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An Epistemic Perspective on Subjective and Objective Time

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Abstract
The article introduces two modalities representing knowledge about the subjective and objective current moment of time. It provides formal semantics of these modalities, shows that the modalities are not definable through each other, and gives a sound and complete axiomatisation of their interplay. The axiomatisation contains an unusual Insertion inference rule generalising the Necessitation rule. The article proves that the Insertion rule is not derivable from the other axioms and inference rules.

Keywords Epistemic logic · Subjective time · Objective time · Temporal logic · Undefinability · Axiomatisation · Non-derivability

1 Introduction

In this article, we study two different concepts of time. As an example, consider the following story:

> A young girl, Ann, is running to catch a train. It's 8am, but Ann does not know the current time, and she does not remember the train's timetable. As Ann enters the station's platform, she can see the train starting to leave the station. She can hear the conductor whistling. She can see the last passengers jumping on the train. She can feel the old platform shaking…

In epistemic logic, the satisfaction $\Vdash$ is usually defined as a relation between a possible world and a formula. However, when discussing the statements whose truth

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value depends on the moment of time, it is natural to define satisfaction as a relation between a world-moment combination and a formula. In our example,

$$u, 8{:}00 \Vdash \text{“train is leaving”},$$

where u refers to the current world. Note that Ann (agent a) can see that the train is leaving, thus she must know this:

$$u, 8{:}00 \Vdash \mathsf{K}_a\text{“train is leaving”}. \tag{1}$$

Let us now get back to the story.

> As she watches the train depart, Ann suddenly realises she forgot to turn off her alarm, which is set to ring at 8:00 every morning.

Since it is 8am at the moment, the alarm must be ringing:

$$u, 8{:}00 \Vdash \text{“alarm is ringing”}.$$

Because Ann remembers her oversight, she knows that the alarm is ringing at 8:00:

$$u, 8{:}00 \Vdash \mathsf{K}_a\text{“alarm is ringing”}. \tag{2}$$

Note, however, that the modality K has two different meanings in Eqs. 1 and 2. To understand the difference, let us consider a simple *epistemic model* of the story depicted in Fig. 1. It has possible worlds w and v, in addition to the current world u. In epistemic logic, the indistinguishability is usually defined as an equivalence relation between possible worlds.

However, in a situation like ours, where an agent does not know the current time, she might not be able to distinguish one moment in one world from another moment in another world. Hence, it is sensible to consider indistinguishability as a relation on world-moment combinations. In our model, see Fig. 1, Ann cannot distinguish the

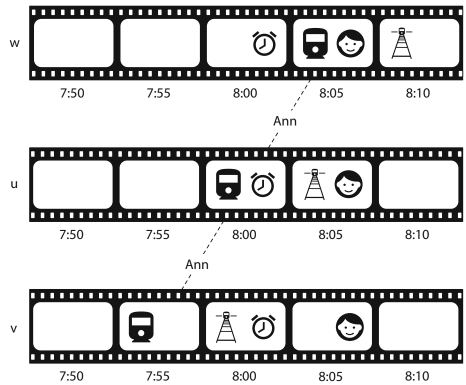

Fig. 1 Three possible worlds in the departing train example

moment 8:00 in the current world u from the moment 8:05 in world w, as well as from the moment 7:55 in world v.

The modality K in Eq. 1 refers to Ann's knowledge derived from what she can see and hear. She knows that the train is leaving in the sense that the train is leaving at each world-moment combination that she cannot distinguish from the current one. We show this in Fig. 1 by placing an image of a train at the moment 8:05 in world w, at the moment 8:00 in world u, and at the moment 7:55 in world v.

The modality K in Eq. 2 refers to Ann's knowledge about the time moment 8:00 itself. She is confused if her current world-moment combination is the moment 8:05 in the world w, the moment 8:00 in world u, or the moment 7:55 in world v. But she is sure that, whichever world it is, the alarm is set to ring at 8:00 in that world! We show this in Fig. 1 by placing a picture of an alarm at the moment 8:00 in each of the worlds w, u, and v.

To say it in another way, in Eq. 1, the modality K_a refers to Ann's *subjective* current moment (as she feels it) and in Eq. 2 the modality K_a refers to Ann's *objective* current moment (as defined by her alarm clock). To differentiate these two forms of knowledge, from now on, we will use two different modalities:

$$u, 8{:}00 \Vdash \mathsf{S}_a\text{“train is leaving”},$$
$$u, 8{:}00 \Vdash \mathsf{O}_a\text{“alarm is ringing”}.$$

It is time to return to our story.

As the train slowly pulls away, Ann finds herself thinking of what comes next. In five minutes, the platform will be empty—except for her, the unlucky one. She knows that she will still be standing there, unsure what to do next, watching the last carriage until it disappears into the distance.

The last carriage will disappear in the distance in five minutes from Ann's subjective current moment. What time it will be, objectively, depends on what world we are in:

$$w, 8{:}10 \Vdash \text{“carriage is in the distance”},$$
$$u, 8{:}05 \Vdash \text{“carriage is in the distance”},$$
$$v, 8{:}00 \Vdash \text{“carriage is in the distance”}.$$

In Fig. 1, we show this by placing a “carriage in the distance” image at those moments. In this article, we use the temporal modality $\mathsf{N}\varphi$ (“next”) to express the fact that a formula φ will be true at the next moment of time. For our example, let us assume that the time is advancing in 5-minute increments. Thus,

$$w, 8{:}05 \Vdash \mathsf{N}\text{“carriage is in the distance”},$$
$$u, 8{:}00 \Vdash \mathsf{N}\text{“carriage is in the distance”},$$
$$v, 7{:}55 \Vdash \mathsf{N}\text{“carriage is in the distance”}.$$

Recall that the moment 8:05 in world w and the moment 7:55 in world v are the only two world-moment combinations that Ann cannot distinguish from the current moment 8:00 in the current world u. Since the formula N"carriage is in the distance" is satisfied at all three of these combinations,

$$u, 8{:}00 \Vdash \mathsf{S}_a\mathsf{N}\text{“carriage is in the distance”}. \tag{3}$$

In other words, as Ann watches the train leave the station, she knows that her subjective current moment has the property "the last carriage will be on the horizon at the next moment".

> Suddenly, Ann's thoughts flash back to the alarm clock. She starts to think about her old, sick grandma, probably cooking breakfast shortly before 8:00. Once the grandma hears the alarm and realises Ann is not there, she will climb the stairs to Ann's room to turn it off. Ann imagines her grandma, tired from the steep stairs, entering Ann's room at 8:05. Ann feels bad about forgetting to turn off the alarm this morning.

Although Ann does not know which of the three possible worlds she is in, she knows that in 5 minutes after 8:00 her grandma will be in her room:

$$w, 8{:}00 \Vdash \mathsf{N}\text{“grandma is in Ann's room”},$$
$$u, 8{:}00 \Vdash \mathsf{N}\text{“grandma is in Ann's room”},$$
$$v, 8{:}00 \Vdash \mathsf{N}\text{“grandma is in Ann's room”}.$$

> Thus, Ann knows that the objective moment 8:00 has the property of N"grandma is in Ann's room":

$$u, 8{:}00 \Vdash \mathsf{O}_a\mathsf{N}\text{“grandma is in Ann's room”}. \tag{4}$$

Note that Eq. 3 expresses Ann's knowledge about her *subjective* next moment, while Eq. 4 expresses Ann's knowledge about her *objective* next moment. More generally, nested modalities $\mathsf{S}_a\mathsf{N}\dots\mathsf{N}$ and $\mathsf{O}_a\mathsf{N}\dots\mathsf{N}$ can be used to express knowledge about subjective and objective *future*.

The concepts of subjective and objective time have been studied in cognitive science [1], neuroscience [2], and the philosophy of time [3–5] literature. In his *The Phenomenology of the Internal Time-Consciousness* [6], Husserl refers to subjective time as *immanent*. He defines the immanent past as the collection of the agent's memories, the immanent present through "immanent temporal objects" (*Zeitobjekte*), and the immanent future as the agent's expectations of what will come. In our case, the leaving train, the whistling conductor, the jumping passengers, and the shaking platform are examples of Zeitobjekte. The empty platform and the carriage in the distance are Ann's expectations that form her immanent (subjective) future.

In this article, we study the interplay between modalities S, O, and N. As we discuss in Section 2, modality S has been proposed and studied before. The modality O is original to this article. We establish several technical results. First, we prove that modalities S and O are *not definable* through each other even if the modality N is used. Second, we give a *sound and complete* logical system describing the interplay between the modalities. One of the inference rules in this system is called the Insertion inference rule. This somewhat unusually looking rule is a generalisation of the standard Necessitation rule for modality O. Naturally, one might ask if the standard Necessitation suffices for the completeness of our logical system. We partially answer this question by showing that the Insertion rule is *not derivable* in the logical system in which this rule is replaced by the standard Necessitation rule for modality O. The admissibility of the Insertion rule in this setting remains an open question.

Of course, in addition to the "next" modality N, one might consider other temporal modalities (such as "past", "future", and "until"). We restrict consideration to temporal modality N because the proposed proof of completeness cannot be easily generalised to other temporal modalities.

The rest of the article is organised as follows. In the next section, we give the formal syntax and semantics of our logical system. We also compare our semantics and modalities with other epistemic temporal logics in the literature. In Section 3, we show that modalities S and O are not definable through each other even in the presence of the modality N. Section 4 lists the axioms and the inference rules of our logical system. Section 5 proves some technical results needed in the proof of the completeness. Section 6 establishes the completeness of our logical system using the "matrix" technique. Section 7 proves that the Insertion inference rule of our system is not derivable. We conclude with Section 8 that contains a discussion of another setting where there is a distinction between subjective and objective time.

2 Syntax and Semantics

Throughout the rest of the article, we fix a set of agents $\mathcal{A}$ and a nonempty set of propositional variables. As usual, we use the symbol ω to represent the smallest infinite ordinal, also known as the set of all non-negative integer numbers. The definition below formally specifies epistemic temporal models that have been informally discussed in the introduction. Note that, as common in the literature on temporal logic, we assume that the time is discrete and use ω to represent the set of all possible "moments".

Definition 1 An epistemic temporal model is a triple $(W, \sim, \pi)$, where

1. W is a set of "possible worlds",
2. $\sim_a$ is an "indistinguishability" equivalence relation on the set $W \times \omega$ for each agent $a \in \mathcal{A}$,
3. $\pi(p) \subseteq W \times \omega$.

In the above definition, the value of the valuation function $\pi(p)$ is a set of world-moment combinations. Intuitively, this is because atomic propositions in our setting express properties of such combinations, not just of possible worlds. The language Φ of our system is defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid \mathsf{N}\varphi \mid \mathsf{S}_a\varphi \mid \mathsf{O}_a\varphi,$$

where p is a propositional variable and $a \in \mathcal{A}$ is an agent. We read $\mathsf{N}\varphi$ as "formula φ is true at the next moment", $\mathsf{S}_a\varphi$ as "agent a knows φ about the subjective current moment", and $\mathsf{O}_a\varphi$ as "agent a knows φ about the objective current moment". We assume that conjunction $\wedge$, implication $\to$, biconditional $\leftrightarrow$, and constants false $\bot$ and true $\top$ are defined in the standard way.

Definition 2 For any epistemic temporal model $(W, \sim, \pi)$, any world $w \in W$, any moment $t \in \omega$, and any formula $\varphi \in \Phi$, the satisfaction relation $w, t \Vdash \varphi$ is defined as follows:

1. $w, t \Vdash p$ if $(w, t) \in \pi(p)$,
2. $w, t \Vdash \neg\varphi$ if $w, t \nVdash \varphi$,
3. $w, t \Vdash \varphi \vee \psi$ if either $w, t \Vdash \varphi$ or $w, t \Vdash \psi$,
4. $w, t \Vdash \mathsf{N}\varphi$ if $w, t+1 \Vdash \varphi$,
5. $w, t \Vdash \mathsf{S}_a\varphi$ if $u, s \Vdash \varphi$ for each world $u \in W$ and each moment $s \in \omega$ such that $(w, t) \sim_a (u, s)$,
6. $w, t \Vdash \mathsf{O}_a\varphi$ if $u, t \Vdash \varphi$ for each world $u \in W$ and each moment $s \in \omega$ such that $(w, t) \sim_a (u, s)$.

Temporal logics [7] have been studied in philosophy and computer science for several decades. Prior [8] and Kamp [9] considered temporal logical systems for modality "now", outside of an epistemic setting. Multiple logical systems that capture the interplay between knowledge and time have been introduced before. They can be divided into four groups based on how such systems specify their semantics.

The logical systems from the first group define semantics in terms of a binary relation $w \Vdash \varphi$ between a state w of a transition system and a formula φ. Such a system can contain a knowledge modality K_a and the "next" modality N. In such a setting, the statement $w \Vdash \mathsf{K}_a\varphi$ means that, each time the transition system is in state w, agent a knows that φ is true. Note that this type of semantics does not use any absolute time at all. Intuitively, the formula $\mathsf{K}_a\varphi$ refers to knowledge of agent a about her subjective current moment. Similarly, the formula $\mathsf{K}_a\mathsf{N}\varphi$ refers to agent a's knowledge about her subjective next moment (state of the system). An example of such a logical system is Epistemic Coalition Logic [10]. Although this system does not explicitly contain modality N, it can be defined in the system as the coalition power modality $[\varnothing]$ for the empty coalition $\varnothing$. A similar setting and modalities also appear in [11] and [12]. The latter work describes a logical system whose language can also be used to define "past" modality through $[\varnothing]^{-1}$.

Another approach is to define the satisfaction as a relation $h \Vdash \varphi$ between a "history" (computational path through a transition system) h and a formula φ. In particular, this allows the same formula to be satisfied during one visit of the transition system to a state and to be not satisfied during another visit. Just like the first type of semantics, the second type can be used to define knowledge about the subjective current moment. Additionally, it can potentially be used to define knowledge about a variation of the current objective moment. Indeed, note that the length of history can serve as a version of an absolute time counter from the beginning of the computation. This can potentially be used to define modality "agent a knows about the current moment from the start of the computation":

$h \Vdash \mathsf{O}_a\varphi$ if $(h'|_h) \Vdash \varphi$ for each history h' such that $h \sim_a h'$,

where $h'|_h$ is the prefix of sequence h' that has the same size as h. For example, $(w_1, w_2, w_3)|_{(u_1,u_2)} = (w_1, w_2)$. Note that this type of semantics treats "time since start of computation" as a substitute for the absolute time. The semantics does not have any computation-independent notion of absolute time. In addition, the above definition has a technical issue: the meaning of $h'|_h$ is not defined when history h' is shorter than history h. We are not aware of any works that use the above modality O_a. Parikh and Ramanujam [13] used the second type of semantics to define an equivalent of our subjective modality S. Logical systems that add knowledge to STIT also usually use this approach [14].

The third approach is used in the epistemic version of Alternating Temporal Logic [15]. Because this logical system contains state formulae as well as path formulae, it essentially combines the first two approaches. [15] only considers the knowledge about subjective time modality S.

In order to be able to capture the true "objective current moment" (as opposed to the moment from the start of the computation), one needs to explicitly add the absolute clock to the semantics of the system. The natural way to do this is to consider the ternary satisfaction relation $w, t \Vdash \varphi$ as we do in the current article. This fourth approach was first suggested in [17]. They proposed a semantics for modality S very similar to the one in our Definition 2. However, they did not consider an equivalent of our modality O. The journal version of the same paper gives a proof of completeness [16]. The S-like modalities have also been used in [18–21]. Formal definition of the knowledge about the objective current moment modality O is original to the current work.

We conclude this section with Theorem 1. Intuitively, it states that there is no difference between subjective and objective time for any agent who knows the current time. To state the theorem precisely, in the definition below, we introduce the notion of "time-awareness". Informally, an agent is time-aware at (w, t) if she knows the current time at moment t in world w.

Definition 3 For any moment $t \in \omega$ and any world $t \in W$ of an epistemic temporal model $(W, \sim, \pi)$, an agent $a \in \mathcal{A}$ is ***time-aware*** at (w, t) when for each world $w' \in W$ and each moment $t' \in \omega$, if $(w, t) \sim_a (w', t')$, then $t = t'$.

Theorem 1 *If agent $a \in \mathcal{A}$ is time-aware at (w, t) in epistemic temporal model $(W, \sim, \pi)$, then $w, t \Vdash \mathsf{S}_a\varphi$ iff $w, t \Vdash \mathsf{O}_a\varphi$ for each formula $\varphi \in \Phi$.*

Proof Consider any world $u \in W$ and any moment $s \in \omega$ such that $(w, t) \sim_a (u, s)$. By items 5 and 6 of Definition 2, it suffices to show that $u, t \Vdash \varphi$ iff $u, s \Vdash \varphi$. Towards this proof, note that, by Definition 3, the assumption $(w, t) \sim_a (u, s)$ and the assumption of the theorem that agent a is time-aware at moment t implies $t = s$. Therefore, $u, t \Vdash \varphi$ iff $u, s \Vdash \varphi$. □

3 Undefinability

In this section, we show that modalities S and O are not definable through each other even in the presence of modality N. Our results are captured in Theorem 2 and Theorem 3. Statements of these theorems use the definition below.

Definition 4 For any given epistemic temporal model M and any formula $\varphi \in \Phi$, the truth set $[\![\varphi]\!]_M$ is the set $\{(w, t) \in W \times \omega \mid w, t \Vdash \varphi\}$.

Without loss of generality, in this section, we assume that set $\mathcal{A}$ consists of a single agent a and that the set of propositional variables contains a single variable p. When the temporal epistemic model M is clear from the context, we use the notation $[\![\varphi]\!]$ instead of $[\![\varphi]\!]_M$.

3.1 Undefinability of Modality O via Modalities S and N

In this and the next section, we use a recently proposed "truth set algebra" technique for proving the undefinability of logical connectives [22]. It is not clear to us how a more traditional "bisimulation" method [23] could be used here because of the complexity introduced by having a possible world and a moment of time on the left-hand side of relation $\Vdash$. Unlike the bisimulation, the "truth set algebra" method uses only a single model. In our case, the model is an epistemic temporal model M. Model M has only two worlds, w and u. Figure 2 contains 10 table-like *diagrams*. The rows of these diagrams represent worlds, and the columns represent moments of time. To keep the figure clean, we only labelled rows and columns on the left-most diagram. The cells of each diagram correspond to world-moment combinations (pairs). For example, the upper-left cell in each diagram corresponds to the pair $(w, 0)$.

The left-most diagram in Fig. 2 captures the equivalence relation $\sim_a$ in model M. Recall, from Definition 1, that $\sim_a$ is a relation on the world-moment combinations. Note that each cell in the left-most diagram in Fig. 2 contains either a dot or nothing. For any $v, v' \in w, u$ and any moments t, t', let $(v, t) \sim_a (v', t')$ if the content of the cells (v, t) and (v', t') is the same on the diagram. For example, $(w, 1) \sim_a (u, 0)$ and $(w, 0) \sim_a (u, 1)$.

Next, we define the valuation function π for model M. Because of our earlier assumption that the language contains a single propositional variable p, it suffices to define the set $\pi(p) = [\![p]\!]$. In this work, we visualise truth sets of various formulae φ by shading cells (v, t) such that $(v, t) \in [\![\varphi]\!]$. The diagram labelled with $[\![p]\!]$ in the second (out of four) column of Fig. 2 specifies the set $\pi(p) = [\![p]\!]$.

This concludes the definition of the epistemic temporal model M that we use in this section to prove the undefinability of modality O through modalities S and N.

Figure 2 is divided into three parts. The middle part contains 8 diagrams. By the above definition of valuation function π, the diagram labelled with $[\![p]\!]$ represents the truth set of propositional variable p. Next, let us show that the remaining seven diagrams in the middle part of Fig. 2 correctly represent the truth sets with which these diagrams are labelled.

By Definition 4 and Definition 2, we have $[\![\bot]\!] = \varnothing$ and $[\![\top]\!] = \{w, u\} \times \omega$. This matches how the cells are shaded in the diagrams labelled with $[\![\bot]\!]$ and $[\![\top]\!]$.

By Definition 4 and item 4 of Definition 2, the diagram for the truth set $[\![\mathsf{N}p]\!]$ is obtained from the diagram for the truth set $[\![p]\!]$ by removing the first column and "shifting" the diagram to the left. This matches the relation between the diagrams

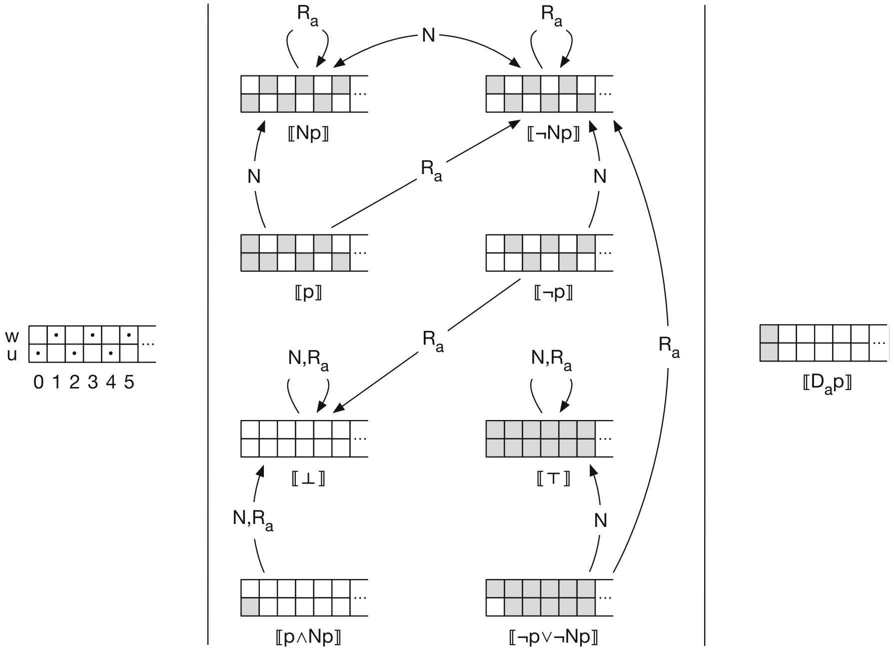

Fig. 2 Towards the proof of Theorem 2: relation $\sim_a$ and truth sets

labelled with $[\![p]\!]$ and $[\![\mathsf{N}p]\!]$ in Fig. 2. Hence, the diagram labelled with $[\![\mathsf{N}p]\!]$ correctly represents the truth set $[\![\mathsf{N}p]\!]$.

By Definition 4 and item 2 of Definition 2, the truth set $[\![\neg\varphi]\!]$ is the complement of the truth set $[\![\varphi]\!]$. Note that the diagram labelled with $[\![\neg p]\!]$ shades exactly the complement of the cells shaded in the diagram labelled with $[\![p]\!]$. Hence, the diagram labelled with $[\![\neg p]\!]$ correctly represents the truth set $[\![\neg p]\!]$. Similarly, the diagram labelled with $[\![\neg\mathsf{N}p]\!]$ correctly represents the truth set $[\![\neg\mathsf{N}p]\!]$.

By Definition 4 and Definition 2, we have $[\![p \wedge \mathsf{N}p]\!] = [\![p]\!] \cap [\![\mathsf{N}p]\!]$ and $[\![\neg p \vee \neg\mathsf{N}p]\!] = [\![\neg p]\!] \cup [\![\neg\mathsf{N}p]\!]$. Observe that the diagram labelled with $[\![p \wedge \mathsf{N}p]\!]$ shades the only cell which is shaded in both the diagram labelled with $[\![p]\!]$ and the diagram labelled with $[\![\mathsf{N}p]\!]$. Similarly, the diagram labelled with $[\![\neg p \vee \neg\mathsf{N}p]\!]$ shades all cells which are shaded in at least one of the diagrams labelled with $[\![\neg p]\!]$ and $[\![\neg\mathsf{N}p]\!]$. Therefore, the diagrams labelled with $[\![p \wedge \mathsf{N}p]\!]$ and $[\![\neg p \vee \neg\mathsf{N}p]\!]$ correctly represents the truth sets $[\![p \wedge \mathsf{N}p]\!]$ and $[\![\neg p \vee \neg\mathsf{N}p]\!]$ respectively.

As we have just observed, the truth sets of formulae $\mathsf{N}p$, $\neg\mathsf{N}p$, p, $\neg p$, $\bot$, $\top$, $p \wedge \mathsf{N}p$, and $\neg p \vee \neg\mathsf{N}p$ are depicted in the middle part of Fig. 2. Surprisingly, a much stronger result holds. Namely, **the truth set $[\![\varphi]\!]$ of any formula φ in language Φ that does not use modality O is depicted in the middle part of Fig. 2**. This observation, proven in Lemma 4, is the key to our proof of undefinability. Before proving Lemma 4, let us make the following three auxiliary observations:

Lemma 1 *For any $\varphi, \psi \in \Phi$, if sets $[\![\varphi]\!]$ and $[\![\psi]\!]$ are represented by the diagrams in the middle part of Fig. 2, then the same is true about the sets $[\![\neg\varphi]\!]$ and $[\![\varphi \vee \psi]\!]$.*

Proof Observe that the family of sets represented by the diagrams in the middle part of Fig. 2 is closed with respect to the complement and the union. Thus, the statement of the lemma follows from Definition 4 as well as item 2 and item 3 of Definition 2. □

Lemma 2 *For any formula $\varphi \in \Phi$, if the set $[\![\varphi]\!]$ is represented by a diagram in the middle part of Fig. 2, then the same is true about the set $[\![\mathsf{N}\varphi]\!]$.*

Proof By Definition 4 and item 4 of Definition 2, the diagram for the set $[\![\mathsf{N}\varphi]\!]$ is obtained by shifting the diagram for the set $[\![\varphi]\!]$ to the left and removing the left-most column. Observe that the family of sets represented by the diagrams in the middle part of Fig. 2 is closed with respect to this operation. For example, the result of such a transformation of the diagram $[\![p \wedge \mathsf{N}p]\!]$ is the diagram $[\![\bot]\!]$. For the benefit of the reader, we have shown the result of each such transformation in the middle part of Fig. 2 by a directed edge labelled with modality N. □

Lemma 3 *For any formula $\varphi \in \Phi$, if the set $[\![\varphi]\!]$ is represented by a diagram in the middle part of Fig. 2, then the same is true about the set $[\![\mathsf{S}_a\varphi]\!]$.*

Proof Recall that the diagram in the left part of Fig. 2 specifies the equivalence relation $\sim_a$ on pairs (v, t). Any two such pairs are equivalent if the content of the corresponding cells is the same. Thus, the equivalence relation $\sim_a$ divides the pairs into two equivalence classes. All cells corresponding to the pairs in one class are empty, and all cells corresponding to the cells in the other class contain a dot. By Definition 4 and item 5 of Definition 2, in order for a pair (v, t) to belong to the set $[\![\mathsf{S}_a\varphi]\!]$, all pairs in its class must belong to the set $[\![\varphi]\!]$.

First, let us consider the case when the set $[\![\varphi]\!]$ is represented by the diagram labelled with $[\![p]\!]$. In this case, the set $[\![\varphi]\!]$ contains all pairs whose corresponding cells are empty and <u>not all</u> pairs whose corresponding cells contain dots, see Fig. 2. Thus, by the observation from the previous paragraph, the set $[\![\mathsf{S}_a\varphi]\!]$ contains all pairs whose corresponding cells are empty and <u>none</u> of the pairs whose corresponding cells contain dots. Thus, the set $[\![\mathsf{S}_a\varphi]\!]$ is represented by the diagram labelled with $[\![\neg\mathsf{N}p]\!]$, see Fig. 2.

The other cases corresponding to the seven remaining diagrams in the middle part of Fig. 2 are similar. In the middle part of Fig. 2, for each such case, we show a directed edge labelled with modality S_a from the diagram representing the set $[\![\varphi]\!]$ to the diagram representing the set $[\![\mathsf{S}_a\varphi]\!]$. □

Lemma 4 *For any formula $\varphi \in \Phi$ that does not contain modality O, the set $[\![\varphi]\!]$ is represented by one of the diagrams in the middle part of Fig. 2.*

Proof We prove the statement of the lemma by induction on the structural complexity of the formula φ. If φ is a propositional variable, then, as we have observed above, the set $[\![\varphi]\!]$ is specified by the diagram labelled with $[\![p]\!]$.

If formula φ has the form $\neg\psi$, then, by the induction hypothesis, the set $[\![\psi]\!]$ is represented by one of the diagrams in the middle part of Fig. 2. In this case, the statement of the lemma follows from Lemma 1.

The cases when formula φ has one of the forms $\psi_1 \vee \psi_2$, $\mathsf{N}\psi$, or $\mathsf{S}_a\psi$ are similar, using Lemma 1, Lemma 2, and Lemma 3, respectively. □

Finally, in the next lemma, we show that the set $[\![\mathsf{O}_a p]\!]$ is not represented in the *middle* part of Fig. 2.

Lemma 5 *The set* $[\![\mathsf{O}_a p]\!]$ *is represented by the diagram in the right part of Fig. 2.*

Proof The statement of the lemma follows from the following two claims:

Claim 1 $(v, 0) \in [\![\mathsf{O}_a p]\!]$ for each world $v \in \{w, u\}$.

Proof-of-claim By Definition 4, it suffices to show that $v, 0 \Vdash \mathsf{O}_a p$. Consider any world-moment pair (v', t'). By item 6 of Definition 2, it suffices to prove that $v', 0 \Vdash p$.

Observe, see the diagram labelled with $[\![p]\!]$ in the middle part of Fig. 2, that $(v', 0) \in [\![p]\!]$ (no matter if $v' = w$ or $v' = u$). Therefore, $v', 0 \Vdash p$ by item 6 of Definition 2.□

Claim 2 $(v, t) \notin [\![\mathsf{O}_a p]\!]$ for each world $v \in \{w, u\}$ and each moment $t > 0$.

Proof-of-claim Consider the diagram labelled with $[\![p]\!]$ in the middle part of Fig. 2. Observe that, because $t > 0$, there must exists world $v' \in \{w, u\}$ such that $(v', t) \notin [\![p]\!]$. Indeed, choose $v' = u$ if t is even and $v' = w$ if t is odd. Hence, by Definition 4,

$$v', t \nVdash p. \tag{5}$$

Next, consider the left-most diagram in Fig. 2. No matter what v' is, there must be a cell in the v'-th row of that diagram that has the same content as cell (v, t). Let such a cell be the cell (v', t'). Thus, $(v, t) \sim_a (v', t')$ by the definition of the relation $\sim_a$. Hence, $v, t \nVdash \mathsf{O}_a p$ by item 6 of Definition 2 and Eq. 5. Therefore, $(v, t) \notin [\![\mathsf{O}_a p]\!]$ by Definition 4. □

This concludes the proof of the lemma. □

The next theorem follows from Lemma 4 and Lemma 5.

Theorem 2 *[undefinability]* $[\![\mathsf{O}_a p]\!]_M \neq [\![\varphi]\!]_M$ *for each formula* $\varphi \in \Phi$ *that does not contain modality* O.

3.2 Undefinability of Modality S via Modalities O and N

In this subsection, we use the same technique as in the previous one to prove the undefinability of S through N and O. The epistemic temporal model M that we consider in this subsection consists of four possible worlds: w, u, v, and y. We show the indistinguishability relation $\sim_a$ on the world-moment pairs in the left part of Fig. 3. Just like before, we define two pairs to be indistinguishable if the corresponding cells have the same content. Note that instead of two classes in the previous subsection, in this subsection the equivalence relation has four classes. Finally, the value of $\pi(p)$ is specified by the diagram labelled with $[\![p]\!]$ in the middle part of Fig. 3.

The proof of the next two lemmas is similar to the proofs of Lemma 1 and Lemma 2 in the previous subsection.

Lemma 6 *For any $\varphi, \psi \in \Phi$, if sets $[\![\varphi]\!]$ and $[\![\psi]\!]$ are represented by the diagrams in the middle part of Fig. 3, then the same is true about the sets $[\![\neg\varphi]\!]$ and $[\![\varphi \vee \psi]\!]$.*

Lemma 7 *For any formula $\varphi \in \Phi$, if the set $[\![\varphi]\!]$ is represented by a diagram in the middle part of Fig. 3, then the same is true about the set $[\![\mathsf{N}\varphi]\!]$.*

Lemma 8 *For any formula $\varphi \in \Phi$, if the set $[\![\varphi]\!]$ is represented by a diagram in the middle part of Fig. 3, then the same is true about the set $[\![\mathsf{O}_a\varphi]\!]$.*

Proof First, let us suppose that the set $[\![\varphi]\!]$ is represented by a diagram in the middle part of Fig. 3 labelled with $[\![p]\!]$, $[\![\neg p]\!]$, or $[\![\bot]\!]$. It suffices to show that the set $[\![\mathsf{O}_a\varphi]\!]$ is represented by the diagram labelled with $[\![\bot]\!]$. Consider any world $z \in \{w, u, v, y\}$ and any moment t. By Definition 4, it suffices to show that $z, t \nVdash \mathsf{O}_a\varphi$.

By analysing the diagram in the left part of Fig. 3 and the diagram labelled with $[\![p]\!]$ in the middle part of the same figure, it is easy to see that no matter what the values of z and t are, there must exist a world $z' \in \{w, u, v, y\}$ and a moment t' such that cells (z, t) and (z', t') have the same content in the diagram in the left part and cell (z', t) is shaded grey in the diagram labelled with $[\![p]\!]$ in the middle part. In other words, $(z, t) \sim_a (z', t')$ and $z', t \nVdash p$. Therefore, $z, t \nVdash \mathsf{O}_a\varphi$ by item 6 of Definition 2.

Let us now assume that the set $[\![\varphi]\!]$ is represented by a diagram in the middle part of Fig. 3 labelled with $[\![\top]\!]$. By Definition 4, this implies that $z, t \Vdash \varphi$ for any world $z \in \{w, u, v, y\}$ and any moment t. Therefore, $z, t \Vdash \varphi$ for any world $z \in \{w, u, v, y\}$ and any moment t by item 6 of Definition 2. Therefore, by Definition 4, the set $[\![\mathsf{O}_a\varphi]\!]$ is also represented by the diagram labelled with $[\![\top]\!]$. □

The proof of the next lemma is similar to the proof of Lemma 4 in the previous subsection, but instead of Lemma 1, Lemma 2, and Lemma 3, it uses Lemma 6, Lemma 7, and Lemma 8, respectively.

Lemma 9 *For any formula $\varphi \in \Phi$ that does not contain modality S, the set $[\![\varphi]\!]$ is represented by one of the diagrams in the middle part of Fig. 3.*

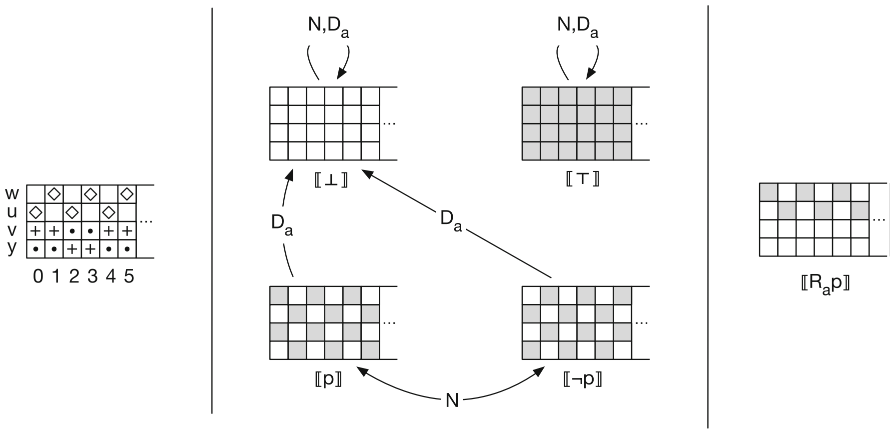

Fig. 3 Towards the proof of Theorem 3: relation $\sim_a$ and truth sets

Lemma 10 *The set $[\![\mathsf{S}_a p]\!]$ is represented by the diagram in the right part of Fig. 3.*

Proof By item 5 of Definition 2 and the definition of the equivalence relation in our model, for any world $z' \in \{w, u, v, y\}$ and any moment t, statement $z, t \Vdash \mathsf{S}_a p$ is true if $z', t' \Vdash p$ for any z', t' such that the cells (z, t) and (z', t') have the same content in the diagram in the middle part of Fig. 3 labelled with $[\![p]\!]$. Observe in Fig. 3 that the latter is true only when cells (z, t) in the left part of Fig. 3 is empty. Therefore, by Definition 4, the set $[\![\mathsf{S}_a p]\!]$ is represented by the diagram in the right part of Fig. 3. □

The next theorem follows from Lemma 9 and Lemma 10.

Theorem 3 *(undefinability) $[\![\mathsf{S}_a p]\!]_M \neq [\![\varphi]\!]_M$ for each formula $\varphi \in \Phi$ that does not contain modality* S.

4 Axioms

In this section, we present a complete logical system describing the properties of modalities S, O, and N. Recall that the semantics of our knowledge modalities, just like in the case of the standard epistemic logic [24], is defined through a transitive, symmetric, and reflexive reachability relation $\sim_a$. In the case of traditional epistemic logic, transitivity, symmetry, and reflexivity of the reachability relation lead to S5 axioms that include the positive and the negative introspection. The situation is different in our case, where only S modality has the properties of positive ($\mathsf{S}_a\varphi \to \mathsf{S}_a\mathsf{S}_a\varphi$) and negative ($\neg\mathsf{S}_a\varphi \to \mathsf{S}_a\neg\mathsf{S}_a\varphi$) introspection.

To see why modality O does *not* satisfy the positive introspection property, let us consider a slightly modified version of our introductory example depicted in Fig. 4. Suppose that Ann cannot distinguish moment 8:00 in world w from moment 7:55 in world u. In addition, she cannot distinguish the moment 8:00 in world u from the moment 7:55 in world v.

To construct a counterexample for the positive introspection, observe that

$$w, 8\!:\!00 \Vdash \text{“The train is leaving now”},$$
$$u, 8\!:\!00 \Vdash \text{“The train is leaving now”}.$$

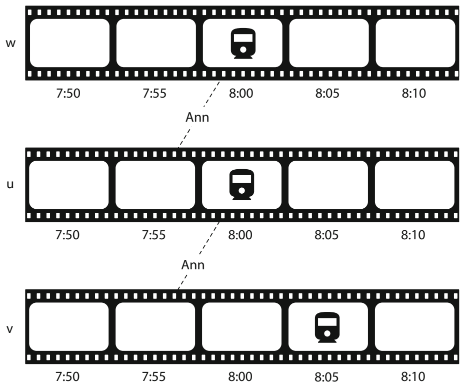

Fig. 4 An example that shows that modality O does not satisfy positive introspection principle

Then, by item 6 of Definition 2,

$$w, 8\!:\!00 \Vdash \mathsf{O}_a\text{“The train is leaving now”}. \tag{6}$$

At the same time,

$$v, 8\!:\!00 \nVdash \text{“The train is leaving now”}.$$

Then, again by item 6 of Definition 2,

$$u, 8\!:\!00 \nVdash \mathsf{O}_a\text{“The train is leaving now”}.$$

Hence, again by item 6 of Definition 2,

$$w, 8\!:\!00 \nVdash \mathsf{O}_a\mathsf{O}_a\text{“The train is leaving now”}.$$

The last statement, considered together with Eq. 6, provides a counterexample for the positive introspection principle for modality O.

In addition to the “pure” positive introspection principle $\mathsf{O}_a\varphi \to \mathsf{O}_a\mathsf{O}_a\varphi$ for modality O, one can also consider various versions of the positive introspection principle that mix the two knowledge modalities: $\mathsf{O}_a\varphi \to \mathsf{S}_a\mathsf{O}_a\varphi$, $\mathsf{O}_a\varphi \to \mathsf{O}_a\mathsf{S}_a\varphi$, $\mathsf{S}_a\varphi \to \mathsf{O}_a\mathsf{S}_a\varphi$, etc. By modifying the above example, one can show that none of these “mixed” principles is valid under our semantics of modalities S and O.

It is well known that the positive introspection principle is derivable in logic S5 from the Truth, the Negative Introspection, and the Distributivity axioms using the Modus Ponens and the Necessitation inference rules. We also prove this in Lemma 14. This fact has two important consequences. First, in our axioms below, we include the negative introspection but not the positive introspection principle for modality S. Second, since modality O satisfies the Truth axiom, the Distributivity axiom, and the Necessitation inference rule, our above counterexample for the positive introspection principle for modality O is also a counterexample for the negative introspection principle for modality O.

In addition to the propositional tautologies in language Φ, the axioms of our system include

1. Truth: $\Box\varphi \to \varphi$, where $\Box \in \{\mathsf{S}_a, \mathsf{O}_a\}$,
2. Distributivity: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, where $\Box \in \{\mathsf{N}, \mathsf{S}_a, \mathsf{O}_a\}$,
3. Negative Introspection: $\neg\mathsf{S}_a\varphi \to \mathsf{S}_a\neg\mathsf{S}_a\varphi$,
4. Functionality: $\neg\mathsf{N}\varphi \leftrightarrow \mathsf{N}\neg\varphi$.

We write $\vdash \varphi$ and say that formula φ is a *theorem* of our logical system if it is derivable from the above axioms using the Modus Ponens, the two forms of Necessitation,

$$\frac{\varphi, \quad \varphi \to \psi}{\psi} \qquad\qquad \frac{\varphi}{\mathsf{S}_a\varphi} \qquad\qquad \frac{\varphi}{\mathsf{N}\varphi}$$

and the Insertion

$$\frac{\mathsf{N}^k\varphi}{\mathsf{N}^k\mathsf{O}_a\varphi}$$

inference rules. In the last rule, by $\mathsf{N}^k\varphi$ we mean the formula $\underbrace{\mathsf{N}\ldots\mathsf{N}}_{k\text{ times}}\varphi$ for any $k \geq 0$. Intuitively, the last rule states that if a formula φ is universally valid (satisfied in each world of each model) starting from moment k, then it is universally known to each agent starting from moment k. Lemma 11 formally proves the soundness of this rule.

Observe that the Necessitation rule for modality O:

$$\frac{\varphi}{\mathsf{O}_a\varphi} \tag{7}$$

is a special case of the Insertion rule for $k = 0$. One might naturally wonder if the Insertion rule is essential. Let us denote by L^- the above axiomatisation in which the Insertion rule is replaced by the rule specified in Eq. 7. In Section 7, we prove that the Insertion rule is not *derivable* in system L^-. Whether it is *admissible* is an open question.

In addition to the unary relation $\vdash \varphi$, we also consider a binary relation $X \vdash \varphi$. For any set of formulae X and any formula φ, let $X \vdash \varphi$ if formula φ is derivable from the set of *theorems* of our logical system and the additional set of formulae X using *only* the Modus Ponens inference rule. Note that the statements $\varnothing \vdash \varphi$ and $\vdash \varphi$ are equivalent. We say that a set X is consistent if $X \nvdash \bot$.

Towards the proof of the soundness of our logical system, let us first show the soundness of the Insertion inference rule.

Lemma 11 *If $w, t \Vdash \mathsf{N}^k\varphi$ for each world w and each moment t of each epistemic temporal model, then $w, t \Vdash \mathsf{N}^k\mathsf{O}_a\varphi$ for each world w and each moment t of each epistemic temporal model.*

Proof Suppose that $w, t \nVdash \mathsf{N}^k\mathsf{O}_a\varphi$ for some world w and moment t of some epistemic temporal model. Thus, $w, t + k \nVdash \mathsf{O}_a\varphi$ by item 4 of Definition 2 applied k times. Hence, by item 6 of Definition 2, there is a world w' and a moment t' such that $(w, t + k) \sim_a (w', t')$ and $w', t + k \nVdash \varphi$. Therefore, $w', t \nVdash \mathsf{N}^k\varphi$ by item 4 of Definition 2 applied k times, which contradicts the assumption of the lemma. $\square$

The soundness of the Modus Ponens inference rule and the two forms of the Necessitation inference rule, as well as of the four axioms, is straightforward. Thus, the above lemma implies the soundness of our logical system stated as a theorem below.

Theorem 4 *[soundness] If $\vdash \varphi$, then $w, t \Vdash \varphi$ for each world and each moment of each epistemic temporal model.*

5 Auxiliary Properties

This section contains several technical results that will be used in the proof of completeness.

Lemma 12 *[Lindenbaum] Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

Proof The standard proof of Lindenbaum's lemma [25, Proposition 2.14] applies. □

The proofs of the next three lemmas are standard.

Lemma 13 *[deduction] If $X, \varphi \vdash \psi$, then $X \vdash \varphi \to \psi$.*

Lemma 14 $\vdash \mathsf{S}_a\varphi \to \mathsf{S}_a\mathsf{S}_a\varphi$.

Lemma 15 *If $\varphi_1, \dots, \varphi_n \vdash \psi$, then $\Box\varphi_1, \dots, \Box\varphi_n \vdash \Box\psi$, where $\Box \in \{\mathsf{N}, \mathsf{S}_a, \mathsf{O}_a\}$.*

Lemma 16 $\vdash (\mathsf{N}\varphi \to \mathsf{N}\psi) \to \mathsf{N}(\varphi \to \psi)$.

Proof First, note that the following two formulae are propositional tautologies:

$$\neg\varphi \to (\varphi \to \psi) \qquad\qquad \psi \to (\varphi \to \psi).$$

Thus, by the Necessitation inference rule,

$$\vdash \mathsf{N}(\neg\varphi \to (\varphi \to \psi)) \qquad \vdash \mathsf{N}(\psi \to (\varphi \to \psi)).$$

Hence, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{N}\neg\varphi \to \mathsf{N}(\varphi \to \psi) \qquad \vdash \mathsf{N}\psi \to \mathsf{N}(\varphi \to \psi).$$

Then, by propositional reasoning using the Functionality axiom,

$$\vdash \neg\mathsf{N}\varphi \to \mathsf{N}(\varphi \to \psi)) \qquad \vdash \mathsf{N}\psi \to \mathsf{N}(\varphi \to \psi)).$$

Thus, again by propositional reasoning,

$$\vdash (\neg\mathsf{N}\varphi \vee \mathsf{N}\psi) \to \mathsf{N}(\varphi \to \psi).$$

Therefore, $\vdash (\mathsf{N}\varphi \to \mathsf{N}\psi) \to \mathsf{N}(\varphi \to \psi)$ by more propositional reasoning. □

Lemma 17 $\vdash \mathsf{N}^n\mathsf{O}_a(\varphi \to \psi) \to (\mathsf{N}^n\mathsf{O}_a\varphi \to \mathsf{N}^n\mathsf{O}_a\psi)$.

Proof Note that the formula

$$\mathsf{O}_a(\varphi \to \psi) \to (\mathsf{O}_a\varphi \to \mathsf{O}_a\psi)$$

is an instance of the Distributivity axiom. Then, by the Necessitation inference rule

$$\vdash \mathsf{N}(\mathsf{O}_a(\varphi \to \psi) \to (\mathsf{O}_a\varphi \to \mathsf{O}_a\psi)).$$

Thus, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{N}\mathsf{O}_a(\varphi \to \psi) \to \mathsf{N}(\mathsf{O}_a\varphi \to \mathsf{O}_a\psi).$$

Hence, by the Distributivity axiom and propositional reasoning,

$$\vdash \mathsf{NO}_a(\varphi \to \psi) \to (\mathsf{NO}_a\varphi \to \mathsf{NO}_a\psi).$$

Therefore, by repeating the previous steps $n-1$ more times,

$$\vdash \mathsf{N}^n\mathsf{O}_a(\varphi \to \psi) \to (\mathsf{N}^n\mathsf{O}_a\varphi \to \mathsf{N}^n\mathsf{O}_a\psi).$$

□

Lemma 18 *If* $\mathsf{N}^k\varphi_1, \ldots, \mathsf{N}^k\varphi_n \vdash \mathsf{N}^k\psi$, *then* $\mathsf{N}^k\mathsf{O}_a\varphi_1, \ldots, \mathsf{N}^k\mathsf{O}_a\varphi_n \vdash \mathsf{N}^k\mathsf{O}_a\psi$.

Proof By Lemma 13, the assumption $\mathsf{N}^k\varphi_1, \ldots, \mathsf{N}^k\varphi_{n-1}, \mathsf{N}^k\varphi_n \vdash \mathsf{N}^k\psi$ implies

$$\mathsf{N}^k\varphi_1, \ldots, \mathsf{N}^k\varphi_{n-1} \vdash \mathsf{N}^k\varphi_n \to \mathsf{N}^k\psi.$$

Hence, by Lemma 16 and the Modus Ponens inference rule,

$$\mathsf{N}^k\varphi_1, \ldots, \mathsf{N}^k\varphi_{n-1} \vdash \mathsf{N}^k(\varphi_n \to \psi).$$

Then, again by Lemma 13,

$$\mathsf{N}^k\varphi_1, \ldots, \mathsf{N}^k\varphi_{n-2} \vdash \mathsf{N}^k\varphi_{n-1} \to \mathsf{N}^k(\varphi_n \to \psi).$$

Thus, again by Lemma 16 and the Modus Ponens inference rule,

$$\mathsf{N}^k\varphi_1, \ldots, \mathsf{N}^k\varphi_{n-2} \vdash \mathsf{N}^k(\varphi_{n-1} \to (\varphi_n \to \psi)).$$

By repeating the previous two steps $n-2$ more times,

$$\vdash \mathsf{N}^k(\varphi_1 \to (\varphi_2 \to \ldots(\varphi_{n-2} \to (\varphi_{n-1} \to (\varphi_n \to \psi)))\ldots)).$$

Hence, by the Insertion inference rule,

$$\vdash \mathsf{N}^k\mathsf{O}_a(\varphi_1 \to (\varphi_2 \to \ldots(\varphi_{n-2} \to (\varphi_{n-1} \to (\varphi_n \to \psi)))\ldots)).$$

Then, by Lemma 17,

$$\vdash \mathsf{N}^k\mathsf{O}_a\varphi_1 \to \mathsf{N}^k\mathsf{O}_a(\varphi_2 \to \ldots(\varphi_{n-2} \to (\varphi_{n-1} \to (\varphi_n \to \psi)))\ldots).$$

Thus, by the Modus Ponens inference rule,

$$\mathsf{N}^k\mathsf{O}_a\varphi_1 \vdash \mathsf{N}^k\mathsf{O}_a(\varphi_2 \to \ldots(\varphi_{n-2} \to (\varphi_{n-1} \to (\varphi_n \to \psi)))\ldots).$$

Therefore, $\mathsf{N}^k\mathsf{O}_a\varphi_1, \ldots, \mathsf{N}^k\mathsf{O}_a\varphi_n \vdash \mathsf{N}^k\mathsf{O}_a\psi$ by repeating the previous two steps $n-1$ more times. □

Lemma 19 $\vdash \neg \mathsf{N}^n\varphi \to \mathsf{N}^n\neg\varphi$ *for any integer* $n \geq 0$.

Proof We prove the lemma by induction on n. In the base case, the formula $\neg\varphi \to \neg\varphi$ is a propositional tautology.

Suppose that $\vdash \neg\mathsf{N}^n\varphi \to \mathsf{N}^n\neg\varphi$. Then, by the Necessitation inference rule, $\vdash \mathsf{N}(\neg\mathsf{N}^n\varphi \to \mathsf{N}^n\neg\varphi)$. Hence, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \mathsf{N}\neg\mathsf{N}^n\varphi \to \mathsf{N}\mathsf{N}^n\neg\varphi$. Therefore, by propositional reasoning using the Functionality axiom, $\vdash \neg\mathsf{N}\mathsf{N}^n\varphi \to \mathsf{N}\mathsf{N}^n\neg\varphi$. $\square$

Lemma 20 $\vdash \neg\mathsf{N}^n\bot$ *for all* $n \geq 0$.

Proof We prove the lemma by induction on n. The statement is true for $n = 0$ because $\neg\bot$ is a tautology. For the induction step, suppose that $\vdash \neg\mathsf{N}^n\bot$. Then, $\vdash \mathsf{N}\neg\mathsf{N}^n\bot$ by the Necessitation inference rule. Therefore, $\vdash \neg\mathsf{N}\mathsf{N}^n\bot$ by the Functionality axiom and propositional reasoning. $\square$

6 Completeness

In this section, we prove the completeness of our logical system. The proof of completeness is using the recently introduced matrix construction [26]. In Section 6.1, we explain the idea behind this method. In Section 6.2, we discuss the modifications that we have made to the method to prove the completeness in our case. In the rest of this section, we present the formal proof.

6.1 The Matrix Method

As usual, the proof of completeness consists of the construction of a canonical model. The matrix method defines the canonical method as a limit of an infinite chain of matrices. Each matrix in the chain represents a partially built canonical model. Intuitively, matrices are formal equivalents of the type of diagrams that can be seen in Fig. 1 and 4. The rows of the matrix represent possible worlds, and the columns represent moments of time. The content of a cell (w, t) of the matrix represents the state of the world w at moment t. In Figs. 1 and 4, the state of the world in each moment is visualised by a picture (a frame of a movie). In the formal setting, the cell (w, t) of the matrix stores a maximal consistent set X_{wt} of formulae that will be satisfied in the world w at moment t of the canonical model when it is completely built. Formally, this connection between set X_{wt} and the satisfaction relation in the completely built canonical model is captured in Claim 17, which is our version of what is usually called the "truth lemma".

In Figs. 1 and 4, some of the pictures are indistinguishable by agents. Accordingly, we also introduce the indistinguishability relation $\sim_a$ on the cells of matrices. A *frame*[1] is a combination of a matrix and a family of indistinguishability relations on cells of the matrix indexed by agents. The formal definition of a frame, given in Definition 6,

[1] A "frame" is not the same as a "movie frame" in Figs. 1 and 4. The latter corresponds to a single cell of a matrix.

requires certain connections between the maximal consistent sets that form the matrix and the indistinguishability relations. For example, item 3(c) of Definition 6 states that if cells (w_1, t_1) and (w_2, t_2) are indistinguishable by an agent a, then the sets $X_{w_1 t_1}$ and $X_{w_2 t_2}$ contain the same S_a-formulae.

Claim 17, our "truth lemma", states that a formula ψ belongs to the set X_{wt} iff this formula is satisfied in world w of the canonical model at moment t. Given item 5 of Definition 2, in order to prove Claim 17 by induction in the case when formula ψ has the form $\mathsf{S}_a\varphi$, we need to guarantee that

$$\mathsf{S}_a\varphi \in X_{wt} \text{ iff } \varphi \in X_{w't'} \text{ for each } w', t' \text{ such that } (w, t) \sim_a (w', t'). \tag{8}$$

The ($\Rightarrow$) part of this statement is already guaranteed by the aforementioned item 3(c) of Definition 6. Indeed, by that item, statements $\mathsf{S}_a\varphi \in X_{wt}$ and $(w, t) \sim_a (w', t')$ imply that $\mathsf{S}_a\varphi \in X_{w't'}$. Then, $\varphi \in X_{w't'}$ by the Truth axiom because $X_{w't'}$ is a maximal consistent set.

The situation is not so simple for the ($\Leftarrow$) part of Eq. 8. Definition 6 does not guarantee that it is true. To make it true, if $\mathsf{S}_a\varphi \notin X_{wt}$, then we might need to extend the frame with a new world w' and a new moment t', see Fig. 5. Formally, the extension consists of adding a new row and a new column to the matrix, defining the maximal consistent sets for the newly added cells of the matrix and extending the indistinguishability relation appropriately. The construction of such an extension is described in Lemma 24. We might need to do such an extension for each w, each t, and each formula $\mathsf{S}_a\varphi$ such that $\mathsf{S}_a\varphi \notin X_{wt}$. Note that once a new row and a new column are added to the matrix, the newly added cells might not contain formula $\mathsf{S}_a\varphi$ and thus the construction will have to be repeated for them too, and so on *ad infinitum*. Modality O requires similar extensions that can be carried out independently or, as we do in Lemma 24, combined with the extensions for modality S. As described so far, the construction is similar to the one in [26], where it was applied to a different pair of modalities, both of which are not related to time.

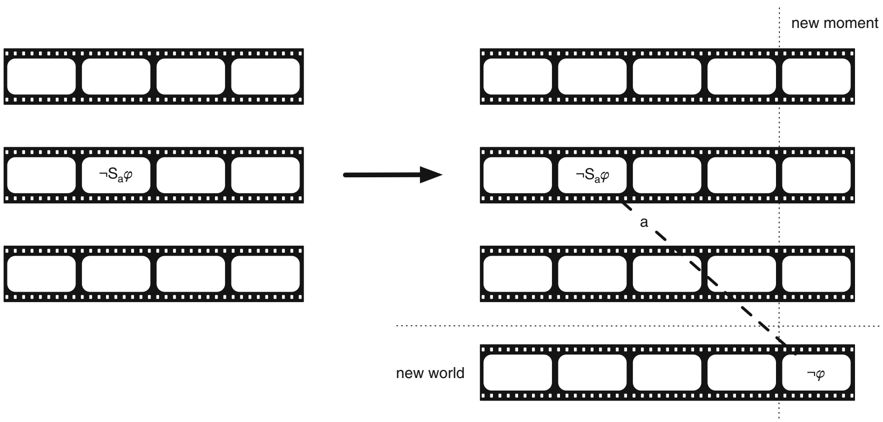

Fig. 5 Frame extension

6.2 Current Contribution

In this article, in addition to modalities S and O, we consider the temporal modality N. The frame extension construction as described above cannot be carried out if all three modalities (S, O, and N) are present in the language. To see what the problem is, let us first note that in order to be able to prove Claim 17 by induction, in addition to Eq. 8 for modality S we also should guarantee similar statements for modalities O and N:

$$\mathsf{O}_a\varphi \in X_{wt} \text{ iff } \varphi \in X_{w't} \text{ for each } w', t' \text{ such that } (w,t) \sim_a (w',t'), \tag{9}$$

$$\mathsf{N}\varphi \in X_{wt} \text{ iff } \varphi \in X_{w,t+1}. \tag{10}$$

The two statements above match items 6 and 4 of Definition 2 in the same way as Eq. 8 matches item 5 of the same definition. To guarantee ($\Rightarrow$) parts of these two statements, we added items 3(a) and 3(c) to Definition 6. This is similar to how we added item 3(b) to Definition 6 in order to guarantee ($\Rightarrow$) part of Eq. 8.

Let us now consider the situation depicted in Fig. 6. Here, set X_{w_1,t_1} contains formula $\neg\mathsf{S}_a\neg\mathsf{O}_a\varphi$. In such a situation, as explained in Section 6.1, the matrix construction might need to add a new world w_2 and new moment t_2 such that $(w_1,t_1) \sim_a (w_2,t_2)$ and $\mathsf{O}_a\varphi \in X_{w_2,t_2}$. Then, $\varphi \in X_{w_1,t_2}$ by item 3(c) of Definition 6, see Fig. 6. Finally, let us suppose that the original frame is such that $\mathsf{N}\neg\varphi \in X_{w_1,t_2-1}$. By item 3(a) of Definition 6, this would require that $\neg\varphi \in X_{w_1,t_2-1}$ in the extended frame. Since the same maximal consistent set X_{w_1,t_2} cannot contain both formulae φ and $\neg\varphi$, there is no consistent way to add the new column t_2 to the frame. Thus, the original matrix method from [26] does not work in language Φ which contains all three modalities: S, O, and N.

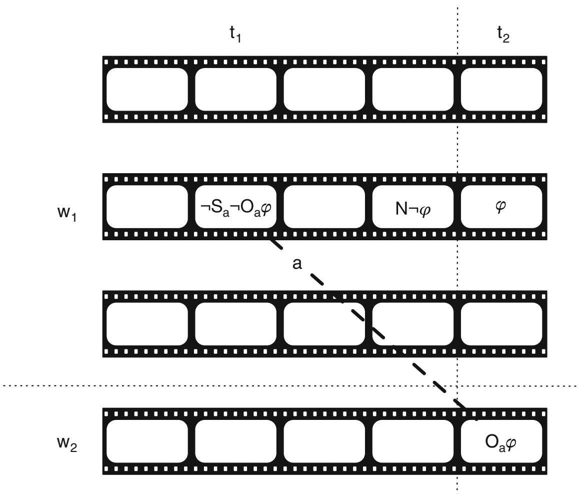

Fig. 6 Towards the explanation why the countdown timer is introduced in the construction when the language contains modalities S, O, and N

In this article, we solve the problem described above by limiting the applicability of item 3(a) of Definition 6. Informally, we assume that the model is equipped with a *countdown timer*. At the moment $t = 0$, the timer is set to some initial non-negative integer value and this value is decremented by 1 at each moment of time. Once the timer reaches value 0, the timer is reset to some new non-negative integer value, and so on. The timer is specific to the model, not to a world. In other words, at any given moment, the value of time is the same in all possible worlds. The values to which the time is reset might change from reset to reset and from model to model.

We use the timer to state Claim 17, our "truth lemma", in a slightly weaker form. Namely, we only require the statement "$w, t \Vdash \psi$ iff $\psi \in X_{wt}$" to be true for formulae ψ whose complexity $|\psi|_{\mathsf{N}}$ is no more than the value of the timer at moment t. The complexity $|\psi|_{\mathsf{N}}$ is defined as the number of times modality N occurs in formula ψ. A similar constraint also applies to formula φ in Eqs. 8, 9, and 10. Item 3(a) of Definition 6 has a related adjustment expressed in the words "belongs to the same block". To see why these changes solve the problem, recall that the issue with Fig. 6 is that formula $\mathsf{N}\neg\varphi$ forces $\neg\varphi$ to be added to the set X_{w_1,t_2}, which should have formula φ. To avoid this conflict, the type of extension shown in Fig. 5 is done only when the value of the countdown timer reaches zero. In other words, in Fig. 6, the value of the timer at moment $t_2 - 1$ is guaranteed to be zero. In this case, because the complexity of formula $\mathsf{N}\neg\varphi$ is more than zero, given the formula complexity constraint in Claim 17, we do not have to guarantee that this formula is true in world w_1 of the canonical model at moment t_2. As a result, we do not have to add $\neg\varphi$ to the set X_{w_1,t_2}. One might wonder what to do if the value of the counter in the right-most column is above zero. In this case, we use Lemma 23 to add additional columns to the matrix until the counter reaches zero, and only after that, apply the type of extension shown in Fig. 5.

6.3 Augmented Matrices

In the rest of this section, we give the formal proof of completeness for our logical system. This proof blends the original matrix technique described in Section 6.1 with the countdown timer introduced in Section 6.2. At the core of this blend is the notion of an *augmented matrix*. An example of an augmented matrix is depicted in Fig. 7.

Informally, an augmented matrix consists of a matrix (X_{wt}) in which each column is labelled with a nonnegative integer number, representing the value of a "timer" at the current column. It is assumed that *if a column is labelled with a* ***positive*** *number r, then the next column is labelled with* $r - 1$. In other words, the timer counts down until zero. Upon reaching zero, the timer is reset to an arbitrary nonnegative number.

Fig. 7 An example of an augmented matrix

$$\left[\begin{array}{ccc|c|cc|ccc}
2 & 1 & 0 & 0 & 1 & 0 & 4 & 3 & 2 \\
\hline
X_{00} & X_{01} & X_{02} & X_{03} & X_{04} & X_{05} & X_{06} & X_{07} & X_{08} \\
X_{10} & X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & X_{17} & X_{18} \\
X_{20} & X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} & X_{28}
\end{array}\right].$$

We have chosen to use variables w and t to represent the coordinates of a position in the matrix because later, in the construction of the canonical model, the rows of the matrix will correspond to possible worlds and columns to moments in time.

The augmented matrix in Fig. 7 has the size of 3 by 9 (not counting the row with integer labels). In this article, we consider augmented matrices of finite and infinite sizes. Thus, formally, we define such matrices using ordinals.

Definition 5 For any two ordinals $\alpha, \beta \leq \omega$, an augmented matrix of size α by β is a pair (X, r), such that

1. X is an arbitrary function on a Cartesian product $\alpha \times \beta$,
2. r is a "timer" function from ordinal β to nonnegative integer numbers such that, for each $t < \beta$, if $r(t) > 0$ and $t + 1 < \beta$, then $r(t + 1) = r(t) - 1$.

Note that $\alpha = 3$ and $\beta = 9$ for the augmented matrix depicted in Fig. 7. To be consistent with our matrix intuition, we write X_{wt} instead of $X(w, t)$ and r_t instead of $r(t)$.

We use the timer to divide columns ("time") into *blocks*. A new block starts with each reset of the timer. We denote the timer by r because it shows how many moments remain before the beginning of the next block. In Fig. 7, blocks are separated by vertical lines. Note that all blocks, with the possible exception of the last one, end with the timer being equal to 0, and all other values of the time inside the block are positive. We capture the second part of the last statement in the following lemma.

Lemma 21 *If f and ℓ are the first and the last column of a block, then $r_t > 0$ for all columns t such that $f \leq t < \ell$.*

6.4 Frames

In this subsection, we define frames and prove their basic properties. Informally, a frame combines an augmented matrix with a set of equivalence relations.

Definition 6 For any two ordinals $\alpha, \beta \leq \omega$, a frame is triple $(X, r, \sim)$ where

1. (X, r) is an augmented matrix of size α by β,
2. $\sim_a$ is an equivalence relation on $\alpha \times \beta$ for each agent $a \in \mathcal{A}$ such that
   (a) if $(w, t) \sim_a (u, s)$, then $r_t = r_s$,
   (b) if $(w, t) \sim_a (u, s)$ and columns t, s belong to the same block, then $(w, t) = (u, s)$,
3. X_{wt} is a maximal consistent set of formulae such that
   (a) if $\mathsf{N}\varphi \in X_{wt}$ and columns t and $t+1$ belong to the same block, then $\varphi \in X_{w,t+1}$,
   (b) if $(w, t) \sim_a (u, s)$, then $\mathsf{S}_a\varphi \in X_{wt}$ iff $\mathsf{S}_a\varphi \in X_{us}$,
   (c) if $\mathsf{O}_a\varphi \in X_{wt}$ and $(w, t) \sim_a (u, s)$, then $\varphi \in X_{ut}$.

Informally, item 2(b) above says that two different cells in the same block cannot be equivalent. If $\alpha, \beta < \omega$, then we say that the frame is *finite*.

Lemma 22 *For any frame $(X, r, \sim)$ of size α by β, any $u < \alpha$ and any $t_1, t_2 < \beta$ if $t_1 \le t_2$, $\varphi \in X_{wt_2}$, and columns t_1, t_2 belong to the same block, then $\mathsf{N}^{t_2-t_1}\varphi \in X_{wt_1}$.*

Proof We prove the statement of the lemma by induction on $t_2 - t_1$. In the base case, $t_2 = t_1$. Then, statement $\mathsf{N}^{t_2-t_1}\varphi \in X_{wt_1}$ follows from the assumption $\varphi \in X_{wt_2}$ of the lemma.

In the induction case, $t_2 > t_1$. Thus, columns t_1 and $t_1 + 1$ must belong to the same block due to the assumption of the lemma that columns t_1 and t_2 belong to the same block.

Towards the proof by contradiction, suppose that $\mathsf{N}^{t_2-t_1}\varphi \notin X_{wt_1}$. Thus, $\neg\mathsf{N}^{t_2-t_1}\varphi \in X_{wt_1}$ because X_{wt_1} is a maximal consistent set. Hence, by Lemma 19 and the Modus Ponens rule, $X_{wt_1} \vdash \mathsf{N}^{t_2-t_1}\neg\varphi$. Then, $\mathsf{N}^{t_2-t_1}\neg\varphi \in X_{wt_1}$ again because X_{wt_1} is a maximal consistent set. Note that, as we have observed earlier, columns t_1 and $t_1 + 1$ must belong to the same block. Thus, $\mathsf{N}^{t_2-t_1-1}\neg\varphi \in X_{w,t_1+1}$ by item 3(a) of Definition 6. Hence, $\neg\varphi \in X_{wt_2}$ by the induction hypothesis. Therefore, $\varphi \notin X_{wt_2}$ because X_{wt_2} is a maximal consistent set. $\square$

Recall from our informal discussion in Section 6.1 that the matrix method consists of extending the frame infinitely many times. At the limit, this method produces what we call a "complete" frame:

Definition 7 A frame $(X, r, \sim)$ of size α by β is ***complete*** when

1. β is the ordinal ω,
2. if $\mathsf{S}_a\varphi \notin X_{wt}$, then there are u, s such that $(w, t) \sim_a (u, s)$ and $\varphi \notin X_{us}$,
3. if $\mathsf{O}_a\varphi \notin X_{wt}$, then there are u, s such that $(w, t) \sim_a (u, s)$ and $\varphi \notin X_{ut}$.

6.5 Frame Extension

In this subsection, we formally define frame extensions informally captured by Fig. 5.

Definition 8 For any frames $F = (X, r, \sim)$ and $F' = (X', r', \sim')$ of sizes α by β and α' by β', respectively, let $F \sqsubseteq F'$ if

1. $\alpha \le \alpha'$,
2. $\beta \le \beta'$,
3. $(w, t) \sim_a (u, s)$ iff $(w, t) \sim'_a (u, s)$ for each $w, u < \alpha$, each $t, s < \beta$, and each $a \in \mathcal{A}$,
4. $r_t = r'_t$ for each $t < \beta$,
5. $X_{wt} = X'_{wt}$ for each $w < \alpha$ and $t < \beta$.

We read $F \sqsubseteq F'$ as frame F' is an *extension* of frame F. The next lemma can be used multiple times to eventually bring the value of the countdown timer $r_{\beta-1}$ to zero. As we discussed in Section 6.2, we need to bring this value to zero before doing an extension shown in Fig. 5.

Lemma 23 *Any finite frame $(X, r, \sim)$ of size α by β such that $r_{\beta-1} > 0$, can be extended to a frame $(X', r', \sim')$ of size α by $\beta + 1$.*

Proof For each integer $w < \alpha$, consider the set of formulae

$$Y_w^- = \{\varphi \mid \mathsf{N}\varphi \in X_{w,\beta-1}\}. \tag{11}$$

Claim 3 Set Y_w^- is consistent for each $w < \alpha$.

Proof-of-claim Suppose that the set Y_w^- is not consistent. Then there are formulae

$$\mathsf{N}\varphi_1, \dots, \mathsf{N}\varphi_n \in X_{w,\beta-1} \tag{12}$$

such that $\varphi_1, \dots, \varphi_n \vdash \bot$. Thus, $\mathsf{N}\varphi_1, \dots, \mathsf{N}\varphi_n \vdash \mathsf{N}\bot$ by Lemma 15. Hence, by Eq. 12,

$$X_{w,\beta-1} \vdash \mathsf{N}\bot. \tag{13}$$

At the same time, the formula $\neg\bot$ is a tautology. Thus, $\vdash \mathsf{N}\neg\bot$ by the Necessitation inference rule. Then, $\vdash \neg\mathsf{N}\bot$ by the Functionality axiom and propositional reasoning. Thus, set $X_{w,\beta-1}$ is inconsistent due to Eq. 13. $\square$

By Lemma 12, consistent sets $Y_0^-, \dots, Y_{\alpha-1}^-$ can be extended to maximal consistent sets $Y_0, \dots, Y_{\alpha-1}$, respectively.

Let (X', r') be the augmented matrix

$$\left(\begin{array}{cccc} r_0 & \dots & r_{\beta-1} & r_{\beta-1}-1 \\ \hline X_{00} & \dots & X_{0,\beta-1} & Y_0 \\ X_{10} & \dots & X_{1,\beta-1} & Y_1 \\ \dots & \dots & \dots & \dots \\ X_{\alpha-1,0} & \dots & X_{\alpha-1,\beta-1} & Y_{\alpha-1} \end{array}\right).$$

In other words, let

$$r'_t = \begin{cases} r_t, & \text{if } t < \beta, \\ r_{\beta-1} - 1, & \text{if } t = \beta, \end{cases} \tag{14}$$

for each $t < \beta + 1$. And let

$$X'_{wt} = \begin{cases} X_{wt}, & \text{if } t < \beta, \\ Y_w, & \text{if } t = \beta, \end{cases} \tag{15}$$

for each $w < \alpha$ and $t < \beta+1$. Note that (X', r') is an augmented matrix by Definition 5 and the assumption of the lemma that $r_{\beta-1} > 0$.

To finish the definition of the frame $(X', r', \sim')$, we define the relation $\sim'_a$ on the set $\alpha \times (\beta + 1)$ as follows:

$$(w, t) \sim'_a (u, s) \text{ iff either } (w, t) \sim_a (u, s) \text{ or } (w, t) = (u, s) \tag{16}$$

for each $a \in \mathcal{A}$, each $w, u < \alpha$, and each $t, s < \beta + 1$. In other words, the relation $\sim'_a$ is a reflexive closure of relation $\sim_a$ on the set $\alpha \times (\beta + 1)$. Note that $\sim'_a$ is an equivalence relation on the set $\alpha \times (\beta + 1)$.

This ends the definition of the triple $(X', r', \sim')$. Next, we use Definition 6 to show that this triple is a frame of size α by $\beta+1$. We start with condition 2(a) of Definition 6.

Claim 4 If $(w, t) \sim'_a (u, s)$, then $r'_t = r'_s$.

Proof-of-claim By Eq. 16, the assumption $(w, t) \sim'_a (u, s)$ implies that either $(w, t) \sim_a (u, s)$ or $(w, t) = (u, s)$. We consider the following two cases separately:

Case 1: $(w, t) \sim_a (u, s)$. Then, $r_t = r_s$ by item 2(a) of Definition 6 and the assumption of the lemma that $(X, r, \sim)$ is a frame. Thus, $r'_t = r_t = r_s = r'_s$ by Eq. 14.
Case 2: $(w, t) = (u, s)$. Thus, $t = s$. Therefore, $r'_t = r'_s$.

□

Next, let us make the following observation that follows from Eq. 14:

Claim 5 For any $s, t < \beta$, columns s and t belong to the same block of the augmented matrix (X, r) iff they belong to the same block of the augmented matrix (X', r').

We are now ready to verify condition 2(b) of Definition 6.

Claim 6 If $(w, t) \sim'_a (u, s)$ and columns t and s belong to the same block of the augmented matrix (X', r'), then $(w, t) = (u, s)$.

Proof-of-claim By Eq. 16, the assumption $(w, t) \sim'_a (u, s)$ implies that either $(w, t) \sim_a (u, s)$ or $(w, t) = (u, s)$. To finish the proof of the claim, it suffices to consider the case $(w, t) \sim_a (u, s)$. Then, $t, s < \beta$. Thus, by Claim 5, the assumption of the claim that columns t and s belong to the same block of the augmented matrix (X', r') implies that these columns belong to the same block of the augmented matrix (X, r). Thus, $(w, t) = (u, s)$ by item 2(b) of Definition 6 and the assumption $(w, t) \sim_a (u, s)$. □

Next, we verify condition 3(a) of Definition 6.

Claim 7 If $r'_t > 0$, $t + 1 < \beta + 1$, and $\mathsf{N}\varphi \in X'_{wt}$, then $\varphi \in X'_{w,t+1}$.

Proof-of-claim We consider the following two cases separately:

Case 1: $t+1 < \beta$. Then, the assumptions $r'_t > 0$ and $\mathsf{N}\varphi \in X'_{wt}$ of the claim imply that $r_t > 0$ and $\mathsf{N}\varphi \in X_{wt}$ by Eqs. 14 and 15, respectively. Thus, $\varphi \in X_{w,t+1}$ by item 3(a) of Definition 6 and the assumption $t + 1 < \beta$ of the case. Therefore, $\varphi \in X'_{w,t+1}$ by Eq. 15 and the assumption $t + 1 < \beta$.
Case 2: $t + 1 = \beta$. Then, the assumption $\mathsf{N}\varphi \in X'_{wt}$ implies that $\mathsf{N}\varphi \in X_{wt}$ by Eq. 15. Then, $\mathsf{N}\varphi \in X_{w,\beta-1}$ by the assumption $t + 1 = \beta$ of the case. Hence, $\varphi \in Y^-_w$ by Eq. 11. Thus, $\varphi \in Y_w$ because $Y^-_w \subseteq Y_w$. Then, $\varphi \in X'_{w\beta}$ by Eq. 15. Therefore, $\varphi \in X'_{w,t+1}$ by the assumption $t + 1 = \beta$ of the case. □

Then, we verify condition 3(b) of Definition 6.

Claim 8 If $\mathsf{S}_a\varphi \in X'_{wt}$ and $(w, t) \sim'_a (u, s)$, then $\varphi \in X'_{us}$.

Proof-of-claim By Eq. 16, the assumption $(w,t) \sim'_a (u,s)$ implies that either $(w,t) \sim_a (u,s)$ or $(w,t) = (u,s)$. We consider these two cases separately:

Case 1: $(w,t) \sim_a (u,s)$. Thus, since relation $\sim_a$ is only defined on $\alpha \times \beta$,

$$t, s < \beta. \tag{17}$$

Hence, $X'_{wt} = X_{wt}$ by Eq. 15. Then, $\mathsf{S}_a\varphi \in X_{wt}$ by the assumption $\mathsf{S}_a\varphi \in X'_{wt}$ of the claim. Thus, $\varphi \in X_{us}$ by item 3(b) of Definition 6 and the assumption $(w,t) \sim_a (u,s)$ of the case. Note that $X_{us} = X'_{us}$ by Eq. 15 and Eq. 17. Therefore, $\varphi \in X'_{us}$.

Case 2: $(w,t) = (u,s)$. Then, $X'_{us} \vdash \varphi$ by the assumption $\mathsf{S}_a\varphi \in X'_{wt}$ of the claim, the Truth axiom, and by the Modus Ponens inference rule. Therefore, $\varphi \in X'_{us}$ as X'_{us} is a maximal consistent set. □

The proof of the next claim is similar to the proof of the previous claim except that it uses item 3(c) of Definition 6 instead of item 3(b).

Claim 9 If $\mathsf{O}_a\varphi \in X'_{wt}$ and $(w,t) \sim'_a (u,s)$, then $\varphi \in X'_{ut}$.

By Definition 8, frame $(X', r', \sim')$ is an extension of frame $(X, r, \sim)$. This concludes the proof of the lemma. □

The next lemma captures the frame extension illustrated in Fig. 5. Note that the same lemma handles modalities S and O simultaneously.

Lemma 24 *For any finite frame $(X, r, \sim)$ of size α by β, where $r_{\beta-1} = 0$, any integers $w < \alpha$, $t < \beta$, any agent $b \in \mathcal{A}$, and any formulae $\mathsf{S}_b\varphi \notin X_{wt}$, $\mathsf{O}_b\psi \notin X_{wt}$, there is an extension $(X', r', \sim')$ of size $\alpha + 1$ by $\beta + 1$ of frame $(X, r, \sim)$ such that $(w,t) \sim'_b (\alpha, \beta)$, $\varphi \notin X'_{\alpha\beta}$, and $\psi \notin X'_{\alpha t}$.*

Proof Consider the set of formulae

$$Y^- = \{\neg\varphi\} \cup \{\chi \mid \mathsf{S}_b\chi \in X_{wt}\}. \tag{18}$$

Claim 10 Set Y^- is consistent.

Proof-of-claim Suppose that Y^- is not consistent. Then, there are formulae

$$\mathsf{S}_b\chi_1, \ldots, \mathsf{S}_b\chi_n \in X_{wt} \tag{19}$$

such that $\chi_1, \ldots, \chi_n \vdash \varphi$. Thus, $\mathsf{S}_b\chi_1, \ldots, \mathsf{S}_b\chi_n \vdash \mathsf{S}_b\varphi$ by Lemma 15. Hence, $X_{wt} \vdash \mathsf{S}_b\varphi$ by Eq. 19. Therefore, $\mathsf{S}_b\varphi \in X_{wt}$ because X_{wt} is a maximal consistent set, which contradicts the assumption $\mathsf{S}_b\varphi \notin X_{wt}$ of the lemma. □

By Lemma 12, the set Y^- can be extended to a maximal consistent set Y. Let $f < \beta$ and $\ell < \beta$ be the first and the last column of the block containing column t. Consider the set

$$Z_0^- = \{\mathsf{N}^{t-f}\neg\psi\} \cup \{\mathsf{N}^{t-f}\chi \mid \mathsf{N}^{t-f}\mathsf{O}_b\chi \in X_{wf}\}. \tag{20}$$

Claim 11 Set Z_0^- is consistent.

Proof-of-claim Suppose the opposite. Thus, there are formulae

$$\mathsf{N}^{t-f}\mathsf{O}_b\chi_1, \ldots, \mathsf{N}^{t-f}\mathsf{O}_b\chi_n \in X_{wf} \tag{21}$$

such that

$$\mathsf{N}^{t-f}\chi_1, \ldots, \mathsf{N}^{t-f}\chi_n \vdash \neg\mathsf{N}^{t-f}\neg\psi.$$

Then, by Lemma 19 applied countrapositively,

$$\mathsf{N}^{t-f}\chi_1, \ldots, \mathsf{N}^{t-f}\chi_n \vdash \neg\neg\mathsf{N}^{t-f}\psi.$$

Thus, by the laws of propositional reasoning,

$$\mathsf{N}^{t-f}\chi_1, \ldots, \mathsf{N}^{t-f}\chi_n \vdash \mathsf{N}^{t-f}\psi.$$

Hence, by Lemma 18,

$$\mathsf{N}^{t-f}\mathsf{O}_b\chi_1, \ldots, \mathsf{N}^{t-f}\mathsf{O}_b\chi_n \vdash \mathsf{N}^{t-f}\mathsf{O}_b\psi.$$

Then, by Eq. 21,

$$X_{wf} \vdash \mathsf{N}^{t-f}\mathsf{O}_b\psi.$$

Thus, $\mathsf{N}^{t-f}\mathsf{O}_b\psi \in X_{wf}$ because X_{wf} is a maximal consistent set of formulae. Recall that $(X, r, \sim)$ is a frame by the assumption of the lemma. Thus,

$$\mathsf{O}_b\psi \in X_{w,f+(t-f)}$$

by item 3(a) of Definition 6 (applied $t-f$ times) and because columns f and t belong to the same block. Therefore, $\mathsf{O}_b\psi \in X_{wt}$, which contradicts the assumption $\mathsf{O}_b\psi \notin X_{wt}$ of the lemma. $\square$

By Lemma 12, set Z_0^- can be extended to a maximal consistent set Z_0.

For each k such that $0 < k \leq \ell - f$, consider the set of formulae

$$Z_k = \{\chi \mid \mathsf{N}^k\chi \in Z_0\}. \tag{22}$$

Claim 12 Z_k is a maximal consistent set for each k, such that $0 < k \leq \ell - f$.

Proof-of-claim Let us first show that the set Z_k is consistent. Suppose the opposite. Thus, $\chi_1, \ldots, \chi_n \vdash \bot$ for some $\mathsf{N}^k\chi_1, \ldots, \mathsf{N}^k\chi_n \in Z_0$. Hence, $\mathsf{N}^k\chi_1, \ldots, \mathsf{N}^k\chi_n \vdash \mathsf{N}^k\bot$ by Lemma 15 applied k times. Then, $Z_0 \vdash \mathsf{N}^k\bot$ by the assumption $\mathsf{N}^k\chi_1, \ldots, \mathsf{N}^k\chi_n \in Z_0$. Then, $Z_0 \nvdash \neg\mathsf{N}^k\bot$ because set Z_0 is consistent, which contrudicts Lemma 20.

To prove the maximality, consider any formula χ such that $\chi \notin Z_k$. It suffices to show that $\neg\chi \in Z_k$. The assumption $\chi \notin Z_k$ implies that $\mathsf{N}^k\chi \notin Z_0$ by Eq. 22. Thus, $\neg\mathsf{N}^k\chi \in Z_0$ because Z_0 is a maximal consistent set. Hence, $Z_0 \vdash \mathsf{N}^k\neg\chi$ by Lemma 19 and the Modus Ponens inference rule. Then, $\mathsf{N}^k\neg\chi \in Z_0$ again because Z_0 is a maximal consistent set. Thus, $\neg\chi \in Z_k$ by Eq. 22. $\square$

Let integers $\{r'_s\}_{s<\beta+1}$ and matrix X'_{us} be defined by the following augmented matrix of the size $(\alpha+1)$ by $(\beta+1)$:

$$\left[\begin{array}{ccc|ccccc|ccc|c}
r_0 & \dots & r_{f-1} & r_f & \dots & r_t & \dots & r_\ell & r_{\ell+1} & \dots & r_{\beta-1} & r_t \\
\hline
X_{01} & \dots & X_{0,f-1} & X_{0f} & \dots & \dots & \dots & X_{0\ell} & X_{0,\ell+1} & \dots & X_{0,\beta-1} & Y \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
X_{w1} & \dots & X_{w,f-1} & X_{wf} & \dots & X_{wt} & \dots & X_{w\ell} & X_{w,\ell+1} & \dots & X_{w,\beta-1} & Y \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
X_{\alpha-1,1} & \dots & X_{\alpha-1,f-1} & X_{\alpha-1,f} & \dots & \dots & \dots & X_{\alpha-1,\ell} & X_{\alpha-1,\ell+1} & \dots & X_{\alpha-1,\beta-1} & Y \\
X_{w1} & \dots & X_{w,f-1} & Z_0 & \dots & \dots & \dots & Z_{\ell-f} & X_{w,\ell+1} & \dots & X_{w,\beta-1} & Y
\end{array}\right]$$

In other words, let

$$r'_s = \begin{cases} r_s, & \text{if } s < \beta, \\ r_t, & \text{if } s = \beta, \end{cases} \tag{23}$$

for each $s < \beta + 1$. And

$$X'_{us} = \begin{cases} X_{us}, & \text{if } u < \alpha \text{ and } s < \beta, \\ Z_{s-f}, & \text{if } u = \alpha \text{ and } f \leq s \leq \ell, \\ X_{ws}, & \text{if } u = \alpha \text{ and either } s < f \text{ or } \ell < s < \beta, \\ Y, & \text{if } s = \beta, \end{cases} \tag{24}$$

for each $u < \alpha + 1$ and $s < \beta + 1$. This ends the definition of the pair (X', r'). The following claim is true by Definition 5 and the assumptions of the lemma that $(X, r, \sim)$ is a frame and $r_{\beta-1} = 0$.

Claim 13 (X', r') is an augmented matrix.

Next, we define relation $\sim'_a$ on the set $(\alpha+1) \times (\beta+1)$ for an arbitrary agent $a \in \mathcal{A}$. If $a \neq b$, then the relation $\sim'_a$ is a reflexive closure of relation $\sim_a$ in set $(\alpha+1) \times (\beta+1)$. If $a = b$, then the relation $\sim'_a$ is a reflexive transitive symmetric closure of relation $\sim_a \cup \{((w,t),(\alpha,\beta))\}$ in set $(\alpha+1) \times (\beta+1)$. Note that relation $\sim'_a$ is an equivalence relation on the set $(\alpha+1) \times (\beta+1)$ for each agent $a \in \mathcal{A}$.

This ends the definition of the tuple $(X', r', \sim')$. Next, we use Definition 6 to show that this tuple is a frame of size $\alpha+1$ by $\beta+1$. By Claim 13, condition 1 of Definition 6 is satisfied. Note that conditions 2(a) and 2(b) of Definition 6 are satisfied by formula Eq. 23, the definition of the relation $\sim'$ and the assumption of the lemma that $(X, r, \sim)$ is a frame. Thus, it suffices to verify conditions 3(a), 3(b), and 3(c) of Definition 6. Due to Eqs. 23 and 24 and the assumption of the lemma that $(X, r, \sim)$ is a frame, to prove condition 3(a), it is enough to establish the following claim.

Claim 14 If $\mathsf{N}\chi \in Z_k$ and $k < \ell - f$, then $\chi \in Z_{k+1}$.

Proof-of-claim Suppose that $\chi \notin Z_{k+1}$. Then, $\mathsf{N}^{k+1}\chi \notin Z_0$ by Eq. 22. Hence, $\neg\mathsf{N}^{k+1}\chi \in Z_0$ because Z_0 is a maximal consistent set of formulae. Thus, $Z_0 \vdash \mathsf{N}^k\neg\mathsf{N}\chi$ by Lemma 19. Then, $\mathsf{N}^k\neg\mathsf{N}\chi \in Z_0$ again because Z_0 is a maximal consistent set of formulae. Hence, $\neg\mathsf{N}\chi \in Z_k$ by Eq. 22. Therefore, $\mathsf{N}\chi \notin Z_k$ because set Z_k is consistent. □

Due to the definition of the relation $\sim'$, Eq. 24, and the assumption of the lemma that $(X, r, \sim)$ is a frame, to prove condition 3(b), it is enough to establish the following claim.

Claim 15 $\mathsf{S}_b\chi \in X'_{wt}$ iff $\mathsf{S}_b\chi \in X'_{\alpha\beta}$.

Proof-of-claim First, observe that, by Eq. 24, it suffices to show that $\mathsf{S}_b\chi \in X_{wt}$ iff $\mathsf{S}_b\chi \in Y$.

$(\Rightarrow)$: Suppose that $\mathsf{S}_b\chi \in X_{wt}$. Thus, $X_{wt} \vdash \mathsf{S}_b\mathsf{S}_b\chi$ by Lemma 14 and the Modus Ponens inference rule. Hence, $\mathsf{S}_b\mathsf{S}_b\chi \in X_{wt}$ because X_{wt} is a maximal consistent set. Then, $\mathsf{S}_b\chi \in Y^-$ by Eq. 18. Thus, $\mathsf{S}_b\chi \in Y$.

$(\Leftarrow)$: Assume that $\mathsf{S}_b\chi \notin X_{wt}$. Then, $\neg\mathsf{S}_b\chi \in X_{wt}$ because because X_{wt} is a maximal consistent set. Thus, $X_{wt} \vdash \mathsf{S}_b\neg\mathsf{S}_b\chi$ by the Negative Introspection axiom and the Modus Ponens inference rule. Hence, $\mathsf{S}_b\neg\mathsf{S}_b\chi \in X_{wt}$ again because X_{wt} is a maximal consistent set. Then, $\neg\mathsf{S}_b\chi \in Y^-$ by Eq. 18. Thus, $\neg\mathsf{S}_b\chi \in Y$. Hence, $\mathsf{S}_b\chi \notin Y$ because set Y is consistent. □

The next claim verifies condition 3(c) of Definition 6.

Claim 16 If $\mathsf{O}_a\chi \in X'_{u_1s_1}$ and $(u_1, s_1) \sim'_a (u_2, s_2)$, then $\chi \in X'_{u_2s_1}$.

Proof-of-claim Due to the definition of the relation $\sim'$, Eq. 24, and the assumption of the lemma that $(X, r, \sim)$ is a frame, it is enough to only consider the following three cases:

Case 1: $s_1 = \beta$. Then, $X'_{u_1s_1} = Y = X'_{u_2s_1}$ by Eq. 24. Thus, $\mathsf{O}_a\chi \in X'_{u_2s_1}$ by the assumption $\mathsf{O}_a\chi \in X'_{u_1s_1}$ of the claim. Hence, $X'_{u_2s_1} \vdash \chi$ by the Truth axiom and the Modus Ponens rule. Thus, $\chi \in X'_{u_2s_1}$ because $X'_{u_2s_1}$ is a maximal consistent set.

Case 2: $u_1 = \alpha$ and $s_1 < \beta$. Then, $(u_1, s_1) = (u_2, s_2)$ by the assumption $(u_1, s_1) \sim'_a (u_2, s_2)$ of the claim and the definition of the relation $\sim'$. Hence, $u_1 = u_2$. Thus, $\mathsf{O}_a\chi \in X'_{u_2s_1}$ by the assumption $\mathsf{O}_a\chi \in X'_{u_1s_1}$ of the claim. Then, $X'_{u_2s_1} \vdash \chi$ by the Truth axiom and the Modus Ponens inference rule. Therefore, $\chi \in X'_{u_2s_1}$ because $X'_{u_2s_1}$ is a maximal consistent set.

Case 3: $a = b$, $(u_1, s_1) \sim_b (w, t)$, and $(u_2, s_2) = (\alpha, \beta)$. We further split this case into two subcases:

Case 3A: cells (u_1, s_1) and (w, t) belong to the same block. Thus, by the assumption $(u_1, s_1) \sim_b (w, t)$ of Case 3 and item 2(b) of Definition 6,

$$(u_1, s_1) = (w, t). \tag{25}$$

Hence, $\mathsf{O}_a\chi \in X'_{wt}$ by the assumption $\mathsf{O}_a\chi \in X'_{u_1s_1}$ of the claim.
Then, $\mathsf{O}_a\chi \in X_{wt}$ by Eq. 24 and the assumptions $w < \alpha$ and $t < \beta$ of the lemma.
Thus, $\mathsf{N}^{t-f}\mathsf{O}_a\chi \in X_{wf}$ by Lemma 22 and because f is the first column of the block containing column t. Hence, $\mathsf{N}^{t-f}\chi \in Z_0^-$ by Eq. 20. Then, $\mathsf{N}^{t-f}\chi \in Z_0$. Thus, $\chi \in Z_{t-f}$ by Eq. 22. Hence, $\chi \in X'_{\alpha t}$ by Eq. 24. Then, the assumption $(u_2, s_2) = (\alpha, \beta)$ of Case 3 implies $\chi \in X'_{u_2t}$. Therefore, $\chi \in X'_{u_2s_1}$ by Eq. 25.
Case 3B: (u_1, s_1) and (w, t) are in different blocks. Thus, by Eq. 24,

$$X'_{\alpha s_1} = X_{ws_1}. \tag{26}$$

At the same time, the assumption $(u_1, s_1) \sim_b (w, t)$ of Case 3 implies that $u_1 < \alpha$ and $s_1 < \beta$. Then, $\mathsf{O}_a\chi \in X_{u_1s_1}$ by the assumption $\mathsf{O}_a\chi \in X'_{u_1s_1}$ of the claim and Eq. 24. Hence, the assumption $(u_1, s_1) \sim_b (w, t)$, by item 3(c) of Definition 6, imply that $\chi \in X_{ws_1}$. Thus, $\chi \in X'_{\alpha s_1}$ by Eq. 26. Therefore, $\chi \in X'_{u_2s_1}$ by the assumption $(u_2, s_2) = (\alpha, \beta)$ of Case 3. □

To finish the proof of the lemma, first notice that $(w, t) \sim'_b (\alpha, \beta)$ by the definition of $\sim'$. Second, $\neg\varphi \in Y^-$ by Eq. 18. Then, $\neg\varphi \in Y$. Hence, $\neg\varphi \in X'_{\alpha\beta}$ by Eq. 24. Thus, $\varphi \notin X'_{\alpha\beta}$ because $X'_{\alpha\beta}$ is a consistent set. Third, $\mathsf{N}^{t-f}\neg\psi \in Z_0^-$ by Eq. 20. Thus, $\mathsf{N}^{t-f}\neg\psi \in Z_0$. Then, $\neg\psi \in Z_{t-f}$ by Eq. 22. Hence, $\neg\psi \in X'_{\alpha t}$ by Eq. 24. Therefore, $\psi \notin X'_{\alpha t}$ because set $X'_{\alpha t}$ is consistent. □

6.6 Complete Frames

In this subsection, we use Lemma 23 and Lemma 24 to extend an arbitrary frame to a complete frame.

Definition 9 For any frame $(X, r, \sim)$ of size α by β, any $w < \alpha$ and $t < \beta$, and any formula $\varphi \in \Phi$, the frame is (w, t, φ)-complete if the following conditions are satisfied:

1. If formula φ has the form $\mathsf{S}_a\psi$ and $\mathsf{S}_a\psi \notin X_{wt}$, then there are $u < \alpha$ and $s < \beta$ such that $(w, t) \sim_a (u, s)$ and $\psi \notin X_{us}$,
2. If formula φ has the form $\mathsf{O}_a\psi$ and $\mathsf{O}_a\psi \notin X_{wt}$, then there are $u < \alpha$ and $s < \beta$ such that $(w, t) \sim_a (u, s)$ and $\psi \notin X_{ut}$.

The next lemma follows from Definition 7 and Definition 9.

Lemma 25 *For any ordinal α, if frame $(X, r, \sim)$ of size α by ω is (w, t, φ)-complete for each $w < \alpha$, each $t < \omega$, and each formula $\varphi \in \Phi$, then the frame is complete.*

The next lemma follows from Definition 8 and Definition 9.

Lemma 26 *For any frame $(X, r, \sim)$ of size α by β, any $w < \alpha$, any $t < \beta$, and any formula $\varphi \in \Phi$, if frame $(X, r, \sim)$ is (w, t, φ)-complete, then any extension of this frame is (w, t, φ)-complete.*

Lemma 27 *Any finite frame $(X, r, \sim)$ of size α by β can be extended to a finite frame $(X', r', \sim')$ of size α' by β' such that $r'_{\beta'-1} = 0$.*

Proof By item 2 of Definition 5, the value of $r_{\beta-1}$ is a nonnegative integer number. We prove the statement of the lemma by induction on $r_{\beta-1}$.

Base Case: $r_{\beta-1} = 0$. Then, let $(X', r', \sim')$ be the frame $(X, r, \sim)$. Note that $(X, r, \sim) \sqsubseteq (X, r, \sim)$ by Definition 8.

Induction Step: let $r_{\beta-1} > 0$. Then, by Lemma 23, frame $(X, r, \sim)$ can be extended to a finite frame $(X', r', \sim')$ of size α by $\beta + 1$. Note that $r_{\beta-1} = r'_{\beta-1}$ by item 4 of Definition 8. At the same time, $r'_{\beta} = r'_{\beta-1} - 1 < r'_{\beta-1}$ by item 2 of Definition 5. Thus, $r'_{\beta} < r_{\beta-1}$. Hence, by the induction hypothesis, finite frame $(X', r', \sim')$ can be extended to a finite frame $(X'', r'', \sim'')$ of size α'' by β'' such that $r''_{\beta''-1} = 0$. Note that $(X, r, \sim) \sqsubseteq (X', r', \sim') \sqsubseteq (X'', r'', \sim'')$. Therefore, $(X, r, \sim) \sqsubseteq (X'', r'', \sim'')$ by Definition 8. □

For any two frames F and F' of sizes α by β and α' by β', respectively, we write $F \sqsubset F'$ if $F \sqsubseteq F'$ and $\beta < \beta'$.

Lemma 28 *If a finite frame F of size α by β is not (w, t, φ)-complete for some $w < \alpha$ and $t < \beta$, then frame F can be extended to a (w, t, φ)-complete finite frame F' such that $F \sqsubset F'$.*

Proof Let $F = (X, r, \sim)$. By Lemma 26 and Lemma 27, it suffices to consider the case when $r_{\beta-1} = 0$.

The assumption of the lemma that frame F is not (w, t, φ)-complete, by Definition 9, implies that formula φ must either have the form $\mathsf{S}_a\psi$ or the form $\mathsf{O}_a\psi$. We consider these two cases separately.

First, suppose that φ has the form $\mathsf{S}_a\psi$. Then, the assumption of the lemma that frame F is not (w, t, φ)-complete, by Definition 9, implies $\mathsf{S}_a\psi \notin X_{wt}$. At the same time, $\vdash \mathsf{O}_a\bot \to \bot$ by the Truth axiom. Hence, $\mathsf{O}_a\bot \notin X_{wt}$ because set X_{wt} is consistent. The statements $\mathsf{S}_a\psi \notin X_{wt}$ and $\mathsf{O}_a\bot \notin X_{wt}$, by Lemma 24, imply that there is an extension $F' = (X', r', \sim')$ of size $\alpha + 1$ by $\beta + 1$ of frame F such that $(w, t) \sim'_b (\alpha, \beta)$ and $\psi \notin X'_{\alpha\beta}$. Note that frame F' is (w, t, φ)-complete by Definition 9 and the assumption of the case that formula φ has the form $\mathsf{S}_a\psi$.

The case when formula φ has the form $\mathsf{O}_a\psi$ is similar except that it uses the Truth axiom $\mathsf{S}_a\bot \to \bot$ instead of $\mathsf{O}_a\bot \to \bot$. The proof is still based on Lemma 24. □

Lemma 29 *Any finite frame F of size $\alpha > 0$ by $\beta > 0$ can be extended to finite frame F' such that $F \sqsubset F'$.*

Proof Suppose that frame $F = (X, r, \sim)$ has the size α by β. We consider the following two cases separately:

Case I: $r_{\beta-1} \neq 0$. Then, by Lemma 27 there exists a finite frame $F' = (X', r', \sim')$ of a size α' by β' such that $F \sqsubseteq F'$ and

$$r'_{\beta'-1} = 0. \tag{27}$$

Note that $r'_{\beta-1} = r_{\beta-1}$ by item 4 of Definition 8 and the statement $F \sqsubseteq F'$. Hence, $r'_{\beta-1} \neq 0$ by the assumption $r_{\beta-1} \neq 0$ of the case. Thus, $\beta \neq \beta'$ by Eq. 27. Therefore, $F \sqsubset F'$ by statement $F \sqsubseteq F'$ and the definition of relation $\sqsubset$.

Case II: $r_{\beta-1} = 0$. Recall that the set of agents $\mathcal{A}$ is nonempty. Let $a \in \mathcal{A}$ be any element of this set. Then, the formulae $\mathsf{S}_a\bot \to \bot$ and $\mathsf{O}_a\bot \to \bot$ are instances of the Truth axiom. Note that $0 < \alpha$ and $0 < \beta$ by the assumption of the lemma. Hence, $\mathsf{S}_a\bot \notin X_{00}$ and $\mathsf{O}_0\bot \notin X_{00}$ because set X_{00} is consistent. Thus, by Lemma 24, there is a frame F' of size $\alpha + 1$ by $\beta + 1$ such that $F \sqsubseteq F'$. Then, $F \sqsubset F'$ because frame F has size α by β. □

For any family of triples of set $T_i = (X_i, Y_i, Z_i)$, let $\bigcup_i T_i$ be the triple $(\cup_i X_i, \cup_i Y_i, \cup_i Z_i)$. Note that any frame $F = (X, r, \sim)$ is a triple consisting of a function X from a Cartesian product of two ordinals into maximal consistent sets, a function r from an ordinal into nonnegative integer numbers, and a relation $\sim$. Functions are formally defined as sets of pairs (binary relations). Although we previously discussed $\sim$ as a family of binary relations $\{\sim_a\}_{a\in\mathcal{A}}$ on set $W \times \omega$, we can also think about $\sim$ as a *ternary* relation. In this case, $\sim$ is a subset of the set $(W\times\omega)\times\mathcal{A}\times(W\times\omega)$. Then, any frame F is a triple of sets. Thus, one can consider a union $\bigcup_i F_i$ of a family of frames.

Lemma 30 *For any infinite chain of finite frames $F_0 \sqsubset F_1 \sqsubset \dots$, triple $\bigcup_i F_i$ is a frame of size α by ω for some ordinal $\alpha \le \omega$ and $F_0 \sqsubset \bigcup_i F_i$.*

Lemma 31 *Any finite frame F can be extended to a complete frame.*

Proof Let $(w_1, t_1, \varphi_1), \dots, (w_n, t_n, \varphi_n)$ be any enumeration of all triples (w_i, t_i, φ_i) such that w_i and t_i are finite ordinals and φ_i is a formula in language Φ. We define a chain of finite frames $F_0 \sqsubset F_1 \sqsubset \dots$. The chain is defined as follows:

1. $F_0 = F$,
2. Suppose that finite chain $F_0 \sqsubset F_1 \sqsubset \cdots \sqsubset F_n$ is already defined and the size of frame F_n is α by β. To define frame F_{n+1} consider the following two cases separately:

   (a) If there is at least one i such that $w_i < \alpha, t_i < \beta$ and frame F_n is not (w_i, t_i, φ_i)-complete, then let $i_{\min}$ be the minimal such i. By Lemma 28, frame F_n can be extended to a $(w_{i_{\min}}, t_{i_{\min}}, \varphi_{i_{\min}})$-complete frame F_{n+1} such that $F_n \sqsubset F_{n+1}$,

   (b) If there is no i such that $w_i < \alpha$, $t_i < \beta$ and frame F_n is not (w_i, t_i, φ_i)-complete, then, by Lemma 29, frame F_n can be extended to frame F_{n+1} such that $F_n \sqsubset F_{n+1}$.

Let $F' = \bigcup_i F_i$. Note that frame F' is an extension of frame $F_0 = F$ by Lemma 30. Also, by the same lemma, the size of frame F' is α by ω for some ordinal $\alpha \le \omega$. Hence, by Lemma 25, to show that frame F' is complete, it suffices to prove that F' is (w, t, φ)-complete for each $w < \alpha$, each $t < \omega$, and each formula $\varphi \in \Phi$. The latter is guaranteed by the construction of the chain $F_0 \sqsubset F_1 \sqsubset \dots$. □

6.7 Final Steps

In this section, we finish the proof of the completeness theorem. As we discussed earlier, Claim 17 below is our version of the "truth lemma".

For any formula $\varphi \in \Phi$, by $|\varphi|_{\mathsf{N}}$ we denote the maximal depth of nestedness of modality N in formula φ. For example, $|p \vee \mathsf{NS}_a p|_{\mathsf{N}} = 1$ and $|\mathsf{N}(\mathsf{N}p \vee \mathsf{NN}p)|_{\mathsf{N}} = 3$.

Theorem 5 *[completeness] If $\nvdash \varphi$, then there is a world $w \in W$ of an epistemic temporal model $(W, \sim, \pi)$ such that $w, 0 \nVdash \varphi$.*

Proof Let X_{00} be any maximal consistent set containing formula $\neg\varphi$. Such a set exists by Lemma 12 and the assumption $\nvdash \varphi$ of the theorem. Consider the augmented matrix $(X^-, r^-) = \begin{bmatrix} |\varphi|_{\mathsf{N}} \\ \hline X_{00} \end{bmatrix}$ of size 1 by 1. That is, the only cell of this matrix contains the maximal consistent set X^-_{00} and the timer function r^- is such that $r^-(0) = |\varphi|_{\mathsf{N}}$. The pair (X^-, r^-) is an augmented matrix under Definition 5. Define equivalence relation $\sim^-_a$ on set 1×1 to be such that $(0, 0) \sim^-_a (0, 0)$ for each agent $a \in \mathcal{A}$. By Definition 6, to show that the triple $(X^-, r^-, \sim^-)$ is a frame of size 1 by 1, it suffices to prove that if $\mathsf{O}_a\psi \in X_{00}$, then $\psi \in X_{00}$ for each formula $\psi \in \Phi$ and each agent $a \in \mathcal{A}$. The latter is true by the Truth axiom and because X_{00} is a maximal consistent set. By Lemma 31, this frame can be extended to a complete frame $(X, r, \sim)$ of size α by ω.

Let $W = \alpha$ and

$$\pi(p) = \{(w, t) \in \alpha \times \omega \mid p \in X_{wt}\} \tag{28}$$

for any propositional variable p. By Definition 1, the triple $(W, \sim, \pi)$ is an epistemic temporal model. $\square$

Claim 17 [truth lemma] $w, t \Vdash \psi$ iff $\psi \in X_{wt}$ for each formula $\psi \in \Phi$, each $w < \alpha$, and each $t < \omega$ such that $|\psi|_{\mathsf{N}} \leq r(t)$.

Proof-of-claim We prove the statement of the lemma by induction on the structural complexity of formula ψ. If formula ψ is a propositional variable, then the statement of the lemma follows from Eq. 28 and item 1 of Definition 2. If formula ψ is a negation or a disjunction, then the statement follows from items 2 and 3 of Definition 2, the maximality and consistency of the set X_{wt}, and the induction hypothesis in the standard way.

Suppose that formula ψ has the form $\mathsf{N}\chi$. Note that $|\psi|_{\mathsf{N}} = |\mathsf{N}\chi|_{\mathsf{N}} = |\chi|_{\mathsf{N}} + 1 \geq 1$. Thus, $r(t) \geq 1$ by the assumption $|\psi|_{\mathsf{N}} \leq r(t)$ of the claim. Thus, columns t and $t+1$ belong to the same block of the frame $(X, r, \sim)$.
$(\Rightarrow)$: Assume that $\mathsf{N}\chi \notin X_{wt}$. Thus, $\neg\mathsf{N}\chi \in X_{wt}$ because X_{wt} is a maximal consistent set. Then, $X_{wt} \vdash \mathsf{N}\neg\chi$ by the Functionality axiom and propositional reasoning. Hence, $\mathsf{N}\neg\chi \in X_{wt}$ because X_{wt} is a maximal consistent set. Then, $\neg\chi \in X_{w,t+1}$ by item 3(a) of Definition 6 because columns t and $t+1$ belong to the same block of the frame $(X, r, \sim)$. Hence, $\chi \notin X_{w,t+1}$ because set $X_{w,t+1}$ is consistent. Thus, $w, t+1 \nVdash \chi$ by the induction hypothesis. Therefore, $w, t \nVdash \mathsf{N}\chi$ by item 4 of Definition 2.
$(\Leftarrow)$: Assume that $\mathsf{N}\chi \in X_{wt}$. Then, $\chi \in X_{w,t+1}$ by item 3(a) of Definition 6 because columns t and $t+1$ belong to the same block. Thus, $w, t+1 \Vdash \chi$ by the induction hypothesis. Therefore, $w, t \Vdash \mathsf{N}\chi$ by item 4 of Definition 2.

Suppose that formula ψ has the form $\mathsf{S}_a\chi$.
$(\Rightarrow)$: Assume that $\mathsf{S}_a\chi \notin X_{wt}$. Thus, by item 2 of Definition 7, there are $u < \alpha$ and $s < \omega$ such that

$$(w, t) \sim_a (u, s) \tag{29}$$

and

$$\chi \notin X_{us}. \tag{30}$$

Hence, $r(t) = r(s)$ by item 2(a) of Definition 6. Then, $|\chi|_\mathsf{N} = |\mathsf{S}_a\chi|_\mathsf{N} = |\psi|_\mathsf{N} \leq r(t) = r(s)$ by the assumption $|\psi|_\mathsf{N} \leq r(t)$ of the claim. Thus, $u, s \nVdash \chi$ by the induction hypothesis and Eq. 30. Therefore, $w, t \nVdash \mathsf{S}_a\chi$ by Eq. 29 and item 5 of Definition 2.

$(\Leftarrow)$: Assume that $\mathsf{S}_a\chi \in X_{wt}$. Consider any $u < \alpha$ and $s < \omega$ such that

$$(w, t) \sim_a (u, s). \tag{31}$$

By item 5 of Definition 2, it suffices to prove that $u, s \Vdash \chi$.

By item 3(b) of Definition 6, the assumption $\mathsf{S}_a\chi \in X_{wt}$ and Eq. 31 imply that $\mathsf{S}_a\chi \in X_{us}$. Then, $X_{us} \vdash \chi$ by the Truth axiom and the Modus Ponens inference rule. Hence, because set X_{us} is maximal consistent,

$$\chi \in X_{us}. \tag{32}$$

Note that $r(t) = r(s)$ by Eq. 31 and item 2(a) of Definition 6. Then, $|\chi|_\mathsf{N} = |\mathsf{S}_a\chi|_\mathsf{N} = |\psi|_\mathsf{N} \leq r(t) = r(s)$ by the assumption $|\psi|_\mathsf{N} \leq r(t)$ of the claim. Therefore, $u, s \Vdash \chi$ by Eq. 32 and the induction hypothesis.

Suppose that formula ψ has the form $\mathsf{O}_a\chi$.

$(\Rightarrow)$: Assume that $\mathsf{O}_a\chi \notin X_{wt}$. Thus, by item 3 of Definition 7, there are $u < \alpha$ and $s < \omega$ such that

$$(w, t) \sim_a (u, s) \tag{33}$$

and

$$\chi \notin X_{ut}. \tag{34}$$

At the same time, $|\chi|_\mathsf{N} = |\mathsf{O}_a\chi|_\mathsf{N} = |\psi|_\mathsf{N} \leq r(t)$ by the assumption $|\psi|_\mathsf{N} \leq r(t)$ of the claim. Thus, $u, t \nVdash \chi$ by Eq. 34 and the induction hypothesis. Therefore, $w, t \nVdash \mathsf{O}_a\chi$ by item 6 of Definition 2 and Eq. 33.

$(\Leftarrow)$: Assume that $\mathsf{O}_a\chi \in X_{wt}$. Consider any $u < \alpha$ and $s < \omega$ such that

$$(w, t) \sim_a (u, s). \tag{35}$$

By item 6 of Definition 2, it suffices to prove that $u, t \Vdash \chi$.

By item 3(c) of Definition 6, the assumption $\mathsf{O}_a\chi \in X_{wt}$ and Eq. 35 imply that

$$\chi \in X_{ut}. \tag{36}$$

Note that $|\chi|_\mathsf{N} = |\mathsf{O}_a\chi|_\mathsf{N} = |\psi|_\mathsf{N} \leq r(t)$ by the assumption $|\psi|_\mathsf{N} \leq r(t)$ of the claim. Therefore, $u, t \Vdash \chi$ by Eq. 36 and the induction hypothesis.

This completes the proof of the claim. □

To finish the proof of the completeness theorem, recall that $\neg\varphi \in X^-_{00}$ and $r^-(0) = |\varphi|_\mathsf{N}$. Thus, $\neg\varphi \in X_{00}$ and $r(0) = |\varphi|_\mathsf{N}$ because the frame $(X, r, \sim)$ is an extension of the frame $(X^-, r^-, \sim^-)$. Then, $\varphi \notin X_{00}$ because X_{00} is a maximal consistent set. Therefore, $0, 0 \nVdash \varphi$ by Claim 17 and because $r(0) = |\varphi|_{\mathsf{N}}$. ◻

Note that Theorem 5 does *not* claim a *strong* completeness of our logical system. To prove strong completeness, one would need to guarantee that Claim 17 holds for all formulae φ, not only those for which $|\psi|_{\mathsf{N}} \leq r(t)$. However, as we discuss in Section 6.2, this restriction on the size of φ is fundamental for the frame extension procedure. The question of the strong completeness of our logical system, as well as a related question of its compactness, remains open.

7 Non-derivability of Insertion Rule

In proof theory, there is a distinction between admissible and derivable rules of a logical system. A rule is *admissible* if an extension of the logical system by this rule does not yield any new theorems. For example, if $Taut$ is the set of all tautologies in propositional language, then the inference rule

$$\frac{\varphi \in Taut}{\varphi} \tag{37}$$

is admissible in any complete axiomatisation of propositional logic. A rule is *derivable* if it can be expressed as a *fixed* combination of the logical system's existing axioms and inference rules. For instance, inference rule $\dfrac{\mathsf{O}\varphi}{\mathsf{ON}\varphi}$ is derivable in the logical system L^- defined in Section 4. Indeed, here is how this inference rule can be expressed through a combination of the Truth axiom, the Modus Ponens inference rule, and the Necessitation inference rules for modalities N and O:

$$\cfrac{\cfrac{\cfrac{\mathsf{O}\varphi \qquad \mathsf{O}\varphi \to \varphi}{\varphi}}{\mathsf{N}\varphi}}{\mathsf{ON}\varphi}$$

At the same time, it is not clear how the inference rule specified by Eq. 37 can be expressed through a single *fixed* combination of inference rules in any of the standard axiomatisations of propositional logic. Thus, the inference rule from Eq. 37 is likely to be non-derivable in most logical systems for propositional logic. In general, any derivable rule of a logical system is admissible, but an admissible rule is not necessarily derivable.

In this section, we prove that the Insertion inference rule is not derivable in the logical system L^-. We do not know if it is admissible in L^-. To prove the non-derivability, we use a new technique introduced in [27]. At the core of the technique is the notion of a *theory* of an arbitrary epistemic temporal model M. By a theory of this model, we mean the set of all formulae that are satisfied at each moment in each world of the model. In addition, following the terminology pattern of Section 4, by a *theorem* of system L^- we mean any formula provable in L^-.

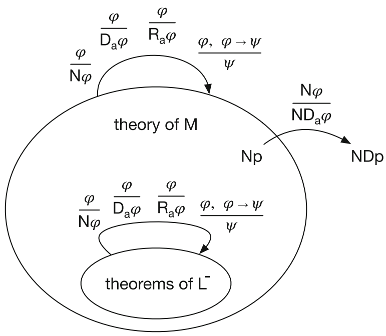

Fig. 8 Towards the proof of non-derivability of the Insertion inference rule

Figure 8 illustrates the application of the technique from [27] to our setting. The two ovals in the figure represent the set of all theorems of L^- and the theory of a model M. Because the logical system L^- is sound, the former set is a subset of the latter. By the definition of a theorem, the set of theorems of L^- is closed with respect to the Necessitation inference rules for modalities N, O, and S as well as the Modus Ponens inference rule. We visualise this in Fig. 8 by a directed loop (labelled with these four inference rules) at the oval representing the set of theorems of L^-. Note that if the same set is also closed with respect to the Insertion inference rule, then this rule is not just derivable, but also admissible in system L^-. We do not know if the set of theorems of L^- is closed with respect to the Insertion rule.

Fortunately, to prove the non-derivability of the Insertion rule in L^-, we do not need to use the set of theorems of L^-. Instead, we can use the theory of a model M. It is easy to see that the set of theorems of an arbitrary model M is closed with respect to the Necessitation inference rules for modalities N, O, and S as well as the Modus Ponens inference rule. We visualise this in Fig. 8 by a directed loop (labelled with these four inference rules) at the oval representing the theory of an *arbitrary* model M.

Note that to prove the non-derivability of the Insertion inference rule, it suffices to find a *single* model M such that the theory of model M is *not* closed with respect to the Insertion inference rule. In this case, because all axioms of L^- also belong to the theory of M, the Insertion rule cannot be represented as a fixed combination of the axioms and rules of L^-. Hence, this would prove that the Insertion rule is not derivable in L^-.

There is, however, a problem: we do not know how to construct such a model M. The solution that we found is, instead of the epistemic temporal models specified in Definition 1, to consider a larger class of models that we call *generalised epistemic temporal models*. The logical system L^- is still sound with respect to this new class of models, and the non-derivability argument sketched above still works. Note that the consideration of a more general class of models does not undermine our completeness result in Section 6. The non-standard "generalised" class of models is only used as a technical tool to prove the non-derivability of the Insertion inference rule in the logical system L^-. The intuitive idea behind generalised models goes back to Albert Einstein's works on relativity theory, where he abandoned Newton's concept of absolute time. In the context of our work, it means *abandoning the assumption that the time advances with the same speed in all possible worlds*. We formally capture this idea in the definition below. Compared to Definition 1, this new definition adds "time increment" Δ_w that, intuitively, specifies how fast the clock ticks in world w. The idea to build a non-standard model M goes back to [27], which considers non-rigid agent designators instead of non-absolute time.

Definition 10 A generalised epistemic temporal model is a tuple $(W, \Delta, \sim, \pi)$, where

1. W is a set of "possible worlds",
2. Δ_w is a "time increment" non-negative integer for each world $w \in W$,
3. $\sim_a$ is an "indistinguishability" equivalence relation on the set $W \times \omega$ for each agent $a \in \mathcal{A}$,
4. $\pi(p) \subseteq W \times \omega$.

The satisfaction relation $w, t \Vdash \varphi$ for the generalised models is defined exactly the same way as Definition 2, except that item 4 of that definition is replaced with:

$$w, t \Vdash \mathsf{N}\varphi \text{ if } w, t + \Delta_w \Vdash \varphi. \tag{38}$$

As a result, if under the standard semantics the modality N at moment t refers to "the next moment" $t + 1$, under the generalised semantics, intuitively, the clock is running faster and "the next moment" means moment $t + \Delta_w$.

The next lemma establishes the soundness of the logical system L^- with respect to the class of generalised models. Informally, this lemma shows that the "theorems of L^-" oval in Fig. 8 is inside the "theory of M" oval for any generalised model M. The proof of this lemma is straightforward (recall that system L^- does not include the Insertion inference rule).

Lemma 32 *$w, t \Vdash \varphi$ for any world w of any generalised epistemic temporal model, any moment $t \in \omega$ and any formula $\varphi \in \Phi$ provable in logical system L^-.*

Earlier in this section, we informally discussed the meaning of the word "theory" in the context of the class of the original (non-generalised) models. The next definition formally specifies this term in the context of the generalised models.

Definition 11 The ***theory*** of a generalised epistemic temporal model $(W, \Delta, \sim, \pi)$ is the set of all formulae $\varphi \in \Phi$ such that $w, t \Vdash \varphi$ for any world $w \in W$ and any moment $t \in \omega$.

The next lemma is one of two major pieces in our proof of the non-derivability. Note that it is stated for all possible generalised models. Later, we introduce a specific model M used in our proof.

Lemma 33 *The theory of any generalised epistemic temporal model contains all axioms of L^- and is closed with respect to the inference rules of L^-.*

Proof Consider any generalised epistemic temporal model $M = (W, \Delta, \sim, \pi)$. Note that $w, t \Vdash \varphi$ for any world $w \in W$, any moment $t \in \omega$, and any axiom $\varphi \in \Phi$ of the logical system L^- by Lemma 32. Thus, by Definition 11, the theory of model M contains all axioms of L^-.

Next, let us show that the theory of model M is closed with respect to the Necessitation rule $\dfrac{\varphi}{\mathsf{O}_a\varphi}$ for modality O_a. Indeed, let formula φ belong to the theory of model M. Thus, by Definition 11,

$$w, t \Vdash \varphi \text{ for any world } w \in W \text{ and any moment } t \in \omega. \tag{39}$$

By the same Definition 11, it suffices to prove that $w', t' \Vdash \mathsf{O}_a\varphi$ for any world $w' \in W$ and any moment $t' \in \omega$. Consider any world $w'' \in W$ and any moment $t'' \in \omega$ such that $(w', t') \sim_a (w'', t'')$. By item 6 of Definition 2, it suffices to show that $w'', t' \Vdash \varphi$, which is true by Eq. 39.

The proofs for the other inference rules of system L^- are similar. □

To finish the proof of the non-derivability, it suffices to construct a generalised model M whose theory is not closed with respect to the Insertion inference rule. Without loss of generality, we assume that the language Φ contains a single agent a and a single propositional variable p.

The generalised model M that we use is depicted in Fig. 9. It contains two possible worlds, w and u. Let $\Delta_w = 1$ and $\Delta_u = 2$. The indistinguishability relation $\sim_a$ is defined as follows: see Fig. 9,

$$(v_1, t_1) \sim_a (v_2, t_2) \text{ iff } t_1 = t_2. \tag{40}$$

Intuitively, it means that agent a always knows the moment, but does not know which of the two possible worlds is the current world. Finally,

$$\pi(p) = \{(w, t) \mid t \geq 1\} \cup \{(u, t) \mid t \geq 2\}, \tag{41}$$

see Fig. 9. Together, the next two lemmas form the second major piece in our proof of non-derivability: they show that the theory of the generalised model M is not closed with respect to the Insertion inference model.

Lemma 34 *Formula* $\mathsf{N}p$ *belongs to the theory of model* M.

Proof By item 1 of Definition 2 (also valid for the semantics of generalised models), Eq. 41 implies that $w, t \Vdash p$ for each moment $t \geq 1$ and $u, t \Vdash p$ for each moment $t \geq 2$. Thus, $v, t + \Delta_v \Vdash p$ for each world $v \in \{w, u\}$ of model M and each moment $t \in \omega$. Hence, $v, t \Vdash \mathsf{N}p$ by Eq. 38. Therefore, formula $\mathsf{N}p$ belongs to the theory of model M by Definition 11. □

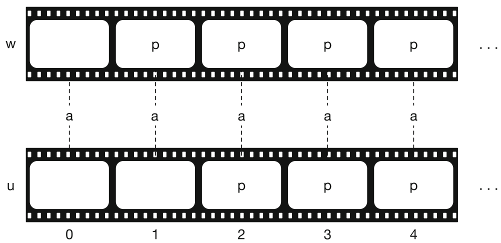

Fig. 9 Generalised epistemic temporal model M

Lemma 35 *Formula* $\mathsf{NO}_a p$ *does not belong to the theory of model* M.

Proof Equation 41 implies that $(u, 1) \notin \pi(p)$. Thus, $(u, 1) \nVdash p$ by item 1 of Definition 2. Observe also that $(u, 1) \sim_a (w, 1)$ by Eq. 40. Hence, $(w, 1) \nVdash \mathsf{O}_a p$ by item 6 of Definition 2 (also valid for the semantics of generalised models). Thus, $(w, 0) \nVdash \mathsf{NO}_a p$ by Eq. 38. Therefore, the formula $\mathsf{NO}_a p$ does not belong to the theory of model M by Definition 11. □

Together, Lemma 33, Lemma 34, and Lemma 35 imply our main non-derivability result stated below.

Theorem 6 *The Insertion inference rule is not derivable in the logical system* L^-.

As mentioned earlier, the admissibility of the Insertion rule in L^- remains an open question.

8 Conclusion

In this article, we propose to formalise the distinction between subjective and objective time. We captured these notions using two modalities, proved their mutual undefinability, and gave the complete axiomatisation of their interplay.

The distinction between subjective and objective time appears in other settings, not considered in this article. For instance, imagine that Ann is a farmer who lives near a train station and never leaves her farm. Although the trains might have a very complicated timetable that Ann does not know, she *knows when* the train is coming because she can *hear* it each time the train is approaching the station. Another person, Brittany, on the other hand, lives far away from the station. She cannot hear the train approaching the station, but she has a *timetable* hanging above her desk. Brittany also *knows when* the train is coming, but in a very different sense. Ann knows when a train is coming subjectively, and Brittany knows it objectively. Note that this distinction exists even if Ann and Brittany always have a clock that shows the correct current time. However, the subjective/objective distinction from the example in the introduction to this article disappears if Ann has access to a clock. Thus, what we just described is indeed a different form of the subjective/objective distinction that we leave for future exploration.

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