Editors' Suggestion

First Law of Binary Black Hole Scattering

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In the last decade, the first law of binary black hole mechanics played an important unifying role in the gravitational two-body problem. More recently, binary black hole scattering and the application of high-energy physics methods have provided a new avenue into this classical problem. In this Letter, we connect these two themes by extending the first law to the case of scattering orbits. We present derivations based on classical S-matrix, Hamiltonian, and pseudo-Hamiltonian methods, the last of which allows us to include dissipative effects for the first time. Finally, a "boundary to bound" map links this first law to the traditional bound-orbit version. Through this map a little-known observable for scatter orbits, the elapsed proper time, is mapped to the Detweiler redshift for bound orbits, which is an invariant building block in gravitational waveform models.

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Introduction—The discovery of gravitational waves (GWs) from compact binary systems opened a new chapter in astronomy. Given the enhanced sensitivity and expanded frequency range of future GW detectors, we expect a dramatic increase in the number and variety of detectable compact binary sources [1–5]. Increasingly accurate waveform models will be needed to detect and analyze these sources [6,7], calling for the development of new tools to study the classical two-body problem.

Motivated by GW modeling, a host of techniques have been developed to solve the two-body problem in general relativity, including numerical relativity, which numerically solves the fully nonlinear Einstein equations [8]; gravitational self-force (GSF) theory, a perturbative method that applies when one body is much smaller than the other [9]; and post-Newtonian (PN) and post-Minkowskian (PM) theory, weak-field expansions that apply when the two bodies are widely separated [10,11]. Historically, focus has been on the bound, inspiraling systems that are the dominant sources for GW detectors. However, the case of hyperbolic, scattering encounters is now of great interest: it is now known that data for scattering orbits can inform bound-orbit models using the effective one-body framework [11–17] or through an analytic continuation from scattering to bound observables [18–23], spurring the development of new particle physics tools [24–28] that have enabled analytical computations of the two-body scattering Hamiltonian and related observables at high PM order [29–50].

In the bound case, synergies between different methods have consistently helped drive progress [51,52]. An important tool in those synergies has been the first law of binary black hole (BH) mechanics [53–60], which describes how a binary system responds to variations of its parameters (see also Ref. [61]). This law has played an important role in the most accurate GSF waveform model [62,63] and in utilizing GSF results within PN, effective one-body, and numerical relativity calculations [64,65]; see Ref. [66] for a review. For spinless particles, the binary's response to variations is determined by a basis of observables $\mathcal{B}^{<}$ consisting of the periastron advance $\Delta\Phi$, the radial frequency Ω_r, and the averaged Detweiler redshift $\langle z\rangle$ [53,57].

To date, a first law for scattering scenarios has not been derived. In this Letter, we establish such a law and find the corresponding basis of scattering observables $\mathcal{B}^{>}$. Two of these observables are well studied: the deflection angle χ and the time delay. We complete the basis with a third observable: the elapsed proper time $\Delta\tau$. Our approach is based on a pseudo-Hamiltonian formulation of GSF theory [58,67–69]. This allows us to include dissipative contributions in the first law, unlike all previous formulations for bound orbits. By comparing to bound-orbit formulations,

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we also establish a novel analytic continuation between the elements of the scattering $\mathcal{B}^{>}$ and bound $\mathcal{B}^{<}$ bases of observables.

Finally, we link our calculations to high-energy physics methods, proving that the exponential representation of the classical S-matrix [37,38] provides a generating functional for the basis of scattering observables and deriving a first law from a PM Hamiltonian.

Conventions: We use geometric units with $G = c = 1$ and the $(-+++)$ metric signature.

First law in the probe limit—In the GSF approach, the smaller body (of mass m_1) is treated as a point particle perturbing the spacetime of the large body, which we take to be a Schwarzschild BH of mass m_2. We first consider the probe limit, also called zeroth self-force (0SF) order, in which the particle moves on a geodesic of the Schwarzschild metric $g^{\rm Schw}_{\alpha\beta}$. The particle's motion is governed by the geodesic Hamiltonian $H_0 = (1/2)g^{\mu\nu}_{\rm Schw}p_\mu p_\nu$, where $p_\alpha = m_1 g^{\rm Schw}_{\alpha\beta}{\rm d}x^\beta/{\rm d}\tau$ is the particle's four-momentum and τ is its proper time. Assuming, without loss of generality, that the motion lies on the equatorial plane $\theta = \pi/2$, we label the position of the particle with $x^\alpha(\tau) = (t(\tau), r(\tau), \pi/2, \varphi(\tau))$. Because of Schwarzschild's Killing symmetries, the particle's energy and angular momentum $E = -p_{t,0}$ and $L = p_{\phi,0}$ are conserved (here and below, a subscript 0 indicates the on-shell geodesic value). We now consider unbound geodesic orbits that begin and end at $r = \infty$; such orbits have $E > m_1$ and $L > L_{\rm crit}(E)$, where $L_{\rm crit}(E)$ is a critical value of the angular momentum [70].

Following Carter's application of Hamilton-Jacobi theory [71], we use the constants of motion $P_i = (m_1, E, L)$ as canonical momenta and transform to canonical coordinates (X^i, P_i) using the type-2 generating function

$$W(t, r, \varphi; P_i) = -Et + L\varphi + I_{r,0}(r; P_i),$$
$$I_{r,0}(r; P_i) = \int_{r_{\rm m}}^{r} {\rm d}r\, p_{r,0}(r; P_i), \tag{1}$$

where $r_{\rm m}(P_i)$ is the geodesic's minimum radius (i.e., closest approach to the BH). $g^{\mu\nu}_{\rm Schw}p_{\mu,0}p_{\nu,0} = -m_1^2$ implies

$$p_{r,0}(r; P_i) = \frac{\sqrt{E^2 r^4 - r(r-2m_2)(L^2 + m_1^2 r^2)}}{r(r-2m_2)}. \tag{2}$$

In the coordinates (X^i, P_i), where $X^i = \partial W/\partial P_i$, the Hamiltonian is simply its on-shell value, $H_0 = -m_1^2/2$, meaning that Hamilton's equations become [72]

$$m_1 \frac{{\rm d}X^i}{{\rm d}\tau} = \frac{\partial H_0}{\partial P_i} = -m_1 \delta^i_1. \tag{3}$$

Therefore X^2 and X^3 are constants, while X^1 is linear in τ. Since $X^i = \partial W/\partial P_i$, this implies

$$\left.\frac{\partial W}{\partial E}\right|_{\rm out} = \left.\frac{\partial W}{\partial E}\right|_{\rm in}, \qquad \left.\frac{\partial W}{\partial L}\right|_{\rm out} = \left.\frac{\partial W}{\partial L}\right|_{\rm in},$$
$$\tau_{\rm out} - \tau_{\rm in} = -\left[\left.\frac{\partial W}{\partial m_1}\right|_{\rm out} - \left.\frac{\partial W}{\partial m_1}\right|_{\rm in}\right], \tag{4}$$

where "in" and "out" denote the initial, incoming state and final, outgoing state.

Equations (1) and (4) imply that the total changes in coordinate time, azimuthal angle, and proper time between initial and final states are

$$\Delta t_0 = t_{\rm out} - t_{\rm in} = \frac{\partial}{\partial E}[I_{r,0}(r_{\rm out}; P_i) - I_{r,0}(r_{\rm in}; P_i)],$$
$$\Delta\varphi_0 = \varphi_{\rm out} - \varphi_{\rm in} = -\frac{\partial}{\partial L}[I_{r,0}(r_{\rm out}; P_i) - I_{r,0}(r_{\rm in}; P_i)],$$
$$\Delta\tau_0 = \tau_{\rm out} - \tau_{\rm in} = -\frac{\partial}{\partial m_1}[I_{r,0}(r_{\rm out}; P_i) - I_{r,0}(r_{\rm in}; P_i)]. \tag{5}$$

We are interested in the limit where the initial and final states are defined at past and future timelike infinity, with $r_{\rm in} = \infty = r_{\rm out}$, passing through the single radial turning point $r_{\rm m}$. In this limit, $\Delta\varphi_0$ remains finite, but $I_{r,0}(r_{\rm in/out}; P_i)$, Δt_0, and $\Delta\tau_0$ all diverge. However, we can define regularized versions. Using a convenient dimensionless regulator ϵ [22], we first define the scattering radial action

$$I^{>,\epsilon}_{r,0}(P_i) = 2\int_{r_{\rm m}}^{+\infty} {\rm d}r\, r^\epsilon\, p_{r,0}(r; P_i). \tag{6}$$

Intermediate results depend on the finite value of ϵ, but we obtain ϵ-independent observables in the limit $\epsilon \to 0$; when necessary, functions are first defined in regions of the complex-ϵ plane where integrals converge [e.g., ${\rm Re}(\epsilon) < -1$ in Eq. (6)] and are then analytically continued to $\epsilon = 0$ [73]. In terms of $I^{>,\epsilon}_{r,0}$ we can write the regularized $r_{\rm in/out} \to \infty$ limit of Eq. (5) for the full scattering path as

$$\Delta\varphi^\epsilon_0 = \pi + \chi^\epsilon_0 = -\frac{\partial}{\partial L}I^{>,\epsilon}_{r,0}(P_i), \tag{7}$$

where $\chi_0 = \lim_{\epsilon\to 0}\chi^\epsilon_0$ is the physical scattering angle, and

$$\Delta t^\epsilon_0 = \frac{\partial}{\partial E}I^{>,\epsilon}_{r,0}(P_i), \qquad \Delta\tau^\epsilon_0 = -\frac{\partial}{\partial m_1}I^{>,\epsilon}_{r,0}(P_i). \tag{8}$$

Unlike χ^ϵ_0, the elapsed times Δt^ϵ_0 and $\Delta\tau^\epsilon_0$ diverge as $\epsilon \to 0$. The associated physical observables, which are well defined when $\epsilon \to 0$, are *relative* measurements—the difference between $\Delta t_0(P_i)$ along the geodesic and $\Delta t_0(P_{i,\rm ref})$ along some reference orbit—and these relative quantities will take the same values as if we had worked consistently with finite $r_{\rm in/out}$ and only taken the limit $r_{\rm in/out} \to \infty$ at the end of the calculation. In Supplemental Material (SM) [75], we provide exact expressions for the geodesic scattering observables as well as the first few terms in their PM expansions (corresponding to $m_1 m_2/L \ll 1$).

Finally, Eqs. (7) and (8) can be immediately combined into a single equation,

$$\delta I_{r,0}^{>,\epsilon} = -(\pi + \chi_0)\delta L + \Delta t_0^\epsilon \delta E - \Delta\tau_0^\epsilon \delta m_1. \tag{9}$$

This is our first law for scattering geodesics. Here and below, we discard terms that vanish at $\epsilon = 0$, and equalities should be understood in this sense.

First law at all SF orders—Beyond leading order in the mass ratio m_1/m_2, the particle generates a metric perturbation $h_{\alpha\beta}$ on the Schwarzschild background. m_1 then moves on a geodesic of a certain *effective* metric $\tilde{g}_{\alpha\beta} = g_{\alpha\beta}^{\rm Schw} + h_{\alpha\beta}^{\rm R}$ [76], where $h_{\alpha\beta}^{\rm R}$ is a certain regular piece of $h_{\alpha\beta}$. At 1SF order, we can write $h_{\alpha\beta}^{\rm R}$ in terms of the Detweiler-Whiting Green's function $G_{\rm R}^{\alpha\beta\alpha'\beta'}$ [77],

$$h_{\rm R}^{\alpha\beta}(x^\mu;\Gamma) = \frac{1}{m_1}\int_\Gamma G_{\rm R}^{\alpha\beta\alpha'\beta'}(x^\mu, x'^\mu(\tilde{\tau}'))\tilde{p}_{\alpha'}\tilde{p}_{\beta'}{\rm d}\tilde{\tau}'. \tag{10}$$

Here, $\tilde{p}_\alpha := \tilde{g}_{\alpha\beta}{\rm d}x^\beta/{\rm d}\tilde{\tau}$, $\tilde{\tau}$ is proper time in $\tilde{g}_{\alpha\beta}$, and Γ denotes the particle's phase-space trajectory. Because of curvature-induced tail effects, $G_{\rm R}^{\alpha\beta\alpha'\beta'}$ is nonzero for all points x'^μ in the past of x^μ, implying $h_{\rm R}^{\alpha\beta}$ at a point on Γ depends on the entire prior history of Γ. At higher SF orders, there is no known Green's-function form analogous to (10), but an appropriate $h_{\alpha\beta}^{\rm R}(x^\mu;\Gamma)$ exists at all SF orders [78].

In this setting, we again consider scattering orbits with initial parameters $P_i = (m_1, E, L)$. The particle's energy and angular momentum evolve due to dissipation, but the orbit remains planar ($\theta = \pi/2$, $\tilde{p}_\theta = 0$). For L above a critical threshold, the orbit remains close to a Schwarzschild geodesic with the same initial P_i [70,79].

Since the motion is geodesic in $\tilde{g}_{\mu\nu}$, it obeys Hamilton's equations with the test-mass pseudo-Hamiltonian $H = (1/2)\tilde{g}^{\mu\nu}\tilde{p}_\mu\tilde{p}_\nu$ [58,67,68]. However, we deviate from [58,67] by restricting to the 4D phase space $(x^A, \tilde{p}_A)$ satisfying the on-shell condition $H = -m_1^2/2$, with $x^A = (r, \varphi)$. Solving the on-shell condition for $\tilde{p}_t = -\mathcal{H}(t, x^A, \tilde{p}_A;\Gamma)$ gives the new pseudo-Hamiltonian

$$\begin{aligned}\mathcal{H} &= \frac{1}{\tilde{g}^{tt}}[\tilde{g}^{tA}\tilde{p}_A - \sqrt{(\tilde{g}^{tA}\tilde{p}_A)^2 - \tilde{g}^{tt}(\tilde{g}^{AB}\tilde{p}_A\tilde{p}_B + m_1^2)}] \\ &= -p_t(r,\tilde{p}_A) - \frac{h_{\rm R}^{\mu\nu}(t, x^A;\Gamma)\tilde{p}_\mu\tilde{p}_\nu}{2g_{\rm Schw}^{tt}p_t} + \mathcal{O}\left(\frac{m_1^2}{m_2^2}\right),\end{aligned} \tag{11}$$

with t now the parameter along the trajectory. $\mathcal{H}$ is referred to as a pseudo-Hamiltonian because it depends on the trajectory $\Gamma = \{(x^A(t), \tilde{p}_A(t))|t\in\mathbb{R}\}$. Hamilton's equations in this context read

$$\frac{{\rm d}x^A}{{\rm d}t} = \left[\frac{\partial\mathcal{H}}{\partial\tilde{p}_A}\right] \quad \text{and} \quad \frac{{\rm d}\tilde{p}_A}{{\rm d}t} = -\left[\frac{\partial\mathcal{H}}{\partial x^A}\right], \tag{12}$$

where $[\cdot]$ indicates specification of Γ as the self-consistent trajectory [80] passing through $(x^A, \tilde{p}_A)$; prior to that specification, Γ is treated as independent, and derivatives do not act on it. We go further by replacing m_1 by m_1' in Eq. (10), setting $m_1' = m_1$ only when $[\cdot]$ is applied. Importantly, Eq. (12) captures the full dynamics, including dissipation, unlike an ordinary Hamiltonian description.

We derive the first law from $\mathcal{H}$. Doing so will require its relationship to the redshift z,

$$z := \frac{{\rm d}\tilde{\tau}}{{\rm d}t} = \left[\frac{\partial\mathcal{H}}{\partial m_1}\right]. \tag{13}$$

To establish this relationship, we consider the normalization condition

$$-m_1 = \tilde{p}_\mu\frac{{\rm d}x^\mu}{{\rm d}\tilde{\tau}} = \left(-\mathcal{H} + \tilde{p}_A\left[\frac{\partial\mathcal{H}}{\partial\tilde{p}_A}\right]\right)z^{-1}, \tag{14}$$

where we used $\tilde{p}_t = -\mathcal{H}$ together with (12). Next we note the first line of (11) shows that, at fixed m_1', $\mathcal{H}$ is a homogeneous function of $(m_1, \tilde{p}_A)$ of order 1. Euler's homogeneous function theorem hence implies $\mathcal{H} = m_1(\partial\mathcal{H}/\partial m_1) + \tilde{p}_A(\partial\mathcal{H}/\partial\tilde{p}_A)$. Comparing this with (14), we obtain (13).

Now, to derive the first law, we loosely follow [57] by considering how $\mathcal{H}$ changes under variations δP_i of the initial data, with $\delta x_{\rm in}^A = 0$. Given (12) and (13), we find

$$\begin{aligned}[\delta\mathcal{H}] &= \left[\frac{\partial\mathcal{H}}{\partial x^A}\delta x^A\right] + \left[\frac{\partial\mathcal{H}}{\partial\tilde{p}_A}\delta\tilde{p}_A\right] + \left[\frac{\partial\mathcal{H}}{\partial m_1}\delta m_1\right] \\ &= -\frac{{\rm d}\tilde{p}_A}{{\rm d}t}\delta x^A + \frac{{\rm d}x^A}{{\rm d}t}\delta\tilde{p}_A + \frac{{\rm d}\tilde{\tau}}{{\rm d}t}\delta m_1.\end{aligned} \tag{15}$$

Since $h_{\mu\nu}^{\rm R}$ vanishes for an inertial particle in Minkowski [76], its contribution to $\tilde{p}_A$ and $\mathcal{H}$ vanishes in the initial state, such that $\tilde{p}_\varphi^{\rm in} = p_{\varphi,0}^{\rm in} = L$ and $\mathcal{H}_{\rm in} = -p_{t,0}^{\rm in} = E$. We isolate δL and δE in (15) by defining "interaction" quantities $\tilde{\wp}_\varphi := \tilde{p}_\varphi - L$, $\tilde{\wp}_r = \tilde{p}_r$, and $\mathscr{H} := \mathcal{H} - E$ such that

$$\delta E + [\delta\mathscr{H}] = -\frac{{\rm d}\tilde{\wp}_A}{{\rm d}t}\delta x^A + \frac{{\rm d}x^A}{{\rm d}t}\delta\tilde{\wp}_A + \frac{{\rm d}\varphi}{{\rm d}t}\delta L + \frac{{\rm d}\tilde{\tau}}{{\rm d}t}\delta m_1. \tag{16}$$

Next, we integrate (16) along the physical scattering trajectory from $t = -\infty$ to $t = +\infty$, introducing the regularized integral $\langle f\rangle_\Gamma := \int_\Gamma {\rm d}t\, r^\epsilon f$ as in the 0SF case. Integrating the term $({\rm d}\tilde{\wp}_A/{\rm d}t)\delta x^A$ by parts, we obtain

$$\Delta t^\epsilon \delta E + \langle[\delta\mathscr{H}]\rangle_\Gamma = \delta\left\langle \tilde{\mathscr{p}}_A \frac{\mathrm{d}x^A}{\mathrm{d}t}\right\rangle_\Gamma + \Delta\varphi^\epsilon \delta L + \Delta\tilde{\tau}^\epsilon \delta m_1, \tag{17}$$

where $\Delta t^\epsilon \coloneqq \langle 1\rangle_\Gamma$, $\Delta\varphi^\epsilon \coloneqq \langle \mathrm{d}\varphi/\mathrm{d}t\rangle_\Gamma$, and $\Delta\tilde{\tau}^\epsilon \coloneqq \langle \mathrm{d}\tilde{\tau}/\mathrm{d}t\rangle_\Gamma$. We have discarded boundary terms by choosing $\mathrm{Re}(\epsilon)$ sufficiently negative and discarded terms that arise from derivatives acting on r^ϵ because they vanish when analytically continued to $\epsilon = 0$. Defining also the regularized interaction action,

$$I^{>,\epsilon} \coloneqq \int_\Gamma \mathrm{d}t\ r^\epsilon\ \tilde{\mathscr{p}}_A \frac{\mathrm{d}x^A}{\mathrm{d}t} = I_r^{>,\epsilon} + \int_\Gamma \mathrm{d}\varphi\ \tilde{\mathscr{p}}_\varphi, \tag{18}$$

with $I_r^{>,\epsilon} \coloneqq \int_\Gamma dr\ r^\epsilon\ \tilde{p}_r$, we rewrite (17) as

$$\Delta t^\epsilon \delta E + \langle[\delta\mathscr{H}]\rangle_\Gamma = \delta I^{>,\epsilon} + \Delta\varphi^\epsilon \delta L + \Delta\tilde{\tau}^\epsilon \delta m_1. \tag{19}$$

We can rewrite Eq. (19) in an alternative form by absorbing $\langle[\delta\mathscr{H}]\rangle_\Gamma$ into "renormalized" variables $\{I_{\mathrm{ren}}^{>,\epsilon}, E_{\mathrm{ren}}, L_{\mathrm{ren}}\}$. Following [58]'s treatment of the bound case, we define renormalized variables,

$$E_{\mathrm{ren}} = \lambda E, \qquad L_{\mathrm{ren}} = \lambda L, \qquad I_{\mathrm{ren}}^{>,\epsilon} = \lambda I^{>,\epsilon}. \tag{20}$$

Choosing $\lambda(P_i)$ appropriately to eliminate $\langle[\delta\mathscr{H}]\rangle$ from Eq. (19), we are left with

$$\delta I_{\mathrm{ren}}^{>,\epsilon} = -(\pi + \chi^\epsilon)\delta L_{\mathrm{ren}} + \Delta t^\epsilon \delta E_{\mathrm{ren}} - \Delta\tilde{\tau}^\epsilon \delta m_1; \tag{21}$$

see SM [75] for more details.

Equation (21) is the first law for scattering orbits, valid at all SF orders and including all dissipative effects. To help understand the renormalization of the variables, we observe that the first law defines a sense of conjugacy between variables and observables, just as in the first law of thermodynamics. In that sense, the renormalized variables are the ones conjugate to the physical observables. In Ref. [81], we show that in the conservative sector, this sense of conjugacy reduces to the usual sense in Hamiltonian mechanics: the renormalized variables are the true, invariant action variables that are canonically conjugate to the system's action angles. The need for this renormalization stems from the fact that, as highlighted in Ref. [68], if a system can be equivalently described by both a pseudo-Hamiltonian and a Hamiltonian, then variables that are conjugate in one description are not generally conjugate in the other. As a consequence, the momenta $\tilde{p}_\mu$, from which E, L, and $I^{>,\epsilon}$ are built, are *not* the canonical momenta in a Hamiltonian description of the conservative sector (i.e., they are not the momenta one would define from a Lagrangian for the conservative sector). We refer to Refs. [69,81] for details.

From scattering to bound—There is a well-known analytic continuation between the deflection angle χ for unbound orbits and the periastron advance $\Delta\Phi$ for bound orbits, as well as between the scattering and bound radial actions [18,19,22]. Here, using the first laws for unbound and bound motion, we extend these analytic continuations to include all the observables in the scattering and bound bases, $\mathcal{B}^> = (\chi, \Delta t^\epsilon, \Delta\tau^\epsilon)$ and $\mathcal{B}^< = (\Delta\Phi, \Omega_r, \langle z\rangle)$. We limit our analysis to 0SF order, as the analytic continuation for the radial action is not known to be valid beyond 0SF order due to nonlocal-in-time tail effects [21,43].

We write the first law for bound geodesics in terms of the bound radial action,

$$I_{r,0}^<(P_i) = 2\int_{r_-(P_i)}^{r_+(P_i)} \mathrm{d}r\, p_{r,0}(r; P_i), \tag{22}$$

where $r_\mp$ are the orbit's minimum and maximum radius (i.e., the radii at periapsis and apoapsis). Following the same arguments as for unbound orbits, one can write the accumulated φ, t, and τ over a single radial period ($T_{r,0} = 2\pi/\Omega_{r,0}$) as derivatives of $I_{r,0}^<$, leading to the first law for bound orbits [56,57],

$$\delta I_{r,0}^< = -(2\pi + \Delta\Phi_0)\delta L + \frac{2\pi}{\Omega_{r,0}}\delta E - \frac{2\pi}{\Omega_{r,0}}\langle z\rangle_0 \delta m_1, \tag{23}$$

where $\langle z\rangle_0 \coloneqq (1/T_{r,0})\int_0^{T_{r,0}} \mathrm{d}t(\mathrm{d}\tau/\mathrm{d}t)_0$.

Knowing the scatter-to-bound map for the radial action [19,22,39],

$$I_{r,0}^<(P_i) = \lim_{\epsilon\to 0}[I_{r,0}^{>,\epsilon}(E, L, m_1) - I_{r,0}^{>,\epsilon}(E, -L, m_1)], \tag{24}$$

and comparing Eq. (23) to Eq. (9), we immediately conclude that there is an analytic continuation between the full set of scattering and bound observables,

$$\begin{aligned} \Delta\Phi_0 &= \chi_0(E, L, m_1) + \chi_0(E, -L, m_1), \\ \frac{2\pi}{\Omega_{r,0}} &= \lim_{\epsilon\to 0}[\Delta t_0^\epsilon(E, L, m_1) - \Delta t_0^\epsilon(E, -L, m_1)], \\ \frac{2\pi\langle z\rangle_0}{\Omega_{r,0}} &= \lim_{\epsilon\to 0}[\Delta\tau_0^\epsilon(E, L, m_1) - \Delta\tau_0^\epsilon(E, -L, m_1)]. \end{aligned} \tag{25}$$

We note that the infrared divergences in Δt_0^ϵ and $\Delta\tau_0^\epsilon$ cancel in these expressions because the divergences are independent of L; see SM [75].

First law from the S-matrix—In this section, we put our first law in the broader context of the quantum S-matrix description of the classical two-body problem [82]. Given the two-body initial state of well-separated massive point particles $|\Psi_{\mathrm{in}}\rangle = |p_1 p_2\rangle$ of mass m_1 and m_2, the action of the unitary S-matrix operator,

$$\hat{S} = \mathcal{T}\exp\left(-\frac{i}{\hbar}\int \mathrm{d}t\, H_{\mathrm{int}}(t)\right), \tag{26}$$

describes the time evolution of the state in terms of the interaction Hamiltonian H_{int}, with $\mathcal{T}$ denoting time ordering. The classical two-body scattering dynamics in the asymptotic $\hbar \to 0$ limit is equivalently obtained by evaluating the action, and therefore H_{int}, on-shell.

Motivated by (26), we define the exponential representation $\hat{S} = \exp(i\hat{N}/\hbar)$ [37,38], where $\hat{N}$ is a Hermitian operator. We then study the *real-valued* two-body matrix element,

$$N(\mathbb{E}, q, m_1, m_2) \coloneqq \langle p_1' p_2' | \hat{N} | p_1 p_2 \rangle, \tag{27}$$

where we defined $|\Psi_{\mathrm{out}}\rangle = |p_1' p_2'\rangle$, the initial *total* energy $\mathbb{E}$ of both particles, and the exchanged momentum $q^\mu = p_1'^\mu - p_1{}^\mu = p_2^\mu - p_2'^\mu$. To make contact with the incoming total angular momentum $\mathbb{L} = (m_1 m_2 \sqrt{\gamma^2 - 1} b)/\mathbb{E}$, where $\gamma \coloneqq (p_1^\mu + p_1'^\mu/2)(p_{2\mu} + p_{2\mu}'/2)$ and b is the impact parameter in the center of mass (c.m.) frame, we perform the Fourier transform

$$\begin{aligned} &N^{>,\epsilon}(\mathbb{E}, \mathbb{L}, \{m_a\}) \\ &= \frac{1}{4 m_1 m_2 \sqrt{\gamma^2 - 1}} \int \frac{\mathrm{d}^{2+2\epsilon} q}{(2\pi)^{2+2\epsilon}} e^{-i\frac{b(\mathbb{L})\cdot q}{\hbar}} N(\mathbb{E}, q, \{m_a\}), \end{aligned} \tag{28}$$

using dimensional regularization with $d = 4 + 2\epsilon$. Infrared, $1/\epsilon$ divergences arise due to the long-range nature of gravity, but their analytic structure is understood [83].

In complete generality, the expectation value (27) is a function of the kinematic data $(\mathbb{E}, \mathbb{L}, \{m_a\})$, and its variation in the phase space is

$$\delta N^{>,\epsilon} = c_{\mathbb{L}}^\epsilon \delta\mathbb{L} + c_{\mathbb{E}}^\epsilon \delta\mathbb{E} + \sum_{a=1,2} c_{m_a}^\epsilon \delta m_a, \tag{29}$$

where $c_{\mathbb{L}}^\epsilon$, $c_{\mathbb{E}}^\epsilon$, and $c_{m_a}^\epsilon$ are gauge-invariant coefficients.

Using insights from the PM Hamiltonian description [30] and the relation with the radial action [38], we now identify the coefficients with the observables $\mathcal{B}^>$. First, by matching the scattering angle χ in the c.m. frame, it was shown that the matrix element (27) in the conservative case agrees with the radial action up to a constant proportional to $\mathbb{L}$ [38],

$$N^{>,\epsilon}|_{\mathrm{cons}} = \underbrace{\int_{\mathcal{C}_r^{>,\epsilon}} \mathrm{d}r \tilde{p}_{r,\mathrm{c.m.}}(r; \mathbb{E}, \mathbb{L}, \{m_a\})}_{\mathbb{I}_r^{>,\epsilon}} + \pi\mathbb{L}, \tag{30}$$

where $\tilde{p}_{r,\mathrm{c.m.}}$ is the radial relative momentum in the c.m. frame and $\mathcal{C}_r^{>,\epsilon}$ is the contour of integration for scattering orbits, which implicitly includes a regulator ϵ inherited from the dimensional regularization. In SM [75], using the PM conservative Hamiltonian and its symmetries in the c.m. frame [19,30], we then provide a proof of the following conservative first law for the two-body scattering problem:

$$\delta N^{>,\epsilon} = -\chi\delta\mathbb{L} + \Delta t^\epsilon \delta\mathbb{E} - \sum_{a=1,2} \Delta\tau_a^\epsilon \delta m_a, \tag{31}$$

where $\Delta\tau_a^\epsilon$ is the elapsed proper time of particle a and Δt^ϵ is the elapsed global time.

If we appeal to $N^{>,\epsilon} = \mathbb{I}_r^{>,\epsilon} + \pi\mathbb{L}$ and restrict to variations with $\delta m_2 = 0$ [84], then we see Eq. (31) is structurally identical to our previous first law (21). However, the quantities in these laws might differ. Even in the conservative sector, the two-body incoming energy $\mathbb{E}$ and angular momentum $\mathbb{L}$ might not agree with the renormalized one-body, SF counterparts E_{ren} and L_{ren}. Moreover, while $N^{>,\epsilon}$ is computed here in the c.m. frame, GSF calculations might be in the initial rest frame of the heavy BH [87–89] or in any "nearby" frame (including the c.m. frame); the frame of a GSF calculation is implicitly determined by the choice of gauge for the metric perturbations. However, at 0SF order, we can trivially identify $\mathbb{E}$ with $E + m_2$, $\mathbb{L}$ with L, and the c.m. frame with the heavy BH frame, as the relative radial momentum $\tilde{p}_{r,\mathrm{c.m.}}$ coincides with the geodesic one $p_{r,0}$. Then (31) identically matches (9), and we have $\chi^\epsilon \to \chi_0^\epsilon$, $\Delta t^\epsilon \to \Delta t_0^\epsilon$, and $\Delta\tau_1^\epsilon \to \Delta\tau_0^\epsilon$. At nSF order, the matching with our SF first law is more challenging as the dynamics of the heavy BH (as well as the choice of frame) is encoded in a nontrivial way in the metric perturbations [90,91]. We leave study of this to future work.

Incorporating dissipation in this framework is possible by combining the in-in expectation value [82] with the exponential representation of $\hat{S}$ [38]: for every observable $\mathcal{O}$ [92],

$$\begin{aligned} \langle \Delta\mathcal{O} \rangle &= \langle \Psi_{\mathrm{in}} | \hat{S}^\dagger \mathcal{O} \hat{S} | \Psi_{\mathrm{in}} \rangle - \langle \Psi_{\mathrm{in}} | \mathcal{O} | \Psi_{\mathrm{in}} \rangle \\ &= \sum_{j=1}^{+\infty} \frac{(-i)^j}{\hbar^j j!} \underbrace{\{\hat{N}, [\hat{N}, \ldots, (\hat{N}, \mathcal{O}) \ldots]\}}_{j\text{times}}, \end{aligned} \tag{32}$$

where now also the $\hat{N}$ matrix elements with on-shell gravitons are relevant. This suggests a physical principle to connect the coefficients $(c_{\mathbb{L}}^\epsilon, c_{\mathbb{E}}^\epsilon, c_{m_a}^\epsilon)$ to observables at all orders, with dissipation included; see for example Eq. (3.48) of [38].

Conclusion—In recent years, the study of unbound orbits through the S-matrix formalism has transformed the gravitational two-body problem. In this Letter, we developed a powerful new tool for such studies: an extension of the first law of binary BH mechanics to the unbound (and dissipative) case. Our derivation at all SF orders utilizes a novel version of the pseudo-Hamiltonian formalism [58,67,69]. More generally, we showed how the first law can be derived from the variation of the classical S-matrix

$$\delta N^{>,\epsilon}(\mathbb{E},\mathbb{L},\{m_a\}) = -\chi\delta\mathbb{L} + \Delta t^{\epsilon}\delta\mathbb{E} - \sum_{a=1,2}\Delta\tau_a^{\epsilon}\delta m_a$$

$$\Big\downarrow\ \delta N^{<}(\mathbb{E},\mathbb{L},\{m_a\}) = \delta N^{>,\epsilon}(\mathbb{E},\mathbb{L},\{m_a\}) - \delta N^{>,\epsilon}(\mathbb{E},-\mathbb{L},\{m_a\})$$

$$\delta N^{<}(\mathbb{E},\mathbb{L},\{m_a\}) = -\Delta\Phi\delta\mathbb{L} + \frac{2\pi}{\Omega_r}\delta\mathbb{E} - \sum_{a=1,2}\frac{2\pi}{\Omega_r}\langle z_a\rangle\delta m_a$$

FIG. 1. Relation between the basis of scattering observables (coordinate time difference Δt^{ϵ}, elapsed proper time $\Delta\tau^{\epsilon}$, and deflection angle χ) and the corresponding bound-orbit ones (radial frequency Ω_r, averaged redshift $\langle z\rangle$, and periastron advance $\Delta\Phi$) for the classical gravitational two-body problem.

in the phase space of the kinematic data $P_i = (\mathcal{E}, L, \{m_a\})$; see Fig. 1. In that context, the S-matrix can be interpreted as a generating functional of classical observables. Among those observables, we have highlighted the elapsed proper time $\Delta\tau^{\epsilon}$ as a new, core element of the (regularized) basis of scattering observables $\mathcal{B}^{>} = (\chi, \Delta t^{\epsilon}, \Delta\tau^{\epsilon})$.

Using the relation between the scattering and bound radial action, we also established a full correspondence (25) between the bases of scattering observables $\mathcal{B}^{>}$ and bound observables $\mathcal{B}^{<}$ at 0SF order, as again summarized in Fig. 1. This extends the well-known map between the deflection angle and periastron advance.

Given the first law's varied applications for bound orbits [62,63,85,95–115], we expect our work to open many new avenues for scattering calculations. We particularly encourage self-force scattering calculations [70,79,116–118] of the observables Δt^{ϵ} and $\Delta\tilde{\tau}^{\epsilon}$. For bound orbits, the 1SF conservative Hamiltonian can be calculated directly from the averaged redshift $\langle z\rangle$, and postadiabatic waveform models [62,119,120] can be written in a gauge-invariant form with $\langle z\rangle$ as an invariant building block [81,121]. This implies that if the analytic continuation between $\Delta\tilde{\tau}^{\epsilon}$ and $\langle z\rangle$ can be extended to 1SF order, then scattering calculations of $\Delta\tilde{\tau}^{\epsilon}$ can provide direct inputs to bound-orbit self-force waveform models.

Natural extensions also present themselves. First, one might consider scattering orbits of spinning BHs [55,60,93]. Second, we considered only two-body matrix elements, but nothing prevents us from studying the variation of matrix elements involving on-shell graviton states, which should be related to the gravitational waveform. Finally, we hope that linking the first laws for scattering and bound orbits beyond 0SF can shed light on the tail effects that have limited the applicability of scatter-to-bound maps [43,49].

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