

Natural neutrino mass hierarchy in a theory of gauge flavour deconstruction

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ABSTRACT: We show how a natural neutrino mass hierarchy with large lepton mixing angles may be achieved in a theory of gauge flavour deconstruction. Hitherto it has been shown that neutrino anarchy may result from such theories, but here we show that this need not necessarily be the case. In particular we consider the minimal tri-hypercharge theory, and show that the decomposition of the family hypercharges into the corresponding $B - L$ gauge groups, together with the charged lepton mass hierarchy, implies the sequential dominance conditions for a neutrino mass hierarchy, where lepton mixing originates from both the neutrino and charged lepton sectors. We present novel and model-independent sequential dominance results applicable to this case, but also useful more generally. We also show how natural quark mass and mixing are included in such a framework.

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1 Introduction

The flavour problem remains one of the most intriguing puzzles of the Standard Model (SM), being responsible for most of its parameters [1]. Under the SM gauge group the three fermion families are identical, but differ greatly in mass showing a hierarchical pattern. The fact that quark mixing is small while lepton mixing is large only adds to the mystery.

An efficient mechanism to generate a hierarchical flavour structure consists of embedding the SM in a larger gauge symmetry that contains a separate gauge group for each fermion family, with the light Higgs doublet(s) originating from the third family group. This general idea was originally proposed in the early 80s [2–16] and has received different names over the years, like “tribal groups”, although the most recent literature is denoting this framework as *flavour deconstruction* [17–38].

At the effective level, flavour-deconstructed models tend to generate small quark mixing, in agreement with the observed structure of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. However, they tend to generate small lepton mixing as well. There are several ways to circumvent this issue and generate large Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing, as required by oscillation data. One can go beyond the effective field theory (EFT), as in [35], or one can also consider particular gauge symmetries where both hierarchical neutrino Yukawa couplings and hierarchical right-handed neutrino masses cancel the overall hierarchies in the effective neutrino mass matrix when applying the seesaw formula [36]. Alternatively, one can introduce extra scalar fields which only participate in the neutrino sector [28], or charge all lepton doublets under the same family group, hence introducing gauge anomalies that need to be canceled by extra fermion content in the UV [34, 38]. However, all these mechanisms achieve an *anarchical* pattern for the neutrino masses and mixing angles: the observed neutrino flavour structure is assumed to have originated as an accidental configuration of $\mathcal{O}(1)$ dimensionless coefficients. While this is a valid approach, it is widely speculated in the literature that the flavour structure of the lepton sector might not necessarily be of anarchic nature, but rather the simultaneous appearance of hierarchical neutrino masses and two large mixing angles calls for further understanding, perhaps hinting at a particular dynamical mechanism.

Going in this direction, the framework of sequential dominance (SD) was proposed (see e.g. [39–42]). Sequential dominance is not in itself a model, but a sub-mechanism within the general framework of the type-I seesaw, that may be applied to constructing different classes of neutrino models beyond anarchy. The starting point of sequential dominance is to assume that one of the right-handed neutrinos contributes dominantly to the heaviest neutrino mass, with the atmospheric mixing angle being determined by a simple ratio of two Yukawa couplings. This is sometimes referred to as single right-handed neutrino dominance. Sequential dominance corresponds to the further assumption that, together with single right-handed neutrino dominance, a second right-handed neutrino contributes dominantly to the second heaviest neutrino mass, with the large solar mixing angle interpreted as a ratio of Yukawa couplings. The third right-handed neutrino is effectively decoupled from the seesaw mechanism, and plays no part in determining the neutrino mass spectrum. If the decoupled right-handed neutrino is also the heaviest one, then sequential dominance is effectively equivalent to having two right-handed neutrinos.

The goal of this paper is to implement sequential dominance in the context of flavour deconstruction in order to go beyond anarchy in the neutrino sector. In particular, we will show that tri-hypercharge [28, 35], which is arguably the simplest theory of flavour deconstruction, naturally

delivers sequential dominance when extended to embrace the right-handed neutrino sector. We will see how the combination of flavour deconstruction with sequential dominance delivers non-trivial features, including the origin of PMNS mixing from both charged lepton and neutrino mixing: the atmospheric angle θ_{23} originates from both the 23 mixing angles of charged leptons and neutrinos, while the reactor angle θ_{13} originates mostly from a Cabibbo-like θ_{12}^e angle in the charged lepton sector, and the solar angle θ_{12} originates mostly from 12 neutrino mixing.

The paper is structured as follows. In Section 2 we discuss the minimal tri-hypercharge theory as an example of anarchy in the neutrino sector of flavour deconstruction. In Section 3 we show how sequential dominance naturally arises when tri-hypercharge is extended to embrace right-handed neutrinos, leading to potential predictivity in contrast to the hypothesis of anarchy. In this framework, we discuss the origin of a natural neutrino mass hierarchy, lepton mixing angles and a natural quark sector. Section 4 outlines our main conclusions. Our conventions are shown in Appendix A. Appendix B.1 contains a review of sequential dominance in the neutrino sector, while Appendix B.2 contains novel model-independent results regarding charged leptons in sequential dominance. Appendix C contains model-independent formulas for the PMNS mixing angles that consider all potential contributions from neutrino and charged lepton mixing angles and phases. Finally, in Appendix D we discuss the scalar potential for the considered model.

2 Tri-hypercharge with anarchic neutrinos

The tri-hypercharge (TH) proposal is one of the simplest theories of flavour deconstruction. It involves just assigning a separate gauge hypercharge to each fermion family at high energies [28],

$$G_{\text{TH}} = U(1)_{Y_1} \times U(1)_{Y_2} \times U(1)_{Y_3}, \quad (2.1)$$

which commutes with $SU(3)_c \times SU(2)_L$ that remains flavour universal as in the SM. The TH symmetry is spontaneously broken down to the diagonal hypercharge $U(1)_Y = U(1)_{Y_1+Y_2+Y_3}$ in two steps by a set of scalar fields charged under different hypercharges which add to zero, denoted as “hyperons”. This diagonal $U(1)_Y$ corresponds to the universal hypercharge of the SM. The Higgs doublet(s) that spontaneously break electroweak symmetry are chosen to carry only third family hypercharge, which allows to write only third family Yukawa couplings at renormalisable level, while the Yukawa couplings of the light families originate from non-renormalisable operators that involve the hyperons. This successfully leads to a dynamical generation of the observed charged fermion mass hierarchies and small quark mixing. In particular, we consider two Higgs doublets $H_{u,d}$ that couple to up-quarks/neutrinos and down-quarks/charged leptons respectively¹, in order to take into account the overall different normalisation between the up sector and the down-quark/charged lepton sectors. Minimal ultraviolet (UV) completions of the non-renormalisable operators have been provided via the inclusion of vector-like fermions and/or heavy Higgs doublets [35].

As shown in Table 1, the minimal but complete realisation of the lepton sector [35] involves only two hyperons, along with two right-handed neutrinos as full singlets of the TH symmetry. Going beyond renormalisable level, this delivers the following Yukawa couplings and Majorana masses for

¹This may be enforced by particular mechanisms such as a softly broken $\mathbb{Z}_2$ symmetry, not specified here.

Field	$U(1)_{Y_1}$	$U(1)_{Y_2}$	$U(1)_{Y_3}$	$SU(3)_c \times SU(2)_L$
ℓ_1	$-\frac{1}{2}$	0	0	$(\mathbf{1}, \mathbf{2})$
ℓ_2	0	$-\frac{1}{2}$	0	$(\mathbf{1}, \mathbf{2})$
ℓ_3	0	0	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{2})$
e_1^c	1	0	0	$(\mathbf{1}, \mathbf{1})$
e_2^c	0	1	0	$(\mathbf{1}, \mathbf{1})$
e_3^c	0	0	1	$(\mathbf{1}, \mathbf{1})$
ν_1^c	0	0	0	$(\mathbf{1}, \mathbf{1})$
ν_2^c	0	0	0	$(\mathbf{1}, \mathbf{1})$
H_u	0	0	$\frac{1}{2}$	$(\mathbf{1}, \mathbf{2})$
H_d	0	0	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{2})$
ϕ_{12}	$\frac{1}{2}$	$-\frac{1}{2}$	0	$(\mathbf{1}, \mathbf{1})$
ϕ_{23}	0	$\frac{1}{2}$	$-\frac{1}{2}$	$(\mathbf{1}, \mathbf{1})$

Table 1: Minimal tri-hypercharge model for the lepton sector [35]. $H_{u,d}$ and ϕ_{ij} are scalars while the rest are the three usual generations of chiral leptons plus two right-handed neutrinos.

the neutrino singlets,

$$\mathcal{L} = a_{3i}^\nu \ell_3 H_u \nu_i^c + a_{2i}^\nu \frac{\phi_{23}}{\Lambda_{23}^\nu} \ell_2 H_u \nu_i^c + a_{1i}^\nu \frac{\phi_{12}}{\Lambda_{12}^\nu} \frac{\phi_{23}}{\Lambda_{23}^\nu} \ell_1 H_u \nu_i^c + M_{ij} \nu_i^c \nu_j^c + \text{h.c.}, \quad (2.2)$$

where $i = 1, 2$ and repeated indices are summed. Once the hyperons get their VEVs, we obtain the following textures for the Dirac and Majorana mass matrices of neutrinos,

$$\mathcal{L}_\nu = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} m_D \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} + \begin{pmatrix} \nu_1^c & \nu_2^c \end{pmatrix} M_M \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} + \text{h.c.} \quad (2.3)$$

$$= \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \begin{pmatrix} a_{11}^\nu \epsilon_{12}^\nu \epsilon_{23}^\nu & a_{12}^\nu \epsilon_{12}^\nu \epsilon_{23}^\nu \\ a_{21}^\nu \epsilon_{23}^\nu & a_{22}^\nu \epsilon_{23}^\nu \\ a_{31}^\nu & a_{32}^\nu \end{pmatrix} \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} H_u + \begin{pmatrix} \nu_1^c & \nu_2^c \end{pmatrix} \begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix} \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} + \text{h.c.}, \quad (2.4)$$

where we have defined $\epsilon_{12}^\nu = \langle \phi_{12} \rangle / \Lambda_{12}^\nu$ and $\epsilon_{23}^\nu = \langle \phi_{23} \rangle / \Lambda_{23}^\nu$. The effective mass matrix for active neutrinos is obtained after applying the seesaw formula,

$$m_\nu \simeq m_D (M_M)^{-1} m_D^T. \quad (2.5)$$

In the charged lepton sector, we obtain the following Yukawa couplings,

$$\mathcal{L}_e = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \begin{pmatrix} a_{11}^e \epsilon_{12}^e \epsilon_{23}^e & a_{12}^e \epsilon_{12}^e \epsilon_{23}^e & a_{13}^e \epsilon_{12}^e \epsilon_{23}^e \\ a_{21}^e (\epsilon_{12}^e)^2 \epsilon_{23}^e & a_{22}^e \epsilon_{23}^e & a_{23}^e \epsilon_{23}^e \\ a_{31}^e (\epsilon_{12}^e)^2 (\epsilon_{23}^e)^2 & a_{32}^e (\epsilon_{23}^e)^2 & a_{33}^e \end{pmatrix} \begin{pmatrix} e_1^c \\ e_2^c \\ e_3^c \end{pmatrix} H_d + \text{h.c.}, \quad (2.6)$$

where we have defined $\epsilon_{12}^e = \langle \phi_{12} \rangle / \Lambda_{12}^e$ and $\epsilon_{23}^e = \langle \phi_{23} \rangle / \Lambda_{23}^e$. Under the validity of the EFT introduced in Eq. (2.2), we expect $\epsilon_{12,23}^{\nu,e} \ll 1$ to imprint hierarchies among the rows of the Yukawa couplings, while M_{ij} are naturally presumed to be of similar order since no symmetry distinguishes

the two singlet neutrinos. This is a general result in theories of flavour deconstruction (see e.g. the discussion of [36]), where one naively expects a hierarchical lepton sector with small mixing angles, similar to the observed pattern of the quark sector, unless the dimensionless coefficients $a^{\nu,e}$ are fine-tuned. This is however in contradiction with the two large mixing angles observed in the PMNS matrix.

So far, the four possibilities to generate large lepton mixing in theories of flavour deconstruction consist on:

- (i) Introduce extra linking scalars (e.g. hyperons) which only participate in the neutrino sector and change the Yukawa texture above [28].
- (ii) Go beyond the validity of the EFT approach of Eq. (2.2) to generate $\epsilon'_{12,23} \sim 1$ in the full UV theory [35].
- (iii) Consider particular gauge symmetries where both hierarchical neutrino Yukawa couplings and hierarchical right-handed neutrino masses cancel the overall hierarchies in the neutrino mass matrix when applying the seesaw formula [36]. The extended gauge theory will then distinguish between the two right-handed neutrinos.
- (iv) Charge all lepton doublets under the same site (or hypercharge). This introduces gauge anomalies that can be canceled with extra fermionic content in the UV [34, 38].

In all these cases, one obtains an *anarchic* effective neutrino mass matrix: all entries are governed by $\mathcal{O}(1)$ coefficients which are simply fitted to neutrino oscillation data. This is successful at the level of reconciling the hypothesis of flavour deconstruction with neutrino data, but it does not give any understanding about the observed neutrino flavour structure, which is assumed to have originated as an accidental configuration of $\mathcal{O}(1)$ dimensionless coefficients. However, the flavour structure of the lepton sector might not necessarily be anarchic in nature. This would be the case if the physical neutrino masses exhibited a strong hierarchy, where one of the right-handed neutrinos dominantly contributes to the heaviest physical neutrino mass, as in sequential dominance [39–42]. The simultaneous appearance of hierarchical neutrino masses and two large (and one small) mixing angles may thus hint towards a dynamical mechanism beyond anarchy.

Such a dynamical mechanism requires us to go beyond tri-hypercharge. The reason why this is necessary is that the two right-handed neutrinos are both singlets under the tri-hypercharge gauge group, and hence are indistinguishable, which results in the two columns of the Dirac matrix being approximately equal, and the heavy Majorana mass matrix being anarchical. In the current framework it is therefore difficult to obtain a natural neutrino mass hierarchy, and this motivates extending tri-hypercharge to a larger gauge group, under which the right-handed neutrinos are no longer indistinguishable singlets.

In the following section, we will show that the conditions of sequential dominance (see Appendix B), which naturally provides a first step to understand the dynamic origin of the observed pattern of neutrino data, are naturally realised when the tri-hypercharge setup is minimally extended such that the right-handed neutrinos are no longer indistinguishable singlets. In this way, it is possible to go beyond anarchy in the neutrino sector of flavour deconstruction.

Field	$U(1)_{Y_1}$	$U(1)_{R_2} \times U(1)_{(B-L)_{2/2}}$	$U(1)_{R_3} \times U(1)_{(B-L)_{3/2}}$	$SU(3)_c \times SU(2)_L$
ℓ_1	$-\frac{1}{2}$	$(0, 0)$	$(0, 0)$	$(\mathbf{1}, \mathbf{2})$
ℓ_2	0	$(0, -\frac{1}{2})$	$(0, 0)$	$(\mathbf{1}, \mathbf{2})$
ℓ_3	0	$(0, 0)$	$(0, -\frac{1}{2})$	$(\mathbf{1}, \mathbf{2})$
e_1^c	1	$(0, 0)$	$(0, 0)$	$(\mathbf{1}, \mathbf{1})$
e_2^c	0	$(\frac{1}{2}, \frac{1}{2})$	$(0, 0)$	$(\mathbf{1}, \mathbf{1})$
e_3^c	0	$(0, 0)$	$(\frac{1}{2}, \frac{1}{2})$	$(\mathbf{1}, \mathbf{1})$
ν_2^c	0	$(-\frac{1}{2}, \frac{1}{2})$	$(0, 0)$	$(\mathbf{1}, \mathbf{1})$
ν_3^c	0	$(0, 0)$	$(-\frac{1}{2}, \frac{1}{2})$	$(\mathbf{1}, \mathbf{1})$
$H_{u,d}$	0	$(0, 0)$	$(\pm\frac{1}{2}, 0)$	$(\mathbf{1}, \mathbf{2})$
χ_2	0	$(1, -1)$	$(0, 0)$	$(\mathbf{1}, \mathbf{1})$
χ_3	0	$(0, 0)$	$(1, -1)$	$(\mathbf{1}, \mathbf{1})$
ϕ_{12}^R	$\frac{1}{2}$	$(-\frac{1}{2}, 0)$	$(0, 0)$	$(\mathbf{1}, \mathbf{1})$
ϕ_{12}^L	$\frac{1}{2}$	$(0, -\frac{1}{2})$	$(0, 0)$	$(\mathbf{1}, \mathbf{1})$
ϕ_{23}^R	0	$(\frac{1}{2}, 0)$	$(-\frac{1}{2}, 0)$	$(\mathbf{1}, \mathbf{1})$
ϕ_{23}^L	0	$(0, \frac{1}{2})$	$(0, -\frac{1}{2})$	$(\mathbf{1}, \mathbf{1})$

Table 2: Field content relevant for the lepton sector. $H_{u,d}$, $\chi_{2,3}$ and $\phi_{ij}^{R,L}$ are scalars while the rest are the three usual generations of chiral leptons plus two right-handed neutrinos.

3 Natural neutrino mass hierarchy from UV extension of tri-hypercharge

It is well known that a natural neutrino mass hierarchy can result from the sequential dominance of three right-handed neutrinos (see e.g. [39–42]), where one of them is decoupled and may be ignored, while the other two contribute sequentially to the seesaw mechanism. In TH theories, this may be achieved by extending two of the hypercharge gauge groups into respective $B - L$ gauge groups, which prevents large Majorana masses, while one hypercharge gauge group remains intact, allowing one heavy decoupled right-handed neutrino. That is the strategy that we follow in this section.

3.1 Gauge symmetry and symmetry breaking

In this section, we will show that the conditions of sequential dominance are naturally achieved when tri-hypercharge is extended to a larger gauge group under which two of the ν_i^c neutrinos are not singlets, thereby preventing their Majorana masses until the larger group is broken. To this end, we consider the tri-hypercharge gauge symmetry in Eq. (2.1) to be an effective low energy theory resulting from the ultraviolet (UV) gauge group,

$$G_{\text{UV}} = U(1)_{Y_1} \times U(1)_{R_2} \times U(1)_{(B-L)_{2/2}} \times U(1)_{R_3} \times U(1)_{(B-L)_{3/2}}, \quad (3.7)$$

while $SU(3)_c$ and $SU(2)_L$ remain universal. The decomposition of hypercharge into an Abelian symmetry for right-handed particles $U(1)_R$ and baryon-minus-lepton number $U(1)_{B-L}$, is well motivated from left-right symmetric embeddings of the SM, such as the Pati-Salam [43] and $SO(10)$ [44, 45] theories. Here we apply this idea to the second and third family hypercharges, with the gauge group in Eq. (3.7) broken down to the TH gauge group in Eq. (2.1) at high energy, as discussed

below, where the hypercharge generators are given by

$$Y_2 = R_2 + \frac{1}{2}(B - L)_2, \quad (3.8)$$

$$Y_3 = R_3 + \frac{1}{2}(B - L)_3, \quad (3.9)$$

noting the $1/2$ normalisation in the $B - L$ charge definition to be consistent with our normalisation of hypercharge.

Two right-handed neutrinos $\nu_{2,3}^c$ are now required by anomaly cancellation, and they are charged under $U(1)_{R_2} \times U(1)_{(B-L)_2/2}$ and $U(1)_{R_3} \times U(1)_{(B-L)_3/2}$ respectively. In general, we may also decompose $U(1)_{Y_1}$ into R_1 and $(B - L)_1$, providing another right-handed neutrino ν_1^c from anomaly cancellation. However, as we will show in the following, our model will naturally generate sequential dominance, where ν_1^c provides subleading² contributions to the seesaw mechanism. Therefore, for simplicity we will assume that Y_1 is not decomposed until higher scales so that ν_1^c is effectively decoupled from the seesaw.

In order to break the symmetry, we introduce the scalars $\chi_i \sim (1, -1)_i$, with $i = 2, 3$ as shown in Table 2, whose VEVs spontaneously break the UV group in Eq. (3.7) down to the TH group in Eq. (2.1),

$$G_{\text{UV}} \xrightarrow{\langle \chi_{2,3} \rangle} G_{\text{TH}} \quad (3.10)$$

corresponding to $U(1)_{R_i} \times U(1)_{(B-L)_i/2} \rightarrow U(1)_{Y_i}$ thereby recovering tri-hypercharge, simultaneously allowing Majorana masses for the right-handed neutrinos $m_{\nu_i^c} \sim \langle \chi_i \rangle$. We expect these VEVs to be large enough to provide most of the suppression for small neutrino masses, in a natural framework.

Finally, we also introduce the scalars $\phi_{12,23}^{R,L}$ as shown in Table 2, which play the role of the hyperons in the TH theory. Indeed after the χ_i scalars get their VEVs, the scalars $\phi_{12,23}^{R,L}$ reduce to the hyperons $\phi_{12,23}$ of TH as shown in Table 1. They get VEVs [28, 35] which break tri-hypercharge down to the SM, i.e. the chain of symmetry breaking is

$$G_{\text{UV}} \xrightarrow{\langle \chi_i \rangle} G_{\text{TH}} \xrightarrow{\langle \phi_{ij} \rangle} U(1)_Y, \quad (3.11)$$

where $\langle \chi_i \rangle > \langle \phi_{ij} \rangle$ and $\langle \chi_i \rangle$ is large enough to provide the required suppression for neutrino masses, as we shall see. We also need to consider Higgs doublet(s) to break spontaneously the electroweak symmetry. Following the same strategy as in the original implementation of tri-hypercharge [28], we consider two Higgs doublets $H_{u,d}$ to take into account the overall different normalisation between the up sector and the down-quark/charged lepton sectors. These are only charged under R_3 to allow for third family renormalisable Yukawa couplings only, as in tri-hypercharge. The scalar potential is discussed in Appendix D, where we also show that all Goldstone modes may be given heavy masses.

²In particular, any contribution from ν_1^c will be higher order in the sequential dominance expansion. Note that we are assuming only mild hierarchies of dimensionless couplings and Majorana masses, which necessitates the gauge extension discussed here, since by inspecting the mass matrix textures in Eq. (2.4) and the sequential dominance conditions in Appendix B, it is clear that unextended tri-hypercharge does not generate sequential dominance unless the dimensionless coefficients or the Majorana mass terms are assumed to be hierarchical.

3.2 Leptons

We start by writing the Yukawa couplings involving charged leptons, along with the Dirac and Majorana mass matrices in the neutrino sector³,

$$\mathcal{L}_e = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \begin{pmatrix} a_{11}^e \epsilon_{12}^R \epsilon_{23}^R & a_{12}^e \epsilon_{12}^L \epsilon_{23}^R & a_{13}^e \epsilon_{12}^L \epsilon_{23}^L \\ a_{21}^e \epsilon_{12}^L \epsilon_{23}^R & a_{22}^e \epsilon_{23}^R & a_{23}^e \epsilon_{23}^L \\ a_{31}^e \epsilon_{12}^L \epsilon_{23}^R & a_{32}^e \epsilon_{23}^L & a_{33}^e \end{pmatrix} \begin{pmatrix} e_1^c \\ e_2^c \\ e_3^c \end{pmatrix} H_d + \text{h.c.}, \quad (3.12)$$

$$\begin{aligned} \mathcal{L}_\nu &= \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} m_D \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} + \begin{pmatrix} \nu_1^c & \nu_2^c \end{pmatrix} M_M \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} + \text{h.c.} \\ &= \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \end{pmatrix} \begin{pmatrix} a_{12}^\nu \epsilon_{12}^L \epsilon_{23}^R & a_{13}^\nu \epsilon_{12}^L \epsilon_{23}^L \\ a_{22}^\nu \epsilon_{23}^R & a_{23}^\nu \epsilon_{23}^L \\ a_{32}^\nu \epsilon_{23}^L & a_{33}^\nu \end{pmatrix} \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} H_u + \begin{pmatrix} \nu_1^c & \nu_2^c \end{pmatrix} \begin{pmatrix} \chi_2 & \epsilon_{23}^L \epsilon_{23}^R \chi_3 \\ \epsilon_{23}^L \epsilon_{23}^R \chi_3 & \chi_3 \end{pmatrix} \begin{pmatrix} \nu_1^c \\ \nu_2^c \end{pmatrix} + \text{h.c.} \end{aligned} \quad (3.13)$$

where $\epsilon_{ij}^{R,L} = \langle \phi_{ij}^{R,L} \rangle / \Lambda_{ij}$ and $a_{ij}^{e,\nu}$ are dimensionless coefficients expected to be of $\mathcal{O}(1)$.

The charged lepton texture above delivers the following approximate scalings for the charged leptons mass eigenvalues,

$$m_e \sim \epsilon_{12}^R \epsilon_{23}^R \langle H_d \rangle, \quad m_\mu \sim \epsilon_{23}^R \langle H_d \rangle, \quad m_\tau \sim \langle H_d \rangle. \quad (3.14)$$

Therefore, we obtain approximate numerical values for the small parameters $\epsilon_{12,23}^R$ in order to reproduce the observed charged lepton mass spectrum,

$$\epsilon_{12}^R \sim \frac{m_e}{m_\mu} \simeq 0.005, \quad \epsilon_{23}^R \sim \frac{m_\mu}{m_\tau} \simeq 0.06. \quad (3.15)$$

The coefficients $\epsilon_{12,23}^L$ are relevant for charged lepton and neutrino mixing, which ultimately will contribute to the PMNS mixing angles. Therefore, we expect $\epsilon_{ij}^R < \epsilon_{ij}^L$ motivated from the observed patterns of charged lepton masses and PMNS mixing. This imprints a hierarchical column structure into the Yukawa couplings of charged leptons and neutrinos, which suggests the presence of sequential dominance [39–42]. In particular, in our model the conditions of sequential dominance in the neutrino and charged lepton sectors (see Eqs. (B.65) and (B.81)) translate into

$$\frac{|a_{33}^\nu|^2, |a_{23}^\nu \epsilon_{23}^L|^2, |a_{33}^\nu a_{23}^\nu \epsilon_{23}^L|}{\langle \chi_3 \rangle} \gg \frac{|x^\nu y^\nu|}{\langle \chi_2 \rangle}, \quad (3.16)$$

$$|a_{33}^e|^2, |a_{23}^e \epsilon_{23}^L|^2, |a_{33}^e a_{23}^e \epsilon_{23}^L| \gg |x^e y^e| \gg |(x^e)'(y^e)'|, \quad (3.17)$$

where

$$x^{\nu,e}, y^{\nu,e} = a_{32}^{\nu,e} \epsilon_{23}^R \epsilon_{23}^L, a_{22}^{\nu,e} \epsilon_{23}^R, a_{12}^{\nu,e} \epsilon_{23}^R \epsilon_{12}^L, \quad (3.18)$$

$$(x^e)', (y^e)' = a_{11}^e \epsilon_{12}^R \epsilon_{23}^R, a_{21}^e \epsilon_{12}^L \epsilon_{12}^R \epsilon_{23}^R, a_{31}^e \epsilon_{12}^L \epsilon_{23}^R \epsilon_{12}^R \epsilon_{23}^R. \quad (3.19)$$

The presence of sequential dominance allows us to derive simple analytical formulas for mass eigen-

³Without loss of generality, we absorb the Yukawa couplings $a_{22}^{\nu,c}$ and $a_{33}^{\nu,c}$ into the definitions of $\langle \chi_2 \rangle$ and $\langle \chi_3 \rangle$ in M_M .

values and mixing angles at leading order in the sequential dominance expansion, following the results in Appendix B. For the neutrino sector, we obtain

$$\tan \theta_{23}^\nu \simeq \frac{|a_{23}^\nu|}{|a_{33}^\nu|} \epsilon_{23}^L, \quad (3.20)$$

$$\theta_{13}^\nu \simeq \frac{|a_{13}^\nu|}{\sqrt{|a_{33}^\nu|^2 + |a_{23}^\nu \epsilon_{23}^L|^2}} \epsilon_{12}^L \epsilon_{23}^L, \quad (3.21)$$

$$\tan \theta_{12}^\nu \simeq \frac{|a_{12}^\nu| \epsilon_{12}^L}{c_{23}^\nu |a_{22}^\nu| \cos(\tilde{\phi}_{a_{22}^\nu}) - s_{23}^\nu |a_{32}^\nu| \cos(\tilde{\phi}_{a_{32}^\nu}) \epsilon_{23}^L}, \quad (3.22)$$

$$m_3 \simeq \frac{|a_{33}^\nu|^2 + |a_{23}^\nu \epsilon_{23}^L|^2}{\langle \chi_3 \rangle} \langle H_u \rangle^2 \quad (3.23)$$

$$m_2 \simeq \frac{|a_{12}^\nu \epsilon_{12}^L \epsilon_{23}^R|^2}{\langle \chi_2 \rangle (s_{12}^\nu)^2} \langle H_u \rangle^2 \quad (3.24)$$

$$m_1 = 0, \quad (3.25)$$

while for the charged lepton sector we obtain

$$\tan \theta_{23}^e \simeq \frac{|a_{23}^e|}{|a_{33}^e|} \epsilon_{23}^L, \quad (3.26)$$

$$\theta_{13}^e \simeq \frac{|a_{13}^e|}{\sqrt{|a_{33}^e|^2 + |a_{23}^e \epsilon_{23}^L|^2}} \epsilon_{12}^L \epsilon_{23}^L, \quad (3.27)$$

$$\tan \theta_{12}^e \simeq \frac{|a_{12}^e| \epsilon_{12}^L}{c_{23}^e |a_{22}^e| \cos(\tilde{\phi}_{a_{22}^e}) - s_{23}^e |a_{32}^e| \cos(\tilde{\phi}_{a_{32}^e}) \epsilon_{23}^L}, \quad (3.28)$$

$$m_\tau \simeq \sqrt{|a_{33}^e|^2 + |a_{23}^e \epsilon_{23}^L|^2} \langle H_d \rangle \quad (3.29)$$

$$m_\mu \simeq \frac{|a_{12}^e \epsilon_{12}^L \epsilon_{23}^R|}{s_{12}^e} \langle H_d \rangle \quad (3.30)$$

$$m_e \simeq \left[|a_{11}^e| c_{12}^e \cos(\tilde{\phi}_{a_{11}^e}) - |a_{21}^e \epsilon_{12}^L| s_{12}^e c_{23}^e \cos(\tilde{\phi}_{a_{21}^e}) \right. \quad (3.31)$$

$$\left. + |a_{31}^e \epsilon_{12}^L \epsilon_{23}^L| s_{12}^e s_{23}^e \cos(\tilde{\phi}_{a_{31}^e}) \right] \epsilon_{12}^R \epsilon_{23}^R \langle H_d \rangle, \quad (3.32)$$

where the phases $\tilde{\phi}$ are defined in Appendix B.

The sequential dominance conditions in the charged lepton sector (3.17) are automatically satisfied by the hierarchical column structure of the Yukawa textures, enforced by the smaller $\epsilon_{12,23}^R$ and larger $\epsilon_{12,23}^L$, which arise as a natural consequence of explaining the charged lepton mass hierarchies and large lepton mixing simultaneously. In the neutrino sector, the strong hierarchy by columns also suggests sequential dominance, although the conditions (3.16) also depend on $\langle \chi_2 \rangle$ and $\langle \chi_3 \rangle$. Due to the $\epsilon_{23}^L \epsilon_{23}^R$ suppression of the off-diagonal entries in M_M , we shall approximate this as a diagonal matrix. By using the formulas (3.23) and (3.24) with $\epsilon_{23}^R \sim 0.06$, $\epsilon_{12,23}^L \gtrsim 0.1$ and setting the dimensionless coefficients to $\mathcal{O}(1)$, we find that in order to reproduce neutrino data we have $\langle \chi_2 \rangle \sim 10^{13} \text{ GeV}$ and $\langle \chi_3 \rangle \sim 10^{14} \text{ GeV}$. Departing from these values would require tuning of dimensionless coefficients in order to be consistent with lepton data. For consistency of the EFT we assume the cutoffs Λ_{ij} to be the largest scales of the theory, i.e. $\Lambda_{ij} \gtrsim \langle \chi_i \rangle, \langle \phi_{ij}^{R,L} \rangle$. This implies

that the VEVs $\langle \phi_{ij}^{R,L} \rangle$ are high scale⁴, most likely $\langle \phi_{23}^{R,L} \rangle \gtrsim 10^{13}$ GeV and $\langle \phi_{12}^{R,L} \rangle \gtrsim 10^{12}$ GeV. The very heavy scales $\langle \chi_2 \rangle$ and $\langle \chi_3 \rangle$ are then responsible for the smallness of active neutrino masses in our model. The fact that both are close to each other may suggest the presence of a cyclic symmetry relating the three families and providing gauge unification of our model in a flavour deconstructed $SO(10)^3$ framework, similar to tri-unification in $SU(5)^3$ [30].

Moreover, the presence of sequential dominance in our model translates into potential predictivity over the origin of the neutrino flavour pattern:

1. We obtain a natural mass hierarchy among normally ordered neutrino mass eigenvalues,

$$m_3^2 \gg m_2^2 \gg m_1^2. \quad (3.33)$$

2. The atmospheric neutrino mass m_3 and the mixing angle θ_{23}' are determined by the couplings of the dominant right-handed neutrino with mass $\langle \chi_3 \rangle$. The solar neutrino mass m_2 and the mixing angle θ_{12}' are determined by the couplings of the subdominant right-handed neutrino of mass $\langle \chi_2 \rangle$. Moreover, these mixing angles and mass eigenvalues are described by the simple analytical formulas given above, at leading order in the sequential dominance expansion, which were derived from the general results in Appendix B.1.
3. In our model the lightest neutrino is massless, $m_1 = 0$, in good agreement with the current bounds on the neutrino scale Σm_ν by cosmological observations [46] and by the KATRIN experiment [47]. However, a tiny mass will be generated if we decompose Y_1 as discussed before, connected to a heavier and effectively decoupled right-handed neutrino with mass $\langle \chi_1 \rangle$. This would be necessary for the potential embedding into a grand unified scenario, such as $SO(10)^3$.
4. Sequential dominance in the charged lepton sector as delivered by our Yukawa texture ensures a naturally hierarchical mass spectrum and suppressed right-handed charged lepton mixing. Mass eigenvalues and mixing angles are described by the simple compact formulas above, derived from the model-independent results in Appendix B.2 at leading order in the sequential dominance expansion.

This potential predictivity can be regarded as a first step towards building a complete theory of neutrino flavour in the framework of flavour deconstruction, and in particular a significant step forward with respect to the hypothesis of anarchy.

3.3 Lepton mixing

The structures of the charged lepton and neutrino Yukawa couplings are similar up to $\mathcal{O}(1)$ coefficients, therefore we expect both to contribute to the PMNS mixing angles. Taking into account contributions from both charged leptons and neutrinos, in all generality the PMNS matrix is given by (see our parametrisations and conventions for the unitary matrices in Appendix A)

$$U_{\text{PMNS}} = U_e U_\nu^\dagger = U_{12}^{e\dagger} U_{13}^{e\dagger} U_{23}^{e\dagger} U_{23}^\nu U_{13}^\nu U_{12}^\nu. \quad (3.34)$$

⁴We note that by going beyond the EFT and specifying the degrees of freedom corresponding to Λ_{ij} as in [35], a hierarchy of scales $\langle \chi_i \rangle \gg \langle \phi_{ij}^{R,L} \rangle$ would be possible without changing our conclusions, and the tri-hypercharge symmetry breaking may be realised at the TeV scale $\langle \phi_{ij}^{R,L} \rangle \sim \mathcal{O}(\text{TeV})$, with the associated low energy phenomenology [28, 35].

In Appendix C we provide fully general formulas for the PMNS mixing angles which take into account both charged lepton and neutrino contributions. In our model, given the structures of Eqs. (3.12) and (3.13), we expect that the 13 angles are doubly suppressed as $\sin \theta_{13}^{\nu,e} \sim \epsilon_{12}^L \epsilon_{23}^L$. As a consequence, the PMNS angle θ_{13} is dominated by 12 mixing in the charged lepton sector, $\theta_{13} \sim \sin \theta_{12}^e \sim \epsilon_{12}^L \sim 0.1$ (see Eq. (C.102)). Interestingly, this suggests that the PMNS angle θ_{13} originates from 12 charged lepton mixing of similar size to the Cabibbo angle in the quark sector. All in all, due to the smallness of the PMNS angle θ_{13} and the parametric suppression of 13 angles in our model, we expand the fully general formulas of Eqs. (C.101), (C.102) and (C.103) to linear order in $\theta_{13}^{\nu,e}$ and θ_{12}^e to achieve simple formulas for the PMNS mixing angles in our model,

$$s_{23} e^{-i\delta_{23}} \approx c_{23}^e s_{23}^\nu e^{-i\delta_{23}^\nu} - c_{23}^\nu s_{23}^e e^{-i\delta_{23}^e}, \quad (3.35)$$

$$\theta_{13} e^{-i\delta_{13}} \approx \theta_{13}^\nu e^{-i\delta_{13}^\nu} - \theta_{13}^e C_{23} e^{-i\delta_{13}^e} - \theta_{12}^e S_{23} e^{-i\delta_{12}^e}, \quad (3.36)$$

$$s_{12} e^{-i\delta_{12}} \approx s_{12}^\nu e^{-i\delta_{12}^\nu} + \theta_{13}^e S_{23}^* c_{12}^\nu e^{-i\delta_{13}^e} - \theta_{12}^e C_{23}^* c_{12}^\nu e^{-i\delta_{12}^e}, \quad (3.37)$$

where $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$, and the complex quantities S_{23} and C_{23} are defined as

$$C_{23} \equiv c_{23}^e c_{23}^\nu + s_{23}^e s_{23}^\nu e^{i(\delta_{23}^e - \delta_{23}^\nu)}, \quad (3.38)$$

$$S_{23} \equiv c_{23}^e s_{23}^\nu e^{-i\delta_{23}^\nu} - c_{23}^\nu s_{23}^e e^{-i\delta_{23}^e}. \quad (3.39)$$

It is illustrative to include explicitly the expressions for these quantities. This leads to

$$s_{23} e^{-i\delta_{23}} \approx c_{23}^e s_{23}^\nu e^{-i\delta_{23}^\nu} - c_{23}^\nu s_{23}^e e^{-i\delta_{23}^e}, \quad (3.40)$$

$$\begin{aligned} \theta_{13} e^{-i\delta_{13}} \approx & \theta_{13}^\nu e^{-i\delta_{13}^\nu} - \theta_{13}^e (c_{23}^e c_{23}^\nu + s_{23}^e s_{23}^\nu e^{i(\delta_{23}^e - \delta_{23}^\nu)}) e^{-i\delta_{13}^e} \\ & - \theta_{12}^e (c_{23}^e s_{23}^\nu e^{-i\delta_{23}^\nu} - c_{23}^\nu s_{23}^e e^{-i\delta_{23}^e}) e^{-i\delta_{12}^e}, \end{aligned} \quad (3.41)$$

$$\begin{aligned} s_{12} e^{-i\delta_{12}} \approx & s_{12}^\nu e^{-i\delta_{12}^\nu} + \theta_{13}^e (c_{23}^e s_{23}^\nu e^{i\delta_{23}^\nu} - c_{23}^\nu s_{23}^e e^{i\delta_{23}^e}) c_{12}^\nu e^{-i\delta_{13}^e} \\ & - \theta_{12}^e (c_{23}^e c_{23}^\nu + s_{23}^e s_{23}^\nu e^{-i(\delta_{23}^e - \delta_{23}^\nu)}) c_{12}^\nu e^{-i\delta_{12}^e}. \end{aligned} \quad (3.42)$$

We observe that the large θ_{23} (atmospheric) angle is generated from both charged lepton and neutrino contributions, which may interfere positively or negatively depending on the value of the phases δ_{23}^ν and δ_{23}^e . The interference is maximal for $\delta_{23}^\nu - \delta_{23}^e = (2n - 1)\pi$ and minimal for $\delta_{23}^\nu - \delta_{23}^e = 2n\pi$ (with n being any integer number), which correspond to the case of real mixing angles. Within the validity of the EFT used to define the ϵ_{ij}^L parameters, we expect some suppression of the 23 angles $s_{23}^{\nu,e} \sim \epsilon_{23}^L$, but this suppression is mild since $\epsilon_{23}^L > \epsilon_{23}^R \sim 0.06$ in order to reproduce lepton data as discussed before. Therefore, we envisage that large PMNS θ_{23} is generated either via a mild tuning (not worst than 10%), which may be distributed among the different dimensionless parameters $a_{23,33}^{\nu,e}$, or by going beyond the EFT to generate $\epsilon_{23}^L \sim \mathcal{O}(1)$ as in [35].

The θ_{13} (reactor) angle is generated from 12 mixing in the charged lepton sector, as $\theta_{13} \sim \theta_{12}^e \sim \epsilon_{12}^L \sim 0.1$, and receives corrections of $\mathcal{O}(\epsilon_{12}^L \epsilon_{23}^L)$ from the 13 angles $\theta_{13}^{\nu,e}$. Interestingly, $\theta_{12}^e \sim 0.1$ and the Cabibbo angle $\theta_c \sim 0.2$ that generates 12 quark mixing are of similar size.

Finally, the θ_{12} (solar) angle is mostly generated from 12 neutrino mixing, $s_{12} \approx s_{12}^\nu$, with leading corrections of $\mathcal{O}(\theta_{12}^e)$. Within the validity of the EFT used to define the ϵ_{ij}^L parameters, we expect a mild suppression of the 12 neutrino mixing angle $s_{12}^\nu \sim \epsilon_{12}^L \sim 0.1$. We consider this acceptable since the solar angle is $s_{12} \sim 0.5$.

Field	$U(1)_{Y_1}$	$U(1)_{R_2} \times U(1)_{(B-L)_2/2}$	$U(1)_{R_3} \times U(1)_{(B-L)_3/2}$	$SU(3)_c \times SU(2)_L$
q_1	$\frac{1}{6}$	$(0, 0)$	$(0, 0)$	$(\mathbf{3}, \mathbf{2})$
q_2	0	$(0, \frac{1}{6})$	$(0, 0)$	$(\mathbf{3}, \mathbf{2})$
q_3	0	$(0, 0)$	$(0, \frac{1}{6})$	$(\mathbf{3}, \mathbf{2})$
u_1^c	$-\frac{2}{3}$	$(0, 0)$	$(0, 0)$	$(\bar{\mathbf{3}}, \mathbf{1})$
u_2^c	0	$(-\frac{1}{2}, -\frac{1}{6})$	$(0, 0)$	$(\bar{\mathbf{3}}, \mathbf{1})$
u_3^c	0	$(0, 0)$	$(-\frac{1}{2}, -\frac{1}{6})$	$(\bar{\mathbf{3}}, \mathbf{1})$
d_1^c	$\frac{1}{3}$	$(0, 0)$	$(0, 0)$	$(\bar{\mathbf{3}}, \mathbf{1})$
d_2^c	0	$(\frac{1}{2}, -\frac{1}{6})$	$(0, 0)$	$(\bar{\mathbf{3}}, \mathbf{1})$
d_3^c	0	$(0, 0)$	$(\frac{1}{2}, -\frac{1}{6})$	$(\bar{\mathbf{3}}, \mathbf{1})$
ϕ_{12}^q	$-\frac{1}{6}$	$(0, \frac{1}{6})$	$(0, 0)$	$(\mathbf{1}, \mathbf{1})$
ϕ_{23}^q	0	$(0, -\frac{1}{6})$	$(0, \frac{1}{6})$	$(\mathbf{1}, \mathbf{1})$

Table 3: Field content relevant for the quark sector. $\phi_{12,23}^q$ are scalars while the rest are the three usual generations of chiral quarks.

3.4 Quarks

In the quark sector, our model generates natural mass hierarchies and small quark mixing as in tri-hypercharge. For completeness we show the explicit results in this section. With the field content shown in Table 3, we obtain the following Yukawa couplings for up-quarks and down-quarks⁵

$$\mathcal{L}_u = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} a_{11}^u \epsilon_{12}^R \epsilon_{23}^R & a_{12}^u \epsilon_{12}^q \epsilon_{23}^R & a_{13}^u \epsilon_{12}^q \epsilon_{23}^q \\ a_{21}^u \epsilon_{12}^q \epsilon_{12}^R \epsilon_{23}^R & a_{22}^u \epsilon_{23}^R & a_{23}^u \epsilon_{23}^q \\ a_{31}^u \epsilon_{12}^q \epsilon_{12}^R \epsilon_{23}^q \epsilon_{23}^R & a_{32}^u \epsilon_{23}^q \epsilon_{23}^R & a_{33}^u \end{pmatrix} \begin{pmatrix} u_1^c \\ u_2^c \\ u_3^c \end{pmatrix} H_u + \text{h.c.}, \quad (3.43)$$

$$\mathcal{L}_d = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} a_{11}^d \epsilon_{12}^R \epsilon_{23}^R & a_{12}^d \epsilon_{12}^q \epsilon_{23}^R & a_{13}^d \epsilon_{12}^q \epsilon_{23}^q \\ a_{21}^d \epsilon_{12}^q \epsilon_{12}^R \epsilon_{23}^R & a_{22}^d \epsilon_{23}^R & a_{23}^d \epsilon_{23}^q \\ a_{31}^d \epsilon_{12}^q \epsilon_{12}^R \epsilon_{23}^q \epsilon_{23}^R & a_{32}^d \epsilon_{23}^q \epsilon_{23}^R & a_{33}^d \end{pmatrix} \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \end{pmatrix} H_d + \text{h.c.}, \quad (3.44)$$

where we have defined $\epsilon_{12}^q = \langle \phi_{12}^q \rangle / \Lambda_{12}$ and $\epsilon_{23}^q = \langle \phi_{23}^q \rangle / \Lambda_{23}$. The quark Yukawa texture above delivers the following approximate scalings for the quark mass eigenvalues,

$$m_u \sim \epsilon_{12}^R \epsilon_{23}^R \langle H_u \rangle, \quad m_c \sim \epsilon_{23}^R \langle H_u \rangle, \quad m_t \sim \langle H_u \rangle, \quad (3.45)$$

$$m_d \sim \epsilon_{12}^R \epsilon_{23}^R \langle H_d \rangle, \quad m_s \sim \epsilon_{23}^R \langle H_d \rangle, \quad m_b \sim \langle H_d \rangle. \quad (3.46)$$

Just like in the charged lepton sector, the scaling with the small parameters $\epsilon_{12,23}^R$ generates a natural hierarchy among quark masses. Small quark mixing is naturally generated thanks to the suppression via the $\epsilon_{12,23}^q$ parameters. In particular, in order to explain CKM mixing we obtain

$$\epsilon_{23}^q \sim V_{cb} \sim 0.04, \quad \epsilon_{12}^q \sim V_{us} \sim 0.2, \quad (3.47)$$

while $V_{ub} \sim \epsilon_{23}^q \epsilon_{12}^q$.

⁵Notice that in the 32 and 31 entries of the down sector, the operators $(\phi_{23}^q)^2 q_3 H_d d_2^c$ and $(\phi_{12}^q)^2 (\phi_{23}^q)^2 q_3 H_d d_1^c$ also contribute at similar order in the EFT expansion, but have been neglected for simplicity.

4 Conclusions

Tri-hypercharge, and more generally the hypothesis of flavour deconstruction, gives a successful and predictive explanation of fermion mass hierarchies and small quark mixing. However, in the lepton sector, the mechanisms presented so far generate an anarchic neutrino flavour structure: the observed neutrino flavour pattern fully originates as an accidental configuration of $\mathcal{O}(1)$ dimensionless coefficients. This allows little predictivity about the origin of neutrino mass eigenvalues or mixing angles; both normal and inverted mass orderings are possible, there is no reason to have hierarchical neutrino masses, and there is no particular understanding of why we have two large and one small mixing angles in the PMNS matrix.

While anarchy is a valid approach, it is widely speculated in the literature that a dynamical mechanism might be responsible for the origin of the lepton flavour structure. For example, such a dynamical mechanism might satisfy the conditions of sequential right-handed neutrino dominance, leading to a natural neutrino mass hierarchy, with the large lepton mixing angles arising from ratios of Yukawa couplings involving particular right-handed neutrinos. Motivated by such considerations, we have shown that the minimal tri-hypercharge theory, when extended to include right-handed neutrinos with $B - L$ gauge charges, satisfies the sequential dominance conditions, but with a new twist: lepton mixing originates from both the neutrino and charged lepton sectors.

In order to obtain suitable analytic results, it was necessary to go beyond the standard sequential dominance results in the literature (see Appendices B.2 and C) in order to allow contributions to lepton mixing angles from both neutrino and charged lepton sectors. The key features of the model-independent results are:

- We use the same parametrisation and conventions for both charged leptons and neutrinos, and provide fully general formulas for the lepton mixing angles in Appendix C, including all potential neutrino and charged lepton mixing angles and phases.
- Within this parametrisation and conventions, we have computed novel formulas for the charged lepton mixing angles and masses at leading order in the sequential dominance expansion.

Armed with these tools, we have shown that, when the hypothesis of tri-hypercharge is extended to embrace the right-handed neutrino sector, by decomposing the family hypercharges into the corresponding $B - L$ gauge groups, then the sequential dominance conditions for right-handed neutrinos and charged leptons are satisfied naturally. The atmospheric neutrino mass m_3 and the mixing angle θ'_{23} are then determined by the couplings of a dominant right-handed neutrino. The solar neutrino mass m_2 and the mixing angle θ'_{12} are further determined by the couplings of a subdominant right-handed neutrino. Moreover, these mixing angles and mass eigenvalues are described by simple analytical formulas at leading order in the sequential dominance expansion.

In the present approach, the PMNS mixing angles generally receive contributions from both charged leptons and neutrinos, in contrast to the usual situation where either neutrinos or charged leptons contribute. In the considered case, the PMNS angle θ_{23} originates from large contributions from charged leptons and neutrinos. Interestingly, the PMNS angle θ_{13} originates mostly from 12 mixing in the charged lepton sector, similar in size to the Cabibbo angle. Finally, the PMNS angle θ_{12} is generated mostly from 12 mixing in the neutrino sector. The present model also explains naturally the origin of charged fermion mass hierarchies and small quark mixing, as in standard

tri-hypercharge. The EFT framework that we have employed points towards the scales of symmetry breaking being high. Potential high-scale signatures of this theory remain to be explored.

In principle, the present approach leads to the prospect of predictivity in the neutrino sector, due to the lepton mixing angles being simple ratios of Yukawa couplings, unlike theories with anarchy where such relations are not present. However, in practice, predictivity is limited by the unknown Yukawa couplings, which would need to be constrained by some model, as for example the Littlest Seesaw Models⁶. Even without such constrained sequential dominance, we find it remarkable that all fermion mass hierarchies, including that of neutrinos, as well as the disparate lepton and quark mixing patterns, can be qualitatively understood without relying on family symmetry, within a theory of gauge flavour deconstruction.

In summary, we have shown how a natural neutrino mass hierarchy with large lepton mixing angles may be achieved in a theory of gauge flavour deconstruction. The particular framework we considered is based on a tri-hypercharge gauge theory, extended to include $B - L$ gauge groups, where we found that the sequential dominance conditions arise naturally, as a consequence of the charged lepton mass hierarchy, and lepton mixing originates from both the neutrino and charged lepton sectors. We have presented new model-independent sequential dominance results applicable to this case but which may also be useful more generally. Finally we showed how natural quark mass and mixing may be included in such a framework.

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A Conventions

Note that instead of 4-component left-handed spinors e_L, ν_L and right-handed spinors e_R, ν_R , we choose to work with 2-component left-handed spinors e, ν and CP-conjugate right-handed spinors e^c, ν^c . In this notation, all spinors are left-handed by construction, and one can drop the chiral indexes. The same notation applies to the quark sector as well.

In the following, we exhibit the equivalence of different conventions used to parametrise the PMNS and the respective neutrino and charged lepton mixing matrices. In all cases we follow Appendix A in [41] for conventions and definitions. In the context of sequential dominance, the following convention for the PMNS is widely used,⁷

$$V_{\text{PMNS}} = V_e V_\nu^\dagger. \quad (\text{A.48})$$

⁶See [48] for a recent phenomenological discussion and references to explicit models.

⁷Note that our convention differs from the usual PDG convention [1] where $V_{\text{PMNS}} = V_e^\dagger V_\nu$.

With this convention, it is convenient to define the parametrisation of $V^\dagger$ rather than V because the PMNS matrix involves $V_\nu^\dagger$ and the neutrino mixing angles typically play a central role,

$$V^\dagger = P_2 R_{23} R_{13} P_1 R_{12} P_3. \quad (\text{A.49})$$

Here R_{ij} are a sequence of real rotations corresponding to real and positive angles θ_{ij} , and P_i are diagonal phase matrices. Note that a 3×3 unitary matrix may be parametrised by 3 angles and 6 phases. Our conventions are as follows

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \quad R_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}, \quad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{A.50})$$

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\chi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\phi_2} & 0 \\ 0 & 0 & e^{i\phi_3} \end{pmatrix}, \quad P_3 = \begin{pmatrix} e^{i\omega_1} & 0 & 0 \\ 0 & e^{i\omega_2} & 0 \\ 0 & 0 & e^{i\omega_3} \end{pmatrix}, \quad (\text{A.51})$$

where $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$. By commuting the phase matrices to the left, it can be shown that the parametrisation in Eq. (A.49) is equivalent to

$$U^\dagger = P U_{23} U_{13} U_{12}, \quad (\text{A.52})$$

where $P = P_1 P_2 P_3$ and

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{-i\delta_{23}} \\ 0 & -s_{23}e^{i\delta_{23}} & c_{23} \end{pmatrix}, \quad U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix}, \quad U_{12} = \begin{pmatrix} c_{12} & s_{12}e^{-i\delta_{12}} & 0 \\ -s_{12}e^{i\delta_{12}} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{A.53})$$

where the phases of the U_{ij} matrices are related to the phases of the P_i matrices as follows,

$$\delta_{23} = \chi + \omega_2 - \omega_3, \quad \delta_{13} = \omega_1 - \omega_3, \quad \delta_{12} = \omega_1 - \omega_2. \quad (\text{A.54})$$

We find the parametrisation of Eq. (A.49) convenient to diagonalise the neutrino and charged lepton mass matrices, in order to obtain expressions for the mixing angles and mass eigenvalues including the phases. In contrast, for the PMNS angles and phases we find more convenient the parametrisation of Eq. (A.52), i.e.

$$U_{\text{PMNS}} = P U_{23} U_{13} U_{12} \quad (\text{A.55})$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13}e^{-i\delta_{12}} & s_{13}e^{-i\delta_{13}} \\ s_{12}c_{23}e^{i\delta_{12}} - c_{12}s_{23}s_{13}e^{-i(\delta_{23}-\delta_{13})} & c_{12}c_{23} - s_{23}s_{23}s_{13}e^{-i(\delta_{12}+\delta_{23}-\delta_{13})} & s_{23}c_{13}e^{-i\delta_{23}} \\ -s_{12}s_{23}e^{i(\delta_{12}+\delta_{23})} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}e^{i\delta_{23}} - s_{12}c_{23}s_{13}e^{-i(\delta_{12}-\delta_{13})} & c_{23}c_{13} \end{pmatrix},$$

where in the second step the matrix P on the left (and an overall minus sign) have been removed

by charged lepton field redefinitions⁸ $\Delta V^e = -P^\dagger$. Finally, an alternative is given by

$$U_{\text{PMNS}} = R_{23}U_{13}R_{12}P_0, \quad (\text{A.56})$$

where $P_0 = \text{diag}(1, e^{i\beta_1}, e^{i\beta_2})$ and the phase δ_{13} in U_{13} is replaced by the Dirac phase δ_{CP} , while the phases β_1 and β_2 are known as Majorana phases. This parametrisation is widely used in the literature in the context of neutrino oscillations, see e.g. the PDG [1]. The parametrisation in Eq. (A.56) can be transformed to that in Eq. (A.55) by commuting the phase matrix P_0 to the left and then removing the phases of the left-hand side via charged lepton field redefinitions. The two parametrisations are then related by the phase relations

$$\delta_{23} = \beta_2, \quad \delta_{13} = \delta_{\text{CP}} + \beta_1, \quad \delta_{12} = \beta_1 - \beta_2. \quad (\text{A.57})$$

When taking into account contributions from both charged leptons and neutrinos, the general expression for the PMNS is simplest when using the parametrisation of Eq. (A.52) for the neutrino and charged lepton mixing matrices,

$$U_{\text{PMNS}} = U_e U_\nu^\dagger = U_{12}^{e\dagger} U_{13}^{e\dagger} U_{23}^{e\dagger} U_{23}^\nu U_{13}^\nu U_{12}^\nu. \quad (\text{A.58})$$

This is why we prefer to describe the PMNS with the parametrisation of Eq. (A.52) rather than with Eq. (A.49) or Eq. (A.56). Note that the expression above has 6 phases, so 3 of them will be removed with charged lepton field redefinitions to match the expression in Eq. (A.55).

The relation between the neutrino and charged lepton phases that appear in the PMNS and those which are relevant for the diagonalisation of the mass matrices is as follows,

$$\delta_{12}^\nu = \omega_1^\nu - \omega_2^\nu, \quad \delta_{13}^\nu = \omega_1^\nu - \omega_3^\nu, \quad \delta_{23}^\nu = \chi^\nu + \omega_2^\nu - \omega_3^\nu. \quad (\text{A.59})$$

$$\delta_{12}^e = \chi^e + \phi_2^e - \phi_2^\nu - \chi^\nu + \omega_1^\nu - \omega_2^\nu, \quad (\text{A.60})$$

$$\delta_{13}^e = \phi_3^e - \phi_3^\nu + \omega_1^\nu - \omega_3^\nu, \quad (\text{A.61})$$

$$\delta_{23}^e = -\phi_2^e + \phi_3^e + \phi_2^\nu - \phi_3^\nu + \chi^\nu + \omega_2^\nu - \omega_3^\nu. \quad (\text{A.62})$$

B Sequential dominance formalism

Sequential dominance was originally proposed as an elegant and natural way of accounting for a neutrino mass hierarchy and two large mixing angles, in contrast to the less appealing idea that these originate from an anarchic framework. The idea of sequential dominance is that one of the right-handed neutrinos contributes dominantly to the seesaw mechanism and determines the atmospheric neutrino mass and mixing. A second right-handed neutrino contributes subdominantly and determines the solar neutrino mass and mixing. The third right-handed neutrino is effectively decoupled from the seesaw mechanism. Subsequently, this concept has been extended to embrace the charged lepton sector as well, which may contribute equally to the origin of the PMNS mixing angles.

⁸Note that this is always possible since right-handed charged lepton phase rotations can always make the charged lepton masses real.

Ultimately, sequential dominance also delivers a simple framework to study the flavour structure of the lepton sector, where simple but accurate analytic formulas for mixing angles and mass eigenvalues are obtained in the form of a perturbative series expansion.

B.1 Neutrinos

The mechanism of sequential dominance is most simply described by assuming three right-handed neutrinos in the basis where the right-handed neutrino mass matrix is diagonal, although it can be also developed in other bases [39, 40]. In this basis, we parametrise the Dirac mass matrix and the Majorana mass matrix of right-handed neutrinos as (see e.g. [39–42])

$$m_D = \begin{pmatrix} a' & a & d \\ b' & b & e \\ c' & c & f \end{pmatrix}, \quad M_M = \begin{pmatrix} X' & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & Y \end{pmatrix}, \quad (\text{B.63})$$

where each right-handed neutrino couples to a column in m_D . By applying the seesaw formula, we obtain the effective active neutrino mass matrix as,⁹

$$m_\nu \simeq m_D M_M^{-1} m_D^T = \begin{pmatrix} \frac{a'^2}{X} + \frac{d^2}{Y} & \frac{ab}{X} + \frac{de}{Y} & \frac{ac}{X} + \frac{df}{Y} \\ \cdot & \frac{b^2}{X} + \frac{e^2}{Y} & \frac{bc}{X} + \frac{ef}{Y} \\ \cdot & \cdot & \frac{c^2}{X} + \frac{f^2}{Y} \end{pmatrix}, \quad (\text{B.64})$$

which is symmetric by construction. Sequential dominance occurs when the right-handed neutrinos dominate the effective neutrino mass matrix sequentially. This translates to the following dominance prescription over the model parameters,

$$\frac{|e|^2, |f|^2, |ef|}{Y} \gg \frac{|xy|}{X} \gg \frac{|x'y'|}{X'}, \quad (\text{B.65})$$

where $x, y = a, b, c$, and $x', y' = a', b', c'$. This prescription naturally delivers normal ordering for the neutrino mass eigenvalues along with a natural neutrino mass hierarchy,

$$m_3^2 \gg m_2^2 \gg m_1^2. \quad (\text{B.66})$$

Without loss of generality, we have chosen the prescription such that Y is the dominant right-handed neutrino, with X and X' being sequentially subdominant. The effective neutrino mass matrix m_ν must be diagonalised by applying a series of unitary transformations,

$$V_\nu m_\nu V_\nu^T = \text{diag}(m_1, m_2, m_3), \quad (\text{B.67})$$

where, by convention, we choose to parametrise V_ν via three subsequent 23, 13 and 12 unitary matrices as described in Appendix A. The prescription of sequential dominance allows to simplify this diagonalisation process by noting that the contributions associated to Y are dominant, and those associated to X and X' can be treated as small perturbations. Therefore, sequential dominance delivers simple approximate formulas for the flavour parameters in the neutrino sector at leading

⁹For simplicity we neglect contributions from the decoupled neutrino X' when writing m_ν .

order in the sequential dominance expansion [41]. In the following, we will neglect contributions from the small θ_{13}^ν angle to show compact formulas.

For the 23 mixing angle we have

$$\tan \theta_{23}^\nu \simeq \frac{e}{f} e^{-i(\phi_2^\nu - \phi_3^\nu)} = \frac{|e|}{|f|}, \quad (\text{B.68})$$

where the phases $\phi_{2,3}^\nu$, which originate from the unitary rotations (see Appendix A, Eq. (A.51)), are fixed as $\phi_2^\nu - \phi_3^\nu = \phi_e^\nu - \phi_f^\nu$ to make the angle real and positive.

For the 13 mixing angle we have

$$\theta_{13}^\nu \simeq \frac{Y}{|e|^2 + |f|^2} e^{i(\phi_2^\nu - 2\phi_e^\nu)} \left[\frac{a(s_{23}^\nu b + c_{23}^\nu c e^{i(\phi_e^\nu - \phi_f^\nu)})}{X} + e^{i\phi_e^\nu} \frac{d\sqrt{|e|^2 + |f|^2}}{Y} \right]. \quad (\text{B.69})$$

In our model of Section 3, beyond the sequential dominance conditions of Eq. (B.65), we find the following conditions to be satisfied,

$$\frac{|de|}{Y}, \frac{|df|}{Y} \gg \frac{|ab|}{X}, \frac{|bc|}{X}. \quad (\text{B.70})$$

In this case, the second term in Eq. (B.69) dominates, and we obtain the simpler result

$$\theta_{13}^\nu \simeq \frac{d}{\sqrt{|e|^2 + |f|^2}} e^{i(\phi_2^\nu - \phi_e^\nu)} = \frac{|d|}{\sqrt{|e|^2 + |f|^2}}, \quad (\text{B.71})$$

where the phase $\phi_2^\nu = \phi_e^\nu - \phi_d^\nu$ is fixed to make θ_{13}^ν real and positive. Together with the previous condition, this fixes also $\phi_3^\nu = \phi_f^\nu - \phi_d^\nu$.

The 12 mixing angle is given by

$$\tan \theta_{12}^\nu \simeq \frac{a}{c_{23}^\nu b - s_{23}^\nu c e^{i(\phi_e^\nu - \phi_f^\nu)}} e^{i(\phi_2^\nu + \chi^\nu)} = \frac{|a|}{c_{23}^\nu |b| \cos \tilde{\phi}_b^\nu - s_{23}^\nu |c| \cos \tilde{\phi}_c^\nu}, \quad (\text{B.72})$$

where the phase χ^ν originates as well from the unitary rotations (see Eq. (A.51)), and is fixed to make θ_{12}^ν real and positive as

$$c_{23}^\nu |b| \sin(\tilde{\phi}_b^\nu) = s_{23}^\nu |c| \sin(\tilde{\phi}_c^\nu), \quad (\text{B.73})$$

where we have defined

$$\tilde{\phi}_b^\nu = \phi_b^\nu - \phi_a^\nu - \phi_2^\nu - \chi^\nu, \quad (\text{B.74})$$

$$\tilde{\phi}_c^\nu = \phi_c^\nu + \phi_e^\nu - \phi_f^\nu - \phi_a^\nu - \phi_2^\nu - \chi^\nu. \quad (\text{B.75})$$

Finally, the mass eigenvalues are

$$m_3 \simeq \frac{|e|^2 + |f|^2}{Y} e^{2i(\phi_e^\nu - \phi_2^\nu - \omega_3^\nu)} = \frac{|e|^2 + |f|^2}{Y}, \quad (\text{B.76})$$

$$\begin{aligned} m_2 &\simeq \left[\frac{a^2}{X} + e^{-2i(\phi_2^\nu + \chi^\nu)} \frac{(c_{23}^\nu b - s_{23}^\nu c e^{i(\phi_e^\nu - \phi_f^\nu)})^2}{X} \right] e^{-2i\omega_2^\nu} \\ &= \left[\frac{|a|^2}{X} + \frac{(c_{23}^\nu |b| \cos \tilde{\phi}_b^\nu - s_{23}^\nu |c| \cos \tilde{\phi}_c^\nu)^2}{X} \right] e^{2i(\phi_a^\nu - \omega_2^\nu)} \end{aligned} \quad (\text{B.77})$$

$$= \frac{|a|^2}{X(s_{12}^\nu)^2},$$

$$m_1 \simeq 0 \tag{B.78}$$

where in the second and third steps of Eq. (B.77) we have used Eq. (B.72). The phases ω_3'' and ω_2'' , which originate as well from the unitary rotations (see Eq. (A.51)), have been fixed to make the mass eigenvalues real as $\omega_3'' = \phi_e'' - \phi_2''$ and $\omega_2'' = \phi_a''$. With this, all the phases from the unitary matrices in the neutrino sector have been fixed to obtain three real and positive mixing angles and mass eigenvalues.

These results show that in sequential dominance the atmospheric neutrino mass m_3 and the mixing angle θ_{23}'' are determined by the couplings of the dominant right-handed neutrino with mass Y . The solar neutrino mass m_2 and the mixing angle θ_{12}'' are determined by the couplings of the subdominant right-handed neutrino of mass X . The third right-handed neutrino of mass X' is effectively decoupled from the seesaw mechanism and leads to the vanishingly small mass m_1 , in good agreement with the current bounds on the neutrino scale $\sum m_\nu$ by cosmological observations [46] and by the KATRIN experiment [47].

We also note that the neutrino mixing angles quoted above correspond to the unitary matrices that diagonalise m_ν . They are equivalent to the physical angles of the PMNS matrix only when charged lepton mixing is neglected. Otherwise, in full generality, one obtains the PMNS matrix as,

$$V_{\text{PMNS}} = V_e V_\nu^\dagger, \tag{B.79}$$

where the charged lepton contribution is described in the next section.

B.2 Charged leptons

Sequential dominance in the context of charged lepton mixing generating the PMNS has been considered in the literature (see e.g. [49–51] for dedicated studies), but all these studies assume that neutrino mixing is subleading, and work with a parametrisation different to the one used when neutrinos dominate. Since we are interested in the case where both charged leptons and neutrinos may contribute significantly to the PMNS, we need to extend the results to allow for this so that we can treat neutrino and charged lepton contributions consistently. We therefore obtain new expressions for the mixing angles with respect to what is found in the literature, in particular using the same parametrisation of unitary matrices that is employed in the neutrino sector.

We define the charged lepton mass matrix as

$$m_e = \begin{pmatrix} a' & a & d \\ b' & b & e \\ c' & c & f \end{pmatrix}, \tag{B.80}$$

with the SD condition

$$|d|, |e|, |f| \gg |a|, |b|, |c| \gg |a'|, |b'|, |c'|. \tag{B.81}$$

This predicts hierarchical charged lepton masses and subleading right-handed charged lepton mixing with respect to left-handed charged lepton mixing. In the calculation, we include right-handed mixing and the associated phases, although in the results presented here the right-handed mixing

angles are neglected. Our parametrisation and conventions are shown in Appendix A, including the definition of mixing angles and phases. The general diagonalisation of the charged lepton mass matrix is done by following the recipe of Appendix E in [41]. In the following, we show compact results at leading order in the sequential dominance expansion and neglecting small corrections of order θ_{13}^e .

For the 23 mixing angle we obtain,

$$\tan \theta_{23}^e \simeq \frac{e}{f} e^{-i(\phi_2^e - \phi_3^e)} = \frac{|e|}{|f|}, \quad (\text{B.82})$$

where the phases $\phi_{2,3}^e$, which originate from the unitary rotations (see Eq. (A.51)), are fixed as $\phi_2^e - \phi_3^e = \phi_e^e - \phi_f^e$ to make the angle real and positive.

For the 13 mixing angle we obtain, in the small angle limit

$$\theta_{13}^e \approx \frac{d}{c_{23}^e f e^{-i\phi_3^e} + s_{23}^e e e^{-i\phi_2^e}} = \frac{|d|}{c_{23}^e |f| + s_{23}^e |e|} \simeq \frac{|d|}{\sqrt{|f|^2 + |e|^2}}. \quad (\text{B.83})$$

where we have fixed the value of the phase ϕ_3^e to make θ_{13}^e real and positive, $\phi_3^e = \phi_f^e - \phi_d^e$, or equivalently $\phi_2^e = \phi_e^e - \phi_d^e$ from the condition $\phi_2^e - \phi_3^e = \phi_e^e - \phi_f^e$.

For the 12 mixing angle we obtain

$$\tan \theta_{12}^e \simeq \frac{a e^{i(\phi_2^e + \chi^e)}}{c_{23}^e b - s_{23}^e c e^{i(\phi_2^e - \phi_3^e)}} = \frac{|a|}{c_{23}^e |b| \cos(\tilde{\phi}_b^e) - s_{23}^e |c| \cos(\tilde{\phi}_c^e)}, \quad (\text{B.84})$$

where the phase χ^e originates as well from the unitary rotations (see Eq. (A.51)), and we have defined

$$\tilde{\phi}_b^e = \phi_b^e - \phi_a^e - \phi_2^e - \chi^e, \quad (\text{B.85})$$

$$\tilde{\phi}_c^e = \phi_c^e + \phi_e^e - \phi_f^e - \phi_a^e - \phi_2^e - \chi^e, \quad (\text{B.86})$$

and fixed the phase χ^e to make $\tan \theta_{12}^e$ real,

$$c_{23}^e |b| \sin(\tilde{\phi}_b^e) = s_{23}^e |c| \sin(\tilde{\phi}_c^e). \quad (\text{B.87})$$

Finally, the mass eigenvalues are

$$m_\tau \simeq \sqrt{|e|^2 + |f|^2} e^{i(\phi_3^e - \phi_2^e + \phi_f^e + \omega_3^e - \omega_2^e)} = \sqrt{|e|^2 + |f|^2}, \quad (\text{B.88})$$

$$\begin{aligned} m_\mu &\simeq \left[c_{12}^e c_{23}^e b - c_{12}^e s_{23}^e c e^{i(\phi_2^e - \phi_3^e)} + s_{12}^e a e^{i(\phi_2^e + \chi^e)} \right] e^{i(\phi_2^e - \phi_3^e + \chi^e - \omega_2^e + \omega_3^e)} \\ &= \frac{|a|}{s_{12}^e} e^{i(\phi_2^e + \phi_a^e + \chi^e + \omega_2^e - \omega_3^e)} = \frac{|a|}{s_{12}^e}, \end{aligned} \quad (\text{B.89})$$

$$\begin{aligned} m_e &\simeq \left[a' c_{12}^e - b' s_{12}^e c_{23}^e e^{-i(\phi_2^e + \chi^e)} + c' s_{12}^e s_{23}^e e^{-i(\phi_3^e + \chi^e)} \right] e^{i(\omega_1^e - \omega_2^e)} \\ &= |a'| c_{12}^e \cos(\tilde{\phi}_{a'}^e) - |b'| s_{12}^e c_{23}^e \cos(\tilde{\phi}_{b'}^e) + |c'| s_{12}^e s_{23}^e \cos(\tilde{\phi}_{c'}^e), \end{aligned} \quad (\text{B.90})$$

where we have defined

$$\tilde{\phi}_{a'}^e = \omega_1^e - \omega_2^e + \phi_{a'}^e, \quad (\text{B.91})$$

$$\tilde{\phi}_{b'}^e = \omega_1^{e^c} - \omega_1^e - \phi_2^e - \chi^e + \phi_{b'}^e, \quad (\text{B.92})$$

$$\tilde{\phi}_{c'}^e = \omega_1^{e^c} - \omega_1^e - \phi_3^e - \chi^e + \phi_{c'}^e, \quad (\text{B.93})$$

and we have fixed the phases $\omega_i^{e^c}$ to make the mass eigenvalues real,

$$\omega_3^{e^c} = \omega_3^e - \phi_3^{e^c} + \phi_3^e - \phi_f^e, \quad (\text{B.94})$$

$$\omega_2^{e^c} = \omega_2^e - \phi_2^{e^c} - \phi_a^e - \chi^{e^c}, \quad (\text{B.95})$$

$$|a'|c_{12}^e \sin(\tilde{\phi}_{a'}^e) = |b'|s_{12}^e c_{23}^e \sin(\tilde{\phi}_{b'}^e) - |c'|s_{12}^e s_{23}^e \sin(\tilde{\phi}_{c'}^e). \quad (\text{B.96})$$

The three phases ω_i^e remain unfixed from the diagonalisation process and can be used to remove three unphysical phases from the PMNS, see Eq. (A.55).

C General formulas for PMNS mixing angles

In this Appendix we provide fully general formulas for the mixing angles of the PMNS matrix, taking into account both charged lepton and neutrino contributions. In full generality, the PMNS matrix is given by

$$U_{\text{PMNS}} = U_e U_\nu^\dagger = U_{12}^{e\dagger} U_{13}^{e\dagger} U_{23}^{e\dagger} U_{23}^\nu U_{13}^\nu U_{12}^\nu, \quad (\text{C.97})$$

where we consider the conventions and parametrisation for the unitary matrices discussed in Appendix A. Note that the expression above has 6 phases, so 3 of them will be removed from charged lepton field redefinitions to match the expression in (A.55).

Let us focus first on the $U_{23}^{e\dagger} U_{23}^\nu$ product,

$$U_{23}^{e\dagger} U_{23}^\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23}^* & S_{23} \\ 0 & -S_{23}^* & C_{23} \end{pmatrix}, \quad (\text{C.98})$$

where we have defined

$$C_{23} \equiv c_{23}^e c_{23}^\nu + s_{23}^e s_{23}^\nu e^{i(\delta_{23}^e - \delta_{23}^\nu)}, \quad (\text{C.99})$$

$$S_{23} \equiv c_{23}^e s_{23}^\nu e^{-i\delta_{23}^\nu} - c_{23}^\nu s_{23}^e e^{-i\delta_{23}^e}. \quad (\text{C.100})$$

We can extract the PMNS $\sin\theta_{23}$ parameter from the 23 element of the PMNS matrix,

$$(U_{\text{PMNS}})_{23} = s_{23} c_{13} e^{-i\delta_{23}} = s_{12}^e e^{i\delta_{12}^e} (c_{13}^e s_{13}^\nu e^{-i\delta_{13}^\nu} - c_{13}^\nu C_{23} s_{13}^e e^{-i\delta_{13}^e}) + c_{12}^e c_{13}^\nu S_{23}. \quad (\text{C.101})$$

Now we extract the PMNS θ_{13} angle from the 13 element of the PMNS matrix,

$$(U_{\text{PMNS}})_{13} = s_{13} e^{-i\delta_{13}} = c_{12}^e \left(c_{13}^e s_{13}^\nu e^{-i\delta_{13}^\nu} - c_{13}^\nu C_{23} s_{13}^e e^{-i\delta_{13}^e} \right) - c_{13}^\nu s_{12}^e S_{23} e^{-i\delta_{12}^e}. \quad (\text{C.102})$$

Finally, we extract the PMNS $\sin\theta_{12}$ parameter from the 12 element of the PMNS matrix,

$$\begin{aligned} (U_{\text{PMNS}})_{12} &= s_{12} c_{13} e^{-i\delta_{12}} \\ &= c_{12}^\nu \left(S_{23}^* c_{12}^e s_{13}^e e^{-i\delta_{13}^e} - C_{23}^* s_{12}^e e^{-i\delta_{12}^e} \right) \end{aligned} \quad (\text{C.103})$$

$$+ s_{12}^\nu e^{-i\delta_{12}^\nu} \left(c_{12}^e c_{13}^e c_{13}^\nu + s_{13}^\nu e^{i\delta_{13}^\nu} \left(S_{23} s_{12}^e e^{-i\delta_{12}^e} + C_{23} c_{12}^e s_{13}^e e^{-i\delta_{13}^e} \right) \right).$$

D Scalar potential

The scalar potential of our model contains the following renormalisable terms,

$$V_\phi = \sum_i m_i^2 |\phi_i|^2 + \sum_{i,j} \lambda_{ij} |\phi_i|^2 |\phi_j|^2 + \left(\lambda_{q_{12}^3 L_{12}} (\phi_{12}^q)^3 \phi_{12}^L + \lambda_{q_{23}^3 L_{23}} (\phi_{23}^q)^3 \phi_{23}^L + \text{h.c.} \right), \quad (\text{D.104})$$

$$V_H = M_{H_u}^2 |H_u|^2 + M_{H_d}^2 |H_d|^2 + \left(M_{H_{ud}}^2 H_u H_d + \text{h.c.} \right) + \lambda_{H_u} |H_u|^4 + \lambda_{H_d} |H_d|^4 \quad (\text{D.105})$$

$$+ \lambda_{H_u H_d} |H_u|^2 |H_d|^2 + \lambda_{\tilde{H}_u \tilde{H}_d} (H_u H_d) (\tilde{H}_d \tilde{H}_u) + (\lambda_{H_u H_u H_d H_d} H_u H_u H_d H_d + \text{h.c.}),$$

$$V_{H\phi} = \sum_i \lambda_i^{H\phi} |H_u|^2 |\phi_i|^2 + \sum_i \lambda_i^{H\phi} |H_d|^2 |\phi_i|^2, \quad (\text{D.106})$$

$$V_\chi = M_{\chi_3}^2 |\chi_3|^2 + M_{\chi_2}^2 |\chi_2|^2 + \lambda_{\chi_3} |\chi_3|^4 + \lambda_{\chi_2} |\chi_2|^4 + \lambda_{\chi_{3,2}} |\chi_3|^2 |\chi_2|^2, \quad (\text{D.107})$$

$$V_{\chi\phi} = \sum_i \lambda_{\chi_3\phi} |\chi_3|^2 |\phi_i|^2 + \sum_i \lambda_{\chi_2\phi} |\chi_2|^2 |\phi_i|^2, \quad (\text{D.108})$$

$$V_{\chi H} = \lambda_{\chi_3 H} |\chi_3|^2 |H_{u,d}|^2 + \lambda_{\chi_2 H} |\chi_2|^2 |H_{u,d}|^2. \quad (\text{D.109})$$

Our model contains 5 gauge $U(1)$ s which are spontaneously broken down to the diagonal gauge hypercharge. This breaking is performed by 8 complex scalar degrees of freedom involving the fields $\phi_{12,23}^{R,L,q}$ and $\chi_{2,3}$. Therefore, one has 8 potential Goldstone modes, out of which 4 are eaten by heavy Z' s. Two of the remaining Goldstone modes acquire a heavy mass at renormalisable level, proportional to the non-trivial couplings $\lambda_{q_{12}^3 L_{12}}$ and $\lambda_{q_{23}^3 L_{23}}$. Given that we consider our model an EFT, the two final modes may acquire mass at non-renormalisable level, e.g. via the couplings

$$V_{d>4} \supset \frac{c_{R12}}{\Lambda} (\phi_{12}^R)^2 (\tilde{\phi}_{12}^L)^2 \chi_2 + \frac{c_{R23}}{\Lambda^2} \chi_3 (\phi_{23}^R)^2 \tilde{\chi}_2 (\tilde{\phi}_{23}^L)^2 + \text{h.c.} \quad (\text{D.110})$$

From the operators above, the remaining Goldstone modes will get a mass suppressed by the factors $\langle \chi_2 \rangle / \Lambda$ and $\langle \chi_3 \rangle \langle \chi_2 \rangle / \Lambda^2$ with respect to the radial modes. Since the radial modes are heavy (around 10^{12} GeV or more) and the VEVs $\langle \chi_{2,3} \rangle \sim 10^{14}$ GeV are large, then we expect the Goldstone modes to acquire a heavy mass, since Λ is not expected to exceed the GUT or Planck scales.

In a realistic UV theory, the Goldstone modes likely get a mass at tree level or loop level. Given that we expect all couplings (including Yukawa) of the model to be $\mathcal{O}(1)$, and that the radial modes are expected to be heavy (as mentioned above), we expect any potential Goldstone mode to be very heavy as well. The electroweak symmetry breaking then proceeds via the two Higgs doublets $H_{u,d}$, where potential Goldstone modes may get heavy masses proportional to non-trivial couplings at renormalisable level, such as $\lambda_{H_u H_u H_d H_d}$.

References

- [1] PARTICLE DATA GROUP collaboration, S. Navas et al., *Review of particle physics*, *Phys. Rev. D* **110** (2024) 030001.

- [2] A. Salam, *A gauge appreciation of developments in particle physics*, in *Proceedings of the European Physical Society International Conference on High Energy Physics*, (CERN, Geneva, 1979), footnote 41 therein.
- [3] S. Rajpoot, *Some Consequences of Extending the $SU(5)$ Gauge Symmetry to the Generation Symmetry $SU(5)_e \times SU(5)_\mu \times SU(5)_\tau$* , *Phys. Rev. D* **24** (1981) 1890.
- [4] H. Georgi, *Composite/Fundamental Higgs Mesons II: Model Building*, *Nucl. Phys. B* **202** (1982) 397.
- [5] X. Li and E. Ma, *Gauge Model of Generation Nonuniversality*, *Phys. Rev. Lett.* **47** (1981) 1788.
- [6] E. Ma, X. Li and S. F. Tuan, *Gauge Model of Generation Nonuniversality Revisited*, *Phys. Rev. Lett.* **60** (1988) 495.
- [7] E. Ma and D. Ng, *Gauge and Higgs Bosons in a Model of Generation Nonuniversality*, *Phys. Rev. D* **38** (1988) 304.
- [8] X.-y. Li and E. Ma, *Gauge model of generation nonuniversality reexamined*, *J. Phys. G* **19** (1993) 1265 [[hep-ph/9208210](#)].
- [9] C. T. Hill, *Topcolor assisted technicolor*, *Phys. Lett. B* **345** (1995) 483 [[hep-ph/9411426](#)].
- [10] R. Barbieri, G. R. Dvali and A. Strumia, *Fermion masses and mixings in a flavor symmetric GUT*, *Nucl. Phys. B* **435** (1995) 102 [[hep-ph/9407239](#)].
- [11] C. D. Carone and H. Murayama, *Third family flavor physics in an $SU(3)^3 \times SU(2)_L \times U(1)_Y$ model*, *Phys. Rev. D* **52** (1995) 4159 [[hep-ph/9504393](#)].
- [12] D. J. Muller and S. Nandi, *Top flavor: A Separate $SU(2)$ for the third family*, *Phys. Lett. B* **383** (1996) 345 [[hep-ph/9602390](#)].
- [13] E. Malkawi, T. M. P. Tait and C. P. Yuan, *A Model of strong flavor dynamics for the top quark*, *Phys. Lett. B* **385** (1996) 304 [[hep-ph/9603349](#)].
- [14] G. R. Dvali and M. A. Shifman, *Families as neighbors in extra dimension*, *Phys. Lett. B* **475** (2000) 295 [[hep-ph/0001072](#)].
- [15] T. Asaka and Y. Takanishi, *Masses and mixing of quarks and leptons in product-group unification*, [[hep-ph/0409147](#)].
- [16] K. S. Babu, S. M. Barr and I. Gogoladze, *Family Unification with $SO(10)$* , *Phys. Lett. B* **661** (2008) 124 [[0709.3491](#)].
- [17] N. Craig, D. Green and A. Katz, *(De)Constructing a Natural and Flavorful Supersymmetric Standard Model*, *JHEP* **07** (2011) 045 [[1103.3708](#)].
- [18] G. Panico and A. Pomarol, *Flavor hierarchies from dynamical scales*, *JHEP* **07** (2016) 097 [[1603.06609](#)].
- [19] M. Bordone, C. Cornella, J. Fuentes-Martin and G. Isidori, *A three-site gauge model for flavor hierarchies and flavor anomalies*, *Phys. Lett. B* **779** (2018) 317 [[1712.01368](#)].
- [20] A. Greljo and B. A. Stefanek, *Third family quark-lepton unification at the TeV scale*, *Phys. Lett. B* **782** (2018) 131 [[1802.04274](#)].
- [21] J. Fuentes-Martín and P. Stangl, *Third-family quark-lepton unification with a fundamental composite Higgs*, *Phys. Lett. B* **811** (2020) 135953 [[2004.11376](#)].
- [22] L. Allwicher, G. Isidori and A. E. Thomsen, *Stability of the Higgs Sector in a Flavor-Inspired Multi-Scale Model*, *JHEP* **01** (2021) 191 [[2011.01946](#)].
- [23] J. Fuentes-Martin, G. Isidori, J. Pagès and B. A. Stefanek, *Flavor non-universal Pati-Salam unification and neutrino masses*, *Phys. Lett. B* **820** (2021) 136484 [[2012.10492](#)].

- [24] J. Fuentes-Martin, G. Isidori, J. M. Lizana, N. Selimovic and B. A. Stefanek, *Flavor hierarchies, flavor anomalies, and Higgs mass from a warped extra dimension*, *Phys. Lett. B* **834** (2022) 137382 [[2203.01952](#)].
- [25] J. Davighi and J. Tooby-Smith, *Electroweak flavour unification*, *JHEP* **09** (2022) 193 [[2201.07245](#)].
- [26] J. Davighi, G. Isidori and M. Pesut, *Electroweak-flavour and quark-lepton unification: a family non-universal path*, *JHEP* **04** (2023) 030 [[2212.06163](#)].
- [27] J. Davighi and G. Isidori, *Non-universal gauge interactions addressing the inescapable link between Higgs and flavour*, *JHEP* **07** (2023) 147 [[2303.01520](#)].
- [28] M. Fernández Navarro and S. F. King, *Tri-hypercharge: a separate gauged weak hypercharge for each fermion family as the origin of flavour*, *JHEP* **08** (2023) 020 [[2305.07690](#)].
- [29] J. Davighi and B. A. Stefanek, *Deconstructed hypercharge: a natural model of flavour*, *JHEP* **11** (2023) 100 [[2305.16280](#)].
- [30] M. Fernández Navarro, S. F. King and A. Vicente, *Tri-unification: a separate $SU(5)$ for each fermion family*, *JHEP* **05** (2024) 130 [[2311.05683](#)].
- [31] J. Davighi, A. Gosnay, D. J. Miller and S. Renner, *Phenomenology of a Deconstructed Electroweak Force*, *JHEP* **05** (2024) 085 [[2312.13346](#)].
- [32] B. Capdevila, A. Crivellin, J. M. Lizana and S. Pokorski, *$SU(2)_L$ deconstruction and flavour (non)-universality*, *JHEP* **08** (2024) 031 [[2401.00848](#)].
- [33] R. Barbieri and G. Isidori, *Minimal flavour deconstruction*, *JHEP* **05** (2024) 033 [[2312.14004](#)].
- [34] J. Fuentes-Martín and J. M. Lizana, *Deconstructing flavor anomalously*, *JHEP* **07** (2024) 117 [[2402.09507](#)].
- [35] M. Fernández Navarro, S. F. King and A. Vicente, *Minimal complete tri-hypercharge theories of flavour*, *JHEP* **07** (2024) 147 [[2404.12442](#)].
- [36] A. Greljo and G. Isidori, *Neutrino anarchy from flavor deconstruction*, *Phys. Lett. B* **856** (2024) 138900 [[2406.01696](#)].
- [37] S. Covone, J. Davighi, G. Isidori and M. Pesut, *Flavour deconstructing the composite Higgs*, *JHEP* **01** (2025) 041 [[2407.10950](#)].
- [38] J. M. Lizana, *A common origin of the Higgs boson and the flavor hierarchies*, *JHEP* **05** (2025) 176 [[2412.14243](#)].
- [39] S. F. King, *Atmospheric and solar neutrinos with a heavy singlet*, *Phys. Lett. B* **439** (1998) 350 [[hep-ph/9806440](#)].
- [40] S. F. King, *Large mixing angle MSW and atmospheric neutrinos from single right-handed neutrino dominance and $U(1)$ family symmetry*, *Nucl. Phys. B* **576** (2000) 85 [[hep-ph/9912492](#)].
- [41] S. F. King, *Constructing the large mixing angle MNS matrix in seesaw models with right-handed neutrino dominance*, *JHEP* **09** (2002) 011 [[hep-ph/0204360](#)].
- [42] S. Antusch and S. F. King, *Sequential dominance*, *New J. Phys.* **6** (2004) 110 [[hep-ph/0405272](#)].
- [43] J. C. Pati and A. Salam, *Lepton Number as the Fourth Color*, *Phys. Rev. D* **10** (1974) 275.
- [44] H. Fritzsch and P. Minkowski, *Unified Interactions of Leptons and Hadrons*, *Annals Phys.* **93** (1975) 193.
- [45] H. Georgi, *The State of the Art—Gauge Theories*, *AIP Conf. Proc.* **23** (1975) 575.

- [46] DESI collaboration, A. G. Adame et al., *DESI 2024 VI: cosmological constraints from the measurements of baryon acoustic oscillations*, *JCAP* **02** (2025) 021 [[2404.03002](#)].
- [47] KATRIN collaboration, M. Aker et al., *Direct neutrino-mass measurement based on 259 days of KATRIN data*, *Science* **388** (2025) adq9592 [[2406.13516](#)].
- [48] F. Costa and S. F. King, *Neutrino Mixing Sum Rules and the Littlest Seesaw*, *Universe* **9** (2023) 472 [[2307.13895](#)].
- [49] S. Antusch and S. F. King, *Neutrino mixing from the charged lepton sector with sequential right-handed lepton dominance*, *Phys. Lett. B* **591** (2004) 104 [[hep-ph/0403053](#)].
- [50] S. F. King, *Predicting neutrino parameters from $SO(3)$ family symmetry and quark-lepton unification*, *JHEP* **08** (2005) 105 [[hep-ph/0506297](#)].
- [51] S. Antusch and S. F. King, *Charged lepton corrections to neutrino mixing angles and CP phases revisited*, *Phys. Lett. B* **631** (2005) 42 [[hep-ph/0508044](#)].