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MODELLING WAVE PROPAGATION IN CYLINDERS USING A WAVE/FINITE ELEMENT TECHNIQUE

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ABSTRACT
The propagation of waves in axisymmetric structures can be modelled using a wave/finite element (WFE) approach. A small, rectangular segment of the structure is modelled using conventional finite element methods, typically using a commercial package. Periodicity conditions are then applied. An eigenvalue problem results, the solutions of which yield the dispersion relations. In this paper the WFE method is applied to cylindrical shells modelled using ANSYS. The circumferential order of the wave can be specified in order to define the phase change a wave experiences as it propagates across the element in the circumferential direction. The resulting eigenproblem then relates the axial wavenumber and frequency. The method is described and illustrated by application to cylinders of different constructions. First a thin, isotropic shell is considered - for this case analytical solutions are available from which the accuracy and efficiency of the method can be demonstrated. A steel cylinder filled with water comprises the second example. The third example concerns a sandwich cylinder with a foam core and orthotropic, laminated skins, for which analytical solutions are not available. The method is seen to be simple in application and provide accurate results with very little computational cost.

INTRODUCTION
The analysis of wave propagation in cylindrical shells is of importance in a number of applications. Examples include structure-borne sound, acoustic wave propagation in fluid-filled ducts, sound transmission in aerospace structures, structural integrity and SEA. Of primary importance is knowledge of the dispersion relation. Analytical expressions can be developed for simple cases – e.g. isotropic cylinders *in-vacuo*, fluid-filled pipes with rigid walls – but for more complex structures analytical approaches become very difficult or even impossible. On the other hand the computational cost of standard finite element (FE) models of the structure as a whole becomes prohibitive at higher frequencies, so that alternative techniques are sought.

This paper concerns the application of a wave/finite element (WFE) method to the analysis of wave propagation in uniform axisymmetric structures, and in particular cylindrical shells with or without internal fluid. This is a special case of WFE analysis of 2-dimensional structures. In summary, a small segment of the structure is modelled using conventional FE methods, typically using a commercial FE package. The mass and stiffness matrices are subsequently post-processed using methods originally developed by Abdel-Rahman [1] for the FE analysis of periodic structures. An eigenvalue problem is formulated whose solutions give the dispersion relations. For 1-dimensional waveguides WFE methods have been developed for free [2] and forced vibration [3] analysis and applied to laminate plates [2], thin-walled structures [4], fluid-filled pipes [5] and tyres [6]. Similar approaches have previously been applied to rail tracks [7,8]. In the application to fluid-filled pipes [5] axisymmetry was not exploited – the WFE models are significantly larger and it is difficult to characterise the motion in terms of circumferential orders *a priori*. In this paper the WFE analysis of structures for which waves can propagate in 2-dimensions [9] is applied to wave propagation in axisymmetric structures. First the general approach is briefly outlined. Then the method is applied to several examples, these being an isotropic cylinder *in-vacuo*, a water-filled steel pipe and a sandwich cylindrical panel.

One of the main advantages of WFE methods is the fact that standard FE routines and commercial FE packages can be used. They can therefore be applied to structures of arbitrary complexity and structural configuration. Furthermore, the computational cost is very small.

WFE ANALYSIS OF AXISYMMETRIC STRUCTURES

In this section the WFE method is briefly described. Further details for 2-dimensional wave propagation can be found in [9]. Consider a uniform axisymmetric structure whose axis is the *y*-axis. A time harmonic disturbance at frequency ω can propagate through the structure as $w(r,\theta,y,t) = W(r)\exp\left(i\left(\omega t - k_\theta \theta - k_y y\right)\right)$, where k_θ, k_y are the components of the wavenumber k and $W(r)$ is the (complex) wave amplitude. In the absence of damping the wavenumber is real for propagating waves, imaginary for evanescent waves or complex for oscillating, decaying waves. For cylindrical shells of mean radius R it is perhaps more convenient to define an axis x around the circumference, for which $x = R\theta$. Such waves propagate as $w(r,x,y,t) = W(r)\exp\left(i\left(\omega t - k_x x - k_y y\right)\right)$ where $k_\theta = Rk_x$.

The aim of the WFE method is to estimate the dispersion relations between k_x, k_y and ω (or k_θ, k_y and ω) from FEA. A short segment of rectangular cross-section, of length L_y and subtending an angle L_θ as shown in Figure 1(a) is taken from the structure and modelled using FEA. For cylindrical shells one might alternatively take a single, rectangular, 4-noded element of length L_x (Figure 1(b)). In the subsequent analysis there are assumed to be only corner degrees of freedom (DOFs) although mid-side nodes can be included straightforwardly. Any internal DOFs are condensed.

Consider the rectangular cross-section as shown in Figure 1(c). The element degrees of freedom (DOFs) $\mathbf{q}$ are given in terms of the nodal DOFs by

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1^T & \mathbf{q}_2^T & \mathbf{q}_3^T & \mathbf{q}_4^T \end{bmatrix}^T \qquad \text{(Eq. 1)}$$

where the superscript T denotes the transpose, with a similar expression for the nodal forces $\mathbf{f}$. The mass and stiffness matrices of the element are found using conventional FE methods. Typically a commercial package might be used so that existing element libraries can be exploited. It is common in FEA to model curved structures as being piecewise flat. In order to model the desired curvature, as shown in Figure 1(b), the DOFs of node 2 and node 4 (defined in local coordinates) must be transformed to global coordinates by a rotation through an angle L_θ. A transformation matrix $\mathbf{R}$ can be defined so that the mass and stiffness matrices for the "curved" element become $\mathbf{M} = \mathbf{R}^T\mathbf{M}_{loc}\mathbf{R}$ and $\mathbf{K} = \mathbf{R}^T\mathbf{K}_{loc}\mathbf{R}$, where $\mathbf{M}_{loc}$ and $\mathbf{K}_{loc}$ are the element matrices in local coordinates.

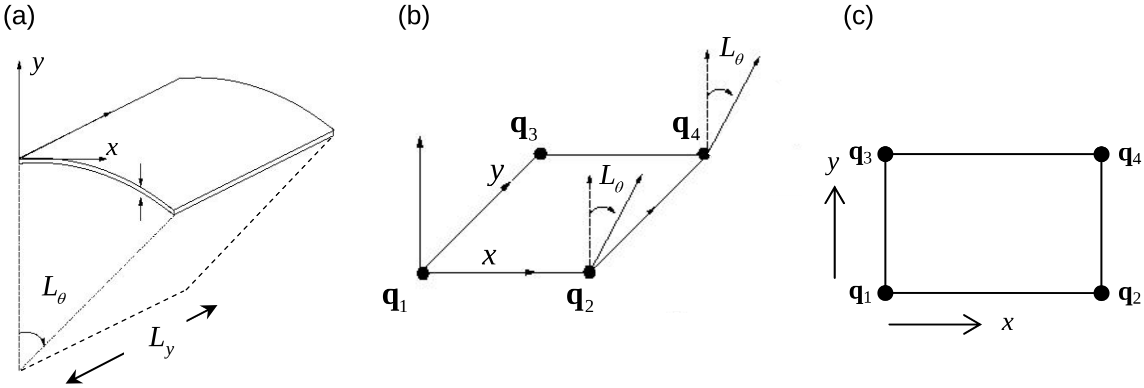

Figure 1.- WFE of axisymmetric structures: (a) segment of axisymmetric structure; (b) plane 4-noded element and coordinate rotation; (c) rectangular element and nodal DOFs.

The equation of motion for the element, assuming time-harmonic behaviour, is

$$\mathbf{Dq} = \mathbf{f}; \quad \mathbf{D} = \mathbf{K} - \omega^2\mathbf{M} \qquad \text{(Eq. 2)}$$

where $\mathbf{D}$ is the dynamic stiffness matrix. Under the propagation of a wave the nodal degrees of freedom in global coordinates are such that

$$\mathbf{q}_2 = \lambda_x \mathbf{q}_1; \; \mathbf{q}_3 = \lambda_y \mathbf{q}_1; \; \mathbf{q}_4 = \lambda_x \lambda_y \mathbf{q}_1 \qquad \text{(Eq. 3)}$$

where

$$\lambda_x = e^{-i\mu_x}; \; \lambda_y = e^{-i\mu_y}; \; \mu_x = k_x L_x; \; \mu_y = k_y L_y \qquad \text{(Eq. 4)}$$

Here μ_x and μ_y are the propagation constants. The nodal forces are related by similar equations. Thus the nodal DOFs can be written in terms of the DOFs $\mathbf{q}_1$ of node 1. In the absence of external excitation, equilibrium at node 1 implies that the sum of the nodal forces of all the elements connected to node 1 is zero. Equation (2) can thus be transformed into

$$\left[\mathbf{K}'\left(\lambda_x, \lambda_y\right) - \omega^2 \mathbf{M}'\left(\lambda_x, \lambda_y\right)\right]\mathbf{q}_1 = \mathbf{0} \qquad \text{(Eq. 5)}$$

where $\mathbf{K}'$ and $\mathbf{M}'$ are the reduced stiffness and mass matrices, i.e. the element matrices projected onto the DOFs of node 1 under the assumption of time harmonic disturbance propagation. The eigenvalue problem of Eq. 5 can also be written as

$$\mathbf{D}'\left(\omega; \lambda_x, \lambda_y\right)\mathbf{q}_1 = \mathbf{0} \qquad \text{(Eq. 6)}$$

where $\mathbf{D}' = \mathbf{K}' - \omega^2 \mathbf{M}'$ is the reduced dynamic stiffness matrix (DSM). If the DSM of the segment of the structure is partitioned into appropriate submatrices then the reduced eigenvalue problem is given by

$$\begin{aligned}&[\left(\mathbf{D}_{11} + \mathbf{D}_{22} + \mathbf{D}_{33} + \mathbf{D}_{44}\right) + \left(\mathbf{D}_{12} + \mathbf{D}_{34}\right)\lambda_x + \left(\mathbf{D}_{21} + \mathbf{D}_{43}\right)\lambda_x^{-1} + \\ &\left(\mathbf{D}_{13} + \mathbf{D}_{24}\right)\lambda_y + \left(\mathbf{D}_{31} + \mathbf{D}_{42}\right)\lambda_y^{-1} + \mathbf{D}_{14}\lambda_x\lambda_y + \mathbf{D}_{41}\lambda_x^{-1}\lambda_y^{-1} + \mathbf{D}_{32}\lambda_x\lambda_y^{-1} + +\mathbf{D}_{23}\lambda_x^{-1}\lambda_y]\mathbf{q}_1 = 0\end{aligned} \qquad \text{(Eq. 7)}$$

If there are n DOFs per node, the nodal displacement and force vectors are $n \times 1$, the element mass and stiffness matrices are $4n \times 4n$ while the reduced matrices are $n \times n$. Eqs. 6 and 7 define an eigenproblem relating λ_x, λ_y and ω for the discretised structure, whose solutions give FE estimates of the dispersion relations of the continuous structure. The form of the eigenproblem may be linear, quadratic, polynomial or transcendental according to the nature of the solution sought. Further details can be found in [9].

The eigenvalue problem for closed axisymmetric structures

Generally, a plane wave in an axisymmetric structure propagates with a helical pattern so that k_x, k_θ can in principle take arbitrary values: real, imaginary or complex. However, in closed structures the phase change of a wave as it propagates around the circumference must be a multiple of 2π so that the circumferential wavenumber can only take the discrete values $k_x = n/R, \; n = 0,1,2...$ which define the order n of the wave mode. Under these circumstances $\lambda_x = \exp\left(-inL_x/R\right)$ is known for a given circumferential order and Eq. 7 becomes either a linear eigenproblem in ω^2 for given λ_y or a quadratic eigenproblem in λ_y for given ω^2.

NUMERICAL EXAMPLES

In this section various numerical examples are presented to illustrate the application of the WFE method to cylinders. The non-dimensionalised frequency $\Omega = \omega/\omega_r$ is introduced**,** where

$$\omega_r = \sqrt{E/\rho(1-\nu^2)}\,, \qquad \text{(Eq. 9)}$$

is the shell ring frequency.

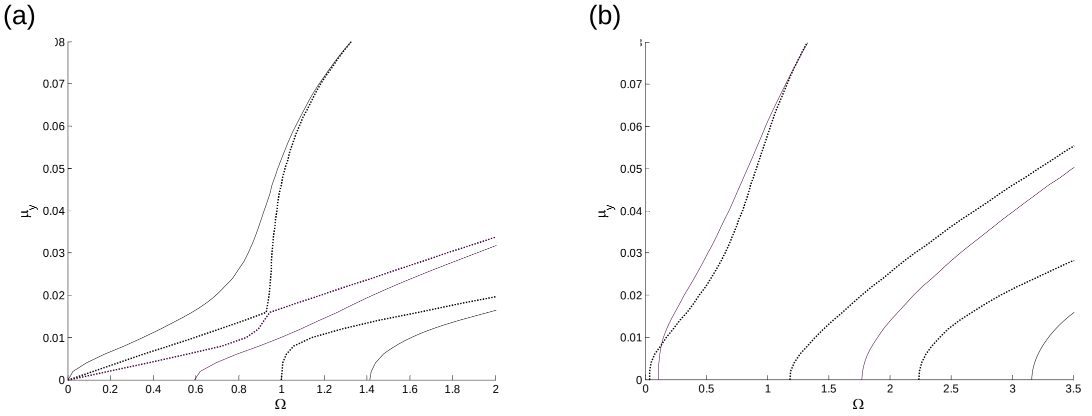

Figure 2.- Dispersion curves for cylindrical steel shell: (a) …… $n=0$, —— $n=1$;
(b) …… $n=2$, —— $n=3$.

Isotropic cylinder *in-vacuo*

An in-vacuo steel cylinder with thickness-to-mean-radius ratio 0.05 is considered. The mass and stiffness matrices were found from a single plane element of type SOLID45 in ANSYS. Analytical express exist for this situation.

Figure 2 shows the real-valued dispersion curves for the circumferential modes of orders $n=0,1,2,3$. The three branches shown in Figure 2 broadly correspond to flat-plate flexural, torsional and extensional waves. This behaviour is particularly clear above the ring frequency, while near and below the ring frequency the effect of the curvature results in a more complicated behaviour.

The dispersion relation for the propagation of helical waves is shown in Figure 3 for $\Omega=0,0.5,1$. The group velocity $c_g = d\Omega/dk$ is in the direction of the normal to the dispersion curves in the (μ_x,μ_y) plane. In Figure 3(a) it can be seen that at low frequencies there exist regions in which a particular value of μ_y corresponds to two distinct values of μ_x, e.g. points A and B. These points represent distinct waves with group velocities in different directions: point B represents a wave having a negative group velocity in the circumferential direction while point A represents a wave having a positive group velocity in the circumferential direction.

Isotropic water-filled cylinder

In this section the WFE method is applied to a water-filled steel pipe with thickness-to-mean radius ratio equal to 0.1. Damping is neglected and the speed of flow of the fluid is assumed to be negligible compared to the speed of sound in both the fluid and the structure. The finite element model of a segment of the structure is shown in Figure 4 and was realised in ANSYS. It

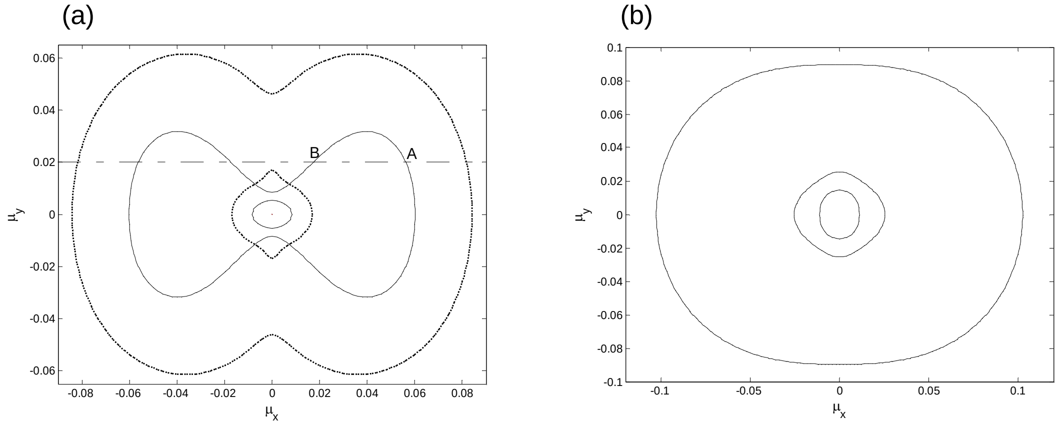

Figure 3.- Dispersion curves for steel shell: (a) —— $\Omega=0.5$, …. $\Omega=1$; (b) $\Omega=1.5$.

comprises 2 solid structural "brick" elements (SOLID45) and 20 fluid elements (FLUID30), resulting in a total number of DOFs equal to 120. The model includes the fluid structure interaction at the interface between the fluid and the structure. Figure 5 shows the non-dimensional wavenumber k_yR corresponding to orders $n = 0,1,2,3,4$. The various waves can be associated with motion that is predominantly structure- (bending, torsion or axial) or fluid-borne.

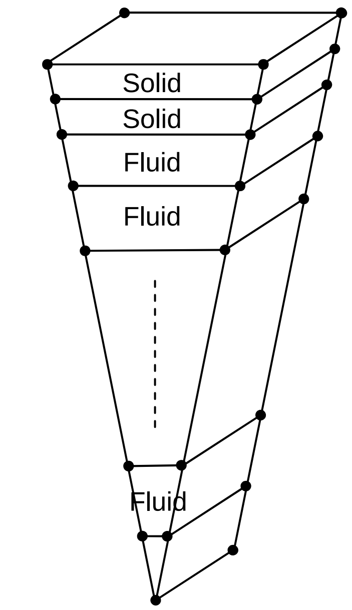

Figure 4.- FE model of segment of water-filled steel cylinder

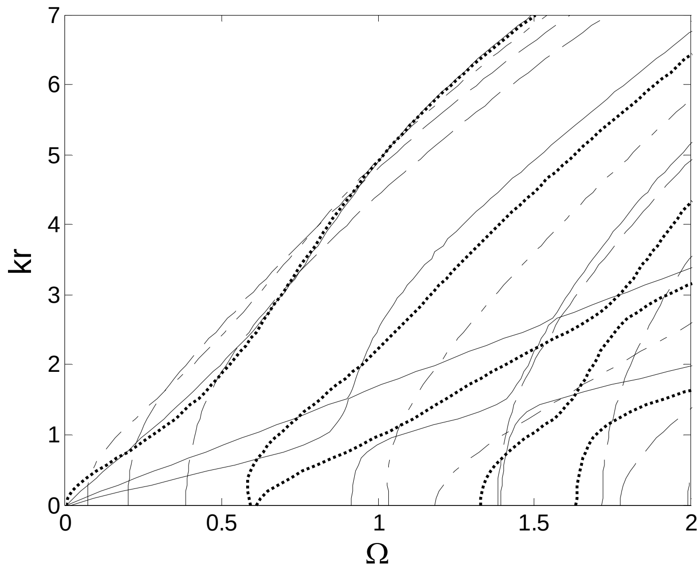

Figure 5.- Dispersion curves for a water-filled cylindrical steel shell: ——— $n=0$, …. $n=1$, -·-·-·· $n=2$, - - - - $n=3$, — — — — — $n=4$.

Sandwich cylindrical shell

As a final example, consider a cylindrical sandwich shell comprising two laminated skins and a foam core. The example panel is very similar to one considered by Heron in [8]. The thickness-to-mean radius ratio of the sandwich construction is 0.018. The two skins each comprise 4 orthotropic sheets of glass/epoxy with a lay-up of [+45/-45/-45/+45] while the core material is a polymethacrylamide ROHACELL foam with 110WF density. A rectangular, 4-node ANSYS finite element SHELL181, is used to obtain the mass and stiffness matrices. The real-valued dispersion curves are shown in Figure 6 for the circumferential modes n = 0,1,2,3. The ring frequency is at approximately 617 Hz. Below the ring frequency the wave behaviour is very complex and cannot be described simply in terms of torsional, extensional and flexural waves

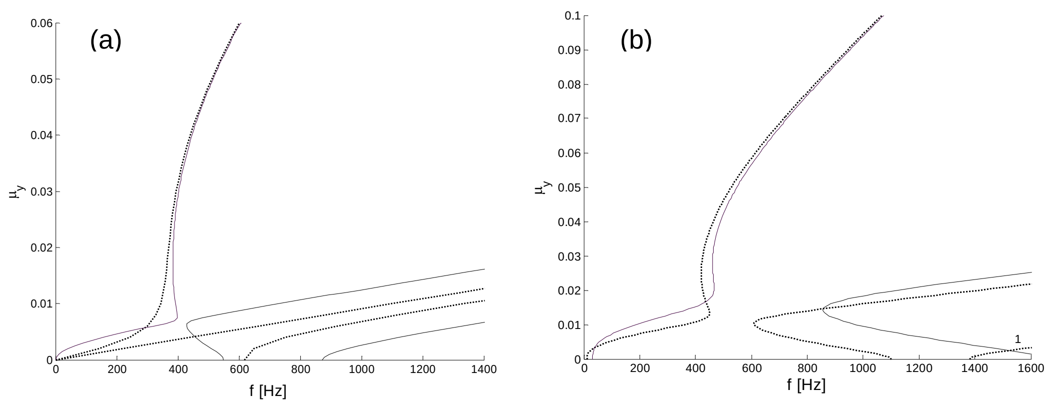

Figure 6.- Dispersion curves for sandwich shell: (a) …. $n=0$, ____ $n=1$; (b) ….. $n=2$, ___ $n=3$.

alone. It can be seen in Figure 6(a) that, for the n = 1 branches 1 and 2, there is more than one possible value of μ_y for a given frequency. For example, there are three different values of μ_y for the n = 1 branch 1 when 384Hz < f < 399Hz. The lower and the higher values correspond to waves with positive group velocities in the y direction while the middle value corresponds to a wave which has a negative group velocity in the y direction, but a positive phase velocity. For the n = 1 branch 2, in the frequency range 428Hz < f < 550Hz, the two values of μ_y correspond to two waves travelling in opposite directions along the shell. In particular, the wave associated with the lower value of μ_y has a negative group velocity in the y direction and a positive phase velocity. Similar results can be seen for higher circumferential mode numbers. Figure 7 shows the dispersion curves in the (μ_x, μ_y) plane for different values of frequency. The frequencies in Figure 7(a) are below the ring frequency, while that in Figure 7(b) is above the ring frequency. It can be seen that there exist regions in which different values of μ_x correspond to the same value of μ_y and different values of μ_y correspond to the same value of μ_x. Considerations about the energy flow can be made for every one of these points. These dispersion curves are very similar to the results obtained by Heron [8].

CONCLUSIONS

A numerical wave/finite element (WFE) method for the analysis of wave propagation in axisymmetric structures was described. Examples of various *in-vacuo* and fluid-filled cylinders were presented. A rectangular segment with 4 nodes is modelled using conventional FEA, typically using a commercial package. The resulting system matrices are post-processed using periodicity conditions to yield the dispersion relations. The computational cost is extremely small and existing FE packages and their extensive element libraries can be exploited. The method typically provides accurate predictions when the size of the FE is less than about 1/6 of the wavelength.

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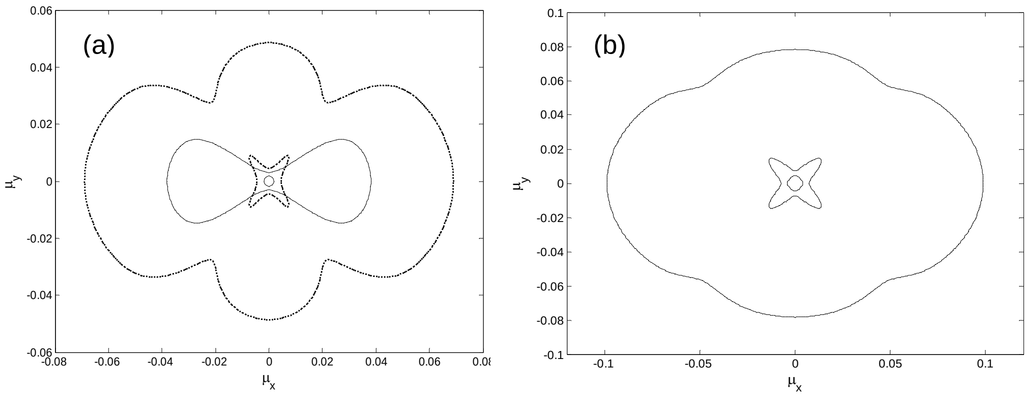

Figure 7.- Dispersion contours for steel shell: (a) ___200 Hz, ….. 500 Hz; (b) 800 Hz.