

Note that, since the monopole is constant at Newtonian order and only acquires time-dependent contributions at subleading 1PN order, $d\mathcal{U}^s/dt$ is actually a 1PN quantity. Thus, the leading contributions to the scalar fluxes for P^i and G^i are of order $\mathcal{O}(1/c^5)$, and read

$$\mathcal{F}_{s,P}^i = \frac{G\phi_0 (3 + 2\omega_0)}{c^5} \left[\frac{2}{3} \frac{E^{(1)s}}{\phi_0} I_i^{(2)s} + \frac{2}{15} \frac{I_j^{(2)s}}{I_{ij}^{(3)s}} \right] + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (5.6a)$$

$$\mathcal{F}_{s,G}^i = \frac{G\phi_0 (3 + 2\omega_0)}{c^5} \left[\frac{2}{3} \frac{E^{(1)s}}{\phi_0} I_i^{(1)s} + \frac{1}{5} \frac{I_j^{(2)s}}{I_{ij}^{(2)s}} - \frac{1}{15} \frac{I_j^{(1)s}}{I_{ij}^{(3)s}} \right] + \mathcal{O}\left(\frac{1}{c^7}\right). \quad (5.6b)$$

The expression (5.5b) of the scalar flux of CM position is analogous to the tensorial flux of CM position found by [94, 96], rather than the one obtained by [97]. For our purposes, both expressions are equivalent, because they differ by a total time derivative, see the discussion in section 5 of [48].

5.2. Passage to the center of mass at 2.5PN order

For the clarity of the discussion, the linear momentum and position of the CM are first decomposed into different PN contributions, namely

$$\mathbf{P} = \mathbf{P}^N + \mathbf{P}^{1PN} + \mathbf{P}^{1.5PN} + \mathbf{P}^{2PN} + \mathbf{P}^{2.5PN} + \mathcal{O}(1/c^6), \quad (5.7a)$$

$$\mathbf{G} = \mathbf{G}^N + \mathbf{G}^{1PN} + \mathbf{G}^{1.5PN} + \mathbf{G}^{2PN} + \mathbf{G}^{2.5PN} + \mathcal{O}(1/c^6). \quad (5.7b)$$

⁵ Just as in [95], the uncontrolled $\mathcal{O}(R^{-2})$ subleading term of the asymptotic multipolar expansion is assumed not to contribute. This assumption was rigorously proven in [94] within the Bondi-Sachs framework in the case of GR, and I will assume without proof that the same arguments hold in the scalar sector.

Before this work, the even (conservative) contributions to $\mathbf{G}$ and $\mathbf{P}$ were derived in [31]. The 1.5PN dissipative contributions have been previously computed in order to obtain (A.2) of [35], but were not published. They read:

$$G_i^{1.5\text{PN}} = \frac{\tilde{G}\alpha m^2 \zeta \mathcal{S}_- \mathcal{S}_+ (2\nu - \delta - 1) v_{12}^i}{3c^3}, \quad (5.8a)$$

$$P_i^{1.5\text{PN}} = \frac{2\tilde{G}^2 \alpha^2 m^3 \zeta \mathcal{S}_- \mathcal{S}_+ (1 + \delta - 3\nu - \nu\delta) n_{12}^i}{3c^3 r_{12}^2}, \quad (5.8b)$$

and are defined such that $dG^i/dt = P_i + \mathcal{O}(1/c^5)$ and $dP^i/dt = \mathcal{O}(1/c^5)$ when using the full 2PN equations of motion, including both conservative and dissipative contributions. The latter were first computed in [27], but the 1.5PN term is given in a more explicit form in (A.1a) of [paper I](#). At 2PN order, the equation for the passage to the CM is then obtained by solving $G^i = \mathcal{O}(1/c^5)$, and is given by (A.2) of [paper I](#).

I am now going to obtain the 2.5PN-accurate equation for the passage to the CM frame, but without needing to determine $\mathbf{G}^{2.5\text{PN}}$ in a general frame [98]. At 2.5PN order enter the associated fluxes for $\mathbf{P}$ and $\mathbf{G}$, so the CM frame (associated to both the matter content and gravitational waves) will now be generically defined by the condition [48]

$$G_i + \Gamma_i + \Gamma_i^s = 0, \quad (5.9)$$

where the radiation contribution is described by

$$\Gamma_i = \int_{-\infty}^t dt' \mathcal{F}_G^i(t') + \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \mathcal{F}_P^i(t''), \quad (5.10a)$$

$$\Gamma_i^s = \int_{-\infty}^t dt' \mathcal{F}_{s,G}^i(t') + \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \mathcal{F}_{s,P}^i(t''). \quad (5.10b)$$

It will also prove useful to introduce

$$\Pi_i = \int_{-\infty}^t dt' \mathcal{F}_P^i(t'), \quad (5.11a)$$

$$\Pi_i^s = \int_{-\infty}^t dt' \mathcal{F}_{s,P}^i(t'), \quad (5.11b)$$

such that $P_i + \Pi_i + \Pi_i^s = 0$ is satisfied in the CM frame (in fact, this equation is the *definition* of a center-of-inertia frame). Note that at 2.5PN, the tensor nonlocal quantities Γ_i and Π_i do not contribute. Now, using the decomposition (5.7), one can rewrite the balance equations for the linear momentum (5.1a) as

$$\frac{dP_i^{2.5\text{PN}}}{dt} = - \left(\frac{dP_i^{\text{N}}}{dt} + \frac{dP_i^{1\text{PN}}}{dt} + \frac{dP_i^{1.5\text{PN}}}{dt} + \frac{dP_i^{2\text{PN}}}{dt} + \mathcal{F}_P^i + \mathcal{F}_{s,P}^i \right). \quad (5.12)$$

Since $dP_i^{\text{N}}/dt = \mathcal{O}(1/c^2)$, all quantities on the right hand side of (5.12) can be rewritten with 2.5PN accuracy in terms of the CM variables with the sole knowledge of the 1.5PN-accurate formula for the passage to the CM frame, which is given by (3.6)–(8) of [31] for the conservative piece and (A.2a) of [35] for the dissipative piece. By construction, the right-hand side will be of order $\mathcal{O}(1/c^5)$ in the CM frame.

For any quantity $\mathcal{Z}$ expressed in terms of (y_1, y_2, v_1, v_2) , I will now define the related quantity $(\mathcal{Z})_{\text{CM}}$, which depends only on (x^i, v^i) and is obtained by applying to $\mathcal{Z}$ the formulas for the passage to the CM (e.g. $y_1^i = \dots$) at the relevant order. Note that in the following, I will often be able to control $(\mathcal{Z})_{\text{CM}}$ without controlling the full expression of $\mathcal{Z}$.

The quantity I am now interested in obtaining is $(P_i^{2.5\text{PN}})_{\text{CM}}$, expressed at leading 2.5PN order. To this end, the equation which should be solved reads

$$\left. \frac{d(P_i^{2.5\text{PN}})_{\text{CM}}}{dt} \right|_{a_i^{\text{CM},N}} = (\text{RHS of (5.12)})_{\text{CM}}, \quad (5.13)$$

where RHS denotes the ‘right hand side’ of the equation, and where I have indicated that only the Newtonian relative acceleration in the CM frame $a_i^{\text{CM},N}$ is needed when taking the time derivative. This is easily solved in the CM frame, and the solution reads:

$$(P_i^{2.5\text{PN}})_{\text{CM}} = \left[\frac{p_1}{r^3} + p_2 \frac{(nv)^2}{r^2} + p_3 \frac{v^2}{r^2} \right] n^i + p_4 \frac{(nv)v^i}{r^2}, \quad (5.14)$$

where the coefficients p_n are given in appendix B. Repeating this process for the CM balance equation (5.1b) leads to

$$\begin{aligned} \frac{dG_i^{2.5\text{PN}}}{dt} &= P_i^N + P_i^{1\text{PN}} + P_i^{1.5\text{N}} + P_i^{2\text{PN}} + P_i^{2.5\text{PN}} \\ &\quad - \left(\frac{dG_i^N}{dt} + \frac{dG_i^{1\text{PN}}}{dt} + \frac{dG_i^{1.5\text{N}}}{dt} + \frac{dG_i^{2\text{PN}}}{dt} + \mathcal{F}_G^i + \mathcal{F}_{s,G}^i \right). \end{aligned} \quad (5.15)$$

Since $P_i^N - dG_i^N/dt = \mathcal{O}(1/c^2)$, and since $(P_i^{2.5\text{PN}})_{\text{CM}}$ has been computed in the previous step, one can again easily express the right hand side of (5.15) in terms of the CM variables using solely the 1.5PN-accurate formula for the passage to the CM. Again, by construction, the right-hand side is thus of order $\mathcal{O}(1/c^5)$ in the CM frame, and one is left to solve

$$\left. \frac{d(G_i^{2.5\text{PN}})_{\text{CM}}}{dt} \right|_{a_i^{\text{CM},N}} = (\text{RHS of (5.15)})_{\text{CM}}, \quad (5.16)$$

Again, this is easily solved for in the CM frame, and I find:

$$(G_i^{2.5\text{PN}})_{\text{CM}} = q_1 \frac{(nv)n^i}{r} + \left[\frac{q_2}{r} + q_3 v^2 \right] v^i, \quad (5.17)$$

where the coefficients q_n are given in appendix B. With this expression, I am now in a position to solve for y_1^i with 2.5PN accuracy in the equation

$$G_i^N + (G_i^{1\text{PN}} + G_i^{1.5\text{PN}} + G_i^{2\text{PN}} + G_i^{2.5\text{PN}} + \Gamma_i + \Gamma_i^s)_{\text{CM}} = \mathcal{O}\left(\frac{1}{c^6}\right), \quad (5.18)$$

where $G_i^N = m_1 y_1^i + m_2 y_2^i = m y_1^i - m_2 r n^i$. This yields the final expression

$$y_1^i = (X_2 + \nu \mathcal{P}) x^i + \nu \mathcal{Q} v^i + \mathcal{R}_i \quad (5.19a)$$

$$y_2^i = (-X_1 + \nu \mathcal{P}) x^i + \nu \mathcal{Q} v^i + \mathcal{R}_i, \quad (5.19b)$$

where $X_1 = m_1/m = (1 + \delta)/2$ and $X_2 = m_2/m = (1 - \delta)/2$. The $\mathcal{R}_i$ corresponds to the contribution arising from the radiation (in line with the conventions of [48]), and simply reads

$$\mathcal{R}^i = -\frac{\Gamma_i^s}{m} + \mathcal{O}\left(\frac{1}{c^7}\right), \quad (5.20)$$

where it should be noted that Γ_i^s is intrinsically a 2.5PN quantity, i.e. $\Gamma_i^s = \mathcal{O}(c^{-5})$. Conversely, the $\mathcal{P}$ and $\mathcal{Q}$ contributions are the contributions arising from matter (i.e. what one would find if one solved $G^i = 0$). They are decomposed as

$$\mathcal{P} = \mathcal{P}_{1\text{PN}} + \mathcal{P}_{1.5\text{PN}} + \mathcal{P}_{2\text{PN}} + \mathcal{P}_{2.5\text{PN}} + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (5.21a)$$

$$\mathcal{Q} = \mathcal{Q}_{1\text{PN}} + \mathcal{Q}_{1.5\text{PN}} + \mathcal{Q}_{2\text{PN}} + \mathcal{Q}_{2.5\text{PN}} + \mathcal{O}\left(\frac{1}{c^6}\right), \quad (5.21b)$$

where the conservative contributions $\mathcal{P}_{1\text{PN}}$, $\mathcal{Q}_{1\text{PN}}$, $\mathcal{P}_{2\text{PN}}$ and $\mathcal{Q}_{2\text{PN}}$, are given by (3.7) and (3.8) of [31], and where the dissipative contributions read⁶

$$\mathcal{P}_{1.5\text{PN}} = 0, \quad (5.22a)$$

$$\mathcal{Q}_{1.5\text{PN}} = \frac{2\tilde{G}\alpha m \zeta \mathcal{S}_- (\mathcal{S}_+ + \mathcal{S}_- \delta)}{3c^3}, \quad (5.22b)$$

$$\begin{aligned} \mathcal{P}_{2.5\text{PN}} = & \frac{(\tilde{G}\alpha m)^2 (nv)}{15c^5 r^2} [10\bar{\beta}_- - 5\zeta \mathcal{S}_- \mathcal{S}_+ + 24\zeta \mathcal{S}_- \mathcal{S}_+ \nu \\ & + \delta (5\bar{\gamma} - 10\bar{\beta}_+ - 5\zeta \mathcal{S}_-^2) + 26\zeta \mathcal{S}_-^2 \nu \delta], \end{aligned} \quad (5.22c)$$

$$\begin{aligned} \mathcal{Q}_{2.5\text{PN}} = & \frac{\tilde{G}\alpha m v^2}{15c^5} [5\zeta \mathcal{S}_- \mathcal{S}_+ + \delta (12 + 5\bar{\gamma} + 5\zeta \mathcal{S}_-^2) - 23\zeta \mathcal{S}_- \mathcal{S}_+ \nu - 17\zeta \mathcal{S}_-^2 \nu \delta] \\ & + \frac{(\tilde{G}\alpha m)^2}{15c^5 r} [-10\bar{\beta}_- - 5\zeta \mathcal{S}_- \mathcal{S}_+ + \delta (-24 - 15\bar{\gamma} + 10\bar{\beta}_+ - 5\zeta \mathcal{S}_-^2) \\ & + \nu (160\bar{\gamma}^{-1} \zeta \mathcal{S}_- (\bar{\beta}_+ \mathcal{S}_+ + \bar{\beta}_- \mathcal{S}_-) - 18\zeta \mathcal{S}_- \mathcal{S}_+) + 28\zeta \mathcal{S}_-^2 \nu \delta]. \end{aligned} \quad (5.22d)$$

Taking the time derivative of (5.19), using the 2.5PN-accurate acceleration [27, 31, 35], as well as the identity $d\Gamma_i^s/dt = \Pi_i^s + \mathcal{F}_{s,G}^i$, one obtains the expressions for the passage to the CM for v_1^i and v_2^i . For the reader's convenience, the expressions of $\mathcal{F}_{s,P}^i$ (which enter the definition of Π^s) and of $\mathcal{F}_{s,G}^i$ (which enters the definitions of Π^s and Γ^s) are given at leading order and in the CM frame as follows:

⁶ See also (A.2) of [35] for the 1.5PN piece, where the 1.5PN contributions of y_1^i and v_1^i are expressed with a less compact choice of ST variables.

$$\mathcal{F}_{s,P}^i = \frac{8(\tilde{G}\alpha m)^3 m v^2 \zeta \mathcal{S}_-}{15c^5 r^4} \left[(20\bar{\gamma}^{-1}(\bar{\beta}_+ \mathcal{S}_+ + \bar{\beta}_- \mathcal{S}_-) - 8\mathcal{S}_+ + 3\mathcal{S}_- \delta)(nv)n^i + 2(\mathcal{S}_+ - \mathcal{S}_- \delta)v^i \right], \quad (5.23a)$$

$$\mathcal{F}_{s,G}^i = \frac{4(\tilde{G}\alpha m)^2 m v^2 \zeta \mathcal{S}_-}{15c^5 r^2} \left[(\mathcal{S}_+ - \mathcal{S}_- \delta) \left(\frac{2\tilde{G}\alpha m}{r} - 3(nv)^2 + 3v^2 \right) n^i - (40\bar{\gamma}^{-1}(\bar{\beta}_+ \mathcal{S}_- + \bar{\beta}_- \mathcal{S}_-) - 12\mathcal{S}_+ + 2\mathcal{S}_- \delta)(nv)v^i \right]. \quad (5.23b)$$

Note that only the 1.5PN-accurate formulas for the passage to the CM frame were required when deriving the 2.5PN equations of motion in the CM frame, so previous results [27, 35] are unaffected by the inclusion of the radiation contributions.

5.3. Expression of Γ_i^s and Π_i^s for an elliptic orbit

The appearance of nonlocal terms in the formula for the passage to the CM frame introduces a practical difficulty, because it depends on the whole past history of the binary. In this section, I will show that assuming the two-body system has eternally been isolated makes it possible to ‘localize’ these terms, in a similar way to other nonlocal effects like tails or memory. I will consider the case of a binary on an elliptic orbit (only the leading Newtonian order is required here), and discuss its reduction to the special case of circular orbits at the end.

Since these quantities are to be computed only at leading order, I assume that the motion has been eternally elliptic, where the orbital parameters adiabatically evolve due to radiation reaction. Thus, at this level of accuracy, the semimajor axis a and the eccentricity e asymptote, respectively, to $+\infty$ and 1 in the infinite past. The fluxes of linear momentum (5.6a) and CM position (5.6b) are controlled at any point in the past, which I subdivide into an orbit-averaged contribution and a zero-average oscillating contribution, namely $\mathcal{F}_{s,P}^i = \langle \mathcal{F}_{s,P}^i \rangle + \tilde{\mathcal{F}}_{s,P}^i$ and $\mathcal{F}_{s,G}^i = \langle \mathcal{F}_{s,G}^i \rangle + \tilde{\mathcal{F}}_{s,G}^i$. Since only the leading-order contribution in Π_i^s and Γ_i^s are required, I will consider that the motion is Keplerian, up to leading-order radiation reaction terms. This leads to the decomposition $\Pi_i^s = \Pi_i^{s,\text{DC}} + \Pi_i^{s,\text{AC}}$ and $\Gamma_i^s = \Gamma_i^{s,\text{DC}} + \Gamma_i^{s,\text{AC}}$, where

$$\begin{aligned} \Pi_i^{s,\text{DC}} &= \int_{-\infty}^t dt' \langle \mathcal{F}_{s,P}^i \rangle(t'), & \Gamma_i^{s,\text{DC}} &= \int_{-\infty}^t dt' \langle \mathcal{F}_{s,G}^i \rangle(t') + \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \langle \mathcal{F}_{s,P}^i \rangle(t''), \\ \Pi_i^{s,\text{AC}} &= \int_{-\infty}^t dt' \tilde{\mathcal{F}}_{s,P}^i(t'), & \Gamma_i^{s,\text{AC}} &= \int_{-\infty}^t dt' \tilde{\mathcal{F}}_{s,G}^i(t') + \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \tilde{\mathcal{F}}_{s,P}^i(t''). \end{aligned} \quad (5.24)$$

5.3.1. The AC part. I will first consider the AC contribution. The oscillatory part of the flux can be written in terms of the Fourier decomposition of the source moments (see section 4), and reads

$$\tilde{\mathcal{F}}_{s,P}^i = \frac{G\phi_0(3+2\omega_0)}{c^5} \sum_{p+q \neq 0} e^{i(p+q)\ell} \left[\frac{2(ipn)(iqn)^2}{3\phi_0} \left({}_p\tilde{E}^s \right) \left({}_q\tilde{\Gamma}_i^s \right) + \frac{2(ipn)^2(iqn)^3}{15} \left({}_p\tilde{\Gamma}_j^s \right) \left({}_q\tilde{\Gamma}_{ij}^s \right) \right], \quad (5.25a)$$

$$\begin{aligned} \tilde{\mathcal{F}}_{s,G}^i &= \frac{G\phi_0(3+2\omega_0)}{c^5} \sum_{p+q \neq 0} e^{i(p+q)\ell} \left[\frac{2}{3} \frac{(ipn)(iqn)}{\phi_0} \left({}_p\tilde{E}^s \right) \left({}_q\tilde{\Gamma}_i^s \right) \right. \\ &\quad \left. + \left(\frac{(ipn)^2(iqn)^2}{5} - \frac{(ipn)(iqn)^3}{15} \right) \left({}_p\tilde{\Gamma}_j^s \right) \left({}_q\tilde{\Gamma}_{ij}^s \right) \right], \end{aligned} \quad (5.25b)$$

where the $p = q$ case has been excluded from the sum, as it corresponds exactly to the DC piece, which will be treated separately. It is then immediate to compute its time integral, namely

$$\Pi_i^{s,AC} = \frac{G\phi_0(3+2\omega_0)}{c^5} \sum_{p+q \neq 0} e^{i(p+q)\ell} \left[\frac{2}{3\phi_0} \frac{(ipn)(iqn)^2}{i(p+q)n} \left({}_p\tilde{E}^s \right) \left({}_q\tilde{\Gamma}_i^s \right) + \frac{2}{15} \frac{(ipn)^2(iqn)^3}{i(p+q)n} \left({}_p\tilde{\Gamma}_j^s \right) \left({}_q\tilde{\Gamma}_{ij}^s \right) \right], \quad (5.26a)$$

$$\begin{aligned} \Gamma_i^{s,AC} &= \frac{G\phi_0(3+2\omega_0)}{c^5} \sum_{p+q \neq 0} e^{i(p+q)\ell} \left[\frac{2}{3\phi_0} \left(\frac{(ipn)(iqn)}{i(p+q)n} + \frac{(ipn)(iqn)^2}{[i(p+q)n]^2} \right) \left({}_p\tilde{E}^s \right) \left({}_q\tilde{\Gamma}_i^s \right) \right. \\ &\quad \left. + \frac{1}{5} \left(\frac{(ipn)^2(iqn)^2}{i(p+q)n} - \frac{1}{3} \frac{(ipn)(iqn)^3}{i(p+q)n} + \frac{2}{3} \frac{(ipn)^2(iqn)^3}{[i(p+q)n]^2} \right) \left({}_p\tilde{\Gamma}_j^s \right) \left({}_q\tilde{\Gamma}_{ij}^s \right) \right], \end{aligned} \quad (5.26b)$$

where the $-\infty$ bound does not contribute and has been discarded (see [99, 100] for a justification). Projecting onto the (x, y, z) components, leveraging the fact that $\Pi_i^{s,AC} \in \mathbb{R}$, and using the real or purely imaginary character of the Fourier coefficients in equation (4.7), I find that $\Gamma_i^{s,AC} = \Gamma_x^{s,AC} n_0^i + \Gamma_y^{s,AC} \lambda_0^i$ and $\Pi_i^{s,AC} = \Pi_x^{s,AC} n_0^i + \Pi_y^{s,AC} \lambda_0^i$, where

$$\begin{aligned} \Pi_x^{s,AC} &= -\frac{2G\phi_0(3+2\omega_0)}{3c^5} \sum_{p+q \neq 0} \cos((p+q)\ell) \\ &\quad \times \left\{ \frac{n^2}{\phi_0} \frac{pq^2}{p+q} {}_p\tilde{E}^s {}_q\tilde{\Gamma}_x^s - \frac{n^4}{5} \frac{p^2q^3}{p+q} \left({}_p\tilde{\Gamma}_x^s {}_q\tilde{\Gamma}_{xx}^s + {}_p\tilde{\Gamma}_y^s {}_q\tilde{\Gamma}_{xy}^s \right) \right\}, \end{aligned} \quad (5.27a)$$

$$\begin{aligned} \Pi_y^{s,AC} &= -\frac{2iG\phi_0(3+2\omega_0)}{3c^5} \sum_{p+q \neq 0} \sin((p+q)\ell) \\ &\quad \times \left\{ \frac{n^2}{\phi_0} \frac{pq^2}{p+q} {}_p\tilde{E}^s {}_q\tilde{\Gamma}_y^s - \frac{n^4}{5} \frac{p^2q^3}{p+q} \left({}_p\tilde{\Gamma}_x^s {}_q\tilde{\Gamma}_{xy}^s + {}_p\tilde{\Gamma}_y^s {}_q\tilde{\Gamma}_{yy}^s \right) \right\}, \end{aligned} \quad (5.27b)$$

$$\begin{aligned} \Gamma_x^{s,AC} &= \frac{G\phi_0(3+2\omega_0)}{c^5} \sum_{p+q \neq 0} \sin((p+q)\ell) \left\{ -\frac{2n}{3\phi_0} \frac{pq(p+2q)}{(p+q)^2} {}_p\tilde{E}^s {}_q\tilde{\Gamma}_x^s \right. \\ &\quad \left. + \frac{n^3}{15} \frac{pq^2(3p^2+4pq-q^2)}{(p+q)^2} \left({}_p\tilde{\Gamma}_x^s {}_q\tilde{\Gamma}_{xx}^s + {}_p\tilde{\Gamma}_y^s {}_q\tilde{\Gamma}_{xy}^s \right) \right\}, \end{aligned} \quad (5.27c)$$

$$\begin{aligned} \Gamma_y^{s,AC} &= -\frac{iG\phi_0(3+2\omega_0)}{c^5} \sum_{p+q \neq 0} \cos((p+q)\ell) \left\{ -\frac{2n}{3\phi_0} \frac{pq(p+2q)}{(p+q)^2} {}_p\tilde{E}^s {}_q\tilde{\Gamma}_y^s \right. \\ &\quad \left. + \frac{n^3}{15} \frac{pq^2(3p^2+4pq-q^2)}{(p+q)^2} \left({}_p\tilde{\Gamma}_x^s {}_q\tilde{\Gamma}_{xy}^s + {}_p\tilde{\Gamma}_y^s {}_q\tilde{\Gamma}_{yy}^s \right) \right\}. \end{aligned} \quad (5.27d)$$

Note that the latter contributions vanish upon orbit averaging. Using the QK parametrization (3.1), it is possible to expand the previous expressions in the small eccentricity. These expressions are quite cumbersome, so I will only include the leading correction in the eccentricity. They read:

$$\begin{aligned} \Pi_i^{s,AC} = & \frac{16}{15} c \zeta m \nu^2 \mathcal{S}_- x^3 \left\{ (\mathcal{S}_+ - \mathcal{S}_- \delta) \left[\cos(\ell) n_0^i + \sin(\ell) \lambda_0^i \right] \right. \\ & \left. + e \left[-\frac{5}{2} \bar{\gamma}^{-1} (\bar{\beta}_+ \mathcal{S}_+ + \bar{\beta}_- \mathcal{S}_-) + \frac{5}{2} \mathcal{S}_+ - \frac{15}{8} \mathcal{S}_- \delta \right] \left(\cos(2\ell) n_0^i + \sin(2\ell) \lambda_0^i \right) + \mathcal{O}(e^2) \right\}, \end{aligned} \quad (5.28a)$$

$$\begin{aligned} \Gamma_i^{s,AC} = & \frac{12}{5c^2} \tilde{G} \alpha m^2 \nu^2 \zeta \mathcal{S}_- x^{3/2} \left\{ (\mathcal{S}_+ - \mathcal{S}_- \delta) \left[\sin(\ell) n_0^i - \cos(\ell) \lambda_0^i \right] \right. \\ & \left. + e \left[-\frac{5}{3} \bar{\gamma}^{-1} (\bar{\beta}_+ \mathcal{S}_+ + \bar{\beta}_- \mathcal{S}_-) + \frac{5}{3} \mathcal{S}_+ - \frac{5}{4} \mathcal{S}_- \delta \right] \left(\sin(2\ell) n_0^i - \cos(2\ell) \lambda_0^i \right) + \mathcal{O}(e^2) \right\}. \end{aligned} \quad (5.28b)$$

I can then take the limit of circular orbits, re-express everything in terms of the corotating basis $(\mathbf{n}, \boldsymbol{\lambda}, \ell)$ and then substitute $\boldsymbol{\lambda}$ and x by their expressions in terms of $\mathbf{v}$ and r . I then find

$$\left[\Pi_i^{s,AC} \right]_{\text{circ}} = \frac{16 (\tilde{G} \alpha m)^3 m \nu^2 \zeta \mathcal{S}_- (\mathcal{S}_+ - \mathcal{S}_- \delta)}{15 c^5 r^3} n^i, \quad (5.29a)$$

$$\left[\Gamma_i^{s,AC} \right]_{\text{circ}} = -\frac{12 (\tilde{G} \alpha m)^3 m \nu^2 \zeta \mathcal{S}_- (\mathcal{S}_+ - \mathcal{S}_- \delta)}{5 c^5 r} v^i. \quad (5.29b)$$

I have checked that the simpler calculation that assumes circular orbits from the start (and does not use the Fourier decomposition of the multipolar moments) yields the same result, which is an important consistency test of the result and of the Fourier decomposition.

5.3.2. The DC part. Injecting the Keplerian parametrization into equations (5.23), and performing the orbit averaging as usual, I find that the orbit-averaged fluxes of linear momentum and CM position read

$$\langle \mathcal{F}_{s,G}^i \rangle = \frac{4 (\tilde{G} \alpha m)^3 m \nu^2 \zeta \mathcal{S}_-}{15 \bar{\gamma} c^5} (20 \bar{\beta}_+ \mathcal{S}_+ + 20 \bar{\beta}_- \mathcal{S}_- - 2 \bar{\gamma} \mathcal{S}_+ - 3 \bar{\gamma} \mathcal{S}_- \delta) \frac{e}{(1-e^2)^{3/2} a^3} n_0^i, \quad (5.30a)$$

$$\langle \mathcal{F}_{s,P}^i \rangle = \frac{(\tilde{G} \alpha m)^{7/2} m \nu^2 \zeta \mathcal{S}_-}{3 \bar{\gamma} c^5} (4 \bar{\beta}_+ \mathcal{S}_+ + 4 \bar{\beta}_- \mathcal{S}_- - \bar{\gamma} \mathcal{S}_- \delta) \frac{e (4+e^2)}{(1-e^2)^3 a^{9/2}} \lambda_0^i. \quad (5.30b)$$

Thanks to the leading-order relations [47]

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{(\tilde{G} \alpha m)^2 \zeta \mathcal{S}_-^2 \nu}{c^3} \times \frac{2e}{a^3 (1-e^2)^{3/2}}, \quad (5.31a)$$

$$a(e) = \frac{c_0 e^{4/3}}{1-e^2}, \quad (5.31b)$$

one can perform a change of variables in the integrals of (5.30), namely transform an integral over time into an integral over eccentricity. In the far past, the eccentricity asymptotes towards 1 (at leading order in the post-Newtonian expansion), and I found that the associated bound

converges in this limit. These integrals can in fact be computed analytically, and I find that $\Pi_i^{s,\text{DC}} = \Pi_y^{s,\text{DC}} \lambda_0^i$ and $\Gamma_i^{s,\text{DC}} = \Gamma_x^{s,\text{DC}} n_0^i + \Gamma_y^{s,\text{DC}} \lambda_0^i$, where

$$\Pi_y^{s,\text{DC}} = \frac{(\tilde{G}\alpha m)^{3/2} m\nu}{6\bar{\gamma}\mathcal{S}_-c^2} (4\bar{\beta}_+\mathcal{S}_+ + 4\bar{\beta}_-\mathcal{S}_- - \bar{\gamma}\mathcal{S}_-\delta) \frac{e(4+e)}{a^{3/2}(1+e)\sqrt{1-e^2}}, \quad (5.32a)$$

$$\Gamma_x^{s,\text{DC}} = \frac{2(\tilde{G}\alpha m)m\nu}{15\bar{\gamma}\mathcal{S}_-c^2} (20\bar{\beta}_+\mathcal{S}_- + 20\bar{\beta}_-\mathcal{S}_- - 2\bar{\gamma}\mathcal{S}_+ - 3\bar{\gamma}\mathcal{S}_-\delta) (1-e), \quad (5.32b)$$

$$\begin{aligned} \Gamma_y^{s,\text{DC}} &= \frac{mc}{24(\tilde{G}\alpha m)^{1/2}\bar{\gamma}\zeta\mathcal{S}_-^3} (4\bar{\beta}_+\mathcal{S}_+ + 4\bar{\beta}_-\mathcal{S}_- - \bar{\gamma}\mathcal{S}_-\delta) \frac{a^{3/2}(1-e^2)}{e^2} \\ &\times \left[12 - 5e - 6e^2 - e^3 - 10\sqrt{1-e^2} \arctan \sqrt{\frac{1-e}{1+e}} \right]. \end{aligned} \quad (5.32c)$$

In the $e \rightarrow 0$ limit (with a kept constant), these functions behave as

$$\Pi_y^{s,\text{DC}} = \mathcal{O}(e), \quad \Gamma_x^{s,\text{DC}} = \mathcal{O}(1), \quad \Gamma_y^{s,\text{DC}} = \mathcal{O}\left(\frac{1}{e^2}\right), \quad (5.33)$$

whereas in the $e \rightarrow 1$ limit, they behave as

$$\Pi_y^{s,\text{DC}} = \mathcal{O}\left(\frac{1}{\sqrt{1-e}}\right), \quad \Gamma_x^{s,\text{DC}} = \mathcal{O}(1-e), \quad \Gamma_y^{s,\text{DC}} = \mathcal{O}\left((1-e)^2\right). \quad (5.34)$$

One can interpret these quantities as the boost and shift associated to the secular piece of the gravitational recoil. The boost term $\Pi^{s,\text{DC}}$ vanishes in the limit of vanishing *current* eccentricity, which smoothly connects with the predictions that, for an eternally quasicircular orbit, there is no secular boost, at least in the post-Newtonian, adiabatic regime. However, the y -component of the shift term $\Gamma^{s,\text{DC}}$ blows up as the current eccentricity goes to zero, whilst the x -component goes to a nonzero finite limit. This behavior appears to be paradoxical, since in the quasicircular limit, the shift is predicted to be strictly zero. This paradox is also observed in the case of GR [101], although with more complicated analytical expressions. Since only $\Pi^{s,\text{DC}}$ will be required in this work, the complete resolution of this paradox is left to future work. However, I will state a few mitigating arguments. First, assuming quasicircular orbits artificially imposes a symmetry which does not exist in the general case, because an ellipse exhibits a preferred direction, but not a circle. Moreover, the quasicircular model implies that the eccentricity was zero in the infinite past—whereas assuming that the current eccentricity is nonzero but arbitrarily small implies that the eccentricity was one in the infinite past (up to small PN corrections due to the radiation). Thus, it is not entirely surprising that there could be a nonsmooth limit when going from the quasielliptic to the quasicircular scenario, although a smooth limit would of course more satisfactory. Second, in most cases, only the center-of-inertia is physically relevant, and a constant shift between two frames is unlikely to affect any physical observable. Thus, it should be possible to renormalize this apparent divergence by a physically irrelevant, constant shift. Instead of working in the CM frame, defined uniquely by $G_i + \Gamma_i^s + \Gamma_i = 0$, one would then work in a center-of-inertia frame, defined up to an arbitrary constant shift by $P_i + \Pi_i^s + \Pi_i = 0$.

5.4. The scalar dipolar moment as 2.5PN in the CM frame

When computing the 1.5PN energy flux in the CM frame [35], the scalar dipole was obtained at 2.5PN order in a generic frame. But only $\partial_t^2 \mathbf{I}_i^s$ was needed with 2.5PN accuracy in the CM frame, and not $\mathbf{I}_i^s$, so only the 1.5PN-accurate formula for the passage to the CM was required (see appendix B1 of [paper I](#) for an explanation). However, the angular momentum flux $\mathbf{I}$ will be computing hereafter involves $\partial_t \mathbf{I}_i^s$, so the 2.5PN-accurate formula for the passage to the CM is necessary⁷. The nonlocal term (5.10) will thus propagate into the expression of the scalar dipole moment in the CM frame, which is thus split as follows:

$$\mathbf{I}_i^s = \mathbf{I}_i^{s,\text{inst}} + \frac{\sqrt{\alpha}\zeta (\mathcal{S}_+ + \mathcal{S}_- \delta)}{(1 - \zeta) \phi_0} \Gamma_i^s, \quad (5.35)$$

where $\mathbf{I}_i^{s,\text{inst}}$ is instantaneous and relegated to appendix F, and Γ_i^s is defined in (5.10). However, the angular momentum flux does not depend directly on $\mathbf{I}_i^s$, but rather on its time derivative $\partial_t \mathbf{I}_i^s$, which is obtained thanks to the identity $d\Gamma_i^s/dt = \Pi_i^s + \mathcal{F}_{s,G}^i$, where Π_i^s is defined in (5.11) and $\mathcal{F}_{s,G}^i$ is explicitly presented in (5.23b). The first derivative of the scalar dipole is then split as follows:

$$\mathbf{I}_i^s = \mathbf{I}_i^{s,\text{inst}} + \frac{\sqrt{\alpha}\zeta (\mathcal{S}_+ + \mathcal{S}_- \delta)}{(1 - \zeta) \phi_0} (\mathcal{F}_{s,G}^i + \Pi_i^s). \quad (5.36)$$

Finally, using the identity $d\Pi_i^s/dt = \mathcal{F}_{s,P}^i$, I find that

$$\mathbf{I}_i^s = \mathbf{I}_i^{s,\text{inst}} + \frac{\sqrt{\alpha}\zeta (\mathcal{S}_+ + \mathcal{S}_- \delta)}{(1 - \zeta) \phi_0} \left(\mathcal{F}_{s,G}^i + \mathcal{F}_{s,P}^i \right), \quad (5.37)$$

which is purely instantaneous. Importantly, I have checked that this result is in agreement with the corresponding expression used in [paper I](#), which was derived without the knowledge of the 2.5PN-accurate formula for the passage to the CM frame.

6. The flux of energy and angular momentum

This section is devoted to the derivation of the orbit-averaged fluxes of energy and angular momentum sourced by quasielliptic binary systems. These quantities are very important as they are required for the derivation of the secular evolution of the orbital elements. They were obtained at Newtonian order in [paper II](#) (i.e. at relative 1PN order beyond to the leading -1 PN order dipolar fluxes), and I will now extend this result to 1.5PN order, which is the same order at which the energy flux generated by *circular orbits* was derived in [paper I](#). Due to the presence of hereditary tail and memory integrals, the difficulty of the computation is comparable to the 2.5PN flux in GR [59, 90, 102].

⁷ *A contrario*, in GR, the quadrupolar moment $\mathbf{I}_{ij}$ has a particular structure that avoids the appearance of this nonlocal term at 3.5PN order, see [48].

6.1. General expressions in terms of the source moments

Recall that the expressions of the fluxes in terms of the radiative moments read [35, 47, 73, 94, 95]

$$\mathcal{F} = \sum_{\ell=2}^{\infty} \frac{G\phi_0}{c^{2\ell+1}} \left(\alpha_{\ell} \mathcal{U}_L^{(1)} \mathcal{U}_L^{(1)} + \frac{\beta_{\ell}}{c^2} \mathcal{V}_L^{(1)} \mathcal{V}_L^{(1)} \right), \quad (6.1a)$$

$$\mathcal{F}^s = \sum_{\ell=0}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}} \gamma_{\ell} \mathcal{U}_L^s \mathcal{U}_L^s, \quad (6.1b)$$

$$\mathcal{G}_i = \epsilon_{ijk} \sum_{\ell=2}^{\infty} \frac{G\phi_0}{c^{2\ell+1}} \left(\alpha'_{\ell} \mathcal{U}_{jL-1}^{(1)} \mathcal{U}_{kL-1}^{(1)} + \frac{\beta'_{\ell}}{c^2} \mathcal{V}_{jL-1}^{(1)} \mathcal{V}_{kL-1}^{(1)} \right), \quad (6.1c)$$

$$\mathcal{G}_i^s = \epsilon_{ijk} \sum_{\ell=1}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}} \gamma'_{\ell} \mathcal{U}_{jL-1}^s \mathcal{U}_{kL-1}^s. \quad (6.1d)$$

where

$$\alpha_{\ell} = \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell!(2\ell+1)!!}, \quad \beta_{\ell} = \frac{4\ell(\ell+2)}{(\ell-1)(\ell+1)!(2\ell+1)!!}, \quad \gamma_{\ell} = \frac{1}{\ell!(2\ell+1)!!}, \quad (6.2)$$

and $\alpha'_{\ell} = \ell \alpha_{\ell}$, $\beta'_{\ell} = \ell \beta_{\ell}$, and $\gamma'_{\ell} = \ell \gamma_{\ell}$.

Anticipating the fact that nonspinning binaries do not undergo precession, I will define $\mathcal{G} = \mathcal{G}_i \ell_i$ and $\mathcal{G}^s = \mathcal{G}_i^s \ell_i$. Indeed, the angular momentum vector will always be along ℓ_i , which is a constant vector of norm 1, and always orthogonal to the orbital plane. Replacing the radiative moments by their expressions in terms of the source moments (given by (3.28) and (29) of [35]), I find that, at 1.5PN order beyond quadrupolar radiation, the fluxes can be divided in different pieces: instantaneous, tails, memory, and the new nonlocal Π^s contribution. This decomposition reads:

$$\mathcal{F} = \mathcal{F}^{\text{inst}} + \mathcal{F}^{\text{tail}}, \quad (6.3a)$$

$$\mathcal{F}^s = \mathcal{F}^{s,\text{inst}} + \mathcal{F}^{s,\text{tail}}, \quad (6.3b)$$

$$\mathcal{G}_i = \mathcal{G}_i^{\text{inst}} + \mathcal{G}_i^{\text{tail}} + \mathcal{G}_i^{\text{mem}}, \quad (6.3c)$$

$$\mathcal{G}_i^s = \mathcal{G}_i^{s,\text{inst}} + \mathcal{G}_i^{s,\Pi^s} + \mathcal{G}_i^{s,\text{tail}}. \quad (6.3d)$$

The memory term, present only in the tensor angular momentum flux, reads

$$\mathcal{G}_i^{\text{mem}} = -\frac{2G^2\phi_0(3+2\omega_0)}{15c^8} \epsilon_{ik(j} \mathbf{I}_{a)k}^{(3)} \int_0^{\infty} d\tau \left[\mathbf{I}_a^s \mathbf{I}_j^s \right] (u - \tau), \quad (6.4)$$

which is in agreement with (4.11) of [62].

The tails terms read:

$$\mathcal{F}^{\text{tail}} = \frac{4G^2M}{5c^8} \text{I}_{ij}^{(3)} \int_0^\infty d\tau \text{I}_{ij}^{(5)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12} \right], \quad (6.5a)$$

$$\begin{aligned} \mathcal{F}^{s,\text{tail}} = & \frac{4G^2M(3+2\omega_0)}{c^6} \left\{ \frac{1}{3} \text{I}_i^{(2)} \int_0^\infty d\tau \text{I}_i^{(4)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1 \right] \right. \\ & + \frac{\text{E}^s}{c^2\phi_0^2} \int_0^\infty d\tau \text{E}^s(u-\tau) \ln\left(\frac{c\tau}{2b_0}\right) + \frac{1}{30c^2} \text{I}_{ij}^{(3)} \int_0^\infty d\tau \text{I}_{ij}^{(5)}(u-\tau) \\ & \left. \times \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{3}{2} \right] \right\}, \quad (6.5b) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_i^{\text{tail}} = & \frac{4G^2M}{5c^8} \epsilon_{ijk} \left\{ \text{I}_{ja}^{(2)} \int_0^\infty d\tau \text{I}_{ka}^{(5)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12} \right] \right. \\ & \left. - \text{I}_{ja}^{(3)} \int_0^\infty d\tau \text{I}_{ka}^{(4)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12} \right] \right\}, \quad (6.5c) \end{aligned}$$

$$\begin{aligned} \mathcal{G}_i^{s,\text{tail}} = & \frac{2G^2M(3+2\omega_0)}{3c^6} \epsilon_{ijk} \left\{ \text{I}_j^{(1)} \int_0^\infty d\tau \text{I}_k^{(4)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1 \right] \right. \\ & - \text{I}_j^{(2)} \int_0^\infty d\tau \text{I}_k^{(3)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1 \right] \\ & + \frac{1}{5c^2} \text{I}_{ja}^{(2)} \int_0^\infty d\tau \text{I}_{ka}^{(5)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{3}{2} \right] \\ & \left. - \frac{1}{5c^2} \text{I}_{ja}^{(3)} \int_0^\infty d\tau \text{I}_{ka}^{(4)}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{3}{2} \right] \right\}, \quad (6.5d) \end{aligned}$$

which is also in agreement with (4.10) and (5.17) of [62].

The nonlocal Π^s contribution to the angular momentum flux arises from the nonlocal contribution in the passage to the CM, see section 5. Note that in the energy flux, both dipolar moments entering at Newtonian order bear two time derivatives, which has the effect of ‘localizing’ the Π^s contribution, see (5.37). However, the angular momentum flux contains a contribution (entering at Newtonian order) from the first time derivative of the dipolar moment, which will thus feature Π^s . This contribution reads:

$$\mathcal{G}_i^{s,\Pi^s} = \frac{G(3+2\omega_0)}{3c^3} \frac{\sqrt{\alpha}\zeta(\mathcal{S}_+ + \mathcal{S}_-\delta)}{(1-\zeta)} \epsilon_{ijk} \Pi_j^s \text{I}_k^{(2)} \quad (6.6)$$

Finally, the instantaneous fluxes are defined as what remains, namely

$$\begin{aligned} \mathcal{F}^{\text{inst}} = & \frac{G\phi_0}{5c^5} \text{I}_{ij}^{(3)} \text{I}_{ij}^{(3)} + \frac{G\phi_0}{c^7} \left[\frac{16}{45} \text{J}_{ij}^{(3)} \text{J}_{ij}^{(3)} + \frac{1}{189} \text{I}_{ijk}^{(4)} \text{I}_{ijk}^{(4)} \right] \\ & - \frac{G^2\phi_0(3+2\omega_0)}{15c^8} \left[4 \text{I}_i^{(1)} \text{I}_j^{(3)} \text{I}_{ij}^{(3)} + 2 \text{I}_i^{(4)} \text{I}_j^{(3)} \text{I}_{ij}^{(3)} + \frac{m_s}{\phi_0} \text{I}_{ij}^{(3)} \text{I}_{ij}^{(4)} \right] + \mathcal{O}\left(\frac{1}{c^9}\right), \quad (6.7a) \end{aligned}$$

$$\begin{aligned}
\mathcal{F}^{s,\text{inst}} = & \frac{G\phi_0(3+2\omega_0)}{3c^3} \mathbf{I}_i^s \mathbf{I}_i^s + \frac{G\phi_0(3+2\omega_0)}{c^5} \left[\frac{1}{30} \mathbf{I}_{ij}^s \mathbf{I}_{ij}^s + \frac{\mathbf{E}^s \mathbf{E}^s}{\phi_0^2} \right] \\
& + \frac{G\phi_0(3+2\omega_0)}{630c^7} \mathbf{I}_{ijk}^s \mathbf{I}_{ijk}^s + \frac{G^2\phi_0(3+2\omega_0)}{45c^8} \left[-180 \frac{m_s \mathbf{E}^s \mathbf{E}^s}{\phi_0^3} - 120 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} \right. \\
& - 60 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} - 240 \mathbf{I}_a^s \mathbf{I}_a^s \mathbf{W}^s - 60 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} - 12 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s + 20 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} \\
& - 120 \mathbf{I}_a^s \mathbf{I}_a^s \mathbf{W}^s + 12 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s + 12 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s - 120 \mathbf{I}_a^s \mathbf{I}_a^s \mathbf{W}^s + 120 \frac{m_s}{\phi_0} \mathbf{I}_a^s \mathbf{Y}_a^s \\
& - 6 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s - 3 \frac{m_s}{\phi_0} \mathbf{I}_{ab}^s \mathbf{I}_{ab}^s + 20 \frac{\mathbf{E}^s}{\phi_0} \mathbf{I}_a^s \mathbf{I}_a^s + 36 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s + 24 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s \\
& + 30 \epsilon_{abi} J_a \mathbf{I}_b^s \mathbf{I}_i^s + 18 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s + 12 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s + \frac{4\omega_0' \phi_0}{3+2\omega_0} \left(45 \frac{m_s \mathbf{E}^s \mathbf{E}^s}{\phi_0^3} \right. \\
& + 30 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} + 15 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} + 15 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} - 5 \frac{\mathbf{E}^s \mathbf{I}_a^s \mathbf{I}_a^s}{\phi_0} - 3 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s \\
& \left. \left. - 5 \frac{\mathbf{E}^s}{\phi_0} \mathbf{I}_a^s \mathbf{I}_a^s - 6 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s - 3 \mathbf{I}_a^s \mathbf{I}_b^s \mathbf{I}_{ab}^s \right) \right] + \mathcal{O}\left(\frac{1}{c^9}\right), \quad (6.7b)
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_i^{s,\text{inst}} = & \frac{2G\phi_0}{5c^5} \epsilon_{ijk} \mathbf{I}_{ja}^s \mathbf{I}_{ka}^s + \frac{G\phi_0}{c^7} \epsilon_{ijk} \left[\frac{32}{45} J_{ja} J_{ka} + \frac{1}{63} \mathbf{I}_{jab}^s \mathbf{I}_{kab}^s \right] + \frac{G^2\phi_0(3+2\omega)}{15c^8} \epsilon_{ijk} \\
& \times \left[-\mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ak}^s + 2 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ak}^s - \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ak}^s + \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ak}^s - \mathbf{I}_a^s \mathbf{I}_j^s \mathbf{I}_{ak}^s + 2 \mathbf{I}_a^s \mathbf{I}_j^s \mathbf{I}_{ak}^s \right. \\
& \left. - \mathbf{I}_a^s \mathbf{I}_j^s \mathbf{I}_{ak}^s + \frac{m_s}{\phi_0} \mathbf{I}_{ak}^s \mathbf{I}_{aj}^s + \mathbf{I}_a^s \mathbf{I}_j^s \mathbf{I}_{ak}^s - \frac{m_s}{\phi_0} \mathbf{I}_{aj}^s \mathbf{I}_{ak}^s \right] + \mathcal{O}\left(\frac{1}{c^9}\right), \quad (6.7c)
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_i^{s,\text{inst}} = & \frac{G\phi_0(3+2\omega_0)}{3c^3} \epsilon_{ijk} \left(\mathbf{I}_j^{s,\text{inst}} + \frac{\sqrt{\alpha}\zeta(\mathcal{S}_+ + \mathcal{S}_- \delta)}{(1-\zeta)\phi_0} \mathcal{F}_{s,G}^j \right) \mathbf{I}_k^s \\
& + \frac{G\phi_0(3+2\omega_0)}{15c^5} \epsilon_{ijk} \mathbf{I}_{ja}^s \mathbf{I}_{ka}^s + \frac{G\phi_0(3+2\omega_0)}{210c^7} \epsilon_{ijk} \mathbf{I}_{jab}^s \mathbf{I}_{kab}^s + \frac{G^2\phi_0(3+2\omega)}{15c^8} \\
& \times \left[\epsilon_{ijk} \left\{ -20 \frac{\mathbf{E}^s}{\phi_0} \mathbf{I}_j^s \mathbf{I}_k^s - 10 \frac{\mathbf{E}^s}{\phi_0} \mathbf{I}_j^s \mathbf{I}_k^s - 60 \mathbf{I}_j^s \mathbf{I}_k^s \mathbf{W}^s - 20 \frac{m_s}{\phi_0} \mathbf{I}_j^s \mathbf{Y}_k^s + 10 \frac{\mathbf{E}^s}{\phi_0} \mathbf{I}_j^s \mathbf{I}_k^s \right. \right. \\
& - 20 \mathbf{I}_j^s \mathbf{I}_k^s \mathbf{W}^s + 20 \frac{m_s}{\phi_0} \mathbf{I}_j^s \mathbf{Y}_k^s + \mathbf{I}_a^s \mathbf{I}_j^s \mathbf{I}_{ka}^s + \mathbf{I}_a^s \mathbf{I}_j^s \mathbf{I}_{ka}^s - \frac{m_s}{\phi_0} \mathbf{I}_{ja}^s \mathbf{I}_{ka}^s - \mathbf{I}_a^s \mathbf{I}_j^s \mathbf{I}_{ka}^s \\
& - \frac{m_s}{\phi_0} \mathbf{I}_{ja}^s \mathbf{I}_{ka}^s - 2 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s + 2 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s - 3 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s - 2 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s + 2 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s \\
& \left. \left. + 6 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s + 4 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s - 3 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s - 2 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s + 3 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s + 2 \mathbf{I}_j^s \mathbf{I}_a^s \mathbf{I}_{ka}^s \right\} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{\omega'_0 \phi_0}{3 + 2\omega_0} \left(20 \frac{E^s}{\phi_0} \overset{(1)}{I}_j^s \overset{(2)}{I}_k^s + 10 \frac{E^s}{\phi_0} \overset{(2)}{I}_j^s \overset{(2)}{I}_k^s - 10 \frac{E^s}{\phi_0} \overset{(3)}{I}_j^s \overset{(1)}{I}_k^s + 2 \overset{(2)}{I}_j^s \overset{(3)}{I}_a^s \overset{(1)}{I}_{ka}^s - 2 \overset{(1)}{I}_j^s \overset{(3)}{I}_a^s \overset{(2)}{I}_{ka}^s \right. \\
& \left. - 4 \overset{(1)}{I}_j^s \overset{(4)}{I}_a^s \overset{(1)}{I}_{ka}^s + 2 \overset{(2)}{I}_j^s \overset{(4)}{I}_a^s \overset{(1)}{I}_{ka}^s - 2 \overset{(1)}{I}_j^s \overset{(5)}{I}_a^s \overset{(1)}{I}_{ka}^s \right) \Bigg\} + 5J_i \overset{(2)}{I}_a^s \overset{(3)}{I}_a^s - 5J_a \overset{(2)}{I}_a^s \overset{(3)}{I}_i^s - 5J_i \overset{(1)}{I}_a^s \overset{(4)}{I}_a^s \\
& + 5J_a \overset{(1)}{I}_a^s \overset{(4)}{I}_i^s \Bigg] + \mathcal{O}\left(\frac{1}{c^9}\right). \tag{6.7d}
\end{aligned}$$

6.2. The instantaneous fluxes

The instantaneous fluxes are first computed at 1.5PN order in terms of the phase space variables $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{v}_1, \mathbf{v}_2)$ by (i) replacing the source source multipolar moments by their expressions in the CM frame, and (ii) computing the time derivatives using the equations of motion in the CM frame. The required source moments are almost all provided in appendix B1 of [paper I](#), except for the scalar dipole moment I_i^s , which is presented at 2PN order instead of 2.5PN (see the discussion therein). The full scalar dipole at 2.5PN order is presented for the first time in appendix F, and it features a nonlocal piece arising from the passage to the CM frame. I also provide all the source moments in machine-readable form in the supplemental material [58]. The equations of motion (both in a general frame and in the CM frame) are given at required order in [27], and I have provided these in a machine-readable form in the supplemental material [58] as well.

The expressions of the energy fluxes in terms of the orbital variables in the CM frame, namely $\mathbf{x} = \mathbf{m}$ and $\mathbf{v}$, were already provided at 1.5PN order in (4.10) and (C2) of [paper I](#), confirming the previously known 1PN result in the tensor sector ((6.5) of [32]) and correcting the 1PN result in the scalar sector ((6.7)–(8) of [32]). Here, I provide for the angular momentum fluxes at 1.5PN in terms of the orbital variables in the CM frame:

$$\begin{aligned}
\mathcal{G}_i^{\text{inst}} = & \frac{16(1-\zeta)m\nu^2}{5\alpha c^5} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 \epsilon_{ijk} n_j v_k \left\{ \frac{\tilde{G}\alpha m}{r} - \frac{3}{2} (nv)^2 + v^2 + \frac{1}{c^2} \left[\left(-\frac{745}{84} - 4\bar{\beta}_+ \right. \right. \right. \\
& \left. \left. - \frac{7}{2}\bar{\gamma} + 4\bar{\beta}_- \delta + \frac{1}{42}\nu \right) \left(\frac{\tilde{G}\alpha m}{r} \right)^2 + \left(\frac{31}{7} + 4\bar{\beta}_+ + \bar{\gamma} - 4\bar{\beta}_- \delta + \frac{197}{84}\nu \right) \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^2 \right. \\
& + \left(\frac{95}{56} - \frac{45}{7}\nu \right) (nv)^4 + \left(-\frac{29}{21} - 2\bar{\beta}_+ + 2\bar{\beta}_- \delta - \frac{95}{42}\nu \right) \left(\frac{\tilde{G}\alpha m}{r} \right) v^2 \\
& + \left(-\frac{37}{14} + \frac{277}{28}\nu \right) (nv)^2 v^2 + \left. \left(\frac{307}{168} + \frac{1}{2}\bar{\gamma} - \frac{137}{42}\nu \right) v^4 \right] \\
& + \frac{1}{c^3} \left[\left(\frac{\tilde{G}\alpha m}{r} \right)^2 \left(-\frac{1}{4}\bar{\gamma} - \zeta S_-^2 \nu \right) (nv) + \frac{5}{8}\bar{\gamma} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^3 - \frac{1}{2}\bar{\gamma} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv) v^2 \right] \Bigg\} \\
& + \mathcal{O}\left(\frac{1}{c^9}\right), \tag{6.8a}
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}_i^{s,\text{inst}} = & \frac{m\nu^2}{3c^3} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 \epsilon_{ijk} n_j v_k \left\{ \tilde{A}^{-1\text{PN}} + \frac{1}{c^2} \left[\tilde{B}_1^{\text{N}} \left(\frac{\tilde{G}\alpha m}{r} \right) + \tilde{B}_2^{\text{N}} (nv)^2 + \tilde{B}_3^{\text{N}} v^2 \right] + \frac{1}{c^3} \tilde{C}^{0.5\text{PN}} \right. \\
& \times \left(\frac{\tilde{G}\alpha m}{r} \right) (nv) + \frac{1}{c^4} \left[\tilde{D}_1^{1\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 + \tilde{D}_2^{1\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^2 + \tilde{D}_3^{1\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) v^2 \right. \\
& + \tilde{D}_4^{1\text{PN}} (nv)^4 + \tilde{D}_5^{1\text{PN}} (nv)^2 v^2 + \tilde{D}_6^{1\text{PN}} v^4 \left. \right] + \frac{1}{c^5} \left[\tilde{E}_1^{1.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 (nv) \right. \\
& + \tilde{E}_2^{1.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^3 + \tilde{E}_3^{1.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv) v^2 \left. \right] \left. \right\} + \mathcal{O} \left(\frac{1}{c^9} \right), \quad (6.8b)
\end{aligned}$$

where the coefficients associated to (6.8b) are given in appendix G. The tensor flux (6.8a) is in full agreement with (4.7) and (8) of [62] at 1.5PN, and the scalar flux (6.8b) is in full agreement with (5.00)–(15) of [62] at 1PN. The 1.5PN piece of $\mathcal{G}_i^{s,\text{inst}}$ was computed for the first time in this work. All these results can also be found in machine-readable form in the supplemental material [58].

Using the identities $(nv) = \dot{r}$, $v^2 = \dot{r}^2 + r^2 \dot{\phi}^2$, and $\epsilon_{ijk} n_j v_k = r \dot{\phi} \ell^i$, I then obtain the fluxes in terms of $(r, \phi, \dot{r}, \dot{\phi})$. Specializing to the case of quasielliptic orbits, I can then use the identities $\dot{r} = \mathcal{R}(r^{-1}, E, J)$ and $\dot{\phi} = \mathcal{S}(r^{-1}, E, J)$, given in (3.1) of paper II, along with the QK representation (3.1), to obtain the fluxes only in terms of the eccentric anomaly u and the QK parameters. When computing $\langle \mathcal{F}^{\text{inst}} \rangle$, $\langle \mathcal{G}^{\text{inst}} \rangle$, $\langle \mathcal{F}^{s,\text{inst}} \rangle$ and $\langle \mathcal{G}^{s,\text{inst}} \rangle$, all the QK parameters are first reexpressed in terms only of x and e_t [using (C1) of paper II], and one can then always use trigonometric relations to write the orbit averaged fluxes as a linear combinations of the following master integrals,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du}{(1 - e_t \cos u)^n} = \frac{1}{(1 - e_t^2)^{n/2}} P_{n-1} \left(\frac{1}{\sqrt{1 - e_t^2}} \right), \quad (6.9a)$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{du \sin u}{(1 - e_t \cos u)^n} = 0, \quad (6.9b)$$

where $n \geq 1$ and $P_n(x)$ is the n th Legendre polynomial. Finally, I find that the instantaneous fluxes read, for the tensor sector,

$$\begin{aligned}
\langle \mathcal{F}^{\text{inst}} \rangle = & \frac{32c^5 (1 + \bar{\gamma}/2) x^5 \nu^2}{5\tilde{G}\alpha} \left\{ \frac{1 + \frac{73}{24}e_t^2 + \frac{37}{96}e_t^4}{(1 - e_t^2)^{7/2}} + \frac{x}{(1 - e_t^2)^{9/2}} \left[-\frac{1247}{336} - \frac{8}{3}\bar{\beta}_+ - \frac{4}{3}\bar{\gamma} + \frac{8}{3}\bar{\beta}_- \delta \right. \right. \\
& - \frac{35}{12}\nu + e_t^2 \left(\frac{10475}{672} - \frac{571}{36}\bar{\beta}_+ + \frac{379}{36}\bar{\gamma} + \frac{571}{36}\bar{\beta}_- \delta - \frac{1081}{36}\nu \right) \\
& + e_t^4 \left(\frac{10043}{384} - \frac{1003}{144}\bar{\beta}_+ + \frac{955}{72}\bar{\gamma} + \frac{1003}{144}\bar{\beta}_- \delta - \frac{311}{12}\nu \right) \\
& \left. + e_t^6 \left(\frac{2179}{1792} + \frac{131}{288}\bar{\gamma} - \frac{851}{576}\nu \right) \right] + \mathcal{O}(x^2) \right\}, \quad (6.10a)
\end{aligned}$$

$$\begin{aligned}
\langle \mathcal{G}^{\text{inst}} \rangle = & \frac{32c^2 (1 + \bar{\gamma}/2) m\nu^2 x^{7/2}}{5} \left\{ \frac{1 + \frac{7}{8}e_t^2}{(1 - e_t^2)^2} + \frac{x}{(1 - e_t^2)^3} \left[-\frac{1247}{336} - \frac{8}{3}\bar{\beta}_+ - \frac{4}{3}\bar{\gamma} + \frac{8}{3}\bar{\beta}_- \delta \right. \right. \\
& - \frac{35}{12}\nu + e_t^2 \left(\frac{3019}{336} - \frac{35}{6}\bar{\beta}_+ + \frac{21}{4}\bar{\gamma} + \frac{35}{6}\bar{\beta}_- \delta - \frac{335}{24}\nu \right) \\
& \left. + e_t^4 \left(\frac{8399}{2688} - \frac{1}{4}\bar{\beta}_+ + \frac{67}{48}\bar{\gamma} + \frac{1}{4}\bar{\beta}_- \delta - \frac{275}{96}\nu \right) \right] + \mathcal{O}(x^2) \right\}, \quad (6.10b)
\end{aligned}$$

and are given in the scalar scalar by

$$\langle \mathcal{F}^{s,\text{inst}} \rangle = \frac{c^5 \nu^2 \zeta x^5}{3\bar{G}\alpha} \left(4\mathcal{S}_-^2 x^{-1} \frac{1 + \frac{1}{2}e_t^2}{(1-e_t^2)^{5/2}} + \frac{\mathfrak{f}_1 + \mathfrak{f}_2 e_t^2 + \mathfrak{f}_3 e_t^4}{(1-e_t^2)^{7/2}} + x \left[\frac{\mathfrak{f}_4 + \mathfrak{f}_5 e_t^2 + \mathfrak{f}_6 e_t^4 + \mathfrak{f}_7 e_t^6}{(1-e_t^2)^{9/2}} + \frac{\mathfrak{f}_8 + \mathfrak{f}_9 e_t^2 + \mathfrak{f}_{10} e_t^4}{(1-e_t^2)^4} \right] + \mathcal{O}(x^2) \right), \quad (6.11a)$$

$$\langle \mathcal{G}^{s,\text{inst}} \rangle = \frac{c^2 m \nu^2 \zeta x^{7/2}}{3} \left(\frac{4\mathcal{S}_-^2 x^{-1}}{1-e_t^2} + \frac{\mathfrak{g}_1^s + \mathfrak{g}_2^s e_t^2}{(1-e_t^2)^2} + x \left[\frac{\mathfrak{g}_3 + \mathfrak{g}_4 e_t^2 + \mathfrak{g}_5 e_t^4}{(1-e_t^2)^3} + \frac{\mathfrak{g}_6 + \mathfrak{g}_6 e_t^2 + \mathfrak{g}_7 e_t^4}{(1-e_t^2)^{7/2}} \right] + \mathcal{O}(x^2) \right), \quad (6.11b)$$

where the expressions of the coefficients $(\mathfrak{f}_n, \mathfrak{g}_n)$ are relegated to appendix D. These fluxes are also presented in machine-readable form the supplemental material [58].

6.3. The Π^s flux

As we know from section 5.3, one can decompose Π_i^s into an AC and a DC contribution. We propagate this decomposition to the flux, namely $\mathcal{G}_i^{s,\Pi^s} = \mathcal{G}_i^{s,\Pi_{\text{AC}}^s} + \mathcal{G}_i^{s,\Pi_{\text{DC}}^s}$ where

$$\mathcal{G}_i^{s,\Pi_{\text{AC}}^s} = \frac{G(3+2\omega_0)}{3c^3} \frac{\sqrt{\alpha}\zeta(\mathcal{S}_+ + \mathcal{S}_-\delta)}{(1-\zeta)} \epsilon_{ijk} \Pi_j^{s,\text{AC}} \mathbf{I}_k^{(2)s}, \quad (6.12a)$$

$$\mathcal{G}_i^{s,\Pi_{\text{DC}}^s} = \frac{G(3+2\omega_0)}{3c^3} \frac{\sqrt{\alpha}\zeta(\mathcal{S}_+ + \mathcal{S}_-\delta)}{(1-\zeta)} \epsilon_{ijk} \Pi_j^{s,\text{DC}} \mathbf{I}_k^{(2)s}. \quad (6.12b)$$

The orbit averaged DC piece of the flux vanishes, because

$$\langle \mathcal{G}_i^{s,\Pi_{\text{DC}}^s} \rangle \propto \epsilon_{ijk} \Pi_j^{s,\text{DC}} \langle \mathbf{I}_k^{(2)s} \rangle = 0 \quad (6.13)$$

where I have use the fact that Π_{DC}^s is approximately constant over an orbit and that time derivatives of multipolar moments vanish upon orbit averaging. Moreover, even before orbit averaging, this contribution is finite and vanishes for $e \rightarrow 0$.

Conversely, the AC flux is best expressed by replacing the multipolar moments by their expression as a Fourier series (see section 4). After performing explicitly the time integral in $\Pi_j^{s,\text{AC}}$, I explicitly find

$$\begin{aligned} \mathcal{G}_i^{s,\Pi_{\text{AC}}^s} &= \frac{2G^2\phi_0(3+2\omega_0)^2\sqrt{\alpha}\zeta(\mathcal{S}_+ + \mathcal{S}_-\delta)}{9(1-\zeta)c^8} \epsilon_{ijk} \sum_{r \in \mathbb{Z}} \sum_{p+q \neq 0} e^{i(p+q+r)\ell} \frac{(im)^2}{i(p+q)n} \left({}_r\tilde{\mathbf{I}}_k^s \right) \\ &\times \left[\frac{(ipn)(iqn)^2}{\phi_0} \left({}_p\tilde{E}^s \right) \left({}_q\tilde{\mathbf{I}}_j^s \right) + \frac{(ipn)^2(iqn)^3}{5} \left({}_p\tilde{\mathbf{I}}_a^s \right) \left({}_q\tilde{\mathbf{I}}_{ja}^s \right) \right]. \end{aligned} \quad (6.14)$$

Orbit averaging the previous equation is quite simple thanks to the identity $\langle e^{i(p+q+r)\ell} \rangle = \delta_{p+q+r}^0$, where δ_b^a is the Kronecker symbol. I then obtain

$$\begin{aligned} \langle \mathcal{G}_i^{s, \Pi_{AC}^s} \rangle &= \frac{2G^2 \phi_0 (3 + 2\omega_0)^2 \sqrt{\alpha} \zeta (\mathcal{S}_+ + \mathcal{S}_- \delta)}{9(1 - \zeta) c^8} \\ &\times \epsilon_{ijk} \sum_{p+q \neq 0} \left[\frac{n^4 p q^2 (p+q)}{\phi_0} \left(\tilde{p} E^s \right) \left(\tilde{q} \tilde{I}_j^s \right) \left(-p-q \tilde{I}_k^s \right) \right. \\ &\quad \left. - \frac{n^6 p^2 q^3 (p+q)}{5} \left(\tilde{p} \tilde{I}_a^s \right) \left(\tilde{q} \tilde{I}_{ja}^s \right) \left(-p-q \tilde{I}_k^s \right) \right]. \end{aligned} \quad (6.15)$$

Since the motion is planar, only the z component of the angular momentum flux is nonvanishing, and a simple analysis involving the explicit expressions of the Fourier coefficients (given by equations (4.7)) leads to the conclusion that $\langle \mathcal{G}_z^{s, \Pi_{AC}^s} \rangle \in i\mathbb{R}$. But of course, it is also true that $\langle \mathcal{G}_z^{s, \Pi_{AC}^s} \rangle \in \mathbb{R}$, which implies that

$$\langle \mathcal{G}_i^{s, \Pi_{AC}^s} \rangle = 0. \quad (6.16)$$

In summary, I have found that the Π^s contribution to the angular momentum flux entire vanishes after orbit averaging, namely

$$\langle \mathcal{G}_i^{s, \Pi^s} \rangle = 0. \quad (6.17)$$

6.4. The tail fluxes

The tail fluxes, given by (6.1), feature the difficulty of being nonlocal in time with a logarithmic kernel. These are best treated [90, 91, 99] by replacing all multipolar moments by their Fourier decomposition, as given in section 4. Once this replacement is performed, the nonlocal integrals can all be performed using the master formula [90, 91]

$$\begin{aligned} \int_0^{+\infty} d\tau e^{i(p+rk)n\tau} \ln \left(\frac{c\tau}{2b_0} \right) &= -\frac{1}{pn} \left(1 - \frac{rk}{p} \right) \left[\frac{\pi}{2} \text{sg}(p) + i\gamma_E \right. \\ &\quad \left. + i \ln \left(\frac{2|p|nb_0}{c} \right) \right] - \frac{irk}{p^2 n} + \mathcal{O}(k^2), \end{aligned} \quad (6.18)$$

which simplifies straightforwardly at Newtonian order (i.e. in the $k \rightarrow 0$ limit). Note that the case $p=0$ never occurs, because the multipolar moments in the nonlocal integrals always bear enough time derivatives, which is equivalent to multiplying the Fourier coefficients by some power of inp . The flux can be written as a sum over the relative integers p, q, r and s , where each term is proportional to $e^{i\ell[(r+s)k+p+q]}$ [$r+s=0$ is allowed]. It is then straightforward to perform the time averaging thanks to the formula [90, 91]

$$\langle e^{i\ell(p+rk)} \rangle = \begin{cases} 1 + i\pi rk & \text{if } p=0 \\ \frac{r}{p} k & \text{if } p \neq 0 \end{cases} + \mathcal{O}(k^2). \quad (6.19)$$

After some manipulating some sums, I find as expected that the flux is real-valued, and that all arbitrary constants b_0 vanish from the final result, which read

$$\langle \mathcal{F}^{\text{tail}} \rangle = \frac{4\pi G^2 M n^7}{5c^8} (\mathcal{I}_2)^2 \rho_2(e_t), \quad (6.20a)$$

$$\langle \mathcal{G}_i^{\text{tail}} \rangle = -\frac{8i\pi G^2 M n^6}{5c^8} (\mathcal{I}_2)^2 \tilde{\rho}_2(e_t), \quad (6.20b)$$

$$\begin{aligned} \langle \mathcal{F}^{s,\text{tail}} \rangle = & \frac{4\pi G^2 M (3+2\omega_0)}{c^6} \left\{ \frac{n^5}{3} (\mathcal{I}_1^s)^2 [\rho_1^s(e_t) + 5k\sigma_1^s(e_t)] + \frac{n^7}{30c^2} (\mathcal{I}_2^s)^2 \rho_2^s(e_t) \right. \\ & \left. + \frac{n^3}{\phi_0^2 c^2} (\mathcal{E}_0^s)^2 \rho_0^s(e_t) \right\}, \end{aligned} \quad (6.20c)$$

$$\langle \mathcal{G}_i^{s,\text{tail}} \rangle = -\frac{4i\pi G^2 M (3+2\omega_0)}{c^6} \left\{ \frac{n^4}{3} (\mathcal{I}_1^s)^2 [\tilde{\rho}_1^s(e_t) + 4k\tilde{\sigma}_1^s(e_t)] + \frac{n^6}{15c^2} (\mathcal{I}_2^s)^2 \tilde{\rho}_2^s(e_t) \right\}, \quad (6.20d)$$

where I have introduced in this intermediate step some *unnormalized enhancement functions*, defined by

$$\begin{aligned} \rho_2(e_t) &= \sum_{p=1}^{\infty} p^7 \left(\widehat{\mathbf{I}}_{ij} \right) \left(\widehat{\mathbf{I}}_{ij} \right)^*, & \tilde{\rho}_2(e_t) &= \epsilon_{ijk} \sum_{p=1}^{\infty} p^6 \left(\widehat{\mathbf{I}}_{ja} \right) \left(\widehat{\mathbf{I}}_{ka} \right)^*, \\ \rho_2^s(e_t) &= \sum_{p=1}^{\infty} p^7 \left(\widehat{\mathbf{I}}_{ij}^s \right) \left(\widehat{\mathbf{I}}_{ij}^s \right)^*, & \tilde{\rho}_2^s(e_t) &= \epsilon_{ijk} \sum_{p=1}^{\infty} p^6 \left(\widehat{\mathbf{I}}_{ja}^s \right) \left(\widehat{\mathbf{I}}_{ka}^s \right)^*, \\ \rho_1^s(e_t) &= \sum_{p=1}^{\infty} p^5 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left(\widehat{\mathbf{I}}_i^s \right) \left(\widehat{\mathbf{I}}_i^s \right)^*, & \tilde{\rho}_1^s(e_t) &= \epsilon_{ijk} \sum_{p=1}^{\infty} p^4 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left(\widehat{\mathbf{I}}_j^s \right) \left(\widehat{\mathbf{I}}_k^s \right)^*, \\ \sigma_1^s(e_t) &= \sum_{p=1}^{\infty} p^4 \sum_{m \in \{-1,1\}} m \left(\widehat{\mathbf{I}}_i^s \right) \left(\widehat{\mathbf{I}}_i^s \right)^*, & \tilde{\sigma}_1^s(e_t) &= \epsilon_{ijk} \sum_{p=1}^{\infty} p^3 \sum_{m \in \{-1,1\}} m \left(\widehat{\mathbf{I}}_j^s \right) \left(\widehat{\mathbf{I}}_k^s \right)^*, \\ \rho_0^s(e_t) &= \sum_{p=1}^{\infty} p^3 \left(\widehat{\mathbf{E}}^s \right) \left(\widehat{\mathbf{E}}^s \right)^*. \end{aligned} \quad (6.21)$$

It will prove useful to decompose the dipolar unnormalized enhancement functions as

$$\begin{aligned} \rho_1^s(e_t) &= \rho_1^{s,00}(e_t) + x \left(\rho_1^{s,01}(e_t) + \nu \rho_1^{s,11}(e_t) \right) \\ \tilde{\rho}_1^s(e_t) &= \tilde{\rho}_1^{s,00}(e_t) + x \left(\tilde{\rho}_1^{s,01}(e_t) + \nu \tilde{\rho}_1^{s,11}(e_t) \right) \end{aligned}$$

where

$$\rho_1^{s,00}(e_t) = \sum_{p=1}^{\infty} p^5 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left(\widehat{\mathbf{I}}_i^{s,00} \right) \left(\widehat{\mathbf{I}}_i^{s,00} \right)^* \quad (6.22a)$$

$$\tilde{\rho}_1^{s,00}(e_t) = \epsilon_{ijk} \sum_{p=1}^{\infty} p^4 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left(\widehat{\mathbf{I}}_j^{s,00} \right) \left(\widehat{\mathbf{I}}_k^{s,00} \right)^*, \quad (6.22b)$$

$$\rho_1^{s,01}(e_t) = \sum_{p=1}^{\infty} p^5 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left[\left(\widehat{\mathbf{I}}_i^{s,00} \right) \left(\widehat{\mathbf{I}}_i^{s,01} \right)^* + \left(\widehat{\mathbf{I}}_i^{s,01} \right) \left(\widehat{\mathbf{I}}_i^{s,00} \right)^* \right], \quad (6.22c)$$

$$\tilde{\rho}_1^{s,01}(e_t) = \epsilon_{ijk} \sum_{p=1}^{\infty} p^4 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left[\left(\widehat{m}_{\Gamma_j}^{s,00} \right) \left(\widehat{m}_{\Gamma_k}^{s,01} \right)^* + \left(\widehat{m}_{\Gamma_j}^{s,01} \right) \left(\widehat{m}_{\Gamma_k}^{s,00} \right)^* \right], \quad (6.22d)$$

$$\rho_1^{s,11}(e_t) = \sum_{p=1}^{\infty} p^5 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left[\left(\widehat{m}_{\Gamma_i}^{s,00} \right) \left(\widehat{m}_{\Gamma_i}^{s,11} \right)^* + \left(\widehat{m}_{\Gamma_i}^{s,11} \right) \left(\widehat{m}_{\Gamma_i}^{s,00} \right)^* \right], \quad (6.22e)$$

$$\tilde{\rho}_1^{s,11}(e_t) = \epsilon_{ijk} \sum_{p=1}^{\infty} p^4 \sum_{\substack{m \in \{-1,1\} \\ s \in \{-1,1\}}} \left[\left(\widehat{m}_{\Gamma_j}^{s,00} \right) \left(\widehat{m}_{\Gamma_k}^{s,11} \right)^* + \left(\widehat{m}_{\Gamma_j}^{s,11} \right) \left(\widehat{m}_{\Gamma_k}^{s,00} \right)^* \right]. \quad (6.22f)$$

The limits of these unnormalized enhancement functions when $e_t \rightarrow 0$ are straightforward to compute, and read:

$$\begin{aligned} \rho_2(0) &= 32, & \rho_2^s(0) &= 32, & \rho_1^{s,00}(0) &= \frac{1}{2} & \rho_1^{s,01}(0) &= -\frac{9}{5}\mathcal{A}, & \rho_1^{s,11}(0) &= \frac{14}{15}, \\ \sigma_1^s(0) &= \frac{1}{2}, & \rho_0^s(0) &= 0, \\ \tilde{\rho}_2(0) &= 16i, & \tilde{\rho}_2^s(0) &= 16i, & \tilde{\rho}_1^{s,00}(0) &= \frac{i}{2} & \tilde{\rho}_1^{s,01}(0) &= -\frac{9i}{5}\mathcal{A}, & \tilde{\rho}_1^{s,11}(0) &= \frac{14i}{15}, \\ \tilde{\sigma}_1^s(0) &= \frac{i}{2}, \end{aligned} \quad (6.23)$$

where I have introduced the recurring parameter

$$\begin{aligned} \mathcal{A} \equiv & 1 + \frac{5}{27}\bar{\gamma} + \frac{10}{27}\bar{\beta}_+ - \frac{10}{9}\frac{\bar{\beta}_+}{\bar{\gamma}} - \frac{10}{9}\frac{\bar{\beta}_-\mathcal{S}_+}{\bar{\gamma}\mathcal{S}_-} \\ & + \delta \left[-\frac{10}{27}\bar{\beta}_- + \frac{10}{9}\frac{\bar{\beta}_-}{\bar{\gamma}} - \frac{4}{9}\frac{\mathcal{S}_+}{\mathcal{S}_-} + \frac{10}{9}\frac{\bar{\beta}_+\mathcal{S}_+}{\bar{\gamma}\mathcal{S}_-} \right]. \end{aligned} \quad (6.24)$$

It will prove useful to note that $\rho_0^s(e_t) \sim e_t/4$ as $e_t \rightarrow 0$. I then express equations (6.20) only in terms of the pair of variable x and e_t . For this, I use the 1PN QK parametrization [47] to expand the ADM mass, the mean motion and the periastron advance:

$$M = m \left(1 - \frac{x\nu}{2} + \mathcal{O}(x^2) \right), \quad (6.25a)$$

$$n = \frac{c^3 x^{3/2}}{\tilde{G}\alpha m} \left(1 - \frac{x\mathcal{B}}{1 - e_t^2} + \mathcal{O}(x^2) \right), \quad (6.25b)$$

$$k = \frac{x\mathcal{B}}{1 - e_t^2} + \mathcal{O}(x^2), \quad (6.25c)$$

and where the reader is reminded that $\mathcal{B}$ was defined in equation (3.6).

The expression for the fluxes thus obtained can finally be written similarly to the fluxes for circular orbits, in which some extra *normalized* enhancement functions of the eccentricity have been inserted. In the tensor sector, they are given by

$$\langle \mathcal{F}^{\text{tail}} \rangle = \frac{32c^5 x^5 \nu^2 (1 + \bar{\gamma}/2)}{5\tilde{G}\alpha} \times 4\pi (1 + \bar{\gamma}/2) x^{3/2} \varphi_2(e_t) + \mathcal{O}(x^7), \quad (6.26a)$$

$$\langle \mathcal{G}^{\text{tail}} \rangle = \frac{32c^2 x^{7/2} m \nu^2 (1 + \bar{\gamma}/2)}{5} \times 4\pi (1 + \bar{\gamma}/2) x^{3/2} \tilde{\varphi}_2(e_t) + \mathcal{O}(x^7), \quad (6.26b)$$

where the normalized enhancement functions read

$$\varphi_2(e_t) = \frac{1}{32}\rho_2(e_t) \quad \text{and} \quad \tilde{\varphi}_2(e_t) = -\frac{i}{16}\tilde{\rho}_2(e_t), \quad (6.27)$$

where the reader is reminded that the unnormalized enhancement functions are given by (6.21).

Similarly, the scalar sector is given by

$$\begin{aligned} \langle \mathcal{F}^{s,\text{tail}} \rangle = & \frac{c^5 x^5 \nu^2 \zeta}{3\tilde{G}\alpha} \times 4\pi (1 + \bar{\gamma}/2) \sqrt{x} \left\{ 2\mathcal{S}_-^2 \varphi_1^s(e_t) \right. \\ & + x \left[\frac{8}{5} \varphi_2^s(e_t) (4\mathcal{S}_-^2 - 4\mathcal{S}_- \mathcal{S}_+ \delta - \bar{\gamma} \zeta^{-1} - 8\mathcal{S}_-^2 \nu) + \frac{41}{15} \mathcal{S}_-^2 \nu \theta_1^s(e_t) \right. \\ & + \frac{4}{15} \alpha_1^s(e_t) (-27\mathcal{S}_-^2 - 10\bar{\beta}_+ \mathcal{S}_-^2 + 30\bar{\beta}_+ \mathcal{S}_-^2 \bar{\gamma}^{-1} - 5\bar{\gamma} \mathcal{S}_-^2 + 30\bar{\beta}_- \mathcal{S}_- \mathcal{S}_+ \bar{\gamma}^{-1} \\ & + \delta [10\bar{\beta}_- \mathcal{S}_-^2 - 30\bar{\beta}_- \mathcal{S}_-^2 \bar{\gamma}^{-1} + 12\mathcal{S}_- \mathcal{S}_+ - 30\bar{\beta}_+ \mathcal{S}_- \mathcal{S}_+ \bar{\gamma}^{-1}]) \\ & + \frac{1}{3} e_t^2 \varphi_0^s(e_t) (48\bar{\beta}_+ \zeta^{-1} - 72\bar{\beta}_+^2 \zeta^{-1} \bar{\gamma}^{-1} - 8\bar{\gamma} \zeta^{-1} + 17\mathcal{S}_-^2 + 144\bar{\beta}_-^2 \mathcal{S}_-^2 \bar{\gamma}^{-2} \\ & + 144\bar{\beta}_+^2 \mathcal{S}_-^2 \bar{\gamma}^{-2} - 96\bar{\beta}_+ \mathcal{S}_-^2 \bar{\gamma}^{-1} + 288\bar{\beta}_- \bar{\beta}_+ \mathcal{S}_- \mathcal{S}_+ \bar{\gamma}^{-2} - 96\bar{\beta}_- \mathcal{S}_- \mathcal{S}_+ \bar{\gamma}^{-1} \\ & \left. \left. + \delta [24\bar{\beta}_- \mathcal{S}_-^2 \bar{\gamma}^{-1} - 8\mathcal{S}_- \mathcal{S}_+ + 24\bar{\beta}_+ \mathcal{S}_- \mathcal{S}_+ \bar{\gamma}^{-1}] - 4\mathcal{S}_-^2 \nu) \right] \right\} + \mathcal{O}(x^7), \quad (6.28a) \end{aligned}$$

$$\begin{aligned} \langle \mathcal{G}^{s,\text{tail}} \rangle = & \frac{c^2 x^{7/2} m \nu^2 \zeta}{3} \times 4\pi (1 + \bar{\gamma}/2) \sqrt{x} \left\{ 2\mathcal{S}_-^2 \tilde{\varphi}_1^s(e_t) \right. \\ & + x \left[\frac{8}{5} \tilde{\varphi}_2^s(e_t) (4\mathcal{S}_-^2 - 4\mathcal{S}_- \mathcal{S}_+ \delta - \bar{\gamma} \zeta^{-1} - 8\mathcal{S}_-^2 \nu) + \frac{41}{15} \mathcal{S}_-^2 \nu \tilde{\theta}_1^s(e_t) \right. \\ & + \frac{4}{15} \tilde{\alpha}_1^s(e_t) (-27\mathcal{S}_-^2 - 10\bar{\beta}_+ \mathcal{S}_-^2 + 30\bar{\beta}_+ \mathcal{S}_-^2 \bar{\gamma}^{-1} - 5\bar{\gamma} \mathcal{S}_-^2 + 30\bar{\beta}_- \mathcal{S}_- \mathcal{S}_+ \bar{\gamma}^{-1} \\ & + \delta [10\bar{\beta}_- \mathcal{S}_-^2 - 30\bar{\beta}_- \mathcal{S}_-^2 \bar{\gamma}^{-1} + 12\mathcal{S}_- \mathcal{S}_+ - 30\bar{\beta}_+ \mathcal{S}_- \mathcal{S}_+ \bar{\gamma}^{-1}]) \left. \right] \left. \right\} + \mathcal{O}(x^7), \quad (6.28b) \end{aligned}$$

where the enhancement functions read

$$\begin{aligned} \varphi_0^s(e_t) &= \frac{4\rho_0^s(e_t)}{e_t^2}, \\ \varphi_2^s(e_t) &= \frac{1}{32}\rho_2^s(e_t), & \tilde{\varphi}_2^s(e_t) &= -\frac{i}{16}\tilde{\rho}_2^s(e_t), \\ \varphi_1^s(e_t) &= 2\rho_1^{s,00}(e_t), & \tilde{\varphi}_1^s(e_t) &= -2i\tilde{\rho}_1^{s,00}(e_t), \\ \alpha_1^s(e_t) &= -\frac{5}{9\mathcal{A}} \left[\rho_1^{s,01}(e_t) + \frac{5\mathcal{B}}{1-e_t^2} \right. \\ & \quad \left. \times \left(\sigma_1^s(e_t) - \rho_1^{s,00}(e_t) \right) \right], & \tilde{\alpha}_1^s(e_t) &= \frac{5i}{9\mathcal{A}} \left[\tilde{\rho}_1^{s,01}(e_t) + \frac{4\mathcal{B}}{1-e_t^2} \right. \\ & \quad \left. \times \left(\tilde{\sigma}_1^s(e_t) - \tilde{\rho}_1^{s,00}(e_t) \right) \right], \\ \theta_1^s(e_t) &= -\frac{30}{41} \left(\rho_1^{s,00}(e_t) - 2\rho_1^{s,11}(e_t) \right), & \tilde{\theta}_1^s(e_t) &= \frac{30i}{41} \left(\tilde{\rho}_1^{s,00}(e_t) - 2\tilde{\rho}_1^{s,11}(e_t) \right), \quad (6.29) \end{aligned}$$

and where the reader is reminded that the unnormalized enhancement functions are given by (6.21). The normalized enhancement functions are defined such that they tend to 1 when $e_t \rightarrow 0$. I chose to normalize it such that $\varphi_0^s(e_t) \sim e_t^2$ when $e_t \rightarrow 0$. Thus, taking the $e_t \rightarrow 0$ limit and setting all enhancement functions to 1 in (6.26) and (6.28) immediately yields the

expressions for the tail sector of the circular energy flux. For the energy fluxes, it can be verified that the circular limit for the tail sector agrees with the odd PN-order terms in (5.3) of [paper I](#). Moreover, it is immediate to verify that in the circular limit, $\langle \mathcal{F}_{\text{circ}}^{\text{tail}} \rangle = \omega \langle \mathcal{G}_{\text{circ}}^{\text{tail}} \rangle$ and $\langle \mathcal{F}_{\text{circ}}^{s,\text{tail}} \rangle = \omega \langle \mathcal{G}_{\text{circ}}^{s,\text{tail}} \rangle$.

Of particular interest will be the small-eccentricity expansions of these enhancement functions:

$$\varphi_2(e_t) = 1 + \frac{2335}{192}e_t^2 + \frac{42955}{768}e_t^4 + \frac{6204647}{36864}e_t^6 + \frac{352891481}{884736}e_t^8 + \frac{286907786543}{353894400}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30a)$$

$$\tilde{\varphi}_2(e_t) = 1 + \frac{209}{32}e_t^2 + \frac{2415}{128}e_t^4 + \frac{730751}{18432}e_t^6 + \frac{10355719}{147456}e_t^8 + \frac{6594861233}{58982400}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30b)$$

$$\varphi_2^s(e_t) = 1 + \frac{2335}{192}e_t^2 + \frac{42955}{768}e_t^4 + \frac{6204647}{36864}e_t^6 + \frac{352891481}{884736}e_t^8 + \frac{286907786543}{353894400}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30c)$$

$$\tilde{\varphi}_2^s(e_t) = 1 + \frac{209}{32}e_t^2 + \frac{2415}{128}e_t^4 + \frac{730751}{18432}e_t^6 + \frac{10355719}{147456}e_t^8 + \frac{6594861233}{58982400}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30d)$$

$$\varphi_1^s(e_t) = 1 + 7e_t^2 + \frac{717}{32}e_t^4 + \frac{7435}{144}e_t^6 + \frac{7305575}{73728}e_t^8 + \frac{103947697}{614400}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30e)$$

$$\tilde{\varphi}_1^s(e_t) = 1 + 3e_t^2 + \frac{179}{32}e_t^4 + \frac{1249}{144}e_t^6 + \frac{99715}{8192}e_t^8 + \frac{29573111}{1843200}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30f)$$

$$\alpha_1^s(e_t) = 1 + \frac{1}{\mathcal{A}^2} \sum_{n=2}^{10} C_n e_t^n + \mathcal{O}(e_t^2), \quad (6.30g)$$

$$\tilde{\alpha}_1^s(e_t) = 1 + \frac{1}{\mathcal{A}^2} \sum_{n=2}^{10} \tilde{C}_n e_t^n + \mathcal{O}(e_t^2), \quad (6.30h)$$

$$\theta_1^s(e_t) = 1 + \frac{317}{41}e_t^2 + \frac{36837}{1312}e_t^4 + \frac{214765}{2952}e_t^6 + \frac{469593535}{3022848}e_t^8 + \frac{7363736527}{25190400}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30i)$$

$$\tilde{\theta}_1^s(e_t) = 1 + \frac{153}{41}e_t^2 + \frac{11179}{1312}e_t^4 + \frac{46327}{2952}e_t^6 + \frac{25733108}{1007616}e_t^8 + \frac{2895685801}{75571200}e_t^{10} + \mathcal{O}(e_t^{12}), \quad (6.30j)$$

$$\varphi_0^s(e_t) = 1 + \frac{31}{4}e_t^2 + \frac{2771}{96}e_t^4 + \frac{177151}{2304}e_t^6 + \frac{62037619}{368640}e_t^8 + \mathcal{O}(e_t^{10}), \quad (6.30k)$$

where $\mathcal{A}$ is given in (6.24), and the coefficients C_n and $\tilde{C}_n$ are given explicitly in [appendix C](#) (they depend in a complicated manner on the scalar-tensor parameters and are therefore very lengthy).

Although these expansions are formally only valid for small eccentricities, they can be resummed by factoring out $(1 - e_t^2)^{-N/2}$, where N is an integer to be determined for each enhancement function. This determination can either be done heuristically, by comparison to the numerically-evaluated function, or using asymptotic analysis in the lines of IV.C. of [\[103\]](#)

(see also [91] for an alternative asymptotic analysis). Despite their simplicity, these resummations can dramatically improve the accuracy of the approximation for high eccentricities.

6.5. The memory flux

Finally, I tackle the case of the memory integral, given by (6.4). Such an integral appears in GR for the first time at 2.5PN, and was first studied in [59]. The latter work found that the magnitude of the memory effect was infinite in the limit where the initial eccentricity of the system is 1, i.e. in the limit where the binary was formed in the infinite past [see their equation (A11)]. To mitigate this problem, the authors then invoke the astrophysical argument that in practice, the binary must have formed by capture with some initial eccentricity. Still, this suggests that the waveform is dominated by the remote past, and the zero-eccentricity limit is not smooth. That method was recently adapted to ST theories in [62], whose equation (4.19) also diverges⁸ as the initial eccentricity goes to 1.

When reexamining this problem, I found that these aforementioned divergences are cured by a careful split between AC and DC pieces. I will now describe in detail the full treatment of the memory integral in the case of ST theory, and the complete treatment of the GR case is relegated to appendix A.

For a nonprecessing orbit, the direction of the angular momentum flux is constant and points in the z -direction. It is then straightforward to explicitly express its norm $\mathcal{G} = |\vec{\mathcal{G}}| = \mathcal{G}_z$ in terms of the (x, y, z) components, namely

$$\mathcal{G}^{\text{mem}}(u) = \frac{G^2 \phi_0 (3 + 2\omega_0)}{15c^8} \left\{ \left(I_{xx}^{(3)}(u) - I_{yy}^{(3)}(u) \right) \int_0^\infty d\tau f(u - \tau) - I_{xy}^{(3)}(u) \int_0^\infty d\tau g(u - \tau) \right\}, \quad (6.31)$$

where I have used the fact that $I_{xz} = I_{yz} = 0$ and defined

$$f = I_x^{(2)s} I_y^{(2)s} \quad \text{and} \quad g = I_x^{(2)s} I_x^{(2)s} - I_y^{(2)s} I_y^{(2)s}. \quad (6.32)$$

In order to distinguish the AC and DC contributions, I decompose these functions into orbit-averaged and zero-average contributions, namely $f = \langle f \rangle + \tilde{f}$ and $g = \langle g \rangle + \tilde{g}$. Consequently, the functions $\tilde{f}$ and $\tilde{g}$ contain purely oscillatory terms which average out to zero over a single orbit. Thus, it is now possible to define precisely the AC and DC contributions:

$$\mathcal{G}_{\text{DC}}^{\text{mem}}(u) = \frac{G^2 \phi_0 (3 + 2\omega_0)}{15c^8} \left\{ \left(I_{xx}^{(3)}(u) - I_{yy}^{(3)}(u) \right) \int_0^\infty d\tau \langle f \rangle(u - \tau) - I_{xy}^{(3)}(u) \int_0^\infty d\tau \langle g \rangle(u - \tau) \right\}, \quad (6.33a)$$

⁸ In (4.19) of [62], replace a by its expression in terms of e given by (5.31b), and perform the change of variables $t \rightarrow e$ using (5.31a). The integral then reads (for $\kappa \equiv (\tilde{G}\alpha m)^2 \zeta \mathcal{S}_-^2 \nu / c^3$):

$$\begin{aligned} \int_{t_1}^{t_0} dt \frac{e^2 (e^2 + 6)}{a^4 (e^2 - 1)^4} &= -\frac{1}{2\kappa c_0} \int_{e_1}^{e_0} de \frac{6 + e^2}{(1 - e^2)^{3/2} e^{1/3}} = -\frac{1}{2\kappa c_0} \left[e^{2/3} \left\{ \frac{7}{\sqrt{1 - e^2}} + 2F\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; e^2\right) \right\} \right]_{e_1}^{e_0} \\ &\sim \frac{7}{2c_0 \kappa} \frac{1}{\sqrt{1 - e_1^2}} \quad \text{as } e_1 \rightarrow 1. \end{aligned}$$

$$\mathcal{G}_{\text{AC}}^{\text{mem}}(u) = \frac{G^2 \phi_0 (3 + 2\omega_0)}{15c^8} \left\{ \left(\mathbf{I}_{xx}^{(3)}(u) - \mathbf{I}_{yy}^{(3)}(u) \right) \int_0^\infty d\tau \tilde{f}(u - \tau) - \mathbf{I}_{xy}^{(3)}(u) \int_0^\infty d\tau \tilde{g}(u - \tau) \right\}. \quad (6.33b)$$

6.5.1. The AC part. First, I will focus on the AC terms. Replacing the components of the quadrupole moments by their Fourier series following the notations of section 4, I find that

$$f = n^4 \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} e^{i(p+q)\ell} p^2 q^2 {}_p\tilde{\mathbf{I}}_x^s {}_p\tilde{\mathbf{I}}_y^s, \quad (6.34a)$$

$$g = n^4 \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} e^{i(p+q)\ell} p^2 q^2 \left({}_p\tilde{\mathbf{I}}_x^s {}_p\tilde{\mathbf{I}}_x^s - {}_p\tilde{\mathbf{I}}_y^s {}_p\tilde{\mathbf{I}}_y^s \right). \quad (6.34b)$$

By definition of the orbital average, the DC component exactly corresponds to the terms for which $p + q = 0$, so I can now simply reexpress the AC term as

$$\tilde{f} = n^4 \sum_{p+q \neq 0} e^{i(p+q)\ell} p^2 q^2 {}_p\tilde{\mathbf{I}}_x^s {}_p\tilde{\mathbf{I}}_x^s, \quad (6.35a)$$

$$\tilde{g} = n^4 \sum_{p+q \neq 0} e^{i(p+q)\ell} p^2 q^2 \left({}_p\tilde{\mathbf{I}}_x^s {}_p\tilde{\mathbf{I}}_x^s - {}_p\tilde{\mathbf{I}}_y^s {}_p\tilde{\mathbf{I}}_y^s \right). \quad (6.35b)$$

Replacing all moments by their expressions as Fourier series (4.1), replacing the normalizations of the Fourier coefficients by their expressions (4.6), performing the time integrals explicitly, and substituting at leading order $n = c^3 x^{3/2} / (\tilde{G}\alpha m)$, I readily obtain

$$\mathcal{G}_{\text{AC}}^{\text{mem}} = -\frac{4}{15} \mathcal{S}_{-}^2 \zeta (1 + \bar{\gamma}/2) m \nu^3 c^2 x^5 \sum_{r \in \mathbb{Z}} \sum_{p+q \neq 0} e^{i(r+p+q)\ell} \frac{p^2 q^2 r^3}{p+q} \mathcal{S}_{r,p,q}, \quad (6.36)$$

where

$$\mathcal{S}_{r,p,q} = \left({}_r\hat{\mathbf{I}}_{xx} - {}_r\hat{\mathbf{I}}_{yy} \right) {}_p\hat{\mathbf{I}}_x^s {}_q\hat{\mathbf{I}}_y^s - {}_r\hat{\mathbf{I}}_{xy} \left({}_p\hat{\mathbf{I}}_x^s {}_q\hat{\mathbf{I}}_x^s - {}_p\hat{\mathbf{I}}_y^s {}_q\hat{\mathbf{I}}_y^s \right). \quad (6.37)$$

Here, I expressed $\mathcal{S}_{r,p,q}$ in terms of the normalized Fourier coefficients, given by (4.7). Thanks to the latter expressions, it is immediate to see that $\mathcal{S}_{r,p,q} \in i\mathbb{R}$. Since $\mathcal{G}_{\text{AC}}^{\text{mem}} \in \mathbb{R}$ by construction, taking the real part of (6.36) yields the explicitly real-valued expression:

$$\mathcal{G}_{\text{AC}}^{\text{mem}} = -\frac{4}{15} \mathcal{S}_{-}^2 \zeta (1 + \bar{\gamma}/2) m \nu^3 c^2 x^5 \sum_{r \in \mathbb{Z}} \sum_{p+q \neq 0} \sin((r+p+q)\ell) \frac{p^2 q^2 r^3}{p+q} i \mathcal{S}_{r,p,q}. \quad (6.38)$$

In order to compute the orbit-average of $\mathcal{G}_{\text{AC}}^{\text{mem}}$, it is in fact more practical to start from the expression (6.36), and to find

$$\langle \mathcal{G}_{\text{AC}}^{\text{mem}} \rangle = \frac{4}{15} \mathcal{S}_{-}^2 \zeta (1 + \bar{\gamma}/2) m \nu^3 c^2 x^5 \sum_{p+q \neq 0} p^2 q^2 (p+q)^2 \mathcal{S}_{-p-q,p,q}. \quad (6.39)$$

Since $\mathcal{S}_{-p-q,p,q}$ is purely imaginary, it follows that $\langle \mathcal{G}_{\text{AC}}^{\text{mem}} \rangle \in i\mathbb{R}$, but of course it is also true by construction that $\langle \mathcal{G}_{\text{AC}}^{\text{mem}} \rangle \in \mathbb{R}$. All this implies that, necessarily,

$$\langle \mathcal{G}_{\text{AC}}^{\text{mem}} \rangle = 0. \quad (6.40)$$

6.5.2. The DC part. I now turn to the DC part. Thanks to the explicit expression of the quadrupole moment and upon careful computation of the orbital average, I obtain $\langle f \rangle = 0$ and

$$\langle g \rangle = \frac{\tilde{G}^2 \alpha^3 \zeta^2 \mathcal{S}_-^2 m^4 \nu^2}{(1 - \zeta)^2 \phi_0^2} \times \frac{e^2}{a^4 (1 - e^2)^{5/2}}. \quad (6.41)$$

Thanks to the equations (5.31), I can reexpress the nonvanishing memory integral as an integral over eccentricity. In the far past, the eccentricity asymptotes towards one, so the memory integral reads

$$\int_0^\infty d\tau \langle g \rangle (u - \tau) = \frac{c^3 \alpha \zeta m^2 \nu}{2(1 - \zeta)^2 \phi_0^2 c_0} \int_e^1 \frac{d\tilde{e}}{\tilde{e}^{1/3}} = \frac{3c^3 \alpha \zeta m^2 \nu}{4(1 - \zeta)^2 \phi_0^2 c_0} (1 - e^{2/3}). \quad (6.42)$$

Finally, plugging (6.42) into (6.33a) leads to

$$\mathcal{G}_{\text{DC}}^{\text{mem}} = - \frac{(\tilde{G} \alpha m)^2 \nu \phi_0 (1 + \bar{\gamma}/2)}{20c^5} \frac{e^{4/3}}{a(1 + e^{2/3} + e^{4/3})} \mathbf{I}_{xy}^{(3)}, \quad (6.43)$$

where c_0 has been replaced by its expression (5.31b) in terms of a and e . Thus, one can now see that $\mathcal{G}_{\text{DC}}^{\text{mem}} \rightarrow 0$ in the circularized limit $e \rightarrow 0$. Remarkably, this conclusion is perfectly consistent with the result one would have obtained if one had instead computed the DC memory integral using an eternally quasicircular model [104].

Finally, since the orbital parameters are (approximately) constant over one orbit, and since $\langle \mathbf{I}_{ij}^{(3)} \rangle = 0$, I finally find that

$$\langle \mathcal{G}_{\text{DC}}^{\text{mem}} \rangle = 0. \quad (6.44)$$

6.6. Recapitulation

Having reviewed the different contributions to the orbit-averaged energy and angular momentum flux, let me summarize the results:

$$\langle \mathcal{F} \rangle = \langle \mathcal{F}^{\text{inst}} \rangle + \langle \mathcal{F}^{\text{tail}} \rangle, \quad (6.45a)$$

$$\langle \mathcal{G} \rangle = \langle \mathcal{G}^{\text{inst}} \rangle + \langle \mathcal{G}^{\text{tail}} \rangle, \quad (6.45b)$$

$$\langle \mathcal{F}^s \rangle = \langle \mathcal{F}^{s,\text{inst}} \rangle + \langle \mathcal{F}^{s,\text{tail}} \rangle, \quad (6.45c)$$

$$\langle \mathcal{G}^s \rangle = \langle \mathcal{G}^{s,\text{inst}} \rangle + \langle \mathcal{G}^{s,\text{tail}} \rangle. \quad (6.45d)$$

The *tensorial instantaneous* fluxes $\langle \mathcal{F}^{\text{inst}} \rangle$ and $\langle \mathcal{G}^{\text{inst}} \rangle$ are given by (6.10), whereas the *scalar instantaneous* fluxes $\langle \mathcal{F}^{s,\text{inst}} \rangle$ and $\langle \mathcal{G}^{s,\text{inst}} \rangle$ are given (6.11). The *tensorial tail* fluxes $\langle \mathcal{F}^{s,\text{tail}} \rangle$ and $\langle \mathcal{G}^{s,\text{tail}} \rangle$ are instead given by (6.26), whereas the *scalar tail* fluxes $\langle \mathcal{F}^{s,\text{tail}} \rangle$ and $\langle \mathcal{G}^{s,\text{tail}} \rangle$ are given by (6.28). Note that the tail fluxes are given in terms of some *enhancement* functions

of the eccentricity, defined exactly in (6.27) and (6.29), and as a small eccentricity expansion in (6.30). All other nonlocal terms (namely, $\mathcal{G}_i^{s,\Pi^s}$ and $\mathcal{G}_{\text{mem}}$) vanish upon orbit averaging. Note that *after orbit averaging* and at this order, there is a clean separation between the instantaneous terms, which enter at integer PN order, and the hereditary tail terms, which appear at half-integer PN orders. This is analogous to the observation made in GR [102], and one can expect this split to be broken at the level of the tails-of-tails.

7. Evolution of the orbital elements

Thanks to both the QK parametrization of section 3 and the fluxes obtained in section 6, it is now possible to obtain from flux-balance arguments the secular evolution of the QK parameters under the effect of radiation reaction at 1.5PN order, i.e. at relative 2.5PN beyond the leading radiation reaction.

Consider an orbital element, which is denoted generically ξ . Its orbit-averaged evolution can be expressed as

$$\left\langle \frac{d\xi}{dt} \right\rangle = \frac{\partial \xi}{\partial E} \left\langle \frac{dE}{dt} \right\rangle + \frac{\partial \xi}{\partial J} \left\langle \frac{dJ}{dt} \right\rangle. \quad (7.1)$$

where $J = |\mathbf{J}|$. Since the expressions of the orbital elements in terms of E and J were computed in section 3 and the energy and angular momentum fluxes were computed in section 6, we now have all the ingredients to compute the expressions of the orbital elements, assuming that the following balance equations hold:

$$\left\langle \frac{dE}{dt} \right\rangle = -\langle \mathcal{F} \rangle - \langle \mathcal{F}^s \rangle \quad \text{and} \quad \left\langle \frac{dJ}{dt} \right\rangle = -\langle \mathcal{G} \rangle - \langle \mathcal{G}^s \rangle. \quad (7.2)$$

Only the evolution equations for the pair of orbital elements (x, e_t) is presented—once these are solved (numerically), the other orbital elements can be straightforwardly deduced through the QK parametrization of paper II. Note that the 2PN-accurate QK parametrization must also be used to express (E, J) in terms of (x, e_t) . The evolution equations read

$$\begin{aligned} \left\langle \frac{dx}{dt} \right\rangle = & \frac{2c^3 \zeta \nu x^4}{3\tilde{G}\alpha m} \left\{ \frac{4S_-^2 (1 + \frac{1}{2}e_t^2)}{(1 - e_t^2)^{5/2}} + \frac{x}{15(1 - e_t^2)^{7/2}} (\mathfrak{X}_1 + e_t^2 \mathfrak{X}_2 + e_t^4 \mathfrak{X}_3) \right. \\ & + 8\pi (1 + \frac{1}{2}\bar{\gamma}) S_-^2 \varphi_1^s(e_t) x^{3/2} \\ & + x^2 \left(\frac{\mathfrak{X}_4 + e_t^2 \mathfrak{X}_5 + e_t^4 \mathfrak{X}_6 + e_t^6 \mathfrak{X}_7}{(1 - e_t^2)^{9/2}} + \frac{\mathfrak{X}_8 + e_t^2 \mathfrak{X}_9 + e_t^4 \mathfrak{X}_{10}}{(1 - e_t^2)^4} \right) \\ & + 4\pi (1 + \frac{1}{2}\bar{\gamma}) x^{5/2} (\mathcal{X}_{11} \varphi_2(e_t) + \mathcal{X}_{12} \varphi_2^s(e_t) + \mathcal{X}_{13} \alpha_1^s(e_t) + \mathcal{X}_{14} \theta_1^s(e_t) \\ & \left. + (\mathcal{X}_{15} + e_t^2 \mathcal{X}_{16}) \frac{\varphi_1^s(e_t)}{1 - e_t^2} + \mathcal{X}_{17} \frac{\tilde{\varphi}_1^s(e_t)}{(1 - e_t^2)^{3/2}} + \mathcal{X}_{18} e_t^2 \varphi_0^s(e_t) \right) + \mathcal{O}(x^3) \Big\}, \quad (7.3a) \end{aligned}$$

$$\begin{aligned}
\left\langle \frac{de_t}{dt} \right\rangle = & -\frac{c^3 \zeta \nu x^3 e_t}{\tilde{G} \alpha m} \left\{ \frac{2S_-^2}{(1-e_t^2)^{3/2}} + \frac{x}{15(1-e_t^2)^{5/2}} (\mathfrak{E}_1 + e_t^2 \mathfrak{E}_2) + \frac{8\pi}{3} (1 + \tfrac{1}{2}\bar{\gamma}) \right. \\
& \times S_-^2 \frac{1-e_t^2}{e_t^2} \left(\varphi_1^s(e_t) - \frac{\tilde{\varphi}_1^s}{\sqrt{1-e_t^2}} \right) x^{3/2} + x^2 \left(\frac{\mathfrak{E}_3 + e_t^2 \mathfrak{E}_4 + e_t^4 \mathfrak{E}_5}{(1-e_t^2)^{7/2}} + \frac{\mathfrak{E}_6 + e_t^2 \mathfrak{E}_7}{(1-e_t^2)^3} \right) \\
& + 4\pi (1 + \tfrac{1}{2}\bar{\gamma}) x^{5/2} \left[\mathfrak{E}_8 \frac{1-e_t^2}{e_t^2} \left(\varphi_2(e_t) - \frac{\tilde{\varphi}_2}{\sqrt{1-e_t^2}} \right) + \mathfrak{E}_9 \frac{1-e_t^2}{e_t^2} \left(\varphi_2^s(e_t) - \frac{\tilde{\varphi}_2^s}{\sqrt{1-e_t^2}} \right) \right. \\
& + \mathfrak{E}_{10} \frac{1-e_t^2}{e_t^2} \left(\alpha_1^s(e_t) - \frac{\tilde{\alpha}_2^s}{\sqrt{1-e_t^2}} \right) + \mathfrak{E}_{11} \frac{1-e_t^2}{e_t^2} \left(\theta_1^s(e_t) - \frac{\tilde{\theta}_1^s}{\sqrt{1-e_t^2}} \right) \\
& + \frac{\mathfrak{E}_{12}}{e_t^2} \left(\varphi_1^s(e_t) - \frac{\tilde{\varphi}_1^s}{\sqrt{1-e_t^2}} \right) + \mathfrak{E}_{13} \varphi_1^s(e_t) + \frac{\mathfrak{E}_{14}}{\sqrt{1-e_t^2}} \tilde{\varphi}_1^s(e_t) \\
& \left. + \mathfrak{E}_{15} (1-e_t^2) \varphi_0^s(e_t) \right] + \mathcal{O}(x^3) \Big\}, \tag{7.3b}
\end{aligned}$$

where the expressions of the coefficients $\mathfrak{X}_n$ and $\mathfrak{E}_n$ are relegated to appendix E.

In equation (7.3b), at first sight, there seems to be some divergent terms when $e_t \rightarrow 0$ for the half-integer PN orders. However, this is fortunately not the case, because the enhancement functions, denoted here generically by $\varphi(e_t)$ and $\tilde{\varphi}(e_t)$, appear as the combination⁹ $e_t^{-1}[\varphi(e_t) - (1-e_t^2)^{-1/2}\tilde{\varphi}(e_t)]$, which behaves as $\mathcal{O}(e_t)$ in the $e_t \rightarrow 0$ limit. This means not only that the limit for quasicircular orbits is well-defined, but also that eccentricity evolution is quenched when going to the quasicircular orbit limit, which means that quasicircular orbits are indeed stable.

8. Conclusion

In this work on ST theories, I obtained the orbit-averaged fluxes of energy and angular momentum for quasielliptic orbits at 1.5PN beyond quadrupolar radiation, i.e. relative 2.5PN beyond the leading -1 PN dipolar radiation. This extends the results of paper II where these results were obtained at Newtonian order, as well as those of paper I, where these results were obtained in the limiting case of circular orbits at 1.5PN order. From these results, I was able to derive the 1.5PN secular evolution of the orbital elements x and e_t (the evolution of the other orbital elements are then straightforward to obtain thanks to the results of paper II). The secular evolution of the orbital parameters under radiation reaction is the eccentric-orbit analog of the frequency chirp in the case of circular orbits, as is hence the main observable in the GW signal. These results are part of a general program aiming at obtaining a full GW template bank for alternative theories of gravity. In the future, these results should be included into EOB models [40–44] and hybridized with NR models [45, 46] in order to obtain full inspiral-merger-ringdown waveform templates. Although these theories are already heavily constrained by binary pulsar experiments, these waveform models could help constrain these theories even further, in particular with third-generation gravitational wave detectors.

Finally, I would like to stress that studying such alternative theories of gravity has a theoretical interest in its own right, and helps with our understanding of GR. This work was the occasion to revisit the classical GR derivation of the memory contribution to the angular momentum flux [59]. It was found that the spurious divergence in the initial eccentricity

⁹ The global prefactor e_t has been included.

is cured by a more careful split between DC and AC contributions. The simplicity of the Peters formula in ST tensor theory (5.31), with respect to its GR counterpart (A15), means that using it to study hereditary effects provides a useful toy model to better understand GR phenomenology—compare the simple ST expression (6.43) with the GR expression (A20), which is expressed in terms of hypergeometric functions. This is particularly true for the study of the hereditary terms arising from the passage to the center of mass, see section 5. These terms were recently discovered in GR [48] and evaluated for a quasicircular inspiral model. In this work, they appeared in the ST context, and were studied for the first time using an quasielliptic inspiral model. In the term associated to the translation between the matter content and the center of mass frame, I found that taking the vanishing eccentricity limit did not smoothly recover the quasicircular model, see section 5.3. This prompted me to revisit this problem in the GR scenario, for which the expressions are much more complicated and require evaluating integrals involving hypergeometric functions, and I found a completely analogous phenomenology [101], which will prompt further investigations. This work was also the occasion to introduce some technology which could prove useful in GR calculations in the future, such as the special function $\Theta_q^p(e_t)$ involved in the 1PN Fourier coefficients of the scalar dipole moment.

Future developments could include: computing the ‘post-adiabatic’ short period contributions to the orbital phasing, in the lines of [105]; completing the full waveform amplitude modes consistently at 1.5PN order; extending these results to the quasihyperbolic case; etc. Another avenue is to extend these templates to more complicated theories, such as Einstein-scalar-Gauss-Bonnet.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Note

While this paper was in preparation, I became aware that a similar computation was being performed by Jain and Retegno [62]. Before the publication of [62], we compared our results for the generic expression of the *instantaneous* part of the angular momentum flux at (i) 1PN in the scalar sector (equations (5.10)–(15) of [62] and the 1PN truncation of equation (6.8b) of this paper); and (ii) 1.5PN in the tensor sector (equations (4.7) and (8) of [62] and equation (6.8a) of this paper), and found perfect agreement.

Appendix A. Revisiting the memory contribution to the angular momentum flux in GR

The present work in scalar-tensor theory has been the occasion to revisit the problem of the angular momentum flux in GR, and correct some inconsistencies in the literature. Thus, in this section of the appendix, I shall assume that the underlying theory of gravity is GR. The first study of the memory contribution to the angular momentum flux was done by Arun *et al* [59]. In that work, it is claimed that the AC contribution vanishes upon orbit averaging, but that the DC contribution is nonvanishing, even after orbit averaging. Strikingly, it is found in their equation (A12) that the orbit-averaged DC contribution depends strongly on some initial eccentricity e_1 at which the binary was formed, and *diverges* as $\sim (1 - e_1^2)^{-1}$ when $e_1 \rightarrow 1$ (see their equation (A11)). Despite the argument that a compact binary cannot have existed eternally, their conclusions seem unphysical from the viewpoint of the formal two-body problem. Later, it was found by Loutrel and Yunes [91] that the orbit-averaged DC contribution should instead vanish upon orbit-averaging (see the end of their section 6.2), using the argument that the DC contribution is proportional to $I_{ij}^{(3)}$, which trivially vanishes after orbit averaging. However, they do not provide a satisfying explanation for the discrepancy (they invoke the choice of the initial phase, but this choice is arbitrary and should not affect the physical result). Moreover, when studying the DC contribution *before* orbit averaging, Loutrel and Yunes use a discrete model, valid for large eccentricities: the orbital parameters (a, e) are modeled as constant over an orbit, and updated discretely at every periastron passage, where most of the gravitational waves are emitted (see their equation (208)). However, they do not address the convergence of their model when summing over the infinite past.

In the following, I will prove the following statements: (i) the AC and DC memory contributions vanish upon orbit averaging, and (ii) the DC contribution is nonvanishing but finite before orbit averaging. Moreover, the finite DC contribution vanishes in the limit of a circularized orbit ($e \rightarrow 0$), regardless of whether one uses a QK or quasi-circular model for the past inspiral. The reason for the discrepancy is that Arun *et al* [59] decomposed their integrand as follows [see their equation (A4)]:

$$\mathcal{G}^{\text{mem}} = \sum_n \int_{t_1}^{t_0} dt f_n(a, e) e^{in\phi}, \quad (\text{A1})$$

and identified the DC contribution with the $n=0$ component [see their equation (A5)]. However, this implicitly assumes that the orbit-averaging procedure is $\frac{1}{2\pi} \int_0^{2\pi} d\phi$, when instead it should be¹⁰

$$\frac{1}{P} \int_0^P dt = \frac{1}{P} \int_0^{2\pi} \frac{d\phi}{\dot{\phi}} = \frac{(1 - e^2)^{3/2}}{2\pi} \int_0^{2\pi} \frac{d\phi}{(1 + e \cos \phi)^2}. \quad (\text{A2})$$

In the following, I revisit the computation of Arun *et al* [59] by carefully implementing of the orbit averaging procedure following (A2). At 3.5PN order, recall that the memory contribution to the angular momentum flux reads [59]

$$\mathcal{G}_i^{\text{mem}} = \frac{4G^2}{35c^{10}} \epsilon_{ijk} I_{ja}^{(3)} \int_0^\infty d\tau \left[I_{kb}^{(3)} I_{ab}^{(3)} \right] (t_{\text{ret}} - \tau). \quad (\text{A3})$$

¹⁰ I have used the expression of $\dot{\phi}$ given by (A3c) of [59], and used the fact that the orbital (radial) period reads $P = \frac{2\pi a^{3/2}}{\sqrt{Gm}}$.

For a nonprecessing orbit, the direction of the angular momentum flux is constant and points in the z -direction. It is then straightforward to explicitly express its norm $\mathcal{G} = |\vec{\mathcal{G}}| = \mathcal{G}_z$ in terms of the (x, y, z) components:

$$\mathcal{G}^{\text{mem}} = \frac{4G^2}{35c^{10}} \left\{ \left(\mathbf{I}_{xx}^{(3)} - \mathbf{I}_{yy}^{(3)} \right) \int_0^\infty d\tau f(t_{\text{ret}} - \tau) - \mathbf{I}_{xy}^{(3)} \int_0^\infty d\tau g(t_{\text{ret}} - \tau) \right\}, \quad (\text{A4})$$

where I used the fact that $\mathbf{I}_{xz} = \mathbf{I}_{yz} = 0$ and where I define

$$f = \mathbf{I}_{xy}^{(3)} \left(\mathbf{I}_{xx}^{(3)} + \mathbf{I}_{yy}^{(3)} \right), \quad (\text{A5a})$$

$$g = \left(\mathbf{I}_{xx}^{(3)} \right)^2 - \left(\mathbf{I}_{yy}^{(3)} \right)^2. \quad (\text{A5b})$$

In order to distinguish the AC and DC contributions, I decompose these functions into $f = \langle f \rangle + \tilde{f}$ and $g = \langle g \rangle + \tilde{g}$, where I have defined the orbit average as $\langle h \rangle = \frac{1}{P} \int_0^P dt h(t) = \frac{1}{2\pi} \int_0^{2\pi} d\ell h(\ell)$. Here, I have introduced the mean anomaly $\ell = n(t - t_0) = \frac{2\pi}{P}(t - t_0)$, which will prove useful for the analysis. Consequently, the functions $\tilde{f}$ and $\tilde{g}$ contain purely oscillatory terms which average out to zero over a single orbit. Thus, I can now define precisely the AC and DC contributions:

$$\mathcal{G}_{\text{DC}}^{\text{mem}} = \frac{4G^2}{35c^{10}} \left\{ \left(\mathbf{I}_{xx}^{(3)} - \mathbf{I}_{yy}^{(3)} \right) \int_0^\infty d\tau \langle f \rangle(t_{\text{ret}} - \tau) - \mathbf{I}_{xy}^{(3)} \int_0^\infty d\tau \langle g \rangle(t_{\text{ret}} - \tau) \right\}, \quad (\text{A6a})$$

$$\mathcal{G}_{\text{AC}}^{\text{mem}} = \frac{4G^2}{35c^{10}} \left\{ \left(\mathbf{I}_{xx}^{(3)} - \mathbf{I}_{yy}^{(3)} \right) \int_0^\infty d\tau \tilde{f}(t_{\text{ret}} - \tau) - \mathbf{I}_{xy}^{(3)} \int_0^\infty d\tau \tilde{g}(t_{\text{ret}} - \tau) \right\}. \quad (\text{A6b})$$

A.1. The AC part

First, let me focus on the AC terms. I replace the components of the quadrupole moments by their Fourier series, which I denote as $\mathbf{I}_{ij} = \sum_{p \in \mathbb{Z}} e^{ip\ell} {}_p\tilde{\mathbf{I}}_{ij}$. I find that

$$f = -n^6 \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} e^{i(p+q)\ell} p^3 q^3 {}_p\tilde{\mathbf{I}}_{xy} \left({}_q\tilde{\mathbf{I}}_{xx} + {}_q\tilde{\mathbf{I}}_{yy} \right) \quad (\text{A7a})$$

$$g = -n^6 \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} e^{i(p+q)\ell} p^3 q^3 \left({}_p\tilde{\mathbf{I}}_{xx} {}_q\tilde{\mathbf{I}}_{xx} - {}_p\tilde{\mathbf{I}}_{yy} {}_q\tilde{\mathbf{I}}_{yy} \right). \quad (\text{A7b})$$

By definition of the orbital average, the DC component is exactly defined by the terms for which $p + q = 0$, so I can now simply reexpress the AC term as

$$\tilde{f} = -n^6 \sum_{p+q \neq 0} e^{i(p+q)\ell} p^3 q^3 {}_p\tilde{\mathbf{I}}_{xy} \left({}_q\tilde{\mathbf{I}}_{xx} + {}_q\tilde{\mathbf{I}}_{yy} \right) \quad (\text{A8a})$$

$$\tilde{g} = -n^6 \sum_{p+q \neq 0} e^{i(p+q)\ell} p^3 q^3 \left({}_p\tilde{\mathbf{I}}_{xx} {}_q\tilde{\mathbf{I}}_{xx} - {}_p\tilde{\mathbf{I}}_{yy} {}_q\tilde{\mathbf{I}}_{yy} \right). \quad (\text{A8b})$$

Using $\int_0^\infty d\tau e^{ikn(t-t_0-\tau)} = e^{ikn(t-t_0)} / (ikn)$ for $k \neq 0$, replacing all moments by their Fourier expansions, and substituting at leading order $n = c^3 x^{3/2} / (Gm)$ I readily obtain

$$\mathcal{G}_{\text{AC}}^{\text{mem}} = \frac{4m\nu^3 c^2 x^6}{35} \sum_{r \in \mathbb{Z}} \sum_{p+q \neq 0} e^{i(p+q+r)\ell} \frac{p^3 q^3 r^3}{p+q} \mathcal{S}_{r,p,q}, \quad (\text{A9})$$

where

$$\mathcal{S}_{r,p,q} = \left(\hat{r}_{xx} - \hat{r}_{yy} \right) \hat{p}_{xy} \left(\hat{q}_{xx} + \hat{q}_{yy} \right) - \hat{r}_{xy} \left(\hat{p}_{xx} \hat{q}_{xx} - \hat{p}_{yy} \hat{q}_{yy} \right). \quad (\text{A10})$$

Here, I expressed $\mathcal{S}_{r,p,q}$ in terms of the normalized Fourier coefficients $\hat{p}_{ij} \equiv (mva^2)^{-1} \tilde{p}_{ij}$, which are explicitly given by (A3) of [90]. Thanks to the latter expressions, it is immediate to see that $\mathcal{S}_{r,p,q} \in i\mathbb{R}$. Since $\mathcal{G}_{AC}^{\text{mem}} \in \mathbb{R}$ by construction, taking the real part of (A9) yields the explicitly real-valued expression:

$$\mathcal{G}_{AC}^{\text{mem}} = \frac{4m\nu^3 c^2 x^6}{35} \sum_{r \in \mathbb{Z}} \sum_{p+q \neq 0} \sin((p+q+r)\ell) \frac{p^3 q^3 r^3}{p+q} i\mathcal{S}_{r,p,q}. \quad (\text{A11})$$

In order to compute the orbit-average of $\mathcal{G}_{AC}^{\text{mem}}$, it is in fact more practical to start directly from the expression (A9), and to find

$$\langle \mathcal{G}_{AC}^{\text{mem}} \rangle = -\frac{4m\nu^3 c^2 x^6}{35} \sum_{p+q \neq 0} p^3 q^3 (p+q)^2 \mathcal{S}_{-p-q,p,q}. \quad (\text{A12})$$

Noting that on the one hand, $\langle \mathcal{G}_{AC}^{\text{mem}} \rangle$ is necessarily a real number, but on the other hand, it also purely imaginary (because $\mathcal{S}_{-p-q,p,q} \in i\mathbb{R}$), I finally conclude that

$$\langle \mathcal{G}_{AC}^{\text{mem}} \rangle = 0. \quad (\text{A13})$$

A.2. The DC part

I now turn to the DC part. Thanks to the explicit expression of the quadrupole moment and upon careful computation of the orbital average, I obtain $\langle f \rangle = 0$ and

$$\langle g \rangle = -\frac{G^3 m^5 \nu^2 e^2 (13 + 2e^2)}{3a^5 (1 - e^2)^{7/2}}. \quad (\text{A14})$$

Thanks to the standard relations derived by Peters [108]

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{304G^3 m^3 \nu e}{15c^5 a^4 (1 - e^2)^{5/2}} \left(1 + \frac{121}{304} e^2 \right), \quad (\text{A15a})$$

$$a(e) = \frac{c_0 e^{12/19}}{1 - e^2} \left(1 + \frac{121}{304} e^2 \right)^{\frac{870}{2299}}. \quad (\text{A15b})$$

I can reexpress the nonvanishing memory integral as an integral over eccentricity. In the far past, the eccentricity asymptotes towards one, so the memory integral reads

$$\int_0^\infty d\tau \langle g \rangle (u - \tau) = -\frac{5c^5 m^2 \nu}{304c_0} \int_e^1 d\tilde{e} \frac{\tilde{e}^{7/19} (13 + 2\tilde{e}^2)}{\left(1 + \frac{121}{304} \tilde{e}^2 \right)^{\frac{3169}{2299}}} = -\frac{5c^5 m^2 \nu}{304c_0} \mathcal{K} \mathcal{M}(e), \quad (\text{A16})$$

where I have introduced the constant

$$\mathcal{K} = \int_0^1 d\tilde{e} \frac{\tilde{e}^{7/19} (13 + 2\tilde{e}^2)}{(1 + \frac{121}{304}\tilde{e}^2)^{\frac{3169}{2299}}} = \frac{19}{2} F\left(\frac{13}{19}, \frac{3169}{2299}, \frac{32}{19}; -\frac{121}{304}\right) + \frac{19}{32} F\left(\frac{3169}{2299}, \frac{32}{19}, \frac{51}{19}; -\frac{121}{304}\right), \quad (\text{A17})$$

as well as the following function of eccentricity:

$$\begin{aligned} \mathcal{M}(e) &= 1 - \frac{1}{\mathcal{K}} \int_e^1 d\tilde{e} \frac{\tilde{e}^{7/19} (13 + 2\tilde{e}^2)}{(1 + \frac{121}{304}\tilde{e}^2)^{\frac{3169}{2299}}} \\ &= 1 - \frac{e^{26/19}}{\mathcal{K}} \left\{ \frac{19}{2} F\left(\frac{13}{19}, \frac{3169}{2299}, \frac{32}{19}; -\frac{121}{304}e^2\right) + \frac{19}{32} e^2 F\left(\frac{3169}{2299}, \frac{32}{19}, \frac{51}{19}; -\frac{121}{304}e^2\right) \right\}. \end{aligned} \quad (\text{A18})$$

In these expressions, $F(a, b; c; d)$ is the ordinary (or Gaussian) hypergeometric function, and I have chosen to normalize $\mathcal{M}(e)$ such that $\mathcal{M}(0) = 1$ and $\mathcal{M}(1) = 0$ (see figure 3). The constant $\mathcal{K}$ takes the numerical value $\mathcal{K} \approx 8.29969$, and it is possible to approximate $\mathcal{M}(e)$ with the following expansion (which leads to absolute errors smaller than 0.0015 over the whole range of $e \in [0, 1]$)

$$\mathcal{M}(e) \approx 1 - e^{26/19} (1.145 - 0.184e^2 + 0.0511e^4 - 0.0163e^6 + 0.00548e^8). \quad (\text{A19})$$

Plugging (A16) into (A4) leads to the final expression

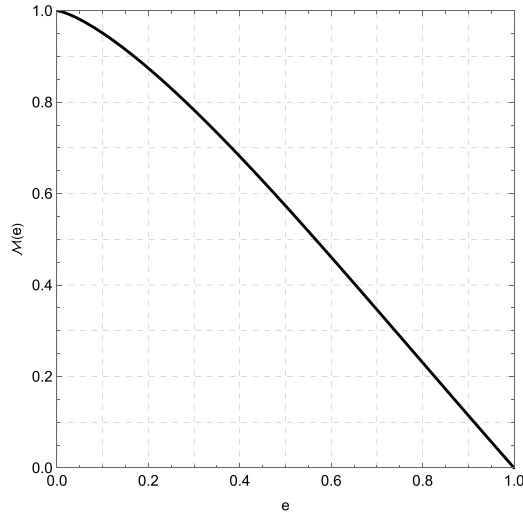
$$\mathcal{G}_{\text{DC}}^{\text{mem}} = \frac{G^2 m^2 \nu \mathcal{K}}{532 c^5} \frac{e^{12/29}}{a(1-e^2)} \left(1 + \frac{121}{304}e^2\right)^{\frac{870}{2299}} \mathcal{M}(e) \mathbf{I}_{xy}^{(3)}, \quad (\text{A20})$$

where c_0 has been replaced by its expression (A15b) in terms of a and e . Thus, one can now see that $\mathcal{G}_{\text{DC}}^{\text{mem}} \rightarrow 0$ in the circularized limit $e \rightarrow 0$. Remarkably, this conclusion is perfectly consistent with the result one would have obtained if one had instead computed the DC memory integral using an eternally quasicircular model, in which case the angular momentum flux is proportional to the energy flux and not does feature any memory-like nonlocal term.

Finally, since the orbital parameters are (approximately) constant over one orbit, and since $\langle \mathbf{I}_{ij}^{(3)} \rangle = 0$, I find that

$$\langle \mathcal{G}_{\text{DC}}^{\text{mem}} \rangle = 0. \quad (\text{A21})$$

Since the memory contribution vanishes under orbit-averaging, it does not lead to any *secular* contributions to the evolution of the orbital parameters. However, *before orbit averaging*, the memory contribution is nonvanishing and finite, so it can lead to some *post-adiabatic*, *oscillatory* contributions to the evolution of the orbital parameters, along the lines of [105]. I will not attempt to compute these contributions here, but let me highlight that the results obtained above prove that these contributions should be finite even for an eternally isolated binary system. Finally, let me mention that a complete treatment of this memory integral in the case of *unbound* quasihyperbolic orbits can be found in [109].

Figure 3. Plot of $\mathcal{M}(e)$.

Appendix B. Details concerning the passage to the center of mass

$$p_1 = \frac{(\tilde{G}\alpha m)^3 m\nu}{15c^5} \left[20\bar{\beta}_- \zeta S_-^2 - 50\zeta S_- S_+ - 20\bar{\beta}_+ \zeta S_- S_+ - 20\zeta \bar{\gamma} S_- S_+ \right. \\ \left. + \delta \left(-24 - 10\bar{\gamma} - 50\zeta S_-^2 - 20\bar{\beta}_+ \zeta S_-^2 - 20\zeta \bar{\gamma} S_-^2 + 20\bar{\beta}_- \zeta S_- S_+ \right) \right. \\ \left. + \nu \left(-80\bar{\beta}_- \zeta S_-^2 + 160\bar{\beta}_- \zeta S_-^2 \bar{\gamma}^{-1} - 26\zeta S_- S_+ + 160\bar{\beta}_+ \zeta S_- S_+ \bar{\gamma}^{-1} \right) + 6\zeta S_-^2 \nu \delta \right] \quad (\text{B1a})$$

$$p_2 = \frac{(\tilde{G}\alpha m)^2 m\nu}{15c^5} \left[30\bar{\beta}_- - 15\zeta S_- S_+ + \delta \left(-30\bar{\beta}_+ + 15\bar{\gamma} - 15\zeta S_-^2 \right) + 45\zeta S_- S_+ \nu + 15\zeta S_-^2 \nu \delta \right] \quad (\text{B1b})$$

$$p_3 = \frac{(\tilde{G}\alpha m)^2 m\nu}{15c^5} \left[-10\bar{\beta}_- + 20\zeta S_- S_+ + 10\zeta \bar{\gamma} S_- S_+ + \delta \left(12 + 10\bar{\beta}_+ + 20\zeta S_-^2 + 10\zeta \bar{\gamma} S_-^2 \right) \right] \quad (\text{B1c})$$

$$p_4 = \frac{(\tilde{G}\alpha m)^2 m\nu}{15c^5} \left[-20\bar{\beta}_- - 30\zeta S_- S_+ - 20\zeta \bar{\gamma} S_- S_+ + \delta \left(20\bar{\beta}_+ - 10\bar{\gamma} - 30\zeta S_-^2 - 20\zeta \bar{\gamma} S_-^2 \right) \right. \\ \left. - 10\zeta S_- S_+ \nu - 10\zeta S_-^2 \nu \delta \right] \quad (\text{B1d})$$

$$q_1 = \frac{(\tilde{G}\alpha m)^2 m\nu}{30c^5} \left[-20\bar{\beta}_- + 5\zeta S_- S_+ + \left(20\bar{\beta}_+ - 10\bar{\gamma} + 5\zeta S_-^2 \right) \delta - 8\zeta S_- S_+ \nu - 12\zeta S_-^2 \nu \delta \right] \quad (\text{B1e})$$

$$q_2 = \frac{(\tilde{G}\alpha m)^2 m\nu}{30c^5} \left[20\bar{\beta}_- + 5\zeta S_- S_+ + \delta \left(48 - 20\bar{\beta}_+ + 30\bar{\gamma} + 5\zeta S_-^2 \right) - 16\zeta S_-^2 \nu \delta \right. \\ \left. + \nu \left(-320\bar{\beta}_- \zeta S_-^2 \bar{\gamma}^{-1} + 76\zeta S_- S_+ - 320\bar{\beta}_+ \zeta S_- S_+ \bar{\gamma}^{-1} \right) \right] \quad (\text{B1f})$$

$$q_3 = \frac{(\tilde{G}\alpha m) m\nu}{30c^5} \left[-5\zeta S_- S_+ + \delta \left(-24 - 10\bar{\gamma} - 5\zeta S_-^2 \right) + 16\zeta S_- S_+ \nu + 4\zeta S_-^2 \nu \delta \right]. \quad (\text{B1g})$$

Appendix C. Coefficients for the small eccentricity expansion of $\alpha_1^s(e_t)$ and $\tilde{\alpha}_1^s(e_t)$

The coefficients associated to $\alpha_1^s(e_t)$ read

$$\begin{aligned}
 c_2 = & \frac{5611}{324} + \frac{350}{729}\bar{\beta}_-^2 - \frac{685}{486}\bar{\beta}_+ + \frac{350}{729}\bar{\beta}_+^2 + \frac{2900\bar{\beta}_-^2}{81\bar{\gamma}^2} + \frac{2900\bar{\beta}_+^2}{81\bar{\gamma}^2} - \frac{200\bar{\beta}_-^2}{27\bar{\gamma}} - \frac{7615\bar{\beta}_+}{162\bar{\gamma}} \\
 & - \frac{200\bar{\beta}_+^2}{27\bar{\gamma}} + \frac{2285}{324}\bar{\gamma} + \frac{1375}{729}\bar{\beta}_+\bar{\gamma} + \frac{200}{243}\bar{\gamma}^2 + \frac{1235\bar{\beta}_+}{162\zeta\mathcal{S}_-^2} - \frac{725\bar{\beta}_-^2}{81\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{725\bar{\beta}_+^2}{81\zeta\bar{\gamma}\mathcal{S}_-^2} \\
 & - \frac{131\bar{\gamma}}{81\zeta\mathcal{S}_-^2} - \frac{1130\bar{\beta}_-\mathcal{S}_+}{243\mathcal{S}_-} + \frac{5800\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} - \frac{7615\bar{\beta}_-\mathcal{S}_+}{162\bar{\gamma}\mathcal{S}_-} - \frac{400\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} \\
 & + \delta \left(\frac{685}{486}\bar{\beta}_- - \frac{700}{729}\bar{\beta}_-\bar{\beta}_+ - \frac{5800\bar{\beta}_-\bar{\beta}_+}{81\bar{\gamma}^2} + \frac{7615\bar{\beta}_-}{162\bar{\gamma}} + \frac{400\bar{\beta}_-\bar{\beta}_+}{27\bar{\gamma}} - \frac{1375}{729}\bar{\beta}_-\bar{\gamma} \right. \\
 & - \frac{1235\bar{\beta}_-}{162\zeta\mathcal{S}_-^2} + \frac{1450\bar{\beta}_-\bar{\beta}_+}{81\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{731\mathcal{S}_+}{54\mathcal{S}_-} + \frac{1130\bar{\beta}_+\mathcal{S}_+}{243\mathcal{S}_-} - \frac{2900\bar{\beta}_-^2\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} - \frac{2900\bar{\beta}_+^2\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} \\
 & \left. + \frac{200\bar{\beta}_-^2\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} + \frac{7615\bar{\beta}_+\mathcal{S}_+}{162\bar{\gamma}\mathcal{S}_-} + \frac{200\bar{\beta}_+^2\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} - \frac{1615\bar{\gamma}\mathcal{S}_+}{486\mathcal{S}_-} \right) \\
 & + \nu \left(-\frac{1048}{81} - \frac{1400}{729}\bar{\beta}_-^2 - \frac{5800\bar{\beta}_-^2}{81\bar{\gamma}^2} - \frac{5800\bar{\beta}_+^2}{81\bar{\gamma}^2} + \frac{800\bar{\beta}_-^2}{27\bar{\gamma}} + \frac{4940\bar{\beta}_+}{81\bar{\gamma}} - \frac{2470\bar{\beta}_+}{81\zeta\mathcal{S}_-^2} \right. \\
 & \left. + \frac{2900\bar{\beta}_+^2}{81\zeta\bar{\gamma}\mathcal{S}_-^2} + \frac{524\bar{\gamma}}{81\zeta\mathcal{S}_-^2} - \frac{1060\bar{\beta}_-\mathcal{S}_+}{81\mathcal{S}_-} - \frac{11600\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} + \frac{4940\bar{\beta}_-\mathcal{S}_+}{81\bar{\gamma}\mathcal{S}_-} + \frac{800\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} \right), \tag{C1a}
 \end{aligned}$$

$$\begin{aligned}
 c_4 = & \frac{89797}{1296} + \frac{5975}{2916}\bar{\beta}_-^2 - \frac{4925}{972}\bar{\beta}_+ + \frac{5975}{2916}\bar{\beta}_+^2 + \frac{56875\bar{\beta}_-^2}{324\bar{\gamma}^2} + \frac{56875\bar{\beta}_+^2}{324\bar{\gamma}^2} - \frac{68825\bar{\beta}_-^2}{1944\bar{\gamma}} \\
 & - \frac{277255\bar{\beta}_+}{1296\bar{\gamma}} - \frac{68825\bar{\beta}_+^2}{1944\bar{\gamma}} + \frac{184855}{7776}\bar{\gamma} + \frac{24125}{3888}\bar{\beta}_+\bar{\gamma} + \frac{60425}{23328}\bar{\gamma}^2 + \frac{16465\bar{\beta}_+}{432\zeta\mathcal{S}_-^2} \\
 & - \frac{56875\bar{\beta}_-^2}{1296\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{56875\bar{\beta}_+^2}{1296\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{5329\bar{\gamma}}{648\zeta\mathcal{S}_-^2} - \frac{3025\bar{\beta}_-\mathcal{S}_+}{216\mathcal{S}_-} + \frac{56875\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{162\bar{\gamma}^2\mathcal{S}_-} \\
 & - \frac{277255\bar{\beta}_-\mathcal{S}_+}{1296\bar{\gamma}\mathcal{S}_-} - \frac{68825\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{972\bar{\gamma}\mathcal{S}_-} + \delta \left(\frac{4925}{972}\bar{\beta}_- - \frac{5975}{1458}\bar{\beta}_-\bar{\beta}_+ \right. \\
 & - \frac{56875\bar{\beta}_-\bar{\beta}_+}{162\bar{\gamma}^2} + \frac{277255\bar{\beta}_-}{1296\bar{\gamma}} + \frac{68825\bar{\beta}_-\bar{\beta}_+}{972\bar{\gamma}} - \frac{24125}{3888}\bar{\beta}_-\bar{\gamma} - \frac{16465\bar{\beta}_-}{432\zeta\mathcal{S}_-^2} \\
 & + \frac{56875\bar{\beta}_-\bar{\beta}_+}{648\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{78397\mathcal{S}_+}{1296\mathcal{S}_-} + \frac{3025\bar{\beta}_+\mathcal{S}_+}{216\mathcal{S}_-} - \frac{56875\bar{\beta}_-^2\mathcal{S}_+}{324\bar{\gamma}^2\mathcal{S}_-} - \frac{56875\bar{\beta}_+^2\mathcal{S}_+}{324\bar{\gamma}^2\mathcal{S}_-} \\
 & \left. + \frac{68825\bar{\beta}_-^2\mathcal{S}_+}{1944\bar{\gamma}\mathcal{S}_-} + \frac{277255\bar{\beta}_+\mathcal{S}_+}{1296\bar{\gamma}\mathcal{S}_-} + \frac{68825\bar{\beta}_+^2\mathcal{S}_+}{1944\bar{\gamma}\mathcal{S}_-} - \frac{50815\bar{\gamma}\mathcal{S}_+}{3888\mathcal{S}_-} \right) \\
 & + \nu \left(-\frac{5329}{81} - \frac{5975}{729}\bar{\beta}_-^2 - \frac{56875\bar{\beta}_-^2}{162\bar{\gamma}^2} - \frac{56875\bar{\beta}_+^2}{162\bar{\gamma}^2} + \frac{68825\bar{\beta}_-^2}{486\bar{\gamma}} + \frac{16465\bar{\beta}_+}{54\bar{\gamma}} \right. \\
 & - \frac{16465\bar{\beta}_+}{108\zeta\mathcal{S}_-^2} + \frac{56875\bar{\beta}_+^2}{324\zeta\bar{\gamma}\mathcal{S}_-^2} + \frac{5329\bar{\gamma}}{162\zeta\mathcal{S}_-^2} - \frac{10475\bar{\beta}_-\mathcal{S}_+}{162\mathcal{S}_-} - \frac{56875\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} \\
 & \left. + \frac{16465\bar{\beta}_-\mathcal{S}_+}{54\bar{\gamma}\mathcal{S}_-} + \frac{68825\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{486\bar{\gamma}\mathcal{S}_-} \right), \tag{C1b}
 \end{aligned}$$

$$\begin{aligned}
C_6 = & \frac{16954805}{93312} + \frac{757075}{104976} \bar{\beta}_-^2 - \frac{787255}{139968} \bar{\beta}_+ + \frac{757075}{104976} \bar{\beta}_+^2 + \frac{3181825 \bar{\beta}_-^2}{5832 \bar{\gamma}^2} + \frac{3181825 \bar{\beta}_+^2}{5832 \bar{\gamma}^2} \\
& - \frac{984725 \bar{\beta}_-^2}{8748 \bar{\gamma}} - \frac{29334965 \bar{\beta}_+}{46656 \bar{\gamma}} - \frac{984725 \bar{\beta}_+^2}{8748 \bar{\gamma}} + \frac{1521485}{31104} \bar{\gamma} + \frac{2693225}{209952} \bar{\beta}_+ \bar{\gamma} + \frac{968075}{209952} \bar{\gamma}^2 \\
& + \frac{5578585 \bar{\beta}_+}{46656 \zeta \mathcal{S}_-^2} - \frac{3181825 \bar{\beta}_-^2}{23328 \zeta \bar{\gamma} \mathcal{S}_-^2} - \frac{3181825 \bar{\beta}_+^2}{23328 \zeta \bar{\gamma} \mathcal{S}_-^2} - \frac{606625 \bar{\gamma}}{23328 \zeta \mathcal{S}_-^2} - \frac{739595 \bar{\beta}_- \mathcal{S}_+}{34992 \mathcal{S}_-} \\
& + \frac{3181825 \bar{\beta}_- \bar{\beta}_+ \mathcal{S}_+}{2916 \bar{\gamma}^2 \mathcal{S}_-} - \frac{29334965 \bar{\beta}_- \mathcal{S}_+}{46656 \bar{\gamma} \mathcal{S}_-} - \frac{984725 \bar{\beta}_- \bar{\beta}_+ \mathcal{S}_+}{4374 \bar{\gamma} \mathcal{S}_-} + \delta \left(\frac{787255}{139968} \bar{\beta}_- \right. \\
& - \frac{757075}{52488} \bar{\beta}_- \bar{\beta}_+ - \frac{3181825 \bar{\beta}_- \bar{\beta}_+}{2916 \bar{\gamma}^2} + \frac{29334965 \bar{\beta}_-}{46656 \bar{\gamma}} + \frac{984725 \bar{\beta}_- \bar{\beta}_+}{4374 \bar{\gamma}} - \frac{2693225}{209952} \bar{\beta}_- \bar{\gamma} \\
& - \frac{5578585 \bar{\beta}_-}{46656 \zeta \mathcal{S}_-^2} + \frac{3181825 \bar{\beta}_- \bar{\beta}_+}{11664 \zeta \bar{\gamma} \mathcal{S}_-^2} - \frac{905435 \mathcal{S}_+}{5184 \mathcal{S}_-} + \frac{739595 \bar{\beta}_+ \mathcal{S}_+}{34992 \mathcal{S}_-} - \frac{3181825 \bar{\beta}_-^2 \mathcal{S}_+}{5832 \bar{\gamma}^2 \mathcal{S}_-} \\
& \left. - \frac{3181825 \bar{\beta}_+^2 \mathcal{S}_+}{5832 \bar{\gamma}^2 \mathcal{S}_-} + \frac{984725 \bar{\beta}_-^2 \mathcal{S}_+}{8748 \bar{\gamma} \mathcal{S}_-} + \frac{29334965 \bar{\beta}_+ \mathcal{S}_+}{46656 \bar{\gamma} \mathcal{S}_-} + \frac{984725 \bar{\beta}_+^2 \mathcal{S}_+}{8748 \bar{\gamma} \mathcal{S}_-} - \frac{4582045 \bar{\gamma} \mathcal{S}_+}{139968 \mathcal{S}_-} \right) \\
& + \nu \left(-\frac{606625}{2916} - \frac{757075}{26244} \bar{\beta}_-^2 - \frac{3181825 \bar{\beta}_-^2}{2916 \bar{\gamma}^2} - \frac{3181825 \bar{\beta}_+^2}{2916 \bar{\gamma}^2} + \frac{984725 \bar{\beta}_-^2}{2187 \bar{\gamma}} + \frac{5578585 \bar{\beta}_+}{5832 \bar{\gamma}} \right. \\
& - \frac{5578585 \bar{\beta}_+}{11664 \zeta \mathcal{S}_-^2} + \frac{3181825 \bar{\beta}_+^2}{5832 \zeta \bar{\gamma} \mathcal{S}_-^2} + \frac{606625 \bar{\gamma}}{5832 \zeta \mathcal{S}_-^2} - \frac{3638785 \bar{\beta}_- \mathcal{S}_+}{17496 \mathcal{S}_-} - \frac{3181825 \bar{\beta}_- \bar{\beta}_+ \mathcal{S}_+}{1458 \bar{\gamma}^2 \mathcal{S}_-} \\
& \left. + \frac{5578585 \bar{\beta}_- \mathcal{S}_+}{5832 \bar{\gamma} \mathcal{S}_-} + \frac{984725 \bar{\beta}_- \bar{\beta}_+ \mathcal{S}_+}{2187 \bar{\gamma} \mathcal{S}_-} \right), \tag{C1c}
\end{aligned}$$

$$\begin{aligned}
C_8 = & \frac{2280254495}{5971968} + \frac{133124075}{6718464} \bar{\beta}_-^2 + \frac{67506595}{8957952} \bar{\beta}_+ + \frac{133124075}{6718464} \bar{\beta}_+^2 + \frac{984057275 \bar{\beta}_-^2}{746496 \bar{\gamma}^2} \\
& + \frac{984057275 \bar{\beta}_+^2}{746496 \bar{\gamma}^2} - \frac{138922825 \bar{\beta}_-^2}{497664 \bar{\gamma}} - \frac{2175881095 \bar{\beta}_+}{1492992 \bar{\gamma}} - \frac{138922825 \bar{\beta}_+^2}{497664 \bar{\gamma}} \\
& + \frac{24773035}{331776} \bar{\gamma} + \frac{539614825}{26873856} \bar{\beta}_+ \bar{\gamma} + \frac{30374075}{5971968} \bar{\gamma}^2 + \frac{433693805 \bar{\beta}_+}{1492992 \zeta \mathcal{S}_-^2} \\
& - \frac{984057275 \bar{\beta}_-^2}{2985984 \zeta \bar{\gamma} \mathcal{S}_-^2} - \frac{984057275 \bar{\beta}_+^2}{2985984 \zeta \bar{\gamma} \mathcal{S}_-^2} - \frac{23688235 \bar{\gamma}}{373248 \zeta \mathcal{S}_-^2} - \frac{48448015 \bar{\beta}_- \mathcal{S}_+}{4478976 \mathcal{S}_-} \\
& + \frac{984057275 \bar{\beta}_- \bar{\beta}_+ \mathcal{S}_+}{373248 \bar{\gamma}^2 \mathcal{S}_-} - \frac{2175881095 \bar{\beta}_- \mathcal{S}_+}{1492992 \bar{\gamma} \mathcal{S}_-} - \frac{138922825 \bar{\beta}_- \bar{\beta}_+ \mathcal{S}_+}{248832 \bar{\gamma} \mathcal{S}_-} \\
& + \delta \left(-\frac{67506595}{8957952} \bar{\beta}_- - \frac{133124075}{3359232} \bar{\beta}_- \bar{\beta}_+ - \frac{984057275 \bar{\beta}_- \bar{\beta}_+}{373248 \bar{\gamma}^2} + \frac{2175881095 \bar{\beta}_-}{1492992 \bar{\gamma}} \right. \\
& + \frac{138922825 \bar{\beta}_- \bar{\beta}_+}{248832 \bar{\gamma}} - \frac{539614825}{26873856} \bar{\beta}_- \bar{\gamma} - \frac{433693805 \bar{\beta}_-}{1492992 \zeta \mathcal{S}_-^2} + \frac{984057275 \bar{\beta}_- \bar{\beta}_+}{1492992 \zeta \bar{\gamma} \mathcal{S}_-^2} \\
& - \frac{66169445 \mathcal{S}_+}{165888 \mathcal{S}_-} + \frac{48448015 \bar{\beta}_+ \mathcal{S}_+}{4478976 \mathcal{S}_-} - \frac{984057275 \bar{\beta}_-^2 \mathcal{S}_+}{746496 \bar{\gamma}^2 \mathcal{S}_-} - \frac{984057275 \bar{\beta}_+^2 \mathcal{S}_+}{746496 \bar{\gamma}^2 \mathcal{S}_-} \\
& \left. + \frac{138922825 \bar{\beta}_-^2 \mathcal{S}_+}{497664 \bar{\gamma} \mathcal{S}_-} + \frac{2175881095 \bar{\beta}_+ \mathcal{S}_+}{1492992 \bar{\gamma} \mathcal{S}_-} + \frac{138922825 \bar{\beta}_+^2 \mathcal{S}_+}{497664 \bar{\gamma} \mathcal{S}_-} - \frac{291555685 \bar{\gamma} \mathcal{S}_+}{4478976 \mathcal{S}_-} \right) \\
& + \nu \left(-\frac{23688235}{46656} - \frac{133124075}{1679616} \bar{\beta}_-^2 - \frac{984057275 \bar{\beta}_-^2}{373248 \bar{\gamma}^2} - \frac{984057275 \bar{\beta}_+^2}{373248 \bar{\gamma}^2} + \frac{138922825 \bar{\beta}_-^2}{124416 \bar{\gamma}} \right.
\end{aligned}$$

$$+ \frac{433693805\bar{\beta}_+}{186624\bar{\gamma}} - \frac{433693805\bar{\beta}_+}{373248\zeta S_-^2} + \frac{984057275\bar{\beta}_+^2}{746496\zeta\bar{\gamma}S_-^2} + \frac{23688235\bar{\gamma}}{93312\zeta S_-^2} - \frac{24177665\bar{\beta}_-S_+}{46656S_-} \\ - \frac{984057275\bar{\beta}_- \bar{\beta}_+ S_+}{186624\bar{\gamma}^2 S_-} + \frac{433693805\bar{\beta}_- S_+}{186624\bar{\gamma} S_-} + \frac{138922825\bar{\beta}_- \bar{\beta}_+ S_+}{124416\bar{\gamma} S_-} \Big), \quad (C1d)$$

$$C_{10} = \frac{417420389873}{597196800} + \frac{1216325399}{26873856}\bar{\beta}_-^2 + \frac{9152412233}{179159040}\bar{\beta}_+ + \frac{1216325399}{26873856}\bar{\beta}_+^2 \\ + \frac{4052305187\bar{\beta}_-^2}{1492992\bar{\gamma}^2} + \frac{4052305187\bar{\beta}_+^2}{1492992\bar{\gamma}^2} - \frac{2634315293\bar{\beta}_-^2}{4478976\bar{\gamma}} - \frac{173559800129\bar{\beta}_+}{59719680\bar{\gamma}} \\ - \frac{2634315293\bar{\beta}_+^2}{4478976\bar{\gamma}} + \frac{30980827673}{358318080}\bar{\gamma} + \frac{147842587}{5971968}\bar{\beta}_+\bar{\gamma} + \frac{28564471}{26873856}\bar{\gamma}^2 \\ + \frac{11949020167\bar{\beta}_+}{19906560\zeta S_-^2} - \frac{4052305187\bar{\beta}_-^2}{5971968\zeta\bar{\gamma}S_-^2} - \frac{4052305187\bar{\beta}_+^2}{5971968\zeta\bar{\gamma}S_-^2} - \frac{19637839753\bar{\gamma}}{149299200\zeta S_-^2} \\ + \frac{611720999\bar{\beta}_-S_+}{14929920S_-} + \frac{4052305187\bar{\beta}_- \bar{\beta}_+ S_+}{746496\bar{\gamma}^2 S_-} - \frac{173559800129\bar{\beta}_-S_+}{59719680\bar{\gamma}S_-} \\ - \frac{2634315293\bar{\beta}_- \bar{\beta}_+ S_+}{2239488\bar{\gamma}S_-} + \delta \left(-\frac{9152412233}{179159040}\bar{\beta}_- - \frac{1216325399}{13436928}\bar{\beta}_- \bar{\beta}_+ \right. \\ \left. - \frac{4052305187\bar{\beta}_- \bar{\beta}_+}{746496\bar{\gamma}^2} + \frac{173559800129\bar{\beta}_-}{59719680\bar{\gamma}} + \frac{2634315293\bar{\beta}_- \bar{\beta}_+}{2239488\bar{\gamma}} \right. \\ \left. - \frac{147842587}{5971968}\bar{\beta}_- \bar{\gamma} - \frac{11949020167\bar{\beta}_-}{19906560\zeta S_-^2} + \frac{4052305187\bar{\beta}_- \bar{\beta}_+}{2985984\zeta\bar{\gamma}S_-^2} - \frac{234588929299S_+}{298598400S_-} \right. \\ \left. - \frac{611720999\bar{\beta}_+S_+}{14929920S_-} - \frac{4052305187\bar{\beta}_+^2 S_+}{1492992\bar{\gamma}^2 S_-} - \frac{4052305187\bar{\beta}_+^2 S_+}{1492992\bar{\gamma}^2 S_-} \right. \\ \left. + \frac{2634315293\bar{\beta}_+^2 S_+}{4478976\bar{\gamma}S_-} + \frac{173559800129\bar{\beta}_+S_+}{59719680\bar{\gamma}S_-} + \frac{2634315293\bar{\beta}_+^2 S_+}{4478976\bar{\gamma}S_-} \right. \\ \left. - \frac{20094871289\bar{\gamma}S_+}{179159040S_-} \right) + \nu \left(-\frac{19637839753}{18662400} - \frac{1216325399}{6718464}\bar{\beta}_-^2 - \frac{4052305187\bar{\beta}_-^2}{746496\bar{\gamma}^2} \right. \\ \left. - \frac{4052305187\bar{\beta}_+^2}{746496\bar{\gamma}^2} + \frac{2634315293\bar{\beta}_-^2}{1119744\bar{\gamma}} + \frac{11949020167\bar{\beta}_+}{2488320\bar{\gamma}} - \frac{11949020167\bar{\beta}_+}{4976640\zeta S_-^2} \right. \\ \left. + \frac{4052305187\bar{\beta}_+^2}{1492992\zeta\bar{\gamma}S_-^2} + \frac{19637839753\bar{\gamma}}{37324800\zeta S_-^2} - \frac{2722571261\bar{\beta}_-S_+}{2488320S_-} - \frac{4052305187\bar{\beta}_- \bar{\beta}_+ S_+}{373248\bar{\gamma}^2 S_-} \right. \\ \left. + \frac{11949020167\bar{\beta}_-S_+}{2488320\bar{\gamma}S_-} + \frac{2634315293\bar{\beta}_- \bar{\beta}_+ S_+}{1119744\bar{\gamma}S_-} \right). \quad (C1e)$$

The coefficients associated to $\tilde{\alpha}_1^s(e_i)$ read

$$\tilde{C}_2 = \frac{2299}{324} + \frac{250}{729}\bar{\beta}_-^2 + \frac{5}{162}\bar{\beta}_+ + \frac{250}{729}\bar{\beta}_+^2 + \frac{1300\bar{\beta}_-^2}{81\bar{\gamma}^2} + \frac{1300\bar{\beta}_+^2}{81\bar{\gamma}^2} - \frac{100\bar{\beta}_-^2}{27\bar{\gamma}} - \frac{3295\bar{\beta}_+}{162\bar{\gamma}} \\ - \frac{100\bar{\beta}_+^2}{27\bar{\gamma}} + \frac{2455}{972}\bar{\gamma} + \frac{175}{243}\bar{\beta}_+\bar{\gamma} + \frac{200}{729}\bar{\gamma}^2 + \frac{185\bar{\beta}_+}{54\zeta S_-^2} - \frac{325\bar{\beta}_-^2}{81\zeta\bar{\gamma}S_-^2} - \frac{325\bar{\beta}_+^2}{81\zeta\bar{\gamma}S_-^2} \\ - \frac{59\bar{\gamma}}{81\zeta S_-^2} - \frac{110\bar{\beta}_-S_+}{81S_-} + \frac{2600\bar{\beta}_- \bar{\beta}_+ S_+}{81\bar{\gamma}^2 S_-} - \frac{3295\bar{\beta}_-S_+}{162\bar{\gamma}S_-} - \frac{200\bar{\beta}_- \bar{\beta}_+ S_+}{27\bar{\gamma}S_-} \\ + \delta \left(-\frac{5}{162}\bar{\beta}_- - \frac{500}{729}\bar{\beta}_- \bar{\beta}_+ - \frac{2600\bar{\beta}_- \bar{\beta}_+}{81\bar{\gamma}^2} + \frac{3295\bar{\beta}_-}{162\bar{\gamma}} + \frac{200\bar{\beta}_- \bar{\beta}_+}{27\bar{\gamma}} - \frac{175}{243}\bar{\beta}_- \bar{\gamma} \right)$$

$$\begin{aligned}
& -\frac{185\bar{\beta}_-}{54\zeta\mathcal{S}_-^2} + \frac{650\bar{\beta}_-\bar{\beta}_+}{81\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{937\mathcal{S}_+}{162\mathcal{S}_-} + \frac{110\bar{\beta}_+\mathcal{S}_+}{81\mathcal{S}_-} - \frac{1300\bar{\beta}_-^2\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} - \frac{1300\bar{\beta}_+^2\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} \\
& + \frac{100\bar{\beta}_-^2\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} + \frac{3295\bar{\beta}_+\mathcal{S}_+}{162\bar{\gamma}\mathcal{S}_-} + \frac{100\bar{\beta}_+^2\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} - \frac{205\bar{\gamma}\mathcal{S}_+}{162\mathcal{S}_-} \Big) + \nu \left(-\frac{472}{81} - \frac{1000}{729}\bar{\beta}_-^2 \right. \\
& - \frac{2600\bar{\beta}_-^2}{81\bar{\gamma}^2} - \frac{2600\bar{\beta}_+^2}{81\bar{\gamma}^2} + \frac{400\bar{\beta}_-^2}{27\bar{\gamma}} + \frac{740\bar{\beta}_+}{27\bar{\gamma}} - \frac{370\bar{\beta}_+}{27\zeta\mathcal{S}_-^2} + \frac{1300\bar{\beta}_+^2}{81\zeta\bar{\gamma}\mathcal{S}_-^2} + \frac{236\bar{\gamma}}{81\zeta\mathcal{S}_-^2} \\
& \left. - \frac{1580\bar{\beta}_-\mathcal{S}_+}{243\mathcal{S}_-} - \frac{5200\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} + \frac{740\bar{\beta}_-\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} + \frac{400\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{27\bar{\gamma}\mathcal{S}_-} \right), \tag{C2a}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{C}}_4 = & \frac{4867}{288} + \frac{10625}{11664}\bar{\beta}_-^2 + \frac{415}{864}\bar{\beta}_+ + \frac{10625}{11664}\bar{\beta}_+^2 + \frac{15025\bar{\beta}_-^2}{324\bar{\gamma}^2} + \frac{15025\bar{\beta}_+^2}{324\bar{\gamma}^2} - \frac{40675\bar{\beta}_-^2}{3888\bar{\gamma}} \\
& - \frac{11875\bar{\beta}_+}{216\bar{\gamma}} - \frac{40675\bar{\beta}_+^2}{3888\bar{\gamma}} + \frac{2335}{486}\bar{\gamma} + \frac{32075}{23328}\bar{\beta}_+\bar{\gamma} + \frac{3575}{7776}\bar{\gamma}^2 + \frac{13075\bar{\beta}_+}{1296\zeta\mathcal{S}_-^2} \\
& - \frac{15025\bar{\beta}_-^2}{1296\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{15025\bar{\beta}_+^2}{1296\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{157\bar{\gamma}}{72\zeta\mathcal{S}_-^2} - \frac{3685\bar{\beta}_-\mathcal{S}_+}{1944\mathcal{S}_-} + \frac{15025\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{162\bar{\gamma}^2\mathcal{S}_-} \\
& - \frac{11875\bar{\beta}_-\mathcal{S}_+}{216\bar{\gamma}\mathcal{S}_-} - \frac{40675\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{1944\bar{\gamma}\mathcal{S}_-} + \delta \left(-\frac{415}{864}\bar{\beta}_- - \frac{10625}{5832}\bar{\beta}_-\bar{\beta}_+ - \frac{15025\bar{\beta}_-\bar{\beta}_+}{162\bar{\gamma}^2} \right. \\
& + \frac{11875\bar{\beta}_-}{216\bar{\gamma}} + \frac{40675\bar{\beta}_-\bar{\beta}_+}{1944\bar{\gamma}} - \frac{32075}{23328}\bar{\beta}_-\bar{\gamma} - \frac{13075\bar{\beta}_-}{1296\zeta\mathcal{S}_-^2} + \frac{15025\bar{\beta}_-\bar{\beta}_+}{648\zeta\bar{\gamma}\mathcal{S}_-^2} \\
& - \frac{19939\mathcal{S}_+}{1296\mathcal{S}_-} + \frac{3685\bar{\beta}_+\mathcal{S}_+}{1944\mathcal{S}_-} - \frac{15025\bar{\beta}_-^2\mathcal{S}_+}{324\bar{\gamma}^2\mathcal{S}_-} - \frac{15025\bar{\beta}_+^2\mathcal{S}_+}{324\bar{\gamma}^2\mathcal{S}_-} \\
& \left. + \frac{40675\bar{\beta}_-^2\mathcal{S}_+}{3888\bar{\gamma}\mathcal{S}_-} + \frac{11875\bar{\beta}_+\mathcal{S}_+}{216\bar{\gamma}\mathcal{S}_-} + \frac{40675\bar{\beta}_+^2\mathcal{S}_+}{3888\bar{\gamma}\mathcal{S}_-} - \frac{3785\bar{\gamma}\mathcal{S}_+}{1296\mathcal{S}_-} \right) \\
& + \nu \left(-\frac{157}{9} - \frac{10625}{2916}\bar{\beta}_-^2 - \frac{15025\bar{\beta}_-^2}{162\bar{\gamma}^2} - \frac{15025\bar{\beta}_+^2}{162\bar{\gamma}^2} + \frac{40675\bar{\beta}_-^2}{972\bar{\gamma}} + \frac{13075\bar{\beta}_+}{162\bar{\gamma}} \right. \\
& - \frac{13075\bar{\beta}_+}{324\zeta\mathcal{S}_-^2} + \frac{15025\bar{\beta}_+^2}{324\zeta\bar{\gamma}\mathcal{S}_-^2} + \frac{157\bar{\gamma}}{18\zeta\mathcal{S}_-^2} - \frac{4595\bar{\beta}_-\mathcal{S}_+}{243\mathcal{S}_-} \\
& \left. - \frac{15025\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{81\bar{\gamma}^2\mathcal{S}_-} + \frac{13075\bar{\beta}_-\mathcal{S}_+}{162\bar{\gamma}\mathcal{S}_-} + \frac{40675\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{972\bar{\gamma}\mathcal{S}_-} \right), \tag{C2b}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{C}}_6 = & \frac{2822579}{93312} + \frac{13450}{6561}\bar{\beta}_-^2 + \frac{365885}{139968}\bar{\beta}_+ + \frac{13450}{6561}\bar{\beta}_+^2 + \frac{568675\bar{\beta}_-^2}{5832\bar{\gamma}^2} + \frac{568675\bar{\beta}_+^2}{5832\bar{\gamma}^2} \\
& - \frac{783875\bar{\beta}_-^2}{34992\bar{\gamma}} - \frac{5129315\bar{\beta}_+}{46656\bar{\gamma}} - \frac{783875\bar{\beta}_+^2}{34992\bar{\gamma}} + \frac{191795}{31104}\bar{\gamma} + \frac{196675}{104976}\bar{\beta}_+\bar{\gamma} \\
& + \frac{89075}{209952}\bar{\gamma}^2 + \frac{998635\bar{\beta}_+}{46656\zeta\mathcal{S}_-^2} - \frac{568675\bar{\beta}_-^2}{23328\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{568675\bar{\beta}_+^2}{23328\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{108739\bar{\gamma}}{23328\zeta\mathcal{S}_-^2} \\
& - \frac{15485\bar{\beta}_-\mathcal{S}_+}{34992\mathcal{S}_-} + \frac{568675\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{2916\bar{\gamma}^2\mathcal{S}_-} - \frac{5129315\bar{\beta}_-\mathcal{S}_+}{46656\bar{\gamma}\mathcal{S}_-} - \frac{783875\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{17496\bar{\gamma}\mathcal{S}_-} \\
& + \delta \left(-\frac{365885}{139968}\bar{\beta}_- - \frac{26900}{6561}\bar{\beta}_-\bar{\beta}_+ - \frac{568675\bar{\beta}_-\bar{\beta}_+}{2916\bar{\gamma}^2} + \frac{5129315\bar{\beta}_-}{46656\bar{\gamma}} + \frac{783875\bar{\beta}_-\bar{\beta}_+}{17496\bar{\gamma}} \right. \\
& - \frac{196675}{104976}\bar{\beta}_-\bar{\gamma} - \frac{998635\bar{\beta}_-}{46656\zeta\mathcal{S}_-^2} + \frac{568675\bar{\beta}_-\bar{\beta}_+}{11664\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{156953\mathcal{S}_+}{5184\mathcal{S}_-} + \frac{15485\bar{\beta}_+\mathcal{S}_+}{34992\mathcal{S}_-} \\
& \left. - \frac{15485\bar{\beta}_-\mathcal{S}_+}{34992\mathcal{S}_-} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{568675\bar{\beta}_-^2\mathcal{S}_+}{5832\bar{\gamma}^2\mathcal{S}_-} - \frac{568675\bar{\beta}_+^2\mathcal{S}_+}{5832\bar{\gamma}^2\mathcal{S}_-} + \frac{783875\bar{\beta}_-^2\mathcal{S}_+}{34992\bar{\gamma}\mathcal{S}_-} + \frac{5129315\bar{\beta}_+\mathcal{S}_+}{46656\bar{\gamma}\mathcal{S}_-} \\
& + \frac{783875\bar{\beta}_+^2\mathcal{S}_+}{34992\bar{\gamma}\mathcal{S}_-} - \frac{686215\bar{\gamma}\mathcal{S}_+}{139968\mathcal{S}_-} \Big) + \nu \left(-\frac{108739}{2916} - \frac{53800}{6561}\bar{\beta}_-^2 - \frac{568675\bar{\beta}_-^2}{2916\bar{\gamma}^2} \right. \\
& - \frac{568675\bar{\beta}_+^2}{2916\bar{\gamma}^2} + \frac{783875\bar{\beta}_-^2}{8748\bar{\gamma}} + \frac{998635\bar{\beta}_+}{5832\bar{\gamma}} - \frac{998635\bar{\beta}_+}{11664\zeta\mathcal{S}_-^2} \\
& + \frac{568675\bar{\beta}_+^2}{5832\zeta\bar{\gamma}\mathcal{S}_-^2} + \frac{108739\bar{\gamma}}{5832\zeta\mathcal{S}_-^2} - \frac{715855\bar{\beta}_-\mathcal{S}_+}{17496\mathcal{S}_-} - \frac{568675\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{1458\bar{\gamma}^2\mathcal{S}_-} \\
& \left. + \frac{998635\bar{\beta}_-\mathcal{S}_+}{5832\bar{\gamma}\mathcal{S}_-} + \frac{783875\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{8748\bar{\gamma}\mathcal{S}_-} \right), \tag{C2c}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{C}}_8 = & \frac{279720785}{5971968} + \frac{52915775}{13436928}\bar{\beta}_-^2 + \frac{10793545}{1492992}\bar{\beta}_+ + \frac{52915775}{13436928}\bar{\beta}_+^2 + \frac{129014575\bar{\beta}_-^2}{746496\bar{\gamma}^2} \\
& + \frac{129014575\bar{\beta}_+^2}{746496\bar{\gamma}^2} - \frac{30321725\bar{\beta}_-^2}{746496\bar{\gamma}} - \frac{140005055\bar{\beta}_+}{746496\bar{\gamma}} - \frac{30321725\bar{\beta}_+^2}{746496\bar{\gamma}} \\
& + \frac{25701095}{4478976}\bar{\gamma} + \frac{3024625}{1492992}\bar{\beta}_+\bar{\gamma} + \frac{1527475}{53747712}\bar{\gamma}^2 + \frac{2108695\bar{\beta}_+}{55296\zeta\mathcal{S}_-^2} - \frac{129014575\bar{\beta}_-^2}{2985984\zeta\bar{\gamma}\mathcal{S}_-^2} \\
& - \frac{129014575\bar{\beta}_+^2}{2985984\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{3113185\bar{\gamma}}{373248\zeta\mathcal{S}_-^2} + \frac{224185\bar{\beta}_-\mathcal{S}_+}{55296\mathcal{S}_-} + \frac{129014575\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{373248\bar{\gamma}^2\mathcal{S}_-} \\
& - \frac{140005055\bar{\beta}_-\mathcal{S}_+}{746496\bar{\gamma}\mathcal{S}_-} - \frac{30321725\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{373248\bar{\gamma}\mathcal{S}_-} + \delta \left(-\frac{10793545}{1492992}\bar{\beta}_- - \frac{52915775}{6718464}\bar{\beta}_-\bar{\beta}_+ \right. \\
& - \frac{129014575\bar{\beta}_-\bar{\beta}_+}{373248\bar{\gamma}^2} + \frac{140005055\bar{\beta}_-}{746496\bar{\gamma}} + \frac{30321725\bar{\beta}_-\bar{\beta}_+}{373248\bar{\gamma}} - \frac{3024625}{1492992}\bar{\beta}_-\bar{\gamma} \\
& - \frac{2108695\bar{\beta}_-}{55296\zeta\mathcal{S}_-^2} + \frac{129014575\bar{\beta}_-\bar{\beta}_+}{1492992\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{76048315\mathcal{S}_+}{1492992\mathcal{S}_-} - \frac{224185\bar{\beta}_+\mathcal{S}_+}{55296\mathcal{S}_-} \\
& - \frac{129014575\bar{\beta}_-^2\mathcal{S}_+}{746496\bar{\gamma}^2\mathcal{S}_-} - \frac{129014575\bar{\beta}_+^2\mathcal{S}_+}{746496\bar{\gamma}^2\mathcal{S}_-} + \frac{30321725\bar{\beta}_-^2\mathcal{S}_+}{746496\bar{\gamma}\mathcal{S}_-} \\
& \left. + \frac{140005055\bar{\beta}_+\mathcal{S}_+}{746496\bar{\gamma}\mathcal{S}_-} + \frac{30321725\bar{\beta}_+^2\mathcal{S}_+}{746496\bar{\gamma}\mathcal{S}_-} - \frac{10479115\bar{\gamma}\mathcal{S}_+}{1492992\mathcal{S}_-} \right) \\
& + \nu \left(-\frac{3113185}{46656} - \frac{52915775}{3359232}\bar{\beta}_-^2 - \frac{129014575\bar{\beta}_-^2}{373248\bar{\gamma}^2} - \frac{129014575\bar{\beta}_+^2}{373248\bar{\gamma}^2} + \frac{30321725\bar{\beta}_-^2}{186624\bar{\gamma}} \right. \\
& + \frac{2108695\bar{\beta}_+}{6912\bar{\gamma}} - \frac{2108695\bar{\beta}_+}{13824\zeta\mathcal{S}_-^2} + \frac{129014575\bar{\beta}_+^2}{746496\zeta\bar{\gamma}\mathcal{S}_-^2} + \frac{3113185\bar{\gamma}}{93312\zeta\mathcal{S}_-^2} \\
& \left. - \frac{41715005\bar{\beta}_-\mathcal{S}_+}{559872\mathcal{S}_-} - \frac{129014575\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{186624\bar{\gamma}^2\mathcal{S}_-} + \frac{2108695\bar{\beta}_-\mathcal{S}_+}{6912\bar{\gamma}\mathcal{S}_-} + \frac{30321725\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{186624\bar{\gamma}\mathcal{S}_-} \right), \tag{C2d}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{C}}_{10} = & \frac{13218658111}{199065600} + \frac{180939719}{26873856}\bar{\beta}_-^2 + \frac{897726671}{59719680}\bar{\beta}_+ + \frac{180939719}{26873856}\bar{\beta}_+^2 \\
& + \frac{410983607\bar{\beta}_-^2}{1492992\bar{\gamma}^2} + \frac{410983607\bar{\beta}_+^2}{1492992\bar{\gamma}^2} - \frac{295961663\bar{\beta}_-^2}{4478976\bar{\gamma}} - \frac{5769772463\bar{\beta}_+}{19906560\bar{\gamma}} \\
& - \frac{295961663\bar{\beta}_+^2}{4478976\bar{\gamma}} + \frac{987177893}{358318080}\bar{\gamma} + \frac{89549603}{53747712}\bar{\beta}_+\bar{\gamma} - \frac{7615843}{8957952}\bar{\gamma}^2 + \frac{3639706241\bar{\beta}_+}{59719680\zeta\mathcal{S}_-^2}
\end{aligned}$$

$$\begin{aligned}
& -\frac{410983607\bar{\beta}_-^2}{5971968\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{410983607\bar{\beta}_+^2}{5971968\zeta\bar{\gamma}\mathcal{S}_-^2} - \frac{665257271\bar{\gamma}}{49766400\zeta\mathcal{S}_-^2} + \frac{560781617\bar{\beta}_-\mathcal{S}_+}{44789760\mathcal{S}_-} \\
& + \frac{410983607\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{746496\bar{\gamma}\mathcal{S}_-} - \frac{5769772463\bar{\beta}_-\mathcal{S}_+}{19906560\bar{\gamma}\mathcal{S}_-} - \frac{295961663\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{2239488\bar{\gamma}\mathcal{S}_-} \\
& + \delta \left(-\frac{897726671}{59719680}\bar{\beta}_- - \frac{180939719}{13436928}\bar{\beta}_-\bar{\beta}_+ - \frac{410983607\bar{\beta}_-\bar{\beta}_+}{746496\bar{\gamma}^2} + \frac{5769772463\bar{\beta}_-}{19906560\bar{\gamma}} \right. \\
& + \frac{295961663\bar{\beta}_-\bar{\beta}_+}{2239488\bar{\gamma}} - \frac{89549603}{53747712}\bar{\beta}_-\bar{\gamma} - \frac{3639706241\bar{\beta}_-}{59719680\zeta\mathcal{S}_-^2} + \frac{410983607\bar{\beta}_-\bar{\beta}_+}{2985984\zeta\bar{\gamma}\mathcal{S}_-^2} \\
& - \frac{23226346279\mathcal{S}_+}{298598400\mathcal{S}_-} - \frac{560781617\bar{\beta}_+\mathcal{S}_+}{44789760\mathcal{S}_-} - \frac{410983607\bar{\beta}_-^2\mathcal{S}_+}{1492992\bar{\gamma}^2\mathcal{S}_-} - \frac{410983607\bar{\beta}_+^2\mathcal{S}_+}{1492992\bar{\gamma}^2\mathcal{S}_-} \\
& + \frac{295961663\bar{\beta}_-^2\mathcal{S}_+}{4478976\bar{\gamma}\mathcal{S}_-} + \frac{5769772463\bar{\beta}_+\mathcal{S}_+}{19906560\bar{\gamma}\mathcal{S}_-} + \frac{295961663\bar{\beta}_+^2\mathcal{S}_+}{4478976\bar{\gamma}\mathcal{S}_-} - \frac{543403783\bar{\gamma}\mathcal{S}_+}{59719680\mathcal{S}_-} \Big) \\
& + \nu \left(-\frac{665257271}{6220800} - \frac{180939719}{6718464}\bar{\beta}_-^2 - \frac{410983607\bar{\beta}_-^2}{746496\bar{\gamma}^2} - \frac{410983607\bar{\beta}_+^2}{746496\bar{\gamma}^2} \right. \\
& + \frac{295961663\bar{\beta}_-^2}{1119744\bar{\gamma}} + \frac{3639706241\bar{\beta}_+}{7464960\bar{\gamma}} - \frac{3639706241\bar{\beta}_+}{14929920\zeta\mathcal{S}_-^2} + \frac{410983607\bar{\beta}_+^2}{1492992\zeta\bar{\gamma}\mathcal{S}_-^2} \\
& + \frac{665257271\bar{\gamma}}{12441600\zeta\mathcal{S}_-^2} - \frac{2719530689\bar{\beta}_-\mathcal{S}_+}{22394880\mathcal{S}_-} - \frac{410983607\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{373248\bar{\gamma}^2\mathcal{S}_-} \\
& \left. + \frac{3639706241\bar{\beta}_-\mathcal{S}_+}{7464960\bar{\gamma}\mathcal{S}_-} + \frac{295961663\bar{\beta}_-\bar{\beta}_+\mathcal{S}_+}{1119744\bar{\gamma}\mathcal{S}_-} \right). \tag{C2e}
\end{aligned}$$

Appendix D. Expression of the orbit-averaged instantaneous scalar fluxes

$$\begin{aligned}
f_1 = & -\frac{8\bar{\gamma}}{5\zeta} - 8\mathcal{S}_-^2 - \frac{16}{3}\bar{\beta}_+\mathcal{S}_-^2 + \frac{16\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} - \frac{8}{3}\bar{\gamma}\mathcal{S}_-^2 + \frac{16\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\
& + \left[\frac{16}{3}\bar{\beta}_-\mathcal{S}_-^2 - \frac{16\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} - \frac{16\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \right] \delta - \frac{16}{3}\mathcal{S}_-^2\nu, \tag{D1a}
\end{aligned}$$

$$\begin{aligned}
f_2 = & \frac{32\bar{\beta}_+}{\zeta} - \frac{48\bar{\beta}_+^2}{\zeta\bar{\gamma}} - \frac{51\bar{\gamma}}{5\zeta} + 72\mathcal{S}_-^2 - \frac{56}{3}\bar{\beta}_+\mathcal{S}_-^2 + \frac{96\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{96\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} \\
& - \frac{16\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} + 36\bar{\gamma}\mathcal{S}_-^2 + \frac{192\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{16\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\
& + \left[\frac{56}{3}\bar{\beta}_-\mathcal{S}_-^2 - \frac{32\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} - \frac{32\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \right] \delta - \frac{124}{3}\mathcal{S}_-^2\nu, \tag{D1b}
\end{aligned}$$

$$\begin{aligned}
f_3 = & \frac{8\bar{\beta}_+}{\zeta} - \frac{12\bar{\beta}_+^2}{\zeta\bar{\gamma}} - \frac{39\bar{\gamma}}{20\zeta} + 13\mathcal{S}_-^2 + \frac{24\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{24\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{10\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} + \frac{14}{3}\bar{\gamma}\mathcal{S}_-^2 \\
& + \frac{48\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{10\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \left[-\frac{2\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} - \frac{2\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \right] \delta - \frac{25}{3}\mathcal{S}_-^2\nu, \tag{D1c}
\end{aligned}$$

$$\begin{aligned}
f_4 = & -\frac{32\bar{\beta}_+}{5\zeta} - \frac{8\bar{\beta}_-^2}{\zeta\bar{\gamma}} - \frac{8\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{97\bar{\gamma}}{14\zeta} + \frac{64\bar{\beta}_+\bar{\gamma}}{15\zeta} + \frac{211\bar{\gamma}^2}{120\zeta} - \frac{3\bar{\gamma}^3}{16\zeta} - 68\mathcal{S}_-^2 - \frac{16}{3}\bar{\beta}_-^2\mathcal{S}_-^2 \\
& - 8\bar{\beta}_+\mathcal{S}_-^2 - \frac{16}{3}\bar{\beta}_+^2\mathcal{S}_-^2 + \frac{32\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{32\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{32\bar{\beta}_-^2\mathcal{S}_-^2}{3\bar{\gamma}} \\
& - \frac{16\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} - \frac{32\bar{\beta}_+^2\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{325}{6}\bar{\gamma}\mathcal{S}_-^2 - \frac{45}{4}\bar{\gamma}^2\mathcal{S}_-^2 - \frac{16}{3}\bar{\chi}_+\mathcal{S}_-^2 + \frac{16\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}} \\
& + \frac{21}{2}\zeta\mathcal{S}_-^4 + \frac{21}{4}\zeta\bar{\gamma}\mathcal{S}_-^4 - \frac{16}{3}\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ + \frac{64\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{16\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\
& - \frac{64\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} + \frac{16\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{3}{2}\zeta\mathcal{S}_+^4 + \frac{3}{4}\zeta\bar{\gamma}\mathcal{S}_+^4 + \left[\frac{32\bar{\beta}_-}{5\zeta} + \frac{16\bar{\beta}_-\bar{\beta}_+}{\zeta\bar{\gamma}} \right. \\
& - \frac{64\bar{\beta}_-\bar{\gamma}}{15\zeta} + 8\bar{\beta}_-\mathcal{S}_-^2 + \frac{32}{3}\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 - \frac{64\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{16\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} + \frac{64\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2}{3\bar{\gamma}} \\
& + \frac{16}{3}\bar{\chi}_-\mathcal{S}_-^2 - \frac{16\bar{\chi}_-\mathcal{S}_-^2}{\bar{\gamma}} + 22\mathcal{S}_-\mathcal{S}_+ + \frac{16}{3}\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ - \frac{32\bar{\beta}_-^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{32\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& + \frac{32\bar{\beta}_-^2\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} + \frac{16\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{32\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} + 15\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{3}{2}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ \\
& \left. - \frac{16\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + 6\zeta\mathcal{S}_-^3\mathcal{S}_+ + 3\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ + 6\zeta\mathcal{S}_-\mathcal{S}_+^3 + 3\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3 \right] \delta \\
& + \left[\frac{32\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{14\bar{\gamma}}{3\zeta} + \frac{\bar{\gamma}^2}{3\zeta} + \frac{\bar{\gamma}^3}{6\zeta} + \frac{142}{3}\mathcal{S}_-^2 + \frac{64}{3}\bar{\beta}_-^2\mathcal{S}_-^2 + \frac{140}{3}\bar{\beta}_+\mathcal{S}_-^2 - \frac{192\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} \right. \\
& + \frac{64\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{320\bar{\beta}_-^2\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{224\bar{\beta}_+\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{64\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}} + \frac{28}{3}\bar{\gamma}\mathcal{S}_-^2 - \frac{2}{3}\bar{\gamma}^2\mathcal{S}_-^2 \\
& + \frac{32}{3}\bar{\chi}_+\mathcal{S}_-^2 - \frac{32\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}} - \frac{28}{3}\zeta\mathcal{S}_-^4 - \frac{14}{3}\zeta\bar{\gamma}\mathcal{S}_-^4 - \frac{128\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& \left. - \frac{224\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} + \frac{128\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} - \frac{32\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} - \frac{4}{3}\zeta\mathcal{S}_+^4 - \frac{2}{3}\zeta\bar{\gamma}\mathcal{S}_+^4 \right] \nu \\
& + \left[-\frac{44}{3}\bar{\beta}_-\mathcal{S}_-^2 + \frac{80\bar{\beta}_-\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{80\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \right] \delta\nu + \frac{8}{3}\mathcal{S}_-^2\nu^2, \tag{D1d}
\end{aligned}$$

$$\begin{aligned}
f_5 = & \frac{4\bar{\beta}_+}{\zeta} - \frac{64\bar{\beta}_+^2}{3\zeta} + \frac{384\bar{\beta}_-^2\bar{\beta}_+}{\zeta\bar{\gamma}^2} - \frac{384\bar{\beta}_+^3}{\zeta\bar{\gamma}^2} - \frac{204\bar{\beta}_-^2}{\zeta\bar{\gamma}} + \frac{164\bar{\beta}_+^2}{\zeta\bar{\gamma}} - \frac{160\bar{\beta}_+^3}{\zeta\bar{\gamma}} \\
& - \frac{5641\bar{\gamma}}{140\zeta} + \frac{1394\bar{\beta}_+\bar{\gamma}}{15\zeta} - \frac{3803\bar{\gamma}^2}{120\zeta} - \frac{5\bar{\gamma}^3}{16\zeta} + \frac{32\bar{\chi}_+}{\zeta} - \frac{96\bar{\beta}_+\bar{\chi}_+}{\zeta\bar{\gamma}} - 29\mathcal{S}_-^2 - 16\bar{\beta}_-^2\mathcal{S}_-^2 \\
& - 96\bar{\beta}_+\mathcal{S}_-^2 - 16\bar{\beta}_+^2\mathcal{S}_-^2 - \frac{768\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^3} + \frac{768\bar{\beta}_+^3\mathcal{S}_-^2}{\bar{\gamma}^3} + \frac{272\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} \\
& + \frac{320\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{112\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{320\bar{\beta}_+^3\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{256\bar{\beta}_-^2\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{264\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} - \frac{160\bar{\beta}_+^2\mathcal{S}_-^2}{3\bar{\gamma}} \\
& + \frac{183}{2}\bar{\gamma}\mathcal{S}_-^2 - \frac{392}{3}\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 + \frac{631}{12}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{192\bar{\beta}_-\bar{\chi}_-\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{160}{3}\bar{\chi}_+\mathcal{S}_-^2 \\
& + \frac{192\bar{\beta}_+\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{16\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}} + \frac{35}{2}\zeta\mathcal{S}_-^4 + \frac{35}{4}\zeta\bar{\gamma}\mathcal{S}_-^4 + 128\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - \frac{768\bar{\beta}_-^3\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3}
\end{aligned}$$

$$\begin{aligned}
& + \frac{768\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3} + \frac{160\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{640\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{264\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\
& - \frac{416\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} + \frac{192\bar{\beta}_+\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{16\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{192\bar{\beta}_-\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& + \frac{5}{2}\zeta\mathcal{S}_+^4 + \frac{5}{4}\zeta\bar{\gamma}\mathcal{S}_+^4 + \left[\frac{92\bar{\beta}_-}{5\zeta} - \frac{32\bar{\beta}_-\bar{\beta}_+}{3\zeta} + \frac{136\bar{\beta}_-\bar{\beta}_+}{\zeta\bar{\gamma}} + \frac{160\bar{\beta}_-\bar{\beta}_+^2}{\zeta\bar{\gamma}} - \frac{198\bar{\beta}_-\bar{\gamma}}{5\zeta} \right. \\
& - \frac{32\bar{\chi}_-}{\zeta} + \frac{96\bar{\beta}_+\bar{\chi}_-}{\zeta\bar{\gamma}} - 32\bar{\beta}_-\mathcal{S}_-^2 + 32\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 - \frac{768\bar{\beta}_-^3\mathcal{S}_-^2}{\bar{\gamma}^3} + \frac{768\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^3} \\
& - \frac{320\bar{\beta}_-^3\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{544\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{320\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{360\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} + \frac{800\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2}{3\bar{\gamma}} \\
& + \frac{392}{3}\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 + \frac{160}{3}\bar{\chi}_-\mathcal{S}_-^2 - \frac{192\bar{\beta}_+\bar{\chi}_-\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{16\bar{\chi}_-\mathcal{S}_-^2}{\bar{\gamma}} - \frac{192\bar{\beta}_-\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}^2} \\
& + 73\mathcal{S}_-\mathcal{S}_+ - 256\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ - \frac{768\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3} + \frac{768\bar{\beta}_+^3\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3} - \frac{80\bar{\beta}_-^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& - \frac{640\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{464\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{352\bar{\beta}_-^2\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} - \frac{360\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{448\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \\
& + 55\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{5}{2}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ - \frac{192\bar{\beta}_-\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{192\bar{\beta}_+\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& - \frac{16\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + 10\zeta\mathcal{S}_-^3\mathcal{S}_+ + 5\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ + 10\zeta\mathcal{S}_-\mathcal{S}_+^3 + 5\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3 \Big] \delta \\
& + \left[-\frac{448\bar{\beta}_+}{3\zeta} + \frac{352\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{1094\bar{\gamma}}{15\zeta} + \frac{10\bar{\gamma}^2}{3\zeta} + \frac{5\bar{\gamma}^3}{3\zeta} - \frac{473}{3}\mathcal{S}_-^2 + 64\bar{\beta}_-^2\mathcal{S}_-^2 + \frac{1228}{3}\bar{\beta}_+\mathcal{S}_-^2 \right. \\
& - \frac{1216\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{192\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{3136\bar{\beta}_-^2\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{872\bar{\beta}_+\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{640\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}} - 172\bar{\gamma}\mathcal{S}_-^2 \\
& - \frac{20}{3}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{320}{3}\bar{\chi}_+\mathcal{S}_-^2 - \frac{128\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}} - \frac{280}{3}\zeta\mathcal{S}_-^4 - \frac{140}{3}\zeta\bar{\gamma}\mathcal{S}_-^4 \\
& - \frac{1408\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{872\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} + \frac{1216\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} - \frac{128\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\
& \left. - \frac{40}{3}\zeta\mathcal{S}_+^4 - \frac{20}{3}\zeta\bar{\gamma}\mathcal{S}_+^4 \right] \nu + \left[-\frac{484}{3}\bar{\beta}_-\mathcal{S}_-^2 + \frac{968\bar{\beta}_-\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{968\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \right] \delta\nu \\
& + \frac{364}{3}\mathcal{S}_-^2\nu^2,
\end{aligned} \tag{D1e}$$

$$\begin{aligned}
f_6 = & \frac{1286\bar{\beta}_+}{5\zeta} - \frac{832\bar{\beta}_+^2}{3\zeta} + \frac{288\bar{\beta}_-^2\bar{\beta}_+}{\zeta\bar{\gamma}^2} - \frac{288\bar{\beta}_+^3}{\zeta\bar{\gamma}^2} - \frac{165\bar{\beta}_-^2}{\zeta\bar{\gamma}} - \frac{273\bar{\beta}_+^2}{\zeta\bar{\gamma}} \\
& - \frac{40\bar{\beta}_+^3}{\zeta\bar{\gamma}} - \frac{7269\bar{\gamma}}{80\zeta} + \frac{5921\bar{\beta}_+\bar{\gamma}}{30\zeta} - \frac{9835\bar{\gamma}^2}{192\zeta} - \frac{43\bar{\gamma}^3}{384\zeta} \\
& + \frac{24\bar{\chi}_+}{\zeta} - \frac{72\bar{\beta}_+\bar{\chi}_+}{\zeta\bar{\gamma}} + 489\mathcal{S}_-^2 + 6\bar{\beta}_-^2\mathcal{S}_-^2 - 337\bar{\beta}_+\mathcal{S}_-^2 + 6\bar{\beta}_+^2\mathcal{S}_-^2 \\
& - \frac{576\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^3} + \frac{576\bar{\beta}_+^3\mathcal{S}_-^2}{\bar{\gamma}^3} + \frac{1020\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{80\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{732\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} \\
& + \frac{80\bar{\beta}_+^3\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{1196\bar{\beta}_-^2\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{206\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} + \frac{1508\bar{\beta}_+^2\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{7835}{16}\bar{\gamma}\mathcal{S}_-^2 - \frac{250}{3}\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 \\
& + \frac{3801}{32}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{144\bar{\beta}_-\bar{\chi}_-\mathcal{S}_-^2}{\bar{\gamma}^2} - 6\bar{\chi}_+\mathcal{S}_-^2 + \frac{144\bar{\beta}_+\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{18\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}} \\
& + \frac{301}{48}\zeta\mathcal{S}_+^4 + \frac{301}{96}\zeta\bar{\gamma}\mathcal{S}_+^4 - \frac{370}{3}\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - \frac{576\bar{\beta}_-^3\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3} + \frac{576\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3} \\
& + \frac{1752\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{160\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{206\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{2704\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \\
& + \frac{144\bar{\beta}_+\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{18\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{144\bar{\beta}_-\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{43}{48}\zeta\mathcal{S}_+^4 + \frac{43}{96}\zeta\bar{\gamma}\mathcal{S}_+^4 \\
& + \left[\frac{136\bar{\beta}_-\bar{\beta}_+}{3\zeta} + \frac{90\bar{\beta}_-\bar{\beta}_+}{\zeta\bar{\gamma}} + \frac{40\bar{\beta}_-\bar{\beta}_+^2}{\zeta\bar{\gamma}} - \frac{307\bar{\beta}_-\bar{\gamma}}{10\zeta} - \frac{24\bar{\chi}_-}{\zeta} + \frac{72\bar{\beta}_+\bar{\chi}_-}{\zeta\bar{\gamma}} \right. \\
& + 69\bar{\beta}_-\mathcal{S}_-^2 - 12\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 - \frac{576\bar{\beta}_-^3\mathcal{S}_-^2}{\bar{\gamma}^3} + \frac{576\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^3} - \frac{80\bar{\beta}_-^3\mathcal{S}_-^2}{\bar{\gamma}^2} \\
& - \frac{360\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{80\bar{\beta}_-\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{214\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} + \frac{80\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{250}{3}\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 \\
& + 6\bar{\chi}_-\mathcal{S}_-^2 - \frac{144\bar{\beta}_+\bar{\chi}_-\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{18\bar{\chi}_-\mathcal{S}_-^2}{\bar{\gamma}} - \frac{144\bar{\beta}_-\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{9}{4}\mathcal{S}_-\mathcal{S}_+ \\
& - \frac{434}{3}\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ - \frac{576\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3} + \frac{576\bar{\beta}_+^3\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^3} - \frac{36\bar{\beta}_-^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& - \frac{160\bar{\beta}_-^2\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{324\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{116\bar{\beta}_-^2\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} - \frac{214\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\
& + \frac{196\bar{\beta}_+^2\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} + \frac{223}{24}\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{43}{48}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ - \frac{144\bar{\beta}_-\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& - \frac{144\bar{\beta}_+\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{18\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{43}{12}\zeta\mathcal{S}_-^3\mathcal{S}_+ + \frac{43}{24}\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ \\
& + \frac{43}{12}\zeta\mathcal{S}_-\mathcal{S}_+^3 + \frac{43}{24}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3 \Big] \delta + \left[-\frac{296\bar{\beta}_+}{\zeta} + \frac{480\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{454\bar{\gamma}}{5\zeta} \right. \\
& + \frac{3\bar{\gamma}^2}{8\zeta} + \frac{3\bar{\gamma}^3}{16\zeta} - \frac{1101}{2}\mathcal{S}_-^2 - 24\bar{\beta}_-^2\mathcal{S}_-^2 + \frac{973}{6}\bar{\beta}_+\mathcal{S}_-^2 - \frac{1104\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{816\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} \\
& + \frac{568\bar{\beta}_-^2\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{152\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} - \frac{72\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}} - \frac{1607}{6}\bar{\gamma}\mathcal{S}_-^2 - \frac{3}{4}\bar{\gamma}^2\mathcal{S}_-^2 + 12\bar{\chi}_+\mathcal{S}_-^2
\end{aligned}$$

$$\begin{aligned}
& -\frac{36\bar{\chi}_+\mathcal{S}_-^2}{\bar{\gamma}} - \frac{21}{2}\zeta\mathcal{S}_-^4 - \frac{21}{4}\zeta\bar{\gamma}\mathcal{S}_-^4 - \frac{1920\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{152\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\
& + \frac{352\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} - \frac{36\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} - \frac{3}{2}\zeta\mathcal{S}_+^4 - \frac{3}{4}\zeta\bar{\gamma}\mathcal{S}_+^4 \Big] \nu \\
& + \left[-\frac{793}{6}\bar{\beta}_-\mathcal{S}_-^2 + \frac{238\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} + \frac{238\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \right] \delta\nu + \frac{397}{2}\mathcal{S}_-^2\nu^2, \tag{D1f}
\end{aligned}$$

$$\begin{aligned}
f_7 = & \frac{137\bar{\beta}_+}{10\zeta} - \frac{8\bar{\beta}_+^2}{\zeta} - \frac{9\bar{\beta}_-^2}{2\zeta\bar{\gamma}} - \frac{57\bar{\beta}_+^2}{2\zeta\bar{\gamma}} - \frac{771\bar{\gamma}}{224\zeta} + \frac{10\bar{\beta}_+\bar{\gamma}}{3\zeta} - \frac{19\bar{\gamma}^2}{20\zeta} + \frac{231}{8}\mathcal{S}_-^2 \\
& - \frac{8}{3}\bar{\beta}_+\mathcal{S}_-^2 + \frac{66\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{66\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{16\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} - \frac{19\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}} \\
& + \frac{16\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}} + \frac{67}{3}\bar{\gamma}\mathcal{S}_-^2 + \frac{14}{3}\bar{\gamma}^2\mathcal{S}_-^2 - \frac{8}{3}\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ + \frac{132\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& - \frac{19\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{32\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \left[-\frac{3\bar{\beta}_-}{10\zeta} + \frac{3\bar{\beta}_-\bar{\beta}_+}{\zeta\bar{\gamma}} + \frac{2}{3}\bar{\beta}_-\mathcal{S}_-^2 \right. \\
& - \frac{12\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}} - \frac{21}{8}\mathcal{S}_-\mathcal{S}_+ + \frac{2}{3}\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ - \frac{6\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{6\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& + \frac{\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} - \frac{5}{4}\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \Big] \delta + \left[-\frac{92\bar{\beta}_+}{3\zeta} + \frac{46\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{299\bar{\gamma}}{40\zeta} - \frac{455}{12}\mathcal{S}_-^2 - \frac{92\bar{\beta}_-\mathcal{S}_-^2}{\bar{\gamma}^2} \right. \\
& - \frac{92\bar{\beta}_+\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{115\bar{\beta}_+\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{37}{3}\bar{\gamma}\mathcal{S}_-^2 - \frac{184\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{115\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \Big] \nu \\
& + \left[\frac{23\bar{\beta}_-\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{23\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \right] \delta\nu + \frac{75}{4}\mathcal{S}_-^2\nu^2, \tag{D1g}
\end{aligned}$$

$$\begin{aligned}
f_8 = & \mathcal{S}_-^2 \left\{ 50 + \frac{155}{3}\bar{\gamma} + \frac{40}{3}\bar{\gamma}^2 - \frac{20}{3}\zeta\mathcal{S}_-^2 - \frac{10}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \right. \\
& + \left[-\frac{20}{3}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{10}{3}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right] \delta + \left[-20 + \frac{20}{3}\bar{\beta}_+ - \frac{40}{3}\bar{\gamma} \right] \nu - \frac{20}{3}\bar{\beta}_-\delta\nu \Big\}, \tag{D1h}
\end{aligned}$$

$$\begin{aligned}
f_9 = & \mathcal{S}_-^2 \left\{ 155 + \frac{961}{6}\bar{\gamma} + \frac{124}{3}\bar{\gamma}^2 - \frac{62}{3}\zeta\mathcal{S}_-^2 - \frac{31}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \right. \\
& + \left[-\frac{62}{3}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{31}{3}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right] \delta + \left[-62 + \frac{62}{3}\bar{\beta}_+ - \frac{124}{3}\bar{\gamma} \right] \nu - \frac{62}{3}\bar{\beta}_-\delta\nu \Big\}, \tag{D1i}
\end{aligned}$$

$$\begin{aligned}
f_{10} = & \mathcal{S}_-^2 \left\{ 20 + \frac{62}{3}\bar{\gamma} + \frac{16}{3}\bar{\gamma}^2 - \frac{8}{3}\zeta\mathcal{S}_-^2 - \frac{4}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \right. \\
& + \left[-\frac{8}{3}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{4}{3}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right] \delta + \left[-8 + \frac{8}{3}\bar{\beta}_+ - \frac{16}{3}\bar{\gamma} \right] \nu - \frac{8}{3}\bar{\beta}_-\delta\nu \Big\}. \tag{D1j}
\end{aligned}$$

$$\begin{aligned} g_1 = & -\frac{8\bar{\gamma}}{5\zeta} - 8S_-^2 - \frac{16}{3}\bar{\beta}_+S_-^2 + \frac{16\bar{\beta}_+S_-^2}{\bar{\gamma}} - \frac{8}{3}\bar{\gamma}S_-^2 + \frac{16\bar{\beta}_-S_-S_+}{\bar{\gamma}} \\ & + \left[\frac{16}{3}\bar{\beta}_-S_-^2 - \frac{16\bar{\beta}_-S_-^2}{\bar{\gamma}} - \frac{16\bar{\beta}_+S_-S_+}{\bar{\gamma}} \right] \delta - \frac{16}{3}S_-^2\nu, \end{aligned} \quad (D2a)$$

$$\begin{aligned} g_2 = & -\frac{7\bar{\gamma}}{5\zeta} + 26S_-^2 - 4\bar{\beta}_+S_-^2 + \frac{8\bar{\beta}_+S_-^2}{\bar{\gamma}} + \frac{40}{3}\bar{\gamma}S_-^2 + \frac{8\bar{\beta}_-S_-S_+}{\bar{\gamma}} \\ & + \left[4\bar{\beta}_-S_-^2 - \frac{8\bar{\beta}_-S_-^2}{\bar{\gamma}} - \frac{8\bar{\beta}_+S_-S_+}{\bar{\gamma}} \right] \delta - \frac{44}{3}S_-^2\nu, \end{aligned} \quad (D2b)$$

$$\begin{aligned} g_3 = & -\frac{32\bar{\beta}_+}{5\zeta} - \frac{8\bar{\beta}_-^2}{\zeta\bar{\gamma}} - \frac{8\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{97\bar{\gamma}}{14\zeta} + \frac{64\bar{\beta}_+\bar{\gamma}}{15\zeta} + \frac{113\bar{\gamma}^2}{60\zeta} - \frac{\bar{\gamma}^3}{8\zeta} \\ & - 38S_-^2 - \frac{16}{3}\bar{\beta}_-^2S_-^2 - 8\bar{\beta}_+S_-^2 - \frac{16}{3}\bar{\beta}_+^2S_-^2 + \frac{32\bar{\beta}_-^2S_-^2}{\bar{\gamma}^2} \\ & + \frac{32\bar{\beta}_+^2S_-^2}{\bar{\gamma}^2} - \frac{32\bar{\beta}_-^2S_-^2}{3\bar{\gamma}} - \frac{16\bar{\beta}_+S_-^2}{\bar{\gamma}} - \frac{32\bar{\beta}_+^2S_-^2}{3\bar{\gamma}} - \frac{71}{3}\bar{\gamma}S_-^2 - \frac{7}{2}\bar{\gamma}^2S_-^2 \\ & - \frac{16}{3}\bar{\chi}_+S_-^2 + \frac{16\bar{\chi}_+S_-^2}{\bar{\gamma}} + 7\zeta S_-^4 + \frac{7}{2}\zeta\bar{\gamma}S_-^4 \\ & - \frac{16}{3}\bar{\beta}_-S_-S_+ + \frac{64\bar{\beta}_-\bar{\beta}_+S_-S_+}{\bar{\gamma}^2} - \frac{16\bar{\beta}_-S_-S_+}{\bar{\gamma}} - \frac{64\bar{\beta}_-\bar{\beta}_+S_-S_+}{3\bar{\gamma}} \\ & + \frac{16\bar{\chi}_-S_-S_+}{\bar{\gamma}} + \zeta S_+^4 + \frac{1}{2}\zeta\bar{\gamma}S_+^4 + \left[\frac{32\bar{\beta}_-}{5\zeta} + \frac{16\bar{\beta}_-\bar{\beta}_+}{\zeta\bar{\gamma}} - \frac{64\bar{\beta}_-\bar{\gamma}}{15\zeta} \right. \\ & + 8\bar{\beta}_-S_-^2 + \frac{32}{3}\bar{\beta}_-\bar{\beta}_+S_-^2 - \frac{64\bar{\beta}_-\bar{\beta}_+S_-^2}{\bar{\gamma}^2} + \frac{16\bar{\beta}_-S_-^2}{\bar{\gamma}} + \frac{64\bar{\beta}_-\bar{\beta}_+S_-^2}{3\bar{\gamma}} + \frac{16}{3}\bar{\chi}_-S_-^2 \\ & - \frac{16\bar{\chi}_-S_-^2}{\bar{\gamma}} + 22S_-S_+ + \frac{16}{3}\bar{\beta}_+S_-S_+ - \frac{32\bar{\beta}_-^2S_-S_+}{\bar{\gamma}^2} - \frac{32\bar{\beta}_+^2S_-S_+}{\bar{\gamma}^2} \\ & + \frac{32\bar{\beta}_-^2S_-S_+}{3\bar{\gamma}} + \frac{16\bar{\beta}_+S_-S_+}{\bar{\gamma}} + \frac{32\bar{\beta}_+^2S_-S_+}{3\bar{\gamma}} + 14\bar{\gamma}S_-S_+ + \bar{\gamma}^2S_-S_+ \\ & \left. - \frac{16\bar{\chi}_+S_-S_+}{\bar{\gamma}} + 4\zeta S_-^3S_+ + 2\zeta\bar{\gamma}S_-^3S_+ + 4\zeta S_-S_+^3 + 2\zeta\bar{\gamma}S_-S_+^3 \right] \delta \\ & + \left[\frac{32\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{14\bar{\gamma}}{3\zeta} + \frac{\bar{\gamma}^2}{3\zeta} + \frac{\bar{\gamma}^3}{6\zeta} + \frac{106}{3}S_-^2 + \frac{64}{3}\bar{\beta}_-^2S_-^2 + \frac{152}{3}\bar{\beta}_+S_-^2 - \frac{192\bar{\beta}_-^2S_-^2}{\bar{\gamma}^2} \right. \\ & + \frac{64\bar{\beta}_+^2S_-^2}{\bar{\gamma}^2} + \frac{320\bar{\beta}_-^2S_-^2}{3\bar{\gamma}} - \frac{224\bar{\beta}_+S_-^2}{3\bar{\gamma}} - \frac{64\bar{\beta}_+^2S_-^2}{\bar{\gamma}} + \frac{4}{3}\bar{\gamma}S_-^2 - \frac{2}{3}\bar{\gamma}^2S_-^2 + \frac{32}{3}\bar{\chi}_+S_-^2 \\ & - \frac{32\bar{\chi}_+S_-^2}{\bar{\gamma}} - \frac{28}{3}\zeta S_-^4 - \frac{14}{3}\zeta\bar{\gamma}S_-^4 - \frac{128\bar{\beta}_-\bar{\beta}_+S_-S_+}{\bar{\gamma}^2} - \frac{224\bar{\beta}_-S_-S_+}{3\bar{\gamma}} \\ & + \frac{128\bar{\beta}_-\bar{\beta}_+S_-S_+}{3\bar{\gamma}} - \frac{32\bar{\chi}_-S_-S_+}{\bar{\gamma}} - \frac{4}{3}\zeta S_+^4 - \frac{2}{3}\zeta\bar{\gamma}S_+^4 \left. \right] \nu + \left[-\frac{56}{3}\bar{\beta}_-S_-^2 + \frac{80\bar{\beta}_-S_-^2}{3\bar{\gamma}} \right. \\ & \left. + \frac{80\bar{\beta}_+S_-S_+}{3\bar{\gamma}} \right] \delta\nu + \frac{8}{3}S_-^2\nu^2, \end{aligned} \quad (D2c)$$

$$\begin{aligned}
g_4 = & -\frac{44\bar{\beta}_+}{5\zeta} - \frac{24\bar{\beta}_-^2}{\zeta\bar{\gamma}} - \frac{24\bar{\beta}_+^2}{\zeta\bar{\gamma}} - \frac{173\bar{\gamma}}{14\zeta} + \frac{28\bar{\beta}_+\bar{\gamma}}{3\zeta} - \frac{1013\bar{\gamma}^2}{120\zeta} - \frac{\bar{\gamma}^3}{48\zeta} - 68S_-^2 - \frac{16}{3}\bar{\beta}_-^2S_-^2 \\
& + \frac{152}{3}\bar{\beta}_+S_-^2 - \frac{16}{3}\bar{\beta}_+^2S_-^2 + \frac{96\bar{\beta}_-^2S_-^2}{\bar{\gamma}^2} + \frac{96\bar{\beta}_+^2S_-^2}{\bar{\gamma}^2} - \frac{112\bar{\beta}_-^2S_-^2}{3\bar{\gamma}} \\
& + \frac{184\bar{\beta}_+S_-^2}{\bar{\gamma}} - \frac{112\bar{\beta}_+^2S_-^2}{3\bar{\gamma}} - \frac{63}{2}\bar{\gamma}S_-^2 - \frac{104}{3}\bar{\beta}_+\bar{\gamma}S_-^2 + \frac{17}{12}\bar{\gamma}^2S_-^2 - 16\bar{\chi}_+S_-^2 \\
& + \frac{24\bar{\chi}_+S_-^2}{\bar{\gamma}} + \frac{7}{6}\zeta S_-^4 + \frac{7}{12}\zeta\bar{\gamma}S_-^4 + 104\bar{\beta}_-S_-S_+ + \frac{192\bar{\beta}_-\bar{\beta}_+S_-S_+}{\bar{\gamma}^2} + \frac{184\bar{\beta}_-S_-S_+}{\bar{\gamma}} \\
& - \frac{224\bar{\beta}_-\bar{\beta}_+S_-S_+}{3\bar{\gamma}} + \frac{24\bar{\chi}_-S_-S_+}{\bar{\gamma}} + \frac{1}{6}\zeta S_+^4 + \frac{1}{12}\zeta\bar{\gamma}S_+^4 + \left[\frac{44\bar{\beta}_-}{5\zeta} + \frac{48\bar{\beta}_-\bar{\beta}_+}{\zeta\bar{\gamma}} \right. \\
& - \frac{28\bar{\beta}_-\bar{\gamma}}{3\zeta} - \frac{152}{3}\bar{\beta}_-S_-^2 + \frac{32}{3}\bar{\beta}_-\bar{\beta}_+S_-^2 - \frac{192\bar{\beta}_-\bar{\beta}_+S_-^2}{\bar{\gamma}^2} - \frac{184\bar{\beta}_-S_-^2}{\bar{\gamma}} \\
& + \frac{224\bar{\beta}_-\bar{\beta}_+S_-^2}{3\bar{\gamma}} + \frac{104}{3}\bar{\beta}_-\bar{\gamma}S_-^2 + 16\bar{\chi}_-S_-^2 - \frac{24\bar{\chi}_-S_-^2}{\bar{\gamma}} + 34S_-S_+ - 104\bar{\beta}_+S_-S_+ \\
& - \frac{96\bar{\beta}_-^2S_-S_+}{\bar{\gamma}^2} - \frac{96\bar{\beta}_+^2S_-S_+}{\bar{\gamma}^2} + \frac{112\bar{\beta}_-^2S_-S_+}{3\bar{\gamma}} - \frac{184\bar{\beta}_+S_-S_+}{\bar{\gamma}} + \frac{112\bar{\beta}_+^2S_-S_+}{3\bar{\gamma}} \\
& + \frac{61}{3}\bar{\gamma}S_-S_+ + \frac{1}{6}\bar{\gamma}^2S_-S_+ - \frac{24\bar{\chi}_+S_-S_+}{\bar{\gamma}} + \frac{2}{3}\zeta S_-^3S_+ + \frac{1}{3}\zeta\bar{\gamma}S_-^3S_+ + \frac{2}{3}\zeta S_-S_+^3 \\
& \left. + \frac{1}{3}\zeta\bar{\gamma}S_-S_+^3 \right] \delta + \left[-\frac{32\bar{\beta}_+}{\zeta} + \frac{96\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{83\bar{\gamma}}{3\zeta} + \frac{\bar{\gamma}^2}{\zeta} + \frac{\bar{\gamma}^3}{2\zeta} - 76S_-^2 + \frac{64}{3}\bar{\beta}_-^2S_-^2 \right. \\
& + 144\bar{\beta}_+S_-^2 - \frac{384\bar{\beta}_-^2S_-^2}{\bar{\gamma}^2} + \frac{1024\bar{\beta}_-^2S_-^2}{3\bar{\gamma}} - \frac{536\bar{\beta}_+S_-^2}{3\bar{\gamma}} - \frac{192\bar{\beta}_+^2S_-^2}{\bar{\gamma}} - \frac{212}{3}\bar{\gamma}S_-^2 \\
& - 2\bar{\gamma}^2S_-^2 + 32\bar{\chi}_+S_-^2 - \frac{48\bar{\chi}_+S_-^2}{\bar{\gamma}} - 28\zeta S_-^4 - 14\zeta\bar{\gamma}S_-^4 - \frac{384\bar{\beta}_-\bar{\beta}_+S_-S_+}{\bar{\gamma}^2} \\
& - \frac{536\bar{\beta}_-S_-S_+}{3\bar{\gamma}} + \frac{448\bar{\beta}_-\bar{\beta}_+S_-S_+}{3\bar{\gamma}} - \frac{48\bar{\chi}_-S_-S_+}{\bar{\gamma}} - 4\zeta S_+^4 - 2\zeta\bar{\gamma}S_+^4 \left. \right] \nu \\
& + \left[-68\bar{\beta}_-S_-^2 + \frac{464\bar{\beta}_-S_-^2}{3\bar{\gamma}} + \frac{464\bar{\beta}_+S_-S_+}{3\bar{\gamma}} \right] \delta\nu + 64S_-^2\nu^2, \tag{D2d}
\end{aligned}$$

$$\begin{aligned}
g_5 = & \frac{\bar{\beta}_+}{5\zeta} - \frac{3\bar{\beta}_-^2}{\zeta\bar{\gamma}} - \frac{3\bar{\beta}_+^2}{\zeta\bar{\gamma}} - \frac{533\bar{\gamma}}{112\zeta} + \frac{2\bar{\beta}_+\bar{\gamma}}{5\zeta} - \frac{521\bar{\gamma}^2}{240\zeta} + \frac{\bar{\gamma}^3}{32\zeta} + \frac{181}{4}S_-^2 \\
& + \frac{4}{3}\bar{\beta}_+S_-^2 + \frac{12\bar{\beta}_-^2S_-^2}{\bar{\gamma}^2} + \frac{12\bar{\beta}_+^2S_-^2}{\bar{\gamma}^2} + \frac{20\bar{\beta}_+S_-^2}{\bar{\gamma}} \\
& + \frac{467}{12}\bar{\gamma}S_-^2 - \frac{8}{3}\bar{\beta}_+\bar{\gamma}S_-^2 + \frac{63}{8}\bar{\gamma}^2S_-^2 - \frac{7}{4}\zeta S_-^4 - \frac{7}{8}\zeta\bar{\gamma}S_-^4 + \frac{28}{3}\bar{\beta}_-S_-S_+ \\
& + \frac{24\bar{\beta}_-\bar{\beta}_+S_-S_+}{\bar{\gamma}^2} + \frac{20\bar{\beta}_-S_-S_+}{\bar{\gamma}} - \frac{1}{4}\zeta S_+^4 \\
& - \frac{1}{8}\zeta\bar{\gamma}S_+^4 + \left[-\frac{\bar{\beta}_-}{5\zeta} + \frac{6\bar{\beta}_-\bar{\beta}_+}{\zeta\bar{\gamma}} - \frac{2\bar{\beta}_-\bar{\gamma}}{5\zeta} - \frac{4}{3}\bar{\beta}_-S_-^2 - \frac{24\bar{\beta}_-\bar{\beta}_+S_-^2}{\bar{\gamma}^2} \right. \\
& - \frac{20\bar{\beta}_-S_-^2}{\bar{\gamma}} + \frac{8}{3}\bar{\beta}_-\bar{\gamma}S_-^2 + \frac{1}{4}S_-S_+ - \frac{28}{3}\bar{\beta}_+S_-S_+ \\
& \left. - \frac{12\bar{\beta}_-^2S_-S_+}{\bar{\gamma}^2} - \frac{12\bar{\beta}_+^2S_-S_+}{\bar{\gamma}^2} - \frac{20\bar{\beta}_+S_-S_+}{\bar{\gamma}} - \frac{1}{4}\bar{\gamma}^2S_-S_+ - \zeta S_-^3S_+ \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2}\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ - \zeta\mathcal{S}_-\mathcal{S}_+^3 - \frac{1}{2}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3 \Big] \delta + \left[-\frac{8\bar{\beta}_+}{\zeta} + \frac{12\bar{\beta}_+^2}{\zeta\bar{\gamma}} + \frac{71\bar{\gamma}}{12\zeta} \right. \\
& -\frac{395}{6}\mathcal{S}_-^2 + \frac{34}{3}\bar{\beta}_+\mathcal{S}_-^2 - \frac{24\bar{\beta}_-^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{24\bar{\beta}_+^2\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{38\bar{\beta}_+\mathcal{S}_-^2}{3\bar{\gamma}} - \frac{98}{3}\bar{\gamma}\mathcal{S}_-^2 \\
& \left. -\frac{48\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} - \frac{38\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \right] \nu + \left[-\frac{34}{3}\bar{\beta}_-\mathcal{S}_-^2 + \frac{74\bar{\beta}_-\mathcal{S}_-^2}{3\bar{\gamma}} + \frac{74\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+}{3\bar{\gamma}} \right] \delta\nu \\
& + \frac{179}{6}\mathcal{S}_-^2\nu^2, \tag{D2e}
\end{aligned}$$

$$\begin{aligned}
g_6 = \mathcal{S}_-^2 \Big\{ 20 + \frac{62}{3}\bar{\gamma} + \frac{16}{3}\bar{\gamma}^2 - \frac{8}{3}\zeta\mathcal{S}_-^2 - \frac{4}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
+ \left[-\frac{8}{3}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{4}{3}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right] \delta + \left[-8 + \frac{8}{3}\bar{\beta}_+ - \frac{16}{3}\bar{\gamma} \right] \nu - \frac{8}{3}\bar{\beta}_-\delta\nu \Big\}, \tag{D2f}
\end{aligned}$$

$$\begin{aligned}
g_7 = \mathcal{S}_-^2 \Big\{ 20 + \frac{62}{3}\bar{\gamma} + \frac{16}{3}\bar{\gamma}^2 - \frac{8}{3}\zeta\mathcal{S}_-^2 - \frac{4}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
+ \left[-\frac{8}{3}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{4}{3}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right] \delta + \left[-8 + \frac{8}{3}\bar{\beta}_+ - \frac{16}{3}\bar{\gamma} \right] \nu - \frac{8}{3}\bar{\beta}_-\delta\nu \Big\}, \tag{D2g}
\end{aligned}$$

$$\begin{aligned}
g_8 = \mathcal{S}_-^2 \Big\{ -40 - \frac{124}{3}\bar{\gamma} - \frac{32}{3}\bar{\gamma}^2 + \frac{16}{3}\zeta\mathcal{S}_-^2 + \frac{8}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
+ \left[\frac{16}{3}\zeta\mathcal{S}_-\mathcal{S}_+ + \frac{8}{3}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right] \delta + \left[16 - \frac{16}{3}\bar{\beta}_+ + \frac{32}{3}\bar{\gamma} \right] \nu + \frac{16}{3}\bar{\beta}_-\delta\nu \Big\}. \tag{D2h}
\end{aligned}$$

Appendix E. Coefficients entering the expression of the evolution of the orbital parameters

The coefficients associated to the frequency evolution equation (7.3a) read

$$\begin{aligned}
\mathfrak{X}_1 = 288\bar{\zeta}^{-1} + 120\bar{\zeta}^{-1}\bar{\gamma} - 30\mathcal{S}_-^2 - 160\bar{\beta}_+\mathcal{S}_-^2 + 40\bar{\gamma}\mathcal{S}_-^2 + 240\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
+ 240\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \delta(160\bar{\beta}_-\mathcal{S}_-^2 - 240\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
- 240\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+) - 70\mathcal{S}_-^2\nu, \tag{E1a}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_2 = 876\bar{\zeta}^{-1} + 480\bar{\beta}_+\bar{\zeta}^{-1} + 285\bar{\zeta}^{-1}\bar{\gamma} - 720\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} + 1095\mathcal{S}_-^2 \\
- 260\bar{\beta}_+\mathcal{S}_-^2 + 540\bar{\gamma}\mathcal{S}_-^2 - 240\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
+ 1440\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 1440\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - 240\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 2880\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
+ \delta(260\bar{\beta}_-\mathcal{S}_-^2 - 480\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 - 480\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+) - 625\mathcal{S}_-^2\nu, \tag{E1b}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_3 = 111\bar{\zeta}^{-1} + 120\bar{\beta}_+\bar{\zeta}^{-1} + \frac{105}{4}\bar{\zeta}^{-1}\bar{\gamma} - 180\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} + 270\mathcal{S}_-^2 \\
+ 110\bar{\gamma}\mathcal{S}_-^2 - 150\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 + 360\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
+ 360\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - 150\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 720\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
+ \delta(-30\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 - 30\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+) - 130\mathcal{S}_-^2\nu, \tag{E1c}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_4 = & -\frac{1486}{35}\bar{\zeta}^{-1} - \frac{416}{5}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{167}{10}\bar{\zeta}^{-1}\bar{\gamma} - 32\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma} + \frac{1}{8}\bar{\zeta}^{-1}\bar{\gamma}^2 \\
& + \frac{1}{16}\bar{\zeta}^{-1}\bar{\gamma}^3 - 8\bar{\beta}_-^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - 8\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} + \frac{59}{2}\mathcal{S}_-^2 \\
& - \frac{64}{9}\bar{\beta}_-^2\mathcal{S}_-^2 - 12\bar{\beta}_+\mathcal{S}_-^2 - \frac{64}{9}\bar{\beta}_+^2\mathcal{S}_-^2 + \frac{391}{6}\bar{\gamma}\mathcal{S}_-^2 - \frac{304}{9}\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 \\
& + \frac{983}{36}\bar{\gamma}^2\mathcal{S}_-^2 - 32\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 8\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 32\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 32\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& + 32\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{64}{3}\bar{\chi}_+\mathcal{S}_-^2 + 16\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - \frac{7}{2}\zeta\mathcal{S}_-^4 - \frac{7}{4}\zeta\bar{\gamma}\mathcal{S}_-^4 + 16\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ \\
& + 8\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 64\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 64\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 16\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \\
& - \frac{1}{2}\zeta\mathcal{S}_+^4 - \frac{1}{4}\zeta\bar{\gamma}\mathcal{S}_+^4 + \delta\left(\frac{416}{5}\bar{\beta}_-\bar{\zeta}^{-1} + 32\bar{\beta}_-\bar{\zeta}^{-1}\bar{\gamma} + 16\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1}\right. \\
& + 12\bar{\beta}_-\mathcal{S}_-^2 + \frac{128}{9}\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 + \frac{304}{9}\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 - 8\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& + 64\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 64\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 + \frac{64}{3}\bar{\chi}_-\mathcal{S}_-^2 - 16\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 + 22\mathcal{S}_-\mathcal{S}_+ \\
& - 16\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ + 11\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - \frac{1}{2}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ + 32\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 8\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 32\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 32\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 32\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& - 16\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ - 2\zeta\mathcal{S}_-^3\mathcal{S}_+ - \zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ - 2\zeta\mathcal{S}_-\mathcal{S}_+^3 - \zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3) \\
& + \nu\left(-\frac{264}{5}\bar{\zeta}^{-1} - 22\bar{\zeta}^{-1}\bar{\gamma} + \frac{4}{3}\bar{\zeta}^{-1}\bar{\gamma}^2 + \frac{2}{3}\bar{\zeta}^{-1}\bar{\gamma}^3 + 32\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - \frac{25}{2}\mathcal{S}_-^2\right. \\
& + \frac{256}{9}\bar{\beta}_-^2\mathcal{S}_-^2 + \frac{1216}{9}\bar{\beta}_+\mathcal{S}_-^2 - \frac{472}{9}\bar{\gamma}\mathcal{S}_-^2 - \frac{8}{3}\bar{\gamma}^2\mathcal{S}_-^2 + 384\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& - 72\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 256\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 192\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 64\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + \frac{128}{3}\bar{\chi}_+\mathcal{S}_-^2 \\
& - 32\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - \frac{112}{3}\zeta\mathcal{S}_-^4 - \frac{56}{3}\zeta\bar{\gamma}\mathcal{S}_-^4 - 72\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 128\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 128\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 32\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ - \frac{16}{3}\zeta\mathcal{S}_+^4 \\
& \left. - \frac{8}{3}\zeta\bar{\gamma}\mathcal{S}_+^4\right) + \delta\nu\left(-\frac{280}{9}\bar{\beta}_-\mathcal{S}_-^2 + 24\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + 24\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right) + \frac{43}{18}\mathcal{S}_-^2\nu^2,
\end{aligned}
\tag{E1d}$$

$$\begin{aligned}
\mathfrak{X}_5 = & \frac{2193}{7}\bar{\zeta}^{-1} - \frac{5308}{15}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{128}{3}\bar{\beta}_+^2\bar{\zeta}^{-1} + \frac{20543}{60}\bar{\zeta}^{-1}\bar{\gamma} - \frac{214}{3}\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma} \\
& + \frac{3769}{48}\bar{\zeta}^{-1}\bar{\gamma}^2 - \frac{13}{32}\bar{\zeta}^{-1}\bar{\gamma}^3 - 204\bar{\beta}_-^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} + 188\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - 128\bar{\beta}_+^3\bar{\zeta}^{-1}\bar{\gamma}^{-1} \\
& + 384\bar{\beta}_-^2\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-2} - 384\bar{\beta}_+^3\bar{\zeta}^{-1}\bar{\gamma}^{-2} + 32\bar{\zeta}^{-1}\bar{\chi}_+ - 96\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1}\bar{\chi}_+ - \frac{49}{4}\mathcal{S}_-^2 \\
& - \frac{8}{9}\bar{\beta}_-^2\mathcal{S}_-^2 - \frac{484}{3}\bar{\beta}_+\mathcal{S}_-^2 - \frac{8}{9}\bar{\beta}_+^2\mathcal{S}_-^2 + \frac{1621}{12}\bar{\gamma}\mathcal{S}_-^2 - \frac{488}{3}\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 + \frac{4885}{72}\bar{\gamma}^2\mathcal{S}_-^2 \\
& - \frac{304}{3}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 296\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{112}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 224\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2
\end{aligned}$$

$$\begin{aligned}
& + 256\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_-^2 - 160\bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 + 256\bar{\beta}_+^3 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 768\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-3} \mathcal{S}_-^2 \\
& + 768\bar{\beta}_+^3 \bar{\gamma}^{-3} \mathcal{S}_-^2 + 192\bar{\beta}_- \bar{\gamma}^{-2} \bar{\chi}_- \mathcal{S}_-^2 - \frac{136}{3} \bar{\chi}_+ \mathcal{S}_-^2 + 16\bar{\gamma}^{-1} \bar{\chi}_+ \mathcal{S}_-^2 \\
& + 192\bar{\beta}_+ \bar{\gamma}^{-2} \bar{\chi}_+ \mathcal{S}_-^2 + \frac{91}{4} \zeta \mathcal{S}_-^4 + \frac{91}{8} \zeta \bar{\gamma} \mathcal{S}_-^4 + \frac{416}{3} \bar{\beta}_- \mathcal{S}_- \mathcal{S}_+ \\
& + 296\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ - \frac{416}{3} \bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ + 64\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ \\
& + 512\bar{\beta}_- \bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ - 768\bar{\beta}_-^3 \bar{\gamma}^{-3} \mathcal{S}_- \mathcal{S}_+ + 768\bar{\beta}_- \bar{\beta}_+^2 \bar{\gamma}^{-3} \mathcal{S}_- \mathcal{S}_+ \\
& + 16\bar{\gamma}^{-1} \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+ + 192\bar{\beta}_+ \bar{\gamma}^{-2} \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+ + 192\bar{\beta}_- \bar{\gamma}^{-2} \bar{\chi}_+ \mathcal{S}_- \mathcal{S}_+ \\
& + \frac{13}{4} \zeta \mathcal{S}_+^4 + \frac{13}{8} \zeta \bar{\gamma} \mathcal{S}_+^4 + \delta \left(\frac{5404}{15} \bar{\beta}_- \bar{\zeta}^{-1} + \frac{32}{3} \bar{\beta}_- \bar{\beta}_+ \bar{\zeta}^{-1} \right. \\
& + \frac{374}{3} \bar{\beta}_- \bar{\zeta}^{-1} \bar{\gamma} + 136\bar{\beta}_- \bar{\beta}_+ \bar{\zeta}^{-1} \bar{\gamma}^{-1} + 128\bar{\beta}_- \bar{\beta}_+^2 \bar{\zeta}^{-1} \bar{\gamma}^{-1} - 32\bar{\zeta}^{-1} \bar{\chi}_- \\
& + 96\bar{\beta}_+ \bar{\zeta}^{-1} \bar{\gamma}^{-1} \bar{\chi}_- + \frac{100}{3} \bar{\beta}_- \mathcal{S}_-^2 + \frac{16}{9} \bar{\beta}_- \bar{\beta}_+ \mathcal{S}_-^2 + \frac{488}{3} \bar{\beta}_- \bar{\gamma} \mathcal{S}_-^2 \\
& - 368\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^2 + \frac{800}{3} \bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^2 - 256\bar{\beta}_-^3 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 544\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_-^2 \\
& - 256\bar{\beta}_- \bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 768\bar{\beta}_-^3 \bar{\gamma}^{-3} \mathcal{S}_-^2 + 768\bar{\beta}_- \bar{\beta}_+^2 \bar{\gamma}^{-3} \mathcal{S}_-^2 + \frac{136}{3} \bar{\chi}_- \mathcal{S}_-^2 \\
& - 16\bar{\gamma}^{-1} \bar{\chi}_- \mathcal{S}_-^2 - 192\bar{\beta}_+ \bar{\gamma}^{-2} \bar{\chi}_- \mathcal{S}_-^2 - 192\bar{\beta}_- \bar{\gamma}^{-2} \bar{\chi}_+ \mathcal{S}_-^2 + 73\mathcal{S}_- \mathcal{S}_+ \\
& - \frac{800}{3} \bar{\beta}_+ \mathcal{S}_- \mathcal{S}_+ + \frac{113}{2} \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ + \frac{13}{4} \bar{\gamma}^2 \mathcal{S}_- \mathcal{S}_+ + \frac{304}{3} \bar{\beta}_-^2 \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \\
& - 368\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ + \frac{496}{3} \bar{\beta}_+^2 \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ - 80\bar{\beta}_-^2 \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ \\
& - 512\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ - 464\bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ - 768\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-3} \mathcal{S}_- \mathcal{S}_+ \\
& + 768\bar{\beta}_+^3 \bar{\gamma}^{-3} \mathcal{S}_- \mathcal{S}_+ - 192\bar{\beta}_- \bar{\gamma}^{-2} \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+ \\
& - 16\bar{\gamma}^{-1} \bar{\chi}_+ \mathcal{S}_- \mathcal{S}_+ - 192\bar{\beta}_+ \bar{\gamma}^{-2} \bar{\chi}_+ \mathcal{S}_- \mathcal{S}_+ + 13\zeta \mathcal{S}_-^3 \mathcal{S}_+ + \frac{13}{2} \zeta \bar{\gamma} \mathcal{S}_-^3 \mathcal{S}_+ \\
& + 13\zeta \mathcal{S}_- \mathcal{S}_+^3 + \frac{13}{2} \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+^3 \Big) + \nu \left(-570\bar{\zeta}^{-1} - 144\bar{\beta}_+ \bar{\zeta}^{-1} - \frac{427}{2} \bar{\zeta}^{-1} \bar{\gamma} \right. \\
& + \frac{17}{6} \bar{\zeta}^{-1} \bar{\gamma}^2 + \frac{17}{12} \bar{\zeta}^{-1} \bar{\gamma}^3 + 344\bar{\beta}_+^2 \bar{\zeta}^{-1} \bar{\gamma}^{-1} - \frac{2095}{12} \mathcal{S}_-^2 + \frac{32}{9} \bar{\beta}_-^2 \mathcal{S}_-^2 \\
& + \frac{3718}{9} \bar{\beta}_+ \mathcal{S}_-^2 - \frac{1682}{9} \bar{\gamma} \mathcal{S}_-^2 - \frac{17}{3} \bar{\gamma}^2 \mathcal{S}_-^2 + \frac{3040}{3} \bar{\beta}_-^2 \bar{\gamma}^{-1} \mathcal{S}_-^2 - 296\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^2 \\
& - 544\bar{\beta}_+^2 \bar{\gamma}^{-1} \mathcal{S}_-^2 - 1200\bar{\beta}_-^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 176\bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 + \frac{272}{3} \bar{\chi}_+ \mathcal{S}_-^2 \\
& - 128\bar{\gamma}^{-1} \bar{\chi}_+ \mathcal{S}_-^2 - \frac{238}{3} \zeta \mathcal{S}_-^4 - \frac{119}{3} \zeta \bar{\gamma} \mathcal{S}_-^4 - 296\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \\
& + \frac{1408}{3} \bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ - 1376\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ - 128\bar{\gamma}^{-1} \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+ \\
& \left. - \frac{34}{3} \zeta \mathcal{S}_+^4 - \frac{17}{3} \zeta \bar{\gamma} \mathcal{S}_+^4 \right) + \delta \nu \left(-\frac{1810}{9} \bar{\beta}_- \mathcal{S}_-^2 + 320\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^2 + 320\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \right) \\
& + \frac{1373}{12} \mathcal{S}_-^2 \nu^2, \tag{E1e}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_6 = & \frac{12217}{20}\bar{\zeta}^{-1} + \frac{3086}{15}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{1040}{3}\bar{\beta}_+^2\bar{\zeta}^{-1} + \frac{120713}{240}\bar{\zeta}^{-1}\bar{\gamma} + \frac{530}{3}\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma} \\
& + \frac{5895}{64}\bar{\zeta}^{-1}\bar{\gamma}^2 - \frac{43}{384}\bar{\zeta}^{-1}\bar{\gamma}^3 - 165\bar{\beta}_-^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - 387\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} \\
& - 32\bar{\beta}_+^3\bar{\zeta}^{-1}\bar{\gamma}^{-1} + 288\bar{\beta}_-^2\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-2} - 288\bar{\beta}_+^3\bar{\zeta}^{-1}\bar{\gamma}^{-2} + 24\bar{\zeta}^{-1}\bar{\chi}_+ \\
& - 72\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1}\bar{\chi}_+ + \frac{1173}{2}\mathcal{S}_-^2 + \frac{10}{3}\bar{\beta}_-^2\mathcal{S}_-^2 - 391\bar{\beta}_+\mathcal{S}_-^2 + \frac{10}{3}\bar{\beta}_+^2\mathcal{S}_-^2 \\
& + \frac{28097}{48}\bar{\gamma}\mathcal{S}_-^2 - \frac{842}{9}\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 + \frac{40865}{288}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{1592}{3}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 257\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& + \frac{1928}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 1248\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 64\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 + 960\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& + 64\bar{\beta}_+^3\bar{\gamma}^{-2}\mathcal{S}_-^2 - 576\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-3}\mathcal{S}_-^2 + 576\bar{\beta}_+^3\bar{\gamma}^{-3}\mathcal{S}_-^2 + 144\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-^2 \\
& - 6\bar{\chi}_+\mathcal{S}_-^2 - 18\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 + 144\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-^2 + \frac{301}{48}\zeta\mathcal{S}_-^4 + \frac{301}{96}\zeta\bar{\gamma}\mathcal{S}_-^4 \\
& - \frac{466}{3}\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - 257\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{3520}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 2208\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + 128\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 576\bar{\beta}_-^3\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& + 576\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 18\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ + 144\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \\
& + 144\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ + \frac{43}{48}\zeta\mathcal{S}_+^4 + \frac{43}{96}\zeta\bar{\gamma}\mathcal{S}_+^4 + \delta\left(\frac{1912}{15}\bar{\beta}_-\bar{\zeta}^{-1} + \frac{152}{3}\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\right. \\
& + \frac{98}{3}\bar{\beta}_-\bar{\zeta}^{-1}\bar{\gamma} + 90\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1} + 32\bar{\beta}_-\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - 24\bar{\zeta}^{-1}\bar{\chi}_- \\
& + 72\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1}\bar{\chi}_- + 59\bar{\beta}_-\mathcal{S}_-^2 - \frac{20}{3}\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 + \frac{842}{9}\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 - 277\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& + \frac{32}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 64\bar{\beta}_-^3\bar{\gamma}^{-2}\mathcal{S}_-^2 - 360\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 - 64\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& - 576\bar{\beta}_-^3\bar{\gamma}^{-3}\mathcal{S}_-^2 + 576\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-3}\mathcal{S}_-^2 + 6\bar{\chi}_-\mathcal{S}_-^2 + 18\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 - 144\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-^2 \\
& - 144\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-^2 - \frac{9}{4}\mathcal{S}_-\mathcal{S}_+ - \frac{530}{3}\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ + \frac{223}{24}\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{43}{48}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ \\
& - \frac{152}{3}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 277\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{184}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 36\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& - 128\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 324\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 576\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& + 576\bar{\beta}_+^3\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 144\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ + 18\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ - 144\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ \\
& + \frac{43}{12}\zeta\mathcal{S}_-^3\mathcal{S}_+ + \frac{43}{24}\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ + \frac{43}{12}\zeta\mathcal{S}_-\mathcal{S}_+^3 + \frac{43}{24}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3) \\
& + \nu\left(-\frac{5061}{10}\bar{\zeta}^{-1} - 300\bar{\beta}_+\bar{\zeta}^{-1} - \frac{1287}{8}\bar{\zeta}^{-1}\bar{\gamma} + \frac{3}{8}\bar{\zeta}^{-1}\bar{\gamma}^2 + \frac{3}{16}\bar{\zeta}^{-1}\bar{\gamma}^3 + 486\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1}\right. \\
& - \frac{3701}{6}\mathcal{S}_-^2 - \frac{40}{3}\bar{\beta}_-^2\mathcal{S}_-^2 + \frac{2819}{18}\bar{\beta}_+\mathcal{S}_-^2 - \frac{5395}{18}\bar{\gamma}\mathcal{S}_-^2 - \frac{3}{4}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{520}{3}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& + 153\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 72\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 1116\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - 828\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 12\bar{\chi}_+\mathcal{S}_-^2 \\
& \left. - 36\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - \frac{21}{2}\zeta\mathcal{S}_-^4 - \frac{21}{4}\zeta\bar{\gamma}\mathcal{S}_-^4 + 153\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{304}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)
\end{aligned}$$

$$\begin{aligned}
& -1944\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 36\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ - \frac{3}{2}\zeta\mathcal{S}_+^4 - \frac{3}{4}\zeta\bar{\gamma}\mathcal{S}_+^4 \Big) \\
& + \delta\nu \left(-\frac{2279}{18}\bar{\beta}_-\mathcal{S}_-^2 + 243\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + 243\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right) + 204\mathcal{S}_-^2\nu^2, \quad (\text{E1f})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_7 = & \frac{11717}{280}\bar{\zeta}^{-1} + \frac{337}{10}\bar{\beta}_+\bar{\zeta}^{-1} - 24\bar{\beta}_+^2\bar{\zeta}^{-1} + \frac{4993}{160}\bar{\zeta}^{-1}\bar{\gamma} + 14\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma} + \frac{23}{4}\bar{\zeta}^{-1}\bar{\gamma}^2 \\
& - \frac{9}{2}\bar{\beta}_-^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - \frac{117}{2}\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} + \frac{561}{8}\mathcal{S}_-^2 - 16\bar{\beta}_+\mathcal{S}_-^2 + \frac{176}{3}\bar{\gamma}\mathcal{S}_-^2 + \frac{112}{9}\bar{\gamma}^2\mathcal{S}_-^2 \\
& + 48\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 44\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 + 48\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 126\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 126\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& - 16\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - 44\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 96\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 252\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left(-\frac{3}{10}\bar{\beta}_-\bar{\zeta}^{-1} + 3\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1} - 2\bar{\beta}_-\mathcal{S}_-^2 - 4\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 \right. \\
& - 12\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{21}{8}\mathcal{S}_-\mathcal{S}_+ - 2\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ - \frac{5}{4}\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 4\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \left. - 6\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 6\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \right) + \nu \left(-\frac{148}{5}\bar{\zeta}^{-1} - 32\bar{\beta}_+\bar{\zeta}^{-1} - 7\bar{\zeta}^{-1}\bar{\gamma} \right. \\
& + 48\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - \frac{353}{6}\mathcal{S}_-^2 - \frac{208}{9}\bar{\gamma}\mathcal{S}_-^2 + 40\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 96\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& \left. - 96\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 40\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 192\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \right) \\
& + \delta\nu \left(8\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + 8\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right) + \frac{184}{9}\mathcal{S}_-^2\nu^2, \quad (\text{E1g})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_8 = & -10\mathcal{S}_-^2 \left[1 + \frac{31}{30}\bar{\gamma} + \frac{4}{15}\bar{\gamma}^2 - \frac{2}{15}\zeta\mathcal{S}_-^2 - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-^2 + \delta \left(-\frac{2}{15}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right) \right. \\
& \left. + \nu \left(-\frac{2}{5} + \frac{2}{15}\bar{\beta}_+ - \frac{4}{15}\bar{\gamma} \right) - \frac{2}{15}\bar{\beta}_-\delta\nu \right], \quad (\text{E1h})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_9 = & 200\mathcal{S}_-^2 \left[1 + \frac{31}{30}\bar{\gamma} + \frac{4}{15}\bar{\gamma}^2 - \frac{2}{15}\zeta\mathcal{S}_-^2 - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-^2 + \delta \left(-\frac{2}{15}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right) \right. \\
& \left. + \nu \left(-\frac{2}{5} + \frac{2}{15}\bar{\beta}_+ - \frac{4}{15}\bar{\gamma} \right) - \frac{2}{15}\bar{\beta}_-\delta\nu \right], \quad (\text{E1i})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{X}_{10} = & 35\mathcal{S}_-^2 \left[1 + \frac{31}{30}\bar{\gamma} + \frac{4}{15}\bar{\gamma}^2 - \frac{2}{15}\zeta\mathcal{S}_-^2 - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-^2 + \delta \left(-\frac{2}{15}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right) \right. \\
& \left. + \nu \left(-\frac{2}{5} + \frac{2}{15}\bar{\beta}_+ - \frac{4}{15}\bar{\gamma} \right) - \frac{2}{15}\bar{\beta}_-\delta\nu \right], \quad (\text{E1j})
\end{aligned}$$

$$\mathfrak{X}_{11} = \frac{96}{5}\bar{\zeta}^{-1} \left(1 + \frac{1}{2}\bar{\gamma} \right), \quad (\text{E1k})$$

$$\mathfrak{X}_{12} = -\frac{8}{5}\bar{\zeta}^{-1}\bar{\gamma} + \frac{32}{5}\mathcal{S}_-^2 - \frac{32}{5}\mathcal{S}_-\mathcal{S}_+\delta - \frac{64}{5}\mathcal{S}_-^2\nu, \quad (\text{E1l})$$

$$\begin{aligned}
\mathfrak{X}_{13} = & -\frac{36}{5}\mathcal{S}_-^2 - \frac{8}{3}\bar{\beta}_+\mathcal{S}_-^2 - \frac{4}{3}\bar{\gamma}\mathcal{S}_-^2 + 8\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 + 8\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + \delta \left(\frac{8}{3}\bar{\beta}_-\mathcal{S}_-^2 - 8\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{16}{5}\mathcal{S}_-\mathcal{S}_+ - 8\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right), \quad (\text{E1m})
\end{aligned}$$

$$\mathfrak{X}_{14} = \frac{41}{15} \mathcal{S}_-^2 \nu, \quad (\text{E1n})$$

$$\mathfrak{X}_{15} = -\mathcal{S}_-^2 \left[1 + \frac{4}{3} \bar{\beta}_+ - \frac{4}{3} \bar{\beta}_- \delta - \frac{1}{3} \nu \right], \quad (\text{E1o})$$

$$\mathfrak{X}_{16} = 5\mathcal{S}_-^2 \left[1 + \frac{8}{15} \bar{\gamma} - \frac{1}{15} \nu \right], \quad (\text{E1p})$$

$$\mathfrak{X}_{17} = 4\mathcal{S}_-^2 \left[1 - \frac{1}{3} \bar{\beta}_+ + \frac{2}{3} \bar{\gamma} + \frac{1}{3} \bar{\beta}_- \delta \right], \quad (\text{E1q})$$

$$\begin{aligned} \mathfrak{X}_{18} = & 16\bar{\beta}_+ \bar{\zeta}^{-1} - \frac{8}{3} \bar{\zeta}^{-1} \bar{\gamma} - 24\bar{\beta}_+^2 \bar{\zeta}^{-1} \bar{\gamma}^{-1} + \frac{17}{3} \mathcal{S}_-^2 - 32\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^2 \\ & + 48\bar{\beta}_-^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 + 48\bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 32\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \\ & + 96\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ + \delta \left(8\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^2 - \frac{8}{3} \mathcal{S}_- \mathcal{S}_+ + 8\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \right) - \frac{4}{3} \mathcal{S}_-^2 \nu. \end{aligned} \quad (\text{E1r})$$

The coefficients associated to the eccentricity evolution equation (7.3b) read

$$\begin{aligned} \mathfrak{E}_1 = & 304\bar{\zeta}^{-1} + 160\bar{\beta}_+ \bar{\zeta}^{-1} + 100\bar{\zeta}^{-1} \bar{\gamma} - 240\bar{\beta}_+^2 \bar{\zeta}^{-1} \bar{\gamma}^{-1} + 165\mathcal{S}_-^2 - 120\bar{\beta}_+ \mathcal{S}_-^2 + 100\bar{\gamma} \mathcal{S}_-^2 \\ & - 40\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^2 + 480\bar{\beta}_-^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 + 480\bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 40\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \\ & + 960\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ + \delta (120\bar{\beta}_- \mathcal{S}_-^2 - 200\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^2 - 200\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+) - 125\mathcal{S}_-^2 \nu, \end{aligned} \quad (\text{E2a})$$

$$\begin{aligned} \mathfrak{E}_2 = & 121\bar{\zeta}^{-1} + 40\bar{\beta}_+ \bar{\zeta}^{-1} + \frac{175}{4} \bar{\zeta}^{-1} \bar{\gamma} - 60\bar{\beta}_+^2 \bar{\zeta}^{-1} \bar{\gamma}^{-1} + 280\mathcal{S}_-^2 - 20\bar{\beta}_+ \mathcal{S}_-^2 + 130\bar{\gamma} \mathcal{S}_-^2 \\ & - 10\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^2 + 120\bar{\beta}_-^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 + 120\bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 10\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \\ & + 240\bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ + \delta (20\bar{\beta}_- \mathcal{S}_-^2 - 50\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_-^2 - 50\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+) - 150\mathcal{S}_-^2 \nu, \end{aligned} \quad (\text{E2b})$$

$$\begin{aligned} \mathfrak{E}_3 = & -\frac{939}{35} \bar{\zeta}^{-1} - \frac{5848}{45} \bar{\beta}_+ \bar{\zeta}^{-1} + \frac{64}{9} \bar{\beta}_+^2 \bar{\zeta}^{-1} + \frac{109}{180} \bar{\zeta}^{-1} \bar{\gamma} - 30\bar{\beta}_+ \bar{\zeta}^{-1} \bar{\gamma} + \frac{289}{144} \bar{\zeta}^{-1} \bar{\gamma}^2 \\ & - \frac{31}{288} \bar{\zeta}^{-1} \bar{\gamma}^3 - \frac{188}{3} \bar{\beta}_-^2 \bar{\zeta}^{-1} \bar{\gamma}^{-1} + 116\bar{\beta}_+^2 \bar{\zeta}^{-1} \bar{\gamma}^{-1} - \frac{128}{3} \bar{\beta}_+^3 \bar{\zeta}^{-1} \bar{\gamma}^{-1} \\ & + 128\bar{\beta}_-^2 \bar{\beta}_+ \bar{\zeta}^{-1} \bar{\gamma}^{-2} - 128\bar{\beta}_+^3 \bar{\zeta}^{-1} \bar{\gamma}^{-2} + \frac{32}{3} \bar{\zeta}^{-1} \bar{\chi}_+ - 32\bar{\beta}_+ \bar{\zeta}^{-1} \bar{\gamma}^{-1} \bar{\chi}_+ - \frac{295}{12} \mathcal{S}_-^2 \\ & - \frac{8}{3} \bar{\beta}_-^2 \mathcal{S}_-^2 - \frac{302}{9} \bar{\beta}_+ \mathcal{S}_-^2 - \frac{8}{3} \bar{\beta}_+^2 \mathcal{S}_-^2 + \frac{69}{4} \bar{\gamma} \mathcal{S}_-^2 - \frac{320}{9} \bar{\beta}_+ \bar{\gamma} \mathcal{S}_-^2 + \frac{1055}{72} \bar{\gamma}^2 \mathcal{S}_-^2 \\ & - \frac{640}{9} \bar{\beta}_-^2 \bar{\gamma}^{-1} \mathcal{S}_-^2 + 52\bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_-^2 - \frac{448}{9} \bar{\beta}_+^2 \bar{\gamma}^{-1} \mathcal{S}_-^2 - \frac{128}{3} \bar{\beta}_-^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 \\ & + \frac{256}{3} \bar{\beta}_-^2 \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_-^2 - \frac{512}{3} \bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_-^2 + \frac{256}{3} \bar{\beta}_+^3 \bar{\gamma}^{-2} \mathcal{S}_-^2 - 256\bar{\beta}_-^2 \bar{\beta}_+ \bar{\gamma}^{-3} \mathcal{S}_-^2 \\ & + 256\bar{\beta}_+^3 \bar{\gamma}^{-3} \mathcal{S}_-^2 + 64\bar{\beta}_- \bar{\gamma}^{-2} \bar{\chi}_- \mathcal{S}_-^2 - \frac{152}{9} \bar{\chi}_+ \mathcal{S}_-^2 + \frac{8}{3} \bar{\gamma}^{-1} \bar{\chi}_+ \mathcal{S}_-^2 + 64\bar{\beta}_+ \bar{\gamma}^{-2} \bar{\chi}_+ \mathcal{S}_-^2 \\ & + \frac{217}{36} \zeta \mathcal{S}_-^4 + \frac{217}{72} \zeta \bar{\gamma} \mathcal{S}_-^4 + \frac{184}{9} \bar{\beta}_- \mathcal{S}_- \mathcal{S}_+ + 52\bar{\beta}_- \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ - \frac{1088}{9} \bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-1} \mathcal{S}_- \mathcal{S}_+ \\ & - \frac{640}{3} \bar{\beta}_- \bar{\beta}_+ \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ + \frac{512}{3} \bar{\beta}_- \bar{\beta}_+^2 \bar{\gamma}^{-2} \mathcal{S}_- \mathcal{S}_+ - 256\bar{\beta}_-^3 \bar{\gamma}^{-3} \mathcal{S}_- \mathcal{S}_+ \\ & + 256\bar{\beta}_- \bar{\beta}_+^2 \bar{\gamma}^{-3} \mathcal{S}_- \mathcal{S}_+ + \frac{8}{3} \bar{\gamma}^{-1} \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+ + 64\bar{\beta}_+ \bar{\gamma}^{-2} \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+ \end{aligned}$$

$$\begin{aligned}
& + 64\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ + \frac{31}{36}\zeta\mathcal{S}_+^4 + \frac{31}{72}\zeta\bar{\gamma}\mathcal{S}_+^4 + \delta\left(\frac{4504}{45}\bar{\beta}_-\bar{\zeta}^{-1} + \frac{32}{9}\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\right. \\
& + \frac{302}{9}\bar{\beta}_-\bar{\zeta}^{-1}\bar{\gamma} + \frac{104}{3}\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1} + \frac{128}{3}\bar{\beta}_-\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - \frac{32}{3}\bar{\zeta}^{-1}\bar{\chi}_- \\
& + 32\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1}\bar{\chi}_- + \frac{110}{9}\bar{\beta}_-\mathcal{S}_-^2 + \frac{16}{3}\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 + \frac{320}{9}\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 - 28\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& + \frac{704}{9}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{256}{3}\bar{\beta}_-^3\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{416}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{256}{3}\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& - 256\bar{\beta}_-^3\bar{\gamma}^{-3}\mathcal{S}_-^2 + 256\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-3}\mathcal{S}_-^2 + \frac{152}{9}\bar{\chi}_-\mathcal{S}_-^2 - \frac{8}{3}\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 - 64\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-^2 \\
& - 64\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-^2 + \frac{61}{3}\mathcal{S}_-\mathcal{S}_+ - \frac{376}{9}\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ + \frac{283}{18}\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{31}{36}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ \\
& + \frac{256}{9}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 28\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{448}{9}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - \frac{16}{3}\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& - \frac{512}{3}\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{400}{3}\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 256\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& + 256\bar{\beta}_+^3\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ - 64\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ - \frac{8}{3}\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ \\
& \left. - 64\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ + \frac{31}{9}\zeta\mathcal{S}_-^3\mathcal{S}_+ + \frac{31}{18}\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ + \frac{31}{9}\zeta\mathcal{S}_-\mathcal{S}_+^3 + \frac{31}{18}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3\right) \\
& + \nu\left(-\frac{4084}{45}\bar{\zeta}^{-1} - \frac{176}{9}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{311}{9}\bar{\zeta}^{-1}\bar{\gamma} + \frac{19}{18}\bar{\zeta}^{-1}\bar{\gamma}^2 + \frac{19}{36}\bar{\zeta}^{-1}\bar{\gamma}^3 + \frac{200}{3}\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1}\right. \\
& + \frac{157}{12}\mathcal{S}_-^2 + \frac{32}{3}\bar{\beta}_-^2\mathcal{S}_-^2 + \frac{1036}{9}\bar{\beta}_+\mathcal{S}_-^2 - \frac{82}{3}\bar{\gamma}\mathcal{S}_-^2 - \frac{19}{9}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{3040}{9}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& - \frac{652}{9}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{608}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{848}{3}\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 16\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + \frac{304}{9}\bar{\chi}_+\mathcal{S}_-^2 \\
& - \frac{112}{3}\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - \frac{266}{9}\zeta\mathcal{S}_-^4 - \frac{133}{9}\zeta\bar{\gamma}\mathcal{S}_-^4 - \frac{652}{9}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{1216}{9}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \left. - \frac{800}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{112}{3}\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ - \frac{38}{9}\zeta\mathcal{S}_+^4 - \frac{19}{9}\zeta\bar{\gamma}\mathcal{S}_+^4\right) \\
& + \delta\nu\left(-\frac{316}{9}\bar{\beta}_-\mathcal{S}_-^2 + \frac{412}{9}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{412}{9}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right) + \frac{223}{36}\mathcal{S}_-^2\nu^2, \quad (\text{E2c})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{E}_4 = & \frac{29917}{105}\bar{\zeta}^{-1} + \frac{2087}{45}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{1184}{9}\bar{\beta}_+^2\bar{\zeta}^{-1} + \frac{44279}{180}\bar{\zeta}^{-1}\bar{\gamma} + \frac{464}{9}\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma} \\
& + \frac{28129}{576}\bar{\zeta}^{-1}\bar{\gamma}^2 - \frac{7}{128}\bar{\zeta}^{-1}\bar{\gamma}^3 - 62\bar{\beta}_-^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - 172\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} \\
& - \frac{32}{3}\bar{\beta}_+^3\bar{\zeta}^{-1}\bar{\gamma}^{-1} + 96\bar{\beta}_-^2\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-2} - 96\bar{\beta}_+^3\bar{\zeta}^{-1}\bar{\gamma}^{-2} + 8\bar{\zeta}^{-1}\bar{\chi}_+ - 24\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1}\bar{\chi}_+ \\
& + \frac{717}{4}\mathcal{S}_-^2 + \frac{10}{9}\bar{\beta}_-^2\mathcal{S}_-^2 - \frac{451}{3}\bar{\beta}_+\mathcal{S}_-^2 + \frac{10}{9}\bar{\beta}_+^2\mathcal{S}_-^2 + \frac{29357}{144}\bar{\gamma}\mathcal{S}_-^2 \\
& - \frac{178}{3}\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 + \frac{16031}{288}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{1736}{9}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 37\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& + \frac{2072}{9}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 + 516\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + \frac{64}{3}\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 + 420\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& + \frac{64}{3}\bar{\beta}_+^3\bar{\gamma}^{-2}\mathcal{S}_-^2 - 192\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-3}\mathcal{S}_-^2 + 192\bar{\beta}_+^3\bar{\gamma}^{-3}\mathcal{S}_-^2 + 48\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-^2
\end{aligned}$$

$$\begin{aligned}
& -\frac{22}{3}\bar{\chi}_+\mathcal{S}_-^2 + 2\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 + 48\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-^2 + \frac{49}{16}\zeta\mathcal{S}_-^4 + \frac{49}{32}\zeta\bar{\gamma}\mathcal{S}_-^4 - \frac{190}{9}\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ \\
& - 37\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{3808}{9}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 936\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& + \frac{128}{3}\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 192\bar{\beta}_-^3\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 192\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 2\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ \\
& + 48\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ + 48\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ + \frac{7}{16}\zeta\mathcal{S}_+^4 + \frac{7}{32}\zeta\bar{\gamma}\mathcal{S}_+^4 \\
& + \delta\left(\frac{3991}{45}\bar{\beta}_-\bar{\zeta}^{-1} + \frac{152}{9}\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1} + \frac{260}{9}\bar{\beta}_-\bar{\zeta}^{-1}\bar{\gamma} + 44\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1} \right. \\
& + \frac{32}{3}\bar{\beta}_-\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - 8\bar{\zeta}^{-1}\bar{\chi}_- + 24\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1}\bar{\chi}_- + \frac{71}{3}\bar{\beta}_-\mathcal{S}_-^2 - \frac{20}{9}\bar{\beta}_-\bar{\beta}_+\mathcal{S}_-^2 \\
& + \frac{178}{3}\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 - 177\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{320}{9}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{64}{3}\bar{\beta}_-^3\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
& - 176\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{64}{3}\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - 192\bar{\beta}_-^3\bar{\gamma}^{-3}\mathcal{S}_-^2 + 192\bar{\beta}_-\bar{\beta}_+^2\bar{\gamma}^{-3}\mathcal{S}_-^2 \\
& + \frac{22}{3}\bar{\chi}_-\mathcal{S}_-^2 - 2\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-^2 - 48\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-^2 - 48\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-^2 + \frac{21}{2}\mathcal{S}_-\mathcal{S}_+ \\
& - \frac{950}{9}\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ + \frac{79}{8}\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{7}{16}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ - \frac{8}{9}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& - 177\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{328}{9}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 40\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& - \frac{128}{3}\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 136\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 192\bar{\beta}_-^2\bar{\beta}_+\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ + 192\bar{\beta}_+^3\bar{\gamma}^{-3}\mathcal{S}_-\mathcal{S}_+ \\
& - 48\bar{\beta}_-\bar{\gamma}^{-2}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ - 2\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ - 48\bar{\beta}_+\bar{\gamma}^{-2}\bar{\chi}_+\mathcal{S}_-\mathcal{S}_+ \\
& \left. + \frac{7}{4}\zeta\mathcal{S}_-^3\mathcal{S}_+ + \frac{7}{8}\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ + \frac{7}{4}\zeta\mathcal{S}_-\mathcal{S}_+^3 + \frac{7}{8}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3\right) \\
& + \nu\left(-\frac{7753}{30}\bar{\zeta}^{-1} - \frac{364}{3}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{2099}{24}\bar{\zeta}^{-1}\bar{\gamma} + \frac{11}{24}\bar{\zeta}^{-1}\bar{\gamma}^2 + \frac{11}{48}\bar{\zeta}^{-1}\bar{\gamma}^3 + 210\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} \right. \\
& - \frac{4165}{18}\mathcal{S}_-^2 - \frac{40}{9}\bar{\beta}_-^2\mathcal{S}_-^2 + \frac{2069}{18}\bar{\beta}_+\mathcal{S}_-^2 - \frac{2321}{18}\bar{\gamma}\mathcal{S}_-^2 - \frac{11}{12}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{1672}{9}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& + \frac{5}{3}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 - 88\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 532\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - 308\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + \frac{44}{3}\bar{\chi}_+\mathcal{S}_-^2 \\
& - 28\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_-^2 - \frac{77}{6}\zeta\mathcal{S}_-^4 - \frac{77}{12}\zeta\bar{\gamma}\mathcal{S}_-^4 + \frac{5}{3}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{880}{9}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& \left. - 840\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 28\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_-\mathcal{S}_+ - \frac{11}{6}\zeta\mathcal{S}_+^4 - \frac{11}{12}\zeta\bar{\gamma}\mathcal{S}_+^4\right) \\
& + \delta\nu\left(-\frac{1433}{18}\bar{\beta}_-\mathcal{S}_-^2 + \frac{415}{3}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{415}{3}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right) + \frac{728}{9}\mathcal{S}_-^2\nu^2, \quad (\text{E2d})
\end{aligned}$$

$$\begin{aligned}
\mathfrak{E}_5 = & \frac{13929}{280}\bar{\zeta}^{-1} + \frac{177}{10}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{40}{3}\bar{\beta}_+^2\bar{\zeta}^{-1} + \frac{57629}{1440}\bar{\zeta}^{-1}\bar{\gamma} + \frac{68}{9}\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma} + \frac{1151}{144}\bar{\zeta}^{-1}\bar{\gamma}^2 \\
& + \frac{1}{96}\bar{\zeta}^{-1}\bar{\gamma}^3 - \frac{5}{2}\bar{\beta}_-^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} - \frac{65}{2}\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} + \frac{1679}{24}\mathcal{S}_-^2 - \frac{86}{9}\bar{\beta}_+\mathcal{S}_-^2 + \frac{2171}{36}\bar{\gamma}\mathcal{S}_-^2 \\
& - \frac{16}{9}\bar{\beta}_+\bar{\gamma}\mathcal{S}_-^2 + \frac{925}{72}\bar{\gamma}^2\mathcal{S}_-^2 + \frac{80}{3}\bar{\beta}_-^2\bar{\gamma}^{-1}\mathcal{S}_-^2 - 14\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{80}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}\mathcal{S}_-^2
\end{aligned}$$

$$\begin{aligned}
& + 70\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 70\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{7}{12}\zeta\mathcal{S}_-^4 - \frac{7}{24}\zeta\bar{\gamma}\mathcal{S}_-^4 - \frac{44}{9}\bar{\beta}_-\mathcal{S}_-\mathcal{S}_+ - 14\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + \frac{160}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 140\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{1}{12}\zeta\mathcal{S}_+^4 - \frac{1}{24}\zeta\bar{\gamma}\mathcal{S}_+^4 \\
& + \delta\left(\frac{43}{30}\bar{\beta}_-\bar{\zeta}^{-1} + \frac{2}{3}\bar{\beta}_-\bar{\zeta}^{-1}\bar{\gamma} + 3\bar{\beta}_-\bar{\beta}_+\bar{\zeta}^{-1}\bar{\gamma}^{-1} - \frac{16}{9}\bar{\beta}_-\mathcal{S}_-^2 + \frac{16}{9}\bar{\beta}_-\bar{\gamma}\mathcal{S}_-^2 \right. \\
& - 14\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 - 12\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{19}{24}\mathcal{S}_-\mathcal{S}_+ - \frac{58}{9}\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+ - \frac{5}{12}\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \\
& - \frac{1}{12}\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ - 14\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - 6\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - 6\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
& \left. - \frac{1}{3}\zeta\mathcal{S}_-^3\mathcal{S}_+ - \frac{1}{6}\zeta\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ - \frac{1}{3}\zeta\mathcal{S}_-\mathcal{S}_+^3 - \frac{1}{6}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3\right) \\
& + \nu\left(-\frac{1664}{45}\bar{\zeta}^{-1} - \frac{160}{9}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{112}{9}\bar{\zeta}^{-1}\bar{\gamma} + \frac{80}{3}\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} \right. \\
& - \frac{1247}{18}\mathcal{S}_-^2 + \frac{44}{9}\bar{\beta}_+\mathcal{S}_-^2 - \frac{280}{9}\bar{\gamma}\mathcal{S}_-^2 + \frac{112}{9}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 \\
& \left. - \frac{160}{3}\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{160}{3}\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + \frac{112}{9}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - \frac{320}{3}\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+\right) \\
& + \delta\nu\left(-\frac{44}{9}\bar{\beta}_-\mathcal{S}_-^2 + \frac{128}{9}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{128}{9}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right) + \frac{80}{3}\mathcal{S}_-^2\nu^2, \tag{E2e}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{E}_6 = 55\mathcal{S}_-^2 \left[1 + \frac{31}{30}\bar{\gamma} + \frac{4}{15}\bar{\gamma}^2 - \frac{2}{15}\zeta\mathcal{S}_-^2 - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-^2 + \delta\left(-\frac{2}{15}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+\right) \right. \\
\left. + \nu\left(-\frac{2}{5} + \frac{2}{15}\bar{\beta}_+ - \frac{4}{15}\bar{\gamma}\right) - \frac{2}{15}\bar{\beta}_-\delta\nu \right], \tag{E2f}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{E}_7 = 35\mathcal{S}_-^2 \left[1 + \frac{31}{30}\bar{\gamma} + \frac{4}{15}\bar{\gamma}^2 - \frac{2}{15}\zeta\mathcal{S}_-^2 - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-^2 + \delta\left(-\frac{2}{15}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{1}{15}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+\right) \right. \\
\left. + \nu\left(-\frac{2}{5} + \frac{2}{15}\bar{\beta}_+ - \frac{4}{15}\bar{\gamma}\right) - \frac{2}{15}\bar{\beta}_-\delta\nu \right], \tag{E2g}
\end{aligned}$$

$$\mathfrak{E}_8 = \frac{32}{5}\bar{\zeta}^{-1}\left(1 + \frac{1}{2}\bar{\gamma}\right), \tag{E2h}$$

$$\mathfrak{E}_9 = -\frac{8}{15}\bar{\zeta}^{-1}\bar{\gamma} + \frac{32}{15}\mathcal{S}_-^2 - \frac{32}{15}\mathcal{S}_-\mathcal{S}_+\delta - \frac{64}{15}\mathcal{S}_-^2\nu, \tag{E2i}$$

$$\begin{aligned}
\mathfrak{E}_{10} = -\frac{12}{5}\mathcal{S}_-^2 - \frac{8}{9}\bar{\beta}_+\mathcal{S}_-^2 - \frac{4}{9}\bar{\gamma}\mathcal{S}_-^2 + \frac{8}{3}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{8}{3}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
+ \delta\left(\frac{8}{9}\bar{\beta}_-\mathcal{S}_-^2 - \frac{8}{3}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 + \frac{16}{15}\mathcal{S}_-\mathcal{S}_+ - \frac{8}{3}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right), \tag{E2j}
\end{aligned}$$

$$\mathfrak{E}_{11} = \frac{41}{45}\mathcal{S}_-^2\nu, \tag{E2k}$$

$$\mathfrak{E}_{12} = -\frac{7}{3}\mathcal{S}_-^2 \left[1 + \frac{4}{21}\bar{\beta}_+ + \frac{8}{21}\bar{\gamma} - \frac{4}{21}\bar{\beta}_-\delta - \frac{11}{21}\nu \right], \tag{E2l}$$

$$\mathfrak{E}_{13} = \frac{11}{3}\mathcal{S}_-^2 \left[1 + \frac{16}{33}\bar{\gamma} - \frac{1}{3}\nu \right], \tag{E2m}$$

$$\mathfrak{E}_{14} = -\mathcal{S}_-^2 \left[1 + \frac{4}{9}\bar{\gamma} - \frac{5}{9}\nu \right], \quad (\text{E2n})$$

$$\begin{aligned} \mathfrak{E}_{15} = & \frac{16}{3}\bar{\beta}_+\bar{\zeta}^{-1} - \frac{8}{9}\bar{\zeta}^{-1}\bar{\gamma} - 8\bar{\beta}_+^2\bar{\zeta}^{-1}\bar{\gamma}^{-1} + \frac{17}{9}\mathcal{S}_-^2 - \frac{32}{3}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-^2 \\ & + 16\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_-^2 + 16\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{32}{3}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\ & + 32\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + \delta \left(\frac{8}{3}\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{8}{9}\mathcal{S}_-\mathcal{S}_+ + \frac{8}{3}\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \right) - \frac{4}{9}\mathcal{S}_-^2\nu. \end{aligned} \quad (\text{E2o})$$

Appendix F. Expression of the scalar dipole at 2.5PN order

Recall that the scalar dipole is decomposed into an instantaneous and a hereditary contribution:

$$\mathbf{I}_i^s = \mathbf{I}_i^{s,\text{inst}} + \mathbf{I}_i^{s,\Gamma}. \quad (\text{F1})$$

The hereditary contribution appears at 2.5PN order and is given by

$$\mathbf{I}_i^{s,\Gamma} = \frac{\sqrt{\alpha}\zeta(\mathcal{S}_+ + \mathcal{S}_-\delta)}{(1-\zeta)\phi_0} \mathbf{I}_i^s, \quad (\text{F2})$$

while the instantaneous contribution is given with 2.5PN accuracy by

$$\begin{aligned} \mathbf{I}_i^{s,\text{inst}} = & -\frac{\alpha^{1/2}\zeta m\nu r}{(1-\zeta)\phi_0} \left(2\mathcal{S}_-n^i + \frac{1}{5c^2} \left\{ \left(\frac{\tilde{G}\alpha m}{r} \right) n^i [-9\mathcal{S}_- + 20\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- + 20\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ \right. \right. \\ & + 13\mathcal{S}_-\nu + \delta(-20\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 4\mathcal{S}_+ - 20\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+) \Big\} \\ & + n^i v^2 [\mathcal{S}_- + 4\mathcal{S}_+\delta - 7\mathcal{S}_-\nu] + v^i (nv) [2\mathcal{S}_- - 2\mathcal{S}_+\delta - 4\mathcal{S}_-\nu] \Big\} \\ & + \frac{1}{3c^3} \left(\frac{\tilde{G}\alpha m}{r} \right) v^i \{ -\bar{\gamma}\mathcal{S}_- + 4\zeta\mathcal{S}_-^3 + 4\zeta\mathcal{S}_-^2\mathcal{S}_+\delta - 8\zeta\mathcal{S}_-^3\nu \} \\ & + \frac{1}{140c^4} \left\{ \left(\frac{\tilde{G}\alpha m}{r} \right)^2 n^i [-38\mathcal{S}_- + 224\bar{\beta}_+\mathcal{S}_- - 56\bar{\gamma}\mathcal{S}_- - 672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- \right. \\ & + 560\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_- + 224\bar{\beta}_-\mathcal{S}_+ - 672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ + 560\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_+ \\ & + \delta(-224\bar{\beta}_-\mathcal{S}_- + 672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- - 560\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_- - 32\mathcal{S}_+ - 224\bar{\beta}_+\mathcal{S}_+ + 56\bar{\gamma}\mathcal{S}_+ \\ & + 672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+ - 560\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_+) + \nu(-1234\mathcal{S}_- + 112\bar{\beta}_+\mathcal{S}_- - 728\bar{\gamma}\mathcal{S}_- \\ & - 224\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 4480\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_- + 4480\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_- - 1120\bar{\gamma}^{-1}\bar{\chi}_+\mathcal{S}_- \\ & - 896\bar{\beta}_-\mathcal{S}_+ - 224\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ - 1120\bar{\gamma}^{-1}\bar{\chi}_-\mathcal{S}_+) \\ & + \nu\delta(448\bar{\beta}_-\mathcal{S}_- - 672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 120\mathcal{S}_+ - 672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+) + 470\mathcal{S}_-\nu^2 \Big\} \\ & + n^i v^4 [33\mathcal{S}_- + 72\mathcal{S}_+\delta - 279\mathcal{S}_-\nu - 312\mathcal{S}_+\delta\nu + 549\mathcal{S}_-\nu^2] \\ & + n^i \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^2 [105\mathcal{S}_- + 56\bar{\gamma}\mathcal{S}_- - 56\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 56\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+ - 245\mathcal{S}_-\nu^2 \\ & + \delta(56\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_- + 140\mathcal{S}_+ + 84\bar{\gamma}\mathcal{S}_+ + 56\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+) \\ & + \nu(-49\mathcal{S}_- - 112\bar{\gamma}\mathcal{S}_- - 112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_- - 112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+) \end{aligned}$$

$$\begin{aligned}
& +\nu\delta\left(-336\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-+224\mathcal{S}_+-336\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+\right)] \\
& +n^i\left(\frac{\tilde{G}\alpha m}{r}\right)v^2\left[65\mathcal{S}_-+28\bar{\gamma}\mathcal{S}_--224\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_--224\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+-675\mathcal{S}_-\nu^2\right. \\
& +\delta\left(224\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-+320\mathcal{S}_++252\bar{\gamma}\mathcal{S}_++224\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+\right) \\
& +\nu\delta\left(336\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-+228\mathcal{S}_++336\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+\right) \\
& \left.+\nu\left(-1303\mathcal{S}_--756\bar{\gamma}\mathcal{S}_-+672\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-+672\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+\right)\right] \\
& +v^i\left(\frac{\tilde{G}\alpha m}{r}\right)(nv)\left[310\mathcal{S}_-+168\bar{\gamma}\mathcal{S}_-+112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-+112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+-514\mathcal{S}_-\nu^2\right. \\
& +\delta\left(-112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_--240\mathcal{S}_+-168\bar{\gamma}\mathcal{S}_+-112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+\right) \\
& +\nu\delta\left(112\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_--192\mathcal{S}_++112\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_+\right) \\
& \left.+\nu\left(1182\mathcal{S}_-+504\bar{\gamma}\mathcal{S}_--336\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_--336\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_+\right)\right] \\
& +v^i(nv)v^2\left[20\mathcal{S}_--20\mathcal{S}_+\delta+-164\mathcal{S}_-\nu+124\mathcal{S}_+\delta+376\mathcal{S}_-\nu^2\right]\Big\} \\
& +\frac{1}{60c^5(1-\zeta)}\left\{\left(\frac{\tilde{G}\alpha m}{r}\right)^2n^i(nv)\left[-40\bar{\beta}_+\mathcal{S}_-+40\bar{\beta}_+\zeta\mathcal{S}_-+30\bar{\gamma}\mathcal{S}_--30\zeta\bar{\gamma}\mathcal{S}_- \right.\right. \\
& -40\zeta\mathcal{S}_-^3+40\zeta^2\mathcal{S}_-^3+40\bar{\beta}_-\mathcal{S}_-+40\bar{\beta}_-\zeta\mathcal{S}_-+\delta\left(40\bar{\beta}_-\mathcal{S}_--40\bar{\beta}_-\zeta\mathcal{S}_- \right. \\
& -40\bar{\beta}_+\mathcal{S}_++40\bar{\beta}_+\zeta\mathcal{S}_++15\bar{\gamma}\mathcal{S}_+-15\zeta\bar{\gamma}\mathcal{S}_+-30\zeta\mathcal{S}_-^2\mathcal{S}_++30\zeta^2\mathcal{S}_-^2\mathcal{S}_+ \\
& \left.-10\zeta\mathcal{S}_+^3+10\zeta^2\mathcal{S}_+^3\right)+\nu\left(-256\mathcal{S}_-+1120\bar{\beta}_+\mathcal{S}_-+256\zeta\mathcal{S}_--1120\bar{\beta}_+\zeta\mathcal{S}_- \right. \\
& -546\bar{\gamma}\mathcal{S}_-+546\zeta\bar{\gamma}\mathcal{S}_--2240\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-+2240\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-+3840\bar{\beta}_-^2\bar{\gamma}^{-2}\mathcal{S}_- \\
& +7680\bar{\beta}_+^2\bar{\gamma}^{-2}\mathcal{S}_--3840\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_--7680\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_- \\
& +816\zeta\mathcal{S}_-^3-816\zeta^2\mathcal{S}_-^3-1920\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^3+1920\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3+3840\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-^3 \\
& -3840\bar{\beta}_+\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^3-15360\bar{\beta}_-^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^3-15360\bar{\beta}_+^2\zeta\bar{\gamma}^{-3}\mathcal{S}_-^3+15360\bar{\beta}_-^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3 \\
& +15360\bar{\beta}_+^2\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^3-800\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_++1280\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_++3840\bar{\beta}_-\bar{\beta}_+\bar{\gamma}^{-2}\mathcal{S}_+ \\
& -3840\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_+-1920\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2\mathcal{S}_++1920\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^2\mathcal{S}_+ \\
& +2880\bar{\beta}_-\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2\mathcal{S}_+-3840\bar{\beta}_-\zeta^2\bar{\gamma}^{-2}\mathcal{S}_-^2\mathcal{S}_+-30720\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-3}\mathcal{S}_-^2\mathcal{S}_+ \\
& +30720\bar{\beta}_-\bar{\beta}_+\zeta^2\bar{\gamma}^{-3}\mathcal{S}_-^2\mathcal{S}_++960\bar{\beta}_-\zeta\bar{\gamma}^{-2}\mathcal{S}_+^3)+\nu\delta\left(18\bar{\gamma}\mathcal{S}_+-18\zeta\bar{\gamma}\mathcal{S}_+ \right. \\
& +108\zeta\mathcal{S}_-^2\mathcal{S}_+-108\zeta^2\mathcal{S}_-^2\mathcal{S}_++36\zeta\mathcal{S}_+^3-36\zeta^2\mathcal{S}_+^3)-448\zeta(1-\zeta)\mathcal{S}_-^3\nu^2 \\
& +\alpha^{1/2}\nu\left(480\bar{\beta}_-+240\bar{\beta}_-\bar{\gamma}+480\bar{\beta}_-\zeta\mathcal{S}_-+960\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-592\zeta\mathcal{S}_-\mathcal{S}_+ \right. \\
& +960\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+-296\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_++1920\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_++480\bar{\beta}_-\zeta\mathcal{S}_+^2 \\
& +960\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_+^2-960\bar{\beta}_-\mathcal{S}_-^2\lambda_1-1920\bar{\beta}_-\bar{\gamma}^{-1}\mathcal{S}_-^2\lambda_1+592\mathcal{S}_-\mathcal{S}_+\lambda_1 \\
& -960\bar{\beta}_+\mathcal{S}_-\mathcal{S}_+\lambda_1+296\bar{\gamma}\mathcal{S}_-\mathcal{S}_+\lambda_1-1920\bar{\beta}_+\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\lambda_1 \\
& \left.+\delta\left(28\bar{\gamma}+14\bar{\gamma}^2+56\zeta\mathcal{S}_-^2+28\zeta\bar{\gamma}\mathcal{S}_-^2+56\zeta\mathcal{S}_+^2+28\zeta\bar{\gamma}\mathcal{S}_+^2\right)\right\}
\end{aligned}$$

$$\begin{aligned}
& -112\mathcal{S}_-^2\lambda_1 - 56\bar{\gamma}\mathcal{S}_-^2\lambda_1) \Bigg] + v^i \left(\frac{\tilde{G}\alpha m}{r} \right)^2 \Big[-96\mathcal{S}_- + 40\bar{\beta}_+\mathcal{S}_- \\
& + 96\zeta\mathcal{S}_- - 40\bar{\beta}_+\zeta\mathcal{S}_- - 50\bar{\gamma}\mathcal{S}_- + 50\zeta\bar{\gamma}\mathcal{S}_- - 40\zeta\mathcal{S}_-^3 + 40\zeta^2\mathcal{S}_-^3 \\
& - 40\bar{\beta}_-\mathcal{S}_+ + 40\bar{\beta}_-\zeta\mathcal{S}_+ + \delta \Big(-40\bar{\beta}_-\mathcal{S}_- + 40\bar{\beta}_-\zeta\mathcal{S}_- - 96\mathcal{S}_+ + 40\bar{\beta}_+\mathcal{S}_+ \\
& + 96\zeta\mathcal{S}_+ - 40\bar{\beta}_+\zeta\mathcal{S}_+ - 60\bar{\gamma}\mathcal{S}_+ + 65\zeta\bar{\gamma}\mathcal{S}_+ \\
& - 40\zeta\mathcal{S}_-^2\mathcal{S}_+ + 30\zeta^2\mathcal{S}_-^2\mathcal{S}_+ + 10\zeta^2\mathcal{S}_+^3 \Big) + \nu \Big(768\mathcal{S}_- - 640\bar{\beta}_+\mathcal{S}_- - 768\zeta\mathcal{S}_- \\
& + 640\bar{\beta}_+\zeta\mathcal{S}_- + 522\bar{\gamma}\mathcal{S}_- - 522\zeta\bar{\gamma}\mathcal{S}_- + 32\zeta\mathcal{S}_-^3 - 32\zeta^2\mathcal{S}_-^3 + 960\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^3 \\
& - 960\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3 + 120\bar{\beta}_-\mathcal{S}_+ - 120\bar{\beta}_-\zeta\mathcal{S}_+ + 720\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2\mathcal{S}_+ \\
& - 720\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^2\mathcal{S}_+ + 240\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_+^3 - 240\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_+^3 \Big) \\
& + \delta\nu \Big(960\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^3 - 960\bar{\beta}_-\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^3 + 120\bar{\beta}_+\mathcal{S}_+ - 120\bar{\beta}_+\zeta\mathcal{S}_+ + 4\zeta\bar{\gamma}\mathcal{S}_+ \\
& - 32\zeta\mathcal{S}_-^2\mathcal{S}_+ + 24\zeta^2\mathcal{S}_-^2\mathcal{S}_+ + 720\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2\mathcal{S}_+ - 720\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_-^2\mathcal{S}_+ + 8\zeta^2\mathcal{S}_+^3 \\
& + 240\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_+^3 - 240\bar{\beta}_+\zeta^2\bar{\gamma}^{-1}\mathcal{S}_+^3 \Big) - 544\zeta(1-\zeta)\mathcal{S}_-^3\nu^2 \\
& + \alpha^{1/2}\nu \Big(16\zeta\mathcal{S}_-\mathcal{S}_+ + 8\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 16\mathcal{S}_-\mathcal{S}_+\lambda_1 - 8\bar{\gamma}\mathcal{S}_-\mathcal{S}_+\lambda_1 \\
& + \delta \Big(-4\bar{\gamma} - 2\bar{\gamma}^2 - 8\zeta\mathcal{S}_-^2 - 4\zeta\bar{\gamma}\mathcal{S}_-^2 - 8\zeta\mathcal{S}_+^2 - 4\zeta\bar{\gamma}\mathcal{S}_+^2 + 16\mathcal{S}_-^2\lambda_1 + 8\bar{\gamma}\mathcal{S}_-^2\lambda_1 \Big) \Bigg] \\
& + v^i \left(\frac{\tilde{G}\alpha m}{r} \right) v^2 \Big[48\mathcal{S}_- - 48\zeta\mathcal{S}_- + 10\bar{\gamma}\mathcal{S}_- - 10\zeta\bar{\gamma}\mathcal{S}_- + 40\zeta\mathcal{S}_-^3 - 40\zeta^2\mathcal{S}_-^3 \\
& + \delta \Big(48\mathcal{S}_+ - 48\zeta\mathcal{S}_+ + 20\bar{\gamma}\mathcal{S}_+ - 20\zeta\bar{\gamma}\mathcal{S}_+ + 40\zeta\mathcal{S}_-^2\mathcal{S}_+ - 40\zeta^2\mathcal{S}_-^2\mathcal{S}_+ \Big) \\
& - 160\zeta\mathcal{S}_-^2\mathcal{S}_+(1-\zeta)\nu\delta + \nu \Big(-192\mathcal{S}_- + 192\zeta\mathcal{S}_- - 36\bar{\gamma}\mathcal{S}_- \\
& + 36\zeta\bar{\gamma}\mathcal{S}_- - 240\zeta\mathcal{S}_-^3 + 240\zeta^2\mathcal{S}_-^3 \Big) + 288\zeta\mathcal{S}_-^3(1-\zeta)\nu^2 \Big] \Big\} + \mathcal{O}\left(\frac{1}{c^6}\right) \quad (F3)
\end{aligned}$$

This result agrees with [77] at 2PN order.

Appendix G. Expression of the instantaneous angular momentum fluxes in terms of the phase space variables

Recall equation (6.8b):

$$\begin{aligned}
\mathcal{G}_i^{s,\text{inst}} &= \frac{m\nu^2}{3c^3} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 \epsilon_{ijk} n_j v_k \left\{ \tilde{A}^{-1\text{PN}} + \frac{1}{c^2} \left[\tilde{B}_1^{\text{N}} \left(\frac{\tilde{G}\alpha m}{r} \right) + \tilde{B}_2^{\text{N}} (nv)^2 + \tilde{B}_3^{\text{N}} v^2 \right] \right. \\
& + \frac{1}{c^3} \tilde{C}^{0.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv) + \frac{1}{c^4} \left[\tilde{D}_1^{1\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 + \tilde{D}_2^{1\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^2 \right. \\
& \left. \left. + \tilde{D}_3^{1\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) v^2 + \tilde{D}_4^{1\text{PN}} (nv)^4 + \tilde{D}_5^{1\text{PN}} (nv)^2 v^2 + \tilde{D}_6^{1\text{PN}} v^4 \right] \right\}
\end{aligned}$$

$$+ \frac{1}{c^5} \left[\tilde{E}_1^{1.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right)^2 (nv) + \tilde{E}_2^{1.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv)^3 + \tilde{E}_3^{1.5\text{PN}} \left(\frac{\tilde{G}\alpha m}{r} \right) (nv) v^2 \right] \Bigg\} \\ + \mathcal{O}\left(\frac{1}{c^9}\right).$$

The coefficients are given by:

$$\tilde{A}^{-1\text{PN}} = 4\zeta\mathcal{S}_-^2 \quad (\text{G1a})$$

$$\tilde{B}_1^{\text{N}} = -\frac{4}{5}\bar{\gamma} - \frac{98}{5}\zeta\mathcal{S}_-^2 - 8\bar{\beta}_+\zeta\mathcal{S}_-^2 + \frac{8\bar{\beta}_+\zeta\mathcal{S}_-^2}{\bar{\gamma}} - 8\zeta\bar{\gamma}\mathcal{S}_-^2 + \frac{8\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\ + \delta \left[8\bar{\beta}_-\zeta\mathcal{S}_-^2 - \frac{8\bar{\beta}_-\zeta\mathcal{S}_-^2}{\bar{\gamma}} - \frac{12}{5}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{8\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \right] + \frac{6}{5}\zeta\mathcal{S}_-^2\nu \quad (\text{G1b})$$

$$\tilde{B}_2^{\text{N}} = \frac{6}{5}\bar{\gamma} + \frac{6}{5}\zeta\mathcal{S}_-^2 - \frac{24\bar{\beta}_+\zeta\mathcal{S}_-^2}{\bar{\gamma}} - \frac{24\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\ + \delta \left[\frac{24\bar{\beta}_-\zeta\mathcal{S}_-^2}{\bar{\gamma}} - \frac{36}{5}\zeta\mathcal{S}_-\mathcal{S}_+ + \frac{24\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \right] + \frac{48}{5}\zeta\mathcal{S}_-^2\nu \quad (\text{G1c})$$

$$\tilde{B}_3^{\text{N}} = -\frac{4}{5}\bar{\gamma} + \frac{38}{5}\zeta\mathcal{S}_-^2 + \frac{8\bar{\beta}_+\zeta\mathcal{S}_-^2}{\bar{\gamma}} + 4\zeta\bar{\gamma}\mathcal{S}_-^2 + \frac{8\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \\ + \delta \left[-\frac{8\bar{\beta}_-\zeta\mathcal{S}_-^2}{\bar{\gamma}} + \frac{12}{5}\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{8\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} \right] - \frac{26}{5}\zeta\mathcal{S}_-^2\nu \quad (\text{G1d})$$

$$\tilde{C}^{0.5\text{PN}} = \frac{1}{4}\bar{\gamma}^2 + \zeta\bar{\gamma}\mathcal{S}_-^2 - 7\zeta^2\mathcal{S}_-^4 - \zeta^2\mathcal{S}_+^4 + \delta \left[-2\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 4\zeta^2\mathcal{S}_-^3\mathcal{S}_+ - 4\zeta^2\mathcal{S}_-\mathcal{S}_+^3 \right] \quad (\text{G1e})$$

$$\tilde{D}_1^{\text{1PN}} = -4\bar{\beta}_+ + \frac{251}{35}\bar{\gamma} + \frac{16}{5}\bar{\beta}_+\bar{\gamma} + \frac{14}{5}\bar{\gamma}^2 - \frac{1}{8}\bar{\gamma}^3 + \frac{1147}{35}\zeta\mathcal{S}_-^2 + \frac{116}{5}\bar{\beta}_+\zeta\mathcal{S}_-^2 - \frac{16\bar{\beta}_-^2\zeta\mathcal{S}_-^2}{\bar{\gamma}} \\ - \frac{104\bar{\beta}_+\zeta\mathcal{S}_-^2}{5\bar{\gamma}} - \frac{16\bar{\beta}_+^2\zeta\mathcal{S}_-^2}{\bar{\gamma}} + \frac{166}{5}\zeta\bar{\gamma}\mathcal{S}_-^2 + 16\bar{\beta}_+\zeta\bar{\gamma}\mathcal{S}_-^2 + \frac{17}{2}\zeta\bar{\gamma}^2\mathcal{S}_-^2 - 8\zeta\bar{\chi}_+\mathcal{S}_-^2 \\ + 8\zeta^2\mathcal{S}_-^4 + \frac{7}{2}\zeta^2\bar{\gamma}\mathcal{S}_-^4 - \frac{144}{5}\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{104\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+}{5\bar{\gamma}} - \frac{32\bar{\beta}_-\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{1}{2}\zeta^2\bar{\gamma}\mathcal{S}_+^4 \\ + \delta \left[4\bar{\beta}_- - \frac{16}{5}\bar{\beta}_-\bar{\gamma} - \frac{116}{5}\bar{\beta}_-\zeta\mathcal{S}_-^2 + \frac{104\bar{\beta}_-\zeta\mathcal{S}_-^2}{5\bar{\gamma}} + \frac{32\bar{\beta}_-\bar{\beta}_+\zeta\mathcal{S}_-^2}{\bar{\gamma}} - 16\bar{\beta}_-\zeta\bar{\gamma}\mathcal{S}_-^2 \right. \\ + 8\zeta\bar{\chi}_-\mathcal{S}_-^2 + \frac{848}{35}\zeta\mathcal{S}_-\mathcal{S}_+ + \frac{144}{5}\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + \frac{16\bar{\beta}_-^2\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{104\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{5\bar{\gamma}} \\ + \frac{16\bar{\beta}_+^2\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{44}{5}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \zeta\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ + 8\zeta^2\mathcal{S}_-^3\mathcal{S}_+ + 2\zeta^2\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ + 2\zeta^2\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3 \Bigg] \\ + \nu \left[\frac{56}{5}\bar{\beta}_+ - \frac{18}{35}\bar{\gamma} + \frac{1}{4}\bar{\gamma}^3 + \frac{1292}{35}\zeta\mathcal{S}_-^2 + \frac{168}{5}\bar{\beta}_+\zeta\mathcal{S}_-^2 + \frac{160\bar{\beta}_-^2\zeta\mathcal{S}_-^2}{\bar{\gamma}} - \frac{208\bar{\beta}_+\zeta\mathcal{S}_-^2}{5\bar{\gamma}} \right. \\ - \frac{96\bar{\beta}_+^2\zeta\mathcal{S}_-^2}{\bar{\gamma}} + \frac{48}{5}\zeta\bar{\gamma}\mathcal{S}_-^2 - \zeta\bar{\gamma}^2\mathcal{S}_-^2 + 16\zeta\bar{\chi}_+\mathcal{S}_-^2 - 16\zeta^2\mathcal{S}_-^4 - 7\zeta^2\bar{\gamma}\mathcal{S}_-^4 \\ + \frac{256}{5}\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{208\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+}{5\bar{\gamma}} + \frac{64\bar{\beta}_-\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} - \zeta^2\bar{\gamma}\mathcal{S}_+^4 \Bigg]$$

$$\begin{aligned}
& + \nu \delta \left[-\frac{8}{5} \bar{\beta}_- \zeta \mathcal{S}_-^2 + \frac{16 \bar{\beta}_- \zeta \mathcal{S}_-^2}{5 \bar{\gamma}} - \frac{156}{35} \zeta \mathcal{S}_- \mathcal{S}_+ + \frac{16 \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{5 \bar{\gamma}} \right] - \frac{191}{35} \zeta \mathcal{S}_-^2 \nu^2 \quad (\text{G1f}) \\
\tilde{D}_2^{\text{IPN}} = & \frac{36}{5} \bar{\beta}_+ + \frac{8 \bar{\beta}_-^2}{\bar{\gamma}} + \frac{8 \bar{\beta}_+^2}{\bar{\gamma}} - \frac{184}{35} \bar{\gamma} - \frac{16}{5} \bar{\beta}_+ \bar{\gamma} - \frac{3}{10} \bar{\gamma}^2 + \frac{1}{4} \bar{\gamma}^3 - \frac{299}{70} \zeta \mathcal{S}_-^2 - \frac{32 \bar{\beta}_-^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} \\
& - \frac{32 \bar{\beta}_+^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{476 \bar{\beta}_+ \zeta \mathcal{S}_-^2}{5 \bar{\gamma}} - \frac{22}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 - \zeta \bar{\gamma}^2 \mathcal{S}_-^2 - \frac{64 \zeta \bar{\chi}_+ \mathcal{S}_-^2}{\bar{\gamma}} - 14 \zeta^2 \mathcal{S}_-^4 - 7 \zeta^2 \bar{\gamma} \mathcal{S}_-^4 \\
& - 32 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+ - \frac{64 \bar{\beta}_- \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} + \frac{476 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{5 \bar{\gamma}} - \frac{64 \zeta \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} - 2 \zeta^2 \mathcal{S}_+^4 - \zeta^2 \bar{\gamma} \mathcal{S}_+^4 \\
& + \delta \left[-\frac{36}{5} \bar{\beta}_- - \frac{16 \bar{\beta}_- \bar{\beta}_+}{\bar{\gamma}} + \frac{16}{5} \bar{\beta}_- \bar{\gamma} + \frac{64 \bar{\beta}_- \bar{\beta}_+ \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{476 \bar{\beta}_- \zeta \mathcal{S}_-^2}{5 \bar{\gamma}} \right. \\
& + \frac{64 \zeta \bar{\chi}_- \mathcal{S}_-^2}{\bar{\gamma}} - \frac{498}{35} \zeta \mathcal{S}_- \mathcal{S}_+ + 32 \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+ + \frac{32 \bar{\beta}_-^2 \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} + \frac{32 \bar{\beta}_+^2 \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} \\
& - \frac{476 \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{5 \bar{\gamma}} - \frac{138}{5} \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ - 2 \zeta \bar{\gamma}^2 \mathcal{S}_- \mathcal{S}_+ \\
& \left. + \frac{64 \zeta \bar{\chi}_+ \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} - 8 \zeta^2 \mathcal{S}_-^3 \mathcal{S}_+ - 4 \zeta^2 \bar{\gamma} \mathcal{S}_-^3 \mathcal{S}_+ - 8 \zeta^2 \mathcal{S}_- \mathcal{S}_+^3 - 4 \zeta^2 \bar{\gamma} \mathcal{S}_- \mathcal{S}_+^3 \right] \\
& + \nu \left[-8 \bar{\beta}_+ - \frac{32 \bar{\beta}_+^2}{\bar{\gamma}} + \frac{89}{35} \bar{\gamma} - \frac{1821}{70} \zeta \mathcal{S}_-^2 - 48 \bar{\beta}_+ \zeta \mathcal{S}_-^2 + \frac{576 \bar{\beta}_-^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} \right. \\
& - \frac{448 \bar{\beta}_+^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{92 \bar{\beta}_+ \zeta \mathcal{S}_-^2}{5 \bar{\gamma}} + \frac{44}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 + \frac{128 \zeta \bar{\chi}_+ \mathcal{S}_-^2}{\bar{\gamma}} \\
& \left. + 128 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+ + \frac{128 \bar{\beta}_- \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} + \frac{92 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{5 \bar{\gamma}} + \frac{128 \zeta \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \right] \\
& + \nu \delta \left[16 \bar{\beta}_- \zeta \mathcal{S}_-^2 + \frac{436 \bar{\beta}_- \zeta \mathcal{S}_-^2}{5 \bar{\gamma}} - \frac{1608}{35} \zeta \mathcal{S}_- \mathcal{S}_+ + \frac{436 \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{5 \bar{\gamma}} \right] + \frac{1341}{70} \zeta \mathcal{S}_-^2 \nu^2 \quad (\text{G1g})
\end{aligned}$$

$$\begin{aligned}
\tilde{D}_3^{\text{IPN}} = & -\frac{4}{5} \bar{\beta}_+ - \frac{8 \bar{\beta}_-^2}{\bar{\gamma}} - \frac{8 \bar{\beta}_+^2}{\bar{\gamma}} + \frac{10}{7} \bar{\gamma} + \frac{8}{5} \bar{\beta}_+ \bar{\gamma} - \frac{887}{70} \zeta \mathcal{S}_-^2 + 8 \bar{\beta}_+ \zeta \mathcal{S}_-^2 + \frac{32 \bar{\beta}_-^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} \\
& + \frac{32 \bar{\beta}_+^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{12 \bar{\beta}_+ \zeta \mathcal{S}_-^2}{\bar{\gamma}} - \frac{32}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 + \frac{16 \zeta \bar{\chi}_+ \mathcal{S}_-^2}{\bar{\gamma}} + 16 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+ \\
& + \frac{64 \bar{\beta}_- \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} - \frac{12 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} + \frac{16 \zeta \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \\
& + \delta \left[\frac{4}{5} \bar{\beta}_- + \frac{16 \bar{\beta}_- \bar{\beta}_+}{\bar{\gamma}} - \frac{8}{5} \bar{\beta}_- \bar{\gamma} - 8 \bar{\beta}_- \zeta \mathcal{S}_-^2 - \frac{64 \bar{\beta}_- \bar{\beta}_+ \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{12 \bar{\beta}_- \zeta \mathcal{S}_-^2}{\bar{\gamma}} \right. \\
& - \frac{16 \zeta \bar{\chi}_- \mathcal{S}_-^2}{\bar{\gamma}} + \frac{286}{35} \zeta \mathcal{S}_- \mathcal{S}_+ - 16 \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+ - \frac{32 \bar{\beta}_-^2 \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} - \frac{32 \bar{\beta}_+^2 \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} \\
& \left. + \frac{12 \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} + \frac{42}{5} \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ - \frac{16 \zeta \bar{\chi}_+ \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \right] \\
& + \nu \left[-\frac{64}{5} \bar{\beta}_+ + \frac{32 \bar{\beta}_+^2}{\bar{\gamma}} + \frac{114}{35} \bar{\gamma} + \frac{1133}{70} \zeta \mathcal{S}_-^2 + 16 \bar{\beta}_+ \zeta \mathcal{S}_-^2 - \frac{192 \bar{\beta}_-^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} \right. \\
& + \frac{64 \bar{\beta}_+^2 \zeta \mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{12 \bar{\beta}_+ \zeta \mathcal{S}_-^2}{\bar{\gamma}} + \frac{4}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{32 \zeta \bar{\chi}_+ \mathcal{S}_-^2}{\bar{\gamma}} - 32 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+ \\
& \left. - \frac{128 \bar{\beta}_- \bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}^2} - \frac{12 \bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} - \frac{32 \zeta \bar{\chi}_- \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \right]
\end{aligned}$$

$$+ \nu \delta \left[-8\bar{\beta}_- \zeta \mathcal{S}_-^2 - \frac{12\bar{\beta}_- \zeta \mathcal{S}_-^2}{\bar{\gamma}} + \frac{408}{35} \zeta \mathcal{S}_- \mathcal{S}_+ - \frac{12\bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \right] - \frac{171}{70} \zeta \mathcal{S}_-^2 \nu^2 \quad (\text{G1h})$$

$$\begin{aligned} \tilde{D}_4^{\text{IPN}} = & -12\bar{\beta}_+ + \frac{33}{14} \bar{\gamma} - \frac{447}{14} \zeta \mathcal{S}_-^2 + \frac{60\bar{\beta}_+ \zeta \mathcal{S}_-^2}{\bar{\gamma}} - 12\zeta \bar{\gamma} \mathcal{S}_-^2 + \frac{60\bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \\ & + \delta \left[12\bar{\beta}_- - \frac{60\bar{\beta}_- \zeta \mathcal{S}_-^2}{\bar{\gamma}} - \frac{144}{7} \zeta \mathcal{S}_- \mathcal{S}_+ - \frac{60\bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} - 18\zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ \right] \\ & + \nu \left[24\bar{\beta}_+ - \frac{36}{7} \bar{\gamma} + \frac{390}{7} \zeta \mathcal{S}_-^2 - \frac{120\bar{\beta}_+ \zeta \mathcal{S}_-^2}{\bar{\gamma}} + 24\zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{120\bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \right] \\ & + \nu \delta \left[\frac{120\bar{\beta}_- \zeta \mathcal{S}_-^2}{\bar{\gamma}} - \frac{258}{7} \zeta \mathcal{S}_- \mathcal{S}_+ + \frac{120\bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{\bar{\gamma}} \right] + \frac{36}{7} \zeta \mathcal{S}_-^2 \nu^2 \end{aligned} \quad (\text{G1i})$$

$$\begin{aligned} \tilde{D}_5^{\text{IPN}} = & 12\bar{\beta}_+ - \frac{12}{7} \bar{\gamma} + \frac{1017}{35} \zeta \mathcal{S}_-^2 - \frac{312\bar{\beta}_+ \zeta \mathcal{S}_-^2}{5\bar{\gamma}} + \frac{54}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{312\bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} \\ & + \delta \left[-12\bar{\beta}_- + \frac{312\bar{\beta}_- \zeta \mathcal{S}_-^2}{5\bar{\gamma}} + \frac{768}{35} \zeta \mathcal{S}_- \mathcal{S}_+ + \frac{312\bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} + \frac{96}{5} \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ \right] \\ & + \nu \left[-\frac{96}{5} \bar{\beta}_+ + \frac{39}{35} \bar{\gamma} - \frac{561}{35} \zeta \mathcal{S}_-^2 + \frac{636\bar{\beta}_+ \zeta \mathcal{S}_-^2}{5\bar{\gamma}} - \frac{48}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 + \frac{636\bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} \right] \\ & + \nu \delta \left[-\frac{852\bar{\beta}_- \zeta \mathcal{S}_-^2}{5\bar{\gamma}} + \frac{1782}{35} \zeta \mathcal{S}_- \mathcal{S}_+ - \frac{852\bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} \right] - \frac{906}{35} \zeta \mathcal{S}_-^2 \nu^2 \end{aligned} \quad (\text{G1j})$$

$$\begin{aligned} \tilde{D}_6^{\text{IPN}} = & -\frac{8}{5} \bar{\beta}_+ - \frac{61}{70} \bar{\gamma} - \frac{2}{5} \bar{\gamma}^2 + \frac{47}{10} \zeta \mathcal{S}_-^2 + \frac{44\bar{\beta}_+ \zeta \mathcal{S}_-^2}{5\bar{\gamma}} + \frac{14}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 + \frac{44\bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} \\ & + \delta \left[\frac{8}{5} \bar{\beta}_- - \frac{44\bar{\beta}_- \zeta \mathcal{S}_-^2}{5\bar{\gamma}} - \frac{16}{5} \zeta \mathcal{S}_- \mathcal{S}_+ - \frac{44\bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} - \frac{14}{5} \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ \right] \\ & + \nu \left[\frac{8}{5} \bar{\beta}_+ + \frac{58}{35} \bar{\gamma} - \frac{273}{10} \zeta \mathcal{S}_-^2 - \frac{92\bar{\beta}_+ \zeta \mathcal{S}_-^2}{5\bar{\gamma}} - \frac{58}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{92\bar{\beta}_- \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} \right] \\ & + \nu \delta \left[\frac{164\bar{\beta}_- \zeta \mathcal{S}_-^2}{5\bar{\gamma}} - \frac{48}{5} \zeta \mathcal{S}_- \mathcal{S}_+ + \frac{164\bar{\beta}_+ \zeta \mathcal{S}_- \mathcal{S}_+}{5\bar{\gamma}} \right] + \frac{131}{10} \zeta \mathcal{S}_-^2 \nu^2 \end{aligned} \quad (\text{G1k})$$

$$\begin{aligned} \tilde{E}_1^{1.5\text{PN}} = & -\frac{12}{5} \bar{\gamma} - \frac{16}{5} \bar{\gamma}^2 - \frac{2}{3} \bar{\beta}_+ \bar{\gamma}^2 - \frac{2}{3} \bar{\gamma}^3 - \frac{48}{5} \zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{8}{3} \bar{\beta}_+ \zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{8}{3} \zeta \bar{\gamma}^2 \mathcal{S}_-^2 + 56\zeta^2 \mathcal{S}_-^4 \\ & + \frac{56}{3} \bar{\beta}_+ \zeta^2 \mathcal{S}_-^4 + \frac{56}{3} \zeta^2 \bar{\gamma} \mathcal{S}_-^4 - \frac{16}{3} \bar{\beta}_- \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ - \frac{32}{3} \bar{\beta}_- \zeta^2 \mathcal{S}_-^3 \mathcal{S}_+ \\ & - \frac{32}{3} \bar{\beta}_- \zeta^2 \mathcal{S}_- \mathcal{S}_+^3 + 8\zeta^2 \mathcal{S}_+^4 + \frac{8}{3} \bar{\beta}_+ \zeta^2 \mathcal{S}_+^4 + \frac{8}{3} \zeta^2 \bar{\gamma} \mathcal{S}_+^4 \\ & + \delta \left[\frac{2}{3} \bar{\beta}_- \bar{\gamma}^2 + \frac{8}{3} \bar{\beta}_- \zeta \bar{\gamma} \mathcal{S}_-^2 - \frac{56}{3} \bar{\beta}_- \zeta^2 \mathcal{S}_-^4 + \frac{72}{5} \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ + \frac{16}{3} \bar{\beta}_+ \zeta \bar{\gamma} \mathcal{S}_- \mathcal{S}_+ \right. \\ & + \frac{16}{3} \zeta \bar{\gamma}^2 \mathcal{S}_- \mathcal{S}_+ + 32\zeta^2 \mathcal{S}_-^3 \mathcal{S}_+ + \frac{32}{3} \bar{\beta}_+ \zeta^2 \mathcal{S}_-^3 \mathcal{S}_+ \\ & + \frac{32}{3} \zeta^2 \bar{\gamma} \mathcal{S}_-^3 \mathcal{S}_+ + 32\zeta^2 \mathcal{S}_- \mathcal{S}_+^3 + \frac{32}{3} \bar{\beta}_+ \zeta^2 \mathcal{S}_- \mathcal{S}_+^3 + \frac{32}{3} \zeta^2 \bar{\gamma} \mathcal{S}_- \mathcal{S}_+^3 - \left. \frac{8}{3} \bar{\beta}_- \zeta^2 \mathcal{S}_+^4 \right] \\ & + \nu \left[24\bar{\beta}_+ - 8\bar{\beta}_+^2 - \frac{96\bar{\beta}_-^2}{\bar{\gamma}} - \frac{96\bar{\beta}_+^2}{\bar{\gamma}} + \frac{227}{120} \bar{\gamma}^2 - \bar{\gamma}^3 - \frac{928}{15} \zeta \mathcal{S}_-^2 \right. \end{aligned}$$

$$\begin{aligned}
& -\frac{80}{3}\bar{\beta}_+\zeta\mathcal{S}_-^2 - \frac{384\bar{\beta}_-^2\zeta\mathcal{S}_-^2}{\bar{\gamma}^2} - \frac{1152\bar{\beta}_+^2\zeta\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{352\bar{\beta}_+\zeta\mathcal{S}_-^2}{\bar{\gamma}} - \frac{96\bar{\beta}_+^2\zeta\mathcal{S}_-^2}{\bar{\gamma}} \\
& + \frac{89}{6}\zeta\bar{\gamma}\mathcal{S}_-^2 + 4\zeta\bar{\gamma}^2\mathcal{S}_-^2 - \frac{1589}{30}\zeta^2\mathcal{S}_-^4 + \frac{2688\bar{\beta}_-^2\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^3} + \frac{2688\bar{\beta}_+^2\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^3} + \frac{256\bar{\beta}_-^2\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^2} \\
& - \frac{672\bar{\beta}_+\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^2} + \frac{224\bar{\beta}_+^2\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^2} + \frac{32\bar{\beta}_+\zeta^2\mathcal{S}_-^4}{\bar{\gamma}} + 28\zeta^2\bar{\gamma}\mathcal{S}_-^4 - \frac{768\bar{\beta}_-\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} \\
& + \frac{256\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{64}{3}\bar{\beta}_-\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{128}{3}\bar{\beta}_-\zeta^2\mathcal{S}_-^3\mathcal{S}_+ + \frac{6144\bar{\beta}_-\bar{\beta}_+\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}^3} \\
& - \frac{768\bar{\beta}_-\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}^2} + \frac{512\bar{\beta}_-\bar{\beta}_+\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}^2} + \frac{32\bar{\beta}_-\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}} + \frac{128}{3}\bar{\beta}_-\zeta^2\mathcal{S}_-\mathcal{S}_+^3 \\
& - \frac{227}{30}\zeta^2\mathcal{S}_+^4 + \frac{384\bar{\beta}_-^2\zeta^2\mathcal{S}_+^4}{\bar{\gamma}^3} + \frac{384\bar{\beta}_+^2\zeta^2\mathcal{S}_+^4}{\bar{\gamma}^3} - \frac{96\bar{\beta}_+\zeta^2\mathcal{S}_+^4}{\bar{\gamma}^2} + \frac{32\bar{\beta}_+^2\zeta^2\mathcal{S}_+^4}{\bar{\gamma}^2} + 4\zeta^2\bar{\gamma}\mathcal{S}_+^4 \Big] \\
& + \nu\delta \left[\bar{\beta}_-\bar{\gamma} - 4\bar{\beta}_-\zeta\mathcal{S}_-^2 - \frac{28\bar{\beta}_-\zeta^2\mathcal{S}_-^4}{\bar{\gamma}} - 8\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ + \frac{7}{15}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + \frac{14}{15}\zeta^2\mathcal{S}_-^3\mathcal{S}_+ \right. \\
& \left. - \frac{16\bar{\beta}_+\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}} + \frac{14}{15}\zeta^2\mathcal{S}_-\mathcal{S}_+^3 - \frac{16\bar{\beta}_+\zeta^2\mathcal{S}_-\mathcal{S}_+^3}{\bar{\gamma}} - \frac{4\bar{\beta}_-\zeta^2\mathcal{S}_+^4}{\bar{\gamma}} \right] \\
& + \nu^2 \left[-\frac{17}{20}\bar{\gamma}^2 + \frac{17}{5}\zeta\bar{\gamma}\mathcal{S}_-^2 + \frac{119}{5}\zeta^2\mathcal{S}_-^4 + \frac{17}{5}\zeta^2\mathcal{S}_+^4 \right] \tag{G1l}
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_2^{1.5\text{PN}} &= 6\bar{\gamma} + \frac{19}{8}\bar{\gamma}^2 + 20\bar{\beta}_+\zeta\mathcal{S}_-^2 - \frac{25}{2}\zeta\bar{\gamma}\mathcal{S}_-^2 + \frac{35}{2}\zeta^2\mathcal{S}_-^4 - 20\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ + \frac{5}{2}\zeta^2\mathcal{S}_+^4 \\
& + \delta \left[-20\bar{\beta}_-\zeta\mathcal{S}_-^2 + 20\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ - 5\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ + 10\zeta^2\mathcal{S}_-^3\mathcal{S}_+ + 10\zeta^2\mathcal{S}_-\mathcal{S}_+^3 \right] \\
& + \nu \left[60\bar{\beta}_+ - \frac{240\bar{\beta}_-^2}{\bar{\gamma}} - \frac{240\bar{\beta}_+^2}{\bar{\gamma}} + \frac{19}{2}\bar{\gamma}^2 + 20\zeta\mathcal{S}_-^2 - 480\bar{\beta}_+\zeta\mathcal{S}_-^2 - \frac{960\bar{\beta}_-^2\zeta\mathcal{S}_-^2}{\bar{\gamma}^2} \right. \\
& - \frac{2880\bar{\beta}_+^2\zeta\mathcal{S}_-^2}{\bar{\gamma}^2} + \frac{880\bar{\beta}_+\zeta\mathcal{S}_-^2}{\bar{\gamma}} + 116\zeta\bar{\gamma}\mathcal{S}_-^2 + 60\zeta\phi_0\mathcal{S}_-^2 + 30\zeta\bar{\gamma}\phi_0\mathcal{S}_-^2 - 266\zeta^2\mathcal{S}_-^4 \\
& + \frac{6720\bar{\beta}_-^2\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^3} + \frac{6720\bar{\beta}_+^2\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^3} - \frac{1680\bar{\beta}_+\zeta^2\mathcal{S}_-^4}{\bar{\gamma}^2} + \frac{960\bar{\beta}_+\zeta^2\mathcal{S}_-^4}{\bar{\gamma}} \\
& - \frac{1920\bar{\beta}_-\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}^2} + \frac{640\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+}{\bar{\gamma}} + \frac{15360\bar{\beta}_-\bar{\beta}_+\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}^3} - \frac{1920\bar{\beta}_-\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}^2} \\
& \left. + \frac{960\bar{\beta}_-\zeta^2\mathcal{S}_-^3\mathcal{S}_+}{\bar{\gamma}} - 38\zeta^2\mathcal{S}_+^4 + \frac{960\bar{\beta}_-^2\zeta^2\mathcal{S}_+^4}{\bar{\gamma}^3} + \frac{960\bar{\beta}_+^2\zeta^2\mathcal{S}_+^4}{\bar{\gamma}^3} - \frac{240\bar{\beta}_+\zeta^2\mathcal{S}_+^4}{\bar{\gamma}^2} \right] \\
& + \nu\delta \left[-4\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ - 8\zeta^2\mathcal{S}_-^3\mathcal{S}_+ - 8\zeta^2\mathcal{S}_-\mathcal{S}_+^3 \right] + \nu^2 \left[-\bar{\gamma}^2 + 4\zeta\bar{\gamma}\mathcal{S}_-^2 + 28\zeta^2\mathcal{S}_-^4 + 4\zeta^2\mathcal{S}_+^4 \right] \tag{G1m}
\end{aligned}$$

$$\begin{aligned}
\tilde{E}_3^{1.5\text{PN}} &= -\frac{24}{5}\bar{\gamma} - \frac{61}{40}\bar{\gamma}^2 + \frac{1}{4}\bar{\gamma}^3 - 12\bar{\beta}_+\zeta\mathcal{S}_-^2 + \frac{103}{10}\zeta\bar{\gamma}\mathcal{S}_-^2 + \zeta\bar{\gamma}^2\mathcal{S}_-^2 - \frac{49}{2}\zeta^2\mathcal{S}_-^4 - 7\zeta^2\bar{\gamma}\mathcal{S}_-^4 \\
& + 12\bar{\beta}_-\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{7}{2}\zeta^2\mathcal{S}_+^4 - \zeta^2\bar{\gamma}\mathcal{S}_+^4 + \delta \left[12\bar{\beta}_-\zeta\mathcal{S}_-^2 - 12\bar{\beta}_+\zeta\mathcal{S}_-\mathcal{S}_+ - \frac{1}{5}\zeta\bar{\gamma}\mathcal{S}_-\mathcal{S}_+ \right. \\
& \left. - 2\zeta\bar{\gamma}^2\mathcal{S}_-\mathcal{S}_+ - 14\zeta^2\mathcal{S}_-^3\mathcal{S}_+ - 4\zeta^2\bar{\gamma}\mathcal{S}_-^3\mathcal{S}_+ - 14\zeta^2\mathcal{S}_-\mathcal{S}_+^3 - 4\zeta^2\bar{\gamma}\mathcal{S}_-\mathcal{S}_+^3 \right] \\
& + \nu \left[-36\bar{\beta}_+ + \frac{144\bar{\beta}_-^2}{\bar{\gamma}} + \frac{144\bar{\beta}_+^2}{\bar{\gamma}} - \frac{59}{10}\bar{\gamma}^2 - \frac{12}{5}\zeta\mathcal{S}_-^2 + 288\bar{\beta}_+\zeta\mathcal{S}_-^2 + \frac{576\bar{\beta}_-^2\zeta\mathcal{S}_-^2}{\bar{\gamma}^2} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1728\bar{\beta}_+^2\zeta S_-^2}{\bar{\gamma}^2} - \frac{528\bar{\beta}_+\zeta S_-^2}{\bar{\gamma}} - \frac{324}{5}\zeta\bar{\gamma}S_-^2 - 36\zeta\phi_0S_-^2 - 18\zeta\bar{\gamma}\phi_0S_-^2 + \frac{826}{5}\zeta^2S_-^4 \\
& - \frac{4032\bar{\beta}_-^2\zeta^2S_+^4}{\bar{\gamma}^3} - \frac{4032\bar{\beta}_+\zeta^2S_+^4}{\bar{\gamma}^3} + \frac{1008\bar{\beta}_+\zeta^2S_-^4}{\bar{\gamma}^2} - \frac{576\bar{\beta}_+\zeta^2S_-^4}{\bar{\gamma}} \\
& + \frac{1152\bar{\beta}_-\bar{\beta}_+\zeta S_-S_+}{\bar{\gamma}^2} - \frac{384\bar{\beta}_-\zeta S_-S_+}{\bar{\gamma}} - \frac{9216\bar{\beta}_-\bar{\beta}_+\zeta^2S_-^3S_+}{\bar{\gamma}^3} \\
& + \frac{1152\bar{\beta}_-\zeta^2S_-^3S_+}{\bar{\gamma}^2} - \frac{576\bar{\beta}_-\zeta^2S_-^3S_+}{\bar{\gamma}} + \frac{118}{5}\zeta^2S_+^4 - \frac{576\bar{\beta}_-^2\zeta^2S_+^4}{\bar{\gamma}^3} \\
& - \frac{576\bar{\beta}_+^2\zeta^2S_+^4}{\bar{\gamma}^3} + \frac{144\bar{\beta}_+\zeta^2S_+^4}{\bar{\gamma}^2} \Big] + \nu\delta \left[\frac{14}{5}\zeta\bar{\gamma}S_-S_+ + \frac{28}{5}\zeta^2S_-^3S_+ + \frac{28}{5}\zeta^2S_-S_-^3 \right] \\
& + \nu^2 \left[\bar{\gamma}^2 - 4\zeta\bar{\gamma}S_-^2 - 28\zeta^2S_-^4 - 4\zeta^2S_+^4 \right] \tag{G1n}
\end{aligned}$$

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References

- [1] Bertotti B, Iess L and Tortora P 2003 A test of general relativity using radio links with the Cassini spacecraft *Nature* **425** 374
- [2] Fomalont E B and Kopeikin S M 2003 The measurement of the light deflection from Jupiter: experimental results *Astrophys. J.* **598** 704
- [3] Kopeikin S M and Fomalont E B 2007 Gravitomagnetism and the Lorentz invariance of gravity in the gravitational light-ray deflection experiments *Gen. Relativ. Gravit.* **39** 1583
- [4] Fomalont E, Kopeikin S, Jones D, Honma M and Titov O 2010 Recent VLBA/VERA/IVS Tests of General Relativity *IAU Symp.* **261** 291
- [5] Williams J G, Turyshev S G and Boggs D H 2004 Progress in lunar laser ranging tests of relativistic gravity *Phys. Rev. Lett.* **93** 261101
- [6] Damour T and Esposito-Farèse G 1992 Tensor multiscalar theories of gravitation *Class. Quantum Grav.* **9** 2093
- [7] Fujii Y and Maeda K 2005 *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press)
- [8] De Felice A and Tsujikawa S 2010 f(R) theories *Living Rev. Relativ.* **13** 3
- [9] Kramer M *et al* 2006 Tests of general relativity from timing the double pulsar *Science* **314** 97
- [10] Kramer M *et al* 2021 Strong-field gravity tests with the double pulsar *Phys. Rev. X* **11** 041050
- [11] Abbott R *et al* (LIGO, Virgo and KAGRA Collaboration) 2021 Tests of general relativity with GWTC-3 (arXiv:2112.06861 [gr-qc])
- [12] Smith T L, Poulin V and Amin M A 2020 Oscillating scalar fields and the Hubble tension: a resolution with novel signatures *Phys. Rev. D* **101** 063523
- [13] Blanchet L, Faye G, Henry Q, Larrouturou F and Trestini D 2023 Gravitational-wave phasing of quasicircular compact binary systems to the fourth-and-a-half post-Newtonian order *Phys. Rev. Lett.* **131** 121402
- [14] Blanchet L, Faye G, Henry Q, Larrouturou F and Trestini D 2023 Gravitational-wave flux and quadrupole modes from quasicircular nonspinning compact binaries to the fourth post-Newtonian order *Phys. Rev. D* **108** 064041
- [15] Ajith P *et al* 2012 The NINJA-2 catalog of hybrid post-Newtonian/numerical-relativity waveforms for non-precessing black-hole binaries *Class. Quantum Grav.* **29** 124001
- Ajith P *et al* 2013 *Class. Quantum Grav.* **30** 199401 erratum
- [16] Ferguson D *et al* 2023 Second MAYA catalog of binary black hole numerical relativity waveforms (arXiv:2309.00262 [gr-qc])
- [17] Boyle M *et al* 2019 The SXS Collaboration catalog of binary black hole simulations *Class. Quantum Grav.* **36** 195006

- [18] Hughes S A, Warburton N, Khanna G, Chua A J K and Katz M L 2021 Adiabatic waveforms for extreme mass-ratio inspirals via multivoice decomposition in time and frequency *Phys. Rev. D* **103** 104014
- [19] Hughes S A, Warburton N, Khanna G, Chua A J K and Katz M L 2023 *Phys. Rev. D* **107** 089901 erratum
- [20] Isoyama S, Fujita R, Chua A J K, Nakano H, Pound A and Sago N 2022 Adiabatic waveforms from extreme-mass-ratio inspirals: an analytical approach *Phys. Rev. Lett.* **128** 231101
- [21] Wardell B, Pound A, Warburton N, Miller J, Durkan L and Le Tiec A 2023 Gravitational waveforms for compact binaries from second-order self-force theory *Phys. Rev. Lett.* **130** 241402
- [22] Buonanno A and Damour T 1999 Effective one-body approach to general relativistic two-body dynamics *Phys. Rev. D* **59** 084006
- [23] Ramos-Buades A, Buonanno A, Estellés H, Khalil M, Mihaylov D P, Ossokine S, Pompili L and Shiferaw M 2023 Next generation of accurate and efficient multipolar precessing-spin effective-one-body waveforms for binary black holes *Phys. Rev. D* **108** 124037
- [24] Gamba R *et al* 2023 Analytically improved and numerical-relativity informed effective-one-body model for coalescing binary neutron stars (arXiv:2307.15125 [gr-qc])
- [25] Pratten G, Husa S, Garcia-Quiros C, Colleoni M, Ramos-Buades A, Estelles H and Jaume R 2020 Setting the cornerstone for a family of models for gravitational waves from compact binaries: the dominant harmonic for nonprecessing quasicircular black holes *Phys. Rev. D* **102** 064001
- [26] Williams N, Schmidt P and Pratten G 2024 PhenomGSF: a phenomenological model of gravitational self-force enhanced tides in inspiralling binary neutron stars (arXiv:2407.08538 [gr-qc])
- [27] Damour T and Esposito-Farèse G 1996 Tensor - scalar gravity and binary pulsar experiments *Phys. Rev. D* **54** 1474
- [28] Mirshekari S and Will C M 2013 Compact binary systems in scalar-tensor gravity: equations of motion to 2.5 post-Newtonian order *Phys. Rev. D* **87** 084070
- [29] Brans C and Dicke R 1961 Mach's principle and a relativistic theory of gravity *Phys. Rev.* **124** 925
- [30] Fierz M 1956 Über die physikalische Deutung der erweiterten Gravitationstheorie P. Jordans *Helv. Phys. Acta* **29** 128
- [31] Bernard L 2018 Dynamics of compact binary systems in scalar-tensor theories: equations of motion to the third post-Newtonian order *Phys. Rev. D* **98** 044004
- [32] Bernard L 2019 Dynamics of compact binary systems in scalar-tensor theories: II. Center-of-mass and conserved quantities to 3PN order *Phys. Rev. D* **99** 044047
- [33] Lang R N 2015 Compact binary systems in scalar-tensor gravity. III. Scalar waves and energy flux *Phys. Rev. D* **91** 084027
- [34] Lang R N 2014 Compact binary systems in scalar-tensor gravity. II. Tensor gravitational waves to second post-Newtonian order *Phys. Rev. D* **89** 084014
- [35] Sennett N and Buonanno A 2016 Modeling dynamical scalarization with a resummed post-Newtonian expansion *Phys. Rev. D* **93** 124004
- [36] Bernard L, Blanchet L and Trestini D 2022 Gravitational waves in scalar-tensor theory to one-and-a-half post-Newtonian order *J. Cosmol. Astropart. Phys.* **JCAP08(2022)008**
- [37] Almeida G L 2024 Binary dynamics to second post-Newtonian order in scalar-tensor and Einstein-scalar-Gauss-Bonnet gravity from effective field theory *Phys. Rev. D* **109** 084060
- [38] Diedrichs R F, Schmitt D and Sagunski L 2023 Binary systems in massive scalar-tensor theories: next-to-leading order gravitational waveform from effective field theory (arXiv:2311.04274 [gr-qc])
- [39] Almeida G L and Zhou S Y 2024 Post-Newtonian dynamics of spinning black hole binaries in Einstein-scalar-gauss-bonnet gravity (arXiv:2408.14196 [gr-qc])
- [40] Bernard L, Dones E and Mougiakakos S 2023 Tidal effects up to next-to-next-to leading post-Newtonian order in massless scalar-tensor theories (arXiv:2310.19679 [gr-qc])
- [41] Julié F-L and Deruelle N 2017 Two-body problem in Scalar-Tensor theories as a deformation of general relativity: an effective-one-body approach *Phys. Rev. D* **95** 124054
- [42] Julié F-L 2018 Reducing the two-body problem in scalar-tensor theories to the motion of a test particle : a scalar-tensor effective-one-body approach *Phys. Rev. D* **97** 024047
- [43] Julié F L, Baibhav V, Berti E and Buonanno A 2022 Third post-Newtonian effective-one-body Hamiltonian in scalar-tensor and Einstein-scalar-Gauss-Bonnet gravity (arXiv:2212.13802 [gr-qc])
- [44] Jain T, Rettengo P, Agathos M, Nagar A and Turco L 2023 Effective-one-body Hamiltonian in scalar-tensor gravity at third post-Newtonian order *Phys. Rev. D* **107** 084017

- [44] Jain T 2023 Nonlocal-in-time effective one body Hamiltonian in scalar-tensor gravity at third post-Newtonian order *Phys. Rev. D* **107** 084018
- [45] Ma S, Varma V, Stein L C, Foucart F, Duez M D, Kidder L E, Pfeiffer H P and Scheel M A 2023 Numerical simulations of black hole-neutron star mergers in scalar-tensor gravity *Phys. Rev. D* **107** 124051
- [46] Corman M and East W E 2024 Black hole-neutron star mergers in Einstein-scalar-Gauss-Bonnet gravity (arXiv:2405.18496 [gr-qc])
- [47] Trestini D 2024 Quasi-Keplerian parametrization for eccentric compact binaries in scalar-tensor theories at second post-Newtonian order and applications *Phys. Rev. D* **109** 104003
- [48] Blanchet L, Faye G and Trestini D 2024 Gravitational radiation reaction for compact binary systems at the fourth-and-a-half post-Newtonian order *Class. Quantum Grav.* **42** 065015
- [49] Barausse E, Yunes N and Chamberlain K 2016 Theory-agnostic constraints on black-hole dipole radiation with multiband gravitational-wave astrophysics *Phys. Rev. Lett.* **116** 241104
- [50] Maggiore M *et al* 2019 Science case for the Einstein telescope *J. Cosmol. Astropart. Phys.* **JCAP03(2020)050**
- [51] Abbott B P *et al* (LIGO Scientific) 2017 Exploring the sensitivity of next generation gravitational wave detectors *Class. Quantum Grav.* **34** 044001
- [52] Magee R, Isi M, Payne E, Chatziioannou K, Farr W M, Pratten G and Vitale S 2023 The impact of selection biases on tests of general relativity with gravitational-wave inspirals (arXiv:2311.03656 [gr-qc])
- [53] Shiralilou B, Hinderer T, Nissanke S M, Ortiz N and Witek H 2022 Post-Newtonian gravitational and scalar waves in scalar-Gauss-Bonnet gravity *Class. Quantum Grav.* **39** 035002
- [54] Julié F-L and Berti E 2019 Post-Newtonian dynamics and black hole thermodynamics in Einstein-scalar-Gauss-Bonnet gravity *Phys. Rev. D* **100** 104061
- [55] Julié F L, Pompili L and Buonanno A 2024 Inspiral-merger-ringdown waveforms in Einstein-scalar-Gauss-Bonnet gravity within the effective-one-body formalism (arXiv:2406.13654 [gr-qc])
- [56] Bošković M and Barausse E 2023 Two-body problem in theories with kinetic screening *Phys. Rev. D* **108** 064033
- [57] de Rham C, Giblin J T and Tolley A J 2024 Scalar radiation with a quartic Galileon *Phys. Rev. D* **109** 104035
- [58] The ancillary file [cqqadedf5suppl.wl](https://arxiv.org/abs/2406.13654) contains machine-readable expressions corresponding to various quantities presented in this paper and in previous works.
- [59] Arun K G, Blanchet L, Iyer B R and Sinha S 2009 Third post-Newtonian angular momentum flux and the secular evolution of orbital elements for inspiralling compact binaries in quasi-elliptical orbits *Phys. Rev. D* **80** 124018
- [60] Will C M and Nordtvedt K 1972 Conservation laws and preferred frames in relativistic gravity. I. Preferred-frame theories and an extended PPN formalism *Astrophys. J.* **177** 757
- [61] Will C M 2018 *Theory and Experiment in Gravitational Physics* (Cambridge University Press)
- [62] Jain T and Retegno P 2024 Angular momentum flux in scalar-tensor theories up to 1.5 post-Newtonian order *Phys. Rev. D* **111** 104038
- [63] Eardley D M 1975 Observable effects of a scalar gravitational field in a binary pulsar *Astrophys. J. Lett.* **196** L59
- [64] Nordtvedt K 1990 $\dot{G}/G$ and a cosmological acceleration of gravitationally compact bodies *Phys. Rev. Lett.* **65** 953
- [65] Blanchet L and Faye G 2000 Hadamard regularization *J. Math. Phys.* **41** 7675
- [66] Blanchet L, Damour T and Esposito-Farèse G 2004 Dimensional regularization of the third post-Newtonian dynamics of point particles in harmonic coordinates *Phys. Rev. D* **69** 124007
- [67] Blanchet L, Damour T, Esposito-Farèse G and Iyer B R 2005 Dimensional regularization of the third post-Newtonian gravitational wave generation of two point masses *Phys. Rev. D* **71** 124004
- [68] Hawking S W 1972 Black holes in the Brans-Dicke theory of gravitation *Commun. Math. Phys.* **25** 167
- [69] Sotiriou T P and Faraoni V 2012 Black holes in scalar-tensor gravity *Phys. Rev. Lett.* **108** 081103
- [70] Healy J, Bode T, Haas R, Pazos E, Laguna P, Shoemaker D and Yunes N 2012 Late inspiral and merger of binary black holes in scalar-tensor theories of gravity *Class. Quantum Grav.* **29** 232002

- [71] Landau L and Lifshitz E 1971 *Course of Theoretical Physics (The Classical Theory of Fields)* vol 2 (Pergamon)
- [72] Blanchet L and Damour T 1986 Radiative gravitational fields in general relativity. I. General structure of the field outside the source *Phil. Trans. R. Soc. A* **320** 379
- [73] Blanchet L 2014 Gravitational radiation from post-Newtonian sources and inspiralling compact binaries *Living Rev. Relativ.* **17** 2
- [74] Blanchet L and Damour T 1992 Hereditary effects in gravitational radiation *Phys. Rev. D* **46** 4304
- [75] Blanchet L 1998 Quadrupole-quadrupole gravitational waves *Class. Quantum Grav.* **15** 89
- [76] Blanchet L 1998 Gravitational-wave tails of tails *Class. Quantum Grav.* **15** 113
Blanchet L 2005 *Class. Quantum Grav.* **22** 3381 (Corrigendum)
- [77] Trestini D, Larrouturou F and Blanchet L 2023 The quadrupole moment of compact binaries to the fourth post-Newtonian order: relating the harmonic and radiative metrics *Class. Quantum Grav.* **40** 055006
- [78] Trestini D and Blanchet L 2023 Gravitational-wave tails of memory *Phys. Rev. D* **107** 104048
- [79] Marchand T, Blanchet L and Faye G 2016 Gravitational-wave tail effects to quartic non-linear order *Class. Quantum Grav.* **33** 244003
- [80] Blanchet L 1987 Radiative gravitational fields in general relativity. II. Asymptotic behaviour at future null infinity *Proc. R. Soc. A* **409** 383
- [81] Sennett N, Marsat S and Buonanno A 2016 Gravitational waveforms in scalar-tensor gravity at 2PN relative order *Phys. Rev. D* **94** 084003
- [82] Blaes O, Lee M H and Socrates A 2002 The kozai mechanism and the evolution of binary super-massive black holes *Astrophys. J.* **578** 775
- [83] Romero-Shaw I M, Lasky P D and Thrane E 2022 Four eccentric mergers increase the evidence that LIGO-Virgo-KAGRA binary black holes form dynamically *Astrophys. J.* **940** 171
- [84] Damour T and Deruelle N 1985 General relativistic celestial mechanics of binary systems I. The post-Newtonian motion *Ann. Inst. Henri Poincaré A* **43** 107 (available at: https://www.numdam.org/item/AIHPA_1985__43_1_107_0/)
- [85] Damour T and Schäfer G 1988 Higher order relativistic periastron advances in binary pulsars *Nuovo Cim.* **B101** 127
- [86] Schäfer G and Wex N 1993 Second post-Newtonian morion of compact binaries *Phys. Lett. A* **174** 196
Schäfer G and Wex N 1993 *Phys. Lett. A* **177** 461 (erratum)
- [87] Memmesheimer R, Gopakumar A and Schäfer G 2004 Third post-Newtonian accurate generalized quasi-Keplerian parametrization for compact binaries in eccentric orbits *Phys. Rev. D* **70** 104011
- [88] Cho G, Tanay S, Gopakumar A and Lee H M 2022 Generalized quasi-Keplerian solution for eccentric, nonspinning compact binaries at 4PN order and the associated inspiral-merger-ringdown waveform *Phys. Rev. D* **105** 064010
- [89] Tucker A and Will C M 2019 Pericenter advance in general relativity: comparison of approaches at high post-Newtonian orders *Class. Quantum Grav.* **36** 115001
- [90] Arun K G, Blanchet L, Iyer B R and Qusailah M S S 2008 Tail effects in the 3PN gravitational wave energy flux of compact binaries in quasi-elliptical orbits *Phys. Rev. D* **77** 064034
- [91] Loutrel N and Yunes N 2017 Hereditary effects in eccentric compact binary inspirals to third post-Newtonian order *Class. Quantum Grav.* **34** 044003
- [92] Gopakumar A, Iyer B R and Iyer S 1997 Second post Newtonian gravitational radiation reaction for two-body systems: nonspinning bodies *Phys. Rev. D* **55** 6030
Gopakumar A, Iyer B R and Iyer S 1998 *Phys. Rev. D* **57** 6562 (erratum)
- [93] Leibovich A K, Pardo B A and Yang Z 2023 Radiation reaction for nonspinning bodies at 4.5PN in the effective field theory approach *Phys. Rev. D* **108** 024017
- [94] Compère G, Oliveri R and Seraj A 2019 The Poincaré and BMS flux-balance laws with application to binary systems *J. High Energy Phys.* **JHEP10(2020)116**
- [95] Thorne K S 1980 Multipole expansions of gravitational radiation *Rev. Mod. Phys.* **52** 299
- [96] Kozameh C, Nieva J and Quirega G 2018 Spin and Center of mass comparison between the PN approach and the asymptotic formulation *Phys. Rev. D* **98** 064032
- [97] Blanchet L and Faye G 2019 Flux-balance equations for linear momentum and center-of-mass position of self-gravitating post-Newtonian systems *Class. Quantum Grav.* **36** 085003
- [98] Blanchet L private communication

- [99] Blanchet L and Schäfer G 1993 Gravitational wave tails and binary star systems *Class. Quantum Grav.* **10** 2699
- [100] Arun K G, Blanchet L, Iyer B R and Qusailah M S S 2004 The 2.5PN gravitational wave polarisations from inspiralling compact binaries in circular orbits *Class. Quantum Grav.* **21** 3771
- Arun K G, Blanchet L, Iyer B R and Qusailah M S S 2005 *Class. Quantum Grav.* **22** 3115 (Corrigendum)
- [101] Trestini D unpublished
- [102] Arun K G, Blanchet L, Iyer B R and Qusailah M S S 2008 Inspiralling compact binaries in quasi-elliptical orbits: the Complete 3PN energy flux *Phys. Rev. D* **77** 064035
- [103] Forseth E, Evans C R and Hopper S 2016 Eccentric-orbit extreme-mass-ratio inspiral gravitational wave energy fluxes to 7PN order *Phys. Rev. D* **93** 064058
- [104] Blanchet L, Faye G, Iyer B R and Joguet B 2002 Gravitational-wave inspiral of compact binary systems to 7/2 post-Newtonian order *Phys. Rev. D* **65** 061501(R)
- Blanchet L, Faye G, Iyer B R and Joguet B 2005 *Phys. Rev. D* **71** 129902 (erratum)
- Blanchet L, Faye G, Iyer B R and Joguet B 2005 *Phys. Rev. D* **71** 129903 (erratum)
- [105] Damour T, Gopakumar A and Iyer B R 2004 Phasing of gravitational waves from inspiralling eccentric binaries *Phys. Rev. D* **70** 064028
- [106] Martín-García J M *et al* xAct: efficient tensor computer algebra for Mathematica (GPL 2002–2012) (available at: www.xact.es/)
- [107] Wolfram Research, Inc. 2023 Mathematica Version 13.3 (available at: www.wolfram.com/mathematica)
- [108] Peters P 1964 Gravitational radiation and the motion of two point masses *Phys. Rev.* **136** B1224
- [109] Bini D and Geralico A 2021 Higher-order tail contributions to the energy and angular momentum fluxes in a two-body scattering process *Phys. Rev. D* **104** 104020