Journal of Philosophical Logic
https://doi.org/10.1007/s10992-025-09821-7

Epistemic Logic with Agentically Non-rigid Designators

Pavel Naumov[1] · **Xinran Zhang**[2]

Received: 8 March 2025 / Accepted: 29 October 2025

Abstract
The article introduces chained designators, like "my buddy's boss's buddy", into epistemic logic and gives a sound and complete axiomatization of a knowledge modality parameterized by such designators. It also studies the expressive power of the proposed modality.

Keywords Epistemic logic · Non-rigid designator · Axiomatization · Completeness · Undefinability

1 Introduction

A designator is a term or an expression used to pick out an object in the world. It can be a proper name, a common name, a definite description, a pronoun, etc. A designator can be used in a rigid way or a non-rigid way. When a designator is non-rigid, it can be non-rigid in different senses.

In the sense of *world non-rigidity*, a non-rigid designator is a term that does not designate the same object in all possible worlds. This is in contrast with a rigid designator [1, 2], which refers to the same object in all possible worlds, regardless of whether the knowledge that the referred object is the same is a priori or a posteriori. For example, Hesperus, which refers to a heavenly body visible in the evening, and Phosphorus, which refers to a heavenly body visible in the morning, both rigidly designate Venus [1]. On the contrary, the definite description "the tallest person in the room" is considered a non-rigid designator because it does not necessarily refer to the same individual in all possible worlds [1].

In addition to world non-rigidity, where a designator picks out different objects in different possible worlds, the designator can be non-rigid in other senses. For example, the referent of a non-rigid designator may change with time [3]. We call this *temporal*

✉ Xinran Zhang
xkz5332@psu.edu

Pavel Naumov
pgn2@cornell.edu

1 University of Southampton, Southampton, UK

2 The Pennsylvania State University, University Park, PA, USA

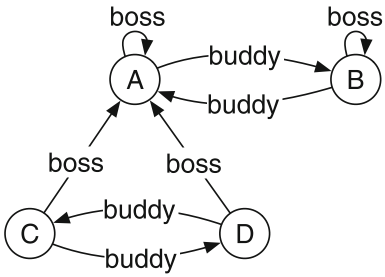

Fig. 1 A start-up company organization chart

non-rigidity. As an example, “the President of the United States” refers to Joseph Biden in 2024 and to Donald Trump in 2026.

In this article, we introduce *agentic non-rigidity*. We adopt a view that an individual’s identity or role is not fixed and can change depending on the *context*. Agentically non-rigid designators capture the context dependent feature of terms that describe social connections between individuals, such as “parent”, “boss”, or “buddy”. When we say “agent A’s boss”, agent A is the context and “boss” is the agentically non-rigid designator. The referent of an agentically non-rigid designator changes depending on which agent is using it to designate a referent. Suppose that “A”(Alice), “B”(Brittany), “C”(Chris), and “D”(Doug) are agents in a given world.[1] As an example, see Fig. 1, Alice is the “boss” of Chris and Doug, and the “buddy” of Brittany. Chris and Doug are the “buddy” of each other. Agentic non-rigidity appears in designators “boss” and “buddy” as follows. The individual whom Chris refers to as the “boss” is Alice, which is different from the individual whom Brittany refers to as the “boss”. Similarly, the individual whom Chris refers to as the “buddy” is Doug, whereas the individual whom Brittany refers to as the “buddy” is Alice. In this example, we assume the world and time are fixed so that we do not consider non-rigidity in the sense of world or time.

In real life, we may refer to a person through a chain of social connections. For example, in Fig. 1, Chris may refer to Brittany as his “boss’s buddy”. This example shows that an agentically non-rigid designator can be constructed out of multiple atomic designators, capturing single social connections. We refer to this type of agentically non-rigid designators as *chained designators*.

Chained designators are common in natural and programming languages. In linguistics, constructions like “the buddy of Chris’ boss” are called *possessive chains* [4]. As a programming language example, consider C structure `Agent` that has variables `boss` and `buddy` of type `Agent`. If `Chris` is a variable of type `Agent`, then one can write `Chris.boss.buddy`. MISRA software development guidelines for the C programming language use the term *chained designators* for such expressions [5].

In traditional logical systems, the sentence “in world w, agent Alice is sick” can be formally expressed by the statement

$$w \Vdash \text{Alice is sick}.$$

[1] Note that we see A, B, C, and D as agents instead of designators in this example. To make our example easier to follow, we call agents A as Alice, B as Brittany, C as Chris, and D as Doug. Alternatively, the reader can think about Alice, Brittany, Chris, and Doug as rigid designators.

Different from traditional logical systems, Prior [6] proposed egocentric logic to capture properties of agents rather than possible worlds. For example, using his approach, "Alice is sick" can be expressed by the statement

$$A \Vdash \text{sick}.$$

In addition to describing the properties of a given agent, egocentric logics can express properties of the agents that are somehow related to the given agent. For example, the logic of friendship uses modality F to talk about properties of an agent's friends [7, 8]. In this logic, statement $A \Vdash \mathsf{F}(\text{poor})$ means "all friends of agent A are poor". The same modality is also used in [9, 10]. In the egocentric logic of preferences, modal formula $\mathsf{L}\varphi$ means that the agent "likes those who have property φ" [11]. This modality (as well as F) can be nested. For example, the statement $A \Vdash \mathsf{LL}(\text{poor})$ means "agent A likes those who like the poor".

In this work, we use modality @ to express the properties of others. Modality @ takes a designator as a subscript and expresses properties of a person whom the given agent refers to by that designator [12]. For example, by using modality @ and an atomic agentically non-rigid designator, we can convey the information "Alice is sick" through agent Chris's social connection using the statement

$$C \Vdash @_{\text{boss}}(\text{sick}).$$

We may use the designator ε to represent the context agent themselves, where ε designates agents by an empty (sequence of) social connection. The designator ε can be seen as a empty string. For example, the statement

$$C \Vdash @_{\varepsilon}(\text{sick}).$$

means "Chris is sick". Moreover, with a chained designator, we may express "Brittany is sick" through Chris's social connections by statement

$$C \Vdash @_{\text{boss,buddy}}(\text{sick}).$$

Modality @ can be nested as well. The statement above can be rewritten as

$$C \Vdash @_{\text{boss}}@_{\text{buddy}}(\text{sick})$$

while maintaining the same meaning.

In addition to expressing the properties of agents, in this work, we also use agentically non-rigid designators in knowledge modalities. To do this, Grove and Halpern [13] combined the traditional and egocentric approach by placing a world parameter and an agent parameter on the left-hand side of "$\Vdash$" to express a property of agent C in a world w. For example, the statement

$$w, C \Vdash @_{\text{boss}}@_{\text{buddy}}(\text{sick})$$

means that Brittany is sick in world w. In the philosophy of language, this type of semantics is known as Two-Dimensional(2D) semantics [14, 15]. 2D semantics are used to discuss topics such as the social character of meanings [16], a priori necessity [17], formalization of epistemic 2D semantics [18], and contents of speech and thoughts [19]. Kaplan [14] used 2D-semantics framework to explain rules for context-dependent referent, such as “I”, “you”, “he”, “here”, “now”, “tomorrow”, and “actual”. In our settings, the agent on the left-hand side of $\Vdash$ is called the *context* because the referents of the designators used in a statement depend on such agent’s social connections. Seligman, Liu, and Girard [7, 8] use modality K to represent “knowledge about yourself” in the 2D setting. For example, the statement $w, C \Vdash \mathsf{K}(\text{sick})$ means that Chris knows that he (himself) is sick.

In this article, we consider epistemic modalities subscripted with designators. There are many ways to interpret such modalities in the 2D setting. For example, the statement

$$w, C \Vdash \mathsf{K}_{\text{boss}}\text{“Buddy is sick”}.$$

can be interpreted in at least three different ways:

1. Chris’s boss knows that Chris’s boss’s buddy is sick.
2. Chris knows that Chris’s boss’s buddy is sick.
3. Chris’s boss knows that Chris’s buddy is sick.

In this article, we use three modalities with subscripts to represent these three different interpretations. Unlike in [12] and [13], a subscript of these modalities can be not only an atomic designator but also a chained designator.

The first interpretation talks about the referent’s self-knowledge. We use modality $[\circlearrowleft]$ to represent it and the statement becomes

$$w, C \Vdash [\circlearrowleft]_{\text{boss}}(\text{“Buddy is sick”}).$$

Modality $[\circlearrowleft]_\varepsilon$ with the empty designator ε is equivalent to the “knowledge about yourself” modality K mentioned before. Modality $[\circlearrowleft]$ with an atomic designator, such as $[\circlearrowleft]_{\text{boss}}$, was discussed by Grove and Halpern [13, 20]. Modality $[\circlearrowleft]$ with a chained designator, introduced in this article, can be expressed through modality @ and “knowledge about yourself” modality:

$$[\circlearrowleft]_{\text{boss,buddy}}\varphi = @_{\text{boss}}@_{\text{buddy}}\mathsf{K}\varphi.$$

The second interpretation talks about the context’s knowledge about the referent. We capture this interpretation by modality $[\rightarrow]$. Then, the interpretation is expressed by the statement

$$w, C \Vdash [\rightarrow]_{\text{boss}}(\text{“Buddy is sick”}).$$

Similar to modality $[\circlearrowleft]_\varepsilon$, modality $[\rightarrow]_\varepsilon$ with the empty designator ε, is equivalent to modality K. The modality $[\rightarrow]$ with a chained designator is also expressible through modalities @ and K. For example, $[\rightarrow]_{\text{boss,buddy}}$ is equivalent to $\mathsf{K}@_{\text{boss}}@_{\text{buddy}}$.

The third interpretation talks about the referent's knowledge about the context. We formalize this interpretation with modality $[\leftarrow]$. Then, the third interpretation can be written as the statement

$$w, C \Vdash [\leftarrow]_{\text{boss}}(\text{“Buddy is sick”}).$$

Note that this modality is able to define modalities K, $[\circlearrowleft]$, and $[\rightarrow]$. For example, $\mathsf{K}\varphi$ is equivalent to $[\leftarrow]_\varepsilon\varphi$, $[\circlearrowleft]_{\text{boss,buddy}}\varphi$ is equivalent to $@_{\text{boss}}@_{\text{buddy}}[\leftarrow]_\varepsilon\varphi$, and $[\rightarrow]_{\text{boss,buddy}}$ is equivalent to $[\leftarrow]_\varepsilon @_{\text{boss}}@_{\text{buddy}}\varphi$.

Modalities $[\circlearrowleft]$, $[\rightarrow]$, and $[\leftarrow]$ can be combined with each other. For example, the statement

$$w, C \Vdash [\leftarrow]_{\text{boss}}[\circlearrowleft]_{\text{buddy}}(\text{sick})$$

means that Chris's boss (Alice) knows that Chris's buddy (Doug) knows that Doug is sick. More examples can be found in Table 1.

The conceptual contribution of this article is the incorporation of the chained designators into the knowledge modalities. The technical contribution has three parts. First, we discover that the expressive powers of the three knowledge modalities mentioned above are not equal. In Section 4, we show that our knowledge modality $[\leftarrow]$ is not expressible through any combination of modalities $@$, $[\circlearrowleft]$, and $[\rightarrow]$, but modalities $[\circlearrowleft]$, and $[\rightarrow]$ are expressible through modalities $[\leftarrow]$ and $@$. This shows that modality $[\leftarrow]$ is expressibly more powerful than the other two knowledge modalities. Hence, in our work, we mainly talk about modality $[\leftarrow]$.

Second, we observe that modality $[\leftarrow]$ with a chained designator is not expressible through $[\leftarrow]$ with atomic designators, but modality $@$ with a chained designator is equivalent to nested $@$ with atomic designators. This is because chained designators on modality $@$ are expressible through nesting the same modality with just a single atomic designator. For example, $@_{\text{boss, buddy}}$ is equivalent to $@_{\text{boss}}@_{\text{buddy}}$. However, in Section 5, we show that modality $[\leftarrow]$ with chained designators is *expressively stronger* than the same modality with just a single atomic designator.

Third, in Sections 6 and 7, we give a sound and complete axiomatization of the interplay between modalities $[\leftarrow]$ and $@$. This axiomatization includes a very non-trivial Insertion inference rule. Note that we do not axiomatize modalities $[\circlearrowleft]$ and $[\rightarrow]$

Table 1 Statements are placed on the left and their meanings are placed on the right

$[\leftarrow]_{\text{boss}}[\leftarrow]_{\text{buddy}}(\text{sick})$	Alice knows that Doug knows that Chris is sick
$[\leftarrow]_{\text{boss}}[\rightarrow]_{\text{buddy}}(\text{sick})$	Alice knows that Chris knows that Doug is sick
$[\leftarrow]_{\text{boss}}[\circlearrowleft]_{\text{buddy}}(\text{sick})$	Alice knows that Doug knows that he is sick
$[\rightarrow]_{\text{boss}}[\leftarrow]_{\text{buddy}}(\text{sick})$	Chris knows that Brittany knows that Alice is sick
$[\rightarrow]_{\text{boss}}[\rightarrow]_{\text{buddy}}(\text{sick})$	Chris knows that Alice knows that Brittany is sick
$[\rightarrow]_{\text{boss}}[\circlearrowleft]_{\text{buddy}}(\text{sick})$	Chris knows that Brittany knows that she is sick
$[\circlearrowleft]_{\text{boss}}[\leftarrow]_{\text{buddy}}(\text{sick})$	Alice knows that Brittany knows that Alice is sick
$[\circlearrowleft]_{\text{boss}}[\rightarrow]_{\text{buddy}}(\text{sick})$	Alice knows that Alice knows that Brittany is sick
$[\circlearrowleft]_{\text{boss}}[\circlearrowleft]_{\text{buddy}}(\text{sick})$	Alice knows that Brittany knows that Brittany is sick

To make the table easier to read and understand, we leave out “$w, C \Vdash$” on the left of each statement because, as we will show in Section 4, they are definable through modalities [←] and @.

Before presenting these three parts, in Section 2, we give a definition of the class of models that we use to specify the semantics of our formal system. In Section 3, we formally present our semantics.

2 Epistemic Models with Extensions

In this section, we define the class of models that will be used to give the semantics of our logical system. Throughout the article, we assume a fixed set of atomic designators Δ and a fixed nonempty set of atomic propositions. In our introductory example, the set Δ consists of atomic designators "boss" and "buddy".

Definition 1 A tuple $(W, \mathcal{A}, \sim, e, \pi)$ is a epistemic model with extensions if

1. W is a set of "worlds",
2. $\mathcal{A}$ is a set of "agents",
3. $\sim_a$ is an "indistinguishibility" equivalence relation on set W for each agent $a \in \mathcal{A}$,
4. e_d is an "extension" function that assigns an agent $e_d(a) \in \mathcal{A}$ to each agent $a \in \mathcal{A}$ for each atomic designator $d \in \Delta$,
5. $\pi(p) \subseteq W \times \mathcal{A}$ for each atomic proposition p.

Generally speaking, sets W and $\mathcal{A}$ can be empty.

The extension function $e_d(a)$ specifies the referent agent to whom the context agent a refers by atomic designator d. In our introductory example, $e_{\text{boss}}(C) = A$.

Note that, unlike traditional modal logic, the value of the valuation function $\pi(p)$ is a set of pairs. This is because under 2D-semantics each statement, including atomic propositions, captures a property of a given agent in a given state.

By a (chained) designator, we mean an arbitrary finite (possibly empty) sequence of atomic designators. The set of all designators is denoted by Δ^*. To keep the notations more compact, we often write a sequence $(d_1, d_2, \ldots, d_k) \in \Delta^*$ as $d_1, d_2, \ldots, d_k$ or even $d_1 d_2 \ldots d_k$.

Definition 2 For any designator $d_1 d_2 \ldots d_k \in \Delta^*$, the extension function $\hat{e}_{d_1 d_2 \ldots d_k}$ is defined as follows: $\hat{e}_{d_1 d_2 \ldots d_k}(a) = e_{d_k}(e_{d_{k-1}}(\ldots(e_{d_1}(a))\ldots))$.

For instance, $\hat{e}_{\text{boss, buddy}}(C) = B$ and $\hat{e}_{\text{buddy,boss}}(C) = A$. Note that $\hat{e}_{\varepsilon}(a) = a$ for any agent $a \in \mathcal{A}$ (recall that ε denotes the empty sequence).

The next lemma follows from Definition 2.

Lemma 1 $\hat{e}_{\tau}(\hat{e}_{\sigma}(a)) = \hat{e}_{\sigma\tau}(a)$ *for any designators* $\tau, \sigma \in \Delta^*$.

3 Syntax and Semantics

In this section, we give the formal syntax and semantics of our logical system. Even though modalities [⟲] and [→] are expressible through modalities [←] and @, we include them in the language for now to provide a more general definition. The language Φ of the system is defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \vee \varphi \mid @_d\varphi \mid [\leftarrow]_\sigma\varphi \mid [\circlearrowleft]_\sigma\varphi \mid [\rightarrow]_\sigma\varphi,$$

where p is a propositional variable, $d \in \Delta$ is an atomic designator, and $\sigma \in \Delta^*$ is a chained designator. We read $@_d\varphi$ as "the referent of an atomic designator d has property φ". Note that we do not include modality $@_\sigma$ for an arbitrary chained designator σ as a primitive construction our language because a formula $@_{d_1,\dots,d_k}\varphi$ would be equivalent to the formula $@_{d_1}\dots@_{d_k}\varphi$. However, throughout the rest of the article, we use $\overline{@}_{d_1,\dots,d_k}\varphi$ as an abbreviation for the formula $@_{d_1}\dots@_{d_k}\varphi$. For a given context, we read $[\leftarrow]_\sigma\varphi$ as "the referent of chained designator σ knows φ about the context". We read $[\circlearrowleft]_\sigma\varphi$ as "the referent of chained designator σ knows φ about the referent themselves". We read $[\rightarrow]_\sigma\varphi$ as "the context knows φ about the referent of the chained designator σ".

We assume the implication $\rightarrow$, the conjunction $\wedge$, the biconditional $\leftrightarrow$, and the constant true $\top$ are defined through the negation and the disjunction in the standard way. For any finite set of formulae $\varphi_1, \dots, \varphi_k$, by $\bigwedge_i \varphi_i$ we denote the formula $\varphi_1 \wedge \dots \wedge \varphi_k$. As usual, if $k = 0$, then $\bigwedge_i \varphi_i$ is the constant $\top$.

Definition 3 For any world $w \in W$ and any agent $a \in \mathcal{A}$ of an epistemic model $(W, \mathcal{A}, \sim, e, \pi)$ with extensions and any formula $\varphi \in \Phi$, the satisfaction relation $w, a \Vdash \varphi$ is defined as follows:

1. $w, a \Vdash p$, if $(w, a) \in \pi(p)$,
2. $w, a \Vdash \neg\varphi$, if $w, a \nVdash \varphi$,
3. $w, a \Vdash \varphi \vee \psi$, if $w, a \Vdash \varphi$ or $w, a \Vdash \psi$,
4. $w, a \Vdash @_d\varphi$, if $w, e_d(a) \Vdash \varphi$,
5. $w, a \Vdash [\leftarrow]_\sigma\varphi$, if $u, a \Vdash \varphi$ for each world $u \in W$ such that $w \sim_{\hat{e}_\sigma(a)} u$.
6. $w, a \Vdash [\circlearrowleft]_\sigma\varphi$, if $u, \hat{e}_\sigma(a) \Vdash \varphi$ for each world $u \in W$ such that $w \sim_{\hat{e}_\sigma(a)} u$,
7. $w, a \Vdash [\rightarrow]_\sigma\varphi$, if $u, \hat{e}_\sigma(a) \Vdash \varphi$ for each world $u \in W$ such that $w \sim_a u$.

As mentioned earlier, the type of semantics given in Definition 3 is often called 2D-semantics. In general, 2D-semantics defines the meaning of a statement based on the possible world w and some other information. That other information could be index [21], counterfactual world [15], possible world [19], or scenario [18]. In our case, the other information is an agent a, and we call it context. We give a polynomial-time model checking algorithm for the above semantics in the appendix.

The next lemma follows from Definition 2 and item 4 of Definition 3.

Lemma 2 *$w, a \Vdash \overline{@}_\sigma\varphi$ iff $w, \hat{e}_\sigma(a) \Vdash \varphi$ for any designator $\sigma \in \Delta^*$.*

The next two definitions introduce the notions used in our undefinability results.

Definition 4 For any epistemic model with extensions, let the truth set of a formula $\varphi \in \Phi$ be defined as

$$[\![\varphi]\!] = \{(w, a) \in W \times \mathcal{A} \mid w, a \Vdash \varphi\}.$$

Note that, technically, the value of $[\![\varphi]\!]$ depends on the choice of the epistemic model with extensions. However, we do not list the model parameter explicitly since its value will always be clear from the context.

Definition 5 Formulae $\varphi, \psi \in \Phi$ are semantically equivalent if $[\![\varphi]\!] = [\![\psi]\!]$ for each epistemic model with extensions.

Alternatively, formulae $\varphi, \psi \in \Phi$ are semantically equivalent when $w, a \Vdash \varphi$ iff $w, a \Vdash \psi$ for each world w and each agent a of each epistemic model with extensions.

4 Undefinability of $[\leftarrow]$ through $[\rightarrow]$, $[\circlearrowleft]$ and @

The theorems below follow from Definitions 3, 4, and 5.

Theorem 1 (definability) *For any formula $\varphi \in \Phi$ and any $d \in \Delta$, the formula $[\circlearrowleft]_{d_1 \dots d_k}\varphi$ is semantically equivalent to formula $\overline{@}_{d_1 \dots d_k}[\leftarrow]_\varepsilon \varphi$.*

Theorem 2 (definability) *For any formula $\varphi \in \Phi$ and any $d \in \Delta$, the formula $[\rightarrow]_{d_1 \dots d_k}\varphi$ is semantically equivalent to formula $[\leftarrow]_\varepsilon \overline{@}_{d_1 \dots d_k} \varphi$.*

In the rest of this section, we prove that modality $[\leftarrow]$ cannot be defined through modalities $[\rightarrow]$, $[\circlearrowleft]$, and @. These would imply that $[\leftarrow]$ is the most expressive out of the three epistemic modalities we consider. Usually, the undefinability results in modal logic are obtained using the bisimulation technique. However, it is not clear how bisimulation can be used in our 2D setting. Instead, use the recently proposed "truth set algebra" technique [22].

To prove the undefinability, let us consider an epistemic model with extensions that has three worlds, w, u and v, as well as two agents a and b. We describe the technique as we present the proof. The indistinguishability relation is represented in Fig. 2.

Furthermore, assume that agents a and b both refer to agent a as "fella": $e_{\text{fella}}(a) = e_{\text{fella}}(b) = a$. Without loss of generality, suppose that "fella" is the only atomic designator in language Φ. Finally, let $\pi(p) = \{(w, b), (u, b)\}$.

By a *truth set* we mean an arbitrary subset of $W \times \mathcal{A} = \{w, u, v\} \times \{a, b\}$. We visualize the truth sets using 3×2 diagrams like the one depicted at the right-most position in Fig. 3. Rows in this diagram are labeled by worlds and columns by agents. The cell is colored gray if the corresponding pair of the world and the agent belongs to the truth set. The diagram in our example represents truth set $R = \{(w, b)\}$. We also consider truth sets S_1, S_2, S_3, and S_4 shown on the left of the same figure.

Lemma 3 *For any formula $\varphi \in \Phi$ and any chained designator $\sigma \in \{\text{fella}\}^*$, if $[\![\varphi]\!] \in \{S_1, S_2, S_3, S_4\}$, then $[\![[\rightarrow]_\sigma \varphi]\!]$, $[\![[\circlearrowleft]_\sigma \varphi]\!]$, $[\![@_{\text{fella}}\varphi]\!] \in \{S_1, S_2, S_3, S_4\}$.*

Proof We start the proof with the following observation:

Claim 1 *If $[\![\varphi]\!] = S_2$ and $\sigma \neq \varepsilon$, then $[\![[\rightarrow]_\sigma \varphi]\!] = S_1$.*

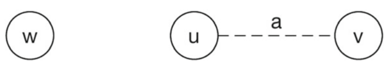

Fig. 2 Indistinguishability relations between worlds

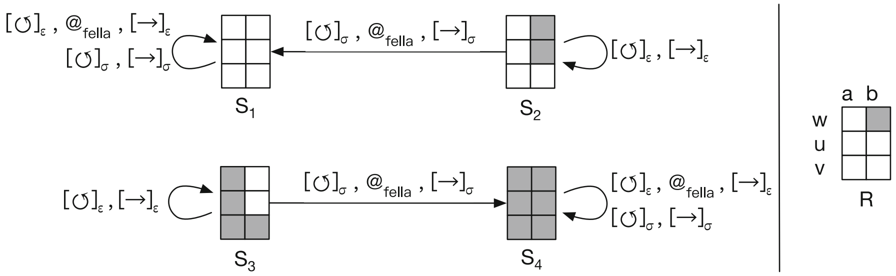

Fig. 3 Towards undefinability of $[\leftarrow]$ modality, $e_{\text{fella}}(a) = e_{\text{fella}}(b) = a$, $u \sim_a v$. We assume $\sigma \neq \varepsilon$

Proof of Claim The assumption $[\![\varphi]\!] = S_2$, see Fig. 3, implies that

$$w, a \nVdash \varphi \qquad u, a \nVdash \varphi \qquad v, a \nVdash \varphi.$$

Recall that $e_{\text{fella}}(a) = e_{\text{fella}}(b) = a$. Hence $\hat{e}_\sigma(a) = a$ by the assumption $\sigma \neq \varepsilon$ of the claim. Thus,

$$w, \hat{e}_\sigma(g) \nVdash \varphi \qquad u, \hat{e}_\sigma(g) \nVdash \varphi \qquad v, \hat{e}_\sigma(g) \nVdash \varphi$$

for any agent $g \in \{a, b\}$. Hence, by item 7 of Definition 3,

$$w, g \nVdash [\rightarrow]_\sigma \varphi \qquad u, g \nVdash [\rightarrow]_\sigma \varphi \qquad v, g \nVdash [\rightarrow]_\sigma \varphi.$$

Thus, $[\![[\rightarrow]_\sigma \varphi]\!] = S_1$, see Fig. 3. □

Claim 2 *If* $[\![\varphi]\!] = S_2$, *then* $[\![[\rightarrow]_\varepsilon \varphi]\!] = S_2$.

Proof of Claim The assumption $[\![\varphi]\!] = S_2$, see Fig. 3, implies that

$$w, a \nVdash \varphi \qquad u, a \nVdash \varphi \qquad v, a \nVdash \varphi$$

$$w, b \Vdash \varphi \qquad u, b \Vdash \varphi \qquad v, b \nVdash \varphi.$$

By Definition 2, $e_\varepsilon(a) = a$ and $e_\varepsilon(b) = b$. Thus,

$$w, e_\varepsilon(a) \nVdash \varphi \qquad u, e_\varepsilon(a) \nVdash \varphi \qquad v, e_\varepsilon(a) \nVdash \varphi$$

$$w, e_\varepsilon(b) \Vdash \varphi \qquad u, e_\varepsilon(b) \Vdash \varphi \qquad v, e_\varepsilon(b) \nVdash \varphi.$$

Hence, by item 7 of Definition 3,

$$w, a \nVdash [\rightarrow]_\varepsilon \varphi \qquad u, a \nVdash [\rightarrow]_\varepsilon \varphi \qquad v, a \nVdash [\rightarrow]_\varepsilon \varphi$$

and, because agent b can distinguish any two of the worlds w, u, and v,

$$w, b \Vdash [\rightarrow]_\varepsilon \varphi \qquad u, b \Vdash [\rightarrow]_\varepsilon \varphi \qquad v, b \nVdash [\rightarrow]_\varepsilon \varphi.$$

Thus, $[\![[\rightarrow]_\varepsilon\varphi]\!] = S_2$, see Fig. 3. □

In Fig. 3, we visualize Claim 1 by the labels on the directed arrow from set S_2 to set S_1 and Claim 2 by the labels from set S_2 to itself. Note that the two above claims are two out of $4 \times 5 = 20$ different facts forming the statement of the lemma. The proofs of the other 18 facts are similar. In Fig. 3, we show the labels for each of these proofs. □

Lemma 4 $[\![\varphi]\!] \in \{S_1, S_2, S_3, S_4\}$ *for any formula* $\varphi \in \Phi$ *that does not contain modalities* $[\leftarrow]$.

Proof We prove the statement of the lemma by induction on the structural complexity of formula φ. Suppose that φ is a propositional variable p. Recall that $\pi(p) = \{(w, b), (u, b)\}$. Thus, by item 1 of Definition 3, we have $x, b \Vdash \varphi$ iff $x \in \{w, u\}$. Hence, $[\![\varphi]\!] = \{(w, b), (u, b)\}$ by Definition 4. Therefore, $[\![\varphi]\!] = S_2$, see Fig. 3.

Suppose that the formula φ has the form $\neg\psi$. Then, by Definition 4 and item 2 of Definition 3, the set $[\![\varphi]\!]$ is the *complement* of the set $[\![\psi]\!]$. Note that $[\![\psi]\!] \in \{S_1, S_2, S_3, S_4\}$ by the induction hypothesis. Observe that the complement of each truth set in the family $\{S_1, S_2, S_3, S_4\}$ belongs to the same family. For example, the complement of set S_2 is set S_3, see Fig. 3. Hence, $[\![\varphi]\!] \in \{S_1, S_2, S_3, S_4\}$.

Suppose that formula φ has the form $\psi_1 \vee \psi_2$. Then, by Definition 4, and item 3 of Definition 3, the set $[\![\varphi]\!]$ is the *union* of the sets $[\![\psi_1]\!]$ and $[\![\psi_2]\!]$. Observe that, by the induction hypothesis, $[\![\psi_1]\!], [\![\psi_2]\!] \in \{S_1, S_2, S_3, S_4\}$. Also, note that the union of any two truth sets in the family $\{S_1, S_2, S_3, S_4\}$ belongs to the same family. For example, the union of sets S_2 and S_3 is set S_4, see Fig. 3. Hence, $[\![\varphi]\!] \in \{S_1, S_2, S_3, S_4\}$.

Finally, suppose that formula φ has one of the following three forms: $[\![[\rightarrow]_\sigma\psi]\!]$, $[\![[\circlearrowleft]_\sigma\psi]\!]$, or $[\![@_{\text{fella}}\psi]\!]$. Note that $[\![\psi]\!] \in \{S_1, S_2, S_3, S_4\}$ by the induction hypothesis. Therefore, $[\![\varphi]\!] \in \{S_1, S_2, S_3, S_4\}$ by Lemma 3. □

Lemma 5 $[\![[\leftarrow]_{\text{fella}}p]\!] \notin \{S_1, S_2, S_3, S_4\}$.

Proof Since $(w, a), (u, a), (v, a) \notin \pi(p)$, we have

$$w, a \nVdash p \qquad u, a \nVdash p \qquad v, a \nVdash p$$

by item 1 of Definition 3. Thus, by item 5 of Definition 3,

$$w, a \nVdash [\leftarrow]_{\text{fella}}p \qquad u, a \nVdash [\leftarrow]_{\text{fella}}p \qquad v, a \nVdash [\leftarrow]_{\text{fella}}p. \tag{1}$$

Since $(w, b) \in \pi(p)$ and $(v, b) \notin \pi(p)$,

$$w, b \Vdash p \qquad v, b \nVdash p \tag{2}$$

by item 1 of Definition 3. Recall that $e_{\text{fella}}(b) = a$. Then, because $w \nsim_a u$, $w \nsim_a v$, and $u \sim_a v$,

$$w \nsim_{e_{\text{fella}}(b)} u \qquad w \nsim_{e_{\text{fella}}(b)} v \qquad u \sim_{e_{\text{fella}}(b)} v.$$

Thus, by item 5 of Definition 3 and Eq. 2,

$$w, b \Vdash [\leftarrow]_{\text{fella}} p \qquad u, b \nVdash [\leftarrow]_{\text{fella}} p \qquad v, b \nVdash [\leftarrow]_{\text{fella}} p. \tag{3}$$

Hence, followed from Eqs. 1 and 3, by Definition 4, we have $[\![[\leftarrow]_{\text{fella}} p]\!] = R$ as shown in Fig. 3. Therefore, $[\![[\leftarrow]_{\text{fella}} p]\!] \notin \{S_1, S_2, S_3, S_4\}$. □

The next theorem follows from the two previous lemmas and Definition 5.

Theorem 3 (undefinability) *Formula* $[\leftarrow]_{\text{fella}} p$ *is not semantically equivalent to any formula that uses only modalities* $[\rightarrow]$, $[\circlearrowleft]$, *and* $@$.

Hence, we have shown that modality $[\leftarrow]$ have the strongest expressive power among the three epistemic modalities we introduced. In the following sections, we only study modality $[\leftarrow]$ and ignore modalities $[\circlearrowleft]$ and $[\rightarrow]$.

5 Necessity of Chained Designators

In this section, through our introductory example, we show that knowledge modality with a chained designator $[\leftarrow]_{d_1\dots d_k}\varphi$ is not expressible through any combinations of modalities with atomic designators $[\leftarrow]_{d_1}\varphi, \dots, [\leftarrow]_{d_k}\varphi, [\leftarrow]_{\varepsilon}\varphi, @_{d_1}\varphi, \dots,$ or $@_{d_k}\varphi$. Suppose that our epistemic model with extensions has three worlds: w, u, and v. The indistinguishibility relations for Alice (A), Brittany (B), Chris (C), and Doug (D) are shown in Fig. 4. Suppose that Chris is sick in worlds u and v, but not in world w. Alice, Brittany, and Doug are not sick in any worlds of u, v, or w. Let propositional variable p denote the property "is sick". In other words, $w, C \nVdash p$ and $u, C \Vdash p$ as well as $v, C \Vdash p$. Also, $x, y \nVdash p$ for any world $x \in \{w, u, v\}$ and any agent $y \in \{A, B, D\}$. Since we have shown in the previous section that modalities $[\rightarrow]$ and $[\circlearrowleft]$ can be defined through modalities $[\leftarrow]$ and $@$, we define a language Φ_0 obtained by removing modalities $[\rightarrow]$ and $[\circlearrowleft]$ from the language Φ. Without loss of generality, in this section, we suppose that p is the only propositional variable in language Φ_0.

Lemma 6 $x_1, y \Vdash \varphi$ *iff* $x_2, y \Vdash \varphi$ *for any formula* $\varphi \in \Phi_0$, *any worlds* $x_1, x_2 \in \{w, u, v\}$, *and any agent* $y \in \{A, B\}$.

Proof We prove the statement of the lemma by induction on the structural complexity of formula φ. If φ is a propositional variable p, then $x_1, y \nVdash \varphi$ and $x_2, y \nVdash \varphi$ because neither Alice nor Brittany is sick in either of the worlds in our model.

If formula φ is a negation or disjunction, then the statement of the lemma follows from the induction hypothesis and item 2 or item 3 of Definition 3 in the standard way.

Suppose that formula φ has the form $@_d\psi$. By Fig. 1, $e_d(y) \in \{A, B\}$. Note that, by induction hypothesis, $x_1, e_d(y) \Vdash \psi$ iff $x_2, e_d(y) \Vdash \psi$. Then, $x_1, y \Vdash @_d\psi$ iff $x_2, y \Vdash @_d\psi$ by item 4 of Definition 3. Therefore, $x_1, y \Vdash \varphi$ iff $x_2, y \Vdash \varphi$.

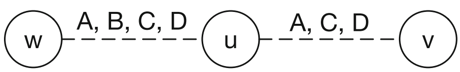

Fig. 4 Indistinguishability relations between worlds

Suppose that formula φ has the form $[\leftarrow]_\sigma \psi$.
$(\Rightarrow)$: Assume that $x_1, y \Vdash [\leftarrow]_\sigma \psi$. Consider any world $x' \in \{w, u, v\}$ such that $x_2 \sim_{\hat{e}_\sigma(y)} x'$. By item 5 of Definition 3, it suffices to show that $x', y \Vdash \psi$. Note that $x_1 \sim_{\hat{e}_\sigma(y)} x_1$. Then, by item 5 of Definition 3, the assumption $x_1, y \Vdash [\leftarrow]_\sigma \psi$ implies that $x_1, y \Vdash \psi$. Therefore, $x', y \Vdash \psi$ by the induction hypothesis.

The other direction can be proved in a similar way. □

Lemma 7 *$u, y \Vdash \varphi$ iff $v, y \Vdash \varphi$ for any agent $y \in \{C, D\}$ and any formula φ that uses only modalities $@_{\text{boss}}$, $@_{\text{buddy}}$, $[\leftarrow]_{\text{boss}}$, $[\leftarrow]_{\text{buddy}}$, and $[\leftarrow]_\varepsilon$.*

Proof We prove the statement of the lemma by induction on the structural complexity of formula φ. If φ is propositional variable p, then

(i) $u, C \Vdash \varphi$ and $v, C \Vdash \varphi$,
(ii) $u, D \nVdash \varphi$ and $v, D \nVdash \varphi$

because Chris is sick in world u and v and Doug is sick in neither of these two worlds. If formula φ is a negation or a disjunction, then the statement of the lemma follows from the induction hypothesis and either item 2 or item 3 of Definition 3.

Suppose that formula φ has the form $@_{\text{buddy}}\psi$. Note that $u, e_{\text{buddy}}(y) \Vdash \psi$ iff $v, e_{\text{buddy}}(y) \Vdash \psi$ by the induction hypothesis. By Fig. 1, $e_{\text{buddy}}(y) \in \{C, D\}$. Hence, $u, y \Vdash @_{\text{buddy}}\psi$ iff $v, y \Vdash @_{\text{buddy}}\psi$ by item 4 of Definition 3.

Suppose that formula φ has the form $@_{\text{boss}}\psi$. Note that $u, e_{\text{boss}}(y) \Vdash \psi$ iff $v, e_{\text{boss}}(y) \Vdash \psi$ by Lemma 6 because $e_{\text{boss}}(y) = A$. Hence, $u, y \Vdash @_{\text{boss}}\psi$ iff $v, y \Vdash @_{\text{boss}}\psi$ by item 4 of Definition 3.

Finally, suppose that formula φ has the form $[\leftarrow]_\sigma \psi$, where σ is either the designator "boss", designator "body", or ε. Then, $\hat{e}_\sigma(y) \in \{A, C, D\}$ by the assumption $y \in \{C, D\}$ of the lemma, see Fig. 1. Observe that agents A, C, and D cannot distinguish worlds u and v, see Fig. 4. Therefore, $u, y \Vdash [\leftarrow]_\sigma \psi$ iff $v, y \Vdash [\leftarrow]_\sigma \psi$ by item 5 of Definition 3. □

Lemma 8 *$u, C \nVdash [\leftarrow]_{\text{boss,buddy}}\, p$ and $v, C \Vdash [\leftarrow]_{\text{boss,buddy}}\, p$.*

Proof Note that $u \sim_B w$. Thus, $u \sim_{\hat{e}_{\text{boss,buddy}}(C)} w$ because $\hat{e}_{\text{boss,buddy}}(C) = B$. Then, $u, C \nVdash [\leftarrow]_{\text{boss,buddy}}\, p$ by item 5 of Definition 3 because $w, C \nVdash p$.

To prove $v, C \Vdash [\leftarrow]_{\text{boss,buddy}}\, p$, let us consider any world $v' \in \{w, u, v\}$ such that $v \sim_{\hat{e}_{\text{boss,buddy}}(C)} v'$. By item 5 of Definition 3, it suffices to prove that $v', C \Vdash p$. The assumption $v \sim_{\hat{e}_{\text{boss,buddy}}(C)} v'$ implies $v \sim_B v'$ because $\hat{e}_{\text{boss,buddy}}(C) = B$. Then, $v = v'$, see Fig. 4. Hence, it suffices to show that $v, C \Vdash p$, which is true because Chris is sick in world v. □

The last two lemmas together imply the following undefinability result.

Theorem 4 *The formula $[\leftarrow]_{\text{boss,buddy}}\, p$ is not semantically equivalent to any formula containing only modalities $@_{\text{boss}}$, $@_{\text{buddy}}$, $[\leftarrow]_{\text{boss}}$, $[\leftarrow]_{\text{buddy}}$, and $[\leftarrow]_\varepsilon$.*

Hence, we have shown that modality $[\leftarrow]$ with a chained designator $[\leftarrow]_{d_1 \dots d_k}\varphi$ is not expressible through any combination of modalities with atomic designators $[\leftarrow]_{d_1}\varphi$, $\dots$, $[\leftarrow]_{d_k}\varphi$, $[\leftarrow]_\varepsilon\varphi$, $@_{d_1}\varphi$, $\dots$, and $@_{d_k}\varphi$.

6 Axioms

In the next two sections, we provide a sound and complete axiomatization of the interplay between modalities $[\leftarrow]$ and $@$. In addition to propositional tautologies in language Φ_0, the axioms of our logical system are:

1. Truth: $[\leftarrow]_\sigma\varphi \to \varphi$,
2. Distributivity: $@_d(\varphi \vee \psi) \leftrightarrow (@_d\varphi \vee @_d\psi)$,
3. Negation: $\neg @_d\varphi \leftrightarrow @_d\neg\varphi$.

For any sign $z \in \{+, -\}$, by $[\leftarrow]^z_\sigma\varphi$ we denote the formula $[\leftarrow]_\sigma\varphi$ if $z =$ "+" and the formula $\neg[\leftarrow]_\sigma\varphi$ if $z =$ "–". We write $\vdash \varphi$ and say that formula φ is a *theorem* if formula φ is provable from the axioms of our logical system using the Modus Ponens,

$$\frac{\varphi, \varphi \to \psi}{\psi}$$

the Necessitation,

$$\frac{\varphi}{@_d\varphi}$$

and the Insertion

$$\frac{\bigwedge_i \overline{@}_{\alpha_i}[\leftarrow]^{z_i}_{\beta_i}\varphi_i \to \overline{@}_\sigma\psi \qquad \forall i\ (\alpha_i\beta_i = \sigma\tau)}{\bigwedge_i \overline{@}_{\alpha_i}[\leftarrow]^{z_i}_{\beta_i}\varphi_i \to \overline{@}_\sigma[\leftarrow]_\tau\psi}$$

inference rules. We call the last rule "Insertion" because it inserts modality $[\leftarrow]_\tau$ in the conclusion of the implication. In the Insertion rule, $\sigma\tau$ represents the concatenation of two (chained) designator strings σ and τ. Below is an example of how this rule is used.

$$\frac{@_{\text{buddy}}@_{\text{boss}}\neg[\leftarrow]_\varepsilon\varphi_1 \wedge [\leftarrow]_{\text{buddy, boss}}\varphi_2 \to @_{\text{buddy}}\psi}{@_{\text{buddy}}@_{\text{boss}}\neg[\leftarrow]_\varepsilon\varphi_1 \wedge [\leftarrow]_{\text{buddy, boss}}\varphi_2 \to @_{\text{buddy}}[\leftarrow]_{\text{boss}}\psi}$$

In this example, $\alpha_1 =$ (buddy, boss), $\beta_1 = \varepsilon$, $\alpha_2 = \varepsilon$, $\beta_2 =$ (buddy, boss), $\sigma =$ (buddy) $\tau =$ (boss). Observe that the application of the rule is valid because $\alpha_1\beta_1 = \alpha_2\beta_2 = \sigma\tau$.

In addition to the unary relation $\vdash \varphi$, we also consider binary relation $X \vdash \varphi$. By definition, $X \vdash \varphi$ is true if the formula φ is derivable from the *theorems* of our logical system using the Modus Ponens inference rule only. Note that statements $\varnothing \vdash \varphi$ and $\vdash \varphi$ are equivalent. We say that set X is consistent if $X \nvdash \neg\top$. The proof of the next standard lemma can be found in the appendix.

Lemma 9 (deduction) *If* $\Gamma, \varphi \vdash \psi$, *then* $\Gamma \vdash \varphi \to \psi$.

Lemma 10 (Lindenbaum) *Any consistent set of formulae can be extended to a maximal consistent set of formulae.*

Proof The standard proof of Lindenbaum's lemma applies here [23, Proposition 2.14]. □

Next, we prove the soundness of our logical system.

Theorem 5 (soundness) *If* $\vdash \varphi$*, then* $w, a \Vdash \varphi$ *for each world* w *and each agent* a *of each epistemic model with extensions.*

The soundness of the axioms and the Modus Ponens inference rule is straightforward. The soundness of the Insertion inference rule is proven in Lemma 12. We start with an auxiliary property which is used in the proof of that lemma.

Lemma 11 $w, a \Vdash [\leftarrow]^z_\beta \varphi$ *iff* $u, a \Vdash [\leftarrow]^z_\beta \varphi$ *for any* $w, u \in W$ *such that* $w \sim_{\hat{e}_\beta(a)} u$.

Proof By item 2 of Definition 3, it suffices to show that $w, a \Vdash [\leftarrow]_\beta \varphi$ iff $u, a \Vdash [\leftarrow]_\beta \varphi$. Furthermore, without loss of generality, it suffices to prove that the statement $w, a \Vdash [\leftarrow]_\beta \varphi$ implies $u, a \Vdash [\leftarrow]_\beta \varphi$. Indeed, the assumption $w, a \Vdash [\leftarrow]_\beta \varphi$ by item 5 of Definition 3, implies that $v, a \Vdash \varphi$ for each world $v \in W$ such that $w \sim_{\hat{e}_\beta(a)} v$. Hence, the assumption $w \sim_{\hat{e}_\beta(a)} u$ of the lemma implies that $v, a \Vdash \varphi$ for each world $v \in W$ such that $u \sim_{\hat{e}_\beta(a)} v$. Therefore, $u, a \Vdash [\leftarrow]_\beta \varphi$ by item 5 of Definition 3. □

Lemma 12 *For any formulae* $\varphi_1, \ldots, \varphi_n, \psi \in \Phi_0$ *and any chained designators* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \sigma, \tau \in \Delta^*$ *such that* $\alpha_i \beta_i = \sigma\tau$ *for each* $i \leq n$*, if*

$$u, b \Vdash \bigwedge_i \overline{@}_{\alpha_i} [\leftarrow]^{z_i}_{\beta_i} \varphi_i \to \overline{@}_\sigma \psi \tag{4}$$

for each world u *and each agent* b *of each epistemic model with extensions, then*

$$w, a \Vdash \bigwedge_i \overline{@}_{\alpha_i} [\leftarrow]^{z_i}_{\beta_i} \varphi_i \to \overline{@}_\sigma [\leftarrow]_\tau \psi$$

for each world u *and each agent* a *of each epistemic model with extensions.*

Proof Consider an arbitrary world $w \in W$ and an arbitrary agent $a \in \mathcal{A}$ of an epistemic model with extensions $(W, \mathcal{A}, \sim, e, \pi)$. Suppose that for each $i \leq n$,

$$w, a \Vdash \overline{@}_{\alpha_i} [\leftarrow]^{z_i}_{\beta_i} \varphi_i. \tag{5}$$

Observe that, by Definition 3, it suffices to show that $w, a \Vdash \overline{@}_\sigma [\leftarrow]_\tau \psi$. Then, by Lemma 2, it suffices to prove that $w, \hat{e}_\sigma(a) \Vdash [\leftarrow]_\tau \psi$. Next, consider any world $u \in W$ such that

$$w \sim_{\hat{e}_\tau(\hat{e}_\sigma(a))} u. \tag{6}$$

By item 5 of Definition 3, it suffices to show $u, \hat{e}_\sigma(a) \Vdash \psi$.

Claim 3 $u, a \Vdash \overline{@}_{\alpha_i} [\leftarrow]^{z_i}_{\beta_i} \varphi_i$ *for each* $i \leq n$.

Proof of Claim By Lemma 2, Eq. 5 implies that

$$w, \hat{e}_{\alpha_i}(a) \Vdash [\leftarrow]^{z_i}_{\beta_i} \varphi_i. \tag{7}$$

The assumption $\alpha_i\beta_i = \sigma\tau$ of the lemma, by the definition of function $\hat{e}$, implies that

$$\hat{e}_{\beta_i}(\hat{e}_{\alpha_i}(a)) = \hat{e}_{\alpha_i\beta_i}(a) = \hat{e}_{\sigma\tau}(a) = \hat{e}_{\tau}(\hat{e}_{\sigma}(a)).$$

Then, $w \sim_{\hat{e}_{\beta_i}(\hat{e}_{\alpha_i}(a))} u$ by Eq. 6. Hence, it follows that $u, \hat{e}_{\alpha_i}(a) \Vdash [\leftarrow]^{z_i}_{\beta_i}\varphi_i$ by Eq. 7 and Lemma 11. Therefore, $u, a \Vdash \overline{@}_{\alpha_i}[\leftarrow]^{z_i}_{\beta_i}\varphi_i$ by Lemma 2. □

Towards the proof of the lemma, note that, by the above claim, Eq. 4 of the lemma (for $b = a$) implies $u, a \Vdash \overline{@}_{\sigma}\psi$. Therefore, $u, \hat{e}_{\sigma}(a) \Vdash \psi$ by Lemma 2. □

Below, we establish several auxiliary lemmas that will be used later in the proof of completeness.

Lemma 13 $\vdash \overline{@}_{d_1d_2\ldots d_n}\varphi \vee \overline{@}_{d_1d_2\ldots d_n}\neg\varphi$.

Proof We prove the statement by induction on n. If $n = 0$, then the formula is a propositional tautology. Suppose that $n > 0$. Then, $\vdash \overline{@}_{d_2\ldots d_n}\varphi \vee \overline{@}_{d_2\ldots d_n}\neg\varphi$ by the induction hypothesis. Thus, $\vdash @_{d_1}(\overline{@}_{d_2\ldots d_n}\varphi \vee \overline{@}_{d_2\ldots d_n}\neg\varphi)$ by the Necessitation inference rule. Therefore, by the Distributivity axiom and the Modus Ponens inference rule, we have $\vdash @_{d_1}\overline{@}_{d_2\ldots d_n}\varphi \vee @_{d_1}\overline{@}_{d_2\ldots d_n}\neg\varphi$. □

Lemma 14 $\vdash \neg\overline{@}_{d_1d_2\ldots d_n}\neg\varphi \vee \neg\overline{@}_{d_1d_2\ldots d_n}\varphi$.

Proof We prove the statement by induction on n. If $n = 0$, then the formula is a propositional tautology. Suppose that $n > 0$. Then, $\vdash \neg\overline{@}_{d_2\ldots d_n}\neg\varphi \vee \neg\overline{@}_{d_2\ldots d_n}\varphi$ by the induction hypothesis. Thus, $\vdash @_{d_1}(\neg\overline{@}_{d_2\ldots d_n}\neg\varphi \vee \neg\overline{@}_{d_2\ldots d_n}\varphi)$ by the Necessitation inference rule. Hence, by the Distributivity axiom and the Modus Ponens inference rule, we have $\vdash @_{d_1}\neg\overline{@}_{d_2\ldots d_n}\neg\varphi \vee @_{d_1}\neg\overline{@}_{d_2\ldots d_n}\varphi$. Therefore, by the Negation axiom and the laws of propositional reasoning, $\vdash \neg @_{d_1}\overline{@}_{d_2\ldots d_n}\neg\varphi \vee \neg @_{d_1}\overline{@}_{d_2\ldots d_n}\varphi$. □

Lemma 15 $\vdash \neg\overline{@}_{d_1d_2\ldots d_n}\varphi \leftrightarrow \overline{@}_{d_1d_2\ldots d_n}\neg\varphi$.

Lemma 15 follows from the two lemmas above by propositional reasoning.

Lemma 16 $\vdash \neg\overline{@}_{d_1d_2\ldots d_n}(\psi \vee \chi) \vee (\overline{@}_{d_1d_2\ldots d_n}\psi \vee \overline{@}_{d_1d_2\ldots d_n}\chi)$.

Proof We prove this statement by induction on n. The formula is a propositional tautology if $n = 0$. Suppose that $n > 0$. Then, by the induction hypothesis,

$$\vdash \neg\overline{@}_{d_2\ldots d_n}(\psi \vee \chi) \vee (\overline{@}_{d_2\ldots d_n}\psi \vee \overline{@}_{d_2\ldots d_n}\chi).$$

Thus, by the Necessitation inference rule,

$$\vdash @_{d_1}(\neg\overline{@}_{d_2\ldots d_n}(\psi \vee \chi) \vee (\overline{@}_{d_2\ldots d_n}\psi \vee \overline{@}_{d_2\ldots d_n}\chi)).$$

Then, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash @_{d_1}\neg\overline{@}_{d_2\ldots d_n}(\psi \vee \chi) \vee @_{d_1}(\overline{@}_{d_2\ldots d_n}\psi \vee \overline{@}_{d_2\ldots d_n}\chi).$$

Hence, by the Negation axiom and propositional reasoning,

$$\vdash \neg @_{d_1} \overline{@}_{d_2 \dots d_n}(\psi \vee \chi) \vee @_{d_1}(\overline{@}_{d_2 \dots d_n}\psi \vee \overline{@}_{d_2 \dots d_n}\chi).$$

Therefore,

$$\vdash \neg @_{d_1} \overline{@}_{d_2 \dots d_n}(\psi \vee \chi) \vee (@_{d_1}\overline{@}_{d_2 \dots d_n}\psi \vee @_{d_1}\overline{@}_{d_2 \dots d_n}\chi)$$

by the Distributivity axiom and propositional reasoning. □

Lemma 17 $\vdash \overline{@}_{d_1 d_2 \dots d_n}(\psi \vee \chi) \leftrightarrow (\overline{@}_{d_1 d_2 \dots d_n}\psi \vee \overline{@}_{d_1 d_2 \dots d_n}\chi)$.

Lemma 17 follows from the two lemmas above by propositional reasoning.

Lemma 18 $\vdash \overline{@}_{d_1 d_2 \dots d_n}(\psi \vee \chi) \vee \neg(\overline{@}_{d_1 d_2 \dots d_n}\psi \vee \overline{@}_{d_1 d_2 \dots d_n}\chi)$.

Proof We prove this statement by induction on n. The formula is a propositional tautology if $n = 0$. Suppose that $n > 0$. Then, by the induction hypothesis,

$$\vdash \overline{@}_{d_2 \dots d_n}(\psi \vee \chi) \vee \neg(\overline{@}_{d_2 \dots d_n}\psi \vee \overline{@}_{d_2 \dots d_n}\chi).$$

Hence, by the Necessitation inference rule,

$$\vdash @_{d_1}(\overline{@}_{d_2 \dots d_n}(\psi \vee \chi) \vee \neg(\overline{@}_{d_2 \dots d_n}\psi \vee \overline{@}_{d_2 \dots d_n}\chi)).$$

Then, by the Distributivity axiom and the Modus Ponens inference rule,

$$\vdash @_{d_1}\overline{@}_{d_2 \dots d_n}(\psi \vee \chi) \vee @_{d_1}\neg(\overline{@}_{d_2 \dots d_n}\psi \vee \overline{@}_{d_2 \dots d_n}\chi).$$

Thus, by the Negation axiom and propositional reasoning,

$$\vdash @_{d_1}\overline{@}_{d_2 \dots d_n}(\psi \vee \chi) \vee \neg @_{d_1}(\overline{@}_{d_2 \dots d_n}\psi \vee \overline{@}_{d_2 \dots d_n}\chi).$$

Therefore, $\vdash @_{d_1}\overline{@}_{d_2 \dots d_n}(\psi \vee \chi) \vee \neg(@_{d_1}\overline{@}_{d_2 \dots d_n}\psi \vee @_{d_1}\overline{@}_{d_2 \dots d_n}\chi)$ by the Distributivity axiom and propositional reasoning. □

Lemma 19 $\vdash \neg\overline{@}_{d_1 d_2 \dots d_n}[\leftarrow]_\tau\psi \vee \overline{@}_{d_1 d_2 \dots d_n}\psi$.

Proof We prove this statement by induction on n. If $n = 0$, the formula follows from the Truth axiom by propositional reasoning. Suppose that $n > 0$. Then, by the induction hypothesis, $\vdash \neg\overline{@}_{d_2 \dots d_n}[\leftarrow]_\tau\psi \vee \overline{@}_{d_2 \dots d_n}\psi$. Thus, by the Neccessitation inference rule, $\vdash @_{d_1}(\neg\overline{@}_{d_2 \dots d_n}[\leftarrow]_\tau\psi \vee \overline{@}_{d_2 \dots d_n}\psi)$. Then, by the Distributivity axiom and the Modus Ponens inference rule, $\vdash @_{d_1}\neg\overline{@}_{d_2 \dots d_n}[\leftarrow]_\tau\psi \vee @_{d_1}\overline{@}_{d_2 \dots d_n}\psi$. Therefore, by the Negation axiom and propositional reasoning, $\vdash \neg @_{d_1}\overline{@}_{d_2 \dots d_n}[\leftarrow]_\tau\psi \vee @_{d_1}\overline{@}_{d_2 \dots d_n}\psi$. □

Lemma 20 $\vdash \overline{@}_\sigma[\leftarrow]_\tau\psi \to \overline{@}_\sigma\psi$.

Proof The formula $[\leftarrow]_\tau\psi \to \psi$ is an instance of the Truth axiom. Then, $\vdash \neg[\leftarrow]_\tau\psi \vee \psi$ by propositional reasoning. Hence, $\vdash \overline{@}_\sigma(\neg[\leftarrow]_\tau\psi \vee \psi)$ by the Necessitation inference rule. Then, $\vdash \overline{@}_\sigma\neg[\leftarrow]_\tau\psi \vee \overline{@}_\sigma\psi$ by Lemma 17. Thus, $\vdash \neg\overline{@}_\sigma[\leftarrow]_\tau\psi \vee \overline{@}_\sigma\psi$ by Lemma 15. Therefore, by propositional reasoning, $\vdash \overline{@}_\sigma[\leftarrow]_\tau\psi \to \overline{@}_\sigma\psi$. □

7 Completeness

As usual in the proofs of completeness, at the center of the proof is a "truth" lemma. In our case, it is Lemma 23. In classical modal logic, the truth lemma usually states that $w \Vdash \varphi$ iff $\varphi \in w$. Since, in this article, we used 2D-semantics, we had to modify the truth lemma as seen in Lemma 23. This way to modify the truth lemma for 2D-semantics was suggested by Sano [24]. In the current work, we extend his approach from single designators to chained designators.

Definition 6 W is the set of all maximal consistent sets of formulae.

In traditional modal logic, each agent designator corresponds to an agent in the canonical model. To handle chained designators, we define agents as sequences of designators. Intuitively, our canonical model contains agent Protos ("first") and other agents related to Protos through chained designators. The empty sequence ε is Protos. The single-element sequence (boss) is Protos' boss. The sequence (boss, buddy) is the buddy of Protos' boss.

Definition 7 $\mathcal{A}$ is the set of all finite sequences of designators.

In canonical models for classical modal logics, two worlds are defined to be indistinguishable by an agent a if they contain the same $[\leftarrow]_a$-formulae. In our model, for example, formula $@_{\text{boss}}[\leftarrow]_{\text{buddy}}\varphi$ means that "Protos' boss' buddy" knows φ about Proto's boss. Thus, the same formula must be true in all worlds that the agent "Protos' boss' buddy" cannot distinguish from the current world. We capture this intuition in the definition below:

Definition 8 $w \sim_\sigma u$ if $\overline{@}_\alpha[\leftarrow]_\beta\varphi \in w$ iff $\overline{@}_\alpha[\leftarrow]_\beta\varphi \in u$ for any α, β such that $\alpha\beta = \sigma$.

Intuitively, extension function e_{spouse} maps the agent "Protos' boss' buddy" to the agent "Protos' boss' buddy's spouse". In other words, $e_{\text{spouse}}(\text{boss}, \text{buddy}) = (\text{boss}, \text{buddy}, \text{spouse})$. This explains the next definition.

Definition 9 $e_d(\sigma) = \sigma d$.

The next lemma is proven by induction on the length of sequence τ using Definitions 2 and 9.

Lemma 21 $\hat{e}_\tau(\sigma) = \sigma\tau$.

The next definition specifies valuation π in a way that guarantees that Lemma 23 is true for propositional variables.

Definition 10 $\pi(p) = \{(w, \sigma) \mid \overline{@}_\sigma p \in w\}$ for each propositional variable p.

To improve readability, we prove the key step in Lemma 23 as a separate lemma below.

Lemma 22 *If* $\overline{@}_\sigma[\leftarrow]_\tau\psi \notin w$*, then* $\overline{@}_\sigma\psi \notin u$ *for some world* $u \in W$ *such that* $w \sim_{\sigma\tau} u$.

Proof Consider the set of formulae

$$u^{-} = \{\neg\overline{@}_{\sigma}\psi\} \cup \{\overline{@}_{\alpha}[\leftarrow]^{z}_{\beta}\chi \mid \overline{@}_{\alpha}[\leftarrow]^{z}_{\beta}\chi \in w, \alpha\beta = \sigma\tau\}. \tag{8}$$

Claim 4 *Set u^{-} is consistent.*

Proof of Claim Suppose the opposite, then there are formulae

$$\overline{@}_{\alpha_1}[\leftarrow]^{z_1}_{\beta_1}\chi_1, \dots, \overline{@}_{\alpha_m}[\leftarrow]^{z_m}_{\beta_m}\chi_m \in w \tag{9}$$

such that $\alpha_i\beta_i = \sigma\tau$ for each $i \geq 1$ and

$$\overline{@}_{\alpha_1}[\leftarrow]^{z_1}_{\beta_1}\chi_1, \dots, \overline{@}_{\alpha_m}[\leftarrow]^{z_m}_{\beta_m}\chi_m \vdash \overline{@}_{\sigma}\psi.$$

Thus, by Lemma 9 and propositional reasoning,

$$\vdash \bigwedge_i \overline{@}_{\alpha_i}[\leftarrow]^{z_i}_{\beta_i}\chi_i \to \overline{@}_{\sigma}\psi.$$

Then, by the Insertion inference rule,

$$\vdash \bigwedge_i \overline{@}_{\alpha_i}[\leftarrow]^{z_i}_{\beta_i}\chi_i \to \overline{@}_{\sigma}[\leftarrow]_{\tau}\psi.$$

Hence $w \vdash \overline{@}_{\sigma}[\leftarrow]_{\tau}\psi$ by Eq. 9 and propositional reasoning. Then, $\overline{@}_{\sigma}[\leftarrow]_{\tau}\psi \in w$ because w is a maximal consistent set of formulae, which contradicts the assumption $\overline{@}_{\sigma}[\leftarrow]_{\tau}\psi \notin w$ of the lemma. □

By Lemma 10, set u^{-} can be extended to a maximal consistent u.

Claim 5 $w \sim_{\sigma\tau} u$.

Proof of Claim Consider any α, β such that $\alpha\beta = \sigma\tau$. By Definition 8, it suffices to show that $\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi \in w$ if and only if $\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi \in u$.
$(\Rightarrow)$: Suppose $\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi \in w$. Therefore, $\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi \in u^{-} \subseteq u$ by Eq. 8.
$(\Leftarrow)$: Suppose $\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi \notin w$. Then, $\neg\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi \in w$ because set w is maximal. Thus, $w \vdash \overline{@}_{\alpha}\neg[\leftarrow]_{\beta}\varphi$ by Lemma 15 and propositional reasoning. Hence, $\overline{@}_{\alpha}\neg[\leftarrow]_{\beta}\varphi \in w$ because set w is maximal. In other words, $\overline{@}_{\alpha}[\leftarrow]^{-}_{\beta}\varphi \in w$. Therefore, $\overline{@}_{\alpha}[\leftarrow]^{-}_{\beta}\varphi \in u^{-} \subseteq u$ by Eq. 8. Then, $\overline{@}_{\alpha}\neg[\leftarrow]_{\beta}\varphi \in u$. Thus, $u \vdash \neg\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi$ by Lemma 15 and propositional reasoning. Therefore, $\overline{@}_{\alpha}[\leftarrow]_{\beta}\varphi \notin u$ because set u is maximal. □

To conclude the proof of the lemma, observe that $\neg\overline{@}_{\sigma}\psi \in u^{-} \subseteq u$. Therefore, $\overline{@}_{\sigma}\psi \notin u$ because set u is consistent. □

Lemma 23 $w, \sigma \Vdash \varphi$ *iff* $\overline{@}_{\sigma}\varphi \in w$.

Proof We prove the lemma by induction on the structural complexity of formula φ. If φ is a propositional variable, then the statement of the lemma follows from Definition 10 and item 1 of Definition 3.

Suppose formula φ has the form $\neg\psi$. The statement $w, \sigma \Vdash \neg\psi$ is equivalent to the statement $w, \sigma \nVdash \psi$ by item 2 of Definition 3. The latter statement, by the induction hypothesis, is equivalent to $\overline{@}_\sigma \psi \notin w$. The last statement is equivalent to $\overline{@}_\sigma \neg\psi \in w$ by Lemma 15 because w is a maximal consistent set.

Suppose formula φ has the form $\psi \vee \chi$. The statement $w, \sigma \Vdash \psi \vee \chi$ is equivalent to the disjunction of the statements $w, \sigma \Vdash \psi$ and $w, \sigma \Vdash \chi$ by item 3 of Definition 3. By the induction hypothesis, the statements in the disjunction are equivalent to $\overline{@}_\sigma \psi \in w$ and $\overline{@}_\sigma \chi \in w$. By Lemma 17, the disjunction of the last two statements is equivalent to $\overline{@}_\sigma (\psi \vee \chi) \in w$ because w is a maximal consistent set.

Suppose formula φ has the form $@_d \psi$. The statement $w, \sigma \Vdash @_d \psi$ is equivalent to the statement $w, e_d(\sigma) \Vdash \psi$ by item 4 of Definition 3. By Definition 9, the latter statement is equivalent to $w, \sigma d \Vdash \psi$. By induction hypothesis, the new statement is equivalent to $\overline{@}_{\sigma d} \psi \in w$. The last statement is equivalent to $\overline{@}_\sigma @_d \psi \in w$ by the definition of $\overline{@}$ notation.

Suppose formula φ has the form $[\leftarrow]_\tau \psi$.
$(\Leftarrow)$: Assume $\overline{@}_\sigma [\leftarrow]_\tau \psi \in w$. Consider any world $u \in W$ such that $w \sim_{\hat{e}_\tau(\sigma)} u$. By item 5 of Definition 3, it suffices to show $u, \sigma \Vdash \psi$. By Lemma 21, the assumption $w \sim_{\hat{e}_\tau(\sigma)} u$ implies $w \sim_{\sigma\tau} u$. Then, $\overline{@}_\sigma [\leftarrow]_\tau \psi \in u$ by Definition 8 and the assumption $\overline{@}_\sigma [\leftarrow]_\tau \psi \in w$. Thus, $\overline{@}_\sigma \psi \in u$ by Lemma 20 and because set u is maximal. Hence, $u, \sigma \Vdash \psi$ by the induction hypothesis.
$(\Rightarrow)$: Assume $\overline{@}_\sigma [\leftarrow]_\tau \psi \notin w$. Then, by Lemma 22, there exists a world $u \in W$ such that $w \sim_{\sigma\tau} u$ and $\overline{@}_\sigma \psi \notin u$. Thus, $u, \sigma \nVdash \psi$ by the induction hypothesis. By Lemma 21, the statement $w \sim_{\sigma\tau} u$ implies $w \sim_{\hat{e}_\tau(\sigma)} u$. Therefore, we have $w, \sigma \nVdash [\leftarrow]_\tau \psi$ by item 5 of Definition 3 and statement $u, \sigma \nVdash \psi$. □

Theorem 6 (strong completeness) *If $X \nvdash \varphi$, then there is a world w and an agent a of an epistemic model with extensions such that $w, a \Vdash \chi$ for each formula $\chi \in X$ and $w, a \nVdash \varphi$.*

Proof Suppose that $X \nvdash \varphi$. Then, set $X \cup \{\neg\varphi\}$ is consistent. Let w be any maximal consistent extension of this set. Such a set exists by Lemma 10. Then, $\overline{@}_\varepsilon \chi \in w$ for each $\chi \in X$ and $\overline{@}\neg\varphi \in w$. Thus, $w, \varepsilon \Vdash \chi$ for each $\chi \in X$ and $w, \varepsilon \Vdash \neg\varphi$ by Lemma 23. Therefore, $w, \varepsilon \nVdash \varphi$ by item 2 of Definition 3. □

8 Conclusion

Agentically non-rigid designators refer to a referent through a sequence of social connections of the context. They are helpful in describing a complicated structure of relations between multiple agents. When such a designator is composed of a chain of social connections, we call it a “chained designator”. The concept of chained designators was inspired by both natural and programming languages. We studied chained designators in knowledge statements and considered three kinds of knowledge modalities: the referent’s self-knowledge about the context [$\circlearrowleft$], the context’s knowledge about the referent [$\rightarrow$], and the referent’s knowledge about the context [$\leftarrow$]. Among these three knowledge modalities, modality [$\leftarrow$] is not definable through the modality @ and modalities [$\circlearrowleft$] and [$\rightarrow$], but modalities [$\circlearrowleft$] and [$\rightarrow$] can be defined through modality [$\leftarrow$] and @. Then, chained designators were proved to enhance the expressiveness of knowledge statements compared to statements using just atomic designators. Lastly, we provided a sound and complete epistemic logic with knowledge modality [$\leftarrow$] that utilizes chained designators.

A Model Checking Algorithm

In this section, we propose a model checking algorithm for our logical system. The algorithm decides if a statement of the form $w, a \Vdash \varphi$ holds for a world $w \in W$ and an agent $a \in \mathcal{A}$ of a given model, and a formula $\varphi \in \Phi$. This algorithm assumes that the set W of worlds, the set $\mathcal{A}$ of agents, and the set Δ of atomic designators are finite. Let $\varphi_1, \varphi_2, \dots, \varphi_n$ be the list of all subformulae of formula φ ordered in non-decreasing order of sizes. Note that φ_n is formula φ. The algorithm pre-computes the Boolean value $sat[u, b, i]$ of the statement $u, b \Vdash \varphi_i$ for each world $u \in W$, each agent $b \in \mathcal{A}$, and each $i \leq n$.

Lemma 24 *The algorithm in* Fig. 5 *has polynomial time complexity.*

Proof The algorithm has three nested “for” loops with five cases that correspond to formula φ_i having different forms. For the cases where φ_i is an atomic proposition, a negation, or an implication, the time complexity is constant. In the case where formula φ_i has the form $@_d\varphi_j$, by Definition 1 item 4, the time complexity is also polynomial. In the case where formula φ_i has the form $[\leftarrow]_\sigma\varphi_j$, by Definition 2, checking $u \sim_{\hat{e}_\sigma(b)} u'$ takes polynomial time. Thus, accounting for the “for” loop inside this case, it also has polynomial time complexity. Therefore, considering checking all cases within the three nested “for” loops, the algorithm has a polynomial time complexity. □

This next lemma can be proven by induction on i using Definition 3.

Lemma 25 $u, b \Vdash \varphi_i$ *iff* $sat[u, b, i] = true$ *for each world* $u \in W$, *each agent* $b \in \mathcal{A}$, *and each* $i \leq n$.

Since φ_n is formula φ, by Lemma 25, the model checking algorithm shown in Fig. 5 assigns the Boolean value true to $sat[w, a, i]$ if and only if formula φ is satisfied in world w with agent a. By Lemma 24, this model checking algorithm has polynomial time complexity.

```
for u ∈ W do
    for b ∈ 𝒜 do
        for i ≤ n do
            switch φ_i do
                case φ_i is an atomic proposition p
                    if u ∈ π(p) then
                        sat[u, b, i] ← true
                    else
                        sat[u, b, i] ← false
                    end if
                case φ_i has the form ¬φ_j
                    sat[u, b, i] ← ¬sat[u, b, j]
                case φ_i has the form φ_j → φ_k
                    sat[u, b, i] ← ¬sat[u, b, j] ∨ sat[u, b, k]
                case φ_i has the form @_d φ_j
                    sat[u, b, i] ← sat[u, e_d(b), j]
                case φ_i has the form [←]_σ φ_j
                    sat[u, a, i] ← true
                    for u′ ∈ W do
                        if u ∼_{ê_σ(b)} u′ and ¬sat[u′, b, j] then
                            sat[u, b, i] ← false
                            break
                        end if
                    end for
            end switch
        end for
    end for
end for
```

Fig. 5 Model checking algorithm

B Proof of Lemma 9

Proof Suppose that sequence $\psi_1, \dots, \psi_n$ is a proof from set $\Gamma \cup \{\varphi\}$ and the theorems of our logical system that uses the Modus Ponens inference rule only. In other words, for each $k \leq n$, either

1. $\vdash \psi_k$, or
2. $\psi_k \in \Gamma$, or
3. ψ_k is equal to φ, or
4. there are $i, j < k$ such that formula ψ_j is equal to $\psi_i \to \psi_k$.

It suffices to show that $\Gamma \vdash \varphi \to \psi_k$ for each $k \leq n$. We prove this by induction on k through considering the four cases above separately.
Case 1: $\vdash \psi_k$. Note that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Hence, $\vdash \varphi \to \psi_k$ by the Modus Ponens inference rule. Therefore, $\Gamma \vdash \varphi \to \psi_k$.
Case 2: $\psi_k \in \Gamma$. Note again that $\psi_k \to (\varphi \to \psi_k)$ is a propositional tautology, and thus, is an axiom of our logical system. Therefore, by the Modus Ponens inference rule, $\Gamma \vdash \varphi \to \psi_k$.

Case 3: formula ψ_k is equal to φ. Thus, $\varphi \to \psi_k$ is a propositional tautology. Therefore, $\Gamma \vdash \varphi \to \psi_k$.

Case 4: formula ψ_j is equal to $\psi_i \to \psi_k$ for some $i, j < k$. Thus, by the induction hypothesis, $\Gamma \vdash \varphi \to \psi_i$ and $\Gamma \vdash \varphi \to (\psi_i \to \psi_k)$. Note that formula $(\varphi \to \psi_i) \to ((\varphi \to (\psi_i \to \psi_k)) \to (\varphi \to \psi_k))$ is a propositional tautology. Therefore, $\Gamma \vdash \varphi \to \psi_k$ by applying the Modus Ponens inference rule twice. □

References

1. Kripke, S. A. (1980). *Naming and Necessity*. Cambridge, Massachusetts, United States: Harvard University Press.
2. Putnam, H. (1975). *The meaning of "meaning"*. Minneapolis, Minnesota: University of Minnesota Press.
3. McGinn, C. (1982). Rigid designation and semantic value. *The Philosophical Quarterly (1950-), 32*(127), 97–115.
4. Comrie, B. (2019). Possessive chains and possessor camouflage. In *Possession in languages of Europe and North and Central Asia* (p. 51). John Benjamins Publishing Company, Amsterdam, Netherlands.
5. Banks, A. (2023). MISRA C: 2012 amendment 4. https://www.misra.org.uk/app/uploads/2023/03/MISRA-C-2012-AMD4.pdf. Accessed 12 04 2024
6. Prior, A. N. (1968). Egocentric logic. *Noûs,* 191–207.
7. Seligman, J., Liu, F., & Girard, P. (2011). Logic in the community. In *Logic and its applications* (pp. 178–188). Springer, Berlin.
8. Seligman, J., Liu, F., & Girard, P. (2013). Facebook and the epistemic logic of friendship. In *14th Conference on Theoretical Aspects of Rationality and Knowledge (TARK '13)*, January 2013, Chennai, India, pp. 229–238.
9. Christoff, Z., & Hansen, J. U. (2015). A logic for diffusion in social networks. *Journal of Applied Logic, 13*(1), 48–77.
10. Christoff, Z., Hansen, J. U., & Proietti, C. (2016). Reflecting on social influence in networks. *Journal of Logic, Language and Information, 25*(3), 299–333.
11. Jiang, J., & Naumov, P. (2024). A logic of higher-order preferences. *Synthese, 203*(6), 210.
12. Naumov, P., & Wu, C. (2025). Egocentric doxastic logic. *Logic and Logical Philosophy,* 1–23.
13. Grove, A. J., & Halpern, J. Y. (1993). Naming and identity in epistemic logics Part I: The propositional case. *Journal of Logic and Computation, 3*(4), 345–378. https://doi.org/10.1093/logcom/3.4.345
14. Kaplan, D. (1979). On the logic of demonstratives. *Journal of Philosophical Logic, 8*, 81–98.
15. Chalmers, D. J. (2004). Epistemic two-dimensional semantics. *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition, 118*(1/2), 153–226. https://doi.org/10.1023/b:phil.0000019546.17135.e0
16. Ball, D. (2014). Two-dimensionalism and the social character of meaning. *Erkenntnis, 79*, 567–595.
17. Biggs, S., & Wilson, J. (2020). Abductive two-dimensionalism: A new route to the a priori identification of necessary truths. *Synthese, 197*, 59–93.
18. Fritz, P. (2013). A logic for epistemic two-dimensional semantics. *Synthese, 190*(10), 1753–1770. https://doi.org/10.1007/s11229-013-0260-x
19. Stalnaker, R. C. (2004). Assertion revisited: On the interpretation of two-dimensional modal semantics. *Philosophical Studies, 118*(1), 299–322. https://doi.org/10.1023/b:phil.0000019550.81145.34

20. Grove, A. J., & Halpern, J. Y. (1991). Naming and identity in a multi-agent epistemic logic. In Allen, J. F., Fikes, R., & Sandewall, E. (Eds.), *Proceedings of the 2nd international conference on principles of knowledge representation and reasoning (KR'91)*. Cambridge, MA, USA, April 22-25, 1991, pp. 301–312. Morgan Kaufmann, San Francisco, California.
21. Lewis, D. (1980) Index, context, and content. In *Philosophy and grammar* (pp. 79–100). Springer, Berlin. https://doi.org/10.1007/978-94-009-9012-8_6
22. Knight, S., Naumov, P., Shi, Q., & Suntharraj, V. (2022). Truth set algebra: A new way to prove undefinability. arXiv:2208.04422
23. Mendelson, E. (2009). *Introduction to mathematical logic*. Boca Raton, Florida: CRC Press.
24. Sano, K. (2010). Axiomatizing hybrid products: How can we reason many-dimensionally in hybrid logic? *Journal of Applied Logic, 8*(4), 459–474. https://doi.org/10.1016/j.jal.2010.08.006