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Two inertial projection-type methods for solving pseudo-monotone variational inequalities with application to image deblurring problem

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Abstract

Our purpose is to propose two different type of inertial algorithms for approximating a solution of pseudo-monotone variational inequality problem in the framework of Banach spaces. The proposed algorithms are established by using Mann's iterative method and single projection type method with adaptive step-size. Strong convergence theorems for minimum-norm solution of the variational inequality problem are established without the prior knowledge of the Lipschitz constant of the mapping. Finally, some numerical experiments are performed to illustrate the advantage of the proposed methods and numerical experiments in image recovery are also presented. Our results generalize and improve some known results existing in the current literature.

Keywords Banach space · Strong convergence · Variational inequality problem · Pseudo-monotone mapping · Minimum-norm

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1 Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E with dual space E^*. The *variational inequality problem* (VIP) is a mathematical model which consists of obtaining a point $x \in C$ such that

$$\langle y - x, \mathcal{F}x \rangle \geq 0, \quad \forall y \in C, \tag{1}$$

where $\mathcal{F} : C \to E^*$ is an operator and $\langle \cdot, \cdot \rangle$ is the duality pairing between the elements of E and E^*. We denote the solution set of the VIP by S. The VIP was introduced independently by Fichera [14, 15] and Stampacchia [39] (see also, Kinderlehrer and Stampacchia [22]). The theory of VIPs has attracted much attention due to its wide applications in both engineering and sciences. It plays an important role as a modelling tool in optimization theory, differential equations and so on. For an extensive literature on this theory, we refer the readers to the excellent books of Fachinei and Pang [13] and Kinderlehrer and Stampacchia [22].

One of the most extensively studied areas in the theory of VIPs is the development of iterative algorithms for approximating their solutions. In recent years, numerous numerical iterative algorithms have been proposed and adopted to solve VIPs involving monotone and pseudo-monotone operators in several settings (see, e.g., [8, 25, 34, 37, 40–43, 52, 55, 58, 59]). The first classical method is the *gradient projection method*, which has been extensively studied by various authors, including Dafermos [12]. This method is of the following scheme: for a given $x_1 \in C$, generate a sequence $\{x_n\}$ by the iterative process:

$$x_{n+1} = P_C(x_n - \lambda \mathcal{F} x_n), \quad n \geq 1, \tag{2}$$

where P_C is the metric projection onto the closed convex subset C of a real Hilbert space H and $\lambda > 0$ is a suitable step-size. It has been shown that if $\mathcal{F}$ is strongly monotone (or inverse strongly monotone), then the algorithm converges globally to a point of S. However, if this monotonicity assumption is dropped, then the algorithm fails to converge. To overcome such a drawback, Korpelevich [23] (see also, Antipin [2]) suggested the *extragradient method* (EGM) for solving the VIP in the finite dimensional Euclidean space $\mathbb{R}^m$. The EGM is given by

$$\begin{cases} x_1 \in C, \\ y_n = P_C(x_n - \lambda \mathcal{F} x_n), \\ x_{n+1} = P_C(x_n - \lambda \mathcal{F} y_n), \quad n \geq 1, \end{cases} \tag{3}$$

where $\mathcal{F}$ is monotone and L-Lipschitz continuous of C into $\mathbb{R}^m$ and $\lambda \in (0, \frac{1}{L})$. It was shown that the sequence $\{x_n\}$ generated by (3) converges to a point of S provided that S is nonempty. We note that in the execution of this method two orthogonal projections onto the feasible set C are calculated. In some cases, if the structure of the set C is not given explicitly or is complicated, then the orthogonal projection onto C might be difficult to calculate. Thus authors have suggested several iterative methods to overcome this drawback. One efficient method to improve the EGM is the *Tseng's extragradient method* (TEGM) introduced by Tseng [46]. The TEGM is given by

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda \mathcal{F} x_n), \\ x_{n+1} = y_n - \lambda(\mathcal{F} y_n - \mathcal{F} x_n), \ \ n \geq 1, \end{cases} \tag{4}$$

The weak convergence of this method was established provided $\lambda \in (0, \frac{1}{L})$. It is worth noting that this method compute only one orthogonal projection in each iteration, which is simpler than the EGM. For this reason, the TEGM has received considerable attention in recent years by many authors (see, e.g., [9, 37, 55–57]). In addition, this approach was also extended for solving the pseudo-monotone VIP.

Thong and Vuong [42] suggested a modification of the TEGM with linesearch procedure for solving pseudo-monotone VIP in a real Hilbert space as follows:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda_n \mathcal{F} x_n), \\ x_{n+1} = y_n - \lambda_n(\mathcal{F} y_n - \mathcal{F} x_n), \ \ n \geq 1, \end{cases} \tag{5}$$

where $\lambda_n := \gamma l^{m_n}$ and m_n is the smallest nonnegative integer satisfies the condition $\gamma l^m \|\mathcal{F} x_n - \mathcal{F} y\| \leq \mu \|x_n - y_n\|$ and $\gamma > 0$, $l \in (0,1)$ and $\mu \in (0,1)$. It was shown that, if $\mathcal{F} : H \to H$ is pseudo-monotone, L-Lipschitz continuous and sequentially weakly continuous, then the sequence $\{x_n\}$ generated by (5) converges weakly to a point of S. Note that above method used a linesearch procedure, which needs an inner loop with a stopping criterion at each iteration and it can be time-consuming.

Thong et al. [43] also proposed a modification of the TEGM (5) with adaptive step-size for pseudo-monotone VIP as follows:

$$\begin{cases} x_1 \in H, \\ y_n = P_C(x_n - \lambda_n \mathcal{F} x_n), \\ z_n = y_n - \lambda_n(\mathcal{F} y_n - \mathcal{F} x_n), \\ x_{n+1} = z_n - \alpha_n \gamma \mathcal{A} z_n, \ \ n \geq 1, \end{cases} \tag{6}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu\|x_n - y_n\|}{\|\mathcal{F} x_n - \mathcal{F} y_n\|}, \lambda_n \right\} & \text{if } \mathcal{F} x_n - \mathcal{F} y_n \neq 0, \\ \lambda_n & \text{otherwise }, \end{cases} \tag{7}$$

where $\gamma > 0$, $\mu \in (0,1)$, $\{\alpha_n\} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. They proved that if in additional $\mathcal{A}$ is strongly monotone and Lipschitz continuous on H, then the sequence $\{x_n\}$ generated by (6) converges strongly to a point of S.

The study of theory of the VIPs is not limited to the finite dimensional space $\mathbb{R}^m$ and the Hilbert spaces, thus extension of iterative methods have been generalized to various Banach spaces (see, e.g., [18, 19, 30, 31]). In 2020, Shehu [36] studied the following Halpern-type TEGM for solving the monotone VIP in a 2-uniformly convex and uniformly smooth Banach space:

$$\begin{cases} x_1 \in E, \\ y_n = \Pi_C J^{-1}(J x_n - \lambda_n \mathcal{F} x_n), \\ z_n = J^{-1}(J y_n - \lambda_n(\mathcal{F} y_n - \mathcal{F} x_n)), \\ x_{n+1} = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J z_n), \ \ n \geq 1, \end{cases} \tag{8}$$

where $\mathcal{F} : E \to E^*$ is monotone and L-Lipschitz continuous, Π_C is the generalized projection from E onto C, J is the normalized duality mapping on E. The strong convergence theorem was established when the sequence of step-size satisfies the condition $0 < a \leq \lambda_n \leq b < \frac{1}{\sqrt{2c\kappa L}}$, where $c > 0$ is the 2-uniform convexity constant of E and $\kappa > 0$ is 2-uniform smoothness constant.

On the another hand, the inertial technique was first introduced by Polyak [33] in 1964 (see also, Nesterov [28]). This technique has been widely used to accelerate performance of iterative algorithms in optimization theory. The inertial technique was motivated by an explicit finite difference discretization of the second-order dynamical systems in time (or heavy ball with friction). In recent years, many authors have established iterative algorithms by employing the inertial technique for solving various optimization problems with different conditions placed on the inertial parameters (see, e.g., [10, 25, 30–32, 38]).

There are many nonlinear problems in the natural sciences as well as engineering problems, can be mathematically solved by finding the minimum-norm solutions, that is, finding a point $x^* \in K$ with the property $x^* = \min\{\|x\| : x \in K\}$, where K is a nonempty, closed, and convex subset of a real Hilbert space. In other words, x^* is exactly the nearest point orthogonal projection of the origin onto K or, equivalently, $x^* = P_K(0)$.

Motivated and inspired by the above mentioned works, we proposed two inertial Tseng's extragradient methods with self adaptive step-size for approximating a minimum-norm solution to the variational inequalities. The main advantages of the proposed methods are that they do not require the prior knowledge of the Lipschitz constant of the cost operator and only requires to compute one orthogonal projection onto a feasible set in each iteration. By using Mann's iterative method, we state and prove the strong convergent results for solving a pseudo-monotone variational inequality in the framework of 2-uniformly smooth Banach space which is also uniformly smooth. The strong convergence under the cost operator is non-Lipschitz continuous assumption of the variational inequality mapping is also obtained. Finally, we perform several numerical experiments to show the performance of both algorithms and provide some applications including its numerical experiments by using our methods to solve the image deblurring problem.

The rest of the paper is organized as follows: In Section 2, we recall some definitions and preliminary results. In Section 3, we prove and analyze the strong convergence results of the proposed algorithms and finally, numerical results of the proposed methods are also presented in Section 4.

2 Preliminaries

In this section, we recall some definitions and results for our main result in this paper. Let E be a real Banach space with dual space E^*. We will denote $\langle x, x^* \rangle$ by the value of a functional x^* in E^* at x in E. For a sequence $\{x_n\}$ in E, the strong convergence and the weak convergence of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let $S_E := \{x \in E : \|x\| = 1\}$ and $B_E := \{x \in E : \|x\| \leq 1\}$. A Banach space E is said to *strictly convex* if $\|(x + y)/2\| < 1$ whenever $x, y \in S_E$ with $x \neq y$. The space E is said to be *uniformly convex* if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$, where δ_E is the modulus of convexity of E defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_E, \|x-y\| \geq \epsilon\right\}$$

for all $\epsilon \in [0,2]$. The space E is said to be 2-*uniformly convex* if there exists $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^2$ for all $\epsilon \in [0,2]$. Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x \in S_E, \|y\| = t\right\}.$$

The space E is said to be *smooth* if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{9}$$

exists for all $x, y \in S_E$ and it is said to be *uniformly smooth* if (9) converges uniformly in $x, y \in S_E$. It is known that a uniformly smooth space is smooth. The space E is said to be 2-*uniformly smooth* if there exists $c > 0$ such that $\rho_E(t) \leq ct^2$ for all $t > 0$. It is known that every 2-uniformly convex (2-uniformly smooth) space is uniformly convex (uniformly smooth) space. If E is uniformly convex, then E is reflexive and strictly convex and if E is uniformly smooth, then E is reflexive and smooth (see [11]).

A concrete examples of both uniformly convex and uniformly smooth Banach spaces are L_p and ℓ_p, where $p > 1$. More specifically, L_p and ℓ_p are $\max\{p,2\}$-uniformly convex and uniformly smooth, while Hilbert spaces 2-uniformly convex and uniformly smooth Banach spaces (see [3, 54] for more details).

The *normalized duality mapping* $J : E \to 2^{E^*}$ is defined by

$$Jx := \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},\ \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between E and E^*. For a duality mapping J, the following properties are well-known [3, 11]:

(i) For each $x \in E$, Jx is nonempty, closed, convex and bounded subset of E^*.
(ii) $J(\lambda x) = \lambda Jx$ for all $x \in E$ and $\lambda \in \mathbb{R}$.
(iii) $J = I$ is the identity mapping on E if and only if E is a Hilbert space.
(iv) If E is uniformly smooth if and only if J is single-valued from E into E^* and it is uniformly continuous on bounded subsets of E.
(v) If E is a reflexive, smooth and strictly convex, then J^{-1} is single-valued, one-to-one, surjective and it is the duality mapping from E^* into E.

Lemma 2.1 *([54]) Let E be a real Banach space.*

(i) *If E is 2-uniformly smooth, then there exists a constant* $\kappa > 0$ *such that*

$$\|x-y\|^2 \leq \|x\|^2 - 2\langle y, Jx\rangle + \kappa\|y\|^2,\quad \forall x, y \in E.$$

(ii) *If E is 2-uniformly convex, then there exists a constant* $c > 0$ *such that*

$$\|x-y\|^2 \geq \|x\|^2 - 2\langle y, Jx\rangle + c\|y\|^2,\ \forall x, y \in E.$$

Remark 1 The 2-uniform smoothness constant can be computed by the formula $\kappa = q-1$ and the 2-uniform convexity constant can be computed by the formula $c = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}$, where t_p is the unique solution of $(p-2)t^{p-1} + (p-1)t^{p-2} - 1 = 0$, where $t \in (0, 1)$ (see [54] for more details). It is well-known that $\kappa = c = 1$ whenever E is a Hilbert space.

Let E be a smooth Banach space. The *Lyapunov function* $\phi : E \times E \to [0, \infty)$ (see [4]) is defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

If E is a Hilbert space, then $\phi(x, y) = \|x-y\|^2$ for all $x, y \in E$. From the definition of ϕ, it is clear that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E.$$

One can see that $\phi(x, y) \geq 0$ and $\phi(x, y) = 0$ if and only if $x = y$. Moreover, we know the following properties:

$$\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz)) \leq \alpha\phi(x, y) + (1-\alpha)\phi(x, z) \tag{10}$$

and

$$\phi(x, y) = \phi(x, z) - \phi(y, z) + 2\langle y - x, Jy - Jz\rangle \tag{11}$$

for all $x, y, z \in E$ and $\alpha \in [0, 1]$.

Lemma 2.2 *([29]) Let E be a uniformly smooth Banach space and $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\phi(x, J^{-1}(\alpha Jy + (1-\alpha)Jz)) \leq \alpha\phi(x, y) + (1-\alpha)\phi(x, z) - \alpha(1-\alpha)g(\|Jy - Jz\|)$$

for all $x \in E$, $y, z \in B_r := \{x \in E : \|x\| \leq r\}$ and $\alpha \in [0, 1]$.

Lemma 2.3 *([5]) Let E be a 2-uniformly convex Banach space, then there exists a constant $c > 0$ such that*

$$c\|x-y\|^2 \leq \phi(x, y),$$

where c is a constant in Lemma 2.1 (ii).

Also, the functional $V : E \times E^* \to [0, \infty)$ (see [4]) is defined by

$$V(x, x^*) := \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2,\ \forall x \in E,\ x^* \in E^*.$$

One can see that $V(x,x^*) = \phi(x, J^{-1}x^*)$.

Lemma 2.4 *([4]) Let E be a reflexive, strictly convex and smooth Banach space. Then*

$$V(x,x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in E$ *and* $x^*, y^* \in E^*$.

Let E be a reflexive, strictly convex and smooth Banach space. Let C be a closed and convex subset of E. Then for each $x \in E$, there exists a unique element $z \in C$ such that

$$\phi(z,x) = \min_{y \in C} \phi(y,x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C(x)$ is called the *generalized projection* of E onto C. If E is a Hilbert space, then Π_C is reduced to the metric projection denoted by P_C.

Lemma 2.5 *([4]) Let E be a reflexive, strictly convex and smooth Banach space and C be a closed and convex subset of E. Let* $x \in E$ *and* $z \in C$*. Then* $z = \Pi_C(x)$ *if and only if* $\langle y - z, Jx - Jz\rangle \leq 0, \forall y \in C$.

Let C be a nonempty subset of a smooth Banach space E. A mapping $\mathcal{F} : C \to E^*$ is said to be:

(1) *monotone* if $\langle x - y, \mathcal{F}x - \mathcal{F}y\rangle \geq 0,\ \forall x, y \in C$;
(2) *pseudo-monotone* if $\langle y - x, \mathcal{F}x\rangle \geq 0 \Longrightarrow \langle y - x, \mathcal{F}y\rangle \geq 0,\ \forall x, y \in C$;
(3) *Lipschitz continuous* if there exists a constant $L > 0$ such that $\|\mathcal{F}x - \mathcal{F}y\| \leq L\|x - y\|,\ \forall x, y \in C$;
(4) *sequentially weakly continuous* if for any sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$ implies that $\mathcal{F}x_n \rightharpoonup^* \mathcal{F}x$.

Remark 2 It is observed that monotone mapping is a pseudo-monotone mapping but the converse may not be true (see [20]).

Example 2.6 ([41]) Let $E = \ell_2$ and $C = \Big\{x = (x_1, x_2, \ldots, x_i, \ldots) \in E : |x_i| \leq \frac{1}{i}, \forall i = 1, 2, \ldots\Big\}$. Define an operator $\mathcal{F} : E \to E$ by

$$\mathcal{F}x := \left(\|x\| + \frac{1}{\|x\| + 1}\right)x,\ \forall x \in E.$$

Then $\mathcal{F}$ is pseudo-monotone and sequentially weakly continuous on C but not Lipschitz continuous on E.

Lemma 2.7 *([27]) Let C be a nonempty, closed and convex subset of a Banach space E. Let* $\mathcal{F} : C \to E^*$ *be a pseudo-monotone and hemicontinuous operator. Then w is a solution of VIP if and only if*

$$\langle y - w, \mathcal{F}y \rangle \geq 0, \quad \forall y \in C.$$

The following lemmas are needed to prove our main results in the next section.

Lemma 2.8 *([26]) Let $\{a_n\}$ be a nonnegative real sequence such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad and \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k := \max\{j \leq k : a_j \leq a_{j+1}\}$.

Lemma 2.9 *([35]) Let $\{a_n\}$ be a nonnegative real sequence, $\{\gamma_n\}$ be a sequence in (0, 1) with $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{b_n\}$ be a real sequence. Assume that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n b_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ with $\liminf_{k\to\infty} (a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10 *([53]) Let $\{a_n\}$ be a nonnegative real sequence, $\{\gamma_n\}$ be a sequence in [0, 1] with $\sum_{n=1}^{\infty} \gamma_n = \infty$, $\{c_n\}$ be a nonnegative real sequence with $\sum_{n=1}^{\infty} c_n < \infty$ and $\{b_n\}$ be a real sequence with $\limsup_{n\to\infty} b_n \leq 0$. Suppose that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n b_n + c_n, \quad \forall n \geq 1.$$

Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we propose two new versions of inertial algorithm for pseudo-monotone variational inequalities in a real 2-uniformly convex and uniformly smooth Banach space E. In what follows, J and J^{-1} denote the the duality mappings on E and E^*, respectively. In the sequel, we will make the following assumptions:

(A1) The feasible set C is a nonempty, closed and convex subset of E.
(A2) The mapping $\mathcal{F} : E \to E^*$ is pseudo-monotone.
(A3) The mapping $\mathcal{F} : E \to E^*$ is L-Lipschitz continuous.
(A4) The mapping $\mathcal{F} : E \to E^*$ is sequentially weakly continuous on C.
(A5) The solution set of VIP is nonempty, that is, $S \neq \emptyset$.

The following control conditions are also assumed.

(C1) The sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{\beta_n\} \subset (a, 1 - \alpha_n)$ for some $a > 0$.
(C2) The sequence $\{\tau_n\}$ is in $(0, \infty)$ satisfies $\lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$.

We now present the first method in Algorithm 1.

Algorithm 1 Inertial Tseng's extragradient method for VIP (I-TEGM)

Initialization: Given $\lambda_1 > 0$, $\tau \geq 0$ and $\mu \in \left(0, \left(\frac{c}{\kappa}\right)^{1/2}\right)$, where c and κ are defined as in Lemma 2.1. Choose $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$.

Iterative Steps: Let $x_0, x_1 \in E$ be arbitrary and calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\tau, \dfrac{\tau_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } \; x_n \neq x_{n-1}, \\ \tau, & \text{otherwise.} \end{cases} \tag{12}$$

Step 2 Set $w_n = J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))$ and compute

$$y_n = \Pi_C J^{-1}(Jw_n - \lambda_n \mathcal{F} w_n).$$

Step 3 Compute

$$x_{n+1} = J^{-1}((1 - \alpha_n - \beta_n)Jw_n + \beta_n(Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n)))$$

and update

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|\mathcal{F}w_n - \mathcal{F}y_n\|}, \lambda_n + s_n\right\} & \text{if } \; \mathcal{F}w_n - \mathcal{F}y_n \neq 0, \\ \lambda_n + s_n & \text{otherwise.} \end{cases} \tag{13}$$

Set $n := n + 1$ go to **Step 1**.

Remark 3 (a) If $y_n = w_n$ or $\mathcal{F}y_n = 0$, then Algorithm 1 stops in finite iterations and y_n is a solution in S, otherwise, go to the next step. In the rest of this paper, we assume that the Algorithm 1 does not stop in any finite iterations and generates an infinite sequence $\{x_n\}$. This implies that $y_n \neq w_n$ or $\mathcal{F}y_n \neq 0$.

(b) If E is a finite-dimensional Hilbert space, then the weak and strong convergences are coincide and, in consequence, the sequential weak continuity of $\mathcal{F}$ is not necessary to assume.

(c) For a sequence of stepsize defined in (13), there exists $\lambda \in \left[\min\{\frac{\mu}{L}, \lambda_1\}, \lambda_1 + s\right]$ such that $\lambda = \lim_{n\to\infty} \lambda_n$, where $s = \sum_{n=1}^{\infty} s_n$ (see [24]). It is observed that this step-size is allowed to increase from iteration to iteration and so the use of this type of step-size reduces the dependence on the initial step-size λ_1.

Remark 4 By the definition of θ_n, we know that $\theta_n\|x_n - x_{n-1}\| \leq \tau_n$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\tau_n}{\alpha_n} = 0$, we have

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0. \tag{14}$$

Since J is uniformly continuous on bounded subsets of E and it is also homogeneous, we have

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|Jx_n - Jx_{n-1}\| = 0. \tag{15}$$

Moreover, since $\alpha_n \in (0,1)$, it follows from (15) that

$$\lim_{n\to\infty} \theta_n\|Jx_n - Jx_{n-1}\| \leq \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|Jx_n - Jx_{n-1}\| = 0. \tag{16}$$

We start with the following lemma which will be useful in proving the convergence result of Algorithm 1.

Lemma 3.1 *Under Assumptions $(A1)-(A5)$ and Conditions $(C1)-(C2)$. Let $\{x_n\}$ be a sequence generated by Algorithm 1. If $\{x_n\}$ is bounded, then for each $p \in S$, we have*

$$\lim_{n\to\infty} \theta_n[\phi(p, x_{n-1}) - \phi(p, x_n)]_+ = 0,$$

where $[a]_+ := \max\{0, a\}$ for $a \in \mathbb{R}$.

Proof From (11), we have

$$\begin{aligned}
\theta_n[\phi(p, x_n) - \phi(p, x_{n-1})]_+ &= -\theta_n\phi(x_n, x_{n-1}) + 2\theta_n\langle x_n - p, Jx_n - Jx_{n-1}\rangle \\
&\leq 2\theta_n\langle x_n - p, Jx_n - Jx_{n-1}\rangle \\
&\leq \theta_n\|Jx_n - Jx_{n-1}\|M_1,
\end{aligned} \tag{17}$$

where $M_1 := \sup_{n\geq 1}\{2\|x_n - p\|\}$. Since $\lim_{n\to\infty} \theta_n\|Jx_n - Jx_{n-1}\| = 0$, it follows that

$$\lim_{n\to\infty} \theta_n[\phi(p, x_{n-1}) - \phi(p, x_n)]_+ = 0.$$

□

Lemma 3.2 *Under Assumptions $(A1)-(A5)$ and Conditions $(C1)-(C2)$. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then $\{x_n\}$ is bounded.*

Proof Let $p \in S$ and $z_n := J^{-1}(Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n))$ for all $n \geq 1$. Using Lemma 2.1, we have

$$
\begin{aligned}
\phi(p, z_n) &= \phi(p, J^{-1}(Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n))) \\
&= V(p, Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n)) \\
&= \|p\|^2 - 2\langle p, Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n)\rangle + \|Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n)\|^2 \\
&= \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda_n\langle p, \mathcal{F}y_n - \mathcal{F}w_n\rangle + \|Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n)\|^2 \\
&\leq \|p\|^2 - 2\langle p, Jy_n\rangle + 2\lambda_n\langle p, \mathcal{F}y_n - \mathcal{F}w_n\rangle + \|Jy_n\|^2 - 2\lambda_n\langle y_n, \mathcal{F}y_n - \mathcal{F}w_n\rangle \\
&\quad + \kappa\lambda_n^2\|\mathcal{F}y_n - \mathcal{F}w_n\|^2 \\
&= \|p\|^2 - 2\langle p, Jy_n\rangle + \|y_n\|^2 - 2\lambda_n\langle y_n - p, \mathcal{F}y_n - \mathcal{F}w_n\rangle + \kappa\lambda_n^2\|\mathcal{F}y_n - \mathcal{F}w_n\|^2 \\
&= \phi(p, y_n) - 2\lambda_n\langle y_n - p, \mathcal{F}y_n - \mathcal{F}w_n\rangle + \kappa\lambda_n^2\|\mathcal{F}y_n - \mathcal{F}w_n\|^2.
\end{aligned}
\tag{18}
$$

From (11), we observe that

$$
\phi(p, y_n) = \phi(p, w_n) - \phi(y_n, w_n) + 2\langle p - y_n, Jw_n - Jy_n\rangle. \tag{19}
$$

Substituting (19) and (18), we have

$$
\begin{aligned}
\phi(p, z_n) &\leq \phi(p, w_n) - \phi(y_n, w_n) + 2\langle p - y_n, Jw_n - Jy_n\rangle - 2\lambda_n\langle y_n - p, \mathcal{F}y_n - \mathcal{F}w_n\rangle \\
&\quad + \kappa\lambda_n^2\|\mathcal{F}y_n - \mathcal{F}w_n\|^2.
\end{aligned}
\tag{20}
$$

By the definition of y_n and the properties of Π_C, we see that $\langle p - y_n,\ Jw_n - \lambda_n \mathcal{F}w_n - Jy_n\rangle \leq 0$and it implies that

$$
\langle p - y_n, Jw_n - Jy_n\rangle \leq \lambda_n\langle p - y_n, \mathcal{F}w_n\rangle. \tag{21}
$$

From (20) and (21), we have

$$
\begin{aligned}
\phi(p, z_n) &\leq \phi(p, w_n) - \phi(y_n, w_n) + 2\lambda_n\langle p - y_n, \mathcal{F}w_n\rangle - 2\lambda_n\langle y_n - p, \mathcal{F}y_n - \mathcal{F}w_n\rangle \\
&\quad + \kappa\lambda_n^2\|\mathcal{F}y_n - \mathcal{F}w_n\|^2 \\
&= \phi(p, w_n) - \phi(y_n, w_n) - 2\lambda_n\langle y_n - p, \mathcal{F}y_n\rangle + \kappa\lambda_n^2\|\mathcal{F}y_n - \mathcal{F}w_n\|^2.
\end{aligned}
$$

Since $\langle y_n - p, \mathcal{F}p\rangle \geq 0$ and $\mathcal{F}$ is pseudo-monotone, we have $\langle y_n - p, \mathcal{F}y_n\rangle \geq 0$. Using this fact and Lemma 2.3, we obtain

$$
\begin{aligned}
\phi(p, z_n) &\leq \phi(p, w_n) - \phi(y_n, w_n) + \kappa\lambda_n^2\|\mathcal{F}y_n - \mathcal{F}w_n\|^2 \\
&\leq \phi(p, w_n) - \phi(y_n, w_n) + \kappa\lambda_n^2\frac{\mu^2}{\lambda_{n+1}^2}\|y_n - w_n\|^2 \\
&\leq \phi(p, w_n) - \left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\phi(y_n, w_n).
\end{aligned}
\tag{22}
$$

By the fact that $\lim_{n\to\infty}\lambda_n$ exists and $\mu \in \left(0, \left(\frac{c}{\kappa}\right)^{1/2}\right)$, we obtain

$$
\lim_{n\to\infty}\left(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \frac{\kappa\mu^2}{c} > 0.
$$

Then there exists $n_0 \in \mathbb{N}$ such that

$$1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \forall n \geq n_0. \tag{23}$$

Combining (22) and (23), we obtain

$$\phi(p, z_n) \leq \phi(p, w_n), \quad \forall n \geq n_0. \tag{24}$$

From (10), we have

$$\begin{aligned} \phi(p, w_n) &= \phi(p, J^{-1}((1-\theta_n)Jx_n + \theta_n Jx_{n-1})) \\ &\leq (1-\theta_n)\phi(p, x_n) + \theta_n\phi(p, x_{n-1}) \\ &\leq \phi(p, x_n) + \theta_n[\phi(p, x_{n-1}) - \phi(p, x_n)]_+. \end{aligned} \tag{25}$$

So we have

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, J^{-1}((1-\alpha_n-\beta_n)Jw_n + \alpha_n J(0) + \beta_n Jz_n)) \\ &\leq (1-\alpha_n-\beta_n)\phi(p, w_n) + \alpha_n\phi(p, 0) + \beta_n\phi(p, z_n) \\ &\leq (1-\alpha_n)\phi(p, w_n) + \alpha_n\phi(p, 0) \\ &\leq (1-\alpha_n)((1-\theta_n)\phi(p, x_n) + \theta_n\phi(p, x_{n-1})) + \alpha_n\phi(p, 0) \\ &\leq (1-\alpha_n)\max\{\phi(p, x_n), \phi(p, x_{n-1})\} + \alpha_n\phi(p, 0) \\ &\leq \max\{\max\{\phi(p, x_n), \phi(p, x_{n-1})\}, \phi(p, 0)\} \\ &\leq \cdots \\ &\leq \max\{\phi(p, x_{n_0}), \phi(p, x_{n_0-1}), \phi(p, 0)\}. \end{aligned}$$

This implies that $\{\phi(p, x_n)\}$ is bounded. Applying Lemma 2.3, we also get $\{x_n\}$ is bounded. □

Lemma 3.3 *Under Assumptions $(A1)$–$(A5)$ and Conditions $(C1)$–$(C2)$. Let $\{w_n\}$ be a sequence generated by Algorithm 1. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to $w \in E$ and $\lim_{k\to\infty}\|w_{n_k} - y_{n_k}\| = 0$, then $w \in S$.*

Proof Let $\{w_{n_k}\}$ be a subsequence of $\{w_n\}$ such that $w_{n_k} \rightharpoonup w \in E$. Then we also get $y_{n_k} \rightharpoonup w \in C$. From the definition of y_{n_k}, we see that

$$\langle y - y_{n_k}, Jy_{n_k} - Jw_{n_k} + \lambda_{n_k}\mathcal{F}w_{n_k}\rangle \geq 0, \quad \forall y \in C.$$

This implies that

$$\lambda_{n_k}\langle y - y_{n_k}, \mathcal{F}w_{n_k},\rangle \geq \langle y - y_{n_k}, Jw_{n_k} - Jy_{n_k}\rangle, \quad \forall y \in C$$

and hence

$$\langle y - y_{n_k}, \mathcal{F}y_{n_k}\rangle \geq \frac{1}{\lambda_{n_k}}\langle y - y_{n_k}, Jw_{n_k} - Jy_{n_k}\rangle - \langle y - y_{n_k}, \mathcal{F}w_{n_k} - \mathcal{F}y_{n_k}\rangle, \quad \forall y \in C. \tag{26}$$

Since $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0$, we have $\lim_{k\to\infty}\|Jw_{n_k}-Jy_{n_k}\|=0$. Moreover, $\mathcal{F}$ is Lipschitz continuous and $\{\lambda_{n_k}\}$ is bounded, then from (26), we obtain

$$\liminf_{k\to\infty}\langle y-y_{n_k},\mathcal{F}y_{n_k}\rangle \geq 0.$$

Now, we choose a decreasing sequence of positive real numbers $\{\epsilon_k\}$ such that $\epsilon_k\to 0$ as $k\to\infty$. For each ϵ_k, we denote by N_k the smallest positive integer such that

$$\langle y-y_{n_j},\mathcal{F}y_{n_j}\rangle+\epsilon_k\geq 0, \quad \forall j\geq N_k. \tag{27}$$

Since $\{\epsilon_k\}$ is decreasing, we have the sequence $\{N_k\}$ is increasing. Moreover, for each k, since $y_{N_k}\in C$, we can suppose $\mathcal{F}y_{N_k}\neq 0$ (otherwise, y_{N_k} is a solution). Setting

$$u_{N_k}:=\frac{1}{\|\mathcal{F}y_{N_k}\|^2}J^{-1}(\mathcal{F}y_{N_k}). \tag{28}$$

Thus we have for each k,

$$\begin{aligned}\langle u_{N_k},\mathcal{F}y_{N_k}\rangle &= \left\langle \frac{1}{\|\mathcal{F}y_{N_k}\|^2}J^{-1}(\mathcal{F}y_{N_k}),\mathcal{F}y_{N_k}\right\rangle\\ &= \frac{1}{\|\mathcal{F}y_{N_k}\|^2}\langle J^{-1}(\mathcal{F}y_{N_k}),\mathcal{F}y_{N_k}\rangle\\ &= \frac{1}{\|\mathcal{F}y_{N_k}\|^2}\cdot\|\mathcal{F}y_{N_k}\|^2=1.\end{aligned}$$

Then (27) can be written as

$$\langle y-y_{N_k},\mathcal{F}y_{N_k}\rangle+\epsilon_k\langle u_{N_k},\mathcal{F}y_{N_k}\rangle\geq 0,$$

or equivalently,

$$\langle y+\epsilon_k u_{N_k}-y_{N_k},\mathcal{F}y_{N_k}\rangle\geq 0.$$

By the pseudo-monotonicity of $\mathcal{F}$, we obtain

$$\langle y+\epsilon_k u_{N_k}-y_{N_k},\mathcal{F}(y+\epsilon_k u_{N_k})\rangle\geq 0.$$

It follows that

$$\langle y-y_{N_k},\mathcal{F}y\rangle\geq\langle y+\epsilon_k u_{N_k}-y_{N_k},\mathcal{F}y-\mathcal{F}(y+\epsilon_k u_{N_k})\rangle-\langle\epsilon_k u_{N_k},\mathcal{F}y\rangle. \tag{29}$$

Now, we show that $\lim_{k\to\infty}\epsilon_k u_{N_k}=0$. Since $y_{n_k}\rightharpoonup w\in C$ and $\mathcal{F}$ is sequentially weakly continuous on C, we have $\mathcal{F}y_{n_k}\rightharpoonup^*\mathcal{F}w$. We assume that $\mathcal{F}w\neq 0$ (otherwise, w is a solution in S). Using the fact that the norm mapping is sequentially weakly lower semicontinuous, we have

$$0 < \|\mathcal{F}w\| \leq \liminf_{k\to\infty} \|\mathcal{F}y_{n_k}\|.$$

Moreover, since $\{y_{N_k}\} \subset \{y_{n_k}\}$ and $\epsilon_k \to 0$ as $k \to \infty$, it follows that

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k u_{N_k}\| = \limsup_{k\to\infty} \left(\frac{\epsilon_k}{\|\mathcal{F}y_{N_k}\|}\right) \leq \frac{\limsup_{k\to\infty} \epsilon_k}{\liminf_{k\to\infty} \|\mathcal{F}y_{N_k}\|} \leq \frac{0}{\|\mathcal{F}w\|} = 0.$$

This implies that $\lim_{k\to\infty} \epsilon_k u_{N_k} = 0$. Taking the limit as $k \to \infty$ in (29), we obtain

$$\langle y - w, \mathcal{F}y \rangle \geq 0, \ \forall y \in C.$$

Thanks to Lemma 2.7, we get $w \in S$. □

Remark 5 (i) In particular, if $\mathcal{F}$ is monotone, then it is not necessary to assume the sequential weak continuity of $\mathcal{F}$. Indeed, from the monotonicity of $\mathcal{F}$ and (26), we see that

$$\begin{aligned}\langle y - y_{n_k}, \mathcal{F}y \rangle \geq & \langle y - y_{n_k}, \mathcal{F}y_{n_k} \rangle \\ \geq & \frac{1}{\lambda_{n_k}} \langle y - y_{n_k}, Jw_{n_k} - Jy_{n_k} \rangle - \langle y - y_{n_k}, \mathcal{F}w_{n_k} - \mathcal{F}y_{n_k} \rangle, \ \ \forall y \in C.\end{aligned}$$

Since $\|Jw_{n_k} - Jy_{n_k}\| \to 0$ and $\|\mathcal{F}w_{n_k} - \mathcal{F}y_{n_k}\| \to 0$, it follows that $\langle y - w, \mathcal{F}y \rangle \geq 0$ for all $y \in C$. In this case, the Assumption ($A4$) can be removed.

(ii) The Lemma 3.3 is different from recent ones in [19, 49–51]. This lemma is quite new for proving the weak limit point belongs to the set of solutions of the pseudo-monotone variational inequalities in Banach spaces. Note that u_{N_k} defined in (28) is new and appropriate in Banach space, while z_{N_k} defined in [49, 50] are neither accurate nor appropriate for Banach spaces.

Next, we state and prove the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 1.

Theorem 3.4 *Under Assumptions $(A1)-(A5)$ and Conditions $(C1)-(C2)$. Let $\{x_n\}$ be a sequence generated by Algorithm 1. Then $\{x_n\}$ converges strongly to the minimum-norm point $z := \Pi_S(0)$.*

Proof Let $z := \Pi_S(0)$. From (22) and (25), we obtain

$$
\begin{aligned}
&\phi(z,x_{n+1})\\
&\quad=\phi(z,J^{-1}((1-\alpha_n-\beta_n)Jw_n+\alpha_nJ(0)+\beta_nJz_n)))\\
&\quad\le(1-\alpha_n-\beta_n)\phi(z,w_n)+\alpha_n\phi(z,0)+\beta_n\phi(z,z_n)\\
&\quad\le(1-\alpha_n-\beta_n)\phi(z,w_n)+\alpha_n\phi(z,0)+\beta_n\phi(z,w_n)-\beta_n\Big(1-\frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n)\\
&\quad=(1-\alpha_n)\phi(z,w_n)+\alpha_n\phi(z,0)-\beta_n\Big(1-\frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n)\\
&\quad\le(1-\alpha_n)\phi(z,x_n)+(1-\alpha_n)\theta_n[\phi(z,x_{n-1})-\phi(z,x_n)]_++\alpha_n\phi(z,0)\\
&\qquad-\beta_n\Big(1-\frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\beta_n\Big(1-\frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,w_n)\\
&\qquad\le\phi(z,x_n)-\phi(z,x_{n+1})+(1-\alpha_n)\theta_n[\phi(z,x_{n-1})-\phi(z,x_n)]_++\alpha_nM_2,
\end{aligned}
\tag{30}
$$

where $M_2 := \sup_{n\ge1}\{|\phi(z,0)-\phi(z,x_n)|\}$. On the other hand, from Lemma 2.4 and (25), we see that

$$
\begin{aligned}
\phi(z,x_{n+1})&=V(z,(1-\alpha_n-\beta_n)Jw_n+\beta_nJz_n))\\
&\le V(z,(1-\alpha_n-\beta_n)Jw_n+\beta_nJz_n+\alpha_nJz)+\alpha_n\langle z-x_{n+1},Jz\rangle\\
&=\phi(z,J^{-1}(\alpha_nJz+(1-\alpha_n-\beta_n)Jw_n+\beta_nJz_n))+\alpha_n\langle z-x_{n+1},Jz\rangle\\
&\le\alpha_n\phi(z,z)+(1-\alpha_n-\beta_n)\phi(z,w_n)+\beta_n\phi(z,z_n)+\alpha_n\langle z-x_{n+1},Jz\rangle\\
&\le(1-\alpha_n)\phi(z,w_n)+\alpha_n\langle z-x_{n+1},Jz\rangle\\
&\le(1-\alpha_n)\phi(z,x_n)+(1-\alpha_n)\theta_n[\phi(z,x_{n-1})-\phi(z,x_n)]_++\alpha_n\langle z-x_{n+1},Jz\rangle\\
&=(1-\alpha_n)\phi(z,x_n)+\alpha_n\delta_n,
\end{aligned}
\tag{31}
$$

where

$$
\delta_n := (1-\alpha_n)\frac{\theta_n}{\alpha_n}[\phi(z,x_{n-1})-\phi(z,x_n)]_++\langle z-x_{n+1},Jz\rangle.
$$

Next, we show that $x_n\to z$. In order to do this, using Lemma 2.9, it is sufficient to show that $\limsup_{k\to\infty}\delta_{n_k}\le 0$ whenever a subsequence $\{\phi(z,x_{n_k})\}$ of $\{\phi(z,x_n)\}$ satisfies

$$
\liminf_{k\to\infty}(\phi(z,x_{n_k+1})-\phi(z,x_{n_k}))\ge 0. \tag{32}
$$

Let $\{\phi(z,x_{n_k})\}$ be a subsequence of $\{\phi(z,x_n)\}$ satisfies (32). It then follows from Lemma 3.1 and (32) that

$$\begin{aligned}
&\limsup_{k\to\infty}\beta_{n_k}\Big(1-\frac{\kappa\mu^2}{c}\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\Big)\phi(y_{n_k},w_{n_k})\\
&\quad\le \limsup_{k\to\infty}(\phi(z,x_{n_k})-\phi(z,x_{n_k+1}))+\limsup_{k\to\infty}(1-\alpha_{n_k})\theta_{n_k}[\phi(z,x_{n_k-1})\\
&\qquad -\phi(z,x_{n_k})]_+ +\limsup_{k\to\infty}\alpha_{n_k}M_2\\
&\quad= -\liminf_{k\to\infty}(\phi(z,x_{n_k+1})-\phi(z,x_{n_k}))\\
&\quad\le 0.
\end{aligned}$$

This implies that

$$\lim_{k\to\infty}\phi(y_{n_k},w_{n_k})=0.$$

Applying Lemma 2.3, we have

$$\lim_{k\to\infty}\|y_{n_k}-w_{n_k}\|=0 \tag{33}$$

and so

$$\lim_{k\to\infty}\|Jy_{n_k}-Jw_{n_k}\|=0. \tag{34}$$

By the fact that $\mathcal{F}$ is Lipschitz continuous, it follows from (33) that

$$\|Jz_{n_k}-Jy_{n_k}\|=\lambda_{n_k}\|\mathcal{F}y_{n_k}-\mathcal{F}w_{n_k}\|\to 0. \tag{35}$$

From (34) and (35), we have

$$\|Jz_{n_k}-Jw_{n_k}\|\le\|Jz_{n_k}-Jy_{n_k}\|+\|Jy_{n_k}-Jw_{n_k}\|\to 0. \tag{36}$$

By the definition of w_{n_k} and (16), we obtain

$$\|Jw_{n_k}-Jx_{n_k}\|=\theta_{n_k}\|Jx_{n_k-1}-Jx_{n_k}\|\to 0. \tag{37}$$

It then follows from (36) and (37) that

$$\begin{aligned}
\|Jx_{n_k+1}-Jx_{n_k}\| &\le\|Jx_{n_k+1}-Jw_{n_k}\|+\|Jw_{n_k}-Jx_{n_k}\|\\
&\le\alpha_{n_k}\|Jw_{n_k}\|+\beta_{n_k}\|Jw_{n_k}-Jz_{n_k}\|+\|Jw_{n_k}-Jx_{n_k}\|\\
&\to 0.
\end{aligned}$$

We know that if E is 2-uniformly convex then its dual E^* is uniformly smooth and, in consequence, J^{-1} is uniformly continuous on bounded subsets of E^*. So we have

$$\lim_{k\to\infty}\|x_{n_k+1}-x_{n_k}\|=0. \tag{38}$$

Since E is reflexive and $\{x_{n_k}\}$ is bounded, we can assume there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ which converges weakly to some $w\in E$. Furthermore,

$$\limsup_{k\to\infty}\langle z-x_{n_k},Jz\rangle=\lim_{i\to\infty}\langle z-x_{n_{k_i}},Jz\rangle=\langle z-w,Jz\rangle.$$

Since $\lim_{i\to\infty}\|w_{n_{k_i}}-x_{n_{k_i}}\|=0$, we also get $w_{n_{k_i}}\rightharpoonup w$. According to Lemma 3.3 implies that $w\in S$. Then from Lemma 2.5, we have

$$\limsup_{k\to\infty}\langle z-x_{n_k},Jz\rangle=\langle z-w,Jz\rangle\le 0. \tag{39}$$

From (38) and (39), we obtain

$$\limsup_{k\to\infty}\langle z-x_{n_k+1},Jz\rangle\le 0. \tag{40}$$

From (17), we note that

$$\frac{\theta_{n_k}}{\alpha_{n_k}}[\phi(z,x_{n_k})-\phi(z,x_{n_k-1})]_+\le\frac{\theta_{n_k}}{\alpha_{n_k}}\|Jx_{n_k}-Jx_{n_k-1}\|\overline{M}_1,$$

where $\overline{M}_1>0$. It then follows from (15) that

$$\lim_{k\to\infty}\frac{\theta_{n_k}}{\alpha_{n_k}}[\phi(z,x_{n_k})-\phi(z,x_{n_k-1})]_+=0. \tag{41}$$

Combining (40) and (41), we obtain

$$\limsup_{k\to\infty}\delta_{n_k}=\limsup_{k\to\infty}\left[(1-\alpha_{n_k})\frac{\theta_{n_k}}{\alpha_{n_k}}[\phi(z,x_{n_k-1})-\phi(z,x_{n_k})]_++\langle z-x_{n_k+1},Jz\rangle\right]\le 0.$$

This together with (31) and Lemma 2.9, yields that $\lim_{n\to\infty}\phi(z,x_n)=0$. Therefore, $x_n\to z:=\Pi_S(0)$. Furthermore, from the property of generalized projection, we have for each $w\in S$,

$$\langle z-w,Jz\rangle\le 0\iff\langle z,Jz\rangle\le\langle w,Jz\rangle\implies\|z\|^2\le\|w\|\|z\|\implies\|z\|\le\|w\|.$$

Hence z is the minimum-norm solution of VIP. □

Next, we propose another algorithm, which is presented below.

Algorithm 2 Relaxed inertial Tseng's extragradient method for VIP (RI-TEGM)

Initialization: Given $\lambda_1 > 0$ and $\mu \in \left(0, \left(\frac{c}{\kappa}\right)^{1/2}\right)$, where c and κ are defined as in Lemma 2.1. Choose $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} s_n < \infty$.
Iterative Steps: Let $x_0, x_1 \in E$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Choose $\{\theta_n\} \subset [a, b] \subset (0, 1)$ for some $a, b > 0$.
Step 2 Set $w_n = J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))$ and compute

$$y_n = \Pi_C J^{-1}(Jw_n - \lambda_n \mathcal{F} w_n).$$

Step 3 Compute

$$x_{n+1} = J^{-1}((1 - \alpha_n - \beta_n)Jw_n + \beta_n(Jy_n - \lambda_n(\mathcal{F}y_n - \mathcal{F}w_n))),$$

where λ_n is the same as in (13).
Set $n := n + 1$ go to **Step 1**.

Remark 6 In another point of view, it is interesting to note that $Jw_n = Jx_n + \theta_n (Jx_{n-1} - Jx_n)$ of Algorithm 2 is a convex combination of Jx_{n-1} and Jx_n, that is, $Jw_n = (1 - \theta_n)\ Jx_n + \theta_n Jx_{n-1}$. This allows, in a simple way, to choose $\{\theta_n\} \subset [a, b] \subset (0, 1)$ which makes it easier to implement. While the inertial factor θ_n used in Algorithm 1 is chosen depends on $\bar{\theta}_n$, where $\bar{\theta}_n$ is defined in (12) which may cause difficulty in numerical computations.

Theorem 3.5 *Under Assumptions* $(A1)-(A5)$ *and Condition (C1). Let* $\{x_n\}$ *be a sequence generated by Algorithm 2. Then* $\{x_n\}$ *converges strongly to the minimum-norm point* $z := \Pi_S(0)$.

Proof As we proved in Lemma 3.2, we have $\{x_n\}$ is bounded. Let $z := \Pi_S(0)$. Using Lemma 2.2, we have

$$\begin{aligned} \phi(z, w_n) &= \phi(z, J^{-1}((1 - \theta_n)Jx_n + \theta_n Jx_{n-1})) \\ &\leq (1 - \theta_n)\phi(z, x_n) + \theta_n \phi(z, x_{n-1}) - \theta_n(1 - \theta_n)g(\|Jx_n - Jx_{n-1}\|) \\ &= \phi(z, x_n) + \theta_n(\phi(z, x_{n-1}) - \phi(z, x_n)) \\ &\quad - \theta_n(1 - \theta_n)g(\|Jx_n - Jx_{n-1}\|). \end{aligned} \tag{42}$$

It follows from (22) and (42) that

$$
\begin{aligned}
&\phi(z, x_{n+1}) \\
&\quad = \phi(z, J^{-1}((1-\alpha_n-\beta_n)Jw_n + \alpha_n J(0) + \beta_n Jz_n))) \\
&\quad \le (1-\alpha_n-\beta_n)\phi(z, w_n) + \alpha_n\phi(z,0) + \beta_n\phi(z, z_n) \\
&\quad \le (1-\alpha_n-\beta_n)\phi(z, w_n) + \alpha_n\phi(z,0) + \beta_n\phi(z, w_n) - \beta_n\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, w_n) \\
&\quad = (1-\alpha_n)\phi(z, w_n) + \alpha_n\phi(z,0) - \beta_n\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, w_n) \\
&\quad \le (1-\alpha_n)\phi(z, x_n) + (1-\alpha_n)\theta_n(\phi(z, x_{n-1}) - \phi(z, x_n)) - (1-\alpha_n)\theta_n(1-\theta_n)g(\|Jx_n - Jx_{n-1}\|) \\
&\qquad + \alpha_n\phi(z,0) - \beta_n\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, w_n).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\beta_n\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, w_n) + (1-\alpha_n)\theta_n(1-\theta_n)g(\|Jx_n - Jx_{n-1}\|) \\
&\quad \le \phi(z, x_n) - \phi(z, x_{n+1}) + (1-\alpha_n)\theta_n(\phi(z, x_{n-1}) - \phi(z, x_n)) + \alpha_n M_2,
\end{aligned} \tag{43}
$$

Next, we show that $x_n \to z$. Let us consider two possible cases as follows:

Case 1. There exists $n_0 \in \mathbb{N}$ such that $\phi(z, x_{n+1}) \le \phi(z, x_n)$ for all $n \ge n_0$. Since $\{\phi(z, x_n)\}$ is bounded, it follows that $\lim_{n\to\infty}\phi(z, x_n)$ exists. Thus $\lim_{n\to\infty}(\phi(z, x_n) - \phi(z, x_{n+1})) = 0$. Moreover, we see that

$$
\sum_{n=n_0}^{\infty}(\phi(z, x_{n-1}) - \phi(z, x_n)) = \lim_{n\to\infty}(\phi(z, x_{n_0-1}) - \phi(z, x_n)) < \infty
$$

and so

$$
\lim_{n\to\infty}(\phi(z, x_{n-1}) - \phi(z, x_n)) = 0.
$$

Thus from (43), we obtain

$$
\lim_{n\to\infty}\phi(y_n, w_n) = \lim_{n\to\infty}g(\|Jx_n - Jx_{n-1}\|) = 0.
$$

This implies that

$$
\lim_{n\to\infty}\|y_n - w_n\| = 0
$$

and so

$$
\lim_{n\to\infty}\|Jy_n - Jw_n\| = 0. \tag{44}
$$

Moreover, by the property of g, we obtain

$$
\lim_{n\to\infty}\|Jx_n - Jx_{n-1}\| = 0. \tag{45}
$$

Since $\mathcal{F}$ is Lipschitz continuous, we have

$$\|Jz_n - Jy_n\| = \lambda_n \|\mathcal{F}y_n - \mathcal{F}w_n\| \to 0. \tag{46}$$

It follows from (44) and (46) that

$$\|Jz_n - Jw_n\| \leq \|Jz_n - Jy_n\| + \|Jy_n - Jw_n\| \to 0. \tag{47}$$

Also from (45), we have

$$\|Jw_n - Jx_n\| = \theta_n \|Jx_{n-1} - Jx_n\| \to 0. \tag{48}$$

Thus we have

$$\begin{aligned}\|Jx_{n+1} - Jx_n\| &\leq \|Jx_{n+1} - Jw_n\| + \|Jw_n - Jx_n\| \\ &\leq \alpha_n \|Jw_n\| + \beta_n \|Jw_n - Jz_n\| + \|Jw_n - Jx_n\| \\ &\to 0.\end{aligned}$$

By utilizing the same arguments as in Theorem 3.4, we can show that the weak cluster point of $\{x_n\}$ is in S and

$$\limsup_{n\to\infty} \langle z - x_{n+1}, Jz\rangle \leq 0. \tag{49}$$

Moreover, from Lemma 2.4 and (42), we have

$$\begin{aligned}\phi(z, x_{n+1}) &= V(z, (1-\alpha_n-\beta_n)Jw_n + \beta_n Jz_n) \\ &\leq V(z, (1-\alpha_n-\beta_n)Jw_n + \beta_n Jz_n + \alpha_n Jz) + 2\alpha_n\langle z - x_{n+1}, Jz\rangle \\ &= \phi(z, J^{-1}(\alpha_n Jz + (1-\alpha_n-\beta_n)Jw_n + \beta_n Jz_n)) + 2\alpha_n\langle z - x_{n+1}, Jz\rangle \\ &\leq \alpha_n\phi(z,z) + (1-\alpha_n-\beta_n)\phi(z,w_n) + \beta_n\phi(z,z_n) + 2\alpha_n\langle z - x_{n+1}, Jz\rangle \\ &\leq (1-\alpha_n)\phi(z,w_n) + 2\alpha_n\langle z - x_{n+1}, Jz\rangle \\ &\leq (1-\alpha_n)\phi(z,x_n) + (1-\alpha_n)\theta_n(\phi(z,x_{n-1}) - \phi(z,x_n)) + 2\alpha_n\langle z - x_{n+1}, Jz\rangle \\ &= (1-\alpha_n)\phi(z,x_n) + \alpha_n\mu_n + c_n,\end{aligned} \tag{50}$$

where $\mu_n := 2\langle z - x_{n+1}, Jz\rangle$ and $c_n := (1-\alpha_n)\theta_n(\phi(z,x_{n-1}) - \phi(z,x_n))$. Then it is easy to see that $\limsup_{n\to\infty}\mu_n \leq 0$ and

$$\sum_{n=n_0}^{\infty} c_n = \sum_{n=n_0}^{\infty}(1-\alpha_n)\theta_n(\phi(z,x_{n-1}) - \phi(z,x_n)) \leq \sum_{n=n_0}^{\infty}(\phi(z,x_{n-1}) - \phi(z,x_n)) < \infty.$$

This together with (50) and Lemma 2.10, yields that $\lim_{n\to\infty}\phi(z,x_n) = 0$. Therefore, $x_n \to z := \Pi_S(0)$.

Case 2. There exists a subsequence $\{n_i\}$ of $\{n\}$ such that $\phi(z, x_{n_i}) < \phi(z, x_{n_i+1})$ for all $i \in \mathbb{N}$. Then by Lemma 2.8 there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all numbers $k \in \mathbb{N}$:

$$\phi(z, x_{m_k}) \le \phi(z, x_{m_k+1}) \quad \text{and} \quad \phi(z, x_k) \le \phi(z, x_{m_k+1}).$$

Then from (43), we have

$$\begin{aligned} &\beta_{m_k}\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_{m_k}^2}{\lambda_{m_k+1}^2}\Big)\phi(y_{m_k}, w_{m_k}) + (1-\alpha_{m_k})\theta_{m_k}(1-\theta_{m_k})g(\|Jx_{m_k} - Jx_{m_k-1}\|) \\ &\quad \le \phi(z, x_{m_k}) - \phi(z, x_{m_k+1}) + (1-\alpha_{m_k})\theta_{m_k}(\phi(z, x_{m_k-1}) - \phi(z, x_{m_k})) + \alpha_{m_k}\overline{M}_2 \\ &\quad \le (1-\alpha_{m_k})\theta_{m_k}(\phi(z, x_{m_k-1}) - \phi(z, x_{m_k})) + \alpha_{m_k}\overline{M}_2, \end{aligned} \tag{51}$$

where $\overline{M}_2 > 0$. Now, we assume that there exists a subsequence $\{m_k\}$ of $\mathbb{N}$ such that

$$\langle x - x_{m_k-1}, Jx_{m_k} - Jx_{m_k-1}\rangle \le 0, \quad \forall x \in E.$$

Thus from (11), we obtain

$$\begin{aligned} \phi(z, x_{m_k-1}) - \phi(z, x_{m_k}) &= -\phi(x_{m_k-1}, x_{m_k}) + 2\langle z - x_{m_k-1}, Jx_{m_k} - Jx_{m_k-1}\rangle \\ &\le 2\langle z - x_{m_k-1}, Jx_{m_k} - Jx_{m_k-1}\rangle \\ &\le 0. \end{aligned} \tag{52}$$

Combining (51) and (52), we obtain

$$\begin{aligned} &\beta_{m_k}\Big(1 - \frac{\kappa\mu^2}{c}\frac{\lambda_{m_k}^2}{\lambda_{m_k+1}^2}\Big)\phi(y_{m_k}, w_{m_k}) + (1-\alpha_{m_k})\theta_{m_k}(1-\theta_{m_k})g(\|Jx_{m_k} - Jx_{m_k-1}\|) \\ &\quad \le \alpha_{m_k}\overline{M}_2. \end{aligned}$$

So we have

$$\lim_{k\to\infty}\phi(y_{m_k}, w_{m_k}) = \lim_{k\to\infty} g(\|Jx_{m_k} - Jx_{m_k-1}\|) = 0.$$

Following the proof line in **Case 1**, we can show that the weak cluster point of $\{x_n\}$ is in *S* and

$$\limsup_{k\to\infty}\langle z - x_{m_k+1}, Jz\rangle \le 0. \tag{53}$$

Moreover, we have

$$\begin{aligned} \phi(z, x_{m_k+1}) &\le (1-\alpha_{m_k})\phi(z, x_{m_k}) + (1-\alpha_{m_k})\theta_{m_k}(\phi(z, x_{m_k-1}) - \phi(z, x_{m_k})) \\ &\quad + \alpha_{m_k}\langle z - x_{m_k+1}, Jz\rangle \\ &\le (1-\alpha_{m_k})\phi(z, x_{m_k}) + \alpha_{m_k}\langle z - x_{m_k+1}, Jz\rangle \\ &\le (1-\alpha_{m_k})\phi(z, x_{m_k+1}) + \alpha_{m_k}\langle z - x_{m_k+1}, Jz\rangle, \end{aligned}$$

which implies that

$$\phi(z, x_{m_k+1}) \le \langle z - x_{m_k+1}, Jz\rangle.$$

From (53), we obtain $\lim_{k\to\infty}\phi(z, x_{m_k+1}) = 0$. Since $\phi(z, x_k) \leq \phi(z, x_{m_k+1})$, it follows that $\lim_{k\to\infty}\phi(z, x_k) = 0$. Therefore, $x_n \to z := \Pi_S(0)$. □

Moreover, we also present the strong convergence of the proposed methods under the cost operator is non-Lipschitz continuous. The following assumption is made here.

$(A3^*)$ The mapping $\mathcal{F} : E \to E^*$ is uniformly continuous.

To obtain the convergence results, the following lemmas are also needed in the sequel.

Lemma 3.6 *([47]) A mapping $\mathcal{F} : E \to E^*$ is uniformly continuous if and only if for every $\epsilon > 0$, there exists a $K < \infty$ such that $\|\mathcal{F}x - \mathcal{F}y\| \leq K\|x - y\| + \epsilon$ for all $x, y \in E$.*

Lemma 3.7 *Under Assumptions $(A1), (A2), (A3^*)$, (A4) and (A5). Let $\{\lambda_n\}$ be a sequence generated by (13). Then $\lim_{n\to\infty}\lambda_n = \lambda$, where $\lambda \in \left[\min\{\frac{\mu}{\mathcal{L}}, \lambda_1\}, \lambda_1 + s\right]$ and $s = \sum_{n=1}^{\infty} s_n$.*

Proof By the continuity of $\mathcal{F}$, it follows from Lemma 3.6 that

$$\frac{\mu\|w_n - y_n\|}{\|\mathcal{F}w_n - \mathcal{F}y_n\|} \geq \frac{\mu\|w_n - y_n\|}{K\|w_n - y_n\| + \epsilon} = \frac{\mu\|w_n - y_n\|}{(K + \bar{\epsilon})\|w_n - y_n\|} = \frac{\mu}{\mathcal{L}},$$

where $\mathcal{L} = K + \bar{\epsilon}$ and $\epsilon = \min_{n\in\mathbb{N}}\{\bar{\epsilon}\|u_n - y_n\|\}$ with $\bar{\epsilon} > 0$. The rest of the proof is similar to the proof of [24, Lemma 3.1]. □

Lemma 3.8 *Under Assumptions $(A1), (A2), (A3^*)$, (A4) and (A5). Let $\{u_n\}$ be a sequence generated by Algorithm 1. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $w \in E$ and $\lim_{k\to\infty}\|u_{n_k} - y_{n_k}\| = 0$, then $w \in S$.*

Proof Using the uniform continuity of $\mathcal{F}$ and Lemma 3.8, we obtain the desired result immediately. □

We obtain the following results for solving non-Lipschitz pseudo-monotone variational inequalities.

Theorem 3.9 *Under Assumptions $(A1), (A2), (A3^*)$, (A4), (A5) and Conditions $(C1)$ –$(C2)$. Then the sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to the minimum-norm point $z := \Pi_S(0)$.*

Proof The proof of this theorem can be done by following the lines of the proof of Theorem 3.4 and, hence, is omitted. □

Theorem 3.10 *Under Assumptions $(A1), (A2), (A3^*)$, (A4), (A5) and Condition (C1). Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to the minimum-norm point $z := \Pi_S(0)$.*

Remark 7 There have been several methods developed for solutions of the non-Lipschitz pseudo-monotone variational inequalities but the most of them required a linesearch procedure (see for instance [7, 34, 40, 52]). The Theorems 3.9 and 3.10 are a new approach to the solutions of the non-Lipschitz pseudo-monotone variational inequalities with adaptive step-size (our algorithms can be implemented without any linesearch procedure).

4 Numerical illustrations

In this section, we provide some numerical examples to show the effectiveness and efficiency of both Algorithm 1 (I-TEGM) and Algorithm 2 (RI-TEGM) in comparisons with Shehu Alg proposed in [36, Algorithm 3.14], Thong and Vuong Alg proposed in [42, Algorithm 1] and Thong et al. Alg proposed in [43, Algorithm 2].

4.1 Numerical examples

Example 4.1 The first example is taken from [16]. Consider the linear operator $\mathcal{F}:\mathbb{R}^m\to\mathbb{R}^m$ $(m=50,100,500,1000)$ defined by

$$\mathcal{F}x := Ax + g \text{ with } g\in\mathbb{R}^m \text{ and } A := BB^{\mathrm{T}} + Q + R,$$

where B is an $m\times m$ matrix, Q is an is an $m\times m$ skew-symmetric matrix, R is an $m\times m$ diagonal matrix whose diagonal terms are nonnegative (hence A is positive symmetric definite) and g is a vector in $\mathbb{R}^m$. The feasible set C is $C=\{x\in\mathbb{R}^m:\|x\|\le 5\}$. It is clear that $\mathcal{F}$ is monotone (hence it is pseudo-monotone) and Lipschitz continuous with $L=\|A\|$. For $g=0$, the solution of the problem is $x^*=0$. All entries of B and R are generated randomly in [0, 2] and Q is generated randomly in $[-2,2]$. The initial points are $x_0=x_1=\underbrace{(1,1,\ldots,1)}_{m}\in\mathbb{R}^m$. We choose the parameters for the experiments as follows:

- I-TEGM and RI-TEGM: $\lambda_1=10^{-5}$, $\mu=10^{-5}$, $s_n=\frac{1}{(n+1)^{1.1}}$, $\alpha_n=\frac{1}{(n+1)^{0.6}}$, $\beta_n=0.8-\alpha_n$. Moreover, take $\tau=0.9$ and $\tau_n=\frac{1}{(n+1)^2}$ in I-TEGM, and $\theta_n=10^{-4}$ in RI-TEGM.
- Shehu Alg: $\alpha_n=\frac{1}{(n+1)^{0.8}}$ and $\lambda_n=\frac{10^{-4}}{L}$.
- Thong and Vuong Alg: $\gamma=2$, $l=0.15$ and $\mu=0.6$.
- Thong et al. Alg: $\lambda_1=10^{-5}$, $\mu=10^{-5}$, $\alpha_n=\frac{1}{n+1}$ and $\gamma=10^{-5}$.

We use $E_n:=\|x_n-x^*\|<\epsilon$ to measure the n-th iteration error and terminate the iterations when $\epsilon=10^{-7}$. The numerical results for each case are displayed in Fig. 1.

Example 4.2 Consider the following fractional programming problem [17]:

$$\begin{cases} \min f(x), \\ \text{subject to } x\in K=\{x\in\mathbb{R}^4: b^Tx+b_0>0\}, \end{cases} \tag{54}$$

where $f(x)=\frac{x^TQx+a^Tx+a_0}{b^Tx+b_0}$ and

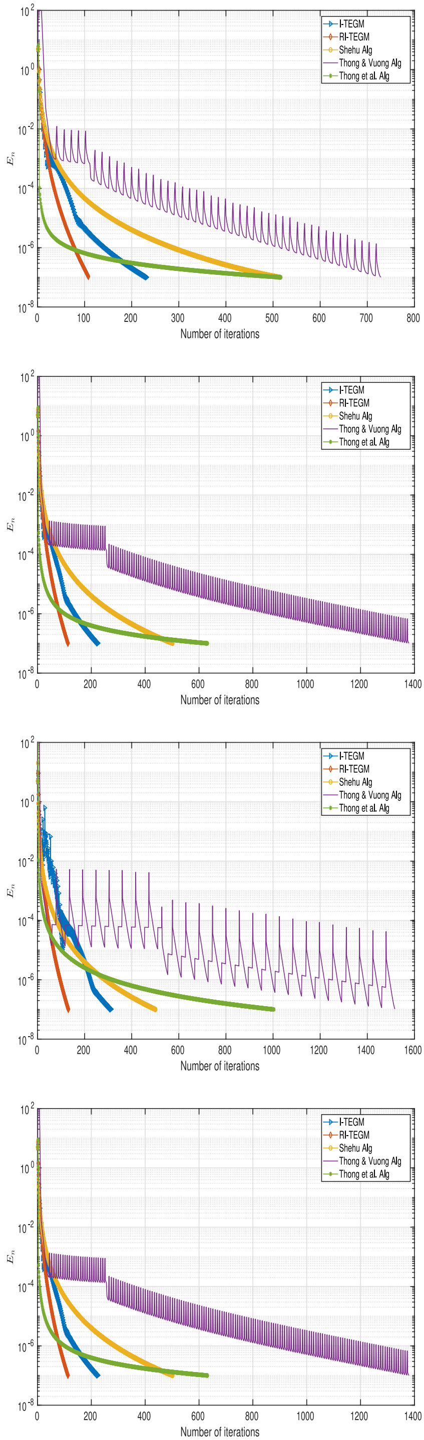


Fig. 1 Numerical results for Example 4.1, Top left: $m = 50$; Top right: $m = 100$; Bottom left: $m = 500$; Bottom right: $m = 1000$

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, \ a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, \ b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \ a_0 = -2, \ b_0 = 4.$$

One can show that Q is symmetric and positive definite and so f is pseudo-convex on K. Then (54) is equivalent to the variational inequality problem (1) with $\mathcal{F} := \nabla f$ (see [21]). By the quotient rule, we have

$$\nabla f(x) = \frac{(b^T x + b_0)(2Qx + a) - b(x^T Q + a^T x + a_0)}{(b^T x + b_0)^2}.$$

Moreover, ∇f is pseudo-monotone and Lipschitz continuous (see [6]). We now minimize f on $C = \{x \in \mathbb{R}^4 : \|x\| \leq 1, \ i = 1, ..., 4\} \subset K$. We choose the parameters for the experiments as follows:

- I-TEGM and RI-TEGM: $\lambda_1 = 3.3$, $\mu = 0.5$, $s_n = \frac{1}{n\sqrt{n}}$, $\alpha_n = \frac{1}{n+2}$, $\beta_n = 0.8 - \alpha_n$. Moreover, take $\tau = 0.5$ and $\tau_n = \frac{1}{n^{1.01}}$ in I-TEGM, and $\theta_n = 10^{-4}$ in RI-TEGM.
- Thong and Vuong Alg: $\gamma = 3.5$, $l = \frac{2}{17}$ and $\mu = 0.25$.
- Thong et al. Alg: $\lambda_1 = 0.01$, $\mu = 0.01$, $\alpha_n = \frac{1}{n+1}$ and $\gamma = 0.0001$.

We use $E_n := \|x_{n+1} - x_n\| < \epsilon$ to measure the n-th iteration error and terminate the iterations when $\epsilon = 10^{-7}$. In this experiment, we test all algorithms for different cases of the initial points x_0 and x_1 as follows:

Case I: $x_0 = (1.2, 0.5, 0.3, 0.9)$ and $x_1 = (1.7, 0.9, 0.7, 1.5)$.
Case II: $x_0 = (2, 2, 0, 0)$ and $x_1 = (2, 1, 2, -4)$.
Case III: $x_0 = (0, -1, 1.2, 0)$ and $x_1 = (0.2, 1.2, 0.2, 1.2)$.
Case IV: $x_0 = (0.01, 0.6, 0.02, 0.50)$ and $x_1 = (0.2, 0.32, 0.02, 0.2)$.
The numerical results for each case are displayed in Fig. 2.

Example 4.3 In this example, we consider pseudo-monotone VIP in infinite dimensional spaces. Let $E := \ell_2 = \{x = (x_1, x_2, x_3, ...) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with the norm $\|x\|_{\ell_2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$ and the inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ for $x, y \in E$. Let us define $\mathcal{F} : E \to E$ by

$$\mathcal{F}(x_1, x_2, x_3, \ldots) := (x_1 e^{-x_1^2}, 0, 0, ...)$$

and the feasible set $C = \{x = (x_1, x_2, x_3, ...) \in E : \|x\|_{\ell_2} \leq 1\}$. It was shown in [6, Example 2.1] that $\mathcal{F}$ is pseudo-monotone (but not monotone), Lipschitz continuous and sequentially weakly continuous on C. We choose the parameters for the experiments as follows:

- I-TEGM and RI-TEGM: $\lambda_1 = 3$, $\mu = 0.5$, $s_n = \frac{1}{n\sqrt{n}}$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.8 - \alpha_n$. Moreover, take $\tau = 0.5$ and $\tau_n = \frac{1}{n^{1.001}}$ in I-TEGM, and $\theta_n = 10^{-4}$ in RI-TEGM.
- Thong and Vuong Alg: $\gamma = \frac{9}{20}$, $l = \frac{3}{10}$, $\mu = 0.05$.
- Thong et al. Alg: $\lambda_1 = 3$, $\mu = 0.03$, $\alpha_n = \frac{1}{n+3}$, $\gamma = 0.1$.

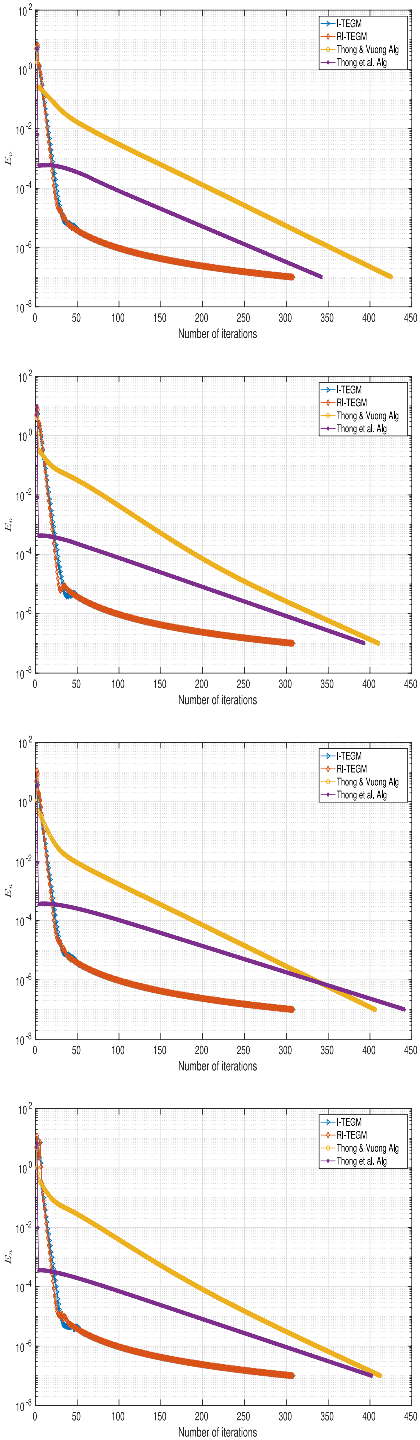


Fig. 2 Numerical results for Example 4.2, Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV

We use $E_n := \|x_{n+1} - x_n\|_{\ell_2} < \epsilon$ to measure the *n*-th iteration error and terminate the iterations when $\epsilon = 10^{-7}$. In this experiment, we test all algorithms for different cases of the initial points x_0 and x_1 as follows:

Case I: $x_0 = (1, 2, 1, 0, 0, 0, ...)$, $x_1 = (2, 2, 4, 0, 0, 0, ...)$.
Case II: $x_0 = (2, 2, 0, 0, 0, 0, ...)$, $x_1 = (3, 1, 0, 0, 0, 0, ...)$.
Case III: $x_0 = (-1, 0, 0, 0, 0, 0, ...)$, $x_1 = (1, 0, 0, 0, 0, 0, ...)$.
Case IV: $x_0 = (5, 5, 0, 0, 0, 0, ...)$, $x_1 = (7, 6, 0, 0, 0, 0, ...)$.
The numerical results for each case are displayed in Fig. 3.

Example 4.4 In this example, we consider VIP in infinite dimensional Banach spaces. Let $E := L_p([0,1])$ $(1 < p < 2)$ with the norm $\|x\|_{L_p} = \left(\int_0^1 |x(s)|^p ds\right)^{1/p}$ and the duality pairing $\langle x, y\rangle = \int_0^1 x(s)y(s)ds$ for all $x \in E$ and $y \in E^*$. Let $\mathcal{F} : E \to E^*$ be defined by

$$\mathcal{F}x(s) := 4Jx(s) + J(1 + \sin(\pi s)),$$

where J is the duality mapping on E. The feasible set C is given by $C = \left\{x \in E : \|x\|_{L_p} \leq 2\right\}$. In this case, J and J^{-1} can be computed by the following formulas ([1, p. 36]):

$$Jx(s) = \|x\|_{L_p}^{2-p}|x(s)|^{p-2}x(s), \quad x \in E \text{ and } J^{-1}y(s) = \|y\|_{L_q}^{2-q}|y(s)|^{q-2}y(s), \quad y \in E^*,$$

where $q = \frac{p}{p-1}$. In this example, let $p = 5/4$. By the properties of *J*, we see that $\mathcal{F}$ is monotone and uniformly continuous on *E*. In this experiment, we only test I-TEGM and RI-TEGM since Thong and Vuong Alg and Thong et al. Alg are established in the Hilbert space setting, and although, Shehu Alg is established in the Banach space setting but we do not know the Lipschitz constant of $\mathcal{F}$. By a direct calculation, we have $\kappa = 4$ and $c \approx 1.2954$. Then we can take $\mu \in (0, 0.5690)$. For both I-TEGM and RI-TEGM, we take $\lambda_1 = 10^{-5}$, $\mu = 10^{-5}$, $s_n = \frac{1}{(n+1)^4}$, $\alpha_n = \frac{1}{n+1}$, $\beta_n = 0.9 - \alpha_n$. Moreover, take $\tau = 0.5$ and $\tau_n = \frac{1}{(n+1)^2}$ in I-TEGM, and $\theta_n = 0.8$ in RI-TEGM.

We use $E_n := \|x_{n+1} - x_n\|_{L_p}$ to measure the *n*-th iteration error. Due to the computational cost of computing J, J^{-1} and $\|\cdot\|$ in L_p, we terminate the iteration process after running the code for hours and generated the first 13 iterations for each algorithm. We consider different cases of the initial points x_0 and x_1 as follows:

Case I: $x_0(s) = 1 + s$, $x_1(s) = s^3 + 2s$.
Case II: $x_0(s) = 1 + e^{-s}$, $x_1(s) = s - 1$.
Case III: $x_0(s) = 2s$, $x_1(s) = \cos(2s)$.
Case IV: $x_0(s) = \sin(\pi s)$, $x_1(s) = \log(1 + s)$.
The numerical results for each case are displayed in Fig. 4.

4.2 Image restoration experiments

The image restoration problem can be modeled in one-dimensional vectors by the following linear equation system:

$$y = Ax + b, \tag{55}$$

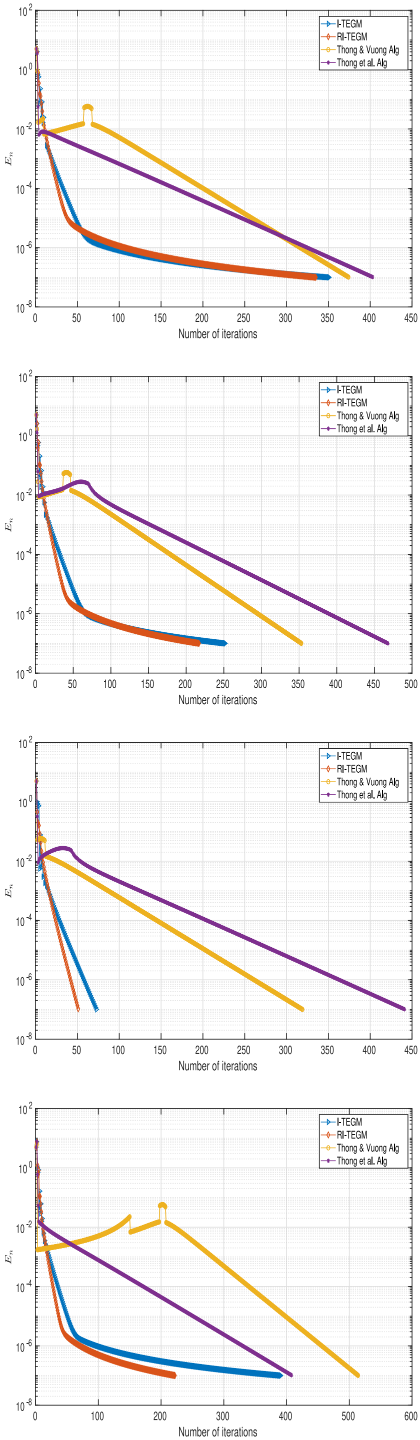
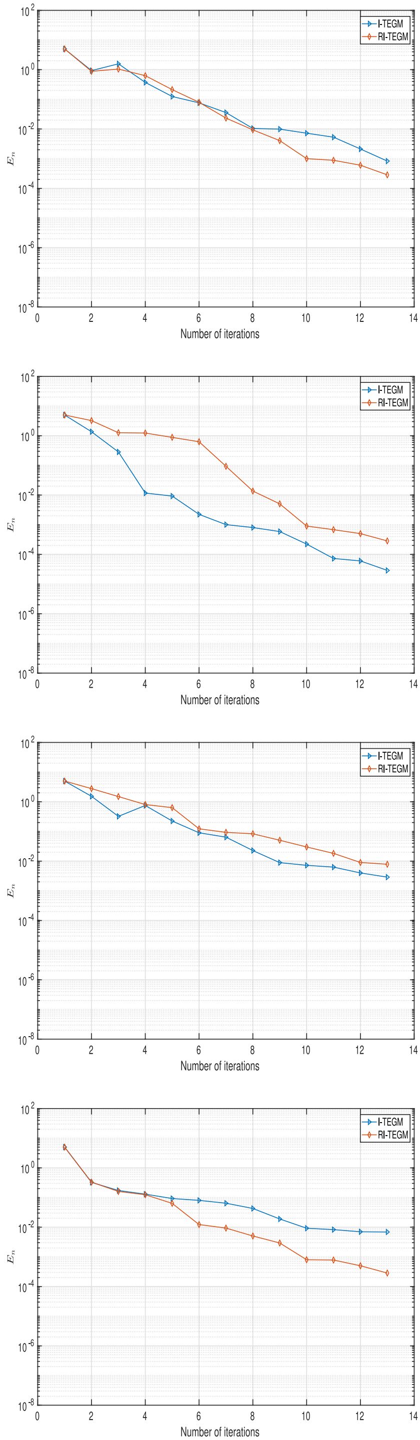


Fig. 3 Numerical results for Example 4.3, Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV

Fig. 4 Numerical results for Example 4.4, Top left: Case I; Top right: Case II; Bottom left: Case III; Bottom right: Case IV



where $x \in \mathbb{R}^{N\times 1}$ is an original image, $y \in \mathbb{R}^{M\times 1}$ is an observed image, b is an additive noise, and $A \in \mathbb{R}^{M\times N}$ is a blurring matrix. For solving above problem, we aim to approximate an original image, vector x, by minimizing an additive noise, which is called LASSO problem [45] as follows:

$$\min_x \frac{1}{2}\|y - Ax\|^2, \tag{56}$$

where $\|\cdot\|$ is the Euclidean norm. The proposed methods can be applied to solve the problem (56) by setting $\mathcal{F}x := A^T(Ax - y)$. Then $\mathcal{F}$ is monotone (hence it is pseudo-monotone) and Lipschitz continuous with $L = \|A^T A\|$.

To measure the quality of restored images, the Peak-Signal-to-Noise-Ratio (PSNR) in decibel (dB) [44] and the Structural-Similarity-Index-Measure (SSIM) [48] are used. Note that larger PSNR value, means the better quality of the restored image. Moreover, the range of SSIM varied in [0, 1], and if SSIM closes to 1, it means the better quality of the restored image.

Now, set $x_0 = (0, 0, 0, ..., 0) \in \mathbb{R}^N$ and $x_1 = (1, 1, 1, ..., 1) \in \mathbb{R}^N$. The parameters of each algorithm are chosen as follows:

- I-TEGM and RI-TEGM: $\lambda_1 = 6$, $\mu = 0.9$, $\tau = 0.95$, $\tau_n = \frac{1}{(10^5 n+10)^7}$, $s_n = \frac{1000}{(n+1)^2}$, $\alpha_n = \frac{1}{10^7 n+1}$, $\beta_n = 0.999(1-\alpha_n)^2$. Moreover, take $\theta_n = \frac{n}{300n+1}$ in RI-TEGM;
- Shehu Alg: $\alpha_n = \frac{1}{10^7 n+1}$, $\lambda_n = \frac{0.1n}{(n+1)\|\mathcal{F}\|^2}$;
- Thong and Vuong Alg: $\gamma = 0.1$, $\ell = 0.8$, $\mu = 0.9$;

Thong et al. Alg: $\lambda_1 = 6$, $\mu = 0.9$, $\alpha_n = \frac{1}{10^7 n+1}$, $f(x) = 0$.

We use the original images of Temple, Fish and Tiger, which are shown in Fig. 5.

We consider the restoration of images corrupted by the following blur types:

- Motion blur with motion length of 45 pixels and motion orientation $180°$;
- Out of focus (disk) with radius 7.

For the results recovering the degraded RGB images, we limit the number of iterations to 1000. The numerical results are reported as follows:

We next show the restored images of Temple, Fish and Tiger images, respectively.

(a) Temple (b) Fish (c) Tiger

Fig. 5 The original images

Table 1 The comparisons of PSNR and SSIM of the restored images for image of Temple

Algorithms	Motion blur		Out of focus	
	PSNR	SSIM	PSNR	SSIM
I-TEGM	33.4687	0.9271	31.2971	0.8602
RI-TEGM	33.4687	0.9271	31.2971	0.8602
Shehu Alg	28.9720	0.8540	28.0544	0.7724
Thoung and Voung Alg	28.9828	0.8542	28.0544	0.7727
Thong et al. Alg	33.2411	0.9246	31.1330	0.8566

Table 2 The comparisons of PSNR and SSIM of the restored images for image of Fish

Algorithms	Motion blur		Out of focus	
	PSNR	SSIM	PSNR	SSIM
I-TEGM	33.7840	0.9554	31.6926	0.9381
RI-TEGM	33.7840	0.9554	31.6926	0.9381
Shehu Alg	27.2200	0.8544	27.4309	0.8598
Thoung and Voung Alg	27.2345	0.8547	27.4403	0.8598
Thong et al. Alg	33.4619	0.9526	31.4973	0.9356

Table 3 The comparisons of PSNR and SSIM of the restored images for image of Tiger

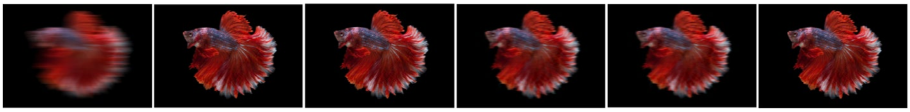
Algorithms	Motion blur		Out of focus	
	PSNR	SSIM	PSNR	SSIM
I-TEGM	26.1131	0.8779	26.4645	0.8716
RI-TEGM	26.1131	0.8779	26.4645	0.8716
Shehu Alg	21.7843	0.7323	22.4079	0.7037
Thoung and Voung Alg	21.7943	0.7327	22.4174	0.7042
Thong et al. Alg	25.9075	0.8734	26.2800	0.8665

(a) blurred image (Motion blur) (b) I-TEGM 33.4687 (c) RI-TEGM 33.4687 (d) Shehu Alg 28.9720 (e) Thong & Vuong Alg 28.9828 (f) Thong et al. Alg 33.2411

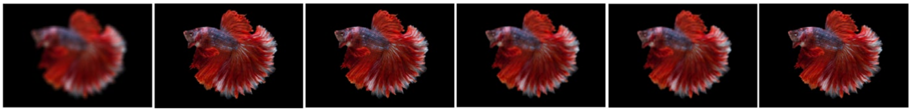
(g) blurred image (Out of focus) (h) I-TEGM 31.2971 (i) RI-TEGM 31.2971 (j) Shehu Alg 28.0544 (k) Thong & Vuong Alg 28.0544 (l) Thong et al. Alg 31.1330

Fig. 6 Recovered images with PSNR of degraded images by Motion blur and Out of focus for image of Temple

Remark 8 (1) It can be seen from numerical results in Tables 1, 2, 3, and Figs. 6, 7, 8 and 9 that our algorithms proposed in this paper have a higher PSNR than Shehu Alg, Thoung and Voung Alg and Thong et al. Alg for the same stopping criterion. Moreover, SSIM of the proposed methods are closer to 1 than other algorithms. This shows that our algorithms have better effectiveness and efficiency than other algorithms in terms of PSNR and SSIM.



(a) blurred image (Motion blur) (b) I-TEGM 33.7840 (c) RI-TEGM 33.7840 (d) Shehu Alg 27.2200 (e) Thong & Vuong Alg 27.2345 (f) Thong et al. Alg 33.4619



(g) blurred image (Out of focus) (h) I-TEGM 31.6926 (i) RI-TEGM 31.6926 (j) Shehu Alg. 27.4309 (k) Thong & Vuong Alg. 27.4403 (l) Thong et al. Alg 31.4973

Fig. 7 Recovered images with PSNR of degraded images by Motion blur and Out of focus for image of Fish

(a) blurred image (Motion blur) (b) I-TEGM 26.1131 (c) RI-TEGM 26.1131 (d) Shehu Alg 21.7843 (e) Thong & Vuong Alg 21.7943 (f) Thong et al. Alg 25.9075

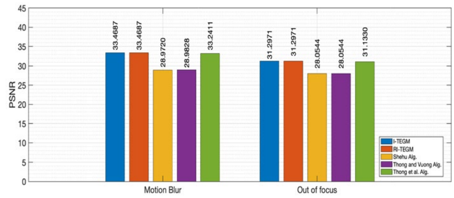
(g) blurred image (Out of focus) (h) I-TEGM 26.4645 (i) RI-TEGM 26.4645 (j) Shehu Alg 22.4079 (k) Thong & Vuong Alg. 22.4174 (l) Thong et al. Alg. 26.2800

Fig. 8 Recovered images with PSNR of degraded images by Motion blur and Out of focus for image of Tiger

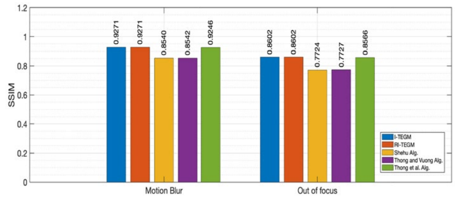
(2) According to the numerical results of I-TEGM and RI-TEGM, one can see that them have similar performance to restored images but the choosing of inertial factor θ_n of I-TEGM is difficult to compute since the value of $\|x_n - x_{n-1}\|$ need to be known before choosing θ_n while the choosing of inertial factor θ_n of RI-TEGM can work without the prior knowledge of the value of $\|x_n - x_{n-1}\|$. It just only chooses $0 < a \leq \theta_n \leq b < 1$ for some $a, b > 0$. In practically, RI-TEGM may be more easily to implemented in the image recovery problems.

5 Conclusions

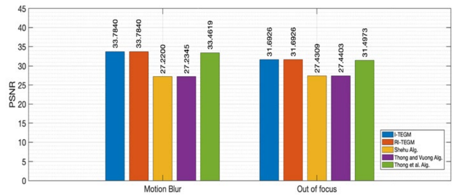
In this work, we have suggested two inertial Mann-type Tseng's extragradient methods for solving variational inequalities involves pseudo-monotone and Lipschitz continuous operator in a 2-uniformly convex Banach space which is also uniformly smooth. The proposed methods used adaptive stepsize which is updated over each iteration by a simple computation and without linesearch procedure. Under suitable conditions imposed on parameters, we have proved the strong convergence results of the sequence generated by the methods to a minimum-norm solution of the variational inequalities. Furthermore, we have provided



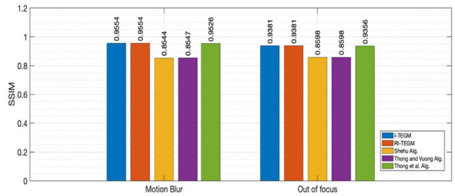
(a) PSNR for image of Temple



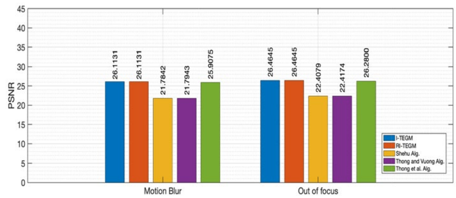
(b) SSIM for image of Temple



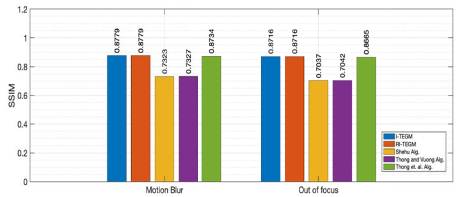
(c) PSNR for image of Fish



(d) SSIM for image of Fish



(e) PSNR for image of Tiger



(f) SSIM for image of Tiger

Fig. 9 Graphs of PSNR and SSIM values for each blurred image and restored images by I-TEGM, RI-TEGM, Shehu Alg, Thong and Vuong Alg and Thong et al. Alg

some numerical experiments in both finite and infinite dimensional spaces to show our algorithms are more effective and efficient than the existing algorithms in the literature

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Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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