Synthese (2026) 207:39
https://doi.org/10.1007/s11229-025-05314-x

ORIGINAL RESEARCH

Information walls and norm-dependent abilities

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Received: 26 April 2025 / Accepted: 6 October 2025

Abstract
The article studies the interplay between obligations, knowledge, and abilities. It introduces the notion of norm-dependent abilities–something that an agent knows how to achieve using a knowingly allowed action and assuming that the other agents also use only knowingly allowed actions. The main technical contribution is a sound and complete logical system that describes the interplay between the modalities representing knowledge, obligations, and norm-dependent abilities in the presence of information walls between the agents.

Keywords Obligations · Knowledge · Norms · Transition system · Normative system · Actions · Strategies

1 Introduction

In this article, we study how knowledge and norms affect the abilities of agents. For example, imagine a setting in which three guys, Ben (b), Charles (c), and David (d), decide to meet for a potluck party, each bringing just a single dish. Not being the greatest chef in the world, each of them knows how to cook only three dishes. Figure 1 shows which dishes each of them can cook. For instance, Ben only learned how to cook bruschetta, beef, and blueberry pie. Suppose that two of them, Charles and David, are vegetarians–we show this in Fig. 1 using the letter "v"–and there are no other dietary restrictions in the group.

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Published online: 08 January 2026

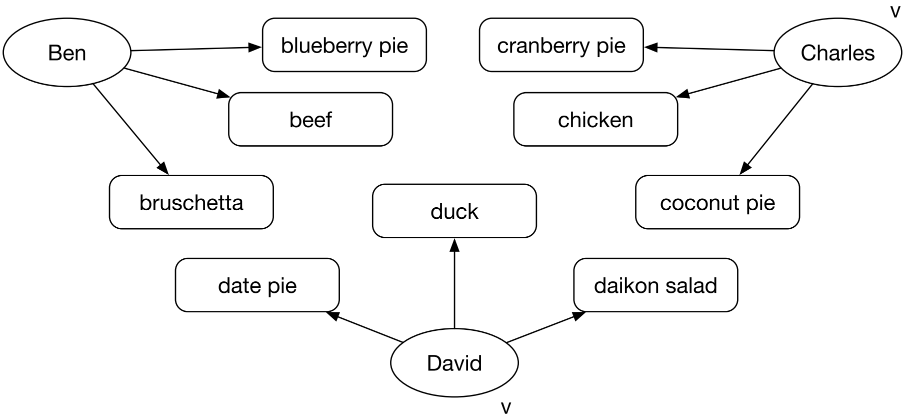

Fig. 1 Potluck party planning

Let us assume that *it is against social norms for any person to bring to a potluck party a dish nobody, except possibly the person himself, can eat*. In our setting, it means that Ben cannot bring beef because Charles and David are vegetarians. It is fine for Charles and David to bring chicken and duck, respectively, because Ben is not a vegetarian.

1.1 Deontic logic models

Several ways to model the above situation and to reason about it have been suggested in the literature. The best known of them is *deontic logic* (Wright, 1951; Anderson, 1956; Prior, 1963; Anderson, 1967; Kanger, 1971; Horty, 2001; Hilpinen, 2012). The semantics of this logical system would consider all possible combinations of dishes at the potluck party as possible worlds. In our case, there are $3 \times 3 \times 3 = 27$ possible worlds. In nine of these worlds, Ben violates the social norm and cooks beef for the party. Such worlds are called *unacceptable*. The other 18 worlds are *acceptable*. The syntax of the deontic logic contains modality $\mathsf{O}\varphi$ (“it ought to be φ”). Intuitively, the formula $\mathsf{O}\varphi$ means that the social norms imply that φ must be true. Formally, $\mathsf{O}\varphi$ is satisfied if φ is true in all acceptable worlds. For example, in our setting,

$$\mathsf{O}\,(\text{There are at most two meat dishes at the party}) \tag{1}$$

because each of the guys is bringing only one dish to the party, and Ben does not cook beef in all admissible worlds.

1.2 Normative systems

Note that the deontic logic approach does not have actions in its semantics. Thus, it cannot be easily used to express what different agents have an *ability* to achieve under the existing social norms. One of the possible ways to model this is to endow agents with sets of possible actions and label some of them as allowed and others as disallowed. To model our running example, suppose that, in the initial state, each of the three agents (Ben, Charles, and David) has three possible actions to choose from. For instance, Ben's actions are "cook bruschetta", "cook beef", and "cook blueberry pie". Based on the actions taken by the three agents, the system transitions from the initial state to one out of 27 final states, representing different combinations of foods brought to the party. Under this approach, the actions, not states, are labelled as allowed and disallowed. In our example, Ben's action "cook beef" is disallowed, and all other actions of all agents are allowed. We refer to such models as normative transition systems or simply *normative systems*.

The semantics of the "ought to" modality O can be defined straightforwardly for normative systems: $\mathsf{O}\varphi$ means that formula φ is guaranteed to be true in the next state as long as *all* agents use allowed actions. Under such semantics, statement (1) is satisfied in the initial state of our "potluck party" normative system.

Wooldridge and van der Hoek proposed a *normative ability* modality that states that agent a has an allowed action that guarantees that formula φ will be true in the next state (Wooldridge & Hoek, 2005). In the current article, we denote this modality by A. For instance, in our running example, David has an allowed action ("cook duck") that guarantees that there will be at least one meat dish at the party:

$$\mathsf{A}_d(\text{There is at least one meat dishes at the party})$$

At the same time, Ben does not have a normative ability to guarantee the same:

$$\neg\mathsf{A}_b(\text{There is at least one meat dishes at the party})$$

because Ben is not allowed to cook beef. Note that modality $\mathsf{A}_a\varphi$ requires the allowed action of agent a to guarantee φ even if *the other agents use disallowed actions*. For example,

$$\neg\mathsf{A}_d(\text{There is at most one meat dishes at the party})$$

because even if David does not cook duck, there is a chance that Ben uses the disallowed action "cook beef". In combination with Charles' allowed action "cook chicken", this would bring the system to a state with two meat dishes at the party.

Logical systems for normative abilities have been widely studied in the literature (Der Hoek et al., 2007; Ågotnes et al., 2009–Herzig et al., 2011); see (Alechina et al., 2018) for an overview of this area. Normative ability is a special case of strategic abilities. The latter have also been studied in Coalition Logic (Pauly, 2002), ATL (Alur et al., 2002), STIT Logic (Horty, 2001, Belnap & Perloff, 1990; Horty & Belnap, 1995; Horty & Pacuit, 2017; Olkhovikov & Wansing, 2019), and Strategy Logic (Chatterjee et al., 2010). Modality $\mathsf{A}_a\varphi$ can also be viewed as a *permission* modality because it expresses the fact that agent a is permitted to use at least one action that guarantees φ. This type of permission modality is often referred to as *weak* permission modality because the Monotonicity inference rule:

$$\frac{\varphi \to \psi}{\mathsf{A}_a\varphi \to \mathsf{A}_a\psi}$$

is valid for this modality (Benthem, 1979). The modalities of the other, *strong* permission type (Benthem, 1979; Asher & Bonevac, 2005), satisfy the Antimonotonicity rule:

$$\frac{\varphi \to \psi}{\Box_a\psi \to \Box_a\varphi}.$$

Shi (2024) gave a sound and complete axiomatization of the interplay between modality A and three other permission modalities.

1.3 Epistemic normative systems

In this article, we consider normative systems with imperfect information in which the agents might not be able to distinguish some of the states. We refer to this more general class of models as *epistemic normative systems*. For instance, suppose that our three guys do not know each other well enough to know which of them is a vegetarian and which is not. Of course, we assume each of them knows this about himself. We model such a situation by 8 “initial” states that represent all possible combinations of who is vegetarian and who is not.

The set of allowed actions changes from one state of an epistemic normative system to another. For example, Ben is allowed to cook beef in any state where at least one of the two other guys is a non-vegetarian, and he is disallowed to cook beef in the current state where Charles and David are vegetarian. In this setting, we interpret $\mathsf{O}\varphi$, as “*formula φ is guaranteed to be true in the next state if all agents* ***knowingly*** *take an allowed action*”.

First, let us assume that there is no *ex ante* (before the actions are taken) communication between Ben, Charles, and David. In this case, David and Charles would not be able to learn that Ben is not a vegetarian. Although both of them are *allowed* by the social norms to bring meat dishes to the party, neither of them would know this. Of course, Ben is not even allowed to bring meat to the party because Charles and David are vegetarians. Hence, if all agents only use actions that they know are allowed, there will be no meat dishes at the party:

$$\mathsf{O}(\text{There are no meat dishes at the party}). \tag{2}$$

1.4 Information walls

Our goal is to study how communication between agents interplays with their obligations. We capture the above setting in which there is no ex-ante communication between all three agents by the assumption that there are *information walls* that completely isolate Ben, Charles, and David. Formally, we model systems of information walls by a partition of the set of agents. Informally, the agents that belong to the same

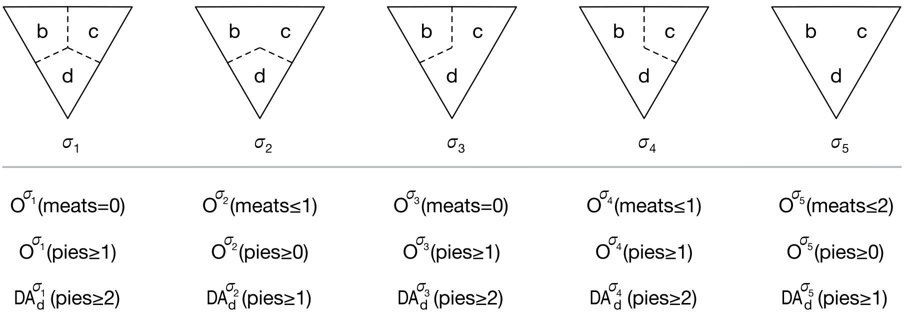

Fig. 2 Five partitions

set of the partition *might* but do not have to share their knowledge[1]. The partition for the setting where there is no ex-ante communication between all three of them is shown as partition σ_1 in Fig. 2. To be able to discuss the properties of different partitions, we add the partition as the superscript of the "ought to" modality O. For example, we will write formula (2) as

$$\mathsf{O}^{\sigma_1}(\text{There are no meat dishes at the party}).$$

Let us now consider the partition σ_2 shown in the same Fig. 2. This partition allows communication between Ben and Charles but not between either of them and David. By talking to Ben, Charles *might* learn that Ben is not a vegetarian. In this case, he will know that he is allowed by the social norms to bring a meat dish to the party. At the same time, due to the existing information wall that separates David from Ben and Charles, David will never learn that Ben is not a vegetarian. Thus, although David is allowed to bring meat according to social norms, he will never learn about this. As a result, David will not bring a meat dish to the party. Of course, Ben is not allowed to bring meat to the party because he is the only non-vegetarian there. Assuming that all communication between the agents is truthful, the conversation between Ben and Charles will not lead to Ben's knowledge of something which is not true. As a result, only Charles *might* potentially bring a meat dish to the party:

$$\mathsf{O}^{\sigma_2}(\text{There is at most one meat dish at the party}).$$

Under partition σ_3, see Fig. 2, Charles and David are guaranteed not to learn that Ben is not a vegetarian. As a result, just like in the case of the partition σ_1, neither of the guys will bring a meat dish:

$$\mathsf{O}^{\sigma_3}(\text{There are no meat dishes at the party}).$$

The case of partition σ_4 is similar to the case of σ_2. More interestingly, under partition σ_5 that allows full ex-ante communication between all three agents, Charles and

[1] A similar partition-based approach to modeling restrictions on communication between agents has been used, for example, in (Lee et al., 2025).

David *might* learn that Ben is not a vegetarian. As a result, potentially, both of them might bring meat dishes. Of course, Ben still will not be able to bring a meat dish:

$$\mathsf{O}^{\sigma_5}(\text{There is at most two meat dishes at the party}).$$

As an additional example, notice that

$$\mathsf{O}^{\sigma_1}(\text{There is at least one pie at the party}). \tag{3}$$

The last statement is true because, under partition σ_1, Charles never learns that Ben is not a vegetarian. As a result, if he only uses actions that he knows are allowed, Charles is guaranteed to bring either a cranberry or a coconut pie to the party. Figure 2 shows what ought to be true about the number of pies under the other four partitions.

1.5 Normative abilities in epistemic setting

One might potentially generalize the normative ability modality $\mathsf{A}_a\varphi$ from normative systems to epistemic normative systems. To do this, it will be convenient to use the ***knowingly allowed*** action of an agent a. By such an action, we mean any action of agent a that is known to agent a to be allowed. Then, modality $\mathsf{A}_a\varphi$ could be interpreted in epistemic normative systems as "agent a has a knowingly allowed action that guarantees φ". In such a modality, knowledge can be considered *ex-ante* (before any possible communication between the agents from the same set of the partition) or *ex-post* (after possible communication). For example,

$$\mathsf{A}_{d,\text{ex-ante}}(\text{There is at least one pie at the party}).$$

because David knows *ex-ante* that he is allowed to bring a date pie to the party. By doing this, he guarantees that there is at least one pie at the party. This result is guaranteed to take place no matter whether the other agents use knowingly allowed actions or not. Note that the semantics of such a modality does not depend on the information walls. As a result, it might be worth considering, but not in the context of the current article.

If one considers ex post knowledge, then the modality $\mathsf{A}_a\varphi$ would describe not an actual ability but a possible ability that an agent *might* acquire as a result of some particular information exchange between the agent in the same set of the partition. For example, under partition σ_4, David *might* learn from Ben that Ben is not a vegetarian. By learning this, David *would* acquire the knowledge that bringing a duck is one of his allowed actions. Hence, if such a communication between Ben and David takes place, then, after that communication, David would have a knowingly allowed action that guarantees that there is at least one meat dish at the party:

$$\mathsf{A}^{\sigma_4}_{d,\text{ex-post}}(\text{There is at least one meat dish at the party}).$$

The same is true for partition σ_5, where the communication between Ben and David can also take place. At the same time, a similar statement is not true. For example, under partition σ_3,

$$\neg\mathsf{A}^{\sigma_3}_{d,\text{ex-post}}(\text{There is at least one meat dish at the party}).$$

because within the information walls defined by partition σ_3, David would *never* learn that bringing a duck to the party is an allowed action. Note that just like in the case of *ex-ante* form of this modality, *ex-post* does not restrict the other agents to only knowingly allowed actions. Unlike the *ex-ante* form, the semantics of the ex-post version of modality A takes into account the structure of the information walls. However, the ex-post form of the modality captures only a *potential* abilities that an agent might never acquire. It captures more of wishful thinking than actual ability.

1.6 Our contribution: norm-dependent abilities

1.6.1 Main idea

In this article, we propose a new concept of *norm-dependent ability*. An agent has such an ability if the agent has a knowingly allowed action that guarantees the result as long as **all other agents also use only knowingly allowed actions**. Note that the bolded part was not present in the definitions of the modalities discussed in the previous subsection. This part makes the definition of the ability consistent with how we have defined the semantics of modality O in epistemic normative systems.

Going back to our example in Fig. 2, note that no matter what information exchange takes place under partition σ_1, Charles will never learn that Ben is not a vegetarian. Hence, Charles will never know that bringing the chicken is one of his allowed actions. Thus, if Charles only uses knowingly allowed actions, he will have to bring a pie to the party. This guarantees that, as we have already observed in statement (3), if everyone uses only knowingly allowed actions, then there will be at least one pie (either cranberry or coconut) at the party. Next, note that bringing a date pie is one of David's knowingly allowed actions. Hence, David has a norm-dependent ability to guarantee that there will be at least *two* pies at the party! We write this as

$$\mathsf{DA}^{\sigma_1}_d(\text{There is at least two pies at the party}). \tag{4}$$

We refer to such an ability as "norm-dependent" because it is conditional on the fact that all other agents only bring knowingly allowed food to the party.

At the same time, under partition σ_2, it ***might*** happen that Ben and Charles communicate. As a result of such communication, Charles *might* learn that Ben eats meat. This would make bringing chicken a knowingly allowed action of Charles. As a result, David has only a norm-dependent ability to guarantee that there will be at least *one* pie at the party:

$$\mathsf{DA}^{\sigma_2}_d(\text{There is at least one pie at the party}). \tag{5}$$

Comparing statements (4) and (5), we can observe an interesting property of norm-dependent strategies: *addition of information walls might increase agents' norm-dependent abilities*. This observation distinguishes norm-dependent abilities from most other types of abilities in imperfect information (Wang, 2018; Li & Wang, 2021; Berthon et al., 2017; 2017; Naumov & Tao, 2017; Ågotnes, 2006), where an exchange of information between agents sometimes increases but never decreases an agent's ability. In Fig. 2, we show the equivalents of statements (4) and (5) for the other three partitions.

1.6.2 Hold yourself to a higher standard

As discussed in Sect. 1.5, there are two possible ways to interpret the term "knowingly allowed action" in our setting. Under *ex-ante* interpretation, it refers to the ex-ante (before communication) knowledge of the acting agent. Under *ex-post* interpretation, it refers to hypothetical knowledge that the acting agent *might* acquire as a result of the communication with the agent from the same set in the partition. Generally speaking, there are fewer "ex-ante knowingly allowed" actions than "ex-post knowingly allowed" actions. The former actions are available to the agent upfront. The latter *might* become available after communication. We believe that the ability should not rely on luck and should guarantee success even if your opponents are lucky. Thus, in the context of modality DA_a, we assume that the actions of agent a are "ex-ante knowingly allowed" while the actions of the other agents are "ex-post knowingly allowed". For example,

$$\neg\mathsf{DA}_d^{\sigma_4}(\text{There is at least one meat dish at the party})$$

in spite of the fact that, under partition σ_4, David might *ex post* learn that Ben is a vegetarian and bring duck to the party. At the same time,

$$\neg\mathsf{DA}_d^{\sigma_2}(\text{There are no meat dishes at the party})$$

because it is not enough for David just to decide to bring date pie (or daikon salad) to the gathering. Indeed, under σ_2, Charles might *ex-post* learn that Ben is not a vegetarian and bring chicken.

1.6.3 The right to remain silent

Although we allow agents to communicate with the other agents in the same partition set, we do not require them to do so. In particular, we allow them to decide not to communicate certain information as a part of their strategy. For example, under any of the five partitions σ,

$$\mathsf{DA}_b^{\sigma}(\text{There are no meat dishes at the party})$$

because Ben can always choose not to tell anyone that he is not a vegetarian. In fact, it is easy to see that in our setting, an agent can never achieve more by "being talkative" (revealing additional information to others). Revealing such information may only make the opponents more powerful by giving them additional *ex-post* knowingly allowed actions. This is the reason why lawyers usually advise their clients to remain silent or to disclose as little information as legally necessary.

1.6.4 Contribution and outline

In this article, we propose a sound and complete logical system that describes the interplay between norm-dependent ability modality DA_a^σ and "ought to" modality O^σ. Our main technical result is the completeness of the proposed logical system. The proof of the completeness uses two key ideas: σ-harmony and distributed key generation. While σ-harmony is a variation of the technique from (Naumov & Tao, 2017, 2018a), distributed key generation is a novel technique that we propose. We are not aware of this technique being used in completeness proofs before, although it is well-known in cryptography (Pedersen, 1991). The preliminary version of this work, without the full proof of the completeness, appeared as (Naumov & Zhang, 2022).

This article is organized as follows. The next section defines the syntax and the semantics of our logical system. Then, we list and discuss its axioms and inference rules. The soundness of these axioms is shown in Sect. 4. The next section presents our main result, the completeness of the system. The last section concludes.

2 Syntax and semantics

In this article, we assume a fixed set of propositional variables and a set $\mathcal{A}$ of agents. A partition of set $\mathcal{A}$ is any family of pairwise disjoint nonempty sets whose union is equal to $\mathcal{A}$. For any agent $a \in \mathcal{A}$ and any partition σ of set $\mathcal{A}$, let $[a]_\sigma$ be the unique set in partition σ that contains a. In Fig. 2, for instance, $[b]_{\sigma_2} = \{b, c\}$.

By σ/a, we mean a modification of partition σ in which set $[a]_\sigma$ is replaced by two sets: $\{a\}$ and $[a]_\sigma \setminus \{a\}$. If $[a]_\sigma = \{a\}$, then σ/a, by definition, is σ. For example,

$$\sigma_2/d = \{\{b, c\}, \{d\}\} = \sigma_2,$$
$$\sigma_3/d = \{\{b\}, \{c\}, \{d\}\} = \sigma_1,$$
$$\sigma_5/d = \{\{b, c\}, \{d\}\} = \sigma_2.$$

Definition 1 *For any two partitions σ and τ of set $\mathcal{A}$, let $\sigma \preceq \tau$ if $[a]_\sigma \subseteq [a]_\tau$ for each agent $a \in \mathcal{A}$. If $\sigma \preceq \tau$, then partition σ is "finer" than partition τ.*

The language Φ of our logical system is defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \to \varphi \mid \mathsf{K}_a\varphi \mid \mathsf{O}^\sigma\varphi \mid \mathsf{DA}_a^\sigma\varphi,$$

where p is a propositional variable, $a \in \mathcal{A}$ is an agent, and σ is a partition of the set of agents. We read $\mathsf{K}_a\varphi$ as "agent a knows φ", $\mathsf{DA}_a^\sigma\varphi$ as "agent a has a norm-dependent ability to achieve φ in the presence of the information walls defined by partition σ", and $\mathsf{O}^{\sigma}\varphi$ as "it ought to be φ in the presence of the information walls defined by partition σ". We assume that the conjunction $\wedge$ and the Boolean constant true $\top$ are defined through $\rightarrow$ and $\neg$ in the standard way. For any finite set of formulae Y, by $\wedge Y$ we mean the conjunction of all formulae in Y. Formula $\wedge\varnothing$, by definition, is $\top$.

Definition 2 *A normative system is a tuple* $(W, \sim, \Delta, S, \ell, M, \pi)$*, where*

1. W *is a (possibly empty) set of* ***states****,*
2. $\sim_a$ *is an* ***indistinguishability*** *equivalence relation on the set of states* W*, for each agent* $a \in \mathcal{A}$*,*
3. Δ_a^w *is a set of all actions (allowed and disallowed) of agent* a *in state* w*; by an* ***action profile*** δ *at state* w*, we mean any function that maps each agent* $a \in \mathcal{A}$ *into an action* $\delta(a) \in \Delta_a^w$*,*
4. $S_a^w \subseteq \Delta_a^w$ *is a set of* ***allowed*** *(or "safe") actions for agent* $a \in \mathcal{A}$ *in state* $w \in W$*,*
5. $\ell_a^w \in S_a^w$ *is a* ***default*** *allowed action of agent* $a \in \mathcal{A}$ *in state* $w \in W$*; we assume that if* $w \sim_a u$*, then* $\ell_a^w = \ell_a^u$*,*
6. ***mechanism*** M *is an arbitrary set of triples* (w, δ, u)*, where* $w, u \in W$ *are states and* δ *is an action profile at state* w*,*
7. $\pi(p) \subseteq W$ *for each propositional variable* p*.*

In our running example from the introduction, the set W consists of $8 + 27 = 35$ states. Intuitively, eight of them are "initial" states in which the three guys have not made their choices yet. In the initial state, each of the guys can be either vegetarian or non-vegetarian. In the actual state, Charles and David are vegetarians, and Ben is not. Additionally, there are 27 "final states" that represent 27 possible outcomes of the choices made by the three guys. To keep the state count low, we do not encode vegetarian/non-vegetarian status into the final states. Note that in the formal setting of Definition 2, we do not distinguish between the initial and the final states. Thus, we potentially allow the transitions to continue. For the sake of generality, item 1 above allows the set W to be empty.

Indistinguishibility relation $\sim_a$ in item 2 above captures *ex-ante* knowledge of agent a. In other words, it captures the knowledge *before* possible information exchange between the agents. For example, the current state (in which Charles is vegetarian) is in the relation $\sim_{\text{Ben}}$ with a hypothetical possible state where Charles eats meat. Because relation $\sim_a$ captures ex-ante knowledge, the presence (or lack of) information walls does not affect this relation.

Δ_a^w is a set of actions of agent a at state w. For instance, in our introductory example, Ben has the same set of available actions $\Delta_{\text{Ben}}^w = \{\text{bruschetta}, \text{beef}, \text{blueberry pie}\}$ in each of the eight "initial" states. The set S_a^w represents allowed actions of agent a in state w. For example, if w_0 is the "current" initial state in which Charles and David are vegetarians, then

$$S_{\text{Ben}}^{w_0} = \{\text{bruschetta}, \text{blueberry pie}\}.$$

Note that the set S_a^w represents allowed actions. It does **not** represent *knowingly* allowed actions. Thus, for example, for the same "current" initial state w_0,

$$S^{w_0}_{\text{Charles}} = \{\text{cranberry pie}, \text{chicken}, \text{coconut pie}\},$$
$$S^{w_0}_{\text{David}} = \{\text{date pie}, \text{duck}, \text{daikon salad}\}.$$

We assume that in each state, each agent has at least one *knowingly* allowed action. To model this, for each agent a and each state w, we identify a "default" allowed action ℓ^w_a. Item 5 of the above definition guarantees that the action ℓ^w_a is *knowingly* allowed in state w. It achieves this by requiring ℓ^w_a to be the default action in all states that agent a cannot distinguish from state w. The existence of a knowingly allowed action is important for the soundness of the Necessitation inference rule for modality DA. The requirement to have at least one knowingly allowed "default" action is our adaptation to social norms of the "safe harbor" provision in law that stipulates that there should be at least one action for an agent to take without violating the law.

Informally, $(w, \delta, u) \in M$ means that the system can transition from state w to state u under action profile δ. In general, a mechanism is a relation, not a function. Thus, transitions might be non-deterministic. If, for some state $w \in W$ and some action profile δ at state w, there is no state u such that $(w, \delta, u) \in M$, then we say that the system terminates in state w under action profile δ.

Recall that, in our setting, the agents know some information *ex ante* (before the communication), and they potentially might learn additional information as a result of the communication. Thus, generally speaking, each agent starts with some initial set of knowingly allowed actions based on ex-ante knowledge and extends this set during the communication. By KS^w_a, we denote the "initial" set of knowingly allowed ("knowingly safe") actions of agent a in state w. Note that this set does not depend on the choice of the partition because the structure of the information walls only plays its role at the communication stage.

Definition 3 *Let KS^w_a be the set of all actions $s \in \Delta^w_a$ such that $s \in S^{w'}_a$ for each state $w' \in W$ such that $w \sim_a w'$.*

In our example, bringing vegetarian dishes are the agents' only knowingly allowed ex-ante actions:

$$KS^{w_0}_{\text{Ben}} = \{\text{bruschetta}, \text{blueberry pie}\},$$
$$KS^{w_0}_{\text{Charles}} = \{\text{cranberry pie}, \text{chicken}, \text{coconut pie}\},$$
$$KS^{w_0}_{\text{David}} = \{\text{date pie}, \text{duck}, \text{daikon salad}\}.$$

Consider now an arbitrary partition σ of the set of all agents. If agents in the same partition communicate, then they *might* learn additional allowed actions. By DS^w_σ, we denote the set of all action profiles δ such that, for each agent $a \in \mathcal{A}$, the set of agents $[a]_\sigma$ *distributively knows* that action $\delta(a)$ is allowed for agent a in state w. Informally, DS^w_σ is the set of all action profiles δ about which each agent $a \in \mathcal{A}$ *might* learn that action $\delta(a)$ is allowed in state w if the agent communicates with the other agents in the set $[a]_\sigma$.

Definition 4 *DS_σ^w consists of all action profiles δ at state w such that for each agent $a \in \mathcal{A}$ and each state $w' \in W$, if $w \sim_b w'$ for each agent $b \in [a]_\sigma$, then $\delta(a) \in S_a^{w'}$.*

In our example, the set $DS_{\sigma_2}^{w_0}$ contains profile $(\mathrm{bruschetta}, \mathrm{chicken}, \mathrm{date\ pie})$ because, under partition σ_2, Charles might learn that Ben is not vegetarian and, thus, he is allowed to bring chicken. At the same time, the same set $DS_{\sigma_2}^{w_0}$ *does not* contains profile $(\mathrm{bruschetta}, \mathrm{cranberry\ pie}, \mathrm{duck})$, because the information walls under partition σ_2 prevent David from learning that Ben is not a vegetarian.

Lemma 1 *If $\sigma \preceq \tau$, then $DS_\sigma^w \subseteq DS_\tau^w$.* □

Next is the key definition of this article. It gives formal semantics of modalities K, O, and DA.

Definition 5 *For any state $w \in W$ and any formula $\varphi \in \Phi$, satisfiability relation $w \Vdash \varphi$ is defined as follows*

1. *$w \Vdash p$ if $w \in \pi(p)$,*
2. *$w \Vdash \neg\varphi$ if $w \nVdash \varphi$,*
3. *$w \Vdash \varphi \to \psi$ if $w \nVdash \varphi$ or $w \Vdash \psi$,*
4. *$w \Vdash \mathsf{K}_a\varphi$ if $u \Vdash \varphi$ for each $u \in W$ such that $w \sim_a u$,*
5. *$w \Vdash \mathsf{O}^\sigma\varphi$ when for each action profile $\delta \in DS_\sigma^w$ and each state $u \in W$, if $(w, \delta, u) \in M$, then $u \Vdash \varphi$,*
6. *$w \Vdash \mathsf{DA}_a^\sigma\varphi$ when there is an action $s \in KS_a^w$ such that for all states $w', u \in W$ and each action profile $\delta \in DS_{\sigma/a}^{w'}$, if $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$, then $u \Vdash \varphi$.*

In item 5, we write $\delta \in DS_\sigma^w$ because we allow agents to use *ex-post* knowingly allowed actions, see Sect. 1.4.

In item 6, we require that $s \in KS_a^w$ to capture that action s must be *ex-ante* knowingly allowed action of agent a in state w, see Sect. 1.6.2. We use partition σ/a instead of partition σ because agent a can only lose power by communicating with others; see Sect. 1.6.3. We assume that $\delta \in DS_{\sigma/a}^{w'}$ because we allow the opponents of a to use *ex-post* knowingly allowed actions, see Sect. 1.6.2.

3 Axioms

In addition to propositional tautologies in language Φ, our logical system contains the following axioms:[2]

1. Truth: $\mathsf{K}_a\varphi \to \varphi$,
2. Negative Introspection: $\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\varphi$,

[2] Notations $\sigma \preceq \tau$ and σ/a have been introduced at the beginning of the previous section.

3. Distributivity: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ where $\Box \in \{\mathsf{K}_a, \mathsf{O}^\sigma\}$ and
4. Monotonicity: $\mathsf{O}^\tau\varphi \to \mathsf{O}^\sigma\varphi$ where $\sigma \preceq \tau$, $\mathsf{DA}_a^\tau\varphi \to \mathsf{DA}_a^\sigma\varphi$ where $\sigma/a \preceq \tau/a$,
5. Strategic Introspection: $\mathsf{DA}_a^\sigma\varphi \to \mathsf{K}_a\mathsf{DA}_a^\sigma\varphi$,
6. Epistemic Monotonicity: $\mathsf{K}_a\mathsf{O}^{\sigma/a}(\varphi \to \psi) \to (\mathsf{DA}_a^\sigma\varphi \to \mathsf{DA}_a^\sigma\psi)$.

The Truth, the Negative Introspection, and the Distributivity axioms are well-known modal properties. The Monotonicity axiom for modality O captures the fact that if something ought to be true under communication walls imposed by partition τ, then the same is also ought to be true under any partition σ that has additional information walls. A similar property is true for modality DA except that assumption $\sigma \preceq \tau$ is replaced with a weaker assumption $\sigma/a \preceq \tau/a$ because formal semantics of modality DA_a^σ excludes communication between agent a and the other agents in class $[a]_\sigma$. The Strategic Introspection axiom states that if an agent has a norm-dependent ability, then she knows that she has such an ability. The Epistemic Monotonicity axiom states that if agent a knows that $\varphi \to \psi$ ought to be true as long as agent a remains silent and the agent also has a norm-dependent ability to achieve φ, then the agent has a norm-dependent ability to achieve φ. Formally, "agent a remains silent" is captured by using partition σ/a instead of partition σ.

We write $\vdash \varphi$, and say that φ is a *theorem* of our logical system if formula φ is provable from the above axioms using the Modus Ponens, the three forms of the Necessitation, and the Monotonicity inference rules:

$$\frac{\varphi, \varphi \to \psi}{\psi} \qquad \frac{\varphi}{\mathsf{DA}_a^\sigma\varphi} \qquad \frac{\varphi}{\mathsf{K}_a\varphi} \qquad \frac{\varphi}{\mathsf{O}^\sigma\varphi} \qquad \frac{\varphi \to \psi}{\mathsf{DA}_a^\sigma\varphi \to \mathsf{DA}_a^\sigma\psi}.$$

In addition to unary relation $\vdash \varphi$, we also consider binary relation $X \vdash \varphi$ which is true if a formula φ is provable from the *theorems* of our logical system and the set of additional axioms X using only the Modus Ponens inference rule. Note that $\varnothing \vdash \varphi$ is equivalent to $\vdash \varphi$.

4 Soundness

In this section, we show the soundness of our logical system. The soundness of the Truth, the Negative Introspection, and the Distributivity axioms is standard. Below, we prove the soundness of each of the remaining axioms as a separate lemma.

Lemma 2 *If $\sigma \preceq \tau$ and $w \Vdash \mathsf{O}^\tau\varphi$, then $w \Vdash \mathsf{O}^\sigma\varphi$.*

Proof Consider any action profile $\delta \in DS_\sigma^w$ and any state $u \in W$ such that $(w, \delta, u) \in M$. By item 5 of Definition 5, it suffices to show $u \Vdash \varphi$. Indeed, the assumption $\delta \in DS_\sigma^w$ and the assumption $\sigma \preceq \tau$ of the lemma imply $\delta \in DS_\tau^w$ by Lemma 1. Hence, $u \Vdash \varphi$ by the assumption $w \Vdash \mathsf{O}^\sigma\varphi$, item 5 of Definition 5, and the assumption $(w, \delta, u) \in M$. □

Lemma 3 *If $\sigma/a \preceq \tau/a$ and $w \Vdash \mathsf{DA}_a^\tau\varphi$, then $w \Vdash \mathsf{DA}_a^\sigma\varphi$.*

Proof By the assumption $w \Vdash \mathsf{DA}^{\tau}_{a}\varphi$ of the lemma and item 6 of Definition 5, there is an action $s \in KS^{w}_{a}$ such that for all states $w', u \in W$ and each action profile $\delta \in DS^{w'}_{\tau/a}$, if $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$, then $u \Vdash \varphi$.

Consider any states $w', u \in W$ and any action profile $\delta \in DS^{w'}_{\sigma/a}$ such that $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$. By item 6 of Definition 5, it suffices to show that $u \Vdash \varphi$.

Notice that the assumption $\delta \in DS^{w'}_{\sigma/a}$ and the assumption $\sigma/a \preceq \tau/a$ of the lemma imply $\delta \in DS^{w'}_{\tau/a}$ by Lemma 1. Therefore, $u \Vdash \varphi$ by the choice of action s using the assumptions $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$. □

Lemma 4 *If* $w \Vdash \mathsf{DA}^{\sigma}_{a}\varphi$*, then* $w \Vdash \mathsf{K}_a\mathsf{DA}^{\sigma}_{a}\varphi$.

Proof Consider any state $v \in W$ such that $w \sim_a v$. By item 4 of Definition 5, it suffices to show that $v \Vdash \mathsf{DA}^{\sigma}_{a}\varphi$.

By item 6 of Definition 5, the assumption $w \Vdash \mathsf{DA}^{\sigma}_{a}\varphi$ of the lemma implies that there is an action $s \in KS^{w}_{a}$ such that for all states $w', u \in W$ and each action profile $\delta \in DS^{w'}_{\sigma/a}$, if $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$, then $u \Vdash \varphi$.

Then, by the assumption $w \sim_a v$, for all states $w', u \in W$ and each action profile $\delta \in DS^{w'}_{\sigma/a}$, if $\delta(a) = s$, $v \sim_a w'$, and $(w', \delta, u) \in M$, then $u \Vdash \varphi$. Therefore, $v \Vdash \mathsf{DA}^{\sigma}_{a}\varphi$ by item 6 of Definition 5. □

Lemma 5 *If* $w \Vdash \mathsf{K}_a\mathsf{O}^{\sigma/a}(\varphi \to \psi)$ *and* $w \Vdash \mathsf{DA}^{\sigma}_{a}\varphi$*, then* $w \Vdash \mathsf{DA}^{\sigma}_{a}\psi$.

Proof By item 6 of Definition 5, the assumption $w \Vdash \mathsf{DA}^{\sigma}_{a}\varphi$ implies that there is an action $s \in KS^{w}_{a}$ such that for all states $w', u \in W$ and each action profile $\delta \in DS^{w'}_{\sigma/a}$, if $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$, then $u \Vdash \varphi$.

Consider any two states $w', u \in W$ and any action profile $\delta \in DS^{w'}_{\sigma/a}$ where $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$. By item 6 of Definition 5, it suffices to show that $u \Vdash \psi$. Indeed, by item 4 of Definition 5, the assumption $w \sim_a w'$ and the assumption of the lemma $w \Vdash \mathsf{K}_a\mathsf{O}^{\sigma/a}(\varphi \to \psi)$ imply that $w' \Vdash \mathsf{O}^{\sigma/a}(\varphi \to \psi)$. Hence, $u \Vdash \varphi \to \psi$ by item 5 of Definition 5 and the assumptions $\delta \in DS^{w'}_{\sigma/a}$ and $(w, \delta, u) \in M$.

At the same time, $u \Vdash \varphi$ by the choice of action s and because $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$. Therefore, $u \Vdash \psi$ by item 3 of Definition 5. □

5 Completeness

In this section, we prove the completeness of our logical system. We start by reviewing the main ideas of this proof. Then, we discuss the harmony construction used in the proof, define the canonical model, and use it to prove completeness.

5.1 Key ideas behind the proof of the completeness

5.1.1 Distributed key generation

The standard proof of completeness for the multiagent version of epistemic logic S5 defines the states of the canonical model as maximal consistent sets of formulae. Two such states are a-indistinguishable if they contain the same K_a-formulae. This construction does not work in our case because we allow formulae that simultaneously use modality DA^σ_a for different partitions σ. Indeed, recall the agents Ben, Charles, and David from one of the introductory examples. Consider any maximal consistent set of formulae w that contains exactly the same K_{Ben}- and $\mathsf{K}_{\text{Charles}}$-formulae. In other words, for any formula $\varphi \in \Phi$,

$$\mathsf{K}_{\text{Ben}}\varphi \in w \quad \text{iff} \quad \mathsf{K}_{\text{Charles}}\varphi \in w.$$

If the indistinguishability relation is defined as in the standard construction, then the equivalence classes of state w with respect to relation $\sim_{\text{Ben}}$ and relation $\sim_{\text{Charles}}$ would be the same. Thus, if the canonical model is defined in the standard way, then agents Ben and Charles will have exactly the same knowledge in state w.

Next, suppose that set w contains formulae $\mathsf{DA}^{\sigma_1}_{\text{David}}\varphi$ and $\neg\mathsf{DA}^{\sigma_2}_{\text{David}}\varphi$ for some formula $\varphi \in \Phi$, where partitions σ_1 and σ_2 are specified in Fig. 2. The key step in the standard proof of completeness is the "truth" (or "induction") lemma that states that a formula belongs to set w if and only if it is satisfied in state w. In our case, this lemma would imply that $w \Vdash \mathsf{DA}^{\sigma_1}_{\text{David}}\varphi$ and $w \Vdash \neg\mathsf{DA}^{\sigma_2}_{\text{David}}\varphi$.

The statements $w \Vdash \mathsf{DA}^{\sigma_1}_{\text{David}}\varphi$ and $w \Vdash \neg\mathsf{DA}^{\sigma_2}_{\text{David}}\varphi$ mean that David has a norm-dependent ability to achieve φ when the wall between Ben and Charles is present and does not have such a strategy otherwise. Informally, this should happen because, in the absence of the wall, information can freely travel between Ben and Charles, and thus, they both have larger sets of knowingly allowed actions. If they use strategies from these larger sets, then David's strategy might no longer work. However, as we have seen above, if the standard construction is used to build the canonical model, then Ben and Charles have exactly the same knowledge, and, thus, there is absolutely nothing new that they can learn by sharing information with each other!

To overcome this issue, we need Ben and Charles to possess some additional knowledge that they do not have under the standard canonical model construction. We add this knowledge to our canonical model using the *distributed key generation*. This is a cryptographic technique consisting of an independent generation of random keys by several agents (Pedersen, 1991). In our running example, each state of the canonical model will be a quadruple (X, b, c, d), where X is a maximal consistent set of formulae and b, c, d are integer "keys" of Ben, Charles, and David respectively. We assume that each agent knows his key, but not the keys of the other agents. The complete infinite set of states consists of all quadruples (X, b, c, d) for all possible maximal consistent set X and all integer values b, c, and d. Incorporation of the distributed key generation into epistemic model construction is a new idea that we introduce in this article.

5.1.2 Harmony

As mentioned earlier, the proofs of completeness usually use a "truth" or "induction" lemma that states that $\varphi \in w$ if and only if $w \Vdash \varphi$ for any formula φ and any state w. In our case, this is Lemma 15. It claims that $\varphi \in X_w$ iff $w \Vdash \varphi$, where X_w is the first component of state w, as discussed in the previous section.

Consider now the case when formula φ has the form $\mathsf{K}_a\psi$. If $\mathsf{K}_a\psi \notin X_w$, then, by item 4 of Definition 5, the canonical model construction must guarantee that $w \nVdash \mathsf{K}_a\psi$. As usual, we achieve this by using Lindenbaum's lemma to construct a new state u such that $w \sim_a u$ and $u \nVdash \psi$.

The situation is more complicated if formula φ has the form $\mathsf{DA}^\sigma_a\psi$. In this case, the canonical model must contain *two* different states, w' and u, satisfying conditions stated in item 6 of Definition 5. An important step in creating these two states is the construction of the corresponding maximal consistent sets $X_{w'}$ and X_u. It turns out that these two sets cannot be created consecutively.

Naumov and Tao (2018a, 2018b) proposed a technique called *harmony* for a simultaneous construction of two maximal consistent sets. Their technique cannot be directly applied in our setting because the original harmony was not designed to deal with information walls in the set of agents. In this article, we propose a variation of their technique that we call σ-harmony.

The technique consists of identifying a certain invariant condition on a pair of sets of formulae, proving that an "initial" pair of sets satisfies this condition, and showing that the sets could be expanded while preserving the invariant. We call the invariant condition σ-harmony, just like the technique itself. The expansion step is repeated infinitely many times to achieve another condition, which we call *complete* σ-harmony. As a final step, Lindenbaum's lemma is used to "top-off" the two sets in complete σ-harmony to maximal consistent sets.

5.2 σ-Harmony

In this section, we define the σ-harmony relation between sets of formulae and prove several properties of this relation. We will use these results in the proof of completeness.

Definition 6 *An arbitrary pair of sets of formulae* (X, Y) *is in* σ*-harmony if* $X \nvdash \mathsf{O}^\sigma\neg\bigwedge Y'$ *for each finite set* $Y' \subseteq Y$.

Lemma 6 *If pair* (X, Y) *is in* σ*-harmony, then sets* X *and* Y *are consistent.*

Proof Assume set X is inconsistent. Thus, $X \vdash \mathsf{O}^\sigma \neg \wedge \varnothing$. Then, by Definition 6, pair (X, Y) is not in σ-harmony.

Suppose that set Y is inconsistent. Thus, there is a finite set $Y' \subseteq Y$ such that $\vdash \neg \wedge Y'$. Hence, by the Necessitation inference rule, $\vdash \mathsf{O}^\sigma \neg \wedge Y'$. Therefore, by Definition 6, pair (X, Y) is not in σ-harmony.

The next lemma shows that two specific "initial" sets are in harmony.

Lemma 7 *Pair of sets $(\{\psi \mid \mathsf{K}_a\psi \in Z\}, \{\neg\varphi, \varphi'\})$ is in (σ/a)-harmony for any consistent set of formulae Z and any formulae $\neg\mathsf{DA}^\sigma_a\varphi, \mathsf{DA}^\sigma_a\varphi' \in Z$.*

Proof Suppose the opposite. Thus, by Definition 6, there are formulae

$$\mathsf{K}_a\psi_1, \dots, \mathsf{K}_a\psi_n \in Z \tag{6}$$

and a finite set of formulae $Y \subseteq \{\neg\varphi, \varphi'\}$ such that

$$\psi_1, \dots, \psi_n \vdash \mathsf{O}^{\sigma/a} \neg \wedge Y. \tag{7}$$

At the same time, note that the formula $\neg \wedge Y \to \neg(\varphi' \wedge \neg\varphi)$ is a propositional tautology because $Y \subseteq \{\neg\varphi, \varphi'\}$. Thus, formula $\neg \wedge Y \to (\varphi' \to \varphi)$ is also a propositional tautology. Hence, by the Necessitation inference rule, $\vdash \mathsf{O}^{\sigma/a}(\neg \wedge Y \to (\varphi' \to \varphi))$. Then,

$$\psi_1, \dots, \psi_n \vdash \mathsf{O}^{\sigma/a}(\varphi' \to \varphi)$$

by the Distributivity axiom and the Modus Ponens inference rule using statement (7). Thus,

$$\mathsf{K}_a\psi_1, \dots, \mathsf{K}_a\psi_n \vdash \mathsf{K}_a\mathsf{O}^{\sigma/a}(\varphi' \to \varphi)$$

by Lemma 17. Hence,

$$Z \vdash \mathsf{K}_a\mathsf{O}^{\sigma/a}(\varphi' \to \varphi)$$

because of statement (6). Then, $Z \vdash \mathsf{DA}^\sigma_a\varphi' \to \mathsf{DA}^\sigma_a\varphi$ by the Epistemic Monotonicity axiom and the Modus Ponens inference rule. Hence, $Z \vdash \mathsf{DA}^\sigma_a\varphi$ by the Modus Ponens inference rule and the assumption $\mathsf{DA}^\sigma_a\varphi' \in Z$ of the lemma. Therefore, $\neg\mathsf{DA}^\sigma_a\varphi \notin Z$ because set Z is consistent, which contradicts the assumption $\neg\mathsf{DA}^\sigma_a\varphi \in Z$ of the lemma. □

The next lemma shows that any two sets in σ-harmony could be further extended while preserving σ-harmony.

Lemma 8 *For any pair (X, Y) in σ-harmony, any formulae $\psi \in \Phi$, either pair $(X \cup \{\neg\mathsf{O}^\sigma\psi\}, Y)$ or pair $(X, Y \cup \{\psi\})$ is in σ-harmony.*

Proof Suppose that both, pair $(X \cup \{\neg \mathsf{O}^\sigma \varphi\}, Y)$ and pair $(X, Y \cup \{\varphi\})$, are not in σ -harmony. Thus, by Definition 6, there are finite sets $Y', Y'' \subseteq Y$ such that

$$X, \neg \mathsf{O}^\sigma \varphi \vdash \mathsf{O}^\sigma \neg \wedge Y' \tag{8}$$

and, for some $Z \subseteq \{\varphi\} \cup Y''$,

$$X \vdash \mathsf{O}^\sigma \neg \wedge Z. \tag{9}$$

Observe that $Z \subseteq \{\varphi\} \cup Y'' \subseteq \{\varphi\} \cup Y' \cup Y''$. Thus, the formula

$$\neg \wedge Z \to (\varphi \to \neg \wedge (Y' \cup Y''))$$

is a tautology. Hence, by the Necessitation inference rule,

$$\vdash \mathsf{O}^\sigma(\neg \wedge Z \to (\varphi \to \neg \wedge (Y' \cup Y''))).$$

Thus, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \mathsf{O}^\sigma \neg \wedge Z \to \mathsf{O}^\sigma(\varphi \to \neg \wedge (Y' \cup Y'')).$$

Then, by the Modus Ponens inference rule and assumption (9),

$$X \vdash \mathsf{O}^\sigma(\varphi \to \neg \wedge (Y' \cup Y'')).$$

Hence, by the Distributivity axiom and the Modus Ponens,

$$X \vdash \mathsf{O}^\sigma \varphi \to \mathsf{O}^\sigma \neg \wedge (Y' \cup Y'').$$

Thus, again by the Modus Ponens inference rule,

$$X, \mathsf{O}^\sigma \varphi \vdash \mathsf{O}^\sigma \neg \wedge (Y' \cup Y''). \tag{10}$$

At the same time, formulae $\neg \wedge Y' \to \neg \wedge (Y' \cup Y'')$ is also a tautology. Then, by the Necessitation inference rule

$$\vdash \mathsf{O}^\sigma(\neg \wedge Y' \to \neg \wedge (Y' \cup Y'')).$$

Hence, by the Distributivity axiom and the Modus Ponens,

$$\vdash \mathsf{O}^\sigma \neg \wedge Y' \to \mathsf{O}^\sigma \neg \wedge (Y' \cup Y'').$$

Thus, using statement (8) and the Modus Ponens rule,

$$X, \neg \mathsf{O}^\sigma \varphi \vdash \mathsf{O}^\sigma \neg \wedge (Y' \cup Y'').$$

Then, $X \vdash \mathsf{O}^\sigma \neg \wedge (Y' \cup Y'')$ by the laws of propositional reasoning using statement (10). Hence, pair (X, Y) is not in σ-harmony by Definition 6. □

Definition 7 *Pair (X, Y) is in complete σ-harmony if, for any formula $\varphi \in \Phi$, either $\neg\mathsf{O}^\sigma\varphi \in X$ or $\varphi \in Y$.*

Lemma 9 *For any pair (X, Y) in σ-harmony, there is a pair (X', Y') in complete σ-harmony where $X \subseteq X'$ and $Y \subseteq Y'$.*

Proof Consider any enumeration $\varphi_1, \varphi_2, \dots$ of all formulae in language Φ. For each integer $i \geq 1$ either add formula $\neg\mathsf{O}^\sigma\varphi_i$ to the first set of the pair in σ-harmony or add formula φ to the second set of the pair in σ-harmony. By Lemma 8, this could be done while maintaining σ-harmony of the pair. Let (X', Y') be the pair obtained after repeating this step for each integer $i \geq 1$. □

5.3 Canonical model

In this section, we define a canonical normative system $(W, \sim, \Delta, S, \ell, M, \pi)$ for any maximal consistent set of formulae X_0.

In our informal discussion of the canonical model, we stated that each state of the model is a tuple containing a maximal consistent set of formulae and a set of integer values representing "keys" of the agents. In our formal definition of the states below, the set of all keys is represented by a function from agents into integers.

Definition 8 *Set W consists of all pairs (X, k) such that X is a maximal consistent subset of Φ and $k \in \mathbb{Z}^{\mathcal{A}}$.*

For any $w = (X, k)$, let $X_w = X$ and $k_w = k$.

Recall from the informal discussion in the Distributed Key Generation subsection that each agent knows her own key, but not the keys of the other agents. Thus, for two states to be indistinguishable by an agent a, the states must have maximal consistent sets with the same K_a-formulae and the same key assigned to agent a.

Definition 9 *For any states $w, u \in W$, let $w \sim_a u$ when*

1. *for each formula $\varphi \in \Phi$, if $\mathsf{K}_a\varphi \in X_w$, then $\varphi \in X_u$,*
2. $k_w(a) = k_u(a)$.

Item 1 of the above definition is equivalent to the statement that sets X_w and X_u have the same K_a-formulae. We use item 1 because it results in shorter proofs of several auxiliary lemmas. Unfortunately, it also requires us to include the following lemma.

Lemma 10 *Relation $\sim_a$ is an equivalence relation on set W.*

Proof **Reflexivity:** Consider any formula $\varphi \in \Phi$. Suppose that $\mathsf{K}_a\varphi \in X_w$. It suffices to show that $\varphi \in X_w$. Indeed, assumption $\mathsf{K}_a\varphi \in X_w$ implies $X_w \vdash \varphi$ by the Truth axiom and the Modus Ponens inference rule. Therefore, $\varphi \in X_w$ because set X_w is maximal.

Symmetry: Consider any states $w, u \in W$ such that $w \sim_a u$ and any formula $\mathsf{K}_a\varphi \in X_u$. It suffices to show $\varphi \in X_w$. Suppose that $\varphi \notin X_w$. Hence, $X_w \nvdash \varphi$ because set X_w is maximal. Thus, $X_w \nvdash \mathsf{K}_a\varphi$ by the contraposition of the Truth axiom. Then, $\neg\mathsf{K}_a\varphi \in X_w$ because set X_w is maximal. Thus, $X_w \vdash \mathsf{K}_a\neg\mathsf{K}_a\varphi$ by the Negative Introspection axiom and the Modus Ponens inference rule. Hence, $\mathsf{K}_a\neg\mathsf{K}_a\varphi \in X_w$ because set X_w is maximal. Then, $\neg\mathsf{K}_a\varphi \in X_u$ by assumption $w \sim_a u$ and Definition 9. Therefore, $\mathsf{K}_a\varphi \notin X_u$ because set X_w is consistent, which contradicts the assumption $\mathsf{K}_a\varphi \in X_u$.

Transitivity: Consider any states $w, u, v \in W$ such that $w \sim_a u$ and $u \sim_a v$ and any formula $\mathsf{K}_a\varphi \in X_w$. It suffices to show $\varphi \in X_v$. Assumption $\mathsf{K}_a\varphi \in X_w$ implies $X_w \vdash \mathsf{K}_a\mathsf{K}_a\varphi$ by Lemma 18 and the Modus Ponens inference rule. Thus, $\mathsf{K}_a\mathsf{K}_a\varphi \in X_w$ because set X_w is maximal. Hence, $\mathsf{K}_a\varphi \in X_u$ by the assumption $w \sim_a u$ and Definition 9. Therefore, $\varphi \in X_v$ by the assumption $u \sim_a v$ and Definition 9.

Definition 10 *For any state $w \in W$ and any agent $a \in \mathcal{A}$, let Δ_a^w be the set consisting of all pairs (φ, C, β) such that $\varphi \in \Phi$ is a formula, $C \subseteq \mathcal{A}$, and $\beta \in \mathbb{Z}$.*

Note that, in our canonical normative system, Δ_a^w does not depend on w and a. Thus, all agents have the same set of actions in all states. Informally, each agent's action consists of specifying a formula φ about the outcome that the agent wants to achieve and the parity β of the sum of keys of members of some group (coalition) C. To be allowed, the action should specify the parity correctly.

Definition 11 *S_a^w is the set of all tuples $(\varphi, C, \beta) \in \Delta_a^w$ such that $\sum_{a \in C} k_w(a) \equiv \beta \pmod{2}$.*

Let $\ell = (\top, \varnothing, 0)$. Then, $\ell \in S_a^w$ for each state $w \in W$ and each agent $a \in \mathcal{A}$ by Definition 11.

The next definition specifies the mechanism of the canonical normative system. Informally, under action profile δ, the system might transition from state u to state v if two conditions are satisfied. To understand these conditions, recall that $\delta \in DS_\sigma^w$ means that each agent a under action profile δ has chosen an action which she *might* learn is allowed in spite of the information walls defined by partition σ. Condition $\delta \in DS_{\sigma/a}^w$ takes into account an additional information wall between agent a and the rest of the agents in the set $[a]_\sigma$.

Definition 12 *For any two states $w, u \in W$ and any action profile δ at state w, let $(w, \delta, u) \in M$ when*

1. *if $\mathsf{O}^\sigma\varphi \in X_w$ and $\delta \in DS_\sigma^w$, then $\varphi \in X_u$.*
2. *if $\mathsf{DA}_a^\sigma\varphi \in X_w$, $\delta \in DS_{\sigma/a}^w$, and $\delta(a) = (\varphi, \{a\}, k_w(a))$, then $\varphi \in X_u$,*

Definition 13 $\pi(p) = \{w \in W \mid p \in X_w\}$.

5.4 The proof

In this section, we prove the strong completeness of our logical system using the σ -harmony construction. As discussed earlier, the key step in the proof of completeness is an "induction" or a "truth" lemma. In our case, this is Lemma 15, which states that $\psi \in X_w$ iff $w \Vdash \psi$. The next four lemmas prove auxiliary statements used in different induction cases of the proof of Lemma 15. The first of them is used in direction $(\Leftarrow)$ when formula ψ has the form $\mathsf{K}_a\varphi$.

Lemma 11 *For any $w \in W$ and any formula $\mathsf{K}_a\varphi \notin X_w$, there exists $w' \in W$ such that $w \sim_a w'$ and $\varphi \notin X_{w'}$.*

Proof Consider set $\widehat{X} = \{\neg\varphi\} \cup \{\psi \mid \mathsf{K}_a\psi \in X_w\}$. First, we show that this set is consistent. Assume the opposite. Then, there are formulae

$$\mathsf{K}_a\psi_1, \dots, \mathsf{K}_a\psi_n \in X_w \tag{11}$$

such that $\psi_1, \dots, \psi_n \vdash \varphi$. Thus, $\mathsf{K}_a\psi_1, \dots, \mathsf{K}_a\psi_n \vdash \mathsf{K}_a\varphi$ by Lemma 17. Hence, $X_w \vdash \mathsf{K}_a\varphi$ because of the assumption (11), which contradicts the assumption $\mathsf{K}_a\varphi \notin X_w$ of the lemma due to the maximality of set X_w. Therefore, set $\widehat{X}$ is consistent.

Let X' be any maximal consistent extension of set $\widehat{X}$. Such an extension exists by Lemma 19. Also, let w' be pair (X', k_w). Note that $w \sim_a w'$ by Definition 9, the choice of sets $\widehat{X}$ and X', and the choice of pair w'. □

The next statement is used in the induction lemma in direction $(\Leftarrow)$ when formula ψ has the form $\mathsf{O}^\sigma\varphi$.

Lemma 12 *For any $w \in W$ and any formula $\mathsf{O}^\sigma\varphi \notin X_w$, there exists an action profile $\delta \in DS^w_\sigma$ and state $u \in W$ such that $(w, \delta, u) \in M$, and $\varphi \notin X_u$.*

Proof Define δ to be an action profile such that[3]

$$\delta(a) = \left(\top, [a]_\sigma, \sum_{x \in [a]_\sigma} k_w(x)\right) \tag{12}$$

[3] Recall that the first component of an action is the formula that the agent wants to achieve. We set this formula to Boolean constant $\top$ to make it easier for us to construct set u. Recall from our discussion before Definition 11, that in order for an action (φ, C, β) to be allowed, the value of β should be congruent to the sum of keys of the coalition C modulo 2. Thus, by defining the second component of the action of agent a as $[a]_\sigma$, we guarantee that this action is distributively known to be allowed under an arbitrary partition τ only if $[a]_\sigma \subseteq [a]_\tau$. This observation is formally stated as Claim 2 in a slightly more general form. We have chosen the second and the third components of $\delta(a)$ to guarantee that the claim holds.

for each agent $a \in \mathcal{A}$.

Claim 1 $\delta \in DS^w_\sigma$.

Proof of Claim. Consider any agent $a \in \mathcal{A}$ and any state $w' \in W$ such that $w \sim_x w'$ for each agent $x \in [a]_w$. By Definition 4, it suffices to show that $\delta(a) \in S^{w'}_a$. By Definition 9, $k_w(x) = k_{w'}(x)$ for each $x \in [a]_\sigma$. Hence,

$$\sum_{x \in [a]_\sigma} k_w(x) \equiv \sum_{x \in [a]_\sigma} k_{w'}(x) \pmod 2.$$

Thus, $\delta(a) \in S^{w'}_a$ by Definition 11 and the choice of δ. □

Claim 2 *For any partition* τ, *if* $\delta \in DS^w_\tau$, *then* $\sigma \preceq \tau$.

Proof of Claim. Suppose that $\delta \in DS^w_\tau$. Consider any agent $a \in \mathcal{A}$. By Definition 1, it suffices to show $[a]_\sigma \subseteq [a]_\tau$. Assume the opposite. Then, there is an agent $b \in [a]_\sigma \setminus [a]_\tau$. Define function $\widehat{k} \in \mathbb{Z}^{\mathcal{A}}$ as follows:

$$\widehat{k}(x) = \begin{cases} k_w(x) + 1, & \text{if } x = b, \\ k_w(x), & \text{otherwise.} \end{cases} \tag{13}$$

Then, $k_w(x) = \widehat{k}(x)$ for each $x \in [a]_\tau$ by the assumption $b \notin [a]_\tau$. Define w' to be set $(X_w, \widehat{k})$. Then, $w \sim_x w'$ for each $x \in [a]_\tau$ by Definition 9. Thus, $\delta(a) \in S^{w'}_a$ by Definition 4 and assumption $\delta \in DS^w_\tau$. Then,

$$\sum_{x \in [a]_\sigma} \widehat{k}(x) = \sum_{x \in [a]_\sigma} k_{w'}(x) \equiv \sum_{x \in [a]_\sigma} k_w(x) \pmod 2$$

by Definition 11, the choice of $w' = (X_w, \widehat{k})$, and Eq. 12. Hence, Eq. 13 and assumption $b \in [a]_\sigma$ imply $1 \equiv 0 \pmod 2$, which is a contradiction. □

Let set $\widehat{X}$ be $\{\neg\varphi\} \cup \{\psi \mid \mathsf{O}^\tau\psi \in X_w, \delta \in DS^w_\tau\}$. First, we show that set $\widehat{X}$ is consistent. Assume the opposite. Then, there are formulae

$$\mathsf{O}^{\tau_1}\psi_1, \ldots, \mathsf{O}^{\tau_n}\psi_n \in X_w \tag{14}$$

such that

$$\delta \in DS^w_{\tau_i}, \tag{15}$$

for all $i \leq n$ and $\psi_1, \ldots, \psi_n \vdash \varphi$. Then, by Lemma 17,

$$\mathsf{O}^\sigma\psi_1, \ldots, \mathsf{O}^\sigma\psi_n \vdash \mathsf{O}^\sigma\varphi.$$

Note $\sigma \preceq \tau_i$ for each $i \leq n$ by Claim 2 and statement (15). Thus, by the Monotonicity axiom applied *n* times,

$$\mathsf{O}^{\tau_1}\psi_1, \ldots, \mathsf{O}^{\tau_n}\psi_n \vdash \mathsf{O}^{\sigma}\varphi.$$

Hence, $X_w \vdash \mathsf{O}^{\sigma}\varphi$ by assumption (14). Then, $\mathsf{O}^{\sigma}\varphi \in X_w$ since set X_w is maximal, which contradicts the assumption of the lemma $\mathsf{O}^{\sigma}\varphi \notin X_w$. Therefore, set $\widehat{X}$ is consistent.

Let set X' be a maximal consistent extension of set $\widehat{X}$. Such an extension exists by Lemma 19. Also, let *u* be pair (X', k), where $k \in \mathbb{Z}^{\mathcal{A}}$ is an arbitrary function. Note that $\neg\varphi \in \widehat{X} \subseteq X' = X_u$. Thus, $\varphi \notin X_u$ because set X_u is consistent.

Claim 3 $(w, \delta, u) \in M$.

Proof of Claim. We will show that conditions 1 and 2 of Definition 12 are satisfied.

1. Suppose that $\mathsf{O}^{\tau}\psi \in X_w$ and $\delta \in DS^w_\tau$. It suffices to show that $\psi \in X_u$. By Claim 2, assumption $\delta \in DS^w_\tau$ implies that $\sigma \preceq \tau$. Thus, $X_w \vdash \mathsf{O}^{\sigma}\psi$ by the Monotonicity axiom and assumption $\mathsf{O}^{\tau}\psi \in X_w$. Hence, $\mathsf{O}^{\sigma}\psi \in X_w$ because set X_w is maximal. Then, $\psi \in \widehat{X} \subseteq X' = X_u$ by the choice of sets $\widehat{X}$ and X' as well as the choice of *u*.
2. If $\delta(a) = (\varphi, \{a\}, k_w(a)))$, then, by Eq. 12, φ is formula $\top$. Thus, $\varphi \in X_u$ because set X_u is maximal.

Therefore, $(w, \delta, u) \in M$. □

This concludes the proof of the lemma. □

The following statement is used in the induction lemma in direction $(\Rightarrow)$, when formula ψ has the form $\mathsf{DA}^{\sigma}_a\varphi$.

Lemma 13 *For any $w \in W$ and any $\mathsf{DA}^{\sigma}_a\varphi \in X_w$, there exists an action $s \in KS^w_a$ such that for all $w', u \in W$ and each profile $\delta \in DS^{w'}_{\sigma/a}$, if $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$, then $\varphi \in X_u$.*

Proof Let action[4] *s* be $(\varphi, \{a\}, k_w(a))$.

Claim 4 $s \in KS^w_a$.

Proof of Claim. Consider any state $v \in [w]_a$. Then, $v \sim_a w$. Hence, $k_v(a) = k_w(a)$ by Definition 9. Thus, $\sum_{x\in\{a\}} k_v(x) \equiv k_w(a) \pmod 2$. Hence, $s \in S^v_a$ by Definition 11. Therefore, $s \in KS^w_a$ by Definition 3. □

[4] We have chosen the first component of the action *s* to be φ so that agent a can use this action to achieve φ. The second component of the action is chosen to be the singleton set $\{a\}$ to guarantee that agent a alone knows that this action is allowed.

Claim 5 For any $w' \in W$, if $w \sim_a w'$, then $\mathsf{DA}^\sigma_a \varphi \in X_{w'}$.

Proof of Claim. By assumption of the lemma, $\mathsf{DA}^\sigma_a \varphi \in X_w$. Thus, $X_w \vdash \mathsf{K}_a \mathsf{DA}^\sigma_a \varphi$ by the Strategic Introspection axiom and the Modus Ponens rule. Then, $\mathsf{K}_a \mathsf{DA}^\sigma_a \varphi \in X_w$ because set X_w is maximal. Hence, $\mathsf{DA}^\sigma_a \varphi \in X_{w'}$ by Definition 9 and the assumption $w \sim_a w'$. □

Finally, consider any states $w', u \in W$ and any action profile $\delta \in DS^{w'}_{\sigma/a}$ such that $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$. Thus, $\mathsf{DA}^\sigma_a \varphi \in X_{w'}$ by Claim 5. Also, $\delta(a) = s = (\varphi, \{a\}, k_w(a))$ by the assumption $\delta(a) = s$ and the choice of action s. Therefore, $\varphi \in X_u$ by item 2 of Definition 12. This concludes the proof of the lemma. □

The last auxiliary lemma is used in the induction lemma in direction ($\Leftarrow$), when formula ψ has the form $\mathsf{DA}^\sigma_a \varphi$. As discussed earlier, its proof simultaneously constructs two maximal consistent sets using σ-harmony technique.

Lemma 14 *For any $w \in W$, any formula $\neg\mathsf{DA}^\sigma_a \varphi \in X_w$, and any action $s \in KS^w_a$, there are states $w', u \in W$ and an action profile $\delta \in DS^{w'}_{\sigma/a}$ such that $\delta(a) = s$, $w \sim_a w'$, $(w', \delta, u) \in M$, and $\varphi \notin X_u$.*

Proof To minimize the number of cases to be considered in this proof, we define an auxiliary formulae

$$\varphi' = \begin{cases} pr_1(s), & \text{if } \mathsf{DA}^\sigma_a(pr_1(s)) \in X_w, \\ \top, & \text{otherwise}, \end{cases} \tag{16}$$

where $pr_1(s)$ is the first component of triple s.

Claim 6 $\mathsf{DA}^\sigma_a \varphi' \in X_w$.

Proof of Claim. We consider the following cases:

Case I: $\mathsf{DA}^\sigma_a(pr_1(s)) \in X_w$. Then, $\varphi' = pr_1(s)$ by Eq. 16. Thus, $\mathsf{DA}^\sigma_a \varphi' \in X_w$ by the assumption of the case.

Case II: $\mathsf{DA}^\sigma_a(pr_1(s)) \notin X_w$. Thus, $\varphi' = \top$ by Eq. 16. Hence, φ' is a tautology. Then, $\vdash \mathsf{DA}^\sigma_a \varphi'$ by the Necessitation rule. Thus, $\mathsf{DA}^\sigma_a \varphi' \in X_w$ since set X_w is maximal. □

The pair of sets $(\{\chi \mid \mathsf{K}_a \chi \in X_w\}, \{\neg\varphi, \varphi'\})$ is in (σ/a)-harmony by Lemma 7, the assumption $\neg\mathsf{DA}^\sigma_a \varphi \in X_w$ of the lemma, and Claim 6. Thus, by Lemma 9, there is a pair of sets $(\widehat{Y}, \widehat{Z})$ in complete (σ/a)-harmony such that

$$\{\chi \mid \mathsf{K}_a \chi \in X_w\} \subseteq \widehat{Y} \quad \text{and} \quad \{\neg\varphi, \varphi'\} \subseteq \widehat{Z}.$$

Let Y and Z be any maximal consistent extensions of sets $\widehat{Y}$ and $\widehat{Z}$, respectively. Such extensions exist by Lemma 19. Let $w' = (Y, k_w)$ and $u = (Z, k_w)$.

Claim 7 $w \sim_a w'$.

Proof of Claim. Suppose that $\mathsf{K}_a\chi \in X_w$. By Definition 9, it suffices to show that $\chi \in X_{w'} = Y$. The latter follows from the choice of sets $\widehat{Y}$ and Y. □

Let action profile δ be defined[5] as follows:

$$\delta(b) = \begin{cases} s, & \text{if } b = a, \\ (\top, [b]_{\sigma/a}, \Sigma_{x\in[b]_{\sigma/a}} k_w(x)), & \text{otherwise.} \end{cases} \tag{17}$$

Claim 8 $\delta \in DS^{w'}_{\sigma/a}$.

Proof of Claim. Consider any agent $b \in \mathcal{A}$, and any state $w'' \in W$ such that $w' \sim_x w''$ for each $x \in [b]_{\sigma/a}$. By Definition 4, it suffices to show that $\delta(b) \in S^{w''}_b$.

Case I: $b \neq a$. By Definition 9, we have $k_{w''}(x) = k_{w'}(x)$ for each agent $x \in [b]_{\sigma/a}$. Also recall that $k_w = k_{w'}$ by the choice of w'. Hence,

$$\sum_{x\in[b]_{\sigma/a}} k_{w''}(x) = \sum_{x\in[b]_{\sigma/a}} k_{w'}(x) = \sum_{x\in[b]_{\sigma/a}} k_w(x).$$

Therefore, $\delta(b) \in S^{w''}_b$ by Definition 11, Eq. 17, and assumption $b\neq a$ of the case.

Case II: $b = a$. Recall that $w' \sim_x w''$ for each $x \in [b]_{\sigma/a}$. Thus, $w' \sim_a w''$ because $a = b$. Hence, $w \sim_a w''$ by Claim 7. Then, $s \in S^{w''}_a$ by Definition 3 and the assumption $s \in KS^w_a$ of the lemma. Thus, $\delta(a) \in S^{w''}_a$ by Eq. 17. Therefore, $\delta(b) \in S^{w''}_b$ because $a = b$. □

Claim 9 For any partition τ, if $\delta \in DS^{w'}_\tau$, then $\sigma/a \preceq \tau$.

Proof of Claim. Suppose that $\delta \in DS^{w'}_\tau$. Consider any agent $b \in \mathcal{A}$. By Definition 1, it suffices to show that $[b]_{\sigma/a} \subseteq [b]_\tau$. Assume the opposite. Then, there is an agent $c \in [b]_{\sigma/a} \setminus [b]_\tau$. Consider the following two cases:

Case I: $b \neq a$. Define function $\widehat{k} \in \mathbb{Z}^{\mathcal{A}}$ as follows:

$$\widehat{k}(x) = \begin{cases} k_w(x) + 1, & \text{if } x = c, \\ k_w(x), & \text{otherwise.} \end{cases} \tag{18}$$

Then, $\widehat{k}(x) = k_w(x)$ for each $x \in [b]_\tau$ by the assumption $c \notin [b]_\tau$. Define state w'' to be pair $(Y, \widehat{k})$.

Next, we show that $w' \sim_x w''$ for each $x \in [b]_\tau$. Since $\widehat{k}(x) = k_w(x)$ for each $x \in [b]_\tau$, by Definition 9, it suffices to show that if $\mathsf{K}_x\psi \in Y_{w'}$, then $\psi \in Y_{w''}$ for

[5] The intuition for the choice of δ is similar to the one described in the footnote on page 20. In the current case, the second component of the action is chosen to guarantee that Claim 9 holds.

each $x \in [b]_\tau$ and each formula $\psi \in \Phi$. Indeed, consider any $x \in [b]_\tau$ and any $\psi \in \Phi$ such that $\mathsf{K}_x\psi \in Y_{w'}$. Thus, $\mathsf{K}_x\psi \in Y$ by the choice of state w'. Hence, $Y \vdash \psi$ by the Truth axiom and the Modus Ponens inference rule. Then, $\psi \in Y = Y_{w''}$ by the maximality of set Y and the choice of state w''.

Thus, $\delta(b) \in S_b^{w''}$ by Definition 4 and the assumption that $\delta \in DS_\tau^{w'}$. Then,

$$\sum_{x\in[b]_{\sigma/a}} \widehat{k}(x) \equiv \sum_{x\in[b]_{\sigma/a}} k_w(x) \pmod 2$$

by Definition 11, Eq. 17, the assumption $b \neq a$, and the choice of $w'' = (Y, \widehat{k})$. Hence, by Eq. 18 and because $c \in [b]_{\sigma/a}$,

$$1 \equiv 0 \pmod 2,$$

which is a contradiction.

Case II: $b = a$. Then, $c \in [a]_{\sigma/a}$ and $c \notin [a]_\tau$ by the assumption $c \in [b]_{\sigma/a} \setminus [b]_\tau$. Statement $c \in [a]_{\sigma/a}$ and the definition of the partition σ/a implies that $c \in \{a\}$. Therefore, $c = a$, which contradicts the statement $c \notin [a]_\tau$. □

Claim 10 For any formula $\mathsf{DA}_b^\tau\psi \in X_{w'}$, if $\delta \in DS_{\tau/b}^{w'}$ and $\delta(b) = (\psi, \{b\}, k_{w'}(b))$, then $\psi \in X_u$.

Proof of Claim. Consider the following two cases:

Case I:$b \neq a$. Thus, $\psi = \top$ by the assumption $\delta(b) = (\psi, \{b\}, k_{w'}(b))$ and Eq. (17). Therefore, $\psi \in X_u$ because set X_u is maximal.

Case II: $b = a$. By Claim 9, assumption $\delta \in DS_{\tau/b}^{w'}$ implies that $\sigma/a \preceq \tau/b$. Thus, $\sigma/a \preceq \tau/a$ because $a = b$. Hence, $X_{w'} \vdash \mathsf{DA}_a^\sigma\psi$ by the Monotonicity axiom, the Modus Ponens inference rule, the assumption $\mathsf{DA}_b^\tau\psi \in X_{w'}$ of the claim, and the assumption $b = a$ of the case. Then, $X_{w'} \vdash \mathsf{K}_a\mathsf{DA}_a^\sigma\psi$ by the Strategic Introspection axiom and the Modus Ponens inference rule. Thus, $\mathsf{K}_a\mathsf{DA}_a^\sigma\psi \in X_{w'}$ by the maximality of set $X_{w'}$. Hence, $\mathsf{DA}_a^\sigma\psi \in X_w$ by Definition 9 and Claim 5. Thus,

$$\begin{aligned}\varphi' &= pr_1(s) = pr_1(\delta(a)) = pr_1(\delta(b)) \\ &= pr_1(\psi, \{b\}, k_{w'}(b)) = \psi\end{aligned}$$

by Eq. 16, Eq. 17, assumption $b = a$ of the case, and assumption $\delta(b) = (\psi, \{b\}, k_{w'}(b))$ of the claim. Therefore, $\psi = \varphi' \in \widehat{Z} \subseteq Z = X_u$ by the choices of $\widehat{Z}$, Z, and u. □

Claim 11 If $\mathsf{O}^\tau\psi \in X_{w'}$ and $\delta \in DS_\tau^{w'}$, then $\psi \in X_u$.

Proof of Claim. The assumption $\delta \in DS^{w'}_{\tau}$ implies that $\sigma/a \preceq \tau$ by Claim 9. Then, $\mathsf{O}^{\sigma/a}\psi \in X_{w'}$ by the assumption $\mathsf{O}^{\tau}\psi \in X_{w'}$ and the Monotonicity axiom. Thus, $\neg\mathsf{O}^{\sigma/a}\psi \notin X_{w'}$, since set $X_{w'}$ is consistent. Hence, $\neg\mathsf{O}^{\sigma/a}\psi \notin \widehat{Y}$ because $\widehat{Y} \subseteq X_{w'}$. Then, $\psi \in \widehat{Z}$ by Definition 7 and the choice of $(\widehat{Y}, \widehat{Z})$ as a pair in complete (σ/a) -harmony. Therefore, $\psi \in \widehat{Z} \subseteq Z \subseteq X_u$ by the choice of set $\widehat{Z}$, set Z, and state u. □

Note that $(w', \delta, u) \in M$ by Definition 12, Claim 10, and Claim 11. Finally, $\neg\varphi \in \widehat{Z} \subseteq Z = X_u$ by the choice of set $\widehat{Z}$, set Z and state u. Therefore, $\varphi \notin X_u$ because set X_u is consistent. This concludes the proof of the lemma. □

Next is the "truth" or "induction" lemma.

Lemma 15 $\varphi \in X_w$ *iff* $w \Vdash \varphi$.

Proof We prove the lemma by structural induction on formula φ. If φ is a propositional variable, then the lemma follows from Definition 13 and item 1 of Definition 5. If formula φ is an implication or a negation, then the required follows from the maximality and the consistency of the set X_w and items 2 and 3 of Definition 5 in the standard way.

Let formula φ have the form $\mathsf{O}^{\sigma}\psi$.

$(\Rightarrow)$: By Definition 12, assumption $\mathsf{O}^{\sigma}\psi \in X_w$ implies that for each state $w \in W$, each action profile $\delta \in DS^{w}_{\sigma}$, and each state $u \in W$, if $(w, \delta, u) \in M$, then $\psi \in X_u$. Thus, by the induction hypothesis, for each state $w \in W$, each action profile $\delta \in DS^{w}_{\sigma}$, and each state $u \in W$, if $(w, \delta, u) \in M$, then $u \Vdash \psi$. Therefore, $w \Vdash \mathsf{O}^{\sigma}\psi$ by item 5 of Definition 5.

$(\Leftarrow)$: By Lemma 12, assumption $\mathsf{O}^{\sigma}\psi \notin X_w$ implies that there exists an action profile $\delta \in DS^{w}_{\sigma}$ and a state $u \in W$ such that $(w, \delta, u) \in M$, and $\varphi \notin X_u$. Thus, by the induction hypothesis, $u \nVdash \psi$. Therefore, $w \nVdash \mathsf{O}^{\sigma}\psi$ by item 5 of Definition 5.

The case when formula φ has the form $\mathsf{K}_a\psi$ is similar to the case $\mathsf{O}^{\sigma}\psi$, but it uses Lemma 11 instead of Lemma 12. Finally, assume that formula φ has the form $\mathsf{DA}^{\sigma}_{a}\psi$.

$(\Rightarrow)$: By Lemma 13, assumption $\mathsf{DA}^{\sigma}_{a}\psi \in X_w$ implies that there is an action $s \in KS^{w}_{a}$ such that for all states $w', u \in W$ and each action profile $\delta \in DS^{w'}_{\sigma/a}$, if $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$, then $\psi \in X_u$. Thus, by the induction hypothesis, for all states $w', u \in W$ and each action profile $\delta \in DS^{w'}_{\sigma/a}$, if $\delta(a) = s$, $w \sim_a w'$, and $(w', \delta, u) \in M$, then $u \Vdash \psi$. Therefore, $w \Vdash \mathsf{DA}^{\sigma}_{a}\psi$ by item 6 of Definition 5.

$(\Leftarrow)$: Suppose that $\mathsf{DA}^{\sigma}_{a}\psi \notin X_w$. Thus, $\neg\mathsf{DA}^{\sigma}_{a}\psi \in X_w$ because set X_w is maximal. Then, by Lemma 14, for any action $s \in KS^{w}_{a}$, there are states $w', u \in W$ and an action profile $\delta \in DS^{w'}_{\sigma/a}$ such that $\delta(a) = s$, $w \sim_a w'$, $(w', \delta, u) \in M$, and $\varphi \notin X_u$. Hence, by the induction hypothesis, for any action $s \in KS^{w}_{a}$, there are states $w', u \in W$ and an action profile $\delta \in DS^{w'}_{\sigma/a}$ such that $\delta(a) = s$, $w \sim_a w'$, $(w', \delta, u) \in M$, and $u \nVdash \varphi$. Therefore, $w \nVdash \mathsf{DA}^{\sigma}_{a}\psi$ by item 6 of Definition 5. □

Finally, we are ready to state and prove a strong completeness theorem.

Theorem 1 *If $Y \nvdash \varphi$, then there is a state w of a normative system such that $w \Vdash \chi$ for each formula $\chi \in Y$ and $w \nVdash \varphi$.*

Proof Suppose $Y \nvdash \varphi$. Thus, set $Y \cup \{\neg\varphi\}$ is consistent. Let Y' be any maximal consistent extension of this set. Such an extension exists by Lemma 19. Also, let function k be an arbitrary function from the set $\mathbb{Z}^{\mathcal{A}}$. Let w be the pair (Y', k), which is a state of the canonical model by Definition 8.

Note that $\gamma \in Y \subseteq Y' = X_w$ for each formula $\gamma \in Y$. Thus, $w \Vdash \chi$ for each formula $\gamma \in Y$ by Lemma 15. Also, $\neg\varphi \in Y \subseteq Y' = X_w$. Hence, $\varphi \notin X_w$ because set X_w is consistent. Therefore, $w \nVdash \varphi$ also by Lemma 15. □

6 Conclusion

In this article, we proposed the concept of norm-dependent abilities and studied it as a modality in the setting of information walls. Perhaps the most interesting observation about norm-dependent abilities is that partial removal of information walls (allowing other agents to communicate more freely) decreases the agent's norm-dependent abilities. On the other hand, the addition of such walls makes the agent more powerful. We are not aware of any other logical systems that capture this "prevailing in the dark" effect.

Perhaps the most natural question about this work is whether the current results could be generalized to group knowledge and coalition norm-dependent abilities. One of the challenges in this direction is finding an intuitively acceptable interpretation of group knowledge in the presence of information walls. Is it sensible to reason about a coalition distributively knowing that a certain action is allowed if the coalition members are on different sides of a wall and explicitly banned from communicating with each other? One might consider only coalitions C located in the same set of a partition, but this makes the syntax confusing, given that we study modalities DA_C^{σ} for different partitions σ_1.

We think that a more interesting direction is to study one-way information walls that only prevent the diffusion of the information in one of two directions. In real-world scenarios, for example, certain groups of people might be banned from spreading information to outsiders, but not from listening to them.

Another possible extension of this work is to consider the interplay between public announcements (Ditmarsch et al., 2007) and norm-dependent abilities in our setting. Any public announcement could simultaneously empower an agent and also weaken the agent by supplying the same information to the opponents.

One can also potentially consider a "trusted friends" setting in which each agent only takes into account the knowledge of the adjacent agents. This would come down to allowing "second-hand knowledge", but not "third-hand", "fourth-hand", etc knowledge. Although significantly more complicated than ours, such a setting is also interesting – it could be used to model the abilities coming from spreading mistrust among opponents. In addition, one can also consider directed communication channels.

Another alternative setting is to consider walls that do not block information diffusion completely but impose costs on it. In such a setting, for instance, one can study modality $\mathsf{DA}_a^m\varphi$ that stands for “agent a knows an allowed action to achieve φ as long as the total cost of communication by all agents is no more than *m*.

Finally, another interesting direction for future research is studying group actions that require *common knowledge* of an action being allowed. For instance, in the famous example with two generals, the generals are not able to start a joint attack on a common enemy because they cannot establish common knowledge of the time to attack. The two general settings are very similar to the setting of this article if the notion of distributively knowingly allowed action from Definition 4 is replaced with commonly knowingly allowed action. In this modified setting, we could, for example, express the fact that the common enemy has a strategy to win the battle with the two generals because they will never be able to start a coordinated counterattack.

Proofs of auxiliary lemmas

The next three lemmas state well-known properties of S5 modality that will be used in the proof of the completeness.

Lemma 16 *[Deduction] If $X, \varphi \vdash \psi$, then $X \vdash \varphi \to \psi$.*

Proof Suppose that sequence $\psi_1, \dots, \psi_n$ is a proof from set $X \cup \{\varphi\}$ and the theorems of our logical system that uses the Modus Ponens inference rule only. In other words, for each $k \leq n$, either

1. $\vdash \psi_k$, or
2. $\psi_k \in X$, or
3. ψ_k is equal to φ, or
4. there are $i, j < k$ such that formula ψ_j is equal to $\psi_i \to \psi_k$.

It suffices to show that $X \vdash \varphi \to \psi_k$ for each $k \leq n$. We prove this by induction on *k* through considering the four cases above separately.

Case I: $\vdash \psi_k$. Note that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Hence, $\vdash \varphi \to \psi_k$ by the Modus Ponens inference rule. Therefore, $X \vdash \varphi \to \psi_k$.

Case II: $\psi_k \in X$. Then, $X \vdash \psi_k$, similarly to the previous case.

Case III: formula ψ_k is equal to φ. Thus, $\varphi \to \psi_k$ is a propositional tautology. Then, $X \vdash \varphi \to \psi_k$.

Case IV: formula ψ_j is equal to $\psi_i \to \psi_k$ for some $i, j < k$. Thus, by the induction hypothesis, $X \vdash \varphi \to \psi_i$ and $X \vdash \varphi \to (\psi_i \to \psi_k)$. Note that formula

$$(\varphi \to \psi_i) \to ((\varphi \to (\psi_i \to \psi_k)) \to (\varphi \to \psi_k))$$

is a propositional tautology. Therefore, $X \vdash \varphi \to \psi_k$ by applying the Modus Ponens inference rule twice. □

Lemma 17 *If* $\varphi_1, \ldots, \varphi_n \vdash \psi$*, then* $\Box\varphi_1, \ldots, \Box\varphi_n \vdash \Box\psi$*, where* $\Box \in \{\mathsf{K}_a, \mathsf{O}^\sigma\}$.

Proof By Lemma 16 applied *n* times, the assumption $\varphi_1, \ldots, \varphi_n \vdash \psi$ implies that

$$\vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots).$$

Thus, by the Necessitation inference rule, $\vdash \Box(\varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots))$. Hence, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash \Box\varphi_1 \to \Box(\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots).$$

Then, $\Box\varphi_1 \vdash \Box(\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots)$, again by the Modus Ponens inference rule. Therefore, $\Box\varphi_1, \ldots, \Box\varphi_n \vdash \Box\psi$ by applying the previous steps $(n-1)$ more times. □

Lemma 18 *Positive Introspection* $\vdash \mathsf{K}_a\varphi \to \mathsf{K}_a\mathsf{K}_a\varphi$.

Proof Formula $\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \neg\mathsf{K}_a\varphi$ is an instance of the Truth axiom. Thus, $\vdash \mathsf{K}_a\varphi \to \neg\mathsf{K}_a\neg\mathsf{K}_a\varphi$ by contraposition. Hence, taking into account the following instance of the Negative Introspection axiom: $\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi$, we have

$$\vdash \mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi. \tag{A1}$$

At the same time, $\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\neg\mathsf{K}_a\varphi$ is an instance of the Negative Introspection axiom. Thus, $\vdash \neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\varphi$ by the law of contrapositive in the propositional logic. Hence, by the Necessitation inference rule, $\vdash \mathsf{K}_a(\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\varphi)$. Thus, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash \mathsf{K}_a\neg\mathsf{K}_a\neg\mathsf{K}_a\varphi \to \mathsf{K}_a\mathsf{K}_a\varphi$. The latter, together with statement (A1), implies the statement of the lemma by propositional reasoning. □

Lemma 19 *[Lindenbaum] Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

Proof The standard proof of Lindenbaum's lemma (Mendelson, 2009) Proposition 2.14 applies here. □

Author contributions The authors contribute equally to this paper.

Funding No funds, grants, or other support was received.

Declarations

Conflict of interests The authors have no relevant financial or non-financial interests to disclose.

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