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The CAT(0) dimension of 3-generator Artin Groups

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ABSTRACT

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The three generator Artin groups A(m,n,2) are known to be have CAT(0) dimension strictly greater than two if both m and n are odd [BC]. In Chapter 1 we introduce the notions of CAT(0) dimension and three generator Artin groups.

In Chapter 2 we show that if one of m or n is even, then the three generator Artin group has CAT(0) dimension two.

In Chapter 3 we extend work by Noel Brady and John Crisp [BC] to enlarge the subclass of groups A(m,n,2) known to have CAT(0) dimension three.

In Chapter 4 we classify the structure of a canonical cell complex which the group A(m,n,2) acts on for the case where m is even, greater or equal to six and not divisible by four and n is prime, greater or equal to five.

Finally, in Chapter 5 we use the results of Chapter 4 to exhibit classes of rank four Artin groups with CAT(0) dimension two, and a class of rank six Artin groups with CAT(0) dimension two.

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Contents

Chapter 1

Dimensions of Groups

In this chapter we review algebraic and geometric dimensions of discrete groups finally considering the CAT(0) dimension of a discrete group. We begin with the cohomological dimension of a group.

1.1 Cohomological dimension

Let R be a ring. A *free* R-module is an R-module which is a direct sum of copies of R. Given any surjective homomorphism π of R-modules, $\pi : A \to B$ and any free module F with a homomorphism $\phi : F \to B$, one may construct a homomorphism $\tilde{\phi} : F \to A$ such that $\tilde{\phi} = \pi\phi$. Any module F (not necessarily free) with this property is said to be *projective.*

Let M be an R-module. A *projective* (respectively free) resolution of M is an exact sequence of the form

$$\ldots P_n \to P_{n-1} \cdots \to P_1 \to P_0 \to M \to 0$$

where each P_i is a projective (respectively free) R-module. We can always construct a projective resolution by choosing a free R-module F_0 with basis in $1-1$ correspondence with a set of generators of M. This projects onto M

with kernel K_0. Repeating the process for K_0 instead of M we obtain a free module F_1 by iterating the process an exact sequence of free R-modules

$$F_n \to \dots F_1 \to F_0 \to M$$

Since each free module is projective this is a projective resolution for M.

By making each element of G act as the identity and extending the action linearly to the group ring $\mathbb{Z}G$ we make $\mathbb{Z}$ a $\mathbb{Z}G$-module. The *cohomological dimension* of G, cd$G$, is the minimal integer n such that there is a projective resolution of length n,

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \twoheadrightarrow \mathbb{Z}$$

If there is no such finite integer n then we say the group has cohomological dimension ∞.

Lemma 1.1.1. *Let G be a discrete group. The cohomological dimension of G is 0 if and only if G is trivial.*

Proof. Suppose G is the trivial group. Then $\mathbb{Z}$ is a free $\mathbb{Z}G$ module, and hence projective. So taking $P_0 = \mathbb{Z}$,

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0$$

is a projective resolution.

Conversely suppose that cd$G = 0$ then there is a projective resolution

$$0 \to P_0 \to \mathbb{Z} \to 0$$

Thus $P_0 = \mathbb{Z}$ is a projective $\mathbb{Z}G$-module. Every projective module is the summand of a free module. Since $\mathbb{Z}$ cannot be a summand of $\mathbb{Z}G$ unless G is trivial, $\mathbb{Z}G = \mathbb{Z}$, and G is trivial. □

If $H \leq G$ then the P_i may also be viewed as $\mathbb{Z}H$-modules, so $\mathrm{cd}H \leq \mathrm{cd}G$.

Note that since free modules are projective, a free resolution will give an upper bound on the value of $\mathrm{cd}G$. If G acts freely on a contractible simplicial complex then the free modules C_i generated by the i-dimensional cells form a free resolution, so the dimension of this complex gives an upper bound on the cohomological dimension of the group.

The Cayley graph of a non-trivial free group with respect to any free basis is a tree [DD]. So any free group acts freely on a 1-dimensional complex, hence for a free group the cohomological dimension is less than or equal to 1. Since the group is non-trivial the cohomological dimension is 1. In the 1960's Stallings and Swan [Swa69, Sta68] showed that if $\mathrm{cd}G = 1$, then G is free. Hence $\mathrm{cd}G = 1$ if and only if G is free.

This geometric upper bound on the cohomological dimension prompts another definition of the dimension of a group.

1.2 Geometric dimension

In this section we give a geometric version of dimension for a discrete group.

A *CW-complex* is a Hausdorff space X with a cell decomposition with the following properties:

1. For each n-cell e, there is a continuous map $\phi_e : D^n \to X$ mapping the unit n-disc into X, such that ϕ_e takes the interior of D^n homeomorphically onto the cell e, and the boundary of D^n into the union of $(n-1)$-cells.

2. The closure of each cell intersects only finitely many other cells.

3. A subset $Y \subset X$ is closed in X if and only if the intersection of Y with the closure of each cell is closed.

Given a discrete group G, a *G-CW-complex* is a CW-complex with a topological G-action which maps cells to cells homeomorphically such that if an element $g \in G$ maps a cell e to itself, then $g.x = x$ for all points $x \in e$.

A complex is said to be a *free* G-CW-complex if the stabiliser of each cell is trivial, $\{g \in G | g.e = e\} = 1$.

The complex X is *contractible* if there is a homotopy equivalence of X to a point.

A complex X is said to be an *Eilenberg-MacLane* complex for G, or $K(G,1)$ if the fundamental group of X is G, $\pi_1(X) \cong G$ and the universal cover of X is contractible.

A free G-CW-complex X is said to be a *model for EG* if X/G is an Eilenberg-MacLane complex $K(G,1)$ for G, or equivalently X is contractible.

Given a discrete group G, the *geometric dimension* of G, gd$G$ is the minimal dimension of a model for EG. From the geometric upper bound on the cohomological dimension, it is clear that cd$G \leq$ gd$G$.

In the 1950's Eilenberg and Ganea showed that for any G,
gd$G \leq max\{$cd$G, 3\}$ [EG57]. It follows from this and the previous results that,

$$\text{cd}G = 1 \Leftrightarrow G \text{ is free} \Leftrightarrow \text{cd}G = 1$$

$$\text{cd}G \geq 3 \Rightarrow \text{gd}G = \text{cd}G$$

$$\text{gd}G > 3 \Rightarrow \text{gd}G = \text{cd}G$$

Or equivalently cd$G =$ gd$G$ unless cd$G = 2$ and gd$G = 3$. The statement that no such a group exists is known as the Eilenberg-Ganea conjecture. Counter examples have been proposed by M. Bestvina and M. Davis [Bes93] and by M. Bestvina and N. Brady [BB97].

It is known that the cohomological dimension of a non-trivial finite cyclic group is infinite [Ev]. It follows from this and the fact gd$G \leq \max\{\text{cd}G, 3\}$ that any group containing torsion has both infinite geometric and cohomological dimension, a drawback as many infinite groups contain 'some' torsion. For example triangle groups.

Example 1.2.1. A (p, q, r)-triangle group has presentation

$$\langle \alpha, \beta, \gamma | \alpha^2 = \beta^2 = \gamma^2 = (\alpha\beta)^p = (\beta\gamma)^q = (\gamma\alpha)^r = 1 \rangle$$

and acts on a 2-complex by reflecting a triangle with internal angles $\frac{\pi}{p}, \frac{\pi}{q}, \frac{\pi}{r}$ along its edges. Thus in some sense the dimension of each triangle group is 2, however these groups contain torsion, so the cohomological dimension and geometric dimension is ∞. Note that the complex formed is hyperbolic (has negative curvature) if $\Sigma = \frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$, Euclidean if $\Sigma = \pi$, and spherical if $\Sigma > \pi$.

1.3 Proper dimension

In order to deal with torsion, we allow a group to act with finite stabilisers.

A group acts on a CW-complex X with finite stabilisers if the stabiliser of each cell in the complex is finite. It follows from this that there are only a finite number of group elements mapping any cell to any other. Given any compact subset $K \subset X$, K is contained in a finite union of cells. If a group element g is such that $g.K \cap K \neq \emptyset$ then g maps at least one cell in this union to another. Since there are only finitely many cells, and a finite number of ways of doing this for each cell, the number of such g's is finite. Hence the finite stabilisers condition is equivalent to proper discontinuity of the action: A group G acts properly discontinuously on a CW-complex if for any compact subset K of this complex, $\{g|g.K \cap K \neq \emptyset\}$ is finite. A proper G-CW-complex is a G-CW-complex in which the action is proper.

We say a proper G-CW-complex is a model for $\underline{E}G$ if for every finite $H \leq G$ the fixed point set of H in X is contractible. Note that since the trivial group is a subgroup, acting as the identity, the complex X itself must be contractible. The proper geometric dimension of G, $\underline{\text{gd}}G$ is the minimal dimension of a model for $\underline{E}G$. Again it is clear that for $H \leq G \Rightarrow \underline{\text{gd}}H \leq \underline{\text{gd}}G$ [BLN01].

Note that we do not have the alternative definition using the quotient complex of a model for $\underline{E}G$ that we had for the definition of a model for EG, as G does not act freely on the complex. An example of this is given by the triangle groups. If the group has no torsion however, it does act freely, and we may consider the universal cover of an Eilenberg-MacLane complex as a model for $\underline{E}G$. Here $\underline{\text{gd}}G = \text{gd}G$.

Example 1.3.1. If a group is finite the stabilisers of any point are necessarily finite, so it acts properly on a single point. Hence for finite G, $\underline{\text{gd}}G = 0 \neq \text{gd}G = \infty$ moreover if G acts properly on a 0-dimensional contractible CW-complex then the complex must be a point, hence G must be finite.

Example 1.3.2. Let $D_\infty = \langle a, b | b^2 = 1, b^{-1}ab = a^{-1} \rangle$ be the infinite dihedral group. Let D_∞ act on the real line by $a.x = a + 1$ and $b.x = -x$. This is a G-action as $b^2.x = b. - x = x$ and $b^{-1}ab.x = b^{-1}a. - x = b^{-1}. - x + 1 = x - 1 = a^{-1}.x$. Since $ab = ba^{-1}$ we can write any group element g as $a^p b^q$ for some $p, q \in \mathbb{Z}$. Hence if $g.x = x$ for some g, $a^p b^q.x = x$ so $p = 0$ and $q = 0$ or 1. Thus the stabiliser of x is finite and $\underline{\text{gd}}D_\infty = 1$.

For each version of geometric dimension a contractible complex is used. In practice it is difficult to construct complexes we know to be contractible. Constructing complexes is in some sense a local operation (gluing cells along faces), however contractibility is a global property. We require local properties which ensure a complex is contractible. The class of CAT(0) spaces allows this since for some given group G we may construct locally CAT(0) complexes with fundamental group G (hopefully) and the universal covers of these complexes are contractible.

1.4 The CAT(0) property

In this section we define the CAT(0) property, give a useful method for checking a cell complex is CAT(0) and observe that CAT(0) spaces are contractible.

Let X be a metric space with metric d_X and x, y, z three points in X. A *geodesic* from x to y in X, is an isometric map γ of the closed interval $[0, d(x,y)]$ into X such that $\gamma(0) = x$ and $\gamma(d(x,y)) = y$. Denote such a geodesic by $[x,y]$, but note that in general the geodesic $[x,y]$ may not be unique. We may form a geodesic triangle with vertices x, y, z by choosing three geodesics $[x,y]$, $[y,z]$ and $[z,x]$ and taking the union of these. Denote this triangle by Δ. A *comparison triangle* for Δ is a geodesic triangle $\bar{\Delta} = [\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$ in the Euclidean plane with the usual metric $d_{\mathbb{E}}$ such that $d_X(x,y) = d_{\mathbb{E}}(\bar{x}, \bar{y})$, $d_X(y,z) = d_{\mathbb{E}}(\bar{y}, \bar{z})$ and $d_X(z,x) = d_{\mathbb{E}}(\bar{z}, \bar{x})$. Note that given any geodesic triangle Δ, $\bar{\Delta}$ is unique (up to translation, rotation and reflection), as geodesics are unique in the Euclidean plane.

Let p be a point on the geodesic $[x,y] \subset \Delta$. The *comparison point* for p is $\bar{p} \in [\bar{x}, \bar{y}] \subset \bar{\Delta}$ such that $d_X(x,p) = d_{\mathbb{E}}(\bar{x}, \bar{p})$. We may similarly define comparison points for other edges of our geodesic triangle.

The geodesic triangle Δ is said to satisfy the *CAT(0) inequality* if for any pair of points $p, q \in \Delta$ and comparison points $\bar{p}$, $\bar{q} \in \bar{\Delta}$ we have $d_X(p,q) \leq d_{\mathbb{E}}(\bar{p}, \bar{q})$. A geodesic metric space is said to be CAT(0) if every geodesic triangle in the space satisfies the CAT(0) inequality.

We form the analogous definition of a CAT(1) space by replacing in the previous definitions instances of $\mathbb{E}$ with $\mathbb{S}^2$ and CAT(0) with CAT(1). We compare triangles in a metric space X with triangles on the unit sphere $\mathbb{S}^2$. However, we note that the maximum perimeter of a geodesic triangle on the unit sphere is restricted, so we compare triangles with perimeter less than or equal to 2π. We define a metric on the sphere by defining the distance between any two points x and y to be the angle between the two lines joining the centre of the sphere to these points.

A metric space X is said to be *locally CAT(κ)* for $\kappa = 0, 1$ if for every point $x \in X$, there is an open neighborhood of x on which the induced metric is CAT(κ).

A simple example of a CAT(0) space is the Euclidean plane. A comparison triangle for any triangle in the Euclidean plane is the triangle itself. A piecewise Euclidian simplicial complex X is a simplicial complex where each simplex is equipped with a Euclidean metric and gluing maps are isometries. If X has only finitely many isometry types of simplices then extending the metric on each simplex gives a complete metric on the complex X (see [BH99]). The Cartan-Hadamard Theorem (see [BH99]) can be used to show that a complete metric space is CAT(0) if and only if it is locally CAT(0) and simply connected. The link condition allows us to check if a complex is locally CAT(0).

Let C be a geodesic n-simplex. The *geometric link* $Lk(x, C)$ of a point x in C is the set of equivalence classes of geodesic segments between x and all other points y in C where two geodesic segments are said to be equivalent if they share a common initial segment. A metric is imposed on $Lk(x, C)$ by taking the distance between two geodesic segments to be the Alexandrov angle between them. The geometric link of a point x in a simplicial complex X is the union of links $Lk(x, C)$ over all simplices C which contain x.

We may think of the link of a simplex σ as a simplicial complex with n-cells corresponding to the $n+1$-cell containing σ. If X is a two dimensional complex, then the link of every vertex is a graph (a union of edges and vertices), the link of every edge is discrete (finite if X is locally finite) and the link of every face is empty.

A piecewise Euclidean complex X is said to satisfy the *link condition* if for each vertex $v \in X$ the link of v, $Lk(v, X)$ is a CAT(1) space with respect to the Alexandrov metric.

Lemma 1.4.1. *[Bri91] A piecewise Euclidian complex (with finitely many isometry types of simplices) is locally CAT(0) if and only if it satisfies the link condition.*

Lemma 1.4.2. *[Bri91] A piecewise spherical complex (for example a link complex) is CAT(1) if and only if it satisfies the link condition and every embedded circle has length at least 2π.*

In the case of a 2-dimensional Euclidean complex, the link of each vertex is a graph so this clearly satisfies the link condition. This enables one to establish the following lemma.

Lemma 1.4.3. *[Bri91] A 2-dimensional complex is CAT(0) if and only if the link of each vertex contains no embedded circles of length less than 2π.*

1.4.1 An algorithm for the link of a 1-dimensional complex

We now produce an algorithm which, in a special case, may be used to verify that a 1-dimensional link complex is CAT(1). The input to this algorithm will be a matrix representing the link, and a choice of diagonal entry on the matrix. On the level of the graph, the operation corresponds to deleting a vertex and possibly adding extra edges between it neighbours to preserve distances between them.

Let $\mathcal{L}$ be a 1-dimensional link complex. The complex $\mathcal{L}$ is a graph with vertex set $V = \{v_1, \ldots, v_n\}$. Suppose there are two edges e_1 and e_2 between vertices v_i and v_j. The edges e_1 and e_2 form a closed non-trivial loop of length $|e_1| + |e_2|$. If this length, $|e_1| + |e_2|$ is strictly less than 2π then the link complex $\mathcal{L}$ is not CAT(1). Suppose $|e_1| + |e_2| \geq 2\pi$. Let $\mathcal{L}'$ denote the metric graph formed by removing the longer of the two edges e_1 and e_2. Suppose e_1 is removed. Any loop in $\mathcal{L}'$ is a loop in $\mathcal{L}$, similarly, any loop in $\mathcal{L}$, not containing the edge e_1 is a loop in $\mathcal{L}'$. If there is a loop in $\mathcal{L}$ containing e_1 then this loop is longer than the loop obtained from it by replacing e_1 with e_2. Thus $\mathcal{L}'$ is CAT(1) if and only if $\mathcal{L}$ is CAT(1). We therefore have the following lemma,

Lemma 1.4.4. *Let $\mathcal{L}$ be a 1-dimensional link complex. Suppose that for every pair of vertices joined by two edges, the sum of the length of these two edges is greater or equal to 2π. Let $\mathcal{L}'$ be the complex formed by removing the longer of each of these pairs of edges. Then $\mathcal{L}$ is CAT(1) if and only if $\mathcal{L}'$ is CAT(1).*

Suppose now that $\mathcal{L}$ is a 1-dimensional link complex with at most 1 edge between each pair of vertices. We construct a weighted adjacency matrix for $\mathcal{L}$. This is an $n \times n$ matrix which records the length of edges between any pair of vertices.

Definition 1.4.5. Let $Dist(\mathcal{L})$ be an $n \times n$ matrix with entries d_{ij} where d_{ij} is the length of the single edge joining v_i and v_j. We take d_{ij} to be 0 if there is no edge.

It is immediate from the definition that $Dist(\mathcal{L})$ is a symmetric matrix. The diagonal entries represent loops for which the initial and terminal vertices coincide. If any of these diagonal entries are less than 2π then $\mathcal{L}$ is not CAT(1).

Given an $n \times n$ symmetric matrix $D = (d_{ij})$ we may construct a graph $\mathcal{L}_D$ with vertex set $\{v_1, \ldots, v_n\}$ by attaching edges between v_i and v_j of length d_{ij} if $d_{ij} \neq 0$. Then $Dist(\mathcal{L}_D) = D$.

Suppose that the diagonal entries of $Dist(\mathcal{L})$ are all zero. Suppose also that the i$^{\text{th}}$ row of $Dist(\mathcal{L})$ contains $r \geq 2$ non-zero entries $d_{ij_1}, \ldots, d_{ij_r}$ say. If in some other row of the matrix there is a non-zero entry $d_{j_s j_t}$ for $1 \leq s, t \leq r$, then there is a triangle in the graph with vertices v_i, v_{j_s} and v_{j_t} and edge length $d_{j_s j_t} + d_{ij_t} + d_{j_s i}$.

Definition 1.4.6. Define the following operation ρ for the matrix $Dist(\mathcal{L})$:

For $s \neq t \in \{1, \ldots, (r-1)\}$,

1. if $d_{j_s j_t} \neq 0$ and $d_{j_s j_t} + d_{ij_s} + d_{ij_t} < 2\pi$ then $\mathcal{L}$ is not CAT(1), terminate operation.

   Else

2. if $d_{j_s j_t} = 0$ then set $d_{j_s j_t} = d_{j_t j_s} = d_{ij_s} + d_{ij_t}$ or,
3. if $d_{j_s j_t} \neq 0$ and $d_{j_s j_t} + d_{ij_s} + d_{ij_t} \geq 2\pi$ set
   $d_{j_s j_t} = d_{j_t j_s} = \min\{d_{j_s j_t}, d_{ij_s} + d_{ij_t}\}$.
4. Then for $k = 1, \ldots, (r-1)$, set $d_{ij_k} = d_{j_k i} = d_{ij_{k+1}} = d_{j_{k+1} i} = 0$.

Denote the new matrix by $Dist(\mathcal{L})'$. This new matrix has zeros in the i$^{\text{th}}$ row and column.

Lemma 1.4.7. *Suppose every edge in $\mathcal{L}$ has length greater or equal to $\frac{\pi}{3}$. Suppose $Dist(\mathcal{L})$ is a weighted adjacency matrix for $\mathcal{L}$ with all diagonal entries zero. Let $Dist(\mathcal{L})'$ be the matrix obtained by an application of ρ on $Dist(\mathcal{L})$ where ρ does not terminate in step 1. Then $\mathcal{L}$ is CAT(1) if and only if $\mathcal{L}' := \mathcal{L}_{Dist(\mathcal{L})'}$ is CAT(1).*

Proof. By a reordering of $\{v_1, \ldots, v_n\}$ we may assume that ρ is applied to the 1^{st} row of $Dist(\mathcal{L})$ with $r \geq 2$ non-zero entries $d_{12}, d_{13}, \ldots, d_{1(r+1)}$.

We will show that if γ is a loop in $\mathcal{L}$ then there exists a loop γ' in $\mathcal{L}'$ of shorter length. Thus $\mathcal{L}'$ is CAT(1) implies $\mathcal{L}$ is CAT(1). Similarly we show that if γ' is a loop in $\mathcal{L}'$ then there exists a loop of shorter length in $\mathcal{L}$.

Let γ be a non-trivial edge loop in $\mathcal{L}$. If γ does not pass through v_1, then γ is a loop in $\mathcal{L}'$. Suppose γ does pass through v_1. Since γ is an edge path, γ passes through 2 edges connecting v_i to v_1 and v_j to v_1 for some $2 \leq i, j \leq r+1$. If $v_i = v_j$ then the removal of these two edges forms a shorted loop in $\mathcal{L}$ and we may reapply this process to that loop. If $v_i \neq v_j$ there is a path in γ between v_i and v_j via v_1 comprising of 2 edges of total length $d_{1i} + d_{1j}$. In $\mathcal{L}'$ there is an edge path between v_i and v_j of length $\min\{d_{ij}, d_{1i} + d_{1j}\}$ if $d_{ij} \neq 0$ and of length $d_{1i} + d_{1j}$ if $d_{ij} = 0$. Substituting this path, we obtain a loop γ' in $\mathcal{L}'$ of shorter length than that of γ. Thus if $\mathcal{L}'$ is CAT(1) then $\mathcal{L}$ is CAT(1).

Now let γ' be a non-trivial edge loop in $\mathcal{L}'$. Suppose γ' does not contain a triangle comprising of the distinct vertices v_i, v_j and v_k $2 \leq i, j, k \leq r+1$ and the single edges between these vertices. Then either γ' is a loop in $\mathcal{L}$ or, replacing the edge between v_s and v_t, $2 \leq s < t \leq r+1$ with the two edges v_s to v_1 and v_t to v_1 we obtain a loop in $\mathcal{L}$ of the same length as γ'.

Consider the triangle $\mathcal{L}'$ with vertices v_i, v_j and v_k $2 \leq i < j < k \leq r+1$ and single edges joining these vertices. Taking $d'_{ij} = \infty$ if $d_{ij} = 0$ and similarly for d'_{jk} and d'_{ik}, the triangle has edge length $\Sigma = \min\{d'_{ij}, d_{1i} + d_{1j}\} + \min\{d'_{jk}, d_{1j} + d_{1k}\} + \min\{d'_{ik}, d_{1i} + d_{1k}\}$. There are four cases to consider;

1. $\Sigma = d'_{ij} + d'_{jk} + d'_{ik}$. Then the triangle is a loop in $\mathcal{L}$.
2. $\Sigma = d'_{ij} + d'_{jk} + d_{1i} + d_{1k}$. Then the edge path in $\mathcal{L}$ comprising of the edges $(v_i, v_1), (v_1, v_k), (v_k, v_j)$ and (v_j, v_i) is a non-trivial loop in $\mathcal{L}$ with the same edge length as γ'.
3. $\Sigma = d'_{ij} + d_{1j} + d_{1k} + d_{1i} + d_{1k}$. Then the triangle in $\mathcal{L}$ with vertices v_1, v_i and v_j is a non-trival loop in $\mathcal{L}$ with edge length $d'_{ij} + d_{1i} + d_{1j} < \Sigma$.
4. $\Sigma = d_{i1} + d_{1j} + d_{1j} + d_{1k} + d_{1i} + d_{1k}$. Then $\Sigma \geq 6 \times \frac{\pi}{3}$ since every edge has length as least $\frac{\pi}{3}$. Hence the length of γ' is greater or equal to 2π.

So if γ' is a loop containing this triangle, the edge length of γ' is at least 2π.

Thus if $\mathcal{L}$ is CAT(1) then so is $\mathcal{L}'$. □

By repeatedly applying the operation ρ a finite number of times we may, if the operation does not terminate in step 1, reduce the matrix $Dist(\mathcal{L})$ to a matrix with at most 1 non-zero entry in each row. If the matrix $Dist(\mathcal{L}$ has at most 1 non-zero entry in each row then each vertex in $\mathcal{L}$ is connected to at most 1 other vertex. Thus $\mathcal{L}$ is CAT(1).

Hence we have the following algorithm.

Let $\mathcal{L}$ be a 1-dimensional link complex with n vertices.

1. For each pair of vertices v_i, v_j, check the length of bigons between v_i and v_j. If the length of the bigon is strictly less than 2π then $\mathcal{L}$ is not CAT(1) and terminate the algorithm. If the length of the bigon is greater or equal to 2π then remove the longer of the 2 edges.

2. Let $D = (d_{ij})$ be a matrix equal to the matrix $Dist(\mathcal{L})$. For each $1 \leq i \leq n$,

   - If $0 < d_{ii} < 2\pi$ then $\mathcal{L}$ is not CAT(1). Terminate the algorithm.
   - If $d_{ii} \geq 2\pi$ or $d_{ii} = 0$, then set $d_{ii} = 0$.

3. Repeatedly apply the operator ρ to D until an application of ρ terminates with the result that $\mathcal{L}$ is not CAT(1), or until there is at most 1 non-zero entry in each row. In this case $\mathcal{L}$ is CAT(1).

In practice when applying this algorithm by hand, it is easier to apply each operation ρ to a row with small numbers of non-zero entries. Hence we always apply ρ to the row containing the least number of entries. Also, a row (and column) containing just zero entries corresponds to a vertex in the link complex which is not the initial or terminal vertex of any edges, hence we may remove this row (and column) from the matrix D.

Recall that for a complex to be a model for $\underline{E}G$ it must be contractible. This is ensured by the Cartan-Hadamard theorem [BH99, II.4.1].

Theorem 1.4.8. *The Cartan-Hadamard Theorem.[BH99]*

Let X be a complete connected metric space. If the metric on X is locally CAT(0) then the induced metric on the universal covering $\tilde{X}$ is (globally) CAT(0). In particular there is a unique geodesic path joining each pair of points in $\tilde{X}$ and geodesic paths in $\tilde{X}$ vary continuously with their endpoints.

It follows from this theorem that there is a geodesic segment connecting every pair of points in a CAT(0) space X. Homotopically retracting along these geodesics shows that a CAT(0) space is contractible.

Just as the CAT(0) property strengthens the notion of contractibility, the notion of a cellular G-action needs to be strengthened to that of a semi-simple G action on a CAT(0) complex.

1.5 Semi-simple actions

In this section we introduce the notion of a semi-simple isometry. We give examples of G-actions which are semi-simple and actions which are not. We also state a theorem which we shall use later.

Let X be a metric space and γ an isometry of X. The *translation length* of γ in X, is defined to be $|\gamma|_X := inf\ \{d_X(x, \gamma.x) | x \in X\}$. The set of points where γ attains this infimum is called the *minset* of γ and is denoted $Min(\gamma)$. The minset of a group is defined to be the set of points lying in the minset of every element in that group $Min(\Gamma) := \bigcap_{\gamma \in \Gamma} Min(\gamma)$.

The following proposition lists some properties of minsets which will be used later.

Proposition 1.5.1. *[BH99] Let X be a metric space and γ an isometry of X. Then,*

1. *$Min(\gamma)$ is γ-invariant.*
2. *Let α be an isometry of X. Then $|\alpha\gamma\alpha^{-1}| = |\gamma|$ and $Min(\alpha\gamma\alpha^{-1}) = \alpha.Min(\gamma)$*
3. *If X is CAT(0), then $Min(\gamma)$ is convex.*
4. *If $C \subset X$ is non-empty, complete, convex and CAT(0), and γ invariant, then $|\gamma|_X = |\gamma|_C$.*

Proof. 1. Since γ is an isometry $d(\gamma.x, \gamma^2.x) = d(x, \gamma.x)$, so $x \in \text{Min}(\gamma) \Rightarrow \gamma.x \in \text{Min}(\gamma)$.

2. $|\alpha\gamma\alpha^{-1}| = \inf\{d(x, \alpha\gamma\alpha^{-1}.x)|x \in X\}$
$= \inf\{d(\alpha^{-1}x, \gamma.\alpha^{-1}x)|x \in X\} = \inf\{d(y, \gamma.y)|y \in X\} = |\gamma|$. The second result follows directly.

3. Let x and y be two points in $\text{Min}(\gamma)$ and let z be a point on the geodesic in X between x and y. The points x, y and $\gamma.x$ form a geodesic triangle, as do the points $y, \gamma.y$ and $\gamma.x$. Let $\bar{x}, \bar{y}, \overline{\gamma.x}, \overline{\gamma.y}$ be comparison points. We may choose comparison triangles in the Euclidean plane such that the triangles share the common edge between $\bar{y}$ and $\overline{\gamma.x}$. The points z and $\gamma.z$ have comparison points $\bar{z}$ and $\overline{\gamma.z}$ respectively. The geodesic in $\mathbb{E}^2$ between $\bar{z}$ and $\overline{\gamma.z}$ intersects the geodesic between $\bar{y}$ and $\overline{\gamma.x}$ at a point $\bar{p}$ say, the comparison point for the point p on the geodesic between y and $\gamma.y$ in X. In $\mathbb{E}^2$, $d(\bar{z}, \bar{p}) + d(\bar{p}, \overline{\gamma.z}) = d(\bar{x}, \overline{\gamma.x}) = |\gamma|$. Since X is CAT(0), we have $d(z, \gamma.z) \leq d(z, p) + d(p, \gamma.z) \leq d(\bar{z}, \bar{p}) + d(\bar{p}, \overline{\gamma.z}) = |\gamma|$. Hence $d(z, \gamma.z) = |\gamma|$ and $z \in \text{Min}(\gamma)$.

4. There is a projection p of X onto C see [BH99, II.2]. It is a fact that $\gamma.p(x) = p(\gamma.x)$ and $d(x, \gamma.x) \geq d(p(x), p(\gamma.x))$ for all x in X. The result follows.

□

An isometry γ is said to be *semi-simple* if $Min(\gamma)$ is non-empty. The action of a group by isometries is said to be semi-simple if all the group elements are semi-simple.

An isometry γ is said to be *hyperbolic* if $|\gamma|_X \gneqq 0$. Given any point $x \in Min(\gamma)$, $\gamma.x \in Min(\gamma)$ and hence by convexity of $Min(\gamma)$ $[x, \gamma.x] \subset Min(\gamma)$. The line formed by $\gamma^n.[x, \gamma.x]$ is a geodesic in $Min(\gamma)$ called a translation axis of γ.

Example 1.5.2. Let $G = \mathbb{Z} \times \mathbb{Z} = \langle a, b | ab = ba \rangle$ act on the Euclidean plane $\mathbb{E}^2$

(with the usual metric) by translations, $a.(x,y) = (x+1,y)$ and $b.(x,y) = (x,y+1)$. Taking unit squares to be a cellularisation of the plane we see that it is a G-CW-complex and the action is free by isometries. If (x_1,y_1) and (x_2,y_2) are any two points in the plane then $d((x_1,y_1),g.(x_1,y_1)) = d((x_2,y_2),g.(x_2,y_2))$. Thus every point in $\mathbb{E}^2$ is in $\mathrm{Min}(g)$ for any $g \in G$. Since the action is free, the only group element which fixes any point is the identity, so every non-trivial group element is hyperbolic.

Example 1.5.3. The previous example can be generalised to show that the natural action of $\mathbb{Z}^n$ on $\mathbb{E}^n$ is a free $\mathbb{Z}$-action by semi-simple isometries with every element hyperbolic.

Example 1.5.4. Let T_n^i be a regular Euclidean trapezium with parallel edges p_n^i and p_{n+1}^i of lengths $\frac{1}{n}$ and $\frac{1}{n+1}$ respectively. Label the non-parallel edges by d_n^i and d_n^{i+1}. Let T_n^i have internal angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ so the height is $\frac{\sqrt{3}}{2n(n+1)}$.

We construct a cell complex by taking the union $\bigcup_{i\in\mathbb{Z}}\bigcup_{n\in\mathbb{N}} T_n^i$ identifying the edges labeled p_n^i in T_{n-1}^i and T_n^i and the edges d_n^i in T_n^{i-1} and T_n^i for all n and i. This forms a CW-complex as each cell is homeomorphic to a disc and the closure of each cell intersects only 8 other cells.

Let $G = \langle g \rangle \cong \mathbb{Z}$ act on this complex by $g.T_n^i = T_n^{i+1}$. This action is free. It is not semi-simple.

Example 1.5.5. By embedding the previous example in the Euclidean plane we see that $g^6.T_n^i = T_n^i$ so here $C_g = \langle g | g^6 = 1\rangle$ acts properly on the complex. Again the action is not semi-simple. See figure 1.1

Example 1.5.6. Suppose K is a connected simplicial complex with only finitely many types of simplices up to isometry. Let γ be an isometry of K.

Let $\{x_n\}_{n\in\mathbb{N}}$ be a set of points in K such that $d(\gamma.x_n, x_n) \to |\gamma|$ in a strictly decreasing sequence. Since there are only finitely many types of simplex, there is an infinite number of points in the sequence contained in simplices

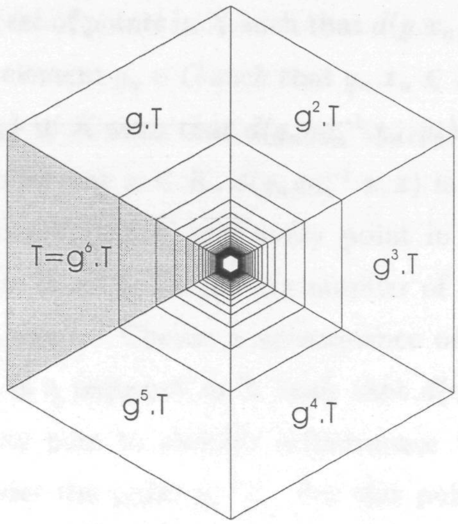

Figure 1.1: A CW-complex with a non-semi-simple G-action

of one type. Choose a subsequence such that the sequence is contained in simplices of that type. Each geodesic $[x_n, \gamma.x_n]$ crosses a finite combination of simplices. Since $d(\gamma.x_n, x_n)$ is bounded above by $d(\gamma.x_1, x_1)$ and there are only finitely many types of simplex there are an infinite number of points with geodesics $[x_n, \gamma.x_n]$ which cross the same combination. Again choose a subsequence so that the same combination is crossed for each point. Hence there is an infinite subsequence where the action on each simplex is the same for each point, so we may assume that there is a subsequence contained in a single simplex. Since each simplex is compact we may choose a subsequence again, which converges to a point y in the simplex. For this point $d(\gamma.y, y) = |\gamma|$. So $\mathrm{Min}(\gamma) \neq \emptyset$, and the action is semi-simple.

Lemma 1.5.7. *Let G, a group, act properly discontinuously and cocompactly by isometries on a metric space X. Then G acts semi-simply.*

Proof. Let K be a compact subset of X such that $G.K = X$. Fix an element

$g \in G$. Let $\{x_n\}$ be a set of points in X such that $d(g.x_n, x_n) \to |g|$ as $n \to \infty$. For each n choose an element $g_n \in G$ such that $g_n.x_n \in K$. This creates a new sequence $\{y_n = g_n.x_n\}$ in K such that $d(g_n g g_n^{-1}.y_n, y_n) = d(g_n g.x_n, g_n.x_n) = d(g.x_n, x_n) \to |g|$. So for any $x \in K$, $d(g_n g g_n^{-1}.x, x)$ is bounded and we can find a compact set containing K and every point in $g_n g g_n^{-1} K$. Since the action is proper, there must be an infinite number of $g_n g g_n^{-1}$ which are the same group element, say γ. Choose a subsequence of $\{y_n\}$ for which each $g_n g g_n^{-1} = \gamma$, so $\{y_n\}$ is a sequence in K such that $d(\gamma.y_n, y_n) \to |g|$. Since K is compact we may pass to another subsequence which converges to a point $y \in K$. Consider the point $g_n^{-1}.y$. For this point $d(g g_n^{-1}.y, g_n^{-1}.y) = d(g_n g g_n^{-1} y, y) = d(\gamma.y, y) = \lim d(\gamma.y_n, y_n) = \lim d(g, x_n, x_n) = |g|$. □

1.6 CAT(0) dimension

We define the CAT(0) dimension of a discrete group G to be the minimal dimension of a CAT(0) proper G-CW-complex where the action is by semi-simple isometries. We take the dimension to be ∞ if no such complex exists.

It is obvious that CAT(0) dimension $\geq$ geometric dimension.

In 1994 Kapovich [K94] produced an example of a hyperbolic group with geometric dimension 2 which does not act properly discontinuously by isometries on a 2-dimensional CAT(0) complex. So in general the geometric dimension of a group G is not equal to the CAT(0) dimension.

Tom Brady [Bra00] showed that Artin groups (see Chapter 3) of finite type with less than or equal to three generators act cocompactly and isometrically on CAT(0) spaces of dimension 3.

Following results of Charney and Davis [CD95] and the results of Tom Brady, Noel Brady and John Crisp showed that an infinite subclass of the Artin groups $A(m, n, 2)$ have geometric dimension 2 and CAT(0) dimension

3. It is these groups that we will study in more detail.

1.7 Coxeter groups and Artin groups

The notion of a triangle group may be generalised to that of a Coxeter group. Let n be a natural number and $S = \{s_1, s_2, \dots, s_n\}$ a set. An $n \times n$ symmetric matrix (m_{ij}), $m_{ij} \in \{2, \dots\} \cup \{\infty\}$ for $i \neq j$ and $m_{ii} = 1$ for each i is said to be a Coxeter matrix. The Coxeter group W has presentation of the form $W = \langle S | (s_i s_j)^{m_{ij}} = 1 \rangle$. If $m_{ij} = \infty$ we take the associated relation to be the empty relation.

Closely related to Coxeter groups are Artin groups. Let $(a, b)_m = aba \dots$ be the alternating product of a and b of length m. A presentation for an Artin group A with generating set S and associated Coxeter matrix (m_{ij}) is

$$A = \left\langle S | (s_i, s_j)_{m_{ij}} = (s_j, s_i)_{m_{ji}} \right\rangle .$$

We say that A is of finite type if the associated Coxeter group (the Coxeter group with the same Coxeter matrix) is finite.

A dihedral group $D(n)$ is a group with presentation

$$\left\langle x, y | x^n = 1, y^2 = 1, y^{-1} x y = x^{-1} \right\rangle .$$

The last relation is equivalent to $yxyx = 1$, so substituting $z = yx$ we see $D(n)$ has presentation

$$\left\langle y, z | y^2 = z^2 = 1, (zy^{-1})^n \right\rangle .$$

Replacing y by y^{-1} we immediately see that $D(n)$ is a Coxeter group. The Artin group associated to $D(n)$ is called a Dihedral Artin group and denoted $A(n)$. This group has presentation

$$A(n) = \langle a, b | (a, b)_n = (b, a)_n \rangle .$$

Observe that if n is odd then $a.(a,b)_n = (a,b)_n.b$ and if n is even then $a.(a,b)_n = (a,b)_n.a$. Hence if n is odd, define z to be $(a,b)_{2n}$ and if n is even $z = (a,b)_n$, then $a.z = z.a$ and $b.z = z.b$.

Let A_1 and A_2 be two Artin groups with generator sets $S = \langle s_1 \ldots, s_n \rangle$ and $T = \langle t_1, \ldots, t_m \rangle$. Let H_1 and H_2 be subsets of S and T of equal size. We may suppose, without loss of generality, that $H_1 = \{s_1 \ldots, s_k\}$ and $H_2 = \{t_1, \ldots, t_k\}$. Suppose that ϕ is a isomorphism from H_1 to H_2 defined by $\phi(s_i) = t_i$. Then the amalgam $A_1 *_\phi A_2$ is also an Artin group.

Define the Artin group $A(p,q,r)$ by the presentation

$$A(p,q,r) = \langle a, b, c | (a,b)_p = (b,a)_p, (b,c)_q = (c,b)_q, (c,a)_r = (a,c)_r \rangle .$$

For any Artin group, it may be show (see [VdL]) that the subgroup generated by a subset of the Artin generators is itself an Artin group, subject only to those Artin relations that contain only those generators. Thus the subgroups $A(p)$, $A(q)$ and $A(r)$ of $A(p,q,r)$ generated by $\{a,b\}$,$\{b,c\}$ and $\{a,c\}$ respectively are dihedral Artin groups with presentations,

$$A(p) = \langle a, b' | (a,b')_p = (b',a)_p \rangle ,$$

$$A(q) = \langle b, c' | (b,c')_q = (c',b)_q \rangle ,$$

$$A(r) = \langle c, a' | (c,a')_r = (a',c)_r \rangle ,$$

Any Artin group is isomorphic to the colimit (or amalgam) of its 1- and 2-generator Artin subgroups, essentially because each relation involves only two generators. This amalgam is not in general a graph of groups decomposition however.

A lot of work has been done on calculating the dimension of 3-generator Artin groups. R. Charney and M. Davis [CD95] showed that the Artin group $A(p,q,r)$ has geometric dimension 2 if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$ and geometric dimension

3 if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$. Three dimensional locally CAT(0) CW-complexes were constructed by T. Brady [Bra00]. He showed these to be Eilenberg-MacLane complexes for 3-generator Artin groups of finite type (ie. when $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$). It follows that these Artin groups have CAT(0) dimension ≤ 3. T. Brady and J. McCammond [BMcC00] showed that the CAT(0) dimension of 3-generator Artin groups is 2 if $p, q, r \geq 3$.

To summarise, for finite type 3-generator Artin groups the CAT(0) dimension and the geometric dimension equal 3. For infinite type 3-generator Artin groups $A(p,q,r)$ the geometric dimension is 2 and if $p, q, r \geq 3$ this coincides with the CAT(0) dimension. This left unresolved the CAT(0) dimension for non-finite Artin groups $A(p,q,r)$ if one of p, q or r equals 2. N. Brady and J.Crisp proved the following 2 theorems,

Theorem 1.7.1. *[BC] Let m, n be odd integers ≥ 3. The Artin group $A(m,n,2)$ does not act properly discontinuously by semi-simple isometries on any 2-dimensional CAT(0) complex.*

Theorem 1.7.2. *[BC] All but finitely many of the 3-generator Artin groups $A(m,n,2)$ are the fundamental groups of compact locally CAT(0) 3-dimensional piecewise Euclidean complexes.*

It is a direct result of this that for all but finitely many of the Artin groups $A(p,q,2)$ where both p and q are odd, the CAT(0) dimension is 3. For all but finitely many of $A(p,q,2)$ where at least one of p and q is even the CAT(0) dimension is ≤ 3.

In fact theorem 1.7.2 deals with all but 65 of the non-finite Artin groups. In chapter 3 we extend theorem 1.7.2 to cover all but 41 of these groups. This leaves, by a result of chapter 2, just 19 groups with unknown CAT(0) dimension: $A(3, 2n+1, 2)$ for $3 \leq n \leq 19$ and $A(5, 2n+1, 2)$ for $2 \leq n \leq 3$.

In chapter 2 we exhibit 2-dimensional locally CAT(0) piecewise Euclidean Eilenberg-MacLane complexes for all non-finite 3-generator Artin groups $A(2p, q, 2)$. Hence for these groups the CAT(0) dimension is 2.

Chapter 2

Locally CAT(0) $K(G, 1)$ 2-complexes for $A(2m, n, 2)$

In this chapter we prove the following theorem:

Theorem 2.0.3. *The 3-generator Artin groups $A(2m, n, 2)$ of non-finite type are the fundamental groups of 2-dimensional locally CAT(0) piecewise Euclidean complexes.*

We prove this by exhibiting 2 classes of piecewise Euclidean simplicial complexes. Both classes are compact and locally CAT(0). One class has fundamental group $A(2m, 3, 2)$, $n \geq 3$ and the other $A(2m, n, 2)$, $m \geq 2$ and $n \geq 4$. Since the geometric dimension of these groups is known to be 2, it follows from this that the CAT(0) dimension is also 2. Recall that the Artin group $A(m, n, 2)$ has the following presentation (see section 1.7) :

$$A(m, n, 2) = \langle a, b, c | (a, b)_m = (b, a)_m, (b, c)_n = (c, b)_n, ac = ca \rangle$$

2.1 A complex for $A(2m, 3, 2)$, $m \geq 3$

We construct a finite simplical complex X_1 which is locally CAT(0) and has fundamental group $A(2m, 3, 2)$, $m \geq 3$. The complex X_1 is the union of $m-1$

vertices, $5m-2$ directed edges of unit length and $4m$ unit equilateral Euclidean triangles. Label the vertices $v_0, \ldots, v_{m-2}$, and the edges $e_0, \ldots, e_{5m-3}$. For each triangle T_i orient the edges clockwise and label them $t_i^{(1)}, t_i^{(2)}$ and $t_i^{(3)}$ respectively. We define a map ϕ from the disjoint union $\coprod T_i \coprod e_j \coprod v_k$ to X_1 such that ϕ restricted to a single cell in the union is an orientation preserving isometry. We use the notation e^{-1} to represent the edge with initial vertex τe and terminal vertex ιe. The map ϕ is defined on the cells in the following way. Note that for clarity we omit the map ϕ from our description, hence $\tau e_i = v_j$ should be read as $\phi(\tau e_i) = \phi(v_j)$.

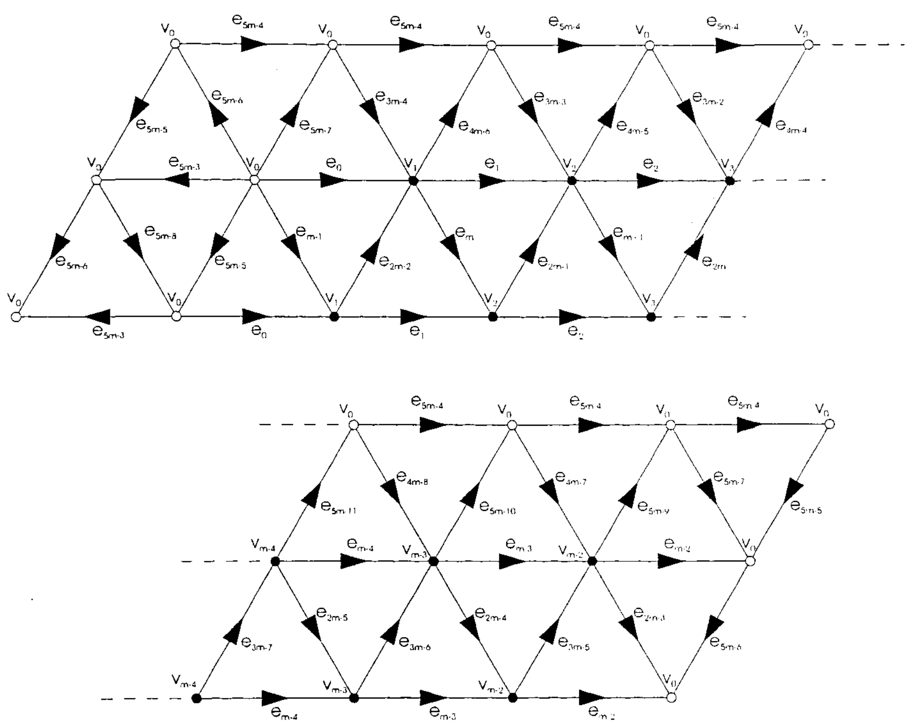

Figure 2.1: Complex for $A(2m, 3, 2)$, $m \geq 3$

Building the 1-skeleton, attaching edges to vertices:

$$
\begin{aligned}
\iota e_i &= v_i && 0 \leq i \leq m-2 \\
&= v_{i-m+1} && m-1 \leq i \leq 2m-3 \\
&= v_{i-2m+3} && 2m-2 \leq i \leq 3m-5 \\
&= v_0 && 3m-4 \leq i \leq 4m-7 \\
&= v_{i-4m+7} && 4m-6 \leq i \leq 5m-9 \\
&= v_0 && 5m-8 \leq i \leq 5m-3 \\
\tau e_i &= v_{i+1} && 0 \leq i \leq m-3 \\
\tau e_{m-2} &= v_0 && \\
\tau e_i &= v_{i-m+2} && m-1 \leq i \leq 2m-4 \\
\tau e_{2m-3} &= v_0 && \\
\tau e_i &= v_{i-2m+3} && 2m-2 \leq i \leq 3m-5 \\
\tau e_i &= v_{i-3m+5} && 3m-4 \leq i \leq 4m-7 \\
\tau e_i &= v_0 && 4m-6 \leq i \leq 5m-3
\end{aligned}
$$

Building the 2-skeleton, attaching triangles to edges:

$$
\begin{aligned}
t_1^{(1)} &= e_{5m-7} && \\
t_1^{(2)} &= (e_{5m-4})^{-1} && \\
t_1^{(3)} &= (e_{5m-6})^{-1} && \\
t_i^{(1)} &= e_{4m-8+i} && 2 \leq i \leq m \\
t_i^{(2)} &= (e_{5m-4})^{-1} && 2 \leq i \leq m-1 \\
t_i^{(3)} &= e_{3m-6+i} && 2 \leq i \leq m-1 \\
t_m^{(1)} &= (e_{5m-5})^{-1} && \\
t_m^{(3)} &= (e_{5m-7})^{-1} && \\
t_{m+1}^{(1)} &= (e_{5m-6})^{-1} && \\
t_{m+1}^{(2)} &= e_{5m-3} && \\
t_{m+1}^{(3)} &= (e_{5m-5})^{-1} && \\
t_{m+2}^{(3)} &= e_{5m-7} &&
\end{aligned}
$$

$$
\begin{array}{rcll}
t_i^{(1)} & = & e_{2m-6+i} & m+2 \le i \le 2m-1 \\
t_i^{(2)} & = & (e_{-m-2+i})^{-1} & m+2 \le i \le 2m \\
t_i^{(3)} & = & e_{3m-9+i} & m+2 \le i \le 2m \\
t_{2m}^{(1)} & = & e_{5m-7} & \\
t_{2m+1}^{(1)} & = & (e_{5m-5})^{-1} & \\
t_{2m+1}^{(2)} & = & e_{5m-3} & \\
t_{2m+1}^{(3)} & = & e_{5m-8} & \\
t_i^{(1)} & = & e_{-4+i} & 2m+2 \le i \le 3m-1 \\
t_i^{(2)} & = & (e_{-2m-2+i})^{-1} & 2m+2 \le i \le 3m \\
t_i^{(3)} & = & e_{-m-3+i} & 2m+2 \le i \le 3m \\
t_{3m}^{(1)} & = & (e_{5m-6})^{-1} & \\
t_{3m+1}^{(1)} & = & e_{5m-8} & \\
t_{3m+1}^{(2)} & = & e_{5m-3} & \\
t_{3m+1}^{(3)} & = & (e_{5m-6})^{-1} & \\
t_{3m+2}^{(1)} & = & e_{m-1} & \\
t_{3m+2}^{(2)} & = & (e_0)^{-1} & \\
t_{3m+2}^{(3)} & = & (e_{5m-5})^{-1} & \\
t_i^{(1)} & = & e_{-2m-3+i} & 3m+3 \le i \le 4m \\
t_i^{(2)} & = & (e_{-3m-2+i})^{-1} & 3m+3 \le i \le 4m \\
t_i^{(3)} & = & e_{-m-5+i} & 3m+3 \le i \le 4m
\end{array}
$$

Checking the CAT(0) property

To prove that this complex is locally CAT(0) we need to check that the link of each vertex is CAT(1) (see lemma 1.4.3). For a given vertex we need to know which edges have ends identified with this vertex, and then for each of these edges which faces of which triangles they are identified with.

For $i \in \{1 \ldots m-2\}$ and $j \in \{1 \ldots 6\}$,

$$\begin{aligned} v_i &= \iota e_i & &= \iota e_{m-1+i} & &= \iota e_{2m-3+i} \\ &= \iota e_{4m-7+i} & &= \tau e_{i-1} & &= \tau e_{m-2+i} \\ &= \tau e_{2m-3+i} & &= \tau e_{3m-5+i} \end{aligned}$$

and

$$\begin{aligned} v_0 &= \iota e_{3m-5+i} & &= \iota e_0 & &= \iota e_{m-1} \\ &= \iota e_{5m-9+j} & &= \tau e_{4m-7+i} & &= \tau e_{m-2} \\ &= \tau e_{2m-3} & &= \tau e_{5m-9+j} \end{aligned}$$

$$\begin{array}{rclcll}
e_i & = & (t^{(2)}_{m+2+i})^{-1} & & & i = 0, \ldots, m-2 \\
 & = & (t^{(2)}_{2m+2+i})^{-1} & & & \\
 & = & (t^{(2)}_{3m+2+i})^{-1} & & & \\
e_i & = & t^{(3)}_{m+3+i} & = & t^{(1)}_{2m+3+i} & i = m-1, \ldots, 2m-3 \\
e_i & = & t^{(1)}_{4+i} & = & t^{(3)}_{m+5+i} & i = 2m-2, \ldots, 3m-5 \\
e_i & = & t^{(1)}_{-2m+6+i} & = & t^{(3)}_{-3m+6+i} & i = 3m-4, \ldots, 4m-7 \\
e_i & = & t^{(3)}_{-3m+9+i} & = & t^{(1)}_{-4m+8+i} & i = 4m-6, \ldots, 5m-9 \\
e_{5m-8} & = & t^{(1)}_{3m+1} & = & t^{(3)}_{2m+1} & \\
e_{5m-7} & = & t^{(1)}_{1} & = & t^{(3)}_{m} & \\
 & = & t^{(3)}_{m+2} & = & t^{(1)}_{2m} & \\
e_{5m-6} & = & (t^{(3)}_{1})^{-1} & = & (t^{(1)}_{m+1})^{-1} & \\
 & = & (t^{(3)}_{3m+1})^{-1} & = & (t^{(1)}_{3m})^{-1} & \\
e_{5m-5} & = & (t^{(3)}_{m+1})^{-1} & = & (t^{(1)}_{m})^{-1} & \\
 & = & (t^{(1)}_{2m+1})^{-1} & = & (t^{(3)}_{3m+2})^{-1} & \\
e_{5m-4} & = & (t^{(2)}_{j})^{-1} & & & j = 1, \ldots, m \\
e_{5m-3} & = & t^{(2)}_{m+1} & = & t^{(2)}_{2m+1} & \\
 & = & t^{(2)}_{3m+1} & & &
\end{array}$$

Consider the vertex v_i for $i \in \{1, \ldots, m-2\}$. From the above data we see that v_i is identified with 8 initial or terminal vertices of edges, these edge vertices correspond to the vertices in the link of v_i. The vertices of the triangles which these edges are identified with are shown below. Note that for a triangle T, $\iota t^{(1)} = \tau t^{(3)}$, $\iota t^{(2)} = \tau t^{(1)}$ and $\iota t^{(3)} = \tau t^{(2)}$.

$$\begin{array}{lllllll}
\iota e_i & = & \tau t^{(2)}_{m+2+i} & = & \tau t^{(2)}_{2m+2+i+} & = & \tau t^{(2)}_{3m+2+i} \\
\iota e_{m-1+i} & = & \iota t^{(3)}_{2m+2+i} & = & \iota t^{(1)}_{3m+2+i} & & \\
\iota e_{2m-3+i} & = & \iota t^{(1)}_{2m+1+i} & = & \iota t^{(3)}_{3m+2+i} & & \\
\iota e_{4m-7+i} & = & \iota t^{(3)}_{m+2+i} & = & \iota t^{(1)}_{1+i} & & \\
\tau e_{i-1} & = & \iota t^{(2)}_{m+1+i} & = & \iota t^{(2)}_{2m+1+i+} & = & \iota t^{(2)}_{3m+1+i} \\
\tau e_{m-2+i} & = & \tau t^{(3)}_{2m+1+i} & = & \tau t^{(1)}_{3m+1+i} & & \\
\tau e_{2m-3+i} & = & \tau t^{(1)}_{2m+1+i} & = & \tau t^{(3)}_{3m+2+i} & & \\
\tau e_{3m-5+i} & = & \tau t^{(1)}_{m+1+i} & = & \tau t^{(3)}_{1+i} & &
\end{array}$$

From this we see that there are 9 edges $l_1, \ldots, l_9$ in the link;

$$l_1 = (\iota e_i, \iota e_{4m-7+i}), l_2 = (\iota e_i, \iota e_{m-1+i}), l_3 = (\iota e_i, \iota e_{2m-3+i})$$

$$l_4 = (\iota e_{m-1+i}, \tau e_{2m-3+i}), l_5 = (\iota e_{2m-3+i}, \tau e_{m-2+i}), l_6 = (\iota e_{4m-7+i}, \tau e_{3m-5+i}),$$

$$l_7 = (\tau e_{i-1}, \tau e_{3m-5+i}), l_8 = (\tau e_{i-1}, \tau e_{2m-3+i}), l_9 = (\tau e_{i-1}, \tau e_{m-2+i})$$

To check that this link is CAT(1) we apply the algorithm developed in section 1.4.1. Since each 2-cell in X_1 is a Euclidean equilateral triangle, each edge has length $\frac{\pi}{3}$ so the algorithm is applicable. There are only single edges between any pair of vertices so we may proceed directly to step 2 of the algorithm. The weighted adjacency matrix $Dist(\mathcal{L})$ for this link is a multiple by $\frac{\pi}{3}$ of the adjacency matrix. Since every entry in $Dist(\mathcal{L})$ is a multiple of $\frac{\pi}{3}$ we consider $\frac{3}{\pi}Dist(\mathcal{L})$ and check for loops of length strictly less than 6.

$$\frac{3}{\pi}Dist(\mathcal{L}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The diagonal entries of $Dist(\mathcal{L})$ are all zero, so we proceed to step 3.

We form a new matrix $D = \frac{3}{\pi}Dist(\mathcal{L})$ and apply the operation ρ to row 2 which has just two non-zero entries in columns 1 and 7. Now $d_{17} = 0$, so we set $d_{17} = d_{71} = d_{21} + d_{27} = 2$, $d_{12} = d_{21} = d_{27} = d_{72} = 0$ and remove the row of zeros (row 2). The resulting matrix is as follows;

$$D = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Similarly we apply the same operation to rows 2 and 3. We obtain the following matrix;

$$D = \begin{pmatrix} 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Now applying the operation ρ to row 3, $d_{31} = 2$, $d_{32} = 1$ and $d_{21} = 0$. We set $d_{21} = d_{12} = d_{31} + d_{32} = 3$ and $d_{31} = d_{13} = d_{32} = d_{23} = 0$ and remove row 3.

$$D = \begin{pmatrix} 0 & 3 & 2 & 2 \\ 3 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

Again we apply ρ, this time to row 3. The entry d_{31} is 2, d_{32} is 1 and d_{12} is 3. Now $d_{31} + d_{32} + d_{12} = 6 \neq 0$. The minimum of d_{12} and $d_{31} + d_{32}$ is 3, so we set $d_{12} = d_{21} = 3$ and $d_{31} = d_{13} = d_{32} = d_{23} = 0$ and remove row 3.

$$D = \begin{pmatrix} 0 & 3 & 2 \\ 3 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

Applying ρ to row 1, $d_{12} = 3$, $d_{13} = 2$ and $d_{23} = 1$. The sum of these edges is equal to 2π so we set d_{23} to $\min\{d_{23}, d_{12} + d_{13}\} = 1$ and $d_{12} = d_{21} = d_{13} = d_{31} = 0$. The matrix D has just two non-zero elements, one in row 2 and one in row 3, so we may conclude that the link complex $\mathcal{L}$ is CAT(1).

We now construct the link of v_0. We observe that the vertex v_0 is identified with the following initial and terminal vertices of edges;

$$For \quad 1 \leq i \leq m-2, \quad 1 \leq j \leq 6$$

$$\iota e_0, \ \iota e_{m-1}, \ \tau e_{m-2}, \ \tau e_{2m-3}$$

$$\iota e_{3m-5+i}, \ \tau e_{4m-7+i}$$

$$\iota e_{5m-9+j}, \ \tau e_{5m-9+j}$$

Thus the link of v_0 in X_1 has $2m+12$ vertices. The vertices of the triangles which these edge vertices are identified with are listed below. We label the vertices in the link l_k, $1 \leq k \leq 2m+12$ and use $\iota t^{(1)} = \tau t^{(3)}$, $\iota t^{(2)} = \tau t^{(1)}$, $\iota t^{(3)} = \tau t^{(2)}$ to show each identity in the form $*e_* = \iota t_*^{(*)}$.

$$\begin{aligned}
l_1 \ &:= \iota e_0 & &= \tau t_{m+2}^{(2)} & &= \iota t_{m+2}^{(3)} \\
& & &= \tau t_{2m+2}^{(2)} & &= \iota t_{2m+2}^{(3)} \\
& & &= \tau t_{3m+2}^{(2)} & &= \iota t_{3m+2}^{(3)} \\
l_2 \ &:= \iota e_{m-1} & &= \iota t_{2m+2}^{(3)} & & \\
& & &= \iota t_{3m+2}^{(1)} & & \\
l_3 \ &:= \tau e_{m-2} & &= \iota t_{2m}^{(2)} & & \\
& & &= \iota t_{3m}^{(2)} & & \\
& & &= \iota t_{4m}^{(2)} & & \\
l_4 \ &:= \tau e_{2m-3} & &= \tau t_{3m}^{(3)} & &= \iota t_{3m}^{(1)} \\
& & &= \tau t_{4m}^{(1)} & &= \iota t_{4m}^{(2)}
\end{aligned}$$

$$
\begin{array}{lllll}
l_{4+i} & := \iota e_{3m-5+i} & = \iota t^{(1)}_{m+1+i} & & 1 \le i \le m-2 \\
 & & = \iota t^{(3)}_{1+i} & & \\
l_{m+2+i} & := \tau e_{4m-7+i} & = \tau t^{(3)}_{m+2+i} & = \iota t^{(1)}_{m+2+i} & 1 \le i \le m-2 \\
 & & = \tau t^{(1)}_{1+i} & = \iota t^{(2)}_{1+i} & \\
l_{2m+1} & := \iota e_{5m-8} & = \iota t^{(1)}_{3m+1} & & \\
 & & = \iota t^{(3)}_{2m+1} & & \\
l_{2m+2} & := \iota e_{5m-7} & = \iota t^{(1)}_{1} & & \\
 & & = \iota t^{(3)}_{m} & & \\
 & & = \iota t^{(1)}_{2m} & & \\
 & & = \iota t^{(3)}_{m+2} & & \\
l_{2m+3} & := \iota e_{5m-6} & = \tau t^{(3)}_{1} & = \iota t^{(1)}_{1} & \\
 & & = \tau t^{(1)}_{m+1} & = \iota t^{(2)}_{m+1} & \\
 & & = \tau t^{(3)}_{3m+1} & = \iota t^{(1)}_{3m+1} & \\
 & & = \tau t^{(1)}_{3m} & = \iota t^{(2)}_{3m} & \\
l_{2m+4} & := \iota e_{5m-5} & = \tau t^{(3)}_{m+1} & = \iota t^{(1)}_{m+1} & \\
 & & = \tau t^{(1)}_{m} & = \iota t^{(2)}_{m} & \\
 & & = \tau t^{(3)}_{3m+2} & = \iota t^{(1)}_{3m+2} & \\
 & & = \tau t^{(1)}_{2m+1} & = \iota t^{(2)}_{2m+1} & \\
l_{2m+5} & := \iota e_{5m-4} & = \tau t^{(2)}_{k} & = \iota t^{(3)}_{k} & 1 \le k \le m \\
l_{2m+6} & := \iota e_{5m-3} & = \iota t^{(2)}_{m+1} & & \\
 & & = \iota t^{(2)}_{2m+1} & & \\
 & & = \iota t^{(2)}_{3m+1} & & \\
l_{2m+7} & := \tau e_{5m-8} & = \tau t^{(1)}_{3m+1} & = \iota t^{(2)}_{3m+1} & \\
 & & = \tau t^{(3)}_{2m+1} & = \iota t^{(1)}_{2m+1} & \\
l_{2m+8} & := \tau e_{5m-7} & = \tau t^{(1)}_{1} & = \iota t^{(2)}_{1} & \\
 & & = \tau t^{(3)}_{m} & = \iota t^{(1)}_{m} & \\
 & & = \tau t^{(1)}_{2m} & = \iota t^{(2)}_{2m} & \\
 & & = \tau t^{(3)}_{m+2} & = \iota t^{(1)}_{m+2} & \\
l_{2m+9} & := \tau e_{5m-6} & = \iota t^{(3)}_{1} & & \\
 & & = \iota t^{(1)}_{m+1} & & \\
 & & = \iota t^{(3)}_{3m+1} & & \\
 & & = \iota t^{(1)}_{3m} & &
\end{array}
$$

$$
\begin{aligned}
l_{2m+10} &:= \tau e_{5m-5} && = \iota t^{(3)}_{m+1} \\
&&& = \iota t^{(1)}_{m} \\
&&& = \iota t^{(3)}_{3m+2} \\
&&& = \iota t^{(1)}_{2m+1} \\
l_{2m+11} &:= \tau e_{5m-4} && = \iota t^{(2)}_{k} && && 1 \le k \le m \\
l_{2m+12} &:= \tau e_{5m-3} && = \tau t^{(2)}_{m+1} && \iota t^{(3)}_{m+1} \\
&&& = \tau t^{(2)}_{2m+1} && \iota t^{(3)}_{2m+1} \\
&&& = \tau t^{(2)}_{3m+1} && \iota t^{(3)}_{3m+1}
\end{aligned}
$$

Using this data we build the adjacency matrix for this link. There are $6m + 44$ triangles listed above, so there are $3m + 22$ edges in the link. To make the matrix easier to read, we replace the zero entries with a point.

$$
\begin{matrix}
1 & & & & & 6 & \ldots & m+2 & m+3 & \ldots & 2m-1 & 2m & & & & & & & & & & & & 2m+12
\end{matrix}
$$

$$
\begin{pmatrix}
. & 1 & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & \ddots & . & . & . & . & . & . & \vdots & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & \ddots & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & \vdots & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 \\
1 & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & 1 & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & 1 & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & 1 & . \\
. & . & . & . & 1 & 1 & \cdots & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . \\
. & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & 1 \\
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & 1 \\
. & . & . & . & . & . & . & . & 1 & \cdots & 1 & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & 1 & . & .
\end{pmatrix}
$$

Again the weighted adjacency matrix is $\frac{\pi}{3}$ times the adjacency matrix, so we work with the adjacency matrix and check for loops of length strictly less than 6.

For $i \in \{6, \ldots, m+2\}$ row i has 2 non-zero entries, $d_{i(m-5+i)} = 1$ and $d_{i(2m+5)} = 1$. Now $d_{(m-3+i)(2m+5)} = 0$, so applying ρ to row i we set $d_{(m-3+i)(2m+5)} = d_{(2m+5)(m-3+i)} = 2$ and $d_{i(m-3+i)} = d_{(m-3+i)i} = d_{i(2m+5)} =$

$d_{(2m+5)i} = 0$. We observe that $(m-3+i), (2m+5) > i$ for $i = 6, \ldots, m+2$, so we may apply ρ in exactly the same way to row i for $i = 6, \ldots, m+2$. This reduces row 6 through to $m+2$ to zeros.

Now, for $i \in \{m+3, \ldots, 2m-1\}$ row i has 2 non-zero entries, $d_{i(2m+5)} = 2$ and $d_{i(2m+11)} = 1$. The entry $d_{(2m+5)(2m+11)}$ is 0. We apply ρ to row $m+3$. We set $d_{(m+3)(2m+5)} = d_{(m+3)(2m+11)} = 0$ and $d_{(2m+5)(2m+11)} = 3$.

Now consider the application of ρ to row $i \in \{m+4, \ldots, 2m-1\}$. We have, $d_{i(2m+5)} = 2$, $d_{i(2m+11)} = 1$ and $d_{(2m+5)(2m+11)} = 3$, so we set $d_{i(2m+5)} = d_{i(2m+11)} = 0$ and $d_{(2m+5)(2m+11)} = 3$. Thus we may apply ρ to all of these rows. The result is a matrix with zeros in rows 6 to $2m-1$. We may therefore remove these rows to produce the following 19×19 matrix:

$$\begin{pmatrix}
. & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . \\
1 & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 \\
1 & . & . & . & . & 1 & . & . & 1 & . & 1 & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & 1 & 1 & . & . & . & 1 & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . & 1 & . \\
. & . & . & . & 1 & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & 1 & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & . \\
. & . & 1 & . & 1 & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . \\
. & . & . & 1 & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . & 1 \\
1 & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & . & 1 \\
. & . & . & . & . & 1 & . & . & . & 1 & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & 1 & . & .
\end{pmatrix}$$

Rows $2, 4, 5, 6, 7$ and 13 have two entries:

Row 2 $\quad d_{2,1} = d_{2,10} = 1, \quad$ and $\quad d_{1,10} = 0,$

Row 4 $\quad d_{4,3} = d_{4,15} = 1, \quad$ and $\quad d_{3,15} = 0,$

Row 5 $d_{5,11} = d_{5,14} = 1$, and $d_{11,14} = 0$,

Row 6 $d_{6,8} = d_{6,17} = 1$, and $d_{8,17} = 0$,

Row 7 $d_{7,9} = d_{7,18} = 1$, and $d_{9,18} = 0$,

Row 13 $d_{13,12} = d_{13,16} = 1$, and $d_{12,16} = 0$.

We apply ρ to each of these rows setting $d_{1,10} = d_{3,15} = d_{11,14} = d_{8,17} = d_{9,18} = d_{12,16} = 0$ and reducing to zero the rows $2, 4, 5, 6, 7$ and 13. The matrix formed by removing these zero rows is as follows,

$$\begin{pmatrix}
. & . & 1 & . & 2 & . & . & . & . & 1 & . & . \\
. & . & . & 1 & . & . & . & 1 & 2 & . & . & . \\
1 & . & . & 1 & . & 1 & . & . & . & . & 2 & . \\
. & 1 & 1 & . & . & . & 1 & . & . & . & . & 2 \\
2 & . & . & . & . & . & 1 & . & 1 & . & 1 & . \\
. & . & 1 & . & . & . & . & 2 & 1 & . & . & . \\
. & . & . & 1 & 1 & . & . & . & . & 2 & . & . \\
. & 1 & . & . & . & 2 & . & . & . & 1 & 1 & . \\
. & 2 & . & . & 1 & 1 & . & . & . & . & . & 1 \\
1 & . & . & . & . & . & 2 & 1 & . & . & . & 1 \\
. & . & 2 & . & 1 & . & . & 1 & . & . & . & . \\
. & . & . & 2 & . & . & . & . & 1 & 1 & . & .
\end{pmatrix}$$

Apply ρ to row 1 and row 2,

Row 1 $d_{1,3} = 1,\ d_{1,5} = 2,\ d_{1,10} = 1$ $\quad d_{3,5} = 0,\ d_{3,10} = 0,\ d_{5,10} = 0$

So set $d_{1,3} = d_{1,5} = d_{1,10} = 0$ and $d_{3,5} = 3,\ d_{3,10} = 2,\ d_{5,10} = 3$.

Row 2 $d_{2,4} = 1,\ d_{2,8} = 1,\ d_{2,9} = 2$ $\quad d_{4,8} = 0,\ d_{4,9} = 0,\ d_{8,9} = 0$

So set $d_{2,4} = d_{2,8} = d_{2,9} = 0$ and $d_{4,8} = 2,\ d_{4,9} = 3,\ d_{8,9} = 3$.

Again, we remove the zero rows (rows 1 and 2) and obtain the following matrix,

$$\begin{pmatrix} . & 1 & 3 & 1 & . & . & . & 2 & 2 & . \\ 1 & . & . & . & 1 & 2 & 3 & . & . & 2 \\ 3 & . & . & . & 1 & . & 1 & 3 & 1 & . \\ 1 & . & . & . & . & 2 & 1 & . & . & . \\ . & 1 & 1 & . & . & . & . & 2 & . & . \\ . & 2 & . & 2 & . & . & 3 & 1 & 1 & . \\ . & 3 & 1 & 1 & . & 3 & . & . & . & 1 \\ 2 & . & 3 & . & 2 & 1 & . & . & . & 1 \\ 2 & . & 1 & . & . & 1 & . & . & . & . \\ . & 2 & . & . & . & . & 1 & 1 & . & . \end{pmatrix}$$

Apply ρ to row 4, 5, 9 and 10,

Row 4 $d_{4,1} = 1,\ d_{4,6} = 2,\ d_{4,7} = 1$ $\quad d_{1,6} = 0,\ d_{1,7} = 0,\ d_{6,7} = 3$

So set $d_{4,1} = d_{4,6} = d_{4,7} = 0$ and $d_{1,6} = 3,\ d_{1,7} = 2,\ d_{6,7} = 3.$

Row 5 $d_{5,2} = 1,\ d_{5,3} = 1,\ d_{5,8} = 2$ $\quad d_{2,3} = 0,\ d_{2,8} = 0,\ d_{3,8} = 3$

So set $d_{5,2} = d_{5,3} = d_{5,8} = 0$ and $d_{2,3} = 2,\ d_{2,8} = 3,\ d_{3,8} = 3.$

Row 9 $d_{9,1} = 2,\ d_{9,3} = 1,\ d_{9,6} = 1$ $\quad d_{1,3} = 3,\ d_{1,6} = 3,\ d_{3,6} = 0$

So set $d_{9,1} = d_{9,3} = d_{9,6} = 0$ and $d_{1,3} = 3,\ d_{1,6} = 3,\ d_{3,6} = 2.$

Row 10 $d_{10,2} = 2,\ d_{10,7} = 1,\ d_{10,8} = 1$ $\quad d_{2,7} = 3,\ d_{2,8} = 3,\ d_{7,8} = 0$

So set $d_{10,2} = d_{10,7} = d_{10,8} = 0$ and $d_{2,7} = 3,\ d_{2,8} = 3,\ d_{7,8} = 2.$

Again remove the zero rows;

$$D = \begin{pmatrix} . & 1 & 3 & 3 & 2 & 2 \\ 1 & . & 2 & 2 & 3 & 3 \\ 3 & 2 & . & 2 & 1 & 3 \\ 3 & 2 & 2 & . & 3 & 1 \\ 2 & 3 & 1 & 3 & . & 2 \\ 2 & 3 & 3 & 1 & 2 & . \end{pmatrix}$$

Rather than repeatedly apply ρ, we recall that the link is CAT(1) if and only if the graph $\mathcal{L}$ associated with D has no non-trivial edge loops of length strictly less than 6. Since $\mathcal{L}$ has no bigons, any loop must contain at least 3 edges. Let γ be an edge path of 3 edges. Every edge in $\mathcal{L}$ has length 1, 2 or

3 so the length of γ is less than 6 if and only if it contains an edge of length 1. Let us suppose the second (or central) edge of γ has length 1. We observe that for $1 \leq i, j \leq 6$ and $k \neq i, j$, if $d_{i,j} = 1$ then $d_{i,k} + d_{k,i} = 5$ thus γ has length at least 6 and hence any non-trivial loop in $\mathcal{L}$ has length at least 6, so the link is CAT(1).

Lastly, we check that the complex has the correct fundamental group. Suppose the triangle T has 2 distinct vertices, say ιt_1 and ιt_2. We may homotopically contract the edge t_1 to ιt_1 forming a bigon with edges t_2 and t_3. We may then homotopically retract the interior of the bigon on to these edges, identifying t_2 with t_3^{-1}. Hence if a cell complex contains a triangle with two distinct vertices, we may contract that triangle to an edge, identifying the two vertices, without changing the fundamental group of the complex.

Observe that the sets of three triangles T_{m+2+i}, T_{2m+2+i} and T_{3m+2+i} for $1 \leq i \leq m-3$ each share an edge e_i, and each have two distinct vertices v_i and v_{i+1}. Homotopically retracting T_{m+2+i}, T_{2m+2+i} and T_{3m+2+i} via the shared edge e_i we identify the edges $e_{4m-7+i} = (e_{3m-4+i})^{-1}$ and $e_{2m-3+i} = (e_{m-1+i})^{-1} = e_{2m-2+i}$. Finally retracting e_{m-2} on to $v_0 = \tau e_{m-2}$ we identify the edges e_{5m-9} with $(e_{5m-7})^{-1}$, e_{3m-5} with $(e_{2m-3})^{-1}$ and with e_{5m-6}. We obtain a complex with 1 vertex, v_0, $m+1$ edges, and $m+5$ triangles (see figure 2.2).

The $m+11$ edges are labeled $e_0, e_{m-1}, e_{3m-3}, e_{3m-2}, \ldots, e_{4m-7}, e_{5m-8}$, $e_{5m-7}, \ldots, e_{5m-3}$. Observe that the edges $e_{m-1}, e_{3m-4}, e_{3m-3}, \ldots, e_{4m-7}$ and e_{5m-8} are each an edge of exactly 2 triangles. Thus we may remove these edges and consider (by extending e_0) the union of the triangles T_{2m+2} and T_{3m+2} as a parallelogram C_1, the triangles T_{m+1} and $T_2, \ldots, T_{m-1}$ as a trapezium C_2 and the triangles T_{2m+1} and $_{3m+1}$ as a parallelogram C_3. We now have

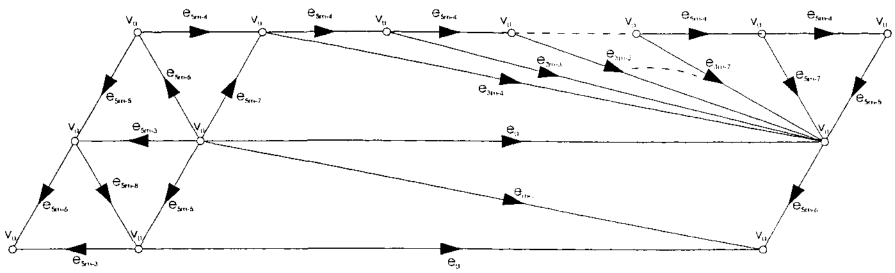

Figure 2.2: Partly reduced complex

a complex with 1 vertex v_0, 6 2-cells $T_1, T_m, T_{m+1}, C_1, C_2$ and C_3, and 5 edges $e_0, e_{5m-7}, e_{5m-6}, e_{5m-5}, e_{5m-4}$ and e_{5m-3}. Relabel the edges $\delta := e_0, \alpha := e_{5m-7}, b := e_{5m-6}, c := e_{5m-5}, x := e_{5m-4}$ and $y := e_{5m-3}$ (see figure 2.3).

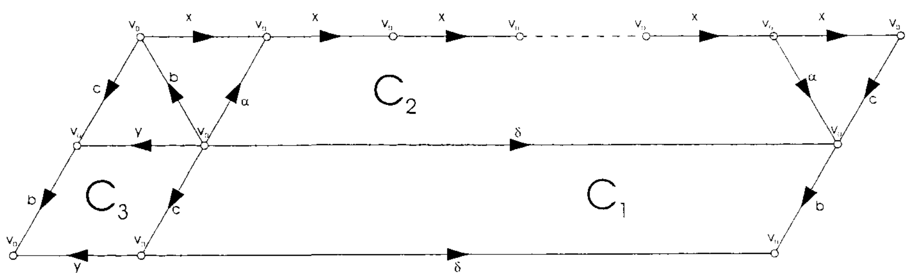

Figure 2.3: Presentation complex for X_1

Since we now have just one vertex, every edge is a loop, so may represent a generator in a presentation for the fundamental group of the complex. The boundary of any 2-cell is a loop which may be homotopically retraced to a point (via the 2-cell), thus the boundary of a 2-cell represents a relator in the presentation. So we may read a presentation for the fundamental group directly from the complex in figure 2.3.

$$\begin{aligned}\pi_1(X_1) \cong \langle \alpha, b, c, \delta, x, y | \quad & xc\alpha^{-1}, bx\alpha^{-1}, bcy^{-1} \\ & yby^{-1}c^{-1}, \delta b\delta^{-1}c^{-1}, \alpha x^{m-2}\alpha\delta^{-1} \rangle\end{aligned}$$

We make the following Tietze transformations to the presentation:

Taking $\alpha = bx$ and $y = bc$ to be definitions of these generators, we may remove them from the presentation;

$$\pi_1(X_1) \cong \langle b, c, \delta, x | xcx^{-1}b^{-1}, bcbc^{-1}b^{-1}c^{-1}, \delta b\delta^{-1}c^{-1}, bx^{m-1}bx\delta^{-1}\rangle$$

Now take $\delta = bx^{m-1}bx$ to be a definition and remove δ from the presentation;

$$\pi_1(X_1) \cong \langle b, c, x | xcx^{-1}b^{-1}, bcbc-1b^{-1}c^{-1}, bx^{m-1}bxb(bx^{m-1}bx)^{-1}c^{-1}\rangle$$

We now add a generator $a := b^{-1}c^{-1}xc$ and then remove the generator $x = cbac^{-1}$. Equivalently we substitute $cbac^{-1}$ for x;

$$\begin{aligned}\pi_1(X_1) \cong \quad \langle a, b, c| \quad & cbac^{-1}cca^{-1}b^{-1}c^{-1}b^{-1}, bcbc^{-1}b^{-1}c^{-1} \\ & b(cbac^{-1}bcbac^{-1}bca^{-1}b^{-1}c^{-1}b^{-1}(cbac^{-1})^{-(m-1)}b^{-1}c^{-1}\rangle\end{aligned}$$

In the group $\pi_1(X_1)$ $bcb = cbc$ (the second relation) so

$$cbac^{-1}cca^{-1}b^{-1}c^{-1}b^{-1} = cbacac^{-1}b^{-1}c^{-1}b^{-1} = cbaca^{-1}c^{-1}b^{-1}c^{-1}$$

and $cbaca^{-1}c^{-1}b^{-1}c^{-1} = 1$ if and only if $aca^{-1}c^{-1} = 1$. Thus the first relation is equivalent to $aca^{-1}c^{-1} = 1$.

Now $(cbac^{-1})^r = c(ba)^r c^{-1}$ for all $r \in \mathbb{Z}$ so the third relation is equivalent to $bc(ba)^{m-1}c^{-1}bcbac^{-1}bca^{-1}b^{-1}c^{-1}b^{-1}c^{-1}(bc)^{-(m-1)}c^{-1}b^{-1}c^{-1}$ and by the second relation this is equivalent to
$bc(ba)^{m-1}c^{-1}cbcac^{-1}bca^{-1}c^{-1}b^{-1}c^{-1}c(ba)^{-(m-1)}b^{-1}c^{-1}b^{-1}$. Cancelling adjacent inverses and using the first relation we find it is equivalent to,
$b^{-1}(ba)^{m-1}baba^{-1}b^{-1}(ba)^{-(m-1)} = (ab)^m(ba)^{-m}$. Thus;

$$\begin{aligned}\pi_1(X_1) \quad & \cong \langle a, b, c | (ab)^m(ba)^{-m}, bcbc^{-1}b^{-1}c^{-1}, aca^{-1}c^{-1}\rangle \\ & \cong \langle a, b, c | (a, b)_{2m} = (b, a)_{2m}, (b, c)_3 = (c, b)_3, (a, c)_2 = (c, a)_2\rangle \\ & \cong A(2m, 3, 2)\end{aligned}$$

as required.

2.2 A complex for $A(2m, n, 2)$, $m \geq 2$, $n \geq 4$

In this section we construct a finite simplicial complex X_2 which is locally CAT(0) and has fundamental group $A(2m, n, 2)$, $m \geq 2$, $n \geq 4$. The complex X_2 is a union of $m+1$ vertices, $5m+n+4$ edges and $4m+n+4$ equilateral triangles. Label the vertices $v_0, \ldots, v_m$, the edges $e_0, \ldots, e_{5m+n+3}$ and the triangles $T_1, \ldots, T_{4m+n+4}$. Label and orientate the edges of each triangle as for the previous complex (see section 2.1). As before we define a map ϕ from the disjoint union $\coprod T_i \coprod e_j \coprod v_k$ to X_2. Again, for simplicity we omit ϕ from our definition.

Attaching the 1-skeleton;

$$
\begin{array}{rcll}
\iota e_0 & = & v_m & \\
\iota e_i & = & v_i & 1 \leq i \leq m-1 \\
\iota e_m & = & v_0 & \\
\iota e_i & = & v_{i-m-1} & m+1 \leq i \leq 2m \\
 & = & v_{i-2m-1} & 2m+1 \leq i \leq 3m \\
 & = & v_m & 3m+1 \leq i \leq 4m \\
 & = & v_{i-4m-1} & 4m+1 \leq i \leq 5m \\
\iota e_{5m+1} & = & v_0 & \\
\iota e_{5m+2} & = & v_m & \\
\iota e_i & = & v_0 & 5m+3 \leq i \leq 5m+4 \\
\iota e_{5m+5} & = & v_m & \\
\iota e_i & = & v_0 & 5m+6 \leq i \leq 5m+n+3 \\
\tau e_0 & = & v_m & \\
\tau e_i & = & v_{i-1} & 1 \leq i \leq m \\
\tau e_i & = & v_m & m+1 \leq i \leq 2m \\
 & = & v_{i-2m-1} & 2m+1 \leq i \leq 3m \\
 & = & v_{i-3m} & 3m+1 \leq i \leq 4m-1 \\
\tau e_i & = & v_{i-4m} & 4m \leq i \leq 5m-1 \\
\tau e_i & = & v_0 & 5m \leq i \leq 5m+1 \\
\tau e_{5m+2} & = & v_m & \\
\tau e_{5m+3} & = & v_0 & \\
\tau e_{5m+4} & = & v_m & \\
\tau e_i & = & v_0 & 5m+5 \leq i \leq 5m+n+3
\end{array}
$$

Attaching the 2-skeleton;

$$\begin{array}{rcll}
t_i^{(1)} & = & e_{i+m+1} & 1 \leq i \leq m-1 \\
t_i^{(2)} & = & e_0 & 1 \leq i \leq m \\
t_i^{(3)} & = & e_{i+3m} & 1 \leq i \leq m \\
t_m^{(1)} & = & e_{m+1} & \\
t_i^{(1)} & = & e_{i+2m} & m+1 \leq i \leq 2m \\
t_i^{(2)} & = & e_{i-m} & m+1 \leq i \leq 2m \\
t_i^{(3)} & = & e_i & m+1 \leq i \leq 2m \\
t_i^{(1)} & = & e_{i+1} & 2m+1 \leq i \leq 3m-1 \\
t_i^{(2)} & = & e_{i-2m} & 2m+1 \leq i \leq 3m \\
t_i^{(3)} & = & e_{i+2m} & 2m+1 \leq i \leq 3m \\
t_{3m}^{(1)} & = & e_{2m+1} & \\
t_i^{(1)} & = & e_{i+m} & 3m+1 \leq i \leq 4m \\
t_i^{(2)} & = & e_{i-3m} & 3m+1 \leq i \leq 4m \\
t_i^{(3)} & = & e_{i-m} & 3m+1 \leq i \leq 4m \\
t_{4m+1}^{(1)} & = & e_{5m+4} & \\
t_{4m+1}^{(2)} & = & (e_{m+1})^{-1} & \\
t_{4m+1}^{(3)} & = & e_{5m+1} & \\
t_{4m+2}^{(1)} & = & e_{5m+2} & \\
t_{4m+2}^{(2)} & = & (e_{m+1})^{-1} & \\
t_{4m+2}^{(3)} & = & e_{5m+4} & \\
t_{4m+3}^{(1)} & = & e_{5m+5} & \\
t_{4m+3}^{(2)} & = & (e_{4m})^{-1} & \\
t_{4m+3}^{(3)} & = & e_{5m+2} & \\
t_{4m+4}^{(1)} & = & e_{5m+3} & \\
t_{4m+4}^{(2)} & = & (e_{4m})^{-1} & \\
t_{4m+4}^{(3)} & = & e_{5m+5} & \\
t_{4m+5}^{(1)} & = & e_{5m+6} & \\
t_{4m+5}^{(2)} & = & (e_{2m+1})^{-1} & \\
t_{4m+5}^{(3)} & = & e_{5m+3} & \\
t_{4m+6}^{(1)} & = & e_{5m+1} & \\
t_{4m+6}^{(2)} & = & (e_{2m+1})^{-1} & \\
t_{4m+6}^{(3)} & = & e_{5m+6} &
\end{array}$$

$$
\begin{array}{rcll}
t_i^{(1)} & = & e_{i+m+k-1} & & 4m+7 \le i \le 4m+k+4 \\
t_{4m+k+5}^{(1)} & = & e_{5m+4} & \text{if n even} & \\
 & = & e_{5m+2k+2} & \text{if n odd} & \\
t_i^{(1)} & = & e_{i+m-k+1} & & 4m+k+6 \le i \le 4m+2k+4 \\
t_{4m+2k+5}^{(1)} & = & e_{5m+4} & \text{if n odd} & \\
t_i^{(2)} & = & e_{5m+6} & & 4m+7 \le i \le 4m+n+4 \\
t_i^{(3)} & = & e_{i-m} & & 4m+7 \le i \le 4m+k+5 \\
t_{4m+k+6}^{(3)} & = & e_{5m+3} & & \\
t_i^{(3)} & = & e_{i+m-1} & & 4m+k+7 \le i \le 4m+2k+4 \\
t_{4m+2k+5}^{(3)} & = & e_{5m+2k+4} & \text{if n odd} &
\end{array}
$$

We now examine the links of the vertices. We begin by listing the edge vertices each v_i is attached to, and then the faces of the triangles which each edge is identified with.

$$
\begin{array}{rclll}
\text{for} & & 5m+6 \le i & \le 5m+n+3 & \\
v_0 & = & \iota e_m & = \iota e_{5m+1} & = \iota e_{5m+3} \\
 & = & \iota e_{5m+4} & = \iota e_i & = \tau e_1 \\
 & = & \tau e_{2m+1} & = \tau e_{3m} & = \tau e_{4m} \\
 & = & \tau e_{5m} & = \tau e_{5m+1} & = \tau e_{5m+3} \\
 & = & \tau e_i & = \iota e_{2m+1} & = \iota e_{m+1} \\
\text{and for} & & 1 \le i \le m-1 & & \\
v_i & = & \iota e_{i+m+1} & = \iota e_i & = \iota e_{4m+i+1} \\
 & = & \iota e_{2m+i+1} & = \tau e_{i+1} & = \tau e_{2m+i+1} \\
 & = & \tau e_{3m+i} & = \tau e_{4m+i} & \\
\text{and} & & & & \\
v_m & = & \iota e_j & & 3m+1 \le j \le 4m \\
 & = & \tau e_j & & m+1 \le j \le 2m \\
 & = & \iota e_0 & = \iota e_{5m+2} & = \iota e_{5m+4} \\
 & = & \tau e_0 & = \tau e_{5m+2} & = \tau e_{5m+5}
\end{array}
$$

$$
\begin{array}{rclcl}
e_0 & = & t_i^{(2)} & & 1 \le i \le m \\
e_i & = & t_{i+m}^{(2)} & & 1 \le i \le m \\
& = & t_{i+2m}^{(2)} & & \\
& = & t_{i+3m}^{(2)} & & \\
e_{m+1} & = & t_m^{(1)} & = & t_{m+1}^{(3)} \\
& = & (t_{4m+1}^{(2)})^{-1} & = & (t_{4m+2}^{(2)})^{-1} \\
e_i & = & t_{i-m-1}^{(1)} & & m+2 \le i \le 2m \\
& = & t_i^{(3)} & & \\
e_{2m+1} & = & t_{3m}^{(1)} & = & t_{3m+1}^{(3)} \\
& = & (t_{4m+5}^{(2)})^{-1} & = & (t_{4m+6}^{(2)})^{-1} \\
e_i & = & t_{i-1}^{(1)} & & 2m+2 \le i \le 3m \\
& = & t_{i+m}^{(3)} & & \\
e_i & = & t_{i-2m}^{(1)} & & 3m+1 \le i \le 4m-1 \\
& = & t_{i-3m}^{(3)} & & \\
e_{4m} & = & t_{2m}^{(1)} & = & t_m^{(3)} \\
& = & (t_{4m+3}^{(2)})^{-1} & = & (t_{4m+4}^{(2)})^{-1} \\
e_i & = & t_{i-m}^{(1)} & & 4m+1 \le i \le 5m \\
& = & t_{i-2m}^{(3)} & & \\
e_{5m+1} & = & t_{4m+6}^{(1)} & & \\
& = & t_{4m+1}^{(3)} & & \\
& = & t_{4m+k+5}^{(1)} & & \text{if n even} \\
& = & t_{4m+2k+5}^{(1)} & & \text{if n odd} \\
e_{5m+2} & = & t_{4m+2}^{(1)} & = & t_{4m+3}^{(3)} \\
e_{5m+3} & = & t_{4m+4}^{(1)} & = & t_{4m+5}^{(3)} \\
e_{5m+4} & = & t_{4m+1}^{(1)} & = & t_{4m+2}^{(3)} \\
e_{5m+5} & = & t_{4m+3}^{(1)} & = & t_{4m+4}^{(3)} \\
e_{5m+6} & = & t_{4m+5}^{(1)} & = & t_{4m+6}^{(3)} \\
& = & t_j^{(2)} & & 4m+7 \le j \le 5m+n+4 \\
e_i & = & t_{i+k-1}^{(1)} & & 5m+7 \le i \le 5m+k+5 \\
& = & t_{i-m}^{(3)} & & \\
e_i & = & t_{i-m-k+1}^{(1)} & & 5m+k+6 \le i \le 5m+n+3 \\
& = & t_{i-m+1}^{(3)} & &
\end{array}
$$

Consider the link of v_i for $1 \le i \le m-1$. We see that this link has 8 vertices.

$$\begin{array}{llllll}
l_1 := & \iota e_i & = \iota t^{(2)}_{i+m} & = \iota t^{(2)}_{i+2m} & = \iota t^{(2)}_{i+3m} \\
l_2 := & \tau e_{2m+i+1} & = \iota t^{(1)}_{3m+i+1} & = \iota t^{(2)}_{2m+i} & \\
l_3 := & \tau e_{3m+i} & = \iota t^{(1)}_{i} & = \iota t^{(2)}_{m+i} & \\
l_4 := & \tau e_{4m+i} & = \iota t^{(1)}_{2m+i} & = \iota t^{(2)}_{3m+i} & \\
l_5 := & \tau e_{i+1} & = \iota t^{(3)}_{m+i+1} & = \iota t^{(3)}_{2m+i+1} & = \iota t^{(3)}_{3m+i+1} \\
l_6 := & \iota e_{i+m+1} & = \iota t^{(1)}_{i} & = \iota t^{(3)}_{i+m+1} & \\
l_7 := & \iota e_{i+4m+1} & = \iota t^{(1)}_{3m+i+1} & = \iota t^{(3)}_{i+2m+1} & \\
l_8 := & \iota e_{i+2m+1} & = \iota t^{(1)}_{2m+i} & = \iota t^{(3)}_{i+3m+1} &
\end{array}$$

The adjacency matrix for this link is;

$$\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}$$

Observe that this adjacency matrix is exactly the same matrix as the first one dealt with in section 2.1. Thus by the same argument this link is CAT(1).

Consider the vertex v_m. The link of this vertex has $6 + 2m$ vertices

$$\begin{array}{lllll}
l_i := & \iota e_{3m+i} & = \iota t^{(1)}_{m+i} & = \iota t^{(3)}_{i} & 1 \le i \le m-1 \\
l_m := & \iota e_{4m} & = \iota t^{(1)}_{2m} & = \iota t^{(3)}_{m} & \\
 & = \iota t^{(3)}_{4m+3} & = \iota t^{(3)}_{4m+4} & & \\
l_{m+1} := & \tau e_{m+1} & = \iota t^{(2)}_{m} & = \iota t^{(3)}_{m+1} & \\
 & = \iota t^{(2)}_{4m+1} & = \iota t^{(2)}_{4m+2} & & \\
l_i := & \tau e_i & = \iota t^{(2)}_{i-m-1} & = \iota t^{(1)}_{i} & m+2 \le i \le 2m
\end{array}$$

$$\begin{array}{llllll}
l_{2m+1} := & \iota e_0 & = \iota t_j^{(2)} & & 1 \leq j \leq m \\
l_{2m+2} := & \tau e_0 & = \iota t_j^{(3)} & & 1 \leq j \leq m \\
l_{2m+3} := & \iota e_{5m+2} & = \iota t_{4m+2}^{(1)} & = \iota t_{4m+3}^{(3)} & \\
l_{2m+4} := & \tau e_{5m+2} & = \iota t_{4m+2}^{(3)} & = \iota t_{4m+3}^{(2)} & \\
l_{2m+5} := & \tau e_{5m+4} & = \iota t_{4m+1}^{(3)} & = \iota t_{4m+2}^{(2)} & \\
l_{2m+6} := & \iota e_{5m+5} & = \iota t_{4m+3}^{(1)} & = \iota t_{4m+4}^{(3)} &
\end{array}$$

Each edge has length $\frac{\pi}{3}$ so we build the adjacency matrix and apply the algorithm to $\frac{3}{\pi}$ this time, checking for loops of length strictly less than 6.

$$\begin{array}{cccccccccccccccccc}
1 & 2 & \dots & m-1 & m & m+1 & m+2 & \dots & 2m-1 & 2m & 2m+1 & 2m+2 & 2m+3 & 2m+4 & 2m+5 & 2m+6
\end{array}$$

$$\begin{pmatrix}
. & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & \ddots & . & . & . & \vdots & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & 1 & 1 & . & . & 1 \\
1 & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & 1 & . \\
. & 1 & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & \ddots & . & . & . & . & . & . & . & \vdots & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & 1 & 1 & \dots & 1 & 1 & . & . & . & . & . & . \\
1 & 1 & \dots & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . & .
\end{pmatrix}$$

Consider row i, for $1 \leq i \leq m-1$. There are 2 non-zero entries $d_{i,(m+i)} = 1$ and $d_{i,(2m+2)} = 1$. The entry $d_{(m+i)(2m+2)}$ is 0, so apply ρ and set $d_{i,(m+i)} = d_{(m+i),i} = d_{i,(2m+2)} = d_{(2m+2),i} = 0$ and $d_{(m+i)(2m+2)} = d_{(2m+2)(m+i)} = 2$. We have not altered the entries in row $i+1$ for $i \leq m-1$, nor in row $m+i$, so we may repeatedly apply ρ to rows $1, \dots, m-1$ obtaining the following matrix:

$$\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \vdots & \vdots & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 2 & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \ldots & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 2 & 2 & \ldots & 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{pmatrix}$$

Row i for $3 \leq i \leq m$ contains 2 non-zero entries, $d_{i,(m+2)} = 1$ and $d_{i(m+3)} = 2$. Now $d_{(m+2)(m+3)} = 0$, so we apply ρ to row i to set $d_{i,(m+2)} = d_{(m+2),i} = d_{i,(m+3)} = d_{(m+3),i} = 0$ and $d_{(m+2)(m+3)} = d_{(m+3)(m+2)} = 3$. Hence we may apply ρ to all row $3, \ldots, m$ setting these rows to zero and entry $d_{(m+2)(m+3)} = d_{(m+3)(m+2)}$ to 3.

$$\begin{pmatrix}
\cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & 1 & 2 & \cdot & 1 & 1 & \cdot \\
1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & 3 & \cdot & \cdot & \cdot & \cdot \\
1 & 2 & \cdot & 3 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{pmatrix}$$

Apply ρ to rows 3, 6 and 7, then 8 and 9; $\begin{pmatrix}
\cdot & 3 & 2 & 1 \\
3 & \cdot & 1 & 2 \\
2 & 1 & \cdot & 3 \\
1 & 2 & 3 & \cdot
\end{pmatrix}$

Lastly apply ρ to row 1 to obtain the matrix; $\begin{pmatrix} . & 1 & 2 \\ 1 & . & 3 \\ 2 & 3 & . \end{pmatrix}$

This matrix represents a CAT(1) graph as it is the complete graph on three points, a loop, with edges of length 1, 2 and 3. So the total length is 6.

Now consider the link of the vertex of v_0. This vertex is the initial and terminal vertex of e_j for $5m+7 \leq j \leq 5m+n+3$. Observe that these edges are each the edges of exactly two triangles. We may therefore amalgamate these pairs of triangles and remove the corresponding edges. We do this in two stages. Firstly we remove edges e_j, $5m+7 \leq j \leq 5m+k+5$, combining T_j and T_{j+k-1} to form a rhombus. We then remove e_j, $5m+k+6 \leq j \leq 5m+2k+3$ so the rhombi form a parallelogram with edges, (clockwise) $(e_{5m+6})^{k-1}, e_{5m+3}, (e_{5m+6})^{-k+1}$ and $(e_{5m+1})^{-1}$ if n is even, or $(e_{5m+n+3})^{-1}$ if n is odd.

We have now simplified our consideration of the link of v_0. This link now has 16 vertices when n is even and 18 when n is odd;

$$\begin{array}{llllll}
l_1 := & \iota e_m & = \iota t^{(2)}_{2m} & = \iota t^{(2)}_{3m} & = \iota t^{(2)}_{4m} & \\
l_2 := & \iota e_{m+1} & = \iota t^{(1)}_{m} & = \iota t^{(3)}_{m+1} & = \iota t^{(3)}_{4m+1} & = \iota t^{(3)}_{4m+2} \\
l_3 := & \iota e_{2m+1} & = \iota t^{(1)}_{3m} & = \iota t^{(3)}_{3m+1} & = \iota t^{(3)}_{4m+5} & = \iota t^{(3)}_{4m+6} \\
l_4 := & \iota e_{4m+1} & = \iota t^{(1)}_{3m+1} & = \iota t^{(3)}_{2m+1} & & \\
l_5 := & \iota e_{5m+1} & = \iota t^{(1)}_{4m+6} & = \iota t^{(3)}_{4m+1} & & \\
l_6 := & \iota e_{5m+3} & = \iota t^{(1)}_{4m+4} & = \iota t^{(3)}_{4m+5} & & \\
l_7 := & \iota e_{5m+4} & = \iota t^{(1)}_{4m+1} & = \iota t^{(3)}_{4m+2} & & \\
l_8 := & \iota e_{5m+6} & = \iota t^{(1)}_{4m+5} & = \iota t^{(3)}_{4m+6} & & \\
l_9 := & \tau e_m & = \iota t^{(3)}_{m+1} & = \iota t^{(3)}_{2m+1} & = \iota t^{(3)}_{3m+1} & \\
l_{10} := & \tau e_{2m+1} & = \iota t^{(2)}_{3m} & = \iota t^{(1)}_{3m+1} & = \iota t^{(2)}_{4m+5} & = \iota t^{(2)}_{4m+6} \\
l_{11} := & \tau e_{4m} & = \iota t^{(2)}_{2m} & = \iota t^{(1)}_{m} & = \iota t^{(2)}_{4m+3} & = \iota t^{(2)}_{4m+4} \\
l_{12} := & \tau e_{5m} & = \iota t^{(2)}_{4m} & = \iota t^{(1)}_{3m} & & \\
l_{13} := & \tau e_{5m+1} & = \iota t^{(2)}_{4m+6} & = \iota t^{(1)}_{4m+1} & & \\
l_{14} := & \tau e_{5m+3} & = \iota t^{(2)}_{4m+4} & = \iota t^{(1)}_{4m+5} & & \\
l_{15} := & \tau e_{5m+5} & = \iota t^{(2)}_{4m+3} & = \iota t^{(1)}_{4m+4} & & \\
l_{16} := & \tau e_{5m+6} & = \iota t^{(2)}_{4m+5} & = \iota t^{(1)}_{4m+6} & &
\end{array}$$

We observe that in addition to the edges in the link given by the above information there are the following edges;

$$\begin{array}{ccc}
2(k-2) & \pi - \text{arcs between} & l_8 \text{ and } l_{16} \\
1 & \frac{2\pi}{3} - \text{arc between} & l_{14} \text{ and } l_{16} \\
1 & \frac{\pi}{3} - \text{arc between} & l_8 \text{ and } l_{13} \\
 & \text{if } n \text{ is even} & \\
1 & \frac{2\pi}{3} - \text{arc between} & l_8 \text{ and } l_5 \\
 & \text{if } n \text{ is odd} & \\
1 & \frac{\pi}{3} - \text{arc between} & l_{18} \text{ and } l_5 \\
1 & \frac{\pi}{3} - \text{arc between} & l_{18} \text{ and } l_8 \\
1 & \frac{\pi}{3} - \text{arc between} & l_{17} \text{ and } l_{16} \\
1 & \frac{2\pi}{3} - \text{arc between} & l_{17} \text{ and } l_8
\end{array}$$

Since each arc between l_8 and l_{16} including the arc via l_{17}, is of length π, any pair makes a loop of length 2π, thus we may remove all but 1 of them. Every pair of vertices now has at most one edge between them. Vertex l_{18} is adjacent to exactly 2 edges so we may remove this vertex and combine the edges to form an arc of length $\frac{2\pi}{3}$ between l_8 and l_5, thus reducing the case to when n is even. Every edge is a multiple of $\frac{\pi}{3}$ so we construct the matrix $\frac{3}{\pi}Dist$ where $Dist$ is the weighted adjacency matrix and check for loops of length strictly less than 6.

$$\begin{pmatrix}
. & . & . & . & . & . & . & . & . & 1 & 1 & 1 & . & . & . & . \\
. & . & . & . & 1 & . & 1 & . & 1 & . & 1 & . & . & . & . & . \\
. & . & . & . & . & 1 & . & 1 & 1 & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & 1 & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & 2 & . & . & . & . & . & . & . & 1 \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & 1 & . & 2 & . & . & . & . & . & . & . & 1 & 1 & . & 3 \\
. & 1 & 1 & 1 & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . & . & 1 \\
1 & 1 & . & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . \\
1 & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & 1 & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & 1 & . & . & . & . & 2 \\
. & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & 1 & . & . & 3 & . & 1 & . & . & . & 2 & . & .
\end{pmatrix}$$

Apply ρ to rows 4,6,7,12, 15 and 13:

$$\begin{pmatrix}
. & . & 2 & . & . & . & 1 & 1 & . & . \\
. & . & . & 1 & . & 1 & . & 1 & 2 & . \\
2 & . & . & . & 1 & 1 & . & 3 & . & . \\
. & 1 & . & . & 2 & . & . & . & . & 1 \\
. & . & 1 & 2 & . & . & 2 & . & 1 & 3 \\
. & 1 & 1 & . & . & . & 2 & . & . & . \\
1 & . & . & . & 2 & 2 & . & . & . & 1 \\
1 & 1 & 3 & . & . & . & . & . & 1 & . \\
. & 2 & . & . & 1 & . & . & 1 & . & 2 \\
. & . & . & 1 & 3 & . & 1 & . & 2 & .
\end{pmatrix}$$

Apply ρ to rows 1,4, and 6:
$$\begin{pmatrix} . & 2 & 3 & 3 & 1 & 2 & 2 \\ 2 & . & 1 & 3 & 3 & . & . \\ 3 & 1 & . & 2 & . & 1 & 3 \\ 3 & 3 & 2 & . & 2 & . & 1 \\ 1 & 3 & . & 2 & . & 1 & . \\ 2 & . & 1 & . & 1 & . & 2 \\ 2 & . & 3 & 1 & . & 2 & . \end{pmatrix}$$

Apply ρ to row 2
$$\begin{pmatrix} . & 1 & 3 & 3 & . & . \\ 1 & . & 2 & 4 & 1 & 3 \\ 3 & 2 & . & 2 & . & 1 \\ 3 & 4 & 2 & . & 1 & . \\ . & 1 & . & 1 & . & 2 \\ . & 3 & 1 & . & 2 & . \end{pmatrix}$$

Apply ρ to row 1,5 and 6
$$\begin{pmatrix} . & 2 & 2 \\ 2 & . & 2 \\ 2 & 2 & . \end{pmatrix}$$

This matrix represents a triangle with each edge length 2, thus the total is 6 and we conclude that the link of v_0 is CAT(1).

We now check the complex has the correct fundamental group. The edges e_i, $1 \leq i \leq m-1$ each have two distinct vertices v_{i-1} and v_i. Contract e_i onto v_{i-1}. This retracts the triangle T_{m+i} identifying the edges e_{m+i} and $(e_{3m+i})^{-1}$, the triangle T_{2m+i} identifying the edge e_{4m+i} with $(e_{2m+1+i})^{-1}$ and the triangle T_{3m+i} identifying the edges e_{2m+i} and $(e_{4m+i})^{-1}$. Contracting e_i identifies the vertices v_i with v_{i-1}, so contracting for each $1 \leq i \leq m-1$ we form a complex with 2 vertices, v_0 and v_m, $m+n+8$ edges and $m+n+7$ triangles and fundamental group $\pi_1(X_2)$. Observe that the edges $e_{m+2}, \ldots, e_{2m}, e_{5m}, e_{5m+2}, e_{5m+4}, e_{5m+5}$ and $e_{5m+7}, \ldots, e_{5m+n+1}$ each are faces of exactly two triangles. We may amalgamate each pair of triangles and remove these edges without altering the fundamental group of the complex. We now have 7 2-cells, 11 edges and 2 vertices v_0 and v_m. Homotopically we contract the edge e_{m+1} which identifies the vertices v_0 and v_m, and identifies the edges e_{4m} and $(e_0)^{-1}$. We now have a complex with 1 vertex, 6 2-cells and 9 edges.

We may take the edge loops as generators for a presentation for the fundamental group. We relabel them, $\alpha := (e_0)^{-1}, z := (e_m)^{-1}, b := (e_{2m+1})^{-1}, c := e_{5m+1}, s := e_{5m+3}$ and $y := (e_{5m+6})^{-1}$ (see figure 2.5. We may read relators for a presentation from the boundaries of 2-cells:

$$\begin{array}{ccc} \alpha^m z^{-1}, & bzb^{-1}z^{-1}, & \\ \alpha s\alpha^{-1}c^{-1} & cby^{-1}, & bsy^{-1}, \end{array}$$

and $sy^{\frac{n}{2}-1}c^{-1}y^{-(\frac{n}{2}-1)}$ if n is even or $sy^{\frac{n-1}{2}-1}c^{-1}y^{-(\frac{n-1}{2})}$ if n is odd.

Taking $z = \alpha^m$ and $s = b^{-1}y$ to be definitions we may replace them in the relators;

$$b\alpha^m b^{-1}\alpha^{-m}, \alpha b^{-1}y\alpha^{-1}c^{-1}, cby^{-1}$$

and $b^{-1}y^{\frac{n}{2}}c^{-1}y^{-(\frac{n}{2}-1)}$ if n is even and $by^{\frac{n-1}{2}}c^{-1}y^{-(\frac{n-1}{2})}$ if n is odd.

Remove y by substituting cb:

$$b\alpha^m b^{-1}\alpha^{-m}, \alpha b^{-1}cb\alpha^{-1}c^{-1} \quad \text{and}$$

$$b^{-1}(cb)^{\frac{n}{2}}c^{-1}(cb)^{-(\frac{n}{2}-1)} \quad \text{if } n \text{ is even or } b(cb)^{\frac{n-1}{2}}c^{-1}(cb)^{-(\frac{n-1}{2})} \quad \text{if } n \text{ is odd.}$$

These last two relators are $(b,c)_n = (c,b)_n$. Adding a new generator $a := \alpha b^{-1}$ and removing the generator α, the first relator becomes $b(ab)^m b^{-1}(ab)^{-m}$ or equivalently $(a,b)_{2m} = (b,a)_{2m}$ and the second relation becomes $aca^{-1}c^{-1}$, or $(a,c)_2 = (c,a)_2$. Thus a presentation for the fundamental group is;

$$\begin{aligned} \pi_1(X_2) &\cong \langle a,b,c | (a,b)_{2m} = (b,a)_{2m}, (b,c)_n = (c,b)_n, (a,c)_2 = (c,a)_2 \rangle \\ &\cong A(2m,n,2) \end{aligned}$$

2.3 Biautomaticity

An important property of some groups is biautomaticity. In particular biautomatic groups have solvable word problem and solvable conjugacy problem. For more information of automatic and biautomatic groups see [E92].

Definition 2.3.1. A finite state automaton over A is a quintuple (S, A, μ, Y, s_0), where S is a finite set called the state set, A a finite set called the alphabet, $\mu : S \times A \to S$ is a function, called the transition function, Y a subset of S called the subset of accept states and $s_0 \in S$ is called the start state.

The idea is that the finite state automaton is a machine into which a string of letters (or a word) of A is fed. The machine has a number of recognised states, S, and starts at state s_0. The machine reads each letter in order, one at a time changing state depending on the letter and the current state (using the map μ). If after reading the whole word the state of the machine is in Y, then the word is accepted. The set of words accepted by a finite state automaton M is called the language accepted by M and denoted $L(M)$. The set of formal inverses of strings in $L(M)$ is denoted by $L(M)^{-1}$.

Definition 2.3.2. Let G be a group. An automatic structure on G consists of a set A of semigroup generators of G, a finite state automaton W over A and a finite state automaton M_x over (A, A) for $x \in A \cup \{\epsilon\}$, satisfying the following conditions:

1. The map $\pi : L(W) \to G$ is surjective.

2. For $x \in A \cup \{\epsilon\}$, we have $(w_1, w_2) \in L(M_x)$ if and only if $w_1 x = w_2$.

Definition 2.3.3. Let G be an automatic group with automatic structure (A, L), where A is closed under inversion. We say that the structure is biautomatic if (A, L^{-1}) is also an automatic structure. The group G is said to be biautomatic if such a biautomatic structure exists.

In [GS90] Gersten and Short showed that groups acting freely and properly discontinuously on CAT(0) 2-complexes, in which every cell is uniformed as an equilateral or isosceles right angled triangle, admits a biautomatic structure. The complexes we build in theorem 2.0.3 all admit such metrics, so we obtain,

Theorem 2.3.4. *The 3-generator Artin groups $A(2m, n, 2)$ of non-finite type are biautomatic.*

Note that none of these groups are covered by the results of Charney [C92], Peifer [Pei96] or Brady and MacCammond [BMcC00].

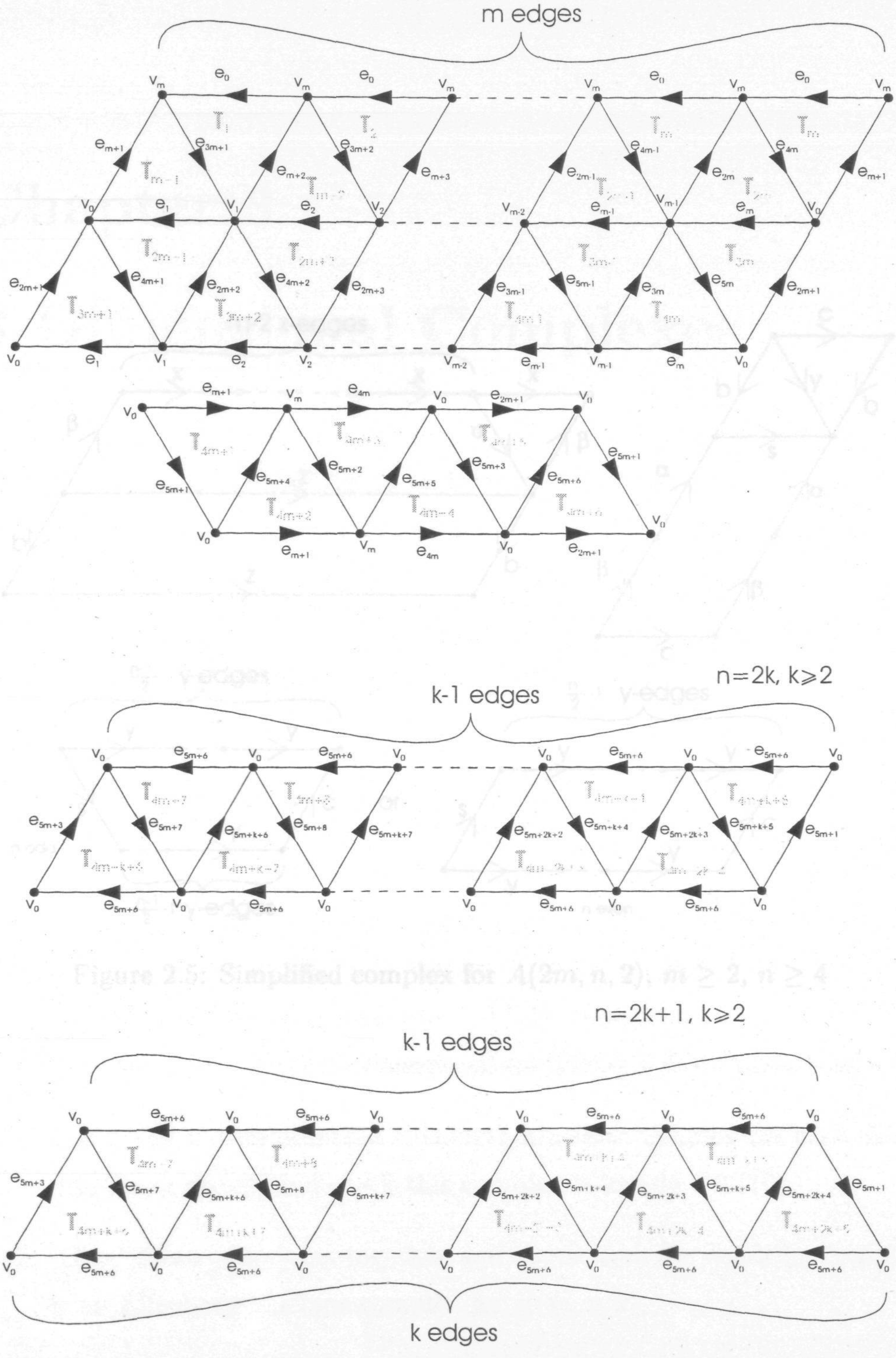

Figure 2.4: Complex for $A(2m, n, 2)$, $m \geq 2$, $n \geq 4$

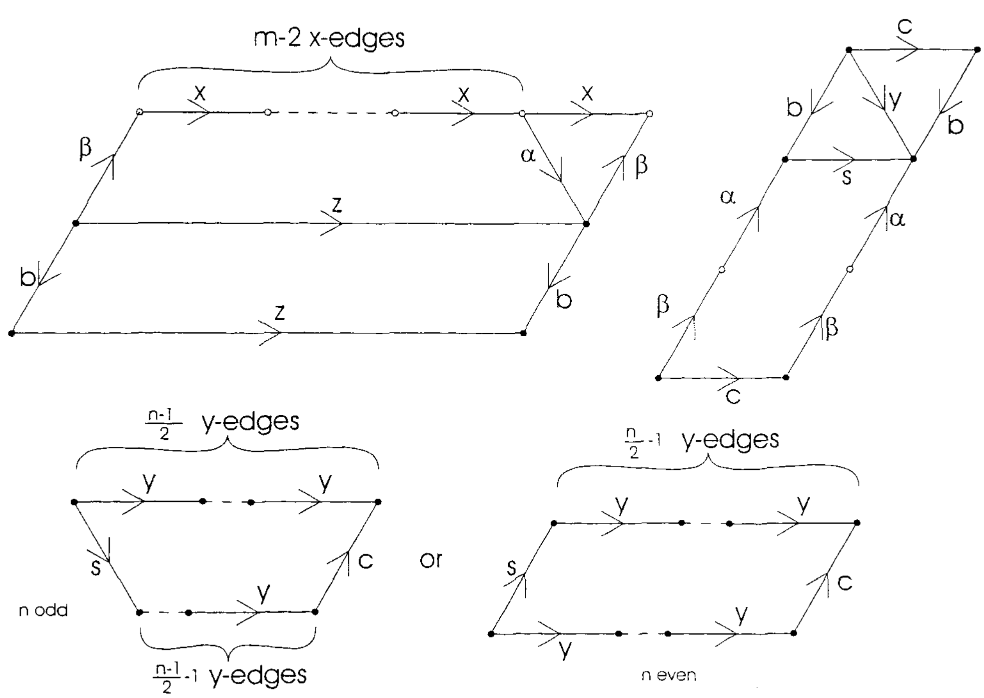

Figure 2.5: Simplified complex for $A(2m, n, 2)$, $m \geq 2$, $n \geq 4$

Chapter 3

3-Dimensional Complexes

In this chapter we explain how to build finite 3-dimensional locally CAT(0) Eilenberg-MacLane complexes for all but finitely many of the groups $A(m,n,2)$. Recall that the Artin group $A(m,n,2)$ has the following presentation (see section 1.7) :

$$A(m,n,2) = \langle a,b,c | (a,b)_m = (b,a)_m, (b,c)_n = (c,b)_n, ac = ca \rangle$$

We begin by recounting the construction given by Brady and Crisp [BC] of a 3-dimensional Eilenberg-Maclane complex for 3-generator Artin groups of type $A(m,n,2)$ and the method used to show that all but 65 are locally CAT(0). We then generalise their construction to cover all but 41 of these groups. As a result of this and chapter 2 there are just 19 non-finite 3-generator Artin groups with unknown CAT(0) dimension.

The construction of the 3-dimensional complexes follows three stages;

1. Construct a 3-dimensional Eilenberg-Maclane complex for each dihedral Artin group, and check this complex is locally CAT(0),

2. 'Glue' these complexes together and check that the resulting complex is an Eilenberg-Maclane complex for $A(m,n,2)$,

3. Check that this complex is locally CAT(0).

We use the following theorem to calculate the fundamental group of each complex.

Theorem 3.0.5 (Van Kampen's theorem). *Suppose X is the union of two subsets U and V, and suppose $U, V, U \cap V$ and X are connected. Let $\langle S_1 | R_1 \rangle$ be a presentation for $\pi_1(U)$, $\langle S_2 | R_2 \rangle$ be a presentation for $\pi_1(V)$ and $\langle S_3 | R_3 \rangle$ be a presentation for $\pi_1(U \cap V)$. The inclusion maps $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ induce homomorphisms $i^* : \pi_1(U \cap V) \to \pi_1(U)$ and $j^* : \pi_1(U \cap V) \to \pi_1(V)$.*

Then a presentation for the fundamental group of X is given by:

$$\pi_1(X) \cong \left\langle S_1 \cup S_2 | R_1 \cup R_2 \cup \left\{ (i^* s)(j^* s)^{-1} : s \in S_3 \right\} \right\rangle$$

Suppose X is a connected complex with disjoint connected subcomplexes U and V and a homeomorphism $f : U \to V$. The homeomorphism induces a homomorphism on the fundamental groups $f^* : \pi_1(U) \to \pi_1(V)$. Define a new complex Y as the quotient complex of the disjoint union $X \coprod (U \times [0, 1])$ by identifying each $u \in U$ with $(u, 0)$ and each $(u, 1)$ with $f(u)$.

Let $\cong \langle S | R \rangle$ be a presentation for $\pi_1(X)$. Then

$$\pi_1(Y) \cong \left\langle S \cup \{p\} \,| R \cup \left\{ p^{-1} u p = f^*(p) \text{ for all } \ u \in \pi_1(U) \right\} \right\rangle$$

The group $\pi_1(Y)$ is a HNN extension of $\pi_1(X)$.

3.1 An Eilenberg-Maclane complex for dihedral Artin groups.

We construct a 3-dimensional locally CAT(0) Eilenberg-Maclane complex X_m for each dihedral Artin group $A(m) = \langle a, b | (a, b)_m = (b, a)_m \rangle$, $m \geq 3$.

We take X_2 to be the flat torus formed by identifying opposite edges of a square. This complex is described by Brady and Crisp [BC].

Let M be a regular m-gon centred at the origin in the xy-plane, $\mathbb{R}^2$. Consider the product $M \times \mathbb{R} \subset \mathbb{R}^3$. Define a homeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ by, $\phi(r, \theta, z) = (r, \theta + \frac{2\pi}{m}, z + k)$ for some constant k with coordinates given as cylindrical polar coordinates. The map ϕ rotates about the z-axis by $\frac{2\pi}{m}$ and translates in the z-direction by k. Let E be the union of subsets of $M \times \mathbb{R}$ of the form $v \times \mathbb{R}$ where v is a vertex of M and $n \in \mathbb{Z}$. Thus E is the set of edges of $M \times \mathbb{R}$. The quotient $M \times \mathbb{R}/\phi^2$ is a solid torus (see fig.3.1). Define T_m to be the quotient of $M \times \mathbb{R}$ by ϕ^2. Let X_m denote the quotient of T_m by the induced action of $\phi|_E$ on T_m.

We now describe the construction and show that the fundamental group is as required. We split into two cases depending on whether m is odd or even.

3.2 The complex X_m for odd m

If m is odd there is a single edge, E/ϕ^2, running around the surface of T_m, m times. The map $\phi|_E$ identifies opposite points on this edge. This identification given by $\phi|_E$ may be obtained homotopically by attaching S^1/ψ to T_m by a cylinder $S^1 \times I$, where $\psi : (1, \theta) \mapsto (1, \theta + \pi)$ is the antipodal map.

Take U to be the union of T_m and the cylinder $S^1 \times I$, and V to be the union of S^1/ψ and the cylinder. Since $U \cap V$ is connected we may apply Van Kampen's theorem to find the fundamental group of $X_m = U \cup V$. The space T_m is a solid torus so $\pi_1(U) \cong \mathbb{Z} = \langle x \rangle$. If we take y to be the loop $\langle (1, \theta) | \theta \in [0, \pi] \rangle$ in S^1/ψ then $\pi_1(V) = \langle y \rangle = \mathbb{Z}$. The intersection $U \cap V$ is a cylinder so $\pi_1(U \cap V) = \mathbb{Z} = \langle z \rangle$. As the edge runs around the torus m times

$x^m = z$ and by the antipodal map $y^2 = z$. So the group has presentation $\pi_1(X_m) \cong \langle x, y | x^m = y^2 \rangle$.

We now demonstrate that this presentation for $\pi_1(X_m)$ is in fact a presentation for $A(m)$ and thus X_m is an Eilenberg-Maclane complex for $A(m)$.

Let v be a vertex of the m-gon M. Denote the geodesic in $M \times \mathbb{R}/\phi^2$ between $\phi^{-1}(v)$ and v by a and the geodesic between v and $\phi(v)$ by b (see fig. 3.1).

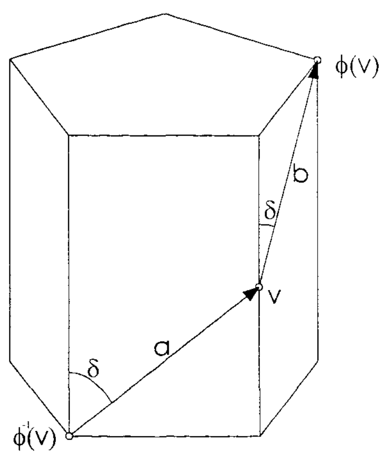

Figure 3.1: $M \times I$

The loop ab in T_m is a generator for $\pi_1(U)$, so it is a representative for $x \in \pi_1(U) = \langle x \rangle$. A representative for the generator $y \in \pi_1(V)$ is the image of $v \times [0, mk]$ (under the map ϕ^2) in T_m. By considering $M \times [0, mk]$ and noting that T_m is a solid torus, we can see that this loop is homotopic to $(a, b)_m$. This motivates an isomorphism between $\langle x, y | x^m = y^2 \rangle$ and $\langle a, b | (a, b)_m = (b, a)_m \rangle$ given by $x \mapsto ab$ and $y \mapsto (a, b)_m$.

3.3 The complex X_m for even $m \geq 4$

In the case of $m \geq 4$ even, T_m is again a solid torus, but with 2 parallel edges, S and T, running $\frac{m}{2}$ times around the core of the solid torus. The map $\phi|_E$ identifies these 2 edges. This identification is homotopy equivalent to attaching a cylinder $S^1 \times [0,1]$ between S and T. So $\pi_1(X_m)$ is a HNN extension with base $\pi_1(X/\phi^2)$ by Van Kampen's theorem.

Choose 2 adjacent vertices of M, v_1 and v_2. The edge S is the image of $v_1 \times [0, \frac{km}{2}]$ under ϕ^2 and T is the image of $v_2 \times [0, \frac{km}{2}]$. The identification of S with $S^1 \times \{0\}$ and T with $S^1 \times \{1\}$ is given by $\rho : S \cup T \to S^1 \times \{0,1\}$ where $\rho|_S : v_1 \times \{z\} \mapsto (1, \frac{2z\pi}{k}) \times \{0\}$ and $\rho|_T : v_2 \times \{z\} \mapsto (1, \frac{2z\pi}{k}) \times \{1\}$.

Now as T_m is a solid torus, $\pi_1(X/\phi^2) \cong \langle x \rangle \cong \mathbb{Z}$. The edges S and T are homotopic to S^1 and each pass round the torus $\frac{m}{2}$ times, so $\pi_1(S) \cong \langle s \rangle \cong \mathbb{Z}$ with $s \approx x^{\frac{m}{2}}$ and $\pi_1(T) \cong \langle t \rangle \cong \mathbb{Z}$ with $t \approx x^{\frac{m}{2}}$. Hence $\pi_1(X_m) \cong \langle x, y | y^{-1} s y = \tau(t) \rangle$ where $\tau : \pi_1(S) \to \pi_1(T)$ is the isomorphism $\tau : s \mapsto t$. So $\pi_1(X_m) \cong \langle x, y | y^{-1} x^{\frac{m}{2}} y = x^{\frac{m}{2}} \rangle$.

We now show that the above presentation for $\pi_1(X_m)$ is a presentation for $A(m)$. As in the case where m is odd, let v be a vertex of M and denote the geodesic of T_m from $\phi^{-1}(v)$ to v by a and the geodesic between v and $\phi(v)$ by b. Again ab represents a generator for $\pi_1(T_m)$. The element y is given by the addition of the cylinder, so a representative for y is the image of the geodesic a under the map $\phi|_E$. We observe that $x \mapsto ab$ and $y \mapsto a$ defines an isomorphism from $\pi_1(X_m)$ to $A(m)$.

In the case where m is 2, we take X_2 to be a torus, formed by identifying the opposite edges of a square. This complex is locally CAT(0).

3.4 Relative sizes of X_m

In the following section we glue the complexes X_m, X_n and X_2 together. To do this we need to ensure that the loops which we will identify have the same length in each space. We do this by showing that we can vary the 'sizes' of the spaces without limitations.

Label the angle in $M \times \mathbb{R}$ between $v \times \mathbb{R}$ and b by δ. Note that δ is also the angle between $\phi^{-1}(v) \times I$ and a.

The complex X_m comes equipped with the metric induced by the Euclidean metric on $M \times \mathbb{R}$. Note this induced metric depends on the area of M and the translation length k of ϕ.

By careful choice of k and the area of M we can guarantee that a and b have unit length and enable δ to take any value in the open interval $(0, \frac{\pi}{2})$. We write $X_m(\delta)$ when we wish to indicate the value of δ used in X_m.

3.5 X_m is locally CAT(0).

In this section we will show that the complex X_m is locally CAT(0). We make use of the link condition. We proceed by identifying the links of points in X_m and show that they are CAT(1). By lemma 1.4.1 and 1.4.2 we need only check that the link of every vertex contains no nontrivial loop of length strictly less than 2π.

Up to isomorphism there are 2 types of link in X_m. These are represented by the link of v, any point in the image of E and the link of w, the image of any point not in the image of E. If w is an interior point of X_m then the link is a sphere. If w is a point on the boundary of X_m then w is an interior point on the face of a 3-cell, so the link is a hemisphere. In each case the link is CAT(1). Locally the vertex v is contained in the common edge of 2 3-cells.

The link is therefore the union of 2 spherical identical 2-cells, see figure 3.2. The metric on the link is given by the angle based at v following a path in the link. So $d(a^+, b^+) = d(a^+, z^+) + d(z^+, b^+)$.

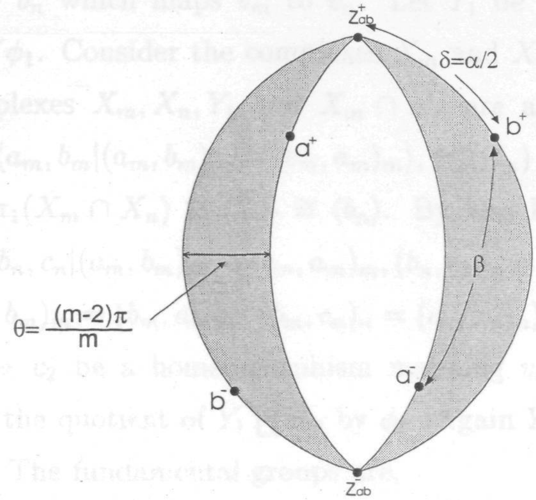

Figure 3.2: Link of v.

We can choose $a^{\pm}, b^{\pm}$ and $z^{\pm}$ so that $Lk(v, X_m)$ lies on a sphere, so $d(z_{ab}^+, z_{ab}^-) = \pi$. Since any locally geodesic loop must pass through both z_{ab}^+ and z_{ab}^-, every loop has length at least 2π. Therefore the link of every vertex in X_m is CAT(1), and hence X_m is locally CAT(0).

3.6 'Gluing' X_m, X_n and X_2.

In this section we construct a 3 dimensional complex whose fundamental group is $A(m, n, 2)$ and in most cases show that this has a locally CAT(0) metric, so is a $K(G, 1)$. We do this by combining the complexes $X_m(\delta_m)$, $X_n(\delta_n)$, and $X_2(\delta_2)$.

The images of the geodesics a and b in X_m are generator curves for

$\Pi_1(X_m, v)$. Label the curves in X_m by a_m and b_m, the curves in X_n by b_n and c_n, and the curves in X_2 by c_2 and a_2.

Consider $b_m \subset X_m$ and $b_n \subset X_n$ as subsets and define a homeomorphism $\phi_1 : b_m \to b_n$ which maps v_m to v_n. Let Y_1 be the quotient complex $(X_m \coprod X_n)/\phi_1$. Consider the complexes X_m and X_n as subcomplexes of Y_1. The complexes X_m, X_n, Y_1 and $X_m \cap X_n$ are all connected. We know $\pi_1(X_m) \cong \langle a_m, b_m | (a_m, b_m)_m = (b_m, a_m)_m \rangle$, $\pi_1(X_n) \cong \langle b_n, c_n | (b_n, c_n)_n = (c_n, b_n)_n \rangle$ and $\pi_1(X_m \cap X_n) \cong \langle b_m \rangle \cong \langle b_n \rangle$. By Van Kampen's theorem $\pi_1(Y_1) \cong \langle a_m, b_m, b_n, c_n | (a_m, b_m)_m = (b_m, a_m)_m, (b_n, c_n)_n = (c_n, b_n)_n, b_m = b_n \rangle \cong \langle a_m, b_m, c_n | (a_m, b_m)_m = (b_m, a_m)_m, (b_m, c_n)_n = (c_n, b_m)_n \rangle$.

Let $\phi_2 : c_n \to c_2$ be a homeomorphism mapping v_n to v_2. Define a complex Y_2 to be the quotient of $Y_1 \coprod X_2$ by ϕ_2. Again $Y_1, X_2, Y_1 \cap X_2$ and Y_2 are connected. The fundamental groups are,

$$\pi_1(Y_1) \cong \langle a_m, b_m, c_n | (a_m, b_m)_m = (b_m, a_m)_m, (b_m, c_n)_n = (c_n, b_m)_n \rangle,$$

$$\pi_1(X_2) \cong \langle a_2, c_2 | a_2 c_2 = c_2 a_2 \rangle, \text{ and}$$

$$\pi_1(Y_1 \cap X_2) \cong \langle c_2 \rangle \cong \langle c_n \rangle$$

Hence, by Van Kampen's theorem,

$$\begin{aligned} \pi_1(Y_2) \cong\ & \langle a_m, b_m, c_n, c_2, a_2 | (a_m, b_m)_m = (b_m, a_m)_m \\ & (b_m, c_n)_n = (c_n, b_m)_n, a_2 c_2 = c_2 a_2, c_n = c_2 \rangle \\ \cong\ & \langle a_m, b_m, c_n, a_2 | (a_m, b_m)_m = (b_m, a_m)_m, \\ & (b_m, c_n)_n = (c_n, b_m)_n, a_2 c_n = c_n a_2 \rangle \end{aligned}$$

In Y_2 the subcomplexes a_m and a_2 are homeomorphic. Let X be the HNN extension formed by identifying a_m and a_2 in Y_2 then the fundamental group of X is the HNN extension of $\pi_1(Y_2)$ formed by quotienting by the subgroup generated by $a_m a_2^{-1}$. Thus $\pi_1(X) \cong \{a_2, b_m, c_n | (a_2, b_m)_m = (b_m, a_2)_m, (b_m, c_n)_n = (c_n, b_m)_n, a_2 c_n = c_n a_2\}$.

3.6.1 When is X locally CAT(0)?

We will show that each link is CAT(1) and hence the space is locally CAT(0).

In the combined complex X the links are of three possible types; links of points lying in a single X_m, links of points common to 2 of the contributing complexes and the link of v the vertex common to all 3 of the contributing complexes. The first type of link is CAT(1) as before. The second type of link corresponds to a vertex of $X_m \cap X_n$. This is the union of the link of v_m in X_m and the link of v_n in X_n with points $b_m^{\pm}$ and $b_n^{\pm}$ identified. Since the distance between b_m^+ and b_m^- is π and the distance between b_n^+ and b_n^- is π any locally geodesic loop must have length at least 2π, so the link is CAT(1).

The link of v in X is the union of the links of v_m in X_m, v_n in X_n and v_2 in X_2 with identifications, $b_m^{\pm} = b_n^{\pm}$, $c_n^{\pm} = c_2^{\pm}$ and $a_2^{\pm} = a_m^{\pm}$. Each contributing link and, by the above, the union of each pair of contributing links is CAT(1), thus any loop of length less than 2π must pass through each contributing link. Since the common points shared by the links are $a^{\pm}$, $b^{\pm}$ and $c^{\pm}$ the loop must pass through these points. It is therefore sufficient to check that the complete graph of geodesics between these points in each contributing link is CAT(1). This graph is a schematic of the distances between these vertices in this link, see figure 3.3.

Let $\alpha_1 = d(a^+, b^+) = d(a^-, b^-)$, $\alpha_2 = d(b^+, c^+) = d(b^-, c^-)$, $\beta_1 = d(a^+, b^-) = d(a^-, b^+)$, and $\beta_2 = d(b^+, c^-) = d(b^-, c^+)$

Lemma 3.6.1 gives a relationship between these distances.

Lemma 3.6.1. *[BC, lemma 4.1] Let $m \geq 3$ be an integer and $\theta = \frac{(m-2)\pi}{m}$. Then $cos\beta + \left(\frac{1+cos\theta}{2}\right) cos\alpha + \left(\frac{1-cos\theta}{2}\right) = 0$.*

For a proof see [BC, Lemma 4.1]

By choosing $\alpha = \beta$ Brady and Crisp reduce the equation to $cos\alpha = cos\beta = \frac{cos\theta - 1}{cos\theta + 3}$ and we need only ensure that $\alpha_1 + \alpha_2 + \frac{\pi}{2} \geq 2\pi$.

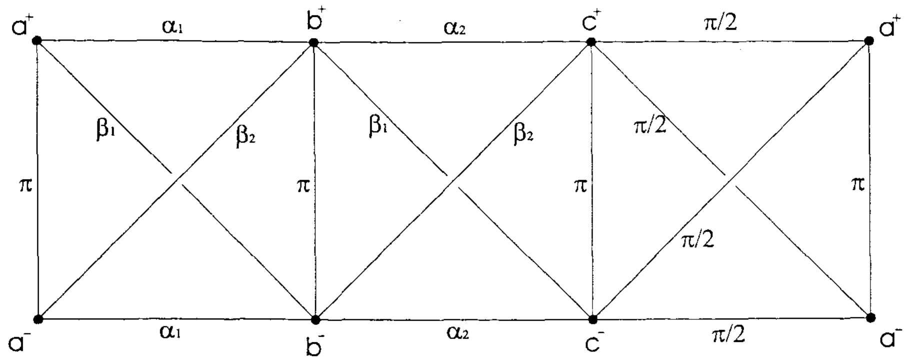

Figure 3.3: Combined link

m	n
3	$6 \leq n \leq 43$
4	$4 \leq n \leq 18$
5	$5 \leq n \leq 11$
6	$6 \leq n \leq 9$
7	7

Table 3.1: Cases not covered by Brady and Crisp.

These inequalities hold in all but the finite number of cases shown in table 3.1.

3.7 Extending the construction.

In the construction of the locally CAT(0) Eilenberg-Maclane 3-dimensional complexes for $A(m, n, 2)$, Brady and Crisp made two restrictions which make the calculations easier and cleaner. In this section we remove these restrictions and thus gain more freedom in our choice of α_i and β_i. In doing this we may choose suitable values not previously available and construct CAT(0) Eilenberg-Maclane 3-complexes for 13 of the remaining 65 groups.

3.7.1 Removing the restrictions

The first restriction made was to construct X_2, a 2-torus, by identifying the edges of a square. By changing the metric we may construct X_2 by identifying the edges of a rhombus. Taking the 2 geodesic loops in X_2 to be the edges of the rhombus, we can define the rhombus by the angle, γ between them, $\gamma \in (0, \frac{\pi}{2})$.

The link of v in X_2, $Lk(v, X_2)$ is a circle divided by 4 vertices a^+, a^-, c^+, c^-. The distances between these vertices are given by, $d(a^+, c^+) = d(a^-, c^-) = \gamma, d(a^+, a^-) = d(c^+, c^-) = \pi$. So $d(a^+, c^-) = d(a^-, c^+) = \pi - \gamma$.

The resulting schematic of the combined link of $X_m(\delta_m), X_n(\delta_n), X_2(\delta_2)$ is shown in figure 3.4 with $\alpha_m = 2\delta_m(= \alpha_1)$, $\alpha_n = 2\delta_n(= \alpha_2)$, and $\gamma = 2\delta_2$.

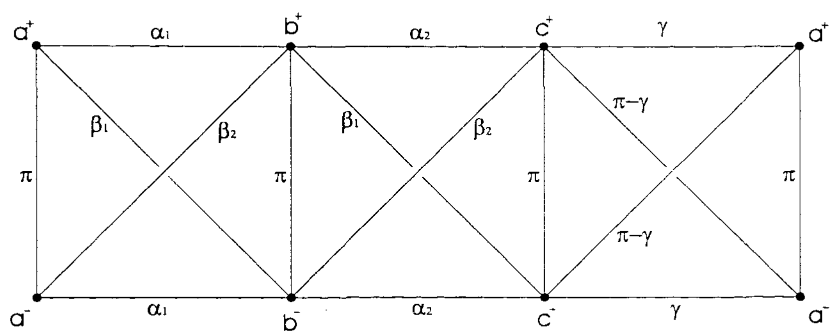

Figure 3.4: New combined link

The second restriction Brady and Crisp made was $\alpha_i = \beta_i$. If we remove this, there are 4 inequalities that we need to satisfy to ensure this link is CAT(1),

$\alpha_m + \alpha_n + \gamma \geq 2\pi$
$\beta_m + \beta_n + \gamma \geq 2\pi$
$\alpha_m + \beta_n + \pi - \gamma \geq 2\pi$
$\beta_m + \alpha_n + \pi - \gamma \geq 2\pi$

We may rearrange the equation given in lemma 3.6.1 to give β in terms

(m,n)	α_m	α_n	β_m	β_n	γ
$(3,41)$	103.30	179.99	≥ 94.44	≥ 171.21	94.35
$(4,16)$	121.00	175.00	≥ 104.00	≥ 157.50	98.50
$(5,9)$	154.00	179.99	≥ 110.11	≥ 140.00	109.89
$(6,7)$	179.00	179.00	≥ 120.00	≥ 128.50	111.50
$(7,7)$	179.00	179.00	≥ 128.50	≥ 128.50	104.00

Table 3.2: Possible solutions to the set of inequalities. Angles given in degrees.

of α and θ,

$\beta = \cos^{-1}[(\frac{\cos\theta-1}{2}) - (\frac{\cos\theta+1}{2})\cos\alpha]$

Table 3.2 shows some purposely selected values of α_i and the corresponding values of β_i.

Each of the choices shown in the table satisfy the inequalities. So for these 5 cases the space may be constructed. The following lemma and corollary show the above results resolve 13 of the 65 cases.

Lemma 3.7.1. *If $\alpha_m, \alpha_n, \beta_m, \beta_n$ can be chosen for given (m,n) which satisfy the following inequalities, then suitable α_{n+1} and β_{n+1} can be chosen so that $\alpha_m, \alpha_{n+1}, \beta_m, \beta_{n+1}$ also satisfy these inequalities.*

$$\begin{aligned} \alpha_m + \alpha_n + \gamma &\geq 2\pi \\ \beta_m + \beta_n + \gamma &\geq 2\pi \\ \alpha_m + \beta_n + \pi - \gamma &\geq 2\pi \\ \beta_m + \alpha_n + \pi - \gamma &\geq 2\pi \end{aligned}$$

Proof. Suppose the above inequalities are satisfied for α_m and α_n and the computed values β_m, β_n and γ.

Let $\alpha_{n+1} = \alpha_n$. Recall by lemma 3.6.1

$$\begin{aligned} \beta_{n+1} &= \cos^{-1}(\tfrac{1-\cos\alpha_{n+1}}{2}\cos\theta_{n+1} - \tfrac{\cos\alpha_{n+1}}{2} - \tfrac{1}{2}) \\ &= \cos^{-1}(\tfrac{1-\cos\alpha_n}{2}\cos\theta_{n+1} - \tfrac{\cos\alpha_n}{2} - \tfrac{1}{2}) \end{aligned}$$

where $\theta_{n+1} = \frac{((n+1)-2)\pi}{n+1}$.

Now $0 < \frac{n-2}{n}\pi < \frac{(n+1-2)}{n+1}\pi < \pi$. So $0 < \theta_n < \theta_{n+1} < \pi$ and $\cos\theta_{n+1} < \cos\theta_n$. Note that $\frac{1-\cos\alpha_n}{2} > 0$ so

$$\begin{aligned}\cos\beta_{n+1} &= \tfrac{1-\cos\alpha_n}{2}\cos\theta_{n+1} - \tfrac{\cos\alpha_n}{2} - \tfrac{1}{2} \\ &< \tfrac{1-\cos\alpha_n}{2}\cos\theta_n - \tfrac{\cos\alpha_n}{2} - \tfrac{1}{2} = cos\beta_n.\end{aligned}$$

But $\beta_n, \beta_{n+1} \in (0, \pi)$, so $\beta_{n+1} > \beta_n$. Hence the values $\alpha_m, \beta_m, \alpha_{n+1}\beta_{n+1}$ and γ satisfy the inequalities. □

This immediately gives the following corollary.

Corollary 3.7.2. *If $A(m, n, 2)$ acts properly discontinuously by semi-simple isometries on a CAT(0) 3-complex, then so does $A(m, n+1, 2)$.*

We have proved the following theorem,

Theorem 3.7.3. *If $(m, n) \neq (1, 2), \ldots, (1, 19)$ and $(m, n) \neq (2, 2), (2, 3)$ then the Artin group $A(2m+1, 2n+1, 2)$ has CAT(0) dimension 3.*

Chapter 4

Classifying the 2-complexes for $A(2m, n, 2)$, $n > 3$, odd

In this chapter we study the minsets for the action of $A(2m, n, 2)$ on CAT(0) 2-complexes.

We recall the following definitions:

Definition 4.0.4. If γ is an isometry of a metric space (X, d_X), the translation length of γ in X, $|\gamma|_X$ is defined to be $\inf\{d_X(x, \gamma.x) | x \in X\}$.

The minset of γ in X is the set $\text{Min}_X(\gamma) := \{x \in X | d_X(x, \gamma.x) = |\gamma|_X\}$.

In this chapter we assume that all actions of groups on complexes are properly discontinuous by semi-simple isometries. We assume that G is a group with such an action on a CAT(0) 2-complex X.

In the case that a free abelian group acts on a CAT(0) complex, the structure of the minsets is particularly nice.

Theorem 4.0.5. *(Flat torus theorem: [BH99, II.7.1])*

Let A be a free abelian group of rank n acting properly discontinuously by semi-simple isometries on a CAT(0) complex X. Then $\text{Min}(A)$ *splits as a metric product $Y \times \mathbb{E}^n$ where each $a \in A$ acts as the identity on Y and by* *translation on the* $\mathbb{E}^n$ *factor. Moreover the quotient of each* $\{y\} \times \mathbb{E}^n$ *by* A *is an* n*-torus.*

It follows directly from this that any group containing a free abelian subgroup of rank n does not act properly discontinuously by semi-simple isometries on a CAT(0) complex of dimension less than n.

Let A be a rank 2 free abelian group acting on a 2-dimensional CAT(0) complex. The set $\mathrm{Min}(A)$ must have dimension at most equal to the dimension of the complex, 2. By the Flat Torus theorem, the dimension of $\mathrm{Min}(A)$ is at least 2, hence $\dim \mathrm{Min}(A) = 2$. If there is any A-invariant plane in X then this plane is a subset of $\mathrm{Min}(A)$, hence we have the following corollary.

Corollary 4.0.6. *If* $\langle a, b\rangle$ *is a rank 2 free abelian group acting properly discontinuously by semi-simple isometries on a 2 dimensional CAT(0) space* X*, then* $\mathrm{Min}(\langle a, b\rangle)$ *is the unique* $\langle a, b\rangle$*-invariant plane in* X*.*

Proof. By the Flat torus theorem $\mathrm{Min}(\langle a, b\rangle)$ is a plane. The 2-dimensionality forces any $\langle a, b\rangle$-invariant plane in X to lie in $\mathrm{Min}(\langle a, b\rangle)$, and hence this plane is the unique $\langle a, b\rangle$-invariant plane in X. □

Definition 4.0.7. If $\langle a, b\rangle$ is a rank 2 free abelian group acting properly discontinuously by semi-simple isometries on a 2 dimensional CAT(0) space X, then denote the unique $\langle a, b\rangle$-invariant plane by,

$$\Pi(a, b) := \mathrm{Min}(a) \cap \mathrm{Min}(b)$$

By proposition 1.5.1 $g.\Pi(a, b) = g(\mathrm{Min}(a) \cap \mathrm{Min}(b)) = g\mathrm{Min}(a) \cap g\mathrm{Min}(b) = \mathrm{Min}(gag^{-1}) \cap \mathrm{Min}(gbg^{-1}) = \Pi(gag^{-1}, gbg^{-1})$. If g commutes with b, then $g.\Pi(a, b) = \Pi(gag^{-1}, b)$.

If Π_1 and Π_2 correspond to the abelian subgroups $\langle h_1, g\rangle$ and $\langle h_2, g\rangle$ of G then both Π_1 and Π_2 are contained in $\mathrm{Min}(g)$. This restricts the possible configurations of these planes. Note that the following lemma does not explain the configuration of planes corresponding to subgroups $\langle g_1, g_2\rangle$, $\langle g_3, g_4\rangle$ where $g_i \neq g_j^k$ for $i \neq j$ and all $k \in \mathbb{Z}$.

Lemma 4.0.8. *[BC, lemma 1.4] Let G be a group acting properly discontinuously by semi-simple isometries on a 2-dimensional CAT(0) complex X. Let $\Pi_1 \neq \Pi_2$ be flat planes corresponding to subgroups $\langle h_1, g\rangle$ and $\langle h_2, g\rangle$ respectively of G. Then the convex closure of $\Pi_1 \cup \Pi_2$ is one of the following,*

1. *H-type. Here $\Pi_1 \cap \Pi_2 = \emptyset$ and the planes are connected by an infinite strip $\mathbb{R} \times I$ (where $I = [p, q] \subset \mathbb{R}$ is a closed interval) with $\{p\} \times \mathbb{R}$ a g-axis in Π_1 and $\{q\} \times \mathbb{R}$ a g-axis in Π_2.*

2. *B-type. Here $\Pi_1 \cap \Pi_2$ is an infinite strip, a closed interval times $\mathbb{R}$ foliated by g-axes.*

3. *X-type. Here $\Pi_1 \cap \Pi_2$ is a single g-axis common to both planes.*

Proof. By the constraint of 2-dimensionality, Y is necessarily an $\mathbb{R}$-tree. The result follows from the observations that both Π_1 and Π_2 are contained in $\mathrm{Min}(g)$ (which is convex) and $\mathrm{Min}(g) \cong Y \times \mathbb{R}$. □

Consider now a subgroup H generated by h_1, h_2 and g. The generators of H satisfy the relations $[h_1, g] = [h_2, g] = 1$. This subgroup acts properly discontinuously by semi-simple isometries on the set of translates of the planes Π_1 and Π_2 by elements of H. This set is a subset of $\mathrm{Min}(g)$ which splits as a metric product $Y \times \mathbb{R}$. We may project along $\mathbb{R}$ on to Y, then $H/\langle g\rangle$ acts on this tree. In the light of this we study the actions of isometries of $\mathbb{R}$-trees.

We use the notation γ_a to refer to a translation axis of a. Since these axes share a common band we may refer to γ_a being transverse (respectively parallel) to γ_b as a and b segments are transverse (respectively parallel) on $B(a, b)$.

Lemma 4.0.9. *Let a and b be hyperbolic isometries of an $\mathbb{R}$-tree T such that ab has a fixed point z. Then,*

1. *The translation axes γ_a and γ_b for a and b intersect.*

2. *This intersection contains an interval J of length $\min\{|a|, |b|\}$.*

3. *a and b translate in opposite directions as defined on J.*

4. *If $|a| \neq |b|$ then $\gamma_a \cap \gamma_b = J$.*

5. *If $|a| \geq |b|$ then there exists a unique point p in γ_a such that $abp = p$ and $[a^{-1}p, p]$ is central about J.*

Proof. If g is a hyperbolic automorphism of T and x any point in T then $[x, g(x)]$ is the union of the intervals $[x, x_g]$, $[x_g, g(x_g)]$ and $[g(x_g), g(x)]$ where x_g is the projection of x on to γ_g, the translation axis of g.

Note that $d(x, x_g) = d(g(x), g(x_g))$, so $[x_g, g(x_g)]$ is central in $[x, g(x)]$.

Let z be a point in T fixed by ab. So $bz = a^{-1}z$.

The interval $[z, a^{-1}z]$ is the union $[z, z_a] \cup [z_a, a^{-1}z_a] \cup [a^{-1}z_a, a^{-1}z]$.

The interval $[z, bz]$ is the union $[z, z_b] \cup [z_b, bz_b] \cup [bz_b, bz]$.

But $[z, a^{-1}z] = [z, bz]$, so $[z, z_a] \cup [z_a, a^{-1}z_a] \cup [a^{-1}z_a, a^{-1}z] = [z, z_b] \cup [z_b, bz_b] \cup [bz_b, bz]$. Since $[z_a, a^{-1}z]$ is central in $[z, a^{-1}z]$ and $[z_b, bz_b]$ is central in $[z, bz]$ we must have either $J := [z_a, a^{-1}z_a] \subseteq [z_b, bz_b]$, or $J := [z_b, bz_b] \subseteq [z_a, a^{-1}z_a]$.

Now $[z_a, a^{-1}z_a] \subset \gamma_a$ and $[z_b, bz_b] \subset \gamma_b$, hence (1). The length of $[z_a, a^{-1}z_a]$ is $l(a)$ and the length of $[z_b, bz_b]$ is $l(b)$, hence (2).

The elements a^{-1} and b translate in the same direction on J, hence (3).

Lastly, suppose $l(a) > l(b)$. Then $[z_a, a^{-1}z_a] = \gamma_a \cap [z, a^{-1}z]$ properly contains $[z_b, bz_b] = \gamma_b \cap [z, a^{-1}z]$. So $\gamma_a \cap \gamma_b = \gamma_a \cap \gamma_b \cap [z, a^{-1}z] = [z_b, bz_b] = J$. Hence (4).

If $|a| \geq |b|$ then $[z, z_b] \supseteq [z, z_a]$ so $b.[z, z_b] \supseteq b.[z, z_a]$ and $[b.z, b.z_b] \supseteq [b.z, b.z_a]$, but $[b.z, b.z_b] \supseteq [b.z, a^{-1}.z_a] = [a^{-1}.z, a^{-1}.a_a]$ so $a^{-1}.z_a, b.z_a \in [b.z, b.z_b]$ and $d(b.z, a^{-1}.z_a) = d(b.z, b.z_a)$. So $a^{-1}.z_a = b.z_a$. Hence 5. $\square$

We use these results in the following section to describe the structure of minsets for the dihedral Artin groups.

4.1 Minsets for dihedral Artin groups

We recall that the 3-generator Artin group $A(m, n, 2)$ contains the dihedral groups $A(m)$, $A(n)$ and $A(2)$ as subgroups [VdL]. With this in mind, we study the structure of minsets for dihedral Artin groups.

Recall that a dihedral Artin group $A(m)$, $m \geq 2$ is a group with two generators and presentation,

$$A(m) = \langle a, b | (a, b)_m = (b, a)_m \rangle$$

We suppose that $A(m)$ is acting properly discontinuously by semi-simple isometries on a CAT(0) 2-complex X.

Let $z_{a,b}$ denote the element $(a, b)_m$ if m is even and $(a, b)_{2m}$ when m is odd. Then $\langle z_{a,b} \rangle$ is the centre of $A(m)$. It follows, by the Flat Torus theorem, that there are two distinguished flat planes $\Pi(a, z_{a,b}) = \mathrm{Min}(a) \cap \mathrm{Min}(z_{a,b})$ and $\Pi(b, z_{a,b}) = \mathrm{Min}(b) \cap \mathrm{Min}(z_{a,b})$ with intersection $B(a, b)$. The following proposition was proved by Brady and Crisp. It shows that $B(a, b)$ is non empty, and gives some strong results particularly in the case that m is odd.

Proposition 4.1.1. *[BC, Proposition 2.2] Suppose that $A(m)$, $m \geq 3$ acts properly discontinuously by semi-simple isometries on a 2-dimensional CAT(0) space X.*

Then

1. *$B(a,b) \cong [p,q] \times \mathbb{R}$ where $[p,q]$ is a non trivial closed interval and each $\mathbb{R}$-fibre is a $z_{a,b}$-axis.*

2. *The a and b-axes intersect $B(a,b)$ transversely.*

   If m is odd we also have:

3. *The edge $l = \{p\} \times \mathbb{R}$ is a ba-axis and $l' = \{q\} \times \mathbb{R}$ is an ab-axis. Moreover $a(l)$ is an ab axis which lies in $B(a,b)$ and $b(l')$ is a ba-axis also in $B(a,b)$.*

4. *For $g, h \in \{a, b, z_{a,b}\}$ write $\theta(g,h)$ for the angle in $B(a,b)$ between a positively oriented g-axis and a positively oriented h-axis. Then*

$$0 < \theta(a,b) = \theta(a, z_{a,b}) + \theta(b, z_{a,b}) < \pi$$

   and $\theta(a, z_{a,b}) = \theta(b, z_{a,b})$. In particular the a-segments are transverse to the b-segments in $B(a,b)$.

Since $(a,b)_m^{-1} a (a,b)_m = b$ for m odd, the translation lengths $|a|$ and $|b|$ are equal in both X and T_m. In order to obtain a bound on the width the band $B(a,b)$ we use the following data.

Definition 4.1.2. We may create a vector basis for $\Pi(a, z_{a,b})$ by taking the vector defined by $z_{a,b}$ and a vector normal to this. Since the translation axes for a are transverse to the translation axes for $z_{a,b}$, we may define the translation lengths of a in the directions of the basis vectors. Denote the translation length of a in the direction of $z_{a,b}$ by $|a|_z$ and the translation length of a normal to $z_{a,b}$ by $|a|^{\perp}_{z_{a,b}}$. When it is clear that we refer to $z_{a,b}$ we may simply use the notion $|a|^{\perp}$ and $|b|^{\perp}$.

Lemma 4.1.3. *The width of $B(a,b)$ is strictly less than the sum $|a|^{\perp} + |b|^{\perp}$ and greater or equal to $|a|^{\perp} = |b|^{\perp}$.*

Proof. If the width of $B(a,b)$ is greater or equal to $|a|^{\perp} + |b|^{\perp}$ there exists a point x in $B(a,b)$ such that $b^{-1}a^{-1}ba.x = x$. But $b^{-1}a^{-1}ba$ has infinite order, contradicting the properly discontinuous action. The second part of the lemma follows directly from proposition 4.1.1 (3). □

Let $\mathcal{O}_m$ denote the subcomplex of X formed by all translates of the planes $\Pi(a, z_{a,b})$ and $\Pi(b, z_{a,b})$ by the elements of $A(m)$. This complex is a connected subcomplex of $\mathrm{Min}(z_{a,b})$. In fact $\mathcal{O}_m$ splits as a product $Y \times \mathbb{R}$ where Y is an $\mathbb{R}$-tree. Hence $\mathcal{O}_m$ is convex in $\mathrm{Min}(z_{a,b})$ and hence in X. Thus $\mathcal{O}_m$ is CAT(0). Any 2-dimensional CAT(0) complex X which the group $A(m)$ acts on properly discontinuously by semi-simple isometries, contains such a subcomplex $\mathcal{O}$.

By projecting $\mathcal{O}_m$ along a $z_{a,b}$ axis we obtain a tree T_m. The quotient group $A(m)/\langle z_{a,b}\rangle$ acts properly discontinuously by semi-simple isometries of T_m. We use this action on T_m to study the action of $A(m)$ on $\mathcal{O}_m$.

4.1.1 A quotient space for $A(m)$ when m is odd

In the case when $m \geq 3$ and odd we can say more about the structure of $\mathcal{O}$.

Lemma 4.1.4. *Let $m \geq 3$ and m odd. Then a and b translate opposite edges of $B(a,b)$ to each other.*

Proof. Throughout this proof we are working in the $\mathbb{R}$-tree and will use $|a|$ and b to denote the translation lengths in this tree. By propostion 4.1.1 (3) we know that $|\gamma_a \cap \gamma_b| \geq |a| = |b|$. Let us suppose that $|\gamma_a \cap \gamma_b| > |a|$. Label the vertex of $\gamma_a \cap \gamma_b$ which a translates into $\gamma_a \cap \gamma_b$ by x. Then the geodesic $[x, ax] \subsetneq \gamma_a \cap \gamma_b$ and $(b[x, ax]) \cap (\gamma_a \cap \gamma_b) = [bx, x] \cap (\gamma_a \cap \gamma_b) = \{x\}$. Choose a point $y \in (x, ax) = [x, ax] \setminus \{x, ax\}$ such that $ay \in \gamma_a \cap \gamma_b$. This is possible since $|\gamma_a \cap \gamma_b| > |a|$.

Observe that since $x, y \in \gamma_a \cap \gamma_b$ and $|a| = |b|$, $ax = b^{-1}x$ and $ay = b^{-1}y$. Hence

$$\begin{aligned}
(a^{-1}, b^{-1})_{m-1}x &= (a^{-1}b^{-1})^{\frac{m-1}{2}}x && \text{since } m \text{ is odd,} \\
&= (a^{-1}b^{-1})^{\frac{m-3}{2}}a^{-1}b^{-1}x \\
&= (a^{-1}b^{-1})^{\frac{m-3}{2}}a^{-1}ax && \text{since } ax = b^{-1}x, \\
&= (a^{-1}b^{-1})^{\frac{m-3}{2}}x \\
&= x
\end{aligned}$$

and similarly $(a^{-1}, b^{-1})_{m-1}y = y$. Also $(b, a)_m x = (a, b)_m x = ax$, so $bx = (a^{-1}, b^{-1})_{m-1}ax$.

Consider the geodesic in T_m, $[x, ax]$. Now $(a^{-1}, b^{-1})_{m-1}[x, ax]$ $= [(a^{-1}, b^{-1})_{m-1}x, (a^{-1}, b^{-1})_{m-1}ax] = [x, bx]$. Since $y \in [x, ax]$ and $(a^{-1}, b^{-1})_{m-1}y = y$, it follows that $y \in [x, bx]$ and $y \in [x, ax] \cap [x, bx] = \{x\}$, a contradiction. □

We immediately obtain the following corollary.

Corollary 4.1.5. *If m is odd then $width(B(a, b)) = |a|_z^{\perp} = |b|_z^{\perp}$.*

By considering the image of the edge $[x, bx]$ under the map $(b, a)_m = (a, b)_m$ we find that $(a, b)_m$ fixes exactly one point in T_m and rotates T_m about this point.

Consider the edge $\gamma_a \cap \gamma_b = [x, bx] \subset T_m$. Suppose there is a point $p \in (x, ax)$ such that $ap \in \gamma_a \setminus (\gamma_a \cap \gamma_b)$ and $bap \in \gamma_a \setminus (\gamma_a \cap \gamma_b)$. Then the same is true for $[p, bx)$. Observe that $[abap, x] = aba[p, bx] \subset (\gamma_a \cap \gamma_b)$

and $(b,a)_m[abap,x] = [p,bx]$, ie $(b,a)_m abap = p$. Hence $p = (b,a)_{m+3}p = (b,a)_{m+3}(b,a)_{m+3}p = (b,a)_{2m+6}p$ so $p = (b,a)_6 p$.

Now $2m = 6k + i$ for some $k \in \mathbb{N}$ and $i = 0, 2, 4$ but (ba) and $baba$ do not fix p, so $2m = 6k$, ie 3 divides m.

Similarly if $p \in \gamma_a \cap \gamma_b$ such that $(a,b)_n p \in \gamma_a \cap \gamma_b$ and $(a,b)_i \notin \gamma_a \cap \gamma_b$ for $i < n$ then $n|m$.

If r is the mid point of $[x,bx]$ fixed by aba then $(aba)^k r = r$ for all $k \in \mathbb{Z}$. Hence, by proper discontinuity of the action, $m = 3$. There is a sequence of integers p_i, $i = 0, \ldots k$ and points $x_i \in [x,r]$ such that $p_0 = 1$, $p_i|m$, $p_i|p_{i+1}$ and $p_k = m$, $x_0 = x$, $x_k = r$ and $(a,b)_{2p_i}$ fixes $[x_0, x_i]$, but $(a,b)_{2p_i}$ does not fix $[x_0, x_k] \setminus [x_0, x_i]$.

We consider first the case where $k = 1$ and $[x,bx]$ is fixed by only $(a,b)_{2m}$. Note that if m is prime, then this is the only possible case.

We begin by describing a fundamental domain for the action of $A(m)$ on $\mathcal{O}_m$. Since $\mathcal{O}_m$ is the set of translates of $\Pi(a, z_{a,b}) \cup \Pi(b, z_{a,b})$ by elements of $A(m)$, a fundamental domain is contained within this union, $\Pi(a, z_{a,b}) \cup \Pi(b, z_{a,b})$. Since both a and b map a side of $B(a,b)$ to the other by translation, $\Pi(a, z_{a,b}) = \langle a \rangle B(a,b)$ and $\Pi(b, z_{a,b}) = \langle b \rangle B(a,b)$. Hence a fundamental domain is contained in $B(a,b)$. Let F_m be a regular trapezium contained in $B(a,b)$ with parallel edges $z_{a,b}$-segments, and non-parallel edges, an a-segment and a b-segment of length $|a| = |b|$. Let the longer of the parallel edges have length $\frac{m+1}{2}|a|\cos\frac{1}{2}\theta(a,b)$. Then F_m is a fundamental domain for the action of $A(m)$ on $\mathcal{O}_m$. By observing that the group element $(a,b)_m$ identifies the non-parallel a and b-segments of F and the group elements a and b identify $z_{a,b}$-segments of length $|a|\cos\frac{1}{2}\theta(a,b)$ on the parallel edges of F_m we form the quotient complex $\mathcal{O}_m/A(m)$ (see figure 4.1).

It is useful to note that the a and b axes translate across F_m with

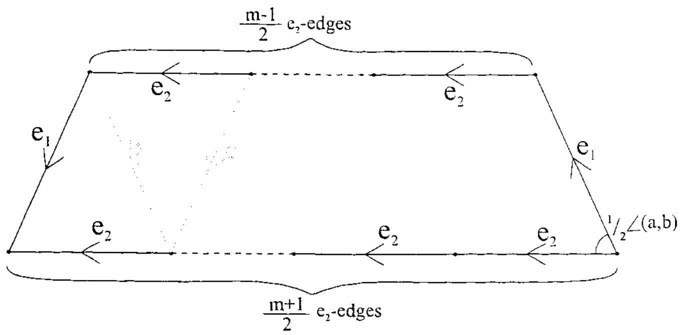

Figure 4.1: The quotient complex $\mathcal{O}_m/A(m)$ for $m \geq 3$ odd

$\theta(a, z_{a,b}) = \theta(b, z_{a,b})$. So we may choose any 2 such intersecting edges in F_m to represent generator loops for a and b in the fundamental group. Examples are shown in figure 4.1.

We check that the complex has the required fundamental group and is locally CAT(0). This complex has one vertex, 2 edges and 1 2-cell, so a presentation for the fundamental group is,

$$\left\langle e_1, e_2 \middle| e_1 e_2^{\frac{m-1}{2}} e_1 = e_2^{\frac{m+1}{2}} \right\rangle$$

The homomorphism defined by $e_1 \mapsto a$ and $e_2 \mapsto ab$ is an isomorphism from this group to $A(m)$.

By lemma 1.4.3 we need only check the link of the vertex is CAT(1). There are two edges, so 4 vertices in the link, $l_1 := \iota e_1$, $l_2 := \tau e_1$, $l_3 := \iota e_2$, and $l_4 := \tau e_2$. The edges are: (l_1, l_3) and (l_2, l_4) of length $\frac{1}{2}\theta(a, b)$, (l_1, l_4) and (l_2, l_3) of length $\pi - \frac{1}{2}\theta(a, b)$ and $m - 2$ edges of length π, $(l_3, l_4$. Since the edges (l_3, l_4) have length π we may remove all but one from our consideration. We construct the weighted adjacency matrix. Let $\theta := \theta(a, b)$.

$$\begin{pmatrix} . & . & \theta & \pi - \theta \\ . & . & \pi - \theta & \theta \\ \theta & \pi - \theta & . & \pi \\ \pi - \theta & \theta & \pi & . \end{pmatrix}$$

Apply ρ to row 1;

$$\begin{pmatrix} . & \pi-\theta & \theta \\ \pi-\theta & . & \pi \\ \theta & \pi & . \end{pmatrix}$$

This represents a triangle with edge lengths θ, $\pi-\theta$ and π. Thus the edge sum is 2π and we may conclude that the link is CAT(1) and the complex, therefore, locally CAT(0). We call this quotient complex and fundamental domain, odd of type 1.

We now consider the situation where $k > 1$ (see definition 4.1.2). Observe that $|ab|_z = |ba|_z = 2|a|\cos\theta(a, z_{a,b})$. Since $[x, r] = [x, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{k-1}, r]$ where each $[x_i, x_{i+1}]$ is fixed by $(ab)^{p_i}$ for some divisor p_i of m, a fundamental domain for this action is a union of rectangles, $R_1 \cup \cdots \cup R_k$ where each R_i is the product of an interval of length $p_i|ab|_z$ with $[x_{i-1}, x_i]$.

By skewing these rectangles we may suppose that they are parallelograms, the union of a (or b) segments. Also we may split each parallelogram into 2 trapeziums, 1 with shorter edge of length $\frac{p_i-1}{2}$ and the other with length $\frac{p_i+1}{2}$. We therefore form a fundamental domain which is a subset of the fundamental domain, 'odd of type 1'.

By identifying the edges we obtain a quotient complex. See figure 4.2 for when $m = 9$.

We call the quotient complex odd of type k.

The fundamental domain always contains a parallelogram with a (or b) edges of length $|a|$ and $z_{a,b}$ edges of length $|a|\cos\frac{1}{2}\theta(a,b)$.

The following is a useful lemma for when $m = 3$.

Lemma 4.1.6. *If $m = 3$, then the group element a maps a strip the same width as $B(a,b)$ in $\Pi(b, z_{a,b})$ adjacent to $B(a,b)$ to a similar strip on the other side of $B(a,b)$ in $\Pi(b, z_{a,b})$. More over if the group element a maps a*

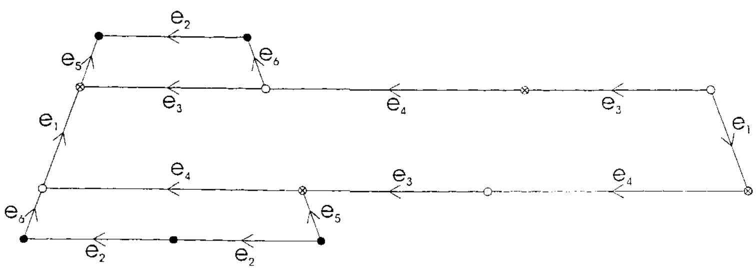

Figure 4.2: A quotient complex for $\mathcal{O}_9/A(9)$ for $k = 2$.

strip in $\Pi(b, z_{a,b})$ *adjacent to* $B(a, b)$ *to a similar strip on the other side of* $B(a, b)$ *in* $\Pi(b, z_{a,b})$ *then* m *is a multiple of* 3.

Proof. Consider the tree T obtained by projecting $\mathcal{O}_m$ in the direction of $z_{a,b}$. Let $[y, ay]$ denote the intersection $\gamma_a \cap \gamma_b$. The point $b.y$ is in $\gamma_b \setminus \gamma_a$. It is sufficient to show that $ab.y = b^{-1}ay$. Since $aba = bab$, $abay = baby$. But $bay = y$, so $baby = ay$, hence result.

We have already demonstrated the second part. □

4.1.2 A quotient space for $A(m)$ when $m \geq 4$ is even

For even values of m we do not have such good constraints on the quotient complex, however if we restrict ourselves to actions where $|a|^{bot} > |b|^{\perp}$ then we are able to describe the complexes in much the same way as for m odd.

Lemma 4.1.7. *If* $m \geq 4$ *is even and* $|a|_z^{\perp} > |b|_z^{\perp}$*, then there exists a* $z_{a,b}$*-axis* l *in* $\Pi(a, z_{a,b}) \setminus B(a, b)$ *fixed by* ba *and* $l' := a.l \subset \Pi(a, z_{a,b}) \setminus B(a, b)$ *fixed by* ab.

This follows from lemma 4.0.9.

Consider the tree T_m. Let x and bx be the vertices of $\gamma_a \cap \gamma_b$, and y the point in γ_a such that $bay = y$. Observe that the stabiliser of the edge $[x, bx]$

is $\langle (a,b)_m \rangle$. However, as for when m is odd, we may divide the edge $[bx, y]$ into intervals $[x_i, x_{i+1}]$, $i = 0, \ldots, k$ where $x_0 = y$, $x_k = bx$ and $[x_0, x_i]$ is fixed by $(b,a)_{2p_i}$ where $p_i | p_{i+1}$, $p_k = \frac{m}{2}$, $p_0 = 2$ and each p_i is even. Also $(b,a)_{2p_i}$ does not fix any point of $[x_i, x_k] \setminus [x_0, x_i]$.

As with the odd case, a fundamental domain F in $\mathcal{O}_m$ for $A(m)$ is the union of rectangles $R_i = [x_i, x_{i+1}] \times [0, 2p_i|ab|_z]$.

Again we may skew these rectangles in the $z_{a,b}$ axis, so each is now a parallelogram formed by a union of a (or b) segments.

We may form a quotient complex for F, see figure 4.3. The $z_{a,b}$ edges of R_i are divided into $\frac{p_i}{p_{i-1}}$ edges labeled z_i on one side and $\frac{p_{i+1}}{p_i}$ edges labeled z_{i+1} on the other, except for R_k which has 1 edge labeled z_k on both sides, and R_0 which has 1 edge labeled z_1 on one side and 2 edges labeled z_0 on the other.

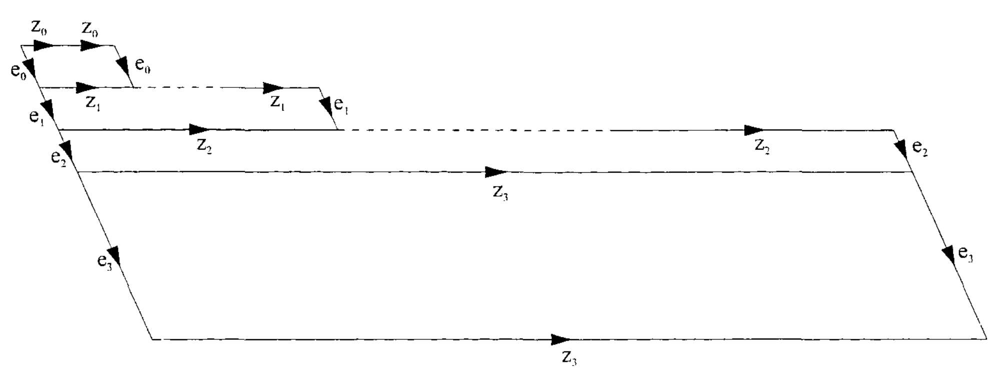

Figure 4.3: A quotient complex for the action of $A(m)$ on $\mathcal{O}_m$ for $m \geq 4$ even.

We call the quotient complex, even of type k.

Note that if there are no even divisors of $\frac{m}{2}$, ie 4 does not divide m, then the quotient complex is necessarily even of type 1. See figure 4.4 for $A(6)$.

The following is a useful lemma specific to the case where m is 4.

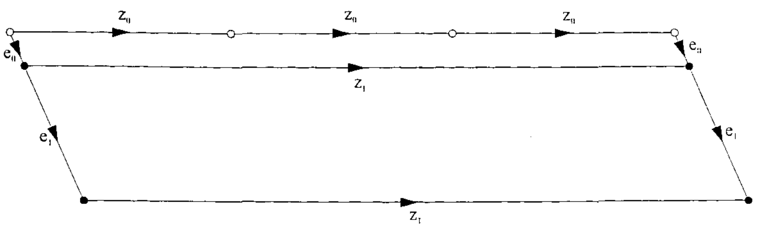

Figure 4.4: A quotient complex for the action of $A(6)$ on $\mathcal{O}_6$.

Lemma 4.1.8. *Suppose $A(4)/\langle z_{a,b}\rangle$ acts properly discontinuously on a tree T such that $|a| > |b|$. Let $[y, b.y] = \gamma_a \cap \gamma_b$. There exists a point $x \in \gamma_a$ such that $d(x, b.y) = d(a.x, y)$. Then there is a $z \in \gamma_a$ such that $d(z, y) = 2d(a.x, y) = 2d(a.x, z)$ and $b.z \in \gamma_a$.*

The proof of this follows directly from the fact that *abab* fixes all points in T. This lemma provides a useful corollary.

Corollary 4.1.9. *For $A(4)$ acting properly discontinuously by semi-simple isometries on $\mathcal{O}_m$ with $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$, b maps a strip of width $|a|^{\perp}_{z_{a,b}} - |b|^{\perp}_{z_{a,b}}$ adjacent to $B(a,b)$ in $\Pi(a, z_{a,b})$ to a similar strip on the other side of $B(a,b)$ in $\Pi(a, z_{a,b})$.*

4.2 Minsets in $A(2m, n, 2)$

We suppose that $A(2m, n, 2)$ acts properly discontinuously by semi-simple isometries on a CAT(0) 2-complex X. In this section we examine the minsets of this action. We begin by showing that for any action of $A(m, n, 2)$ on a CAT(0) 2-complex there is a canonical convex subcomplex invariant under the action of $A(m, n, 2)$.

Throughout this section we will have the following standard presentation for $A(m,n,2)$:

$$\langle a,b,c|(a,b)_m=(b,a)_m,(b,c)_n=(c,b)_n,ac=ca\rangle$$

Since $A(m)=\langle a,b|(a,b)_m=(b,a)_m\rangle$, $A(n)=\langle b,|(b,c)_n=(c,b)_n\rangle$, and $A(2)=\langle a,c|ac=ca\rangle$ are subgroups of $A(m,n,2)$ these groups act properly discontinuously by semi-simple isometries of X. Thus X contains the subcomplexes $\mathcal{O}_m$, $\mathcal{O}_n$ and $\mathcal{O}_2$ as defined in the previous sections. Note that since $A(2)\cong\mathbb{Z}\times\mathbb{Z}$, $\mathcal{O}_2$ is a single $\langle a,c\rangle$-invariant plane.

We recall that $\mathcal{O}_m=A(m).(\Pi(a,z_{a,b})\cup\Pi(b,z_{a,b}))$, $\mathcal{O}_n=A(n).(\Pi(b,z_{b,c})\cup\Pi(c,z_{b,c}))$, and $\mathcal{O}_2=A(2).\Pi(a,c)=\Pi(a,c)$.

Let $\mathcal{O}_A$ denote the convex closure of $A(m,n,2).(\Pi(a,z_{a,b})\cup\Pi(b,z_{a,b})\cup\Pi(b,z_{b,c})\cup\Pi(c,z_{b,c})\cup\Pi(a,c))\subset X$. The group $A(m,n,2)$ acts properly discontinuously by semi-simple isometries on the CAT(0) 2-complex $\mathcal{O}_A$. We examine the structure of $\mathcal{O}_A$ and obtain restrictions on this structure and the actions of $A(m,n,2)$ for m even.

We make use of the following useful definitions:

Definition 4.2.1.

1. $H(a)$ is defined to be the convex closure of $\Pi(a,z_{a,b})\cup\Pi(a,c)$ in X.
   $H(b)$ is defined to be the convex closure of $\Pi(b,z_{a,b})\cup\Pi(b,z_{b,c})$ in X.
   $H(c)$ is defined to be the convex closure of $\Pi(c,z_{b,c})\cup\Pi(a,c)$ in X.

2. $\Sigma(a,b):=\Pi(a,z_{a,b})\cup\Pi(b,z_{a,b})$
   $\Sigma(b,c):=\Pi(b,z_{b,c})\cup\Pi(c,z_{b,c})$
   By proposition 4.1.1 these sets are connected.

3. If $\Pi(a,z_{a,b})\cap\Pi(a,c)\neq\emptyset$ then denote this intersection by $S(a)$. If $\Pi(a,z_{a,b})\cap\Pi(a,c)=\emptyset$ then let $S(a)$ denote the closure of $H(a)\setminus(\Pi(a,z_{a,b})\cup\Pi(a,c))$ in X. In both cases, by lemma 4.0.8, $S(a)$ is an infinite strip. Similarly define $S(b)$ and $S(c)$.

4. Let Y denote the abstract union of $H(a)$, $H(b)$ and $H(c)$ identified along the flat planes which are common between them in X. Note that Y is a connected subset of X.

4.3 Y is simply connected

In this section we show that $H(a)$ and $H(c)$ are not H-type. We deduce from this that $\Pi(a, z_{a,b}) \cap \Pi(b, z_{a,b}) \cap \Pi(b, z_{b,c}) \cap \Pi(c, z_{b,c}) \cap \Pi(a, c) \neq \emptyset$.

Suppose that $H(a)$ is H-type. Here $\Pi(a, c)$ and $\Pi(a, z_{a,b})$ are joined by an infinite strip $[-1, 1] \times \mathbb{R}$ with each line $\{\pm 1\} \times \mathbb{R}$ identified with an a-axes in the two planes $\Pi(a, c)$ and $\Pi(a, z_{a,b})$. Let μ denote the line $\{0\} \times \mathbb{R}$ in the strip and express $H(a)$ and the union $T \cup_\mu T'$ where $T = \Pi(a, z_{a,b}) \cup ([0, 1] \times \mathbb{R})$ and $T' = \Pi(a, c) \cup ([-1, 0] \times \mathbb{R})$. Let μ and μ' denote the preimages of μ in T and T' respectively and $h : \mu \mapsto \mu'$ denote the identifying homeomorphism.

Cutting Y along μ we obtain a piecewise Euclidean space $Z = T \cup_{B(a,b)} H(b) \cup_{B(b,c)} H(c) \cup_{\Pi(a,c)} T'$, such that $Y = Z/(x\ h(x), \text{for all } x \in \mu)$.

Brady and Crisp observe that both Z and $\tilde{Y}$ are $CAT(0)$, and Y is locally $CAT(0)$. Also $\pi_1(Y) \cong \mathbb{Z}$. They use this to prove the following lemma.

Lemma 4.3.1. *[BC]*

There exists a closed geodesic γ in Y which lifts to a geodesic path $\tilde{\gamma}$: $[0, 1] \mapsto Z$ with the following properties.

1. $\tilde{\gamma}(0) \in \mu$ *and* $\tilde{\gamma}(1) = h(\tilde{\gamma}(0)) \in \mu'$,

2. *There exist* $0 \leq t_{a,b} \leq t_{b,c} \leq t_{c,a} \leq 1$ *such that*

$$\tilde{\gamma}(t_{a,b}) \in B(a, b), \ \ \tilde{\gamma}(t_{b,c}) \in B(b, c), \ \ \tilde{\gamma}(t_{c,a}) \in \Pi(a, c)$$

   and

$$\tilde{\gamma}([0, t_{a,b}]) \in T, \ \ \tilde{\gamma}([t_{a,b}, t_{b,c}]) \in H(b), \ \ \tilde{\gamma}([t_{b,c}, t_{c,a}]) \in H(c), \ \ \tilde{\gamma}([t_{c,a}, 1]) \in T'.$$

The path $\tilde{\gamma}$ is a geodesic path in Z. Each path $\tilde{\gamma}([0, t_{a,b}] \cup [t_{c,a}, 1]) \subset H(a)$, $\tilde{\gamma}([t_{a,b}, t_{b,c}]) \subset H(b)$, $\tilde{\gamma}([t_{b,c}, t_{c,a}]) \subset H(c)$ is contained in a convex subset of X so is mapped to a geodesic path in X. For $s \in \{a, b, c\}$ denote by δ_s the above geodesic contained in $H(s)$. The three geodesics δ_a, δ_b, and δ_c form either a geodesic triangle (if all δ_s have nonzero length) or a geodesic bigon (if one of δ_s has zero length). Denote the vertices by $t_{r,s}$, as the endpoint of the two geodesics δ_r and δ_s.

Observe that since $H(a)$ is H-type, δ_a has nonzero length, thus we cannot have $t_{a,b} = t_{b,c} = t_{c,a}$.

Brady and Crisp consider the links of the points $t_{a,b}$, $t_{b,c}$, and $t_{c,a}$ and show that for $t_{a,b}$ and $t_{b,c}$ the diameter of the link at most $\leq \frac{5\pi}{3}$ and for $t_{c,a}$ the diameter of the link at most $\leq \frac{3\pi}{2}$.

This together with the following lemma implies that the triangle $(\delta_a, \delta_b, \delta_c)$ in X has angle sum $> \pi$ in the nondegenerate case, and is otherwise a geodesic bigon with at least one non-zero angle. Each of these outcomes contradicts the assumption that X is CAT(0), thus completing their proof that $H(a)$ in not H-type.

Lemma 4.3.2. *[BC, lemma 3.4] Suppose that $f : U \to V$ is a locally isometric map from a geodesic space U to a CAT(1) space V. Suppose that $p, q \in U$ are such that $d_U(p, q) \geq \pi$. Then*

$$d_V(f(p), f(q)) \geq 2\pi - d_U(p, q).$$

The important part of Brady and Crisp's proof is contained in the examination of the links in Y of the three vertices $t_{a,b}$, $t_{b,c}$ and $t_{c,a}$ and the fact that these links, considered as subspaces of Y embed locally isometrically in X.

The link of $t_{r,s}$ in Y is the union of the links $Lk(t_{r,s}, H(r))$, $Lk(t_{r,s}, \Sigma(r, s))$ and $Lk(t_{r,s}, H(s))$ identified along the common arcs.

Every point of $Lk(t_{r,s}, Y)$ is contained in a neighbourhood which is a subset of one of these links. This assertion uses the fact that the translation axes are transverse since both m and n are odd.

The embedding of $H(r)$, $H(s)$ and $\Sigma(r, s)$ as convex subspaces shows the link $Lk(t_{r,s}, Y)$ embeds locally isometrically in X.

We begin with a short lemma which shows we may suppose δ_a, δ_b and δ_c form a nondegenerate geodesic triangle. We then begin to build information about the space Y. We show that parts of $H(a)$ fold on to $H(b)$ and break into cases depending on the 'amount' of folding. In the more interesting case, where the least folding occurs, we examine the links of $t_{c,a}$ and $t_{b,c}$ to obtain inconsistent inequalities, and hence a contradiction.

Lemma 4.3.3. *If $H(a)$ is H-type and n is odd, then $t_{a,b}$, $t_{a,c}$ and $t_{c,a}$ are distinct points in Y.*

Proof. By [BC, lemma 3.3] there is a geodesic triangle (possibly degenerate), Δ with vertices $t_{a,b}, t_{b,c}, t_{a,c}$ and edges $\delta_r = [t_{r,s}, t_{p,r}]$. We know that $t_{a,b} \in B(a, b)$, $t_{b,c} \in B(b, c)$ and $t_{a,c} \in \Pi(a, c)$.

We show that $|\delta_r| \neq 0$ for all $r \in \{a, b, c\}$.

Since $H(a)$ is H-type, $|\delta_a| > 0$.

Suppose $|\delta_b| = 0$, so $t_{a,b} = t_{b,c}$. We know that $t_{a,b} \in B(a, b) \subset Min(a)$, and $t_{b,c} \in B(b, c) \subset Min(c)$. As $t_{a,b} = t_{b,c}$, it follows that $t_{a,b} \in Min(a) \cap Min(c) = \Pi(a, c)$. Hence $t_{a,b} \in \Pi(a, z_{a,b}) \cap \Pi(a, c)$, contradicting $H(a)$ as H-type.

The remainder of this proof shows that $|\delta_c| > 0$.

If $H(c)$ is H-type, then we are done, so we may assume that $H(c)$ is X or B-type.

We suppose for a contradiction that $|\delta_c| = 0$, so $t_c := t_{a,c} = t_{b,c}$. The triangle Δ is therefore degenerate and forms a bigon with edges δ_a and δ_b.

It is sufficient to show that at least one of the internal angles (θ_{ab} or θ_c the angle at t_c) in this bigon is non-zero.

We begin by showing that $H(b)$ is not X or B-type. Suppose that $\theta_{ab} = \theta_{ac} = 0$, so $\delta_a = \delta_b$ in X, hence $|\delta_a| = |\delta_b|$. It also follows from $\theta_{ab} = 0$ that $\delta_a \perp \gamma_a$ and $\delta_b \perp \gamma_b$. Suppose that $t_c \notin \partial S(b)$. The link $Lk(t_c, Y)$ is isomorphic to a sublink of one of th two links shown in figure 4.5. If in case 1 $\pi - \theta - \rho \neq 0$, then these two links have diameter strictly greater than π and strictly less than 2π and these links embed locally isometrically in $Lk(t_c, X)$. Hence the angle $\theta_c > 0$, a contradiction. If $\pi = \theta + \rho$, then $z^+_{b,c} = a^-$ and this link may collapse in X. However a segment if maximum length $\theta < \frac{\pi}{2}$ in $[a^+, a^-]$ and $[z^+_{b,c}, z^-_{b,c}]$ may be identified. Since $\delta_a \perp \gamma_a$ and the corresponding points in this link are distinct, hence $\theta_c > 0$.

We may therefore assume that $t_c \in \partial S(b)$. Suppose $H(b)$ is X-type. We know $t_c \in \partial S(b)$ and $t_{a,b} \in \partial S(b)$ so δ_b (with end points t_c and $t_{a,b}$) is parallel to γ_b.

Now suppose $H(b)$ is B-type. The above argument shows that t_c and $t_{a,b}$ are on opposite sides of $S(b)$. We know that $t_{a,b} \in intB(a, b)$, so since $\gamma_{z_{a,b}}$ is not parallel to γ_b there exist points $p \in \delta_b \setminus (t_{a,b} \cup t_{a,c}) \cap B(a, b)$ and $q \in \delta_a \setminus (t_{a,b} \cup t_{a,c})$ such that p is identified with q in X. Since $p \in \Pi(a, z_{a,b})$, $q \in \Pi(a, z_{a,b})$, a contradiction as $\delta_a \cap \Pi(a, z_{a,b}) = t_{a,b}$

So $H(b)$ must be H-type.

We now examine the link $Lk(t_c, Y)$ for each case $H(c)$ being H, X or B-type. Again we suppose for a contradiction that $\theta_{ab} = \theta_c = 0$.

Consider the link $Lk(t_c, Y)$. This link is the union of the three sublinks $Lk(t_c, H(r))$, $r \in \{a, b, c\}$.

The link $Lk(t_c, H(a))$ is a sublink of a θ-graph with valance 3 vertices labeled by a^+ and a^- and 2 other distinguised antipodal points c^+ and c^-.

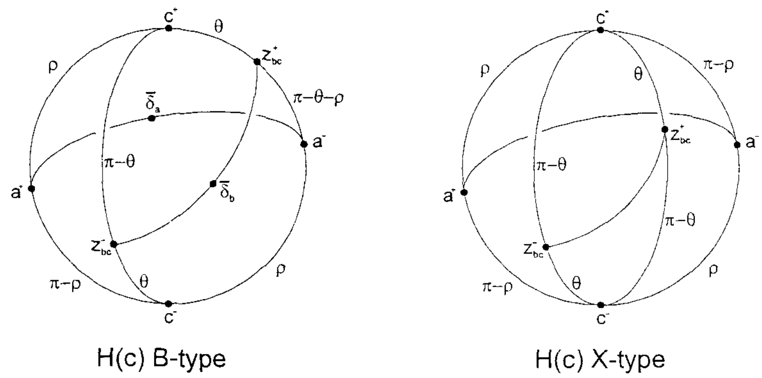

Figure 4.5: Possible links of t_c with largest diameter.

The link $Lk(t_c, H(b))$ is a θ-graph or (similarly) a union of 4 π arcs with valence 3 (or 4) vertices b^+ and b^-, 2 antipodal points z_{bc}^+ and z_{bc}^- and we may assume a third point on the arc $[b^+, z_{bc}^+, b^-]$ labeled c^+ with $d(c^+, z_{bc}^+) = d(b^+, z_{bc}^+)$, see figure 4.6.

If $H(c)$ is X-type then $Lk(t_c, H(c))$ is a union of 4π arcs joined at 2 points labeled c^+ and c^- with a^+, a^-, z_{bc}^+ and z_{bc}^- each on separate π arcs such that a^+ and a^- are antipodal, as are z_{bc}^+ and z_{bc}^-.

If $H(c)$ is B-type then $Lk(t_c, H(c))$ is a θ-graph with valance 3 vertices labeled by c^+ and c^- and 2 pairs of antipodal points, $a^\pm$ and $z_{bc}^\pm$ such that exactly one of $z_{bc}^\pm$ is on the π arc not shared by a^+ or a^-.

The link $Lk(t_c, Y)$ is formed by identifying these links $Lk(t_c, H(r))$ for $r \in \{a, b, c\}$ along the common arcs in each.

We first consider the link if $H(c)$ is B-type. The link takes one of two forms shown in figure 4.7.

We may assume that θ_{ab} is 0, so the angle between δ_a and γ_a is $\frac{\pi}{2}$ as is the angle between δ_b and γ_b. Hence the points in the link $Lk(t_c, Y)$ corresponding to δ_a and δ_b are the mid points of the π-arcs joining a^+ and a^-, and b^+ and b^-. We will label these points $\bar{\delta}_a$ and $\bar{\delta}_b$.

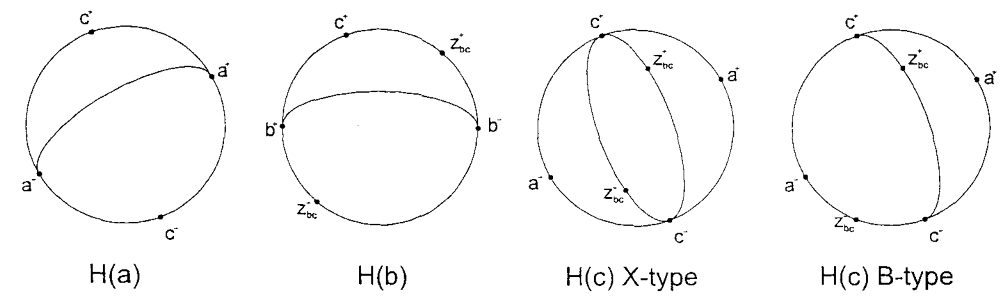

Figure 4.6: Links of t_c in $H(a)$, $H(b)$ and $H(c)$.

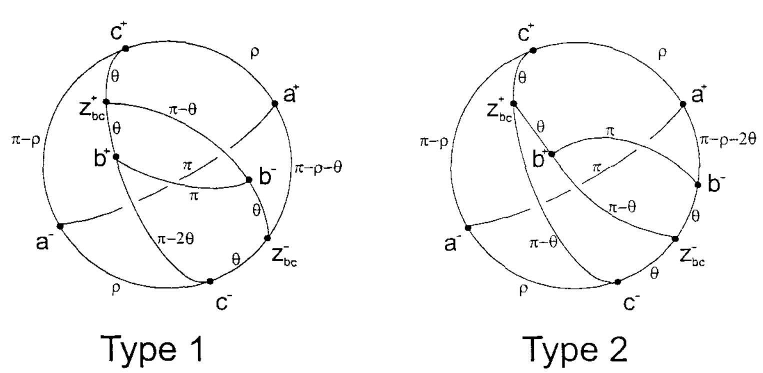

Figure 4.7: The two possible forms of $Lk(t_c, Y)$ when $H(c)$ is B-type.

We initially suppose $d(a^+, z_{bc}^-) > 0$ and in type 2 $d(a^+, b^-) > 0$ so these links embed locally isometrically in the link in X.

The angle θ_{ac} is the distance $d(\bar{\delta}_a, \bar{\delta}_b)$ if $d(\bar{\delta}_a, \bar{\delta}_b) \leq \pi$, or $\theta_{a,c} \geq 2\pi - d(\bar{\delta}_a, \bar{\delta}_b)$ if $d(\bar{\delta}_a, \bar{\delta}_b) > \pi$. Now, $d(\bar{\delta}_a, \bar{\delta}_b) > d(\bar{\delta}_a, a^\pi) + d(b^\pm, \bar{\delta}_b) = \pi$, so $\theta_{a,c} \geq 2\pi - d(\bar{\delta}_a, \bar{\delta}_b)$. So it is sufficient to show that $min\{d(a^\pm, b^\pm)\} < \pi$ for both types of $Lk(t_c, Y)$.

For link of type 1.

$$\min\left\{d(a^\pm, b^\pm)\right\} \leq \min\left\{\pi - 2\theta + \rho, 2\theta + \pi - \rho, 2\theta + \rho, \theta + d(a^+, z_{bc}^-)\right\}$$

This is at least π if and only if $\theta = \frac{\pi}{4}$ and $\rho = \frac{\pi}{2}$ and $\frac{\pi}{4} + d(a^+, z_{bc}^-) = \pi$. Since $\rho < \pi - \theta$, $d(a^+, z_{bc}^-) = \pi - \rho - \theta = \frac{\pi}{2}$, so $\theta + d(a^+, z_{bc}^-) = \frac{3\pi}{4} < \pi$. Hence $\min\{d(a^\pm, b^\pm)\} < \pi$.

For link of type 2 it is immediate that

$$\min\left\{d(a^\pm, b^\pm)\right\} \leq d(a^+, b^-) < \pi$$

Now consider type 1 and type 2 for $a^+ = z_{bc}^-$. In this case a segment of the $[a^+, a^-]$ π-arc at a^+ may collapse on to a segment of the $[z_{bc}^+, z_{bc}^-]$ π-arc at z_{bc}^- in $Lk(t_c, X)$. The loop $[z_{bc}^-, b^-, z_{bc}^+, c^+, a^-, c^-, z_{bc}^-]$ has length $2\pi + 2\theta$, so at most these segments have length $\theta < \frac{\pi}{2}$. Hence $d(\bar{\delta}_a, \bar{\delta}_b) \geq \pi - 2\theta > 0$ since $\bar{\delta}_a$ and $\bar{\delta}_b$ are the mid points of arcs $[a^+, a^-]$ and $[b^+, b^-]$.

Now suppose that $d(a^+, b^-) = 0$ in type 2 with a segment $[a^+, s]$ of length γ in arcs $[b^+, b^-]$ and $[a^+, a^-]$ identified, see figure 4.8. Since $\bar{\delta}_a$ and $\bar{\delta}_b$ are the mid points of $[a^+, a^-]$ and $[b^+, b^-]$, if $\gamma < \frac{\pi}{2}$ then we are done. So suppose that $\gamma \geq \frac{\pi}{2}$.

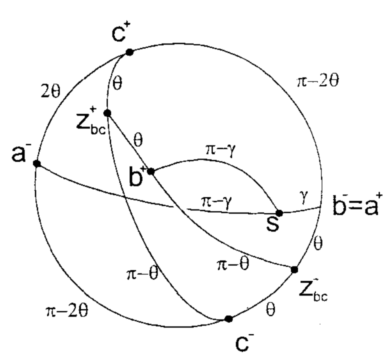

Figure 4.8: Link at t_c when $a^+ = b^-$.

We must ensure that the link remains CAT(1) so the loops $[s, b^+, z_{bc}^+, c^+, a^-, s]$ and $[s, b^+, z_{bc}^-, c^-, a^-, s]$ of lengths $2\pi - 2\gamma + 4\theta$ and $4\pi - 2\theta - 2\gamma$ are greater than 2π. So, $2\theta \geq \gamma$ and $\pi \geq \theta + \gamma$.

Let t_2 denote the other vertex of $\partial S(a) \cap S(c)$ (recall that we are assuming $H(c)$ is B-type). Suppose $t_2 \in intB(b,c)$. Then the link of t_2 is as shown in figure 4.9.

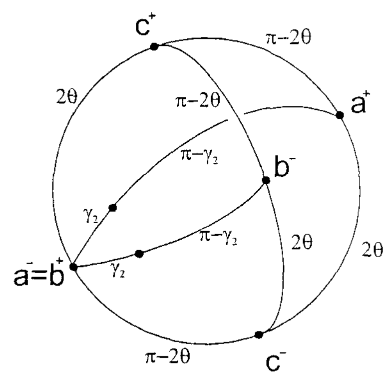

Figure 4.9: The link of $t_2 \in intB(b,c)$.

Since the common convex region in $H(a)$ and $H(b)$ in X contains δ_a and $\partial S(a) \cap S(c)$, a segment of this link $Lk(t_2, Y)$ contained on the arcs $[a^+, a^-]$ and $[b^+, b^-]$ must collapse in X. Say of length γ_2. Hence $2\pi - 2\gamma_2 + 4\theta \geq 2\pi$ and $4\pi - 2\gamma_2 - 4\theta \geq 2\pi$, so $2\theta \geq \gamma_2$ and $\pi \geq \gamma_2 + 2\theta$, hence $\frac{\pi}{2} \geq \gamma_2$. See figure 4.10

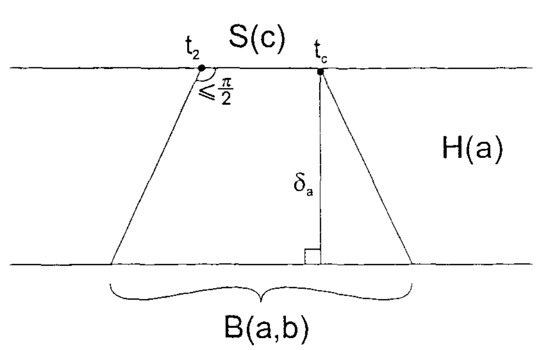

Figure 4.10: The identified region of $H(a)$.

Since $\theta_{ab} = 0$, we know that a convex region containing $\partial S(a) \cap B(a,b)$ in $H(a)$ is identified with a similar region in $H(b)$. By studying the links of the vertices of $\partial S(a) \cap B(a,b)$ we know that the internal angles of this region at the vertices of $\partial S(a) \cap B(a,b)$ are strictly less than $\frac{\pi}{2}$. This contradicts $\frac{\pi}{2} \geq \gamma_2$.

Hence both vertices of $\partial S(a) \cap S(c)$ are contained in the boundary of $B(b,c)$.

The above argument regarding the convexity of the identified regions in $H(a)$ and $H(b)$ shows that the internal angles of the region at the vertices of $\partial S(a) \cap S(c)$ are strictly greater than $\frac{\pi}{2}$, so $\gamma_2 > \frac{\pi}{2}$. Hence $2\theta > \gamma_2 > \frac{\pi}{2}$ and $\pi - 2\theta \geq \gamma_2 > \frac{\pi}{2}$, therefore $\theta > \frac{\pi}{4} \geq \theta$, a contradiction. Thus $H(c)$ is not B-type.

Now we suppose that $H(c)$ is X-type. The link of t_c is as shown in figure 4.11.

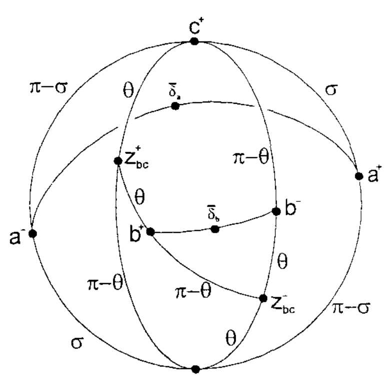

Figure 4.11: The link of t_c if $H(c)$ is X-type.

We consider $d(\bar{\delta}_a, \bar{\delta}_b) = \pi +, \min\{d(a^{\pm}, b^{\pm})\} = \pi + \min\{\sigma, \pi - \sigma\} + \min\{d(b^{\pm}, c^{\pm})\}$ $= \pi + \min\{\sigma, \pi - \sigma\} + \min\{2\theta, \pi - 2\theta\}$. Hence $\pi < d(\bar{\delta}_a, \bar{\delta}_b) \leq 2\pi$. Thus we are done, unless $\sigma = \frac{\pi}{2}$ and $\theta = \frac{\pi}{4}$.

Suppose that $\sigma = \frac{\pi}{2}$ and $\theta = \frac{\pi}{4}$. We are again finished unless $\delta_a = \delta_b$ in $Lk(t_{ac}, X)$. Observe that if δ_a and δ_b are be identified in this link then just these points are identified. So the region on $H(a)$ identified with a region in $H(b)$ in X near t_c is a segment of δ_a, but this contradicts this region being convex. Thus $H(c)$ is not X-type.

□

If the a and b-axes are parallel in $B(a, b)$ as is possible when m is even, we may still apply Brady and Crisp's work to show the angle at $t_{c,a}$, $\theta_{a,c}$ is at least $\frac{\pi}{2}$ and the angle at $t_{b,c}$, $\theta_{b,c}$ is at least $\frac{\pi}{3}$.

However we cannot apply this to find the angle at $t_{a,b}$.

We now prove an initial lemma regarding the intersection of $H(a)$ and $H(b)$.

Lemma 4.3.4. *Suppose γ_a is parallel to γ_b in $B(a, b)$, γ_b is transverse to γ_c in $B(b, c)$, γ_c is transverse to γ_a in $\Pi(a, c)$ and $H(a)$ is H-type. Then*

$$B(a, b) \cap \partial S(a) \cap \partial S(b) \neq \emptyset.$$

Proof. Suppose the lemma is false.

Case 1 Suppose $S(b) \cap \partial S(a) \cap B(a, b) = \emptyset$. Then we may choose $t_{a,b} \in B(a, b) \setminus (S(b) \cup s(a))$. The link of $t_{a,b}$ will then be wholly contained in $\Sigma(a, b)$, so embeds locally isometrically in X. Thus Brady and Crisp's proof applies to show the angle $\theta_{a,b} \geq \frac{\pi}{3}$ and the required contradiction.

Case 2 Suppose $B(a, b) \cap \partial S(a) \cap \text{int}(S(b)) \neq \emptyset$. So $H(b)$ is B-type and $S(a)$ intersects the interior of this band. Again we may choose $t_{a,b} \in B(a, b) \cap \text{int}(S(b)) \setminus \partial S(a)$. Again the link of $t_{a,b}$ is wholly contained in $\Sigma(a, b)$ and the contradiction is obtained as before. □

As a result of the previous lemma we may assume that $t_{a,b}$ is an element of $B(a,b) \cap \partial S(a) \cap \partial S(b)$. The link of this point is shown in fig 4.12.

It is clear that at least one of the points $a^{\pm}$ in $Lk(t_{a,b}, Y)$ considered as a subspace of Y is an element of both $H(a)$ and $H(b)$ ie, in $B(a,b)$. This allows for the possibility that $Lk(t_{a,b}, X)$ does not embed locally isometrically in $Lk(t_{a,b}, X)$. The following lemma shows that if this is the case then 'folding' does occur in $H(a) \cup H(b)$. If the link does embed locally isometrically, then we may apply Brady and Crisp's work to show $H(a)$ is not H-type.

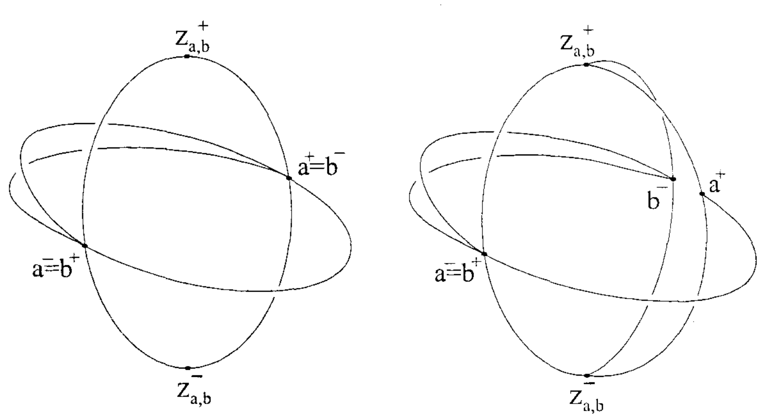

Figure 4.12: Possible links of $t_{a,b}$

Lemma 4.3.5. *Suppose $H(a)$ is H-type, γ_a is parallel to γ_b in $B(a,b)$ and γ_b is transverse to γ_c in $B(b,c)$. Suppose the link of $t_{a,b}$ in Y does not embed locally isometrically in the link of $t_{a,b}$ in X. Then some region of $S(a)$ containing $t_{a,b}$ and properly containing $B(a,b) \cap S(a)$ is identified with a region of $H(b) \setminus \Pi(b, z_{z_{a,b}})$ in X.*

Proof. By lemma 4.3.4, $t_{a,b} \in \partial S(a) \cap \partial S(b)$. The link of $t_{a,b}$ in Y is the union of the links in $H(a)$ and $H(b)$ identified along common arcs in $\Sigma(a,b)$.

Each point of $Lk(t_{a,b}, Y)$ embeds locally isometrically in $Lk(t_{a,b}, X)$ except both the points a^+ and a^- if $t_{a,b}$ is an interior point of $B(a, b)$, or one of these if $t_{a,b} \in \partial B(a, b)$.

Suppose a^+ is one of these points. A neighbourhood of a^+ in $Lk(t_{a,b}, Y)$ is a cross with two opposite arms in $B(a, b)$ one arm in $S(a)$ and one arm in $H(b) \setminus \Pi(b, z_{z_{a,b}})$.

If this neighbourhood does not map isometrically into $Lk(t_{a,b}, X)$ then, since $B(a, b)$ is convex in X the arms of the cross $S(a)$ and $H(b) \setminus \Pi(b, z_{z_{a,b}})$ must be identified.

The result follows directly. □

So we know in the situation we are considering some region of $S(a)$ is identified with a region in $H(b) \setminus \Pi(b, z_{z_{a,b}})$. These regions extend to a maximal pair of connected regions $\Delta_a \subset H(a)\setminus\Pi(a, z_{a,b})$ and $\Delta_b \subset H(b)\setminus\Pi(b, z_{a,b})$ which are identified in X.

We use the following two useful lemmas to locate the positions of $t_{a,c}$ and $t_{b,c}$.

Lemma 4.3.6. $t_{c,a} \in \partial S(a) \cap \partial S(c) \subset \Pi(a, c)$.

Proof. We know from lemma 4.3.1 that $t_{c,a} \in \Pi(a, c)$. Suppose $t_{c,a} \in \Pi(a, c)\setminus (\partial S(a) \cup \partial S(c))$. Then the link of $t_{c,a}$ in Y is a single circle. The diameter of this circle is π, so the maximum distance between any two points of it is π. Hence by lemma 4.3.2 the angle $\theta_{a,c} \geq \pi$. Since $\theta_{b,c} \geq \frac{\pi}{3}$ the sum $\theta_{a,b} + \theta_{b,c} + \theta_{a,c} > \pi$, a contradiction.

Suppose then that $t_{c,a} \in (\Pi(a, c) \setminus \partial S(c)) \cap \partial S(a)$. Here the link of $t_{c,a}$ in Y is theta graph of three π arcs (if $H(a)$ is H-type or B-type) or the union of four π arcs (if $H(a)$ is X-type) joined at two common points. Again the diameter is π and $\theta_{a,c} \geq \pi$.

The case where $t_{c,a} \in (\Pi(a,c) \setminus \partial S(a)) \cap \partial S(c)$ is dealt with in exactly the same way observing that we may assume that $t_{a,c} \notin B(b,c)$ since $t_{a,c} \neq t_{b,c}$, and we are left with the possibility given in the statement of the lemma. □

Lemma 4.3.7. *If b-axes are transverse to c-axes in $B(b,c)$ we may assume that $t_{b,c} \in \partial S(b) \cap \partial S(c) \cap B(b,c)$. Moreover if $t_{b,c} \in \partial S(b) \cap \partial S(c) \cap intB(b,c)$ then $\theta_{b,c} = \frac{\pi}{2} = \theta_{a,c}$ and $\theta_{a,b} = 0$.*

Proof. Suppose $t_{b,c} \in B(b,c) \setminus (S(b) \cup S(c))$. Then the link of $t_{b,c}$ in Y is a theta graph or a circle. The diameter of which is π hence the angle at $t_{b,c}$ is π.

Suppose $t_{b,c} \in (B(b,c) \cap S(b)) \setminus S(c)$. Then the link of $t_{b,c}$ is a circle if $t_{b,c} \in \mathrm{int}B(b,c) \cap S(b)$ and a theta graph if $t_{b,c} \in \mathrm{int}B(b,c) \cap \partial S(b)$ or $\mathrm{int}S(b) \cap B(b,c)$.

Suppose $t_{b,c} \in \partial B(b,c) \cap \partial S(b) \setminus S(c)$. Then the link of $t_{b,c}$ is a theta graph with an additional π-arc joined at antipodal points on a great circle at a distance strictly less than $\frac{\pi}{2}$ from the valency 3 vertices. The diameter of this is strictly less than $\frac{3\pi}{2}$ and hence $\theta_{b,c} > \frac{\pi}{2}$. Thus $\theta_{b,c} + \theta_{a,c} > \pi$, a contradiction.

Similarly $t_{b,c} \notin (B(b,c) \cap S(c)) \setminus S(b)$.

Therefore $t_{b,c} \in B(b,c) \cap S(b) \cap S(c)$. This is the intersection of three pairwise transverse strips ($S(b)$ and $S(c)$ may have zero width).

The above arguments apply if $t_{b,c} \in \mathrm{int}S(b)$ or $t_{b,c} \in \mathrm{int}S(c)$, so $t_{b,c} \in \partial S(b) \cap \partial S(c) \cap B(b,c)$.

If $t_{b,c} \in \partial S(b) \cap \partial S(c) \cap \mathrm{int}B(b,c)$ then the link of $t_{b,c}$ is a circle with four distinct points on it; $b^{\pm}$, $c^{\pm}$, with b^{+} antipodal to b^{-} and c^{+} antipodal to c^{-}, and one or two π arcs joining $b^{\pm}$ and one or two π arcs joining $c^{\pm}$. In this case the diameter of this link is $\leq \frac{3\pi}{2}$ see [BC, lemma 3.6] so $\theta_{b,c} \geq \frac{\pi}{2}$. Thus $\theta_{a,c} + \theta_{b,c} \geq \pi$ so must equal π and $\theta_{a,c} = \theta_{b,c} = \frac{\pi}{2}$ and $\theta_{a,b} = 0$. □

Lemma 4.3.8. *If $H(a)$ is type, then H-$H(c)$ is H-type.*

Proof. The link of $t_{c,a}$ is circle a with two pairs of distinguished antipodal points $a^{\pm}$ and $c^{\pm}$ and a π arc joining $a^{\pm}$ and one or two arcs joining $c^{\pm}$. Since $t_{b,c} \in \partial S(c)$, the point s_c representing the geodesic δ_c lies on the arc of length π containing $c^{\pm}$ and just one of $a^{\pm}$. The point s_a representing the geodesic δ_a lies on the π arc joining $a^{\pm}$. The maximum distance between s_a and s_c in this link is π, so by lemma 4.3.2, $\theta_{a,c} = \pi$. This cannot happen since $\theta_{b,c} > 0$, hence $H(c)$ cannot be X or B-type. □

Let v denote the vertex of the edge $S(a) \cap B(a,b)$ in Δ_a. If Δ_a and Δ_b are identified locally about v, then the link of v embeds locally isometrically in the link of v in X since Δ_a and Δ_b are maximal. This link is shown in figure 4.13. We assume that $\sigma < \frac{\pi}{2}$.

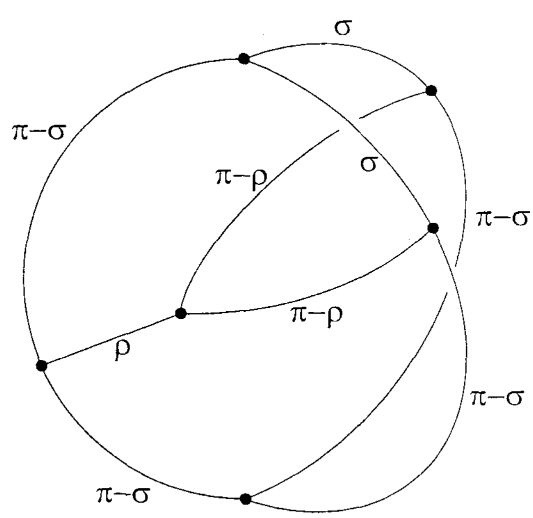

Figure 4.13: Showing the lengths of arcs in the link of v.

Since X is CAT(0), this link is CAT(1), so $2(\pi - \rho) + 2\sigma \geq 2\pi$, ie. $\sigma \geq \rho$. The diameter of this link is $\max\{\pi, \min\{2\pi - \sigma - \rho, 2\sigma + 2\rho\}\}$ which is maximised at $\frac{4\pi}{3}$.

Thus we have the following lemma

Lemma 4.3.9. *The above link has diameter at most $\frac{4\pi}{3}$.*

We now split into four cases depending on the size of Δ_a and Δ_b.

Case 1. Suppose $\Delta_a \cap \Pi(a,c) \neq \emptyset$ and $\Delta_b \subset H(b) \setminus B(b,c)$.

Denote a subset $\Delta'_a \subset \Delta_a$ to be $\Delta'_a := S(a) \cap \Delta_a$. Let Δ'_b the points in Δ_b identified with Δ'_a. Let Y' be the subset of X formed by identifying Δ'_a with Δ'_b in Y. Since $\Delta_b \subset H(b) \setminus B(b,c)$, $\pi_1(Y') \cong \mathbb{Z}$. Note that Y' is also locally CAT(0). We construct a geodesic triangle with vertices $t'_{a,b}$, $t'_{b,c}$ and $t'_{a,c}$ as in the proof of Brady and Crisp, observing that $t'_{a,c} \in \Pi(a,c)$, $t'_{b,c} \in B(b,c)$ and $t'_{a,b} \in \partial\Delta'_a$.

If $t'_{a,b} \in \Pi(a,c)$ then the union of the geodesics $\delta'_a \cup \delta'_c$ is a local geodesic in $H(c)$ which is convex in X and hence CAT(0). So $\delta'_a \cup \delta'_c$ forms a geodesic and we may assume $t'_{a,b} = t'_{a,c}$. Thus $t'_{a,c}$ and $t'_{b,c}$ are the vertices of a bigon with nonzero angle at $t'_{b,c}$. Hence $t'_{a,b} \in \partial\Delta'_a \setminus \Pi(a,c)$.

Since we have made the identification $\Delta'_a \sim \Delta'_b$ we know that locally the link of $t'_{a,b}$ in Y' embeds locally isometrically in the link in X. This link is one of two types. If $t'_{a,b} \notin B(a,b)$ then the link is a theta graph and the diameter is π. Hence the angle at $t'_{a,b}$ is π. If $t'_{a,b} \in B(a,b)$ then by lemma 4.3.9 the diameter of the link is at most $\frac{4\pi}{3}$ and so the angle is $\frac{2\pi}{3}$.

Case 2. Suppose $\Delta_a \subset S(a) \setminus \Pi(a,c)$ and $\Delta_b \cap B(b,c) \neq \emptyset$.

Denote by Δ'_b the closure in Y of $\Delta_b \cap (H(b) \setminus \Pi(b, z_{z_{b,c}}))$. Proceed as in case 1.

Case 3. Suppose $\Delta_a \cap \Pi(a,c) \neq \emptyset$, $\Delta_b \cap B(b,c) \neq \emptyset$. Observe that in X, $\Pi(a,c) \cap B(b,c) \neq \emptyset$. So $H(c)$ is not H-type. However by lemma 4.3.8 this is a contradiction.

We are left with the following case.

Case 4. Suppose $\Delta_a \subset S(a) \setminus \Pi(a,c)$ and $\Delta_b \subset H(b) \setminus B(b,c)$. This is a slightly more interesting case and will be dealt with by the next few lemmas. The method is to make a similar identification as made in cases 1-3 and construct the geodesic triangle, $\delta_a \cup \delta_b \cup \delta_c$. We show exactly where the vertices of this triangle are and examination of the links of these vertices allows us to deduce that $H(c)$ is H-type. The resulting possibilities for the links of two of the vertices and the fact that $H(a)$, $H(b)$ and $H(c)$ are all H-type allow us to construct inconsistent inequalities, a contradiction.

Let us now construct a space Y' by identifying the regions Δ_a and Δ_b in Y to form the region $\Delta \in Y'$. Since $H(a)$ is H-type and $\Delta_a \subset S(a)$, $\pi_1(Y') \cong \mathbb{Z}$ and we may construct a geodesic triangle in Y'. Again we have vertices $t_{b,c} \in B(b,c)$ and $t_{a,c} \in \Pi(a,c)$, but this time $t'_{a,b} \in \partial\Delta$. We observe that our previous work applies for $t_{b,c}$ and $t_{c,a}$ and now consider the link of $t_{a,b}$.

Lemma 4.3.10. Δ *is convex.*

Proof. Suppose there exists points $p, q \in \Delta$ such that the geodesic $[p,q] \not\subseteq \Delta$. Clearly p, q are elements of $H(a)$ and $H(b)$, so the geodesic $[p,q]$ is contained in $H(a)$ and $H(b)$. Hence these points are identified in X, and thus must be in Δ. □

In the following lemma we use the link of $t_{a,b}$ to show that in fact $\theta_{b,c} < \frac{\pi}{2}$.

Lemma 4.3.11. $t_{b,c} \in \partial S(b) \cap \partial S(c) \cap B(b,c)$ *and* $\theta_{b,c} < \frac{\pi}{2}$.

Proof. It is sufficient to show that $\theta_{a,b} > 0$.

Now $t_{a,b} \in \partial\Delta \cap (S(a) \setminus \Pi(a,c)) \cap (H(b) \setminus B(b,c))$. So the link of $t_{a,b}$ is the union of two circles (one in $S(a) \setminus \Pi(a,c)$ and one in $H(b) \setminus B(b,c)$) identified along the common points in Δ. Note that since Δ is convex the identified region is a non trivial interval. A point on each circle denotes the geodesics δ_a and δ_b. These points are not in Δ as the geodesics lie outside this region. If $\theta_{a,b} = 0$ then these two points are equal. Clearly this produces a short loop in this link, contradicting X being CAT(0). □

We have a space Y' containing a geodesic triangle which embeds in X as a non trivial geodesic triangle. This triangle must bound a solid convex triangle T in X.

Lemma 4.3.12. $T \cap Y' = \delta_a \cup \delta_b \cup \delta_c$.

Proof. $H(r) \cap T$ is convex for any $r \in \{a, b, c\}$.

Suppose for some $s \in \{a, b, c\}$, $H(s) \cap T \neq \delta_s$. Let p be a point on the boundary of $(H(r) \cap T) \setminus \delta_r$ $r \in \{a, b\}$ and q on the boundary of $(H(c) \cap T) \setminus \delta_c$. If one of p or q does not exist then take it to be on the boundary of T. However at least one of p and q does exist.

Consider the geodesic $[p, q]$ in Y'. Let u be a point in $[p, q] \cap H(c) \cap H(r)$. Then $[p, u]$ is a geodesic in $H(r)$ and $[u, q]$ a geodesic in $H(c)$, hence both are geodesics in X. So p, q, u are the vertices of a geodesic triangle in X. Note that the link of u in Y' embeds isometrically in the link of u in X. The diameter of this link must be $\geq \pi$ so the internal angle of this triangle at u is π. Note also that the angles at p and q are non zero, as otherwise these would be internal points of $T \cap H(r)$ or $T \cap H(c)$. Hence this triangle has internal angle sum $> \pi$, a contradiction.

Hence either $T \cap Y' = \delta_a \cup \delta_b \cup \delta_c$ or $T \cap Y' = T$. The latter is impossible as in Y' $\delta_a \cup \delta_b \cup \delta_c$ bounds a non trivial loop. □

We know $t_{a,c} \in \Pi(a,c) \cap \partial S(a) \cap \partial S(c)$ and both $H(a)$ and $H(c)$ are H-type. The link of $t_{c,a}$ in Y embeds locally isometrically in X. However the link of $t_{c,a}$ in X has an additional arc corresponding to the triangle T. This link is shown in figure 4.14.

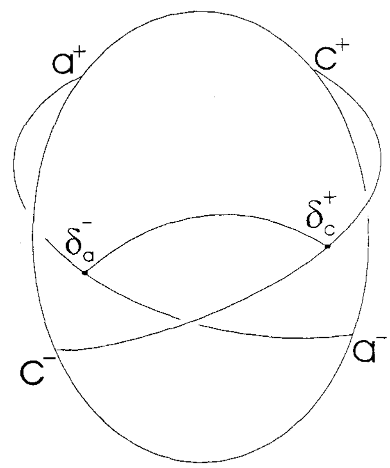

Figure 4.14: Link of $t_{c,a}$

From this link we establish the following inequalities:

1. $\theta(\delta_a^-, \delta_c^+) + \theta(\delta_a^-, a^+) - \theta(a^+, c^+) + \theta(\delta_c^+, c^-) \geq \pi$

2. $\theta(\delta_a^-, \delta_c^+) - \theta(\delta_a^-, a^+) + \theta(a^+, c^+) + \theta(\delta_c^+, c^-) \geq \pi$

Adding these and dividing by two we obtain, $\theta(\delta_a^-, \delta_c^+) + \theta(\delta_c^+, c^-) \geq \pi$. From the link of $t_{b,c}$ we obtain the following two inequalities,

1. $\theta(z_{b,c}^+, c^+) + \theta(z_{b,c}^+, b^+) + \theta(\delta_c^-, \delta_b^+) - \theta(\delta_c^-, c^+) + \theta(\delta_b^+, b^-) \geq \pi$

2. $\pi - \theta(z_{b,c}^+, c^+) - \theta(z_{b,c}^+, b^+) + \theta(\delta_c^-, \delta_b^+) - \theta(\delta_c^-, c^+) - \theta(\delta_b^+, b^-) \geq 0$

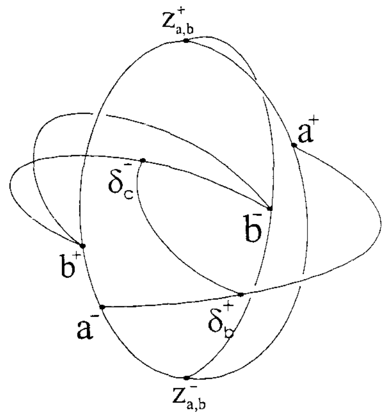

Figure 4.15: Link of $t_{b,c}$

Adding we get, $\theta(\delta_c^-, \delta_b^+) \geq \theta(\delta_c^-, c^+)$.

Figure 4.16 shows $H(c)$ which we know is H-type from lemma 4.3.8. Thus $\theta(\delta_c^-, c^+) = \theta(\delta_c^+, c^-)$.

We also know that since T is a triangle in a CAT(0) space and $\theta(\delta_a^+, \delta_b^-) > 0$ that $\theta(\delta_a^-, \delta_c^+) + \theta(\delta_c^-, \delta_b^+) < \pi$.

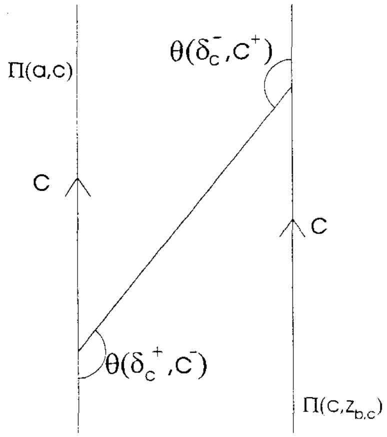

Figure 4.16: $H(c)$

So $\theta(\delta_c^-, \delta_b^+) \geq \theta(\delta_c^-, c^+) = \theta(\delta_c^+, c^-) \geq \pi - \theta(\delta_a^-, \delta_c^+) > \theta(\delta_c^- \delta_b^+)$, a contradiction.

We have proved that $H(a)$ is not H-type if m is even and n odd. It remains to show that $H(b)$ and $H(c)$ are not H-type. The following two lemmas prove this.

Lemma 4.3.13. *$H(c)$ is not H-type.*

Proof. We know that $H(a)$ is not H-type. We may assume $t_{a,b} \in S(a)$ and $t_{a,c} \in S(a)$. Consider the link of $t_{a,c}$. Suppose $H(c)$ is H-type. Then this link is a theta graph with the point representing δ_a at a valency three vertex and the point representing δ_c on a π-arc. Since the geodesics $\delta_a \cup \delta_c$ form a local geodesic, the distance in the link between these two points must be π, hence the angle at $t_{a,c}$ is π. $\square$

Lemma 4.3.14. *$H(b)$ is not H-type.*

Proof. If $\gamma_a \nparallel \gamma_b$ and $\gamma_b \nparallel \gamma_c$ then [BC] applies and we are done.

Suppose that $\gamma_a \parallel \gamma_b$.

If $S(b) \cap B(a,b) \not\subseteq \partial S(a) \cap B(a,b)$ then again done, by [BC].

So $S(b) \cap B(a,b) \subset \partial S(a) \cap B(a,b)$.

We also know that $\emptyset \neq S(c) \cap B(b,c) \subset \Pi(a,c) \cap \Pi(b, z_{b,c})$.

$S(b) \cap B(a,b) \cap S(a)$ is a b-segment.

$S(c) \cap B(b,c)$ is a union of c-segments.

Let $R = \partial S(b) \cap \Pi(b, z_{b,c}) \cap \Pi(a,c)$, a b-segment. $R \subseteq S(c) \cap B(b,c)$.

The convex closure of $(S(b) \cap B(a,b) \cap S(a)) \cup (S(c) \cap B(b,c))$ in $\Pi(a, z_c)$ is identified with the convex closure in $H(b)$.

Let Δ_b denote the region of $S(b)$ identified with a region of $\Pi(a, z_c)$, denote this by Δ_{ac}. Then $\Delta_b \cap S(c) \cap B(b,c) = R$ in $H(b)$.

$R \subsetneq \Delta_{ac} \cap S(c) \cap B(b,c)$ in $\Pi(a, z_c)$.

Either R is a point or $p \in (\Delta_{ac} \cap S(c) \cap B(b,c)) \setminus R \neq \emptyset$.

If R is a point the $\Delta_b \cup (S(c) \cap B(b,c))$ is not convex in $H(b)$.

If $p \in (\Delta_{ac} \cap S(c) \cap B(b,c)) \setminus R \neq \emptyset$ the $p \in S(b) \cap (S(c) \cap B(b,c)) \subset \Pi(a, z_c)$, so $p \in S(c) \cap \Pi(b, z_{z_{b,c}}) \cap \Pi(a, z_c) \subset R$ a contradiction. Hence result. □

So we have shown that $H(a)$, $H(b)$ and $H(c)$ are each either X-type or B-type. In the following section we use this to examine the widths of $S(a)$, $S(b)$, $S(c)$, $B(a,b)$ and $B(b,c)$ and the angles between the translation axes.

4.4 Limits on the structure of Y

In this section we assume that $H(a)$, $H(b)$ and $H(c)$ are not H-type and that n is odd. We know by the previous section that this is true for the group $A(2m, n, 2)$. Since $H(a)$ is not H-type, we know that the intersection $\Pi(a, z_{a,b}) \cap \Pi(a,c)$ is non empty. So, by definintion, $S(a)$ is this intersection. Similarly $S(b) = \Pi(b, z_{a,b}) \cap \Pi(b, z_{b,c})$ and $S(c) = \Pi(c, z_{b,c}) \cap \Pi(a,c)$. We make the following definitions.

Definition 4.4.1.

$$\begin{aligned} R &:= \operatorname{Min}(a) \cap \operatorname{Min}(b) \cap \operatorname{Min}(c) \\ R(a,b) &:= R \cap \operatorname{Min}(z_{a,b}) \\ R(b,c) &:= R \cap \operatorname{Min}(z_{b,c}) \end{aligned}$$

Observe that

$$\begin{aligned} R(a,b) &= \operatorname{Min}(a) \cap \operatorname{Min}(b) \cap \operatorname{Min}(c) \cap \operatorname{Min}(z_{a,b}) \\ &= \Pi(a, z_{a,b}) \cap \Pi(b, z_{a,b}) \cap \Pi(a,c) \\ &= S(a) \cap B(a,b) \end{aligned}$$

This set is non empty as both $S(a)$ and $B(a,b)$ are transverse bands ($S(a)$ may have zero width) in $\Pi(a, z_{a,b})$. Thus $R(a,b)$ is the union of non trivial a-segments. Similarly $R(b,c)$ is the non empty union of c-segments, $S(c) \cap B(b,c)$. It is clear that R contains both $R(a,b)$ and $R(b,c)$, hence R contains at least one non trivial a-segment and one non trivial c-segment. Since a and c axes are transverse, R must be 2-dimensional. This enables us to prove the following lemma.

Lemma 4.4.2. *If $H(a)$, $H(b)$ and $H(c)$ are all not H-type, then $R(a,b) \cap R(b,c)$ is non empty.*

Proof. Let $p \in R(a,b)$ and $q \in R(b,c)$. Then $p \in \Pi(b, z_{a,b})$ and $q \in \Pi(b, z_{b,c})$. Thus the geodesic $[p,q]$ is contained in $H(b)$ and there exists a point $r \in [p,q]$ such that $r \in S(b)$. However $p \in R(a,b) \subset \Pi(a,c)$ so $p \in \Pi(a,c)$. Similarly $q \in \Pi(a,c)$ and hence $[p,q] \subset \Pi(a,c)$ and $r \in \Pi(a,c)$. So $S(b) \cap \Pi(a,c) \neq \emptyset$, ie. $\mathrm{Min}(b) \cap \mathrm{Min}(z_{a,b}) \cap \mathrm{Min}(z_{b,c}) \cap \mathrm{Min}(a) \cap \mathrm{Min}(c) \neq \emptyset$, hence result. □

We include the following lemma as a matter of interest. We are concentrating on the groups $A(2m,n,2)$, n odd, for which we have shown that $H(a)$, $H(b)$ and $H(c)$ are not H-type. We have not, however, shown this for the groups $A(2m,2n,2)$ although it does seem likely. A lot of our results rely on the fact that if a-segments and b-segments are parallel, then b-segments and c-segments are transverse. This is proved in the following lemma.

Lemma 4.4.3. *If $H(a)$, $H(b)$, and $H(c)$ are all not H-type, and γ_a is parallel to γ_b in $B(a,b)$ then γ_b is not parallel to γ_c in $B(b,c)$.*

Proof. Let $r \in S(b) \cap \Pi(a,c) = R(a,b) \cap R(b,c)$.

Suppose $\gamma_a \parallel \gamma_b$ in $B(a,b)$ and $\gamma_b \parallel \gamma_c$ in $B(b,c)$.

Consider the link of r in Y. This is the union of the links $Lk(r, \Sigma(a,b))$, $Lk(r, \Sigma(b,c))$ and $Lk(r, \Pi(a, z_c))$ identified along the appropriate arcs.

The link $Lk(r, \Pi(a,c))$ is a circle with two pairs of distinguished antipodal points, $a^{\pm}$ and $c^{\pm}$. Note that these points are all distinct as γ_a is not parallel to γ_c in $\Pi(a,c)$.

The link $Lk(r, \Sigma(s,b))$ for $s \in \{a,c\}$ is either a circle with a pair of distinguished points $s^+ = b^-$ and $s^- = b^+$, or a theta graph with valence three vertices $z^{\pm}_{s,b}$ and three distinguished points, one on each arc with two of these points antipodal on a great circle to the other point. See figure 4.17.

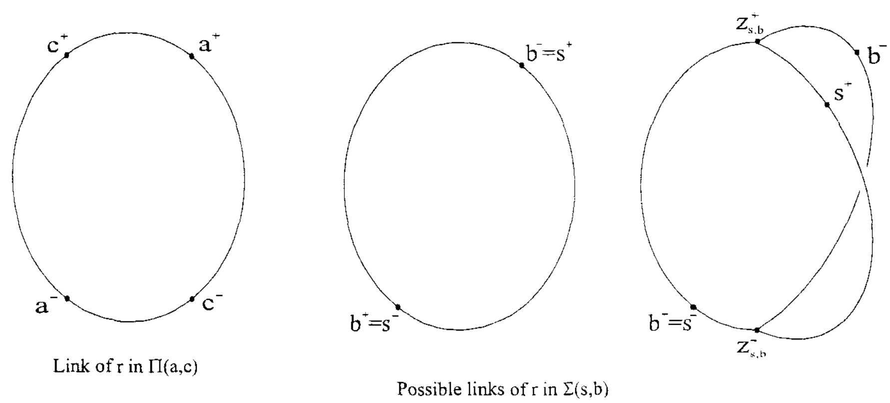

Figure 4.17: Links of r.

If both $S(a)$ and $S(c)$ are X-type then $Lk(r, \Sigma(a,b))$ is identified with $Lk(r, \Pi(a,c))$ only at the points $a^{\pm}$. Similarly $Lk(r, \Sigma(b,c))$ is identified with $Lk(r, \Pi(a, z_c))$ only at the points $c^{\pm}$.

The links $Lk(r, \Sigma(a,b))$ and $Lk(r, \Sigma(b,c))$ are identified at at least the points $b^{\pm}$. Hence the endpoints of one of the arcs $[a^+, c^+], [a^-, c^+], [a^+, c^-]$ or $[a^-, c^-]$ in $Lk(r, \Pi(a, z_c))$ are identified, forming a loop of length $< \frac{\pi}{2}$, contradicting this space being CAT(0).

Suppose that $S(a)$ is B-type. Then $Lk(r, \Pi(a,c))$ is a subset of $Lk(r, \Sigma(a,b))$ and hence the points $c^{\pm} \in Lk(r, \Sigma(a,b))$. These two points are distinct from the points $a^{\pm}$ and $b^{\pm}$ in $Lk(r, \Sigma(a,b))$. If $S(b)$ is not X-type this gives an immediate contradiction, otherwise a short loop is formed in $Lk(r, \Sigma(a,b))$ as either $c^+ = b^-$ or $c^- = b^+$ in $Lk(r, \Sigma(b,c))$.

Hence result. □

Because of lemma 4.4.3 we may assume that γ_b is not parallel to γ_c in $B(b,c)$. We begin now to establish results concerning the width of $S(a)$ and $S(b)$. The following lemma is taken from lemma 3.7 [BC].

Lemma 4.4.4. *If γ_b is not parallel to γ_c in $B(b,c)$ then $R = R(b,c)$.*

Proof. $R(b,c)$ is a union of c-segments. If it is two dimensional it is bounded by c-segments and $z_{b,c}$-segments.

We know that R is a two dimensional subset of $\Pi(a, z_c)$. Suppose that there exists a point $x \in R(b,c)$ in the closure of $R \setminus R(b,c)$. Then $Lk(x,R)$ contains a theta graph with valence three vertices at either $c^{\pm}$ or $z_{b,c}^{\pm}$. However $R \subset \mathrm{Min}(b)$ contradicting [BC, proposition 1.2(2)] since the b-axes are transverse to both the c and $z_{b,c}$-axes. □

Corollary 4.4.5. *$S(c)$ is B-type.*

Proof. This result is immediate from previous lemma, $R(b,c) = S(c) \cap B(b,c)$, c and $z_{b,c}$-segments are transverse and R is two dimensional. □

Since n is odd, γ_c is not perpendicular to $\gamma_{z_{b,c}}$, so $R = R(b,c)$ is a parallelogram (not a rectangle). Similarly, if $R(a,b)$ is 2-dimensional it is a parallelogram. Thus we may refer to the acute angled vertices of $R(b,c)$ or $R(a,b)$. We know that $R(a,b) \subset R(b,c)$. Suppose that a vertex of $R(a,b)$ is contained in the interior of $R(b,c)$. Consider the link of this vertex in Y. This link contains a non-trivial loop of length $2\theta(a, z_{a,b})$ (or $2(\pi - \theta(a, z_{a,b}))$) formed from two arcs of length $\theta(a, z_{a,b})$ (or $2(\pi - \theta(a, z_{a,b}))$) in $\Pi(a, z_{a,b}) \setminus (S(a) \cup B(a,b))$ and $\Pi(b, z_{a,b}) \setminus B(a,b)$. This contradicts the CAT(0) property since both $\theta(a, z_{a,b})$ and $\pi - \theta(a, z_{a,b})$ are strictly less than π. So we have the following lemma.

Lemma 4.4.6. *The vertices of $R(a,b)$ are contained in the boundry of $R(b,c)$.*

Lemma 4.4.7. *$S(a)$ is X-type if and only if $S(b)$ is X-type. Moreover if $S(a)$ is X-type then $\gamma_a \parallel \gamma_b$ and if $\gamma_a \parallel \gamma_b$ then width $S(a) =$ width $S(b)$.*

Proof. Observe that $S(a) \cap B(a,b) = R(a,b) \subset R(b,c) = S(c) \cap B(b,c) = \mathrm{Min}(a) \cap \mathrm{Min}(b) \cap \mathrm{Min}(c) \cap \mathrm{Min}(z_{b,c})$. So $S(a) \cap B(a,b) = \mathrm{Min}(a) \cap \mathrm{Min}(b) \cap \mathrm{Min}(c) \cap \mathrm{Min}(z_{a,b}) \cap \mathrm{Min}(z_{b,c}) = S(b) \cap \Pi(a,c)$, ie. $R(a,b) = S(a) \cap B(a,b) = S(b) \cap \Pi(a,c)$. Hence if $S(b)$ is X-type then $S(a)$ is X-type.

We know that $\gamma_b \nparallel \gamma_c$, $\gamma_b \nparallel \gamma_{z_{b,c}}$ and $\gamma_c \nparallel \gamma_{z_{b,c}}$ and $S(c)$ and $B(b,c)$ are B-type, so $S(b) \cap \Pi(a,c) = S(b) \cap B(b,c) \cap S(c)$ is a union of b-segments. It follows that if $R(a,b)$ is 1-dimensional then it is a b-segment.

If $S(a)$ is X-type then $S(a) \cap B(a,b)$ is a single non trivial a-segment, so $S(b) \cap \Pi(a,c)$ is also a single non trivial b-segment.

□

4.4.1 The structure of Y when $\gamma_a \nparallel \gamma_b$

Let us consider the structure of Y when $\gamma_a \nparallel \gamma_b$ and $\gamma_b \nparallel \gamma_c$. In this case we know, by [BC, Lemma 3.7], that $R = R(a,b) = R(b,c)$. Since $\gamma_a \nparallel \gamma_c$ we must have $\gamma_a \parallel \gamma_{z_{b,c}}$ and $\gamma_c \parallel \gamma_{z_{b,c}}$. As $R(a,b) = S(a) \cap B(a,b)$ and $R(b,c) = S(c) \cap B(b,c)$ we also know that $S(a)$ and $S(c)$ are bands and moreover, the widths of $S(a)$ and $B(b,c)$ are equal and the widths of $S(c)$ and $B(a,b)$ are equal.

Now consider $S(b) = \Pi(b, z_{a,b}) \cap \Pi(b, z_{b,c})$. Since $R = R(a,b) \subset \Pi(b, z_{a,b})$ and $R = R(b,c) \subset \Pi(b, z_{b,c})$, $S(b)$ must contain R. Suppose v is an acute angled vertex of R with angle θ and suppose v is contained in the interior of $S(b)$. Consider the link of v in Y as a subspace of Y. There are four arcs in this link of length θ forming a non trivial loop. One arc lies in $\Pi(a, z_{a,b}) \setminus (S(a) \cup B(a,b))$, one in $\Pi(a,c) \setminus (S(a) \cup S(c))$, one in $\Pi(c, z_{b,c}) \setminus (S(c) \cup B(b,c))$ and one in $S(b)$. Since $\theta < \frac{\pi}{2}$ this loop has length strictly less than 2π, contradicting the CAT(0) property. Hence the acute angled vertices of R are contained in the boundary of $S(b)$.

Suppose now that $|a| = |b|$. Since $|b| = |c|$, v is fixed by the group element cba (or $c^{-1}b^{-1}a^{-1}$). This element has infinite order (see [BC, 3.2] for a proof), so this contradicts proper discontinuity, hence $|a| \neq |b|$.

Since $\theta(b, z_{b,c}) = \theta(c, z_{b,c})$, a and b act transversely across $B(a,b)$ and b and c act transversly across $B(b,c)$, it follows that the a and $z_{b,c}$ axes are oppositely oriented, so are the c and $z_{a,b}$ axes. Also $\theta(a, z_{a,b})$ is acute. Thus $\theta(a, z_{a,b}) = \theta(c, z_{b,c}) = \theta(b, z_{b,c})$, $\theta(a,c) = \pi - \theta(a, z_{a,b})$ and $\theta(b, z_{a,b}) = \pi - 2\theta(a, z_{a,b})$.

So we are able to describe Y in terms of $|a|^{\perp}/|b|^{\perp} \neq 1$ and $\theta(a, z_{a,b})$.

4.4.2 The structure of Y when $\gamma_a \parallel \gamma_b$

We now consider the structure of Y when $\gamma_a \parallel \gamma_b$ and $\gamma_b \nparallel \gamma_c$. we begin with a lemma concerning the relationship between $R(a,b)$ and $R(b,c)$.

Lemma 4.4.8. *If $\gamma_a \parallel \gamma_b$ and $\gamma_b \nparallel \gamma_c$, then the acute angled vertices (or, the vertices, if $R(a,b)$ is 1-dimensional) of $R(a,b)$ coincide with the obtuse angled vertices of $R(b,c)$.*

Proof. Let v be an acute angled vertex of $R(a,b)$.

If $v \in \text{int}S(c) \cap \partial B(b,c)$ then there exists a loop of length $2\min\{\theta(a, z_{a,b}), \pi - \theta(a, z_{a,b})\} + 2\min\{\theta(b, z_{b,c}), \pi - \theta(b, z_{b,c})\} < 2\pi$ since γ_a is not perpendicular to $\gamma_{z_{a,b}}$.

If $v \in \text{int}B(b,c) \cap \partial S(c)$ then there exists a loop of length $2\min\{\theta(a, z_{a,b}), \pi - \theta(a, z_{a,b})\} + 2\min\{\theta(a,c), \pi - \theta(a,c)\} < 2\pi$ since γ_a is not perpendicular to $\gamma_{z_{a,b}}$. So v is a vertex of $R(b,c)$.

If v is an acute angled vertex of $R(b,c)$ then there is a loop of length $2\min\{\theta(a, z_{a,b}), \pi - \theta(a, z_{a,b})\} + 2\min\{\theta(c, z_{b,c}), \pi - \theta(c, z_{b,c})\} < 2\pi$ □

It follows that if $S(a)$ and $S(b)$ are B-type, then $\gamma_{z_{a,b}} \parallel \gamma_c$ and $\gamma_a \nparallel \gamma_{z_{b,c}}$ or $\gamma_{z_{a,b}} \parallel \gamma_{z_{b,c}}$ and $\gamma_a \nparallel \gamma_c$.

Lemma 4.4.9. *Suppose $\gamma_a \parallel \gamma_b$ and $\gamma_b \nparallel \gamma_c$. Then,*

1. *If $S(a)$ is X-type then $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}} \Leftrightarrow |b|^{\perp}_{z_{b,c}} > |c|^{\perp}_{z_{b,c}}$.*

2. *If $S(a)$ is B-type and $\gamma_{z_{a,b}} \parallel \gamma_{z_{b,c}}$ then $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}} \Leftrightarrow |b|^{\perp}_{z_{b,c}} > |c|^{\perp}_{z_{b,c}}$.*

3. *If $S(a)$ is B-type and $\gamma_c \parallel \gamma_{z_{a,b}}$ then $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$ and $|b|^{\perp}_{z_{b,c}} \leq |c|^{\perp}_{z_{b,c}}$.*

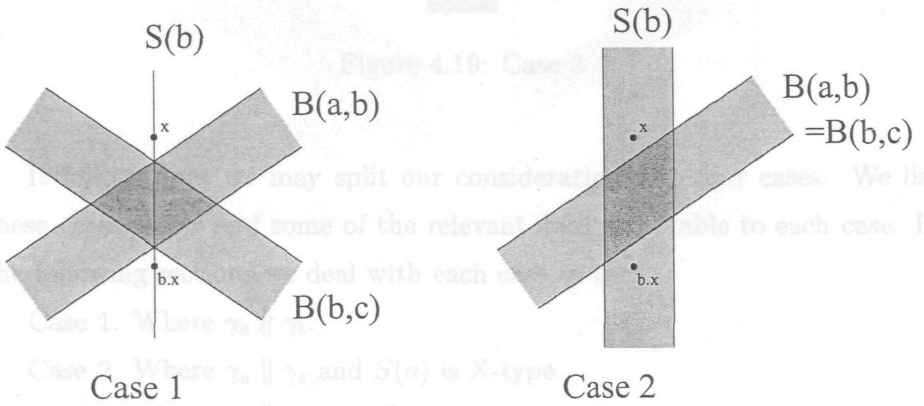

Figure 4.18: Showing $S(b)$, $B(a,b)$ and $B(b,c)$ with $\Pi(b, z_{a,b})$ overlayed by $\Pi(b, z_{b,c})$ for $|a|^{\perp} < |b|^{\perp}$.

Proof. In cases 1 and 2 if $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$ then b does not map one side of $B(a,b)$ to the other, and hence b does not map one side of $B(b,c)$ to the other. So $|b|^{\perp}_{z_{b,c}} > |c|^{\perp}_{z_{b,c}}$. See figure 4.18.

In case 3 it is clear that if $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$ then $|b|$ is too short to map one side of $B(b,c)$ to the other. This is a contradiction. A similar contradiction is reached if $|b|^{\perp}_{z_{b,c}} < |c|^{\perp}_{z_{b,c}}$. See figure 4.19. □

Note that since $\gamma_a \parallel \gamma_b$, $|a|_z^\perp \neq |b|_z^\perp$, so if $S(a)$ is B-type and $\gamma_{z_{a,b}} \parallel \gamma_{z_{b,c}}$ and $|b|^\perp = |c|^\perp$ then $|a|_z^\perp > |b|_z^\perp$.

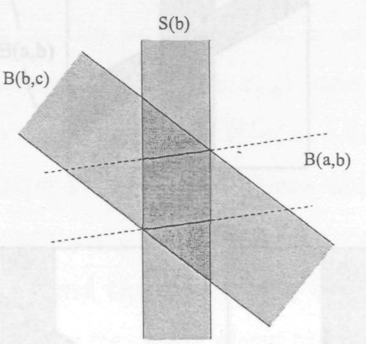

Figure 4.19: Case 3

It follows that we may split our consideration into four cases. We list these cases below and some of the relevant data applicable to each case. In the following sections we deal with each case in turn

Case 1. Where $\gamma_a \nparallel \gamma_b$.

Case 2. Where $\gamma_a \parallel \gamma_b$ and $S(a)$ is X-type.

Case 3. Where $\gamma_a \parallel \gamma_b$, $\gamma_{z_{a,b}} \parallel \gamma_c$ and $S(a)$ is B-type.

Case 4. Where $\gamma_a \parallel \gamma_b$, $\gamma_{z_{a,b}} \parallel \gamma_{z_{b,c}}$ and $S(a)$ is B-type.

In the following 4 sections we explore these 4 cases. We restrict ourselves to the situation where m is not divisible by 4 and n is prime.

4.5 Case 1

We list the follow facts concerning the structure of Y for this case which we have already proved. We assume that n is odd.

- $\gamma_a \nparallel \gamma_b$, $\gamma_b \nparallel \gamma_c$, $\gamma_a \parallel \gamma_{z_{b,c}}$, $\gamma_c \parallel \gamma_{z_{a,b}}$.

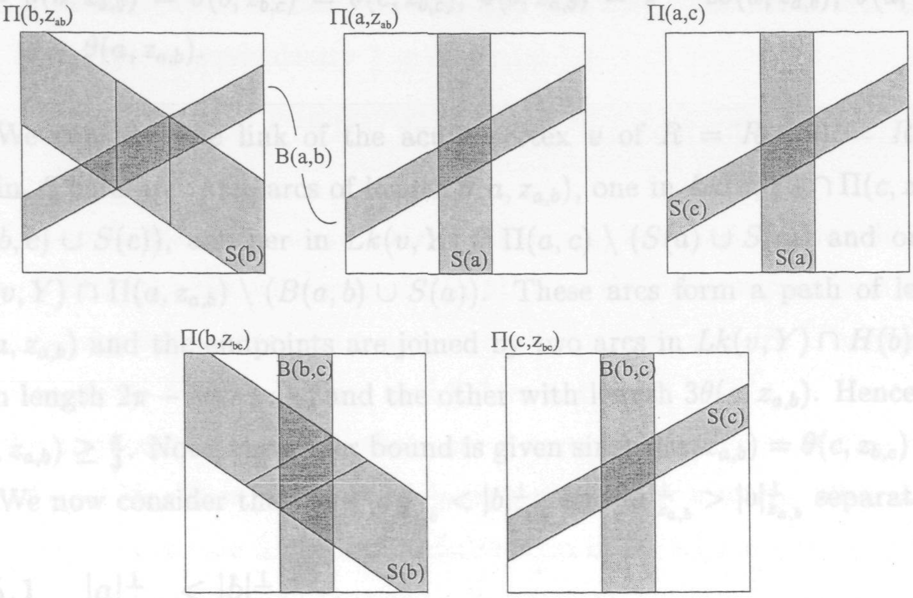

Figure 4.20: Showing the structure of Y for Case 1.

- $S(a)$, $S(b)$, $S(c)$, $B(a,b)$ and $B(b,c)$ are all bands.
- $\text{width}S(a) = \text{width}B(b,c) = |b|^{\perp}_{z_{b,c}} = |c|^{\perp}_{z_{b,c}}$.
- $\text{width}S(c) = \text{width}B(a,b) = \min\left\{|a|^{\perp}_{z_{a,b}}, |b|^{\perp}_{z_{a,b}}\right\}$.
- $R = R(a,b) = R(b,c)$ is a parallelogram, not a rectangle.
- $\text{width}S(b)$ is such that the acute angled vertices of $R = R(a,b) = R(b,c)$ are contained in $\partial S(b)$.
- $|a|^{\perp}_{z_{a,b}} \neq |b|^{\perp}_{z_{a,b}}$.
- The a and $z_{b,c}$ axes are oppositely oriented, as are the c and $z_{a,b}$ axes.

- $\theta(a, z_{a,b}) = \theta(b, z_{b,c}) = \theta(c, z_{b,c})$, $\theta(b, z_{a,b}) = \pi - 2\theta(a, z_{a,b})$, $\theta(a, c) = \pi - \theta(a, z_{a,b})$.

We consider the link of the acute vertex v of $R = R(a, b) = R(b, c)$ again. There are three arcs of length $\theta(a, z_{a,b})$, one in $Lk(v, Y) \cap \Pi(c, z_{b,c}) \setminus (B(b, c) \cup S(c))$, another in $Lk(v, Y) \cap \Pi(a, c) \setminus (S(a) \cup S(c))$ and one in $Lk(v, Y) \cap \Pi(a, z_{a,b}) \setminus (B(a, b) \cup S(a))$. These arcs form a path of length $3\theta(a, z_{a,b})$ and the endpoints are joined by two arcs in $Lk(v, Y) \cap H(b)$, one with length $2\pi - 3\theta(a, z_{a,b})$ and the other with length $3\theta(a, z_{a,b})$. Hence $\frac{\pi}{2} > \theta(a, z_{a,b}) \geq \frac{\pi}{3}$. Note, the upper bound is given since $\theta(a, z_{a,b}) = \theta(c, z_{b,c}) < \frac{\pi}{2}$.

We now consider the cases $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$ and $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$ separately.

4.5.1 $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$

Suppose $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$. Then a maps one side of $S(c)$ to the other hence there are three cases to consider. The plane $\Pi(c, z_{b,c})$ is split into a strip, $S(c)$ and two half planes H_1 and H_2 on either side of $S(c)$.

1. $aH_1 = H_2$ (or $a^{-1}H_1 = H_2$)

2. for all $x \in H_1$, $ax \notin H_2$ (or for all $x \in H_1, a^{-1}x \notin H_2$)

3. there exists $x, y \in H_1$ such that $ax \notin H_2$ and $ay \in H_2$ (or $a^{-1}x \in H_2$ and $a^{-1}y \in H_2$)

We use the following lemma.

Lemma 4.5.1. *If* $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$, *then* $\theta(a, z_{a,b}) = \frac{\pi}{3}$.

Proof. It suffices to show $\frac{\pi}{3} \geq \theta(a, z_{a,b})$. Let z be a point in $H_1 \cap \partial B(b, c) \cap S(b)$ in such that $a.z \in B(b, c)$. Note that $z \in \Pi(b, z_{a,b})$ so we may choose z sufficiently close to $B(a, b)$ such that $a.z \in \Pi(b, z_{a,b})$. But $\Pi(b, z_{a,b}) \cap B(b, c) \subset$

$S(b)$ so $a.z \in S(b)$. It follows that $\theta(a, z_{a,b}) \leq \theta(b, z_{a,b})$ and hence $\theta(a, z_{a,b}) \leq \pi - 2\theta(a, z_{a,b})$, or equivalently $\frac{\pi}{3} \geq \theta(a, z_{a,b})$. □

In case 1, a vertex v of R is fixed by $ba^{-1}bac$, contradicting proper discontinuity.

The group element a maps a strip of $\Pi(b, z_{a,b})$ bounded on one side by an edge of $B(a, b)$ to the other side of $B(a, b)$. This strip intersects $S(b) \cap B(b, c)$ non trivially, hence there are points in both this strip and H_1. These points are mapped to H_2 by a. Hence case 2 does not occur.

We may assume therefore that case 3 occurs. Let y be as in case 3 and $z \in \partial S(c) \cap H_1$. Then the geodesic $[y, z]$ is mapped by a to H_2. Thus the set of points x as in case 3, form an infinite strip in H_1 adjacent to $S(c)$ which is mapped by a to an infinite strip in H_2 adjacent to $S(c)$. Label the strip in H_1 by $\mathcal{S}_c$.

It follows that $|a|^{\perp}_{z_{a,b}} = |a| \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}|a|$ and $|b|^{\perp}_{z_{a,b}} = |b|^{\perp}_{z_{b,c}} = |c|^{\perp}_{z_{b,c}} = \frac{\sqrt{3}}{2}|b|$ and $|a| < |b| = |c|$.

Lemma 4.5.2. *If $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$ then c maps an infinite strip in $\Pi(a, z_{a,b})$ adjacent to $S(a)$ to the other side of $S(a)$ in $\Pi(a, z_{a,b})$ only if $n = 3k$ for some $k \in \mathbb{N}$.*

Proof. If c maps an infinite strip in $\Pi(a, z_{a,b})$ adjacent to $S(a)$ to the other side of $S(a)$ in $\Pi(a, z_{a,b})$, then it follows that c maps a strip in $\Pi(b, z_{b,c})$ adjacent to $B(b, c)$ to the other in $\Pi(b, z_{b,c})$. Hence by lemma 4.1.6, $n = 3k$ for some $k \in \mathbb{N}$. □

We now begin to build a quotient complex for the action of $A(m, n, 2)$, m even and n odd, on $\mathcal{O}_A$. We begin by considering fundamental domains for the actions of $A(m)$, $A(n)$ and $A(2)$ on the subcomplexes $\mathcal{O}_m$, $\mathcal{O}_n$ and

$\Pi(a, c)$ and examine how these fundamental domains intersect in $\mathcal{O}_A$. We then use these to form the quotient complex for $A(m, n, 2)$.

Let F_2 be a fundamental domain for $A(2)$ in $\Pi(a, c)$. We may choose the set F_2 to be a parallelogram bounded by 2 c-segments and by 2 a-segments. Thus we may assume $F_2 = R$.

Since n is prime, we know that F_n, a fundamental domain for $A(n)$ is of type 1. This is a trapezium contained in $B(b, c)$ with parallel edges 2 $z_{b,c}$-segments of length $\frac{n+1}{2}|b|$ and $\frac{n-1}{2}|b|$. The diagonal edges are a b-segment and a c-segment each on length $|b| = |c|$. We may choose F_n to be centred on R, so that $F_2 = R \subset F_n$.

As m is not a multiple of 4, we know that F_m is of type 1. We may assume $F_m \subset \Pi(b, z_{a,b})$. The fundamental domain F_m is as described in section 4.1.2. It contains a parallelogram in $B(a, b)$ with width equal to $B(a, b)$ and length $\frac{m}{2}\frac{|a|+|b|}{2}$. Since $m \geq 6$, $\frac{m}{2}\frac{|a|+|b|}{2} \geq 3\frac{|a|+|b|}{2} > |b| = |c|$ so we may assume that $F_2 = R \subset F_m$.

Figure 4.21 shows the parts of F_n, F_m and F_2 which intersect in $\mathcal{O}_A$ and gives identifications on them given by the action of $A(m, n, 2)$. The similarly cross hatched regions are identified in $\mathcal{O}_A/A(m, n, 2)$.

From this we produce the quotient complex shown in figure 4.22.

We observe that the complex has 4 vertices. We check that each link is CAT(1) (see appendix A.1) to show that the complex is locally CAT(0). By homotopically retracting the edges labeled t_2, x_2 and s_1 we obtain a complex with just one vertex (see figure 4.23). From this we read a presentation for the fundamental group of this complex:

$$\begin{aligned}\pi_1(X) \cong\ & \langle a, c, \delta_1, \delta_2, \epsilon, x, s, t, y, y' | t\delta_2 a^{-1}, yxc, aca^{-1}c^{-1}\delta_2 y^{-1},\\ & x\delta_1 y^{-1}, y\epsilon^{-1}, \delta_1 s a^{-1}, s\delta_1 \epsilon t^{-1}\delta_2^{-1}\epsilon^{-1}, y'x^{-1}\\ & y'xaxaty'^{-1}a^{-1}x^{-1}a^{-1}x^{-1}s^{-1}\rangle\end{aligned}$$

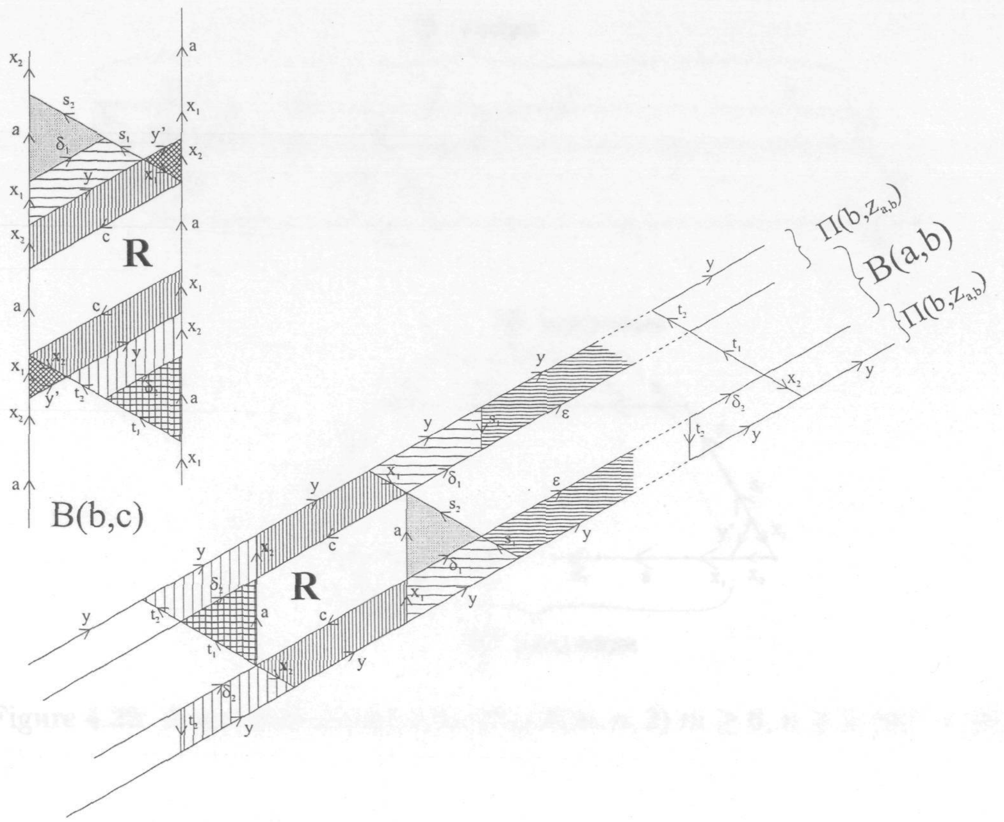

Figure 4.21: Showing parts of $B(b,c)$ and $\Pi(b, z_{a,b})$ for case 1, $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$

By adding a generator $b = a^{-1}y$ and applying Tietze transforms we are able to reduce this to the standard presentation for $A(m,n,2)$.

Note that for this case where $|a|^{\perp} < |b|^{\perp}$, m is not a multiple of 4, and $n \geq 5$ prime, these complexes are unique up to scale.

Observe that all of these complexes may be triangulated by equilateral triangles so each gives a biautomatic structure on $A(m,n,2)$.

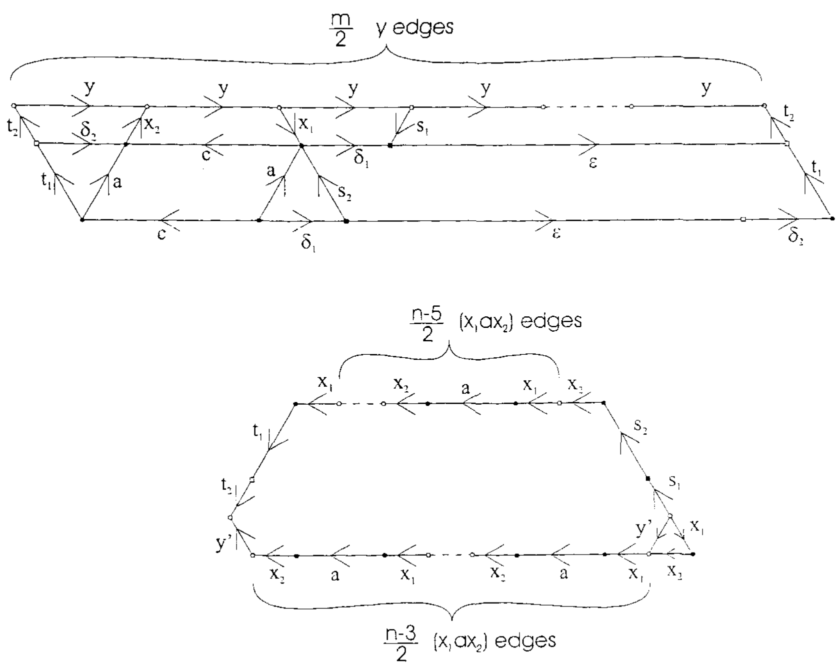

Figure 4.22: A quotient complex for $\mathcal{O}_A/A(m,n,2)$ $m \geq 6$, $n \geq 5$, $|a|^{\perp} < |b|^{\perp}$

4.5.2 $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$

Now suppose that $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$. Again we let $m \geq 6$ be even and m not divisible by 4, and $n \geq 5$ a prime.

The set $R = R(a,b) = R(b,c)$ is a parallelogram with 2 c-segment edges (length $|c|_X$) and 2 $z_{b,c}$-segment edges such that b maps an obtuse vertex of R to the other.

As before F_2 is a parallelogram centred on R bounded by 2 a-segments and 2 c-segments of lengths $|a|$ and $|c|$ respectively. Since $|a|^{\perp} > |b|^{\perp}$, F_2 properly contains R. The domain F_n is a trapeszium as described in section 4.1.1, and F_m is the domain described in section 4.1.2 for $|a| > |b|$. Again, both F_m and F_n are of type 1. We observe that $F_2 \cap F_m = F_2$ and $F_2 \cap F_n = R$. The intersection $S(b) \cap B(a,b)$ is a parallelogram consisting of R and two

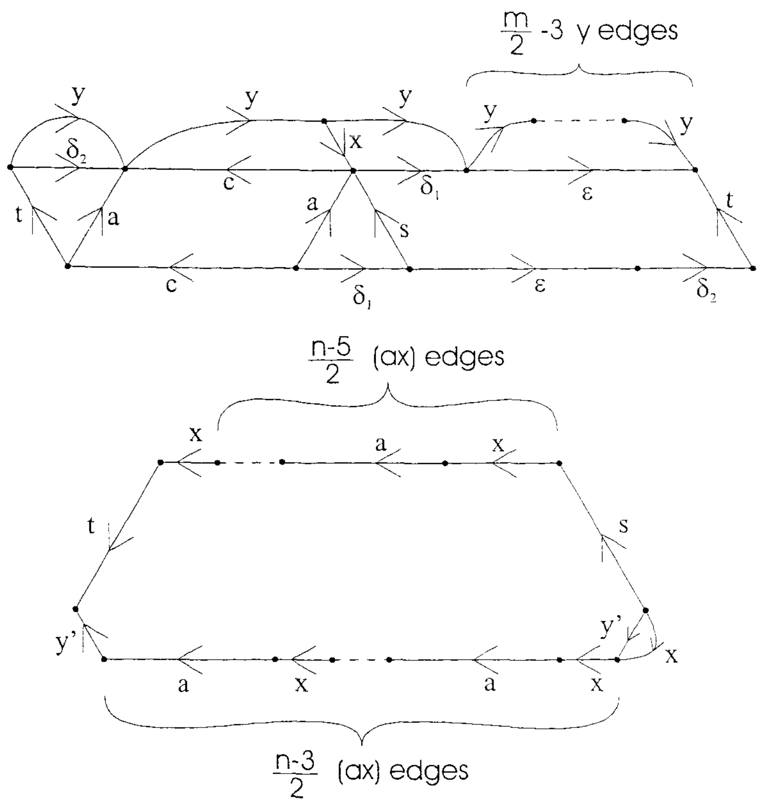

Figure 4.23: A presentation complex for $\mathcal{O}_A/A(m,n,2)$, case 1, $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$

triangles T_1 and T_2 such that $S(b) \cap B(b,c) = R \cup b.T_1 \cup b^{-1}.T_2$. Hence we may choose F_n such that $F_n \cap F_m = R \cup T_1 \cup T_2$. By identifying the edges of these domains we form a quotient complex for $\mathcal{O}_A/A(m,n,2)$ for $n \geq 5$, see figure 4.24.

Again we check that it has the correct fundamental group, see appendix B.1, and the links of the 2 vertices are CAT(1), appendix A.2.

We briefly consider the case when $n = 3$.

Lemma 4.5.3. *If $n = 3$ then $\theta(a, z_{a,b}) = \frac{\pi}{3}$.*

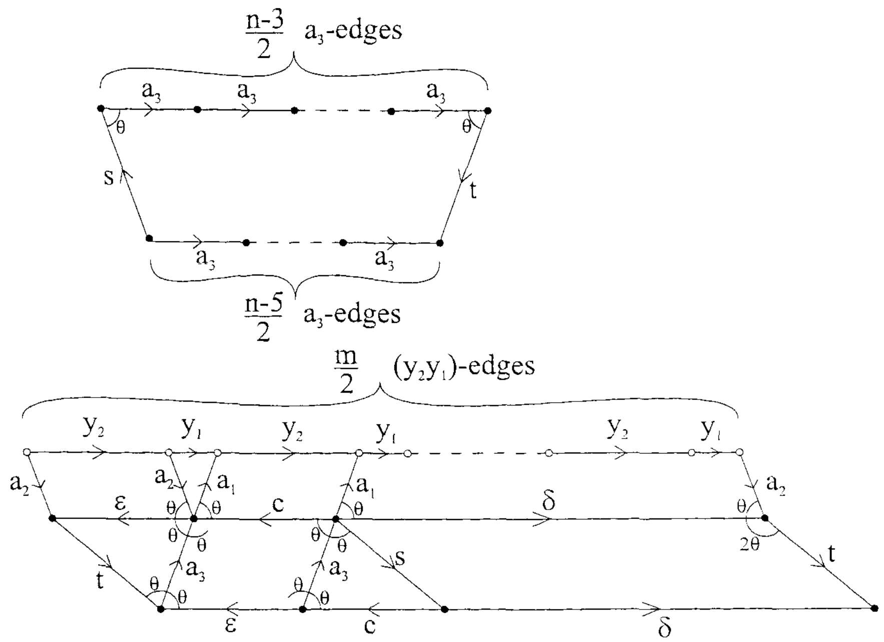

Figure 4.24: Showing a quotient complex for $\mathcal{O}_A/A(m,n,2)$ in case 1 where $m \geq 4$ even, $n \geq 5$ odd, and $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$

Proof. Let α_2 be an a-segment in $\partial S(a)$ such that $|\alpha_2|^{\perp}_{z_{a,b}} = \frac{1}{2}(|a|^{\perp}_{z_{a,b}} - |b|^{\perp}_{z_{a,b}})$, $\alpha_2 \cap B(a,b)$ is the terminal vertex of α_2 and $c.\alpha_2 \subset \partial S(a)$. By construction, α_2 is oriented in the same direction as the a-axes and $\theta(\alpha_2, z_{a,b}) = \theta(a, z_{a,b}) = \theta$. The segment $c.\alpha_2$ is also an a-segment so $\theta(c.\alpha_2, z_{a,b}) = \theta$. The group element b^{-1} reflects $c\alpha_2$ in the $z_{a,b}$ axis and translates to the other side of $B(a,b)$ in $\Pi(a, z_{a,b})$. Thus $\theta(b^{-1}c\alpha_2, z_{a,b}) = \theta$ and $\theta((b^{-1}c\alpha_2)^{-1}, z_{a,b}) = \pi - \theta$.

Since $n = 3$ c^{-1} maps $(b^{-1}c\alpha_2)$ to the other side of $S(a)$ via a reflection in the a-axis. Thus $\theta((c^{-1}b^{-1}c\alpha_2)^{-1}, z_{a,b}) = 3\theta - \pi$ and $c^{-1}b^{-1}c\alpha_2 \subset B(a,b)$.

Now $\theta((bc^{-1}b^{-1}\alpha_2)^{-1}, z_{a,b}) = 3\theta - \pi$ and $bc^{-1}b^{-1}\alpha_2 \subset B(a,b)$ if and only if $3\theta - \pi \leq 0$. But $n = 3$, so $bc^{-1}b^{-1} = c^{-1}b^{-1}c$, so $3\theta - \pi = 0$ and $\theta = \frac{\pi}{3}$. □

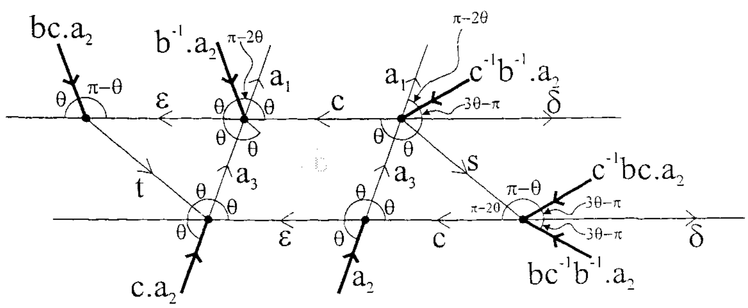

Figure 4.25: Showing the translations of α_2

Note that in case 1 for $|a|^{\perp} > |b|^{\perp}$ with n prime and when m is not a multiple of 4, the family of quotient complexes (up to isometry) is a 2 parameter family, parameterised by $\frac{\pi}{3} \leq \theta(a, z_{a,b}) < \frac{\pi}{2}$ and $|a|/|b|$.

4.6 Case 2

Recall the following data concerning the structure of Y for case 2 when n is odd.

- $\gamma_a \parallel \gamma_b$, $\gamma_b \nparallel \gamma_c$.
- width$S(a)$ =width$S(b) = 0$.
- width$B(a, b) = |b|^{\perp}_{z_{a,b}}$.
- width$B(b, c) = |b|^{\perp}_{z_{b,c}} = |c|^{\perp}_{z_{b,c}}$.
- $R(a, b)$ is a single line.
- width$S(c)$ is such that the vertices of $R(a, b)$ are contained in $\partial S(c)$.
- $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$, $|b| = |c|$.

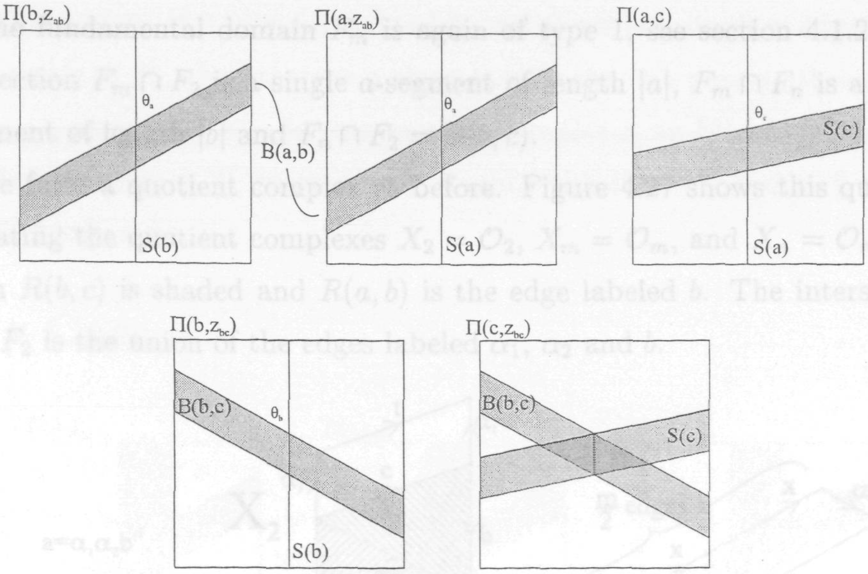

Figure 4.26: Showing the structure of Y for Case 2.

- $\theta(b, z_{b,c}) = \angle(c, z_{b,c})$.

Thus Y may be defined by the data $\theta(a, z_{a,b})$, $\theta(b, z_{b,c})$ and $|a|/|b|$.

Observing that $\theta(a, c) + 2\theta(b, z_{b,c}) = \pi$, an examination of the link of an end vertex of $R(b, c)$ gives the inequalities, $\theta(a, z_{a,b}) \geq \theta(b, z_{b,c})$ and $\theta(a, z_{a,b}) \geq \pi - \theta(b, z_{b,c})$. Thus $\frac{\pi}{2} > \theta(a, z_{a,b}) \geq \frac{\pi}{3}$ and $\theta(a, z_{a,b}) \geq \theta(b, z_{b,c}) \geq \frac{1}{2}(\pi - \theta(a, z_{a,b}))$.

Letting F_2, F_m and F_n denote fundamental domains for the actions of $A(2)$, $A(m)$ and $A(n)$ on $\mathcal{O}_2$, $\mathcal{O}_m$ and $\mathcal{O}_n$ respectively, then F_2 may be chosen to be a parallelogram bounded by 2 a and c segments of length $|a|$ and $|c|$ respectively. We may choose F_2 to contain $R(b, c)$ centrally. The set F_n is a trapezium in $B(b, c)$ with edges a b segment of length $|b|$, a c segment of length $|c| = |b|$ and parallel edges $z_{b,c}$-segments of length $\frac{n-1}{2}|b|\sin(\frac{\angle(b,z_{b,c})}{2})$, and $\frac{n+1}{2}|b|\sin(\frac{\angle(b,z_{b,c})}{2})$. We may choose F_n so that it contains $R(b, c)$.

The fundamental domain F_m is again of type 1, see section 4.1.2. The intersection $F_m \cap F_2$ is a single a-segment of length $|a|$, $F_m \cap F_n$ is a single b-segment of length $|b|$ and $F_n \cap F_2 = R(b, c)$.

We form a quotient complex as before. Figure 4.27 shows this quotient separating the quotient complexes $X_2 = \mathcal{O}_2$, $X_m = \mathcal{O}_m$, and $X_n = \mathcal{O}_n$. The region $R(b, c)$ is shaded and $R(a, b)$ is the edge labeled b. The intersection $F_m \cap F_2$ is the union of the edges labeled α_1, α_2 and b.

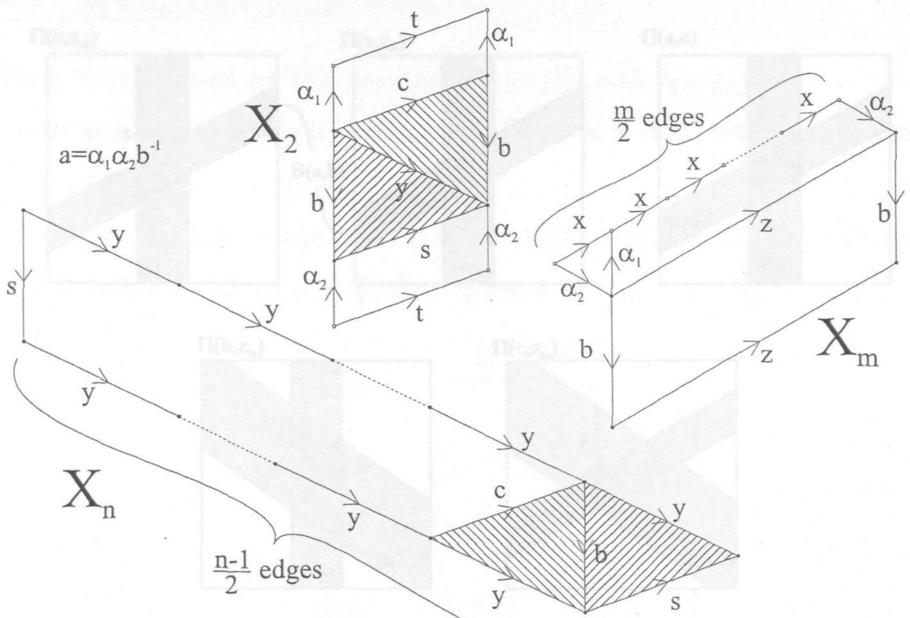

Figure 4.27: A quotient complex for $\mathcal{O}_A/A(m, n, 2)$. Shaded region is $R(b, c)$

We check the link and fundamental group as before see appendix A.3 and B.2. We observe that $\theta(a, z_{a,b}) = \theta(b, z_{a,b}) = \frac{\pi}{3}$ so in fact $\mathcal{O}_A/A(m, n, 2)$ is defined by $|a|_X/|b|_X$.

Observe that the complex is also CAT(0) if we use F_m of type k for any $k \in \mathbb{N}$. Also the complex is CAT(0) if F_n has at least 2 edges labeled y along its shorter edge. If n is a multiple of 3, and there is a single y edge on the shorter side of F_n, then there is a short loop in the link passing through the points $s^+, c^-, \alpha_1^-, \alpha_2^+, s^+$. So this quotient complex is an Eilenberg MacLane complex for $A(m, n, 2)$, m even, $n \geq 5$ odd. Also if $n = 3$ and $\gamma_a \parallel \gamma_b$ then $S(a)$ is not X-type.

4.7 Case 3

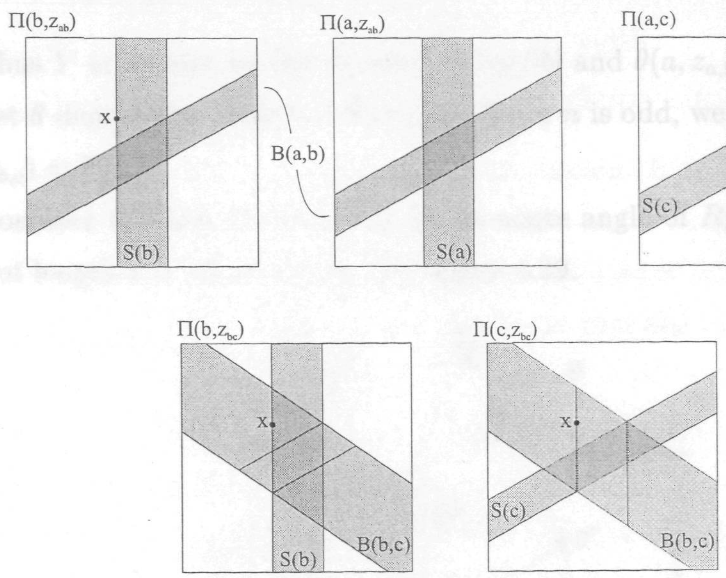

Figure 4.28: Showing the structure of Y for Case 3.

Recall the following data concerning case 3 when n is odd.

- $\gamma_a \parallel \gamma_b$, $\gamma_b \nparallel \gamma_c$, $\gamma_c \parallel \gamma_{z_{a,b}}$.
- $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$, $|b| = |c|$.
- width $B(a, b) = |a|^{\perp}_{z_{a,b}}$.

- $\text{width}S(c) = \text{width}B(a,b) = |b|^{\perp}_{z_{b,c}}$.
- $\text{width}S(a) = \text{width}S(b)$ are such that the acute angled vertices of $R(a,b)$ coincide with the obtuse angled vertices of $R(b,c)$.
- $\theta(a, z_{a,b}) > \frac{\pi}{2}$.
- $\frac{\pi}{2}\theta(a, z_{a,b}) = \theta(b, z_{b,c}) = \theta(c, z_{b,c})$.
- $\pi - \theta(a, z_{a,b}) = \theta(b, z_{a,b}) = \theta(a, c)$.

Thus Y is defined by the parameters $|a|/|b|$ and $\theta(a, z_{a,b})$.

Let θ denote the angle $\pi - \theta(a, z_{a,b})$. Since n is odd, we know $\theta(b, z_{b,c}) = \theta(c, z_{b,c}) = \frac{\pi-\theta}{2}$.

Consider the link of the vertex at an acute angle of $R(a,b)$. There is a loop of length $\pi + 3\theta$, so $\theta \geq \frac{\pi}{3}$. See figure 4.29.

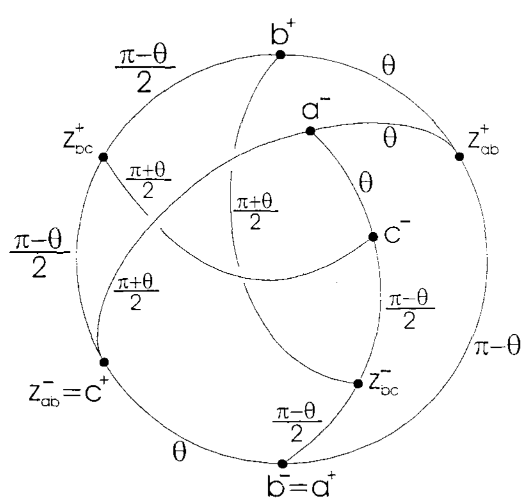

Figure 4.29: The link of a vertex of $R(a,b)$ in Y

Since $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$, a maps a strip in $\Pi(b, z_{a,b})$ adjacent to $B(a,b)$ to the other side of $B(a,b)$ in $\Pi(b, z_{a,b})$. Label this strip $\mathcal{S}_a$. Since $\mathcal{S}_a$ is a union of $z_{a,b}$-axes, $\mathcal{S}_a \cap S(b) \neq \emptyset$, and $a\mathcal{S}_a \cap S(b) \neq \emptyset$. Also $\mathcal{S}_a \cap B(b,c) \neq \emptyset$ and $a\mathcal{S}_a \cap B(b,c) \neq \emptyset$. It follows that a maps a strip, $\mathcal{S}_c$ in $\Pi(c, z_{b,c})$ adjacent to $B(b,c)$ to a similar strip in $\Pi(c, z_{bc})$ adjacent to $B(b,c)$.

If $\theta > \frac{\pi}{3}$ then we may choose $x \in B(b,c) \cap \partial S(b)$ with $a.x \notin B(b,c)$, such that x is close to $R(a,b)$ in $\Pi(c, z_{b,c})$ so $a.x \in \Pi(c, z_{b,c})$ since $a.\mathcal{S}_c \subset \Pi(c, z_{b,c})$. Also $x \in \Pi(b, z_{a,b})$, so we may choose x sufficiently close to $R(a,b)$ so that $a.x \in \Pi(b, z_{a,b})$. Hence $a.x \in B(b,c)$, a contradiction. Thus $\theta \leq \frac{\pi}{3}$, and hence $\theta = \frac{\pi}{3}$.

We now examine $\mathcal{S}_c$ and find bounds on its width.

Lemma 4.7.1. *The minimal width of $\mathcal{S}(c)$ is $\frac{\sqrt{3}}{4}(l(b) - l(a))$.*

Proof. The region $B(b,c) \cap S(b)$ is a parallelogram. It is divided into three regions by $S(b) \cap S(c)$, two equilateral triangles and a parallelogram $(S(b) \cap S(c))$. We divide each triangle into a smaller equilateral triangle (T_1 and T_2) and a trapezium ($\mathcal{T}_1$ and $\mathcal{T}_2$) of equal height, so that the trapezium has base on $S(c)$.

Since $B(b,c) \cap S(b) \subset \Pi(b, z_{a,b})$, a maps $\mathcal{T}_1$ onto $\mathcal{T}_2$ by a reflection in the $z_{a,b}$-axis and a translation.

The region $c^{-1}.\Pi(c, z_{b,c}) \cap \Pi(c, z_{b,c})$ is a union of an edge of $S(c)$ and $\mathcal{T}_2$. Hence this region is not convex and hence the complex is not CAT(0), a contradiction.

The height of T_1 is $\frac{\sqrt{3}}{2}(l(b) - l(a))$ and the height if $\mathcal{T}_1$ is $\frac{\sqrt{3}}{4}(l(b) - l(a))$. Hence the minimal width if $\mathcal{S}(c)$ is $\frac{\sqrt{3}}{4}(l(b) - l(a))$.

□

Lemma 4.7.2. *If $m = 4$ then $\mathcal{S}(c)$ has width $\geq \frac{\sqrt{3}}{2}(l(b) - l(a))$.*

Proof. A fundamental domain for X_m is given by $S(b) \cap S(c)$, T_1 and $\mathcal{T}_1$. It is easy to check that $baba.(a^{-1}b^{-1}.T_1) = b^{-1}.T_2$ and hence $a.T_1 = T_2$. The result follows from a similar argument to the previous lemma.

□

Lemma 4.7.3. *If the width of $\mathcal{S}(c)$ is strictly greater than $\frac{\sqrt{3}}{4}(l(b) - l(a))$ then $m = 4k$ for some $k \in \mathbb{Z}$.*

Proof. Suppose the width of $\mathcal{S}(c)$ is strictly greater than $\frac{\sqrt{3}}{4}(l(b)-l(a))$. Then by the same argument as the previous lemmas $a^{-1}b^{-1}.T_1 \cap \mathcal{S}(b)$ is translated along $\mathcal{S}(b)$ by the group element *baba*, ie abab fixes a point in T_m. It follows that m is therefore a multiple of 4. □

It follows from these lemmas that if m is not a multiple of 4, then the width of $\mathcal{S}_c$ is fixed at $\frac{\sqrt{3}}{4}(l(b) - l(a))$.

We assume m is not divisible by 4 and $n \geq 5$ is prime. The fundamental domains F_m and F_n are both of type 1. As before, F_2 is a parallelogram bounded by a-segments of length $|a|$ and c-segments of length $|c|$. We may choose each to be centred on $R(a, b)$. Observe that $F_n \cap F_2 = F_2 = R(b, c)$, $F_m \cap F_2 = R(a, b) \subset R(b, c)$, and $F_m \cap F_n = S(b) \cap B(b, c)$.

The common regions of F_m, F_n and F_2 and identifications are shown in figure 4.30, and a quotient complex is displayed in figure 4.31. The links are shown to be CAT(1) in appendix A.4 and the fundamental group is shown to be $A(m, n, 2)$ in appendix B.3.

Thus for case 3 with $n \geq 5$ prime and m even, not divisible by 4, the family of quotient complexes for a given $A(m, n, 2)$ is a one parameter family, up to scale. The parameter is $|a|/|b|$.

4.8 Case 4

We recall the follow facts concerning the structure of Y for this case.

- $\gamma_a \parallel \gamma_b$, $\gamma_b \nparallel \gamma_c$, $\gamma_{z_{a,b}} \parallel \gamma_{z_{b,c}}$.

- $S(a)$, $S(b)$, $S(c)$, $B(a,b)$ and $B(b,c)$ all bands.
- width$B(a,b)$ = width$B(b,c) = |b|^{\perp}_{z_{b,c}}$,
- width$S(a)$ = width$S(b)$,
- width$S(c)$ is minimal containing $R = R(a,b)$.
- $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$, $|b|^{\perp}_{z_{b,c}} = |c|^{\perp}_{z_{b,c}}$,
- $\theta(a, z_{a,b}) = \theta(b, z_{b,c}) = \theta(c, z_{b,c})$, $\theta(a, c) = \pi - 2\theta(a, z_{a,b})$ and $\theta(b, z_{a,b}) = \pi - \theta(a, z_{a,b})$.

Hence Y is defined by the parameters $\theta(a, z_{a,b})$, $|a|/|b|$ and the width of $S(a)$. We will show that in fact $\theta(a, z_{a,b}) = \frac{\pi}{3}$

Consider. There is a non trival loop in the link of an acute vertex of $R(a,b)$ of length $6\theta(a, z_{a,b})$, hence $\theta(a, z_{a,b}) \geq \frac{\pi}{3}$.

Lemma 4.8.1. *If $A(2m, n, 2)$ acts properly discontinuously by semi-simple isometries on a CAT(0) 2-complex satisfying the conditions of case 4, then $n = 3k$ for some $k \in \mathbb{N}$.*

Proof. Consider the set $R(b,c) = S(c) \cap B(b,c)$. This is the union of $R = R(a,b)$ and 2 triangles T_1 and T_2 such that $c^{-1}.T_1$ and $c.T_2$ are contained in $S(a) \cap S(c)$. Since $|a|^{\perp}_{z_{a,b}} > |b|^{\perp}_{z_{a,b}}$, b maps a strip $\mathbb{S}$ in $\Pi(a, z_{a,b})$ adjacent to $B(a,b)$ to the other side of $B(a,b)$ in $\Pi(a, z_{a,b})$. Now $c.T_2$ shares an edge with $R(a,b)$, so $\mathbb{S} \cap c.T_2$ is a non empty intersection. Hence $S(a) \cap (\mathbb{S} \cap c.T_2)$ is non empty and is contained in $S(c)$ sharing an edge with $B(b,c)$. Also $b.(S(a) \cap (\mathbb{S} \cap c.T_2))$ is contained in $S(c)$ sharing an edge with $B(b,c)$. Hence b maps a strip adjacent to $B(b,c)$ in $\Pi(c, z_{b,c})$ to a similar strip on the other side of $B(b,c)$. Thus by lemma 4.1.6 $n = 3k$ for some $k \in \mathbb{N}$. ☐

Lemma 4.8.2. *If $A(2m, n, 2)$ acts properly discontinuously by semi-simple isometries on a CAT(0) 2-complex satisfying the conditions of case 4, then $\theta(a, z_{a,b}) = \frac{\pi}{3}$.*

Proof. We may choose $x \in (\partial S(a)) \cap S(c) \setminus R(a, b)$ close to $R(a, b)$ such that $b.x \in S(a) \subset \Pi(a, c)$. If $\theta(a, z_{a,b}) > \frac{\pi}{3}$ then $b.x \notin S(c)$. But $x \in S(a) \cap S(c) \subset \Pi(c, z_{b,c})$, so $b.x \in \Pi(c, z_{b,c})$ and hence $b.x \in \Pi(a, c) \cap \Pi(c, z_{b,c}) = S(c)$, a contradiction. □

4.9 Results

We finally state some theorems we have proved in this chapter.

Theorem 4.9.1. *For the three generator Artin group $A(m, n, 2)$, $m = 4j + 2$ for some $j \geq 1$ and $n \geq 5$ a prime, the complexes $\mathcal{O}_A$ may be classified by the following two parameters:*

- *$\theta(a, z_{a,b})$, the angle between positively oriented a and $z_{a,b}$-axes, and*
- *$|a|^{\perp}/|b|^{\perp}$, the ratio of the translation lengths perpendicular to the $z_{a,b}$-axis.*

Theorem 4.9.2. *For the three generator Artin group $A(m, 3, 2)$, with $m = 4j + 2$ for some $j \geq 1$, the angle $\theta(a, z_{a,b})$ is either $\frac{\pi}{3}$ or $\frac{2\pi}{3}$,*

More over $\theta(a, z_{a,b}) = \frac{2\pi}{3}$ if and only if $\gamma_a \parallel \gamma_b$ and $|a|_X < |b|_X$.

Note that in each case $\theta(a, z_{a,b})$ determines the angles between each of the translation axes for a, b, c, $z_{a,b}$, and $z_{b,c}$. We therefore may deduce the following corollary.

Corollary 4.9.3. *For the three generator Artin group $A(m, 3, 2)$, with $m = 4j + 2$ for some $j \geq 1$, if $|a|_X/|b|_X \in \mathbb{Q}$ then $\mathcal{O}_A$ may be triangulated by a single isometry type of equilateral triangle.*

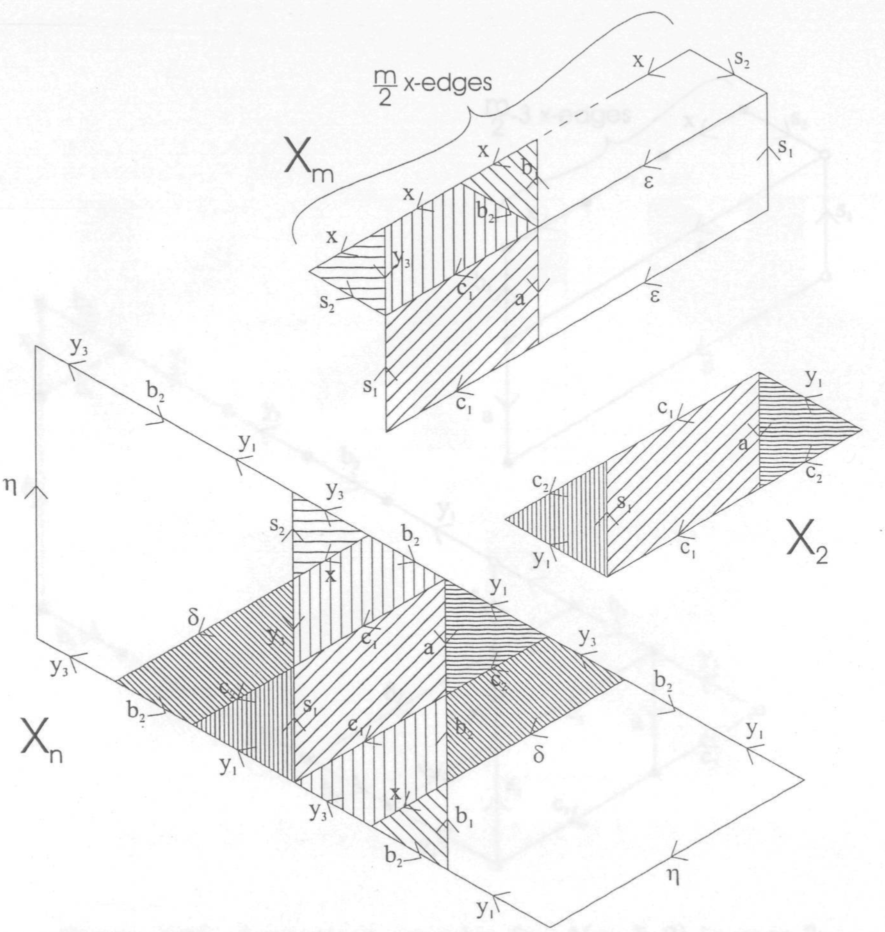

Figure 4.30: Showing identified regions for a complex for $A(m, 5, 2)$ in case 3

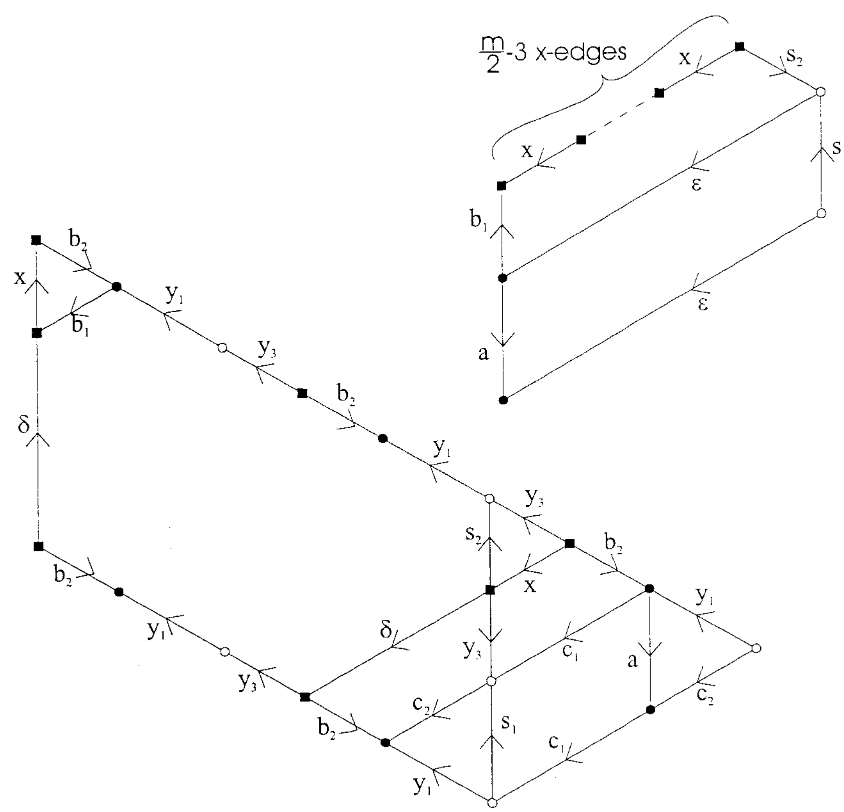

Figure 4.31: A quotient complex for $A(m, 5, 2)$ in case 3

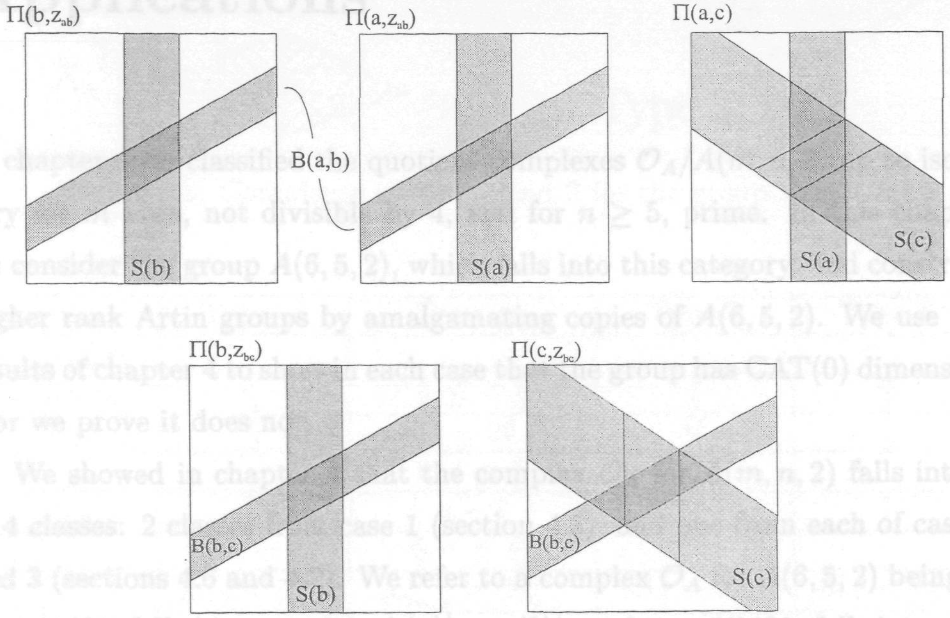

Figure 4.32: Showing the structure of Y for Case 4

Chapter 5

Applications

In chapter 4 we classified the quotient complexes $\mathcal{O}_A/A(m,n,2)$ up to isometry for m even, not divisible by 4, and for $n \geq 5$, prime. In this chapter we consider the group $A(6,5,2)$, which falls into this category, and construct higher rank Artin groups by amalgamating copies of $A(6,5,2)$. We use the results of chapter 4 to show in each case that the group has CAT(0) dimension 2 or we prove it does not.

We showed in chapter 4 that the complex $\mathcal{O}_A$ for $A(m,n,2)$ falls into 1 of 4 classes: 2 classes from case 1 (section 4.5), and one from each of case 2 and 3 (sections 4.6 and 4.7). We refer to a complex $\mathcal{O}_A$ for $A(6,5,2)$ being of type 1i if it falls into case 1 and $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$, of type 1ii if it falls into case 1 and $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$ and of type 2 or 3 if it falls into case 2 or 3 respectively. These types are shown in figure 5.1.

We recall the following information about each type of complex for $m \geq 6$, $4 \nmid m$ and $n \geq 5$ prime. Let θ denote the angle $\theta(a, z_{a,b})$ and $\omega_i = \cos(i\theta)$.

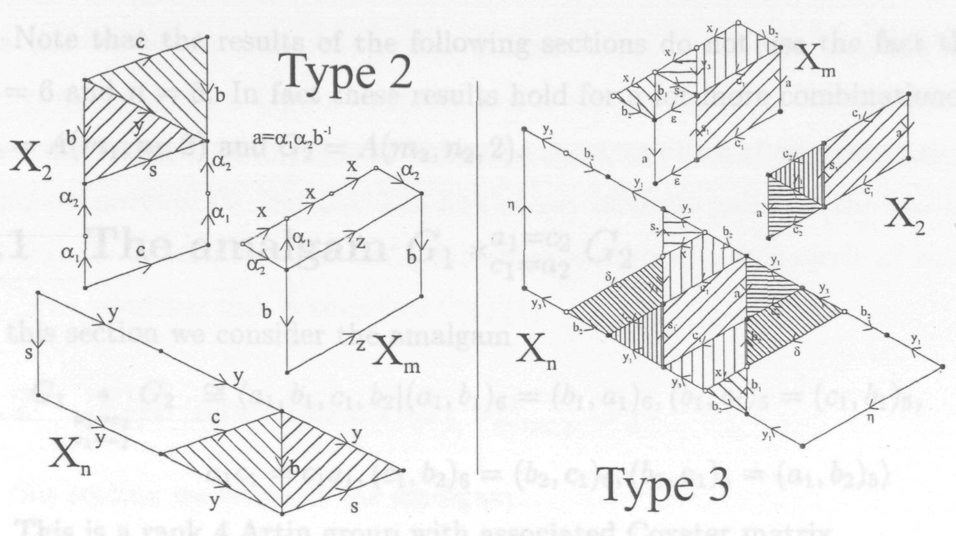

Figure 5.1: Complexes of type 2 and 3 for the group $A(6,5,2)$

	Type 1i	Type 1ii	Type 2	Type 3
Angles				
$\theta(a, z_{a,b})$	$\frac{\pi}{3}$	$\frac{\pi}{3} \leq < \frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$
$\theta(b, z_{a,b})$	$\frac{\pi}{3}$	$\pi - 2\theta$	$\frac{2\pi}{3}$	$\frac{\pi}{3}$
		$0 < \leq \frac{\pi}{3}$		
$\theta(a, c)$	$\frac{2\pi}{3}$	$\pi - \theta$	$\frac{\pi}{3}$	$\frac{\pi}{3}$
		$\frac{\pi}{2} < \leq \frac{2\pi}{3}$		
	$\lvert a\rvert_X < \lvert b\rvert_X$	$\lvert a\rvert^{\perp}_{z_{a,b}} > \lvert b\rvert^{\perp}_{z_{a,b}}$	$\lvert a\rvert_X > \lvert b\rvert_X$	$\lvert a\rvert_X < \lvert b\rvert_X$
$\lvert z_{a,b}\rvert_X$	$\frac{m}{4}(\lvert a\rvert_X + \lvert b\rvert_X)$	$\frac{m}{2}(\lvert a\rvert_X \omega_1 - \lvert b\rvert_X \omega_2)$	$\frac{m}{4}(\lvert a\rvert_X - \lvert b\rvert_X)$	$\frac{m}{4}(\lvert b\rvert_X - \lvert a\rvert_X)$

Let $G_i = \langle a_i, b_i, c_i | (a_i, b_i)_6 = (b_i, a_i)_6, (b_i, c_i)_3 = (c_i, b_i)_3, a_i c_i = c_i a_i \rangle$ be a family of the Artin groups $A(6,5,2)$ indexed by $i \in \mathbb{N}$. In this section let $\mathcal{O}_i$ denote the quotient complex $\mathcal{O}_A / G_i$ for each G_i.

In the following 4 sections we consider the following amalgams;

$$G_1 \underset{\substack{a_1=a_2 \\ b_1=b_2}}{*} G_2,\ G_1 \underset{\substack{a_1=b_2 \\ b_1=a_2}}{*} G_2,\ G_1 \underset{\substack{a_1=a_2 \\ c_1=c_2}}{*} G_2,\ G_1 \underset{\substack{a_1=c_2 \\ c_1=a_2}}{*} G_2$$

We show that these groups each have CAT(0) dimension 2.

Note that the results of the following sections do not use the fact that $m = 6$ and $n = 5$. In fact these results hold for a lot more combinations of $G_1 = A(m_1, n_1, 2)$ and $G_2 = A(m_2, n_2, 2)$.

5.1 The amalgam $G_1 *^{a_1=c_2}_{c_1=a_2} G_2$

In this section we consider the amalgam

$$G_1 \underset{\substack{a_1=c_2 \\ c_1=a_2}}{*} G_2 \cong \langle a_1, b_1, c_1, b_2 | (a_1, b_1)_6 = (b_1, a_1)_6, (b_1, c_1)_5 = (c_1, b_1)_5,$$
$$a_1 c_1 = c_1 a_1, (c_1, b_2)_6 = (b_2, c_1)_6, (b_2, a_1)_5 = (a_1, b_2)_5 \rangle$$

This is a rank 4 Artin group with associated Coxeter matrix,

$$\begin{pmatrix} 1 & 6 & 2 & 5 \\ 6 & 1 & 5 & \infty \\ 2 & 5 & 1 & 6 \\ 5 & \infty & 6 & 1 \end{pmatrix}$$

We may form a complex for this group by identifying generator loops for a with generator loops for c in the quotient complexes $\mathcal{O}_1$ and $\mathcal{O}_2$. Since $|a_1| = |c_2|$ and $|a_2| = |c_1|$ we may not choose 2 types of complex where $|a|_X > |b|_X$ for both types or $|a|_X < |b|_X$ for both types. Thus the ordered pair $(\mathcal{O}_1, \mathcal{O}_2)$ cannot be of type $(1i, 1i), (2, 2), (3, 3), (1i, 3)$ or $(3, 1i)$. Similarly we require $\theta(a_1, c_1) = \theta(a_2, c_2)$, so $(\mathcal{O}_1, \mathcal{O}_2)$ cannot be of type $(1i, 2), (1i, 3), (1ii, 2), (1ii, 3), (2, 1i), (3, 1i), (2, 1ii)$ or $(3, 1i)$. Thus we may use the types $(1i, 1ii)$, $(1ii, 1i)$, $(1ii, 1ii)$,$(2, 3)$ or $(3, 2)$. Note that if $\theta(a, z_{a,b}) = \frac{\pi}{3}$ for type $1ii$ then $|a|_X > |b|_X$, so the combination $(1ii, 1ii)$ is only possible if $\theta(a, z_{a,b}) > \frac{\pi}{3}$. By symmetry we have the possible combinations $(1i, 1ii)$, $(1ii, 1ii)$ and $(2, 3)$.

Amalgamating these groups in this way is equivalent to identifying the subcomplexes X_2 in $\mathcal{O}_1$ and $\mathcal{O}_2$, by identifying a_1 loops with c_2 loops and c_1 loops with a_2 loops. By Van Kampen's theorem the amalgamated complex has the correct fundamental group. As an example, consider constructing the complex from a type 2 complex and a type 3 complex. To check that this complex is CAT(0), it is sufficient to check the link of any point in the common parts of $\mathcal{O}_1$ and $\mathcal{O}_2$. The link of any of these points is the union of the link in $\mathcal{O}_1$ with the link in $\mathcal{O}_2$ identified along a common circle of length 2π. The resulting link is therefore CAT(1).

5.2 The amalgam $G_1 *_{c_1=c_2}^{a_1=a_2} G_2$

In this section we consider the amalgam

$$\begin{aligned} G_1 \underset{\substack{a_1=a_2\\c_1=c_2}}{*} G_2 \cong \langle a_1, b_1, c_1, b_2 | &(a_1, b_1)_6 = (b_1, a_1)_6, (b_1, c_1)_5 = (c_1, b_1)_5, \\ &a_1c_1 = c_1a_1, (a_1, b_2)_6 = (a_2, c_1)_6, (b_2, c_1)_5 = (c_1, b_2)_5\rangle \end{aligned}$$

This is a rank 4 Artin group with associated Coxeter matrix;

$$\begin{pmatrix} 1 & 6 & 2 & 6 \\ 6 & 1 & 5 & \infty \\ 2 & 5 & 1 & 5 \\ 6 & \infty & 5 & 1 \end{pmatrix}$$

As in the previous section if we ensure that $\mathcal{O}_1$ and $\mathcal{O}_2$ may be identified on $\Pi(a_1, c_1)$ and $\Pi(a_2, c_2)$ via an isometry, we know the complex has the correct fundamental group and is CAT(0). To ensure the identification is via an isometry we require $|a_1|_X = |a_2|_X$, $|c_1|_X = |c_2|_X$ and $\theta(a_1, c_1) = \theta(a_2, c_2)$. As before, consideration of the angle shows the following combinations are not applicable; $(1i, 2), (1i, 3), (1ii, 2), (1ii, 3)$. We list these as unordered pairs, as the amalgam is symmetrical. The restriction on lengths implies that $|a_1|_X < |b_1|_X$ if and only if $|a_2|_X < |b_2|_X$. Hence the following combinations are not usable; $(1i, 2), (1i, 1ii), (1ii, 3)$, and $(2, 3)$. This leaves the following combinations $(1i, 1i), (1ii, 1ii), (2, 2)$ and $(3, 3)$. For these 4 combinations the identification is via an isometry. Thus this amalgam has CAT(0) dimension 2.

5.3 The amalgam $G_1 *_{b_1=b_2}^{a_1=a_2} G_2$

Now we consider the amalgam

$$\begin{aligned} G_1 \underset{\substack{a_1=a_2 \\ b_1=b_2}}{*} G_2 \cong \langle a_1, b_1, c_1, c_2 | (a_1, b_1)_6 = (b_1, a_1)_6, (b_1, c_1)_5 = (c_1, b_1)_5, \\ a_1c_1 = c_1a_1, (b_1, c_2)_5 = (c_2, b_1)_5, a_1c_2 = c_2a_1 \rangle \end{aligned}$$

This is the rank 4 Artin group with associated Coxeter matrix,

$$\begin{pmatrix} 1 & 6 & 2 & 2 \\ 6 & 1 & 5 & 5 \\ 2 & 5 & 1 & \infty \\ 2 & 5 & \infty & 1 \end{pmatrix}$$

Here $z_{a_1b_1} = (a_1, b_1)_m = (a_2, b_2)_m = z_{a_2b_2}$. Hence $\theta(a_1, z_{a_1b_1}) = \theta(a_2, z_{a_2b_2})$ and $\theta(b_1, z_{a_1b_1}) = \theta(b_2, z_{a_2b_2})$. This eliminates the following as possible combinations for $(\mathcal{O}_1, \mathcal{O}_2)$: $(1ii, 3), (1i, 3), (2, 3), (3, 1ii), (3, 1i), (3, 1)$, and $(1ii, 2)$, $(1i, 2)$,$(2, 1ii), (2, 1i)$.

Consider the pair $(1i, 1ii)$. Here $\theta(a_i, z_{a_ib_i}) = \frac{\pi}{3}$ and $\theta(b_i, z_{a_ib_i}) = \frac{\pi}{3}$, so $l_X(a_1 > l_X(b_1) = l_X(b_2) > l_X(a_2) = l_X(a_1)$, a contradiction. By symmetry the same contradiction applies for $(1ii, 1i)$.

This leaves the following possible combinations for this amalgam: $(1i, 1i)$, $(1ii, 1ii)$, $(1, 1)$, and $(2, 2)$.

Again, as in the previous sections, since we are attaching $\mathcal{O}_1$ to $\mathcal{O}_2$ along the connected regions $\mathcal{O}_6$ (the complex associated to $A(6)$), by Van Kampen's theorem the amalgamated complex has the correct fundamental group. As an example consider the complex formed by 2 copies of type 2. We may identify the cells as shown in figure 5.1. Here the link of any point in the identified region is the union of the links in $\mathcal{O}_1$ and $\mathcal{O}_2$ joined along either a circle or a θ-graph. Again the link is CAT(1).

5.4 The amalgam $G_1 *_{b_1=a_2}^{a_1=b_2} G_2$

We now consider the amalgam

$$G_1 \underset{\substack{a_1=a_2 \\ b_1=b_2}}{*} G_2 \cong \langle a_1, b_1, c_1, c_2 | (a_1, b_1)_6 = (b_1, a_1)_6, (b_1, c_1)_5 = (c_1, b_1)_5,$$
$$a_1c_1 = c_1a_1, (a_1, c_2)_5 = (c_2, a_1)_5, b_1c_2 = c_2b_1\rangle$$

This is the rank 4 Artin group with associated Coxeter matrix,

$$\begin{pmatrix} 1 & 6 & 2 & 5 \\ 6 & 1 & 5 & 2 \\ 2 & 5 & 1 & \infty \\ 5 & 2 & \infty & 1 \end{pmatrix}$$

Again we ensure the identification of the complexes $\mathcal{O}_1$ and $\mathcal{O}_2$ is via an isometry. Again $z_{a_1b_1} = (a_1, b_1)_m = (b_2, a_2)_m = z_{a_2,b_2}$, so $\theta(a_1, z_{a_1,b_1}) = \theta(b_2, z_{a_2,b_2})$ and $\theta(b_1, z_{a_1,b_1}) = \theta(a_2, z_{a_2,b_2})$. This shows the following unordered pairs for $(\mathcal{O}_1, \mathcal{O}_2)$ do not work: $(1i, 2), (1ii, 2), (2, 2), (3, 1i), (3, 1ii)$, $(3, 3)$. Consideration of the translation lengths shows $|a_1|_X < |b_1|_X$ if and only if $|a_2|_X > |b_2|_X$. So $(1i, 1i)$ cannot be used. Also if the pair $(1ii, 1ii)$ is used then $\theta(a_1, z_{a_1,b_1}) = \pi - 2\theta(a_2, z_{a_2,b_2})$ and $\theta(a_2, z_{a_2,b_2}) = \pi - 2\theta(a_1, z_{a_1,b_1})$, so $\theta(a_1, z_{a_1,b_1}) = \theta(a_2, z_{a_2,b_2}) = \frac{\pi}{3}$. Hence $|a_i|_X > |b_i|_X$. But this contradicts the translation length restriction. Hence we are left with the following unordered pairs which provide an identification via an isometry: $(1i, 1ii)$ and $(2, 3)$.

Consider constructing the complex from a type 2 and a type 3 complex. As in the previous section the link of the common subcomplex is the union of two CAT(1) links identified along either a circle or θ-graph, so the link is CAT(1).

The results of these four sections may easily be generalised to larger classes of groups. We have proved in these four sections the following theorem.

Theorem 5.4.1. *For $m_1, m_2 \geq 6$, even and not divisible by 4, and $n_1, n_2 \geq 5$, prime the four generator Artin groups with the following associated Coxeter matrices have CAT(0) dimension 2.*

$$\begin{pmatrix} 1 & m_1 & 2 & 2 \\ m_1 & 1 & n_1 & n_2 \\ 2 & n_1 & 1 & \infty \\ 2 & n_2 & \infty & 1 \end{pmatrix} \quad A(m_1, n_1, 2) *_{b_1=b_2}^{a_1=a_2} A(m_1, n_2, 2)$$

$$\begin{pmatrix} 1 & m_1 & 2 & n_2 \\ m_1 & 1 & n_1 & 2 \\ 2 & n_1 & 1 & \infty \\ n_1 & 2 & \infty & 1 \end{pmatrix} \quad A(m_1, n_1, 2) *_{b_1=a_2}^{a_1=b_2} A(m_1, n_2, 2)$$

$$\begin{pmatrix} 1 & m_1 & 2 & m_2 \\ m_1 & 1 & n_1 & \infty \\ 2 & n_1 & 1 & n_2 \\ m_2 & \infty & n_2 & 1 \end{pmatrix} \quad A(m_1, n_1, 2) *_{c_1=c_2}^{a_1=a_2} A(m_2, n_2, 2)$$

$$\begin{pmatrix} 1 & m_1 & 2 & n_2 \\ m_1 & 1 & n_1 & \infty \\ 2 & n_1 & 1 & m_2 \\ n_2 & \infty & m_2 & 1 \end{pmatrix} \quad A(m_1, n_1, 2) *_{c_1=a_2}^{a_1=c_2} A(m_2, n_2, 2)$$

5.5 A CAT(0) dimension 2 Artin group of rank 6

In this section we use these rank 4 Artin groups to create an Artin group H of rank 6 with the following associated Coxeter group for the generators

$a_1, b_1, c_1, b_2, c_3, b_4$.

$$\begin{pmatrix} 1 & 6 & 2 & 5 & \infty & \infty \\ 6 & 1 & 5 & \infty & \infty & \infty \\ 2 & 5 & 1 & 6 & 2 & 5 \\ 5 & \infty & 6 & 1 & 5 & \infty \\ \infty & \infty & 2 & 5 & 1 & 6 \\ \infty & \infty & 5 & \infty & 6 & 1 \end{pmatrix}$$

Consider the amalgam H';

$$H' := G_1 \underset{c(1)=a(2)}{\overset{a(1)=c(2)}{\bigstar}} G_2 \underset{b(2)=b(3)}{\overset{a(2)=a(3)}{\bigstar}} G_3 \underset{c(3)=a(4)}{\overset{a(3)=c(4)}{\bigstar}} G_4$$

with $G_i = \{a_i, b_i, c_i | (a_i, b_i)_6 = (b_i, a_i)_6, (b_i, c_i)_5 = (c_i, b_i)_5, a_i c_i = c_i a_i\}$. This group has six generators: $a_i(= c_2)$, b_1, $c_1(= a_2 = a_3 = c_4)$, $b_2(= b_3)$, $c_3(= a_4)$ and b_4. By examining the relations between these generators one may observe that this group H' is isomorphic to the Artin group H. We show this for the generators a_1 and b_1. From G_1 we know that $(a_1, b_1)_6 = (b_1, a_1)_6$. However $a_1 = c_2$, but there is no relationship given between c_2 and b_1, hence the only relationship between a_1 and b_1 is $(a_1, b_1)_6 = (b_1, a_1)_6$. In other cases the relationships given by the contributing G_i match, eg. $a_3 = c_4$ and $c_3 = a_4$.

By the previous work there are just four possible types for each $\mathcal{O}_i$. We have $(\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4)$ of type $(1i, 1ii, 1ii, 1i)$, $(1ii, 1i, 1i, 1ii)$, $(2, 3, 3, 2)$ or $(3, 2, 2, 3)$.

We will consider the complex of type $(3, 2, 2, 3)$. We know that pairwise, the subcomplexes forming this amalgamated complex are locally CAT(0). It suffices then, to check the link of the common vertex. This is the endpoint of the edge labeled c in $\mathcal{O}_1$. Denote this vertex by v.

The link of this vertex restricted to a subcomplex of type 2 is shown in figure A.9 and restricted to type 3 is the union of the links shown in figures A.12 and A.13. Figure 5.2 shows the link of v in $\mathcal{O}_H$.

A schematic for the link of v in $\mathcal{O}_H$ is shown in figure 5.3. It is easy to check from this that the link is CAT(1).

Note that the complexes of type $1ii$ are a family with one of the parameters given by $\theta(a, z_{a,b})$. In this amalgamated group, this angle is fixed at $\frac{\pi}{3}$, hence if $|a_i|_X/|b_i|_X$ is rational, the complex may be triangulated by a finite number of isometry types of equilateral triangle. Moreover, the identifications given by the amalgam give the following restriction of the translation lengths:

$$|a_1|_X = |b_2|_X = |b_3|_X = |a_4|_X \text{ and } |b_1|_X = |a_2|_X = |a_3|_X = |b_4|_X$$

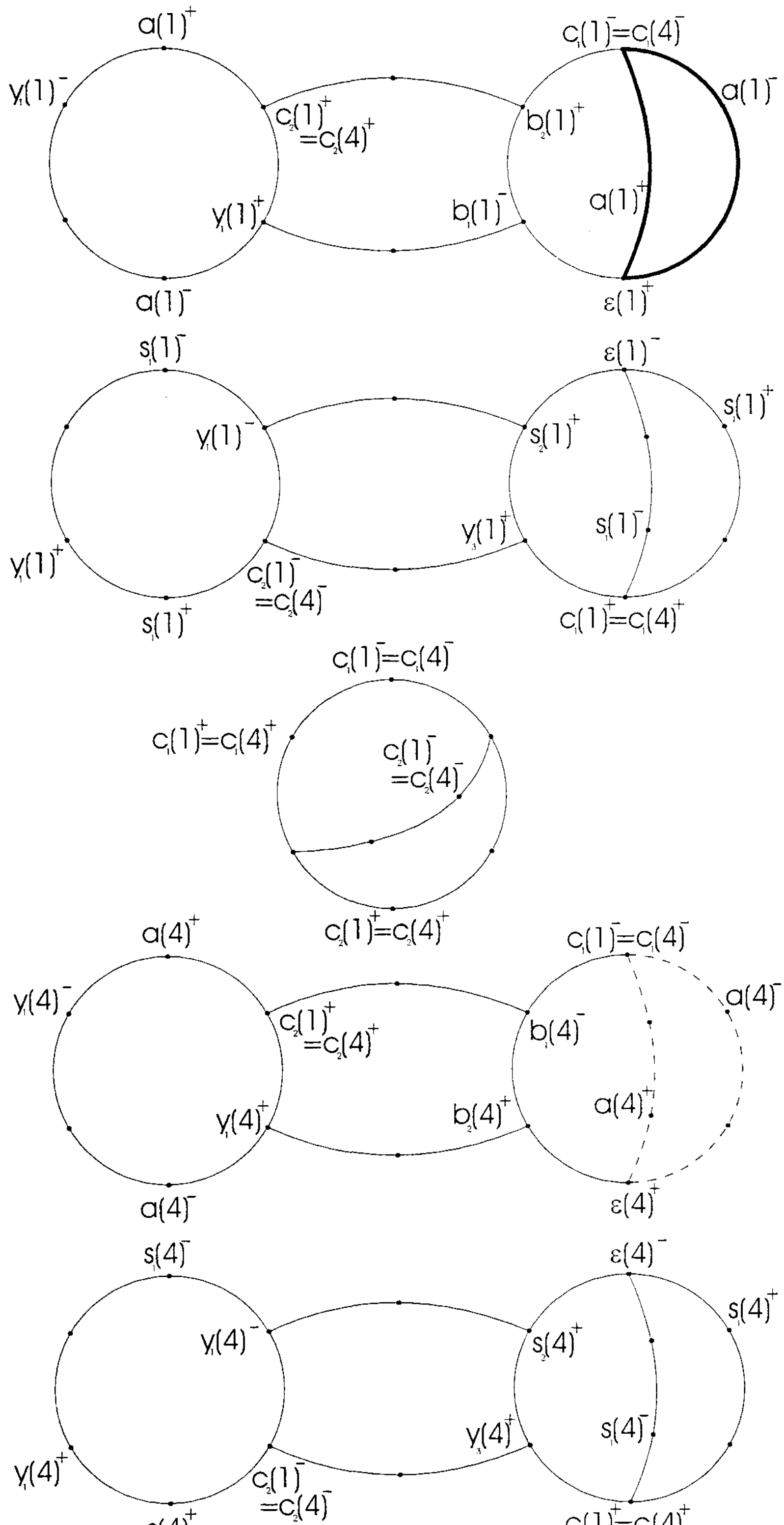

Figure 5.2: Link of v in $\mathcal{O}_H$

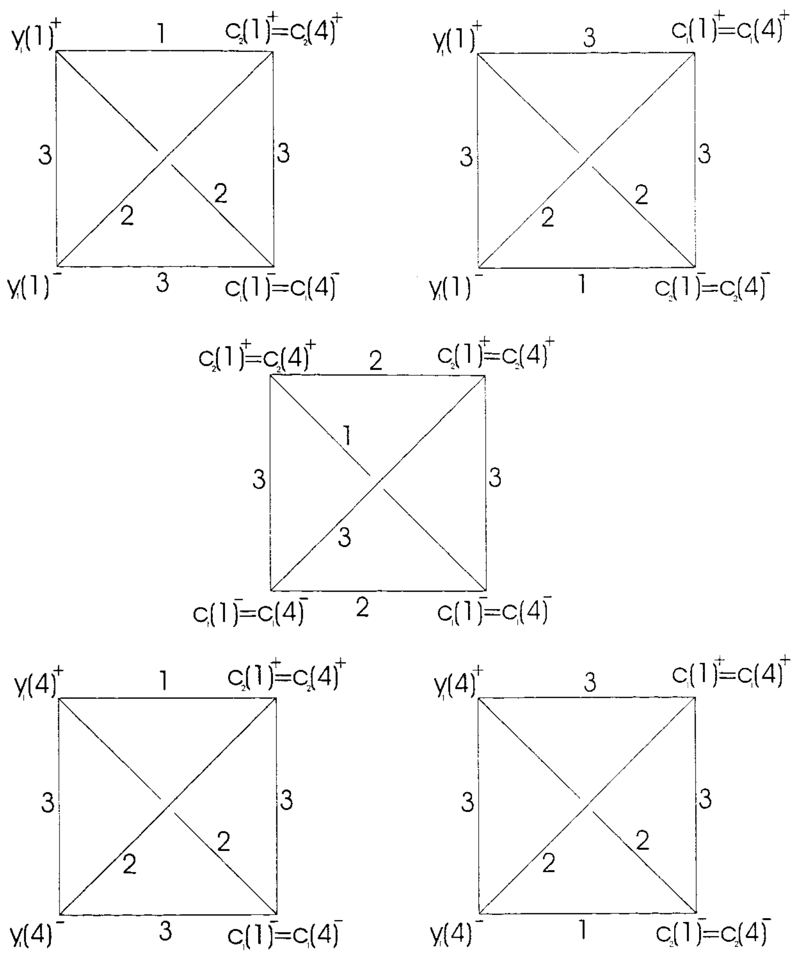

Figure 5.3: A schematic of the Link of v in $\mathcal{O}_H$. Edge lengths are multiples of $\frac{\pi}{3}$

Appendix A

Links

We show that certain links are CAT(1). Each link is a union of CAT(1) sublinks joined on common vertices. For each sublink we produce a graph showing the distances between these common vertices and deduce from this graph that the link has no loops of length strictly less than 2π, and hence the link is CAT(1).

A.1 Links from Case 1i

In this section we check the links of a quotient complex for $\mathcal{O}_A/A(m,n,2)$ for $m \geq 6$, $n \geq 5$ and $|a|^{\perp}_{z_{a,b}} < |b|^{\perp}_{z_{a,b}}$, see figure 4.22. There are 4 links to consider. In figure 4.22 they are represented by a black circle, a white circle, a black square and a white square. The black circle v_1 is the end point of the edge labeled by a, the white circle v_2 the endpoints of the edge y, the black square v_3 the initial vertex of ϵ, and the white square v_4 the terminal vertex of ϵ.

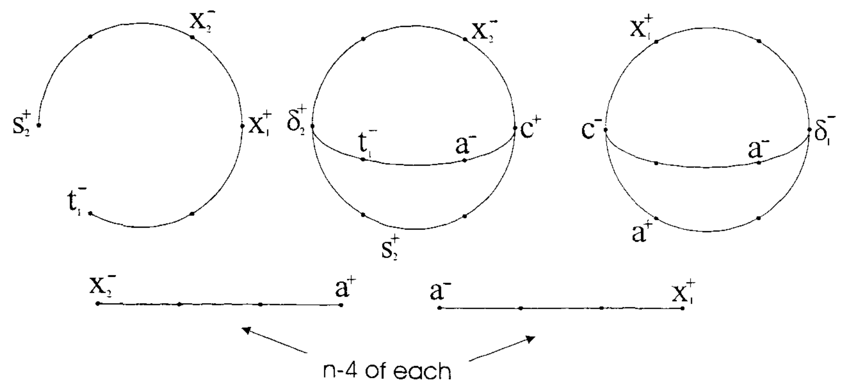

Figure A.1: The link of v_1 in case 1i. All edges have length $\frac{\pi}{2}$

A.1.1 The link of v_1

The link of v_1 is shown in figure A.1. We may remove from our consideration the $2n-8$ edges of length π, $[a^-, x_1^+]$ and $[x_2^-, a^+]$ since there are paths of length π between these points in the other parts of the link. There are now three parts to the link, each CAT(1), with common vertices x_2^-, t_1^- and a^+, a^- and s_2^+, x_1^+. A schematic diagram showing the lengths of the shortest paths in each link between these vertices is shown in figure A.2. From this diagram it is clear that there are no non trivial loops with length strictly less than 2π.

A.1.2 The link of v_2

The link of v_2 is shown in figure A.3. We may remove from our consideration the $n-4$ edges of length π, $[x_1^-, x_2^+]$. There are two parts to the link. We reduce each to a diagram showing the distances between the common vertices: x_2^+, x_1^-, s_1^- and t_2^+, see figure A.4.

Since the distance between each pair of points is at least π, any loop must have length at least 2π, so the link is CAT(1).

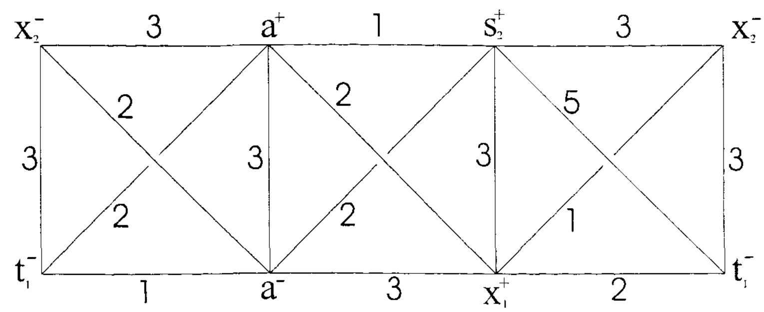

Figure A.2: Showing the shortest paths between the common vertices in each of the 3 parts of the link of v_1. Edge lengths given in multiples of $\frac{\pi}{3}$

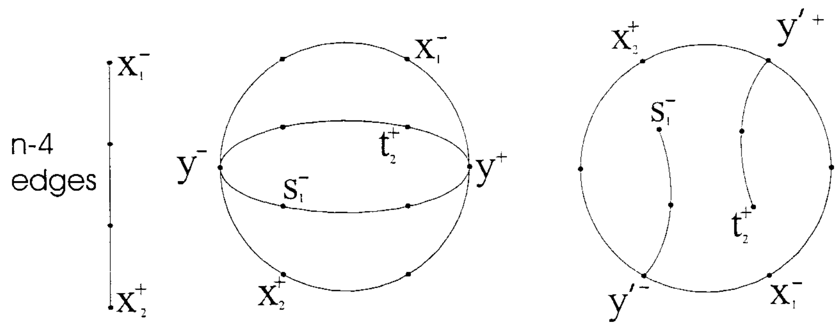

Figure A.3: The link of v_2 in case 1i. All edges have length $\frac{\pi}{2}$

A.1.3 The links of v_3 and v_4

These two links are the union of two π arcs joined at antipodal points to a circle. This construction is always CAT(1).

A.2 Links from Case 1ii

In this section we show the links of a quotient complex for $\mathcal{O}_A/A(m,n,2)$ are CAT(1) for case 1ii, see figure 4.24. There are two links, v_1 represented by a black circle, the end points of the edge labeled c, and v_2 represented

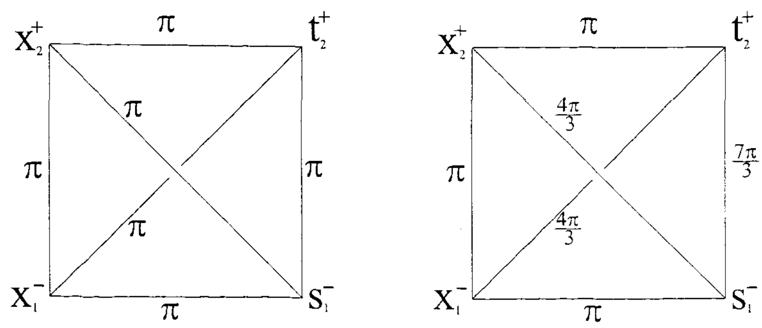

Figure A.4: Showing the shortest paths between the common vertices in the 2 parts of the link of v_2

by a white circle, the end points of the edge y_1. We denote by θ the angle $\theta(a, z_{a,b})$ between positively oriented a and $z_{a,b}$ segments.

A.2.1 The link of v_1

We may removed from our consideration the $n-6$ edges $[a_3^+, a_3^-]$ of length π as there are other paths between these points with this length. There are now three parts to the link, each CAT(1). We reduce these to diagrams showing the distances between the common vertices a_1^-, a_2^+, a_3^- and a_3^+, see figure A.7.

From this diagram one can see that the link is CAT(1) if and only if $\theta \geq \frac{\pi}{3}$.

A.2.2 The link of v_2

This link is formed by attaching π arcs to antipodal points on a circle. This construction is always CAT(1).

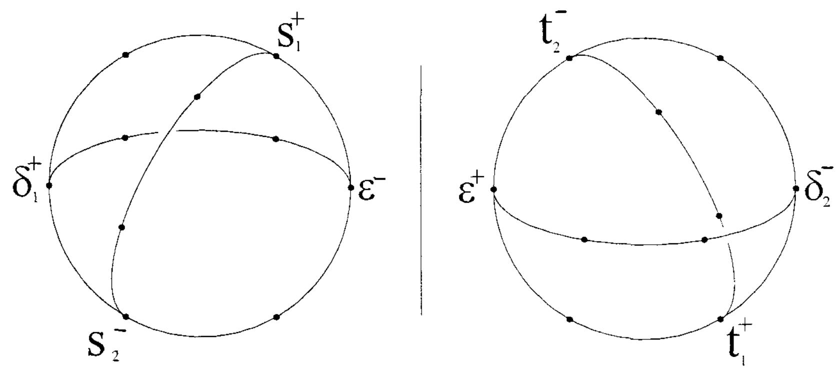

Figure A.5: The links of v_3 on the left and v_4 on the right in case1i. All edges have length $\frac{\pi}{2}$

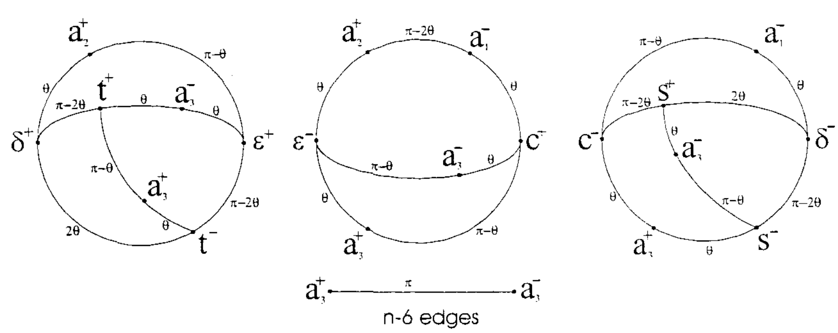

Figure A.6: The links of v_1 in case1ii

A.3 Links from Case 2

In this section we show the links of a quotient complex for $\mathcal{O}_A/A(m,n,2)$ are CAT(1) for case 2, see figure 4.27. There are two links, v_1 represented by a black circle, the end points of the edge labeled b, and v_2 represented by a white circle, the end points of the edge x. We denote by θ_a the angle $\theta(a, z_{a,b})$ between positively oriented a and $z_{a,b}$ segments and θ_b the angle $\theta(b, z_{b,c})$ between positively oriented b and $z_{b,c}$ segments.

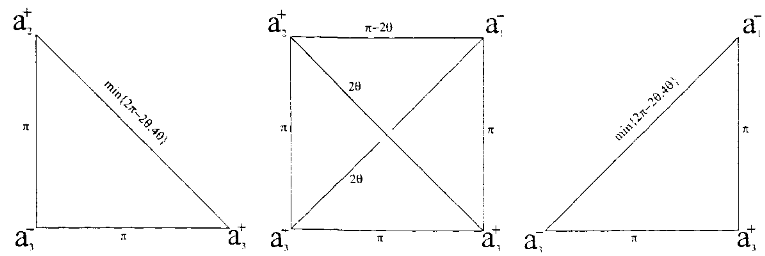

Figure A.7: Showing the distances between common points in each part of the link of v_1 in case1ii

A.3.1 The link of v_1

We may remove from our consideration the $n-4$ edges $[y^+, y^-]$ with length π.

The link is now formed from three CAT(1) sublinks with common vertices, b^+, α_2^+, and α_1^-, b^- and y^+, y^-. We reduce the link to a diagram showing the distances between these common points in each sublink, see figure A.10.

We see that this link is CAT(1) if and only if

$$
\begin{aligned}
&2\theta_a + 3\theta_b + \theta_b \geq 2\pi \\
&2\theta_a + 2\pi - 3\theta_b + \theta_b \geq 2\pi \\
&\pi - 2\theta_a + 3\theta_b + \pi - \theta_b \geq 2\pi \\
&\pi - 2\theta_a + 2\pi - 3\theta_b + \pi - \theta_b \geq 2\pi
\end{aligned}
$$

It follows that this link is CAT(1) if and only if $\theta_a = \theta_b = \frac{\pi}{3}$.

A.3.2 The link of v_2

This link is formed by attaching π arcs to antipodal points on a circle. This construction is always CAT(1).

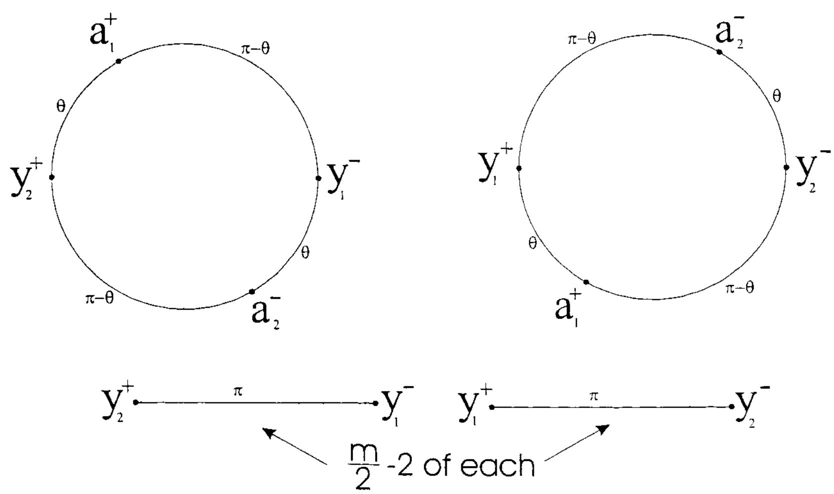

Figure A.8: The links of v_2 in case 1ii

A.4 Links from Case 3

In this section we show the links of a quotient complex for $\mathcal{O}_A/A(m,n,2)$ are CAT(1) for case 3, see figure 4.31. There are three links, v_1 represented by a black circle, the end points of the edge labeled a, v_2 represented by a white circle, the end points of the edge s_1, and v_3 represented by a black square, the end points of x.

A.4.1 The link of v_1

We may remove from our consideration the $n-3$ edges, $[y_1^+, b_2^+]$, with length π. We are left with a link made from two parts with common vertices a^+, a^- and b_1^-. We observe that in the first part the distances between each pair are as follows: $d(a^+,a^-)=\pi$, $d(a^+,b_1^-)=\frac{4\pi}{3}$ and $d(b_1^-,a^-)=\pi$. In the second, the distances are the same, except for $d(a^+,b_1^-)=\frac{2\pi}{3}$, hence every loop has length at least 2π.

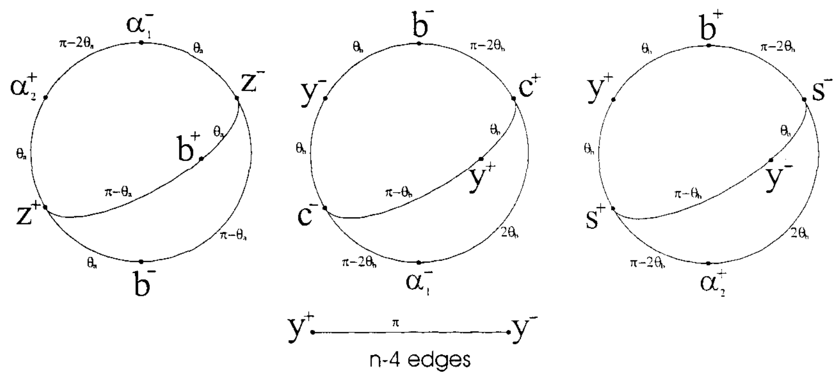

Figure A.9: The links of v_1 in case 2

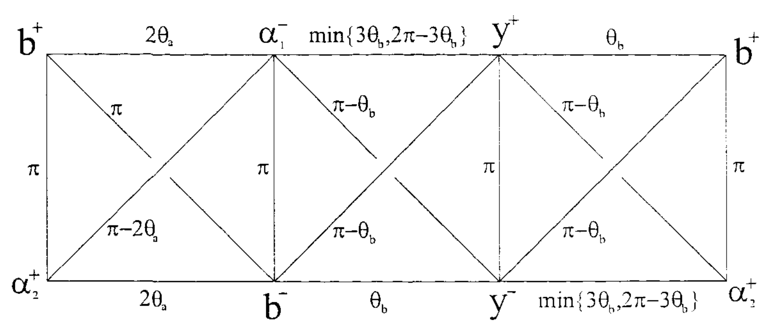

Figure A.10: Showing the distances between common points in each part of the link of v_1 in case 2

A.4.2 The link of v_2

We may remove from our consideration the $n-3$ edges, $[y_1^-, y_3^+]$, with length π. We are left with a link made from two parts with common vertices s_1^-, s_1^+,

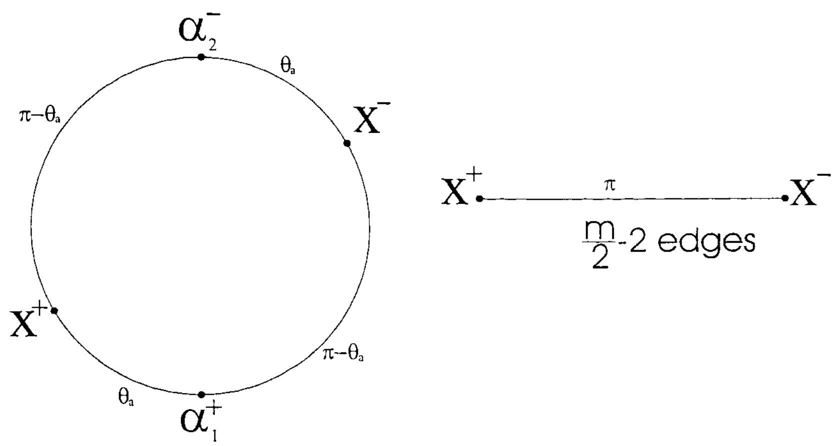

Figure A.11: The links of v_2 in case 2

y_1^- and y_3^+. The distances between these points in each link are as follows:

	first part	second part	total
$d(s_1^-, s_1^+) =$	π	π	2π
$d(s_1^-, y_1^-) =$	$\frac{\pi}{3}$	$\frac{5\pi}{3}$	2π
$d(s_1^-, y_3^+) =$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	2π
$d(s_1^+, y_1^-) =$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	2π
$d(s_1^+, y_3^+) =$	π	π	2π
$d(y_1^-, y_3^+) =$	π	π	2π

Hence every loop has length at least 2π.

A.4.3 The link of v_3

We may remove from our consideration the $n-4$ edges, $[y_3^-, b_2^-]$, with length π and the $\frac{m}{2}-4$ edges, $[x^-, x^+]$, with length π as there are other paths between these vertices with length π. We are left with two parts to the link, each CAT(1), with common vertices x^-, x^+, b_2^- and y_3^-. The distances between

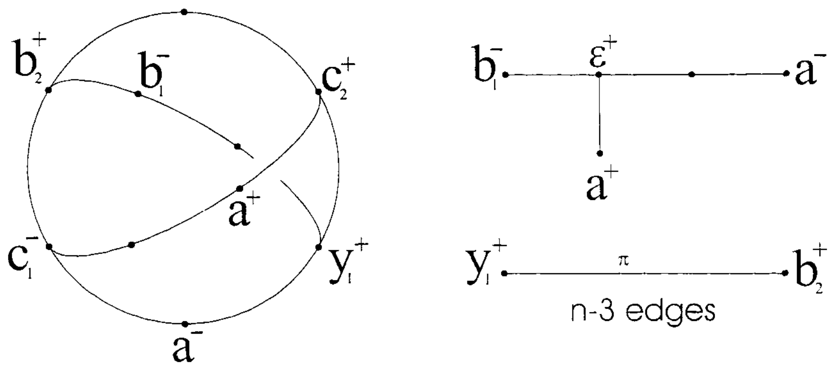

Figure A.12: The links of v_1 in case 3

these points in each link are as follows:

	first part	second part	total
$d(x^-, x^+) =$	π	π	2π
$d(x^-, y_3^-) =$	$\frac{5\pi}{3}$	$\frac{\pi}{3}$	2π
$d(x^-, b_2^-) =$	$\frac{4\pi}{3}$	$\frac{2\pi}{3}$	2π
$d(x^+, y_3^-) =$	$\frac{2\pi}{3}$	$\frac{4\pi}{3}$	2π
$d(x^+, b_2^-) =$	$\frac{\pi}{3}$	$\frac{5\pi}{3}$	2π
$d(y_3^-, b_2^-) =$	π	π	2π

Hence every loop has length at least 2π.

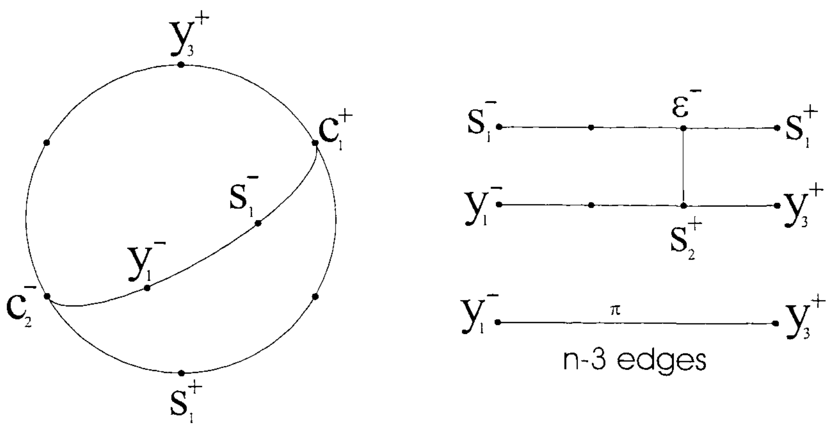

Figure A.13: The links of v_2 in case 3

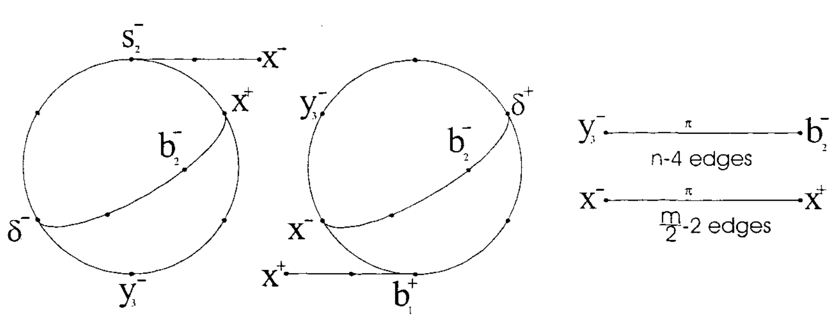

Figure A.14: The links of v_3 in case 3

Appendix B

Tietze Transforms

We show that the fundamental groups of the quotient complexes produced in chapter 4 are the groups $A(m,n,2)$. We use Tietze transforms to change the presentation of the group. To help choose the correct transforms we use GAP on specific instances, eg $A(10,11,2)$ and then generalise to $A(m,n,2)$.

B.1 Case 1ii

In this section we show the fundamental group of the complex produced in section 4.5.2 is $A(m,n,2)$.

The complex is shown in figure 4.24. There are two vertices, the end points of α_2. We may homotopically retract α_2 to a point and contract any bigons formed to single edges. This identifies the edges ϵ with y_2^{-1} and y_1 with a_1. We may then read a presentation from the complex:

$$\begin{aligned}\langle a_1, s_3, c, s, t, y_2, \delta | \quad & s a_3^{\frac{n-3}{2}} t a_3^{n-5} 2, \quad t a_3 y_2^{-1}, \quad y_2 a_3 c a_3^{-1}, \quad a_3 s c \\ & \delta t \delta^{-1} s^{-1}, \quad c a_1 y_2 a_1^{-1}, \quad a_1^2 (y_2 a_1)^{\frac{m}{2}-2} \delta^{-1} \rangle\end{aligned}$$

We number our generators $1 := a_1$, $2 := a_3$, $3 := c$, $4 := s$, $5 := t$, $6 := y_2$, and $7 := \delta$. And enter the presentation into GAP. We make the substitutions $a = a_1 a_3$ and $b = c^{-1} a_3^{-1}$.

The following is the output for $m = 10$ and $n = 11$.

```
gap > m:=10;n:=11;
10
11
gap > F:=FreeGroup( 7, "G");;
G:=F/[
F.4*F.2 ∧ ((n-3)/2)*F.5*F.2 ∧ ((5-n)/2),
F.5*F.2*F.6 ∧ -1,
F.6*F.2*F.3*F.2 ∧ -1,
F.2*F.4*F.3,
F.7*F.5*F.7 ∧ -1*F.4 ∧ -1,
F.3*F.1*F.6*F.1 ∧ -1,
F.1 ∧ 2*(F.6*F.1) ∧ ((m/2)-2)*F.7 ∧ -1
];
H:=SimplifiedFpGroup(G);
RelatorsOfFpGroup(H);
P:=PresentationFpGroup(G);
TzPrintPresentation(P);
a:=GeneratorsOfPresentation(P)[1];
A:=GeneratorsOfPresentation(P)[2];
c:=GeneratorsOfPresentation(P)[3];
s:=GeneratorsOfPresentation(P)[4];
t:=GeneratorsOfPresentation(P)[5];
y:=GeneratorsOfPresentation(P)[6];
d:=GeneratorsOfPresentation(P)[7];
# I generators:
# I 1. G1 7 occurrences
```

```
# I 2. G2 11 occurrences
# I 3. G3 3 occurrences
# I 4. G4 3 occurrences
# I 5. G5 3 occurrences
# I 6. G6 6 occurrences
# I 7. G7 3 occurrences
# I relators:
# I 1. G4*G2 ∧ 4*G5*G2 ∧ -3
# I 2. G5*G2*G6 ∧ -1
# I 3. G6*G2*G3*G2 ∧ -1
# I 4. G2*G4*G3
# I 5. G7*G5*G7 ∧ -1*G4 ∧ -1
# I 6. G3*G1*G6*G1 ∧ -1
# I 7. G1 ∧ 2*G6*G1*G6*G1*G6*G1*G7 ∧ -1
gap > TzSubstitute(P,a*A);
# I there are 7 generators and 7 relators of total length 36
# I now the presentation has 8 generators, the new generator is _x8
# I substituting new generator _x8 defined by G1*G2
gap > TzSubstitute(P,c ∧ -1*A ∧ -1);
# I there are 8 generators and 8 relators of total length 39
# I now the presentation has 9 generators, the new generator is _x9
# I substituting new generator _x9 defined by G3 ∧ -1*G2 ∧ -1
# I there are 9 generators and 9 relators of total length 42
gap > TzEliminate(P,d);TzPrintRelators(P);
# I there are 8 generators and 8 relators of total length 47
# I 1. G5*G2*G6 ∧ -1
# I 2. G2*G4*G3
```

I 3. _x8 ∧ -1*G1*G2

I 4. _x9 ∧ -1*G3 ∧ -1*G2 ∧ -1

I 5. G6*G2*G3*G2 ∧ -1

I 6. G3*G1*G6*G1 ∧ -1

I 7. G4*G2 ∧ 4*G5*G2 ∧ -3

I 8. G1 ∧ 2*G6*G1*G6*G1*G6*G1*G5*G1 ∧ -1*G6 ∧ -1*G1 ∧ -1*G6 ∧ -1*G1 ∧ -1*G6 ∧ -1*

G1 ∧ -2*G4 ∧ -1

gap > TzEliminate(P,t);TzPrintRelators(P);

I there are 7 generators and 7 relators of total length 46

I 1. G2*G4*G3

I 2. _x8 ∧ -1*G1*G2

I 3. _x9 ∧ -1*G3 ∧ -1*G2 ∧ -1

I 4. G6*G2*G3*G2 ∧ -1

I 5. G3*G1*G6*G1 ∧ -1

I 6. G4*G2 ∧ 4*G6*G2 ∧ -4

I 7. G1 ∧ 2*G6*G1*G6*G1*G6*G1*G6*G2 ∧ -1*G1 ∧ -1*G6 ∧ -1*G1 ∧ -1*G6 ∧ -1*G1 ∧ -1*

G6 ∧ -1*G1 ∧ -2*G4 ∧ -1

gap > TzEliminate(P,s);TzPrintRelators(P);

I there are 6 generators and 6 relators of total length 45

I 1. _x8 ∧ -1*G1*G2

I 2. _x9 ∧ -1*G3 ∧ -1*G2 ∧ -1

I 3. G6*G2*G3*G2 ∧ -1

I 4. G3*G1*G6*G1 ∧ -1

I 5. G2 ∧ -1*G3 ∧ -1*G2 ∧ 4*G6*G2 ∧ -4

I 6. G1 ∧ 2*G6*G1*G6*G1*G6*G1*G6*G2 ∧ -1*G1 ∧ -1*G6 ∧ -1*G1 ∧ -1*

G6 ∧ -1*G1 ∧ -1*G6 ∧ -1*G1 ∧ -2*G3*G2

gap > TzEliminate(P,A);TzPrintRelators(P);

I there are 5 generators and 5 relators of total length 54

I 1. _x9 ∧ -1*G3 ∧ -1*_x8 ∧ -1*G1

I 2. G3*G1*G6*G1 ∧ -1

I 3. G6*G1 ∧ -1*_x8*G3*_x8 ∧ -1*G1

I 4. _x8 ∧ -1*G1*G3 ∧ -1*G1 ∧ -1*_x8*G1 ∧ -1*_x8*G1 ∧ -1*_x8*

G1 ∧-1*_x8*G6*_x8 ∧ -1*G1*_x8 ∧ -1*G1*_x8 ∧ -1*G1*_x8 ∧ -1*G1

I 5. G1 ∧ 2*G6*G1*G6*G1*G6*G1*G6*_x8 ∧ -1*G6 ∧ -1*G1 ∧ -1*G6 ∧ -1*

G1 ∧ -1*G6 ∧ -1*G1 ∧ -2*G3*G1 ∧ -1*_x8

gap > TzEliminate(P,y);TzPrintRelators(P);

I there are 4 generators and 4 relators of total length 50

I 1. _x9 ∧ -1*G3 ∧ -1*_x8 ∧ -1*G1

I 2. G3 ∧ -1*_x8*G3*_x8 ∧ -1

I 3. G1*G3 ∧ -1*G1*G3 ∧ -1*G1*G3 ∧ -1*G1*G3 ∧ -1*G1*_x8 ∧ -1*

G1 ∧ -1*G3*G1 ∧ -1*G3*G1 ∧ -1*G3*G1 ∧ -1*G3*G1 ∧ -1*_x8

I 4. _x8 ∧ -1*G1*G3 ∧ -1*G1 ∧ -1*_x8*G1 ∧ -1*_x8*G1 ∧ -1*_x8*

G1 ∧ -1*_x8*G1 ∧ -1*G3 ∧ -1*G1*_x8 ∧ -1*G1*_x8 ∧ -1*G1*_x8 ∧ -1*G1*_x8 ∧

-1*G1

gap > TzEliminate(P,a);TzPrintRelators(P);

I there are 3 generators and 3 relators of total length 68

I 1. G3 ∧ -1*_x8*G3*_x8 ∧ -1

I 2. G3*_x9*G3 ∧ -1*_x9 ∧ -1*G3 ∧ -1*_x9 ∧ -1*G3 ∧ -1*_x9 ∧ -1*G3 ∧ -1*_x9

∧ -1*G3 ∧ -1*

_x9 ∧ -1*G3 ∧ -1*_x8 ∧ -1*G3 ∧ -1*_x8*G3*_x9*G3*_x9*G3*_x9*G3*_x9*G3*_x9

I 3. _x8*G3*_x9*G3 ∧ -1*_x8*G3*_x9*G3 ∧ -1*_x8*G3*_x9*G3 ∧ -1*_x8*G3*_x9*G3

∧

```
-1*_x8*G3*_x9*_x8 ∧ -1*_x9 ∧ -1*G3 ∧ -1*_x8 ∧ -1*G3*_x9 ∧ -1*G3 ∧ -1*_x8 ∧
-1*G3*_x9 ∧ -1*G3
∧ -1*_x8 ∧ -1*G3*_x9 ∧ -1*G3 ∧ -1*_x8 ∧ -1*G3*_x9 ∧ -1*G3 ∧ -1
gap > TzGo(P);
# I there are 3 generators and 3 relators of total length 46
gap > TzPrintPresentation(P);
# I generators:
# I 1. G3 13 occurrences
# I 2. _x8 12 occurrences
# I 3. _x9 21 occurrences
# I relators:
# I 1. G3 ∧ -1*_x8*G3*_x8 ∧ -1
# I 2. _x9*_x8*_x9*_x8*_x9*_x8*_x9*_x8*_x9*_x8 ∧ -1*_x9 ∧ -1*_x8 ∧ -1*
_x9 ∧ -1*_x8 ∧ -1*_x9 ∧ -1*_x8 ∧ -1*_x9 ∧ -1*_x8 ∧ -1*_x9 ∧ -1*_x8
# I 3. G3*_x9*G3 ∧ -1*_x9 ∧ -1*G3 ∧ -1*_x9 ∧ -1*G3 ∧ -1*_x9 ∧ -1*G3 ∧ -1*_x9
∧ -1*G3 ∧ -1*
_x9 ∧ -1*G3 ∧ -1*_x9*G3*_x9*G3*_x9*G3*_x9*G3*_x9
# I there are 3 generators and 3 relators of total length 46
```

This produces the correct relations for $A(10,11,2)$.

Using this we may trace $G2^4$ through and replace it with $G2^{\frac{n-3}{2}}$ and replace $(G6*G1)^3$ with $(G6*G1)^{\frac{m}{2}-2}$ at the stage before we remove $G2 = a_3$. We revert back to the original letter for easier reading:

I 1. $a^{-1}a_1a_3$

I 2. $b^{-1}c^{-1}a_3^{-1}$

I 3. $y_2a_3ca_3^{-1}$

I 4. $ca_1y_2a_1^{-1}$

I 5. $a_3^{-1}c^{-1}a_3^{\frac{n-3}{2}}y_2a_3^{-\frac{n-3}{2}}$

I 6. $a_1^2(y_2a_1)^{\frac{m}{2}-2}y_2a_3^{-1}(y_2a_1)^{-(\frac{m}{2}-2)}a_1^{-2}ca_3$

Remove $a_3 = a_1^{-1}a$:

I 1. $b^{-1}c^{-1}a^{-1}a_1$

I 2. $y_2a_1^{-1}aca^{-1}a_1$

I 3. $ca_1y_2a_1^{-1}$

I 4. $a^{-1}a_1c^{-1}(a_1^{-1}a)^{\frac{n-3}{2}}y_2(a_1^{-1}a)^{-\frac{n-3}{2}}$

I 5. $a_1^2(y_2a_1)^{\frac{m}{2}-2}y_2a^{-1}a_1(y_2a_1)^{-(\frac{m}{2}-2)}a_1^{-2}ca_1^{-1}a$

Eliminate $y_2 = a_1^{-1}ac^{-1}a^{-1}a_1$:

I 1. $b^{-1}c^{-1}a^{-1}a_1$

I 2. $cac^{-1}a^{-1}$

I 3. $a^{-1}a_1c^{-1}(a_1^{-1}a)^{\frac{n-1}{2}}c^{-1}(a_1^{-1}a)^{-\frac{n-1}{2}}$

I 4. $a_1(ac^{-1}a^{-1}a_1)^{\frac{m}{2}-1}a^{-1}(ac^{-1}a^{-1}a_1)^{-(\frac{m}{2}-2)}a_1^{-1}ca_1^{-1}a$

Reduce using I 2. :

I 1. $b^{-1}c^{-1}a^{-1}a_1$

I 2. $cac^{-1}a^{-1}$

I 3. $a^{-1}a_1c^{-1}(a_1^{-1}a)^{\frac{n-1}{2}}c^{-1}(a_1^{-1}a)^{-\frac{n-1}{2}}$

I 4. $a_1(c^{-1}a_1)^{\frac{m}{2}-1}a^{-1}(c^{-1}a_1)^{-(\frac{m}{2}-1)}a_1^{-1}a$

Eliminate $a_1 = acb$

I 1. $cac^{-1}a^{-1}$

I 2. $cbc^{-1}(b^{-1}c^{-1})^{\frac{n-1}{2}}c^{-1}(b^{-1}c^{-1})^{-\frac{n-1}{2}}$

I 3. $acb(c^{-1}acb)^{\frac{m}{2}-1}a^{-1}(c^{-1}acb)^{-(\frac{m}{2}-1)}(acb)^{-1}a$

And reduce using I1. :

I 1. $cac^{-1}a^{-1}$

I 2. $c^{-1}(b^{-1}c^{-1})^{\frac{n-1}{2}}(c^{-1}b^{-1})^{-\frac{n-1}{2}}b$

I 3. $(ab)^{\frac{m}{2}}(ba)^{-\frac{m}{2}}$

These are the required relators.

B.2 Case 2

There are 2 vertices. Homotopically contract α_2 and contract the resulting bigons identifying $t = s$ and $x = \alpha_1$. We read a presentation from the resulting complex:

$$\langle \alpha_1, b, c, s, y, z \mid \quad \alpha_1 s \alpha_1^{-1} c^{-1}, cby^{-1}, bsy^{-1}, \\ bzb^{-1}z^{-1}, \alpha_1^{\frac{m}{2}} z^{-1}, sy^{\frac{n-3}{2}} cy^{\frac{1-n}{2}} \rangle$$

We apply Tietze transforms to the relators.

Remove $y = cb$:

$\alpha_1 s \alpha_1^{-1} c^{-1}, \quad bsb^{-1}c^{-1},$
$bzb^{-1}z^{-1}, \quad \alpha_1^{\frac{m}{2}} z^{-1}, \quad s(cb)^{\frac{n-3}{2}} c(cb)^{\frac{1-n}{2}}$

Eliminate $z = \alpha_1^{\frac{m}{2}}$:

$\alpha_1 s \alpha_1^{-1} c^{-1}, \quad bsb^{-1}c^{-1},$
$b\alpha_1^{\frac{m}{2}} b^{-1} \alpha_1^{-\frac{m}{2}}, \quad s(cb)^{\frac{n-3}{2}} c(cb)^{\frac{1-n}{2}}$

Remove $s = b^{-1}cb$:

$\alpha_1 b^{-1} cb \alpha_1^{-1} c^{-1},$
$b\alpha_1^{\frac{m}{2}} b^{-1} \alpha_1^{-\frac{m}{2}}, \quad b^{-1}(cb)^{\frac{n-1}{2}} c(cb)^{\frac{1-n}{2}}$

Add generator $a = \alpha_1 b^{-1}$ and remove $\alpha_1 = ab$:

$abb^{-1}cbb^{-1}a^{-1}c^{-1}, \quad b(ab)^{\frac{m}{2}}b^{-1}(ab)^{-\frac{m}{2}} \quad b^{-1}(cb)^{\frac{n-1}{2}}c(cb)^{\frac{1-n}{2}}$

Rearrange:

$aca^{-1}c^{-1}, \quad (ba)^{\frac{m}{2}}(ab)^{-\frac{m}{2}}, \quad b^{-1}(cb)^{\frac{n-1}{2}}c(cb)^{\frac{1-n}{2}}$

These are the required relations.

B.3 Case 3

There are three vertices. Homotopically contract b_2 and y_3. Contract the resulting bigons identifying $x = c_1 = s_2^{-1} = b_1^{-1}$ and $\delta = c_2$. We read a presentation for the fundamental group from the resulting complex:

$$\langle a, b_1, c_2, s_1, y_1, \epsilon | \quad b_1^{\frac{m}{2}-1}\epsilon, s_1\epsilon a\epsilon^{-1}, y_1ac_2^{-1} \\ s_1b_1ab_1^{-1}, s_1c_2y_1^{-1}, c_2y_1^{\frac{n-2}{2}}c_2b_1^{-1}y_1^{\frac{1-n}{2}}b_1^{-1}\rangle$$

We apply Tietze transforms:

Remove $\epsilon = b_1^{\frac{m}{2}-1}$: $s_1b_1^{1-\frac{m}{2}}a = b_1^{1-\frac{m}{2}}$

$y_1a = c_2$

$s_1b_1a = b_1$

$s_1c_2 = y_1$

$c_2y_1^{\frac{n-2}{2}}c_2 = b_1y_1^{\frac{n-1}{2}}b_1$

Eliminate $y = s_1c_2$:

$s_1b_1^{1-\frac{m}{2}}a = b_1^{1-\frac{m}{2}}$

$s_1c_2a = c_2$

$s_1b_1a = b_1$

$s_1c_2 = s_1c_2$

$c_2(s_1c_2)^{\frac{n-2}{2}}c_2 = b_1(s_1c_2)^{\frac{n-1}{2}}b_1$

Eliminate $s_1 = c_2 a^{-1} c_2^{-1}$:

$c_2 a^{-1} c_2^{-1} b_1^{1-\frac{m}{2}} a = b_1^{1-\frac{m}{2}}$

$c_2 a^{-1} c_2^{-1} b_1 a = b_1$

$c_2 (c_2 a^{-1} c_2^{-1} c_2)^{\frac{n-2}{2}} c_2 = b_1 (c_2 a^{-1} c_2^{-1} c_2)^{\frac{n-1}{2}} b_1$

Rearrange:

$c_2 a^{-1} c_2^{-1} b_1^{1-\frac{m}{2}} a = b_1^{1-\frac{m}{2}}$

$c_2 a^{-1} c_2^{-1} b_1 a = b_1$

$c_2 (c_2 a^{-1})^{\frac{n-2}{2}} c_2 = b_1 (c_2 a^{-1})^{\frac{n-1}{2}} b_1$

Add generator $c = b_1^{-1} c_2$ and remove $c_2 = b_1 c$:

$a b_1^{\frac{m}{2}} = b_1^{\frac{m}{2}} a$

$c^{-1} a^{-1} = a c^{-1}$

$b_1 c (b_1 c a^{-1})^{\frac{n-2}{2}} b_1 c = b_1 (b_1 c a^{-1})^{\frac{n-1}{2}} b_1$

Add generator $b = b_1 a^{-1}$ and remove $b_1 = ba$:

$a (ba)^{\frac{m}{2}} = (ba)^{\frac{m}{2}} a$

$c^{-1} a^{-1} = a c^{-1}$

$bac (baca^{-1})^{\frac{n-2}{2}} bac = ba (baca^{-1})^{\frac{n-1}{2}} ba$

Rearrange:

$a (ba)^{\frac{m}{2}} = (ba)^{\frac{m}{2}} a$

$c^{-1} a^{-1} = a c^{-1}$

$c (bc)^{\frac{n-2}{2}} bc = (bc)^{\frac{n-1}{2}} b$

These are the required relators.

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