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Gramian-Based Data-Driven ILC for Continuous-Time Systems

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Abstract: We present a data-driven Iterative Learning Control (ILC) scheme for continuous-time systems using a 'Gramian' approach. We present some numerical experiments using Chebyshev Polynomial Orthogonal Bases (CPOB) in both model-driven and data-driven ILC for continuous-time systems. We show that in the model-driven ILC case, the utilisation of a CPOB framework results in improved performance over discrete-time methods for applications requiring high precision. In the data-driven case, the advantages of a CPOB approach are less evident and we discuss some of the open problems being investigated.

Keywords: Data-driven control, Iterative learning control

1. INTRODUCTION

Dynamical systems with continuous-time behaviours, advances in approximation theory as well as the analysis techniques of classical control theory have inspired the developments of continuous-time analogues to the 'fundamental lemma' - see Willems et al. (2005); Markovsky et al. (2005); De Persis and Tesi (2020). There currently exist at least four frameworks for data-driven control of continuous-time systems; these are: Rapisarda et al. (2023b); Schmitz et al. (2024); Lopez et al. (2024); Ohta and Rapisarda (2024). An appropriate question to ask is that of which framework to select for the control of a dynamical system. The advantages of these approaches over each other and discrete-time techniques have not yet been deeply explored.

Iterative Learning Control (ILC) is the predominant control design problem for systems that operate iteratively, i.e. those that are repeatedly reset to the same initial conditions and are required to satisfy the same fixed objective over the same time interval. There exist many types of ILC such as the traditional PID type (see e.g. Arimoto et al. (1984)) and adaptive ILC (see e.g. Tayebi (2004)).

In the continuous-time domain, concerns have emerged over so-called *intersample behaviour* of ILC schemes that use samples of data, i.e. the tracking performance at time instants between the samples. For highly precise systems, failure to consider intersample behaviour can result in undesirable tracking performance (see e.g. Oomen et al. (2007)). A 'multirate' ILC scheme is presented in Oomen et al. (2009) that samples the tracking error at a faster rate. However, by definition, when considering continuous-time systems, this procedure does not consider intersample behaviour in its entirety.

The final two authors of the present paper introduced in Chu and Rapisarda (2023) a data-driven Norm-Optimal ILC (NOILC) for continuous-time systems framework. Control input signals are computed with consideration of the entire intersample and 'at-sample' behaviour by using Chebyshev polynomials.

In this paper, we present preliminary results on a new data-driven ILC scheme for continuous-time systems that uses the data-driven approach in Schmitz et al. (2024). We then use ILC as a vehicle to investigate the numerical behaviours of this approach in comparison to that in Rapisarda et al. (2023b) as well as to compare the performance of continuous-time ILC schemes with those in discrete-time in problems concerning continuous-time intersample behaviour.

This paper is organised as follows. In Section 2, we recall some basic definitions and state two of the continuous-time versions of the 'fundamental lemma' that relate Chebyshev Polynomial Orthogonal Bases (CPOB) representations of input-output data to those of all system trajectories. We present in Section 3 an alternative framework to that presented in Chu and Rapisarda (2023) that utilises the data-driven approach presented in Schmitz et al. (2024). In Section 4, we present the results of some numerical investigations that we use to compare our framework using two data-driven approaches and a model driven approach and the multirate framework from Oomen et al. (2009). We discuss the results in Section 5; we conclude the paper and discuss open problems for research in Section 6.

Notation

We denote by $\mathbb{N}^n$, $\mathbb{R}^n$ the spaces of n-dimensional vectors with natural and real entries respectively; $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices with real entries. The i^{th} row of a matrix M is denoted by $M_{i,:}$, and its i^{th} column by $M_{:,i}$. If A and B are two matrices with the same number of columns, we define $\text{col}(A,B) := \begin{bmatrix} A^\top & B^\top \end{bmatrix}^\top$. For a continuous-time trajectory $u(\cdot)$, we denote its i^{th} derivative with respect to time t by $u^{(i)}$. For a linear operator L, its range is denoted by $\mathcal{R}[L]$. We denote by $\mathcal{L}_2(\mathbb{I},\mathbb{R})$ the space of square-integrable real-valued

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functions defined on a finite interval $\mathbb{I} := [\tau_a, \tau_b] \subset \mathbb{R}$, by $\ell_2(\mathbb{N}, \mathbb{R})$ the space of real square-summable sequences and by $H^k(\mathbb{I}, \mathbb{R})$ the k^{th} order Sobolev space associated with $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$. The space of infinitely differentiable functions from $\mathbb{I}$ to $\mathbb{R}$ are denoted by $\mathcal{C}^\infty(\mathbb{I}, \mathbb{R})$. For $k \in \mathbb{N}\backslash\{0\}$, we define the 'jet' of order k of a function f by

$$\Lambda_k(f) := \begin{bmatrix} f \\ \vdots \\ f^{(k-1)} \end{bmatrix} . \tag{1}$$

2. BACKGROUND

2.1 Norm-optimal iterative learning control

Consider the linear time-invariant continuous-time system

$$\begin{aligned} \frac{d}{dt}x_k(t) &= Ax_k(t) + Bu_k(t) \\ y_k(t) &= Cx_k(t) + Du_k(t),\ x_k(\tau_a) = x_0 , \end{aligned} \tag{2}$$

where $x_k(t) \in \mathbb{R}^n, u_k(t) \in \mathbb{R}^m$ and $y_k(t) \in \mathbb{R}^p$ denote the state, input and output on the k^{th} iteration at time t; A, B, C, D are systems matrices with proper dimensions. The system output y_k is required to track a given reference signal r defined over a finite interval $\mathbb{I} := [\tau_a, \tau_b]$. At each time $t = \tau_b$, the time index and state are reset to $t = \tau_a$ and $x_{k+1}(\tau_a) = x_0$ respectively. The system output y_{k+1} is then required to track r over $\mathbb{I} = [\tau_a, \tau_b]$ again.

With the *tracking error on trial* k defined by $e_k := r - y_k$, the ILC problem is to find a control-updating law $u_{k+1} = f(u_k, e_k)$, such that tracking performance is improved over the iterations k, i.e., $\lim_{k\to\infty} e_k = 0$.

A squared 2-norm is defined by the following, with $Q \succ 0$ and $R \succ 0$ having compatible dimensions, in the output and input spaces $y_k \in \mathcal{L}_2(\mathbb{I}, \mathbb{R}^p)$ and $u_k \in \mathcal{L}_2(\mathbb{I}, \mathbb{R}^m)$ respectively

$$\|y_k\|_Q^2 := \int_{\mathbb{I}} y_k^\top(t) Q y_k(t) dt \ ; \ \|u_k\|_R^2 := \int_{\mathbb{I}} u_k^\top(t) R u_k(t) dt \ . \tag{3}$$

NOILC computes the control input signal for the next iteration as

$$\begin{aligned} u_{k+1} = \operatorname{argmin}_{u_{k+1}} &\{\|e_{k+1}\|_Q^2 + \|u_{k+1} - u_k\|_R^2\} \\ \text{s.t. } &\frac{d}{dt}x_k(t) = Ax_k(t) + Bu_k(t) \\ &y_k(t) = Cx_k(t) + Du_k(t), \\ &x_k(\tau_a) = x_0, \end{aligned} \tag{4}$$

and guarantees monotonic convergence of the tracking error norm, see e.g. Proposition 2 p. 4629 in Chu and Rapisarda (2023).

2.2 Chebyshev polynomial orthogonal bases

The *Chebyshev polynomials* on an interval $\mathbb{I} = [\tau_a, \tau_b] \subset \mathbb{R}$ are defined by

$$C_i(t) := \cos\left(i \arccos\left(\frac{2}{\tau_b - \tau_a}\left(t - \frac{\tau_a + \tau_b}{2}\right)\right)\right), i \in \mathbb{N}, \tag{5}$$

see e.g. Chapter 3 p. 14 in Trefethen (2019).

We define the *weight function* $w(\cdot)$ at time t by

$$w(t) := \frac{1}{\sqrt{1 - (\frac{2}{\tau_b - \tau_a}(t - \frac{\tau_a + \tau_b}{2}))^2}} \ , \ t \in \mathbb{I} \ . \tag{6}$$

The Chebyshev polynomials on $\mathbb{I}$ are orthogonal to each other with respect to the inner product defined by $\langle f, g\rangle_w := \int_{\tau_a}^{\tau_b} f(t)g(t)w(t)dt$, and they form a complete basis for $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$, i.e. their linear span is dense in $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$. Hence, every $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ can be written as a series

$$f = \sum_{k=0}^{\infty} \widetilde{f}_k C_k \tag{7}$$

where $\widetilde{f}_0 := \frac{2}{(\tau_b - \tau_a)\pi}\langle f, C_0\rangle_w$ and $\widetilde{f}_k := \frac{4}{(\tau_b - \tau_a)\pi}\langle f, C_k\rangle_w$, $k \in \mathbb{N}_{>0}$. The convergence of this series to f depends on the smoothness of f, see Remark 3 p. 4309 in Rapisarda et al. (2024). The set of coefficients $\{\widetilde{f}_k\}_{k\in\mathbb{N}}$ is square-summable, see Theorem 23 p. 23 in Sansone (1959). The notation of Section II.B p. 4628 in Chu and Rapisarda (2023) is utilised for *vector* functions.

We define the infinite vector of Chebyshev polynomials $\mathfrak{C} := [C_0\ C_1\ \ldots]^\top$ and the infinite vector of coefficients $\widetilde{f} := [\widetilde{f}_0\ \widetilde{f}_1\ \ldots]$ such that (7) can be written as

$$f = \sum_{k=0}^{\infty} \widetilde{f}_k C_k = \widetilde{f}\mathfrak{C} \ . \tag{8}$$

If $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ is differentiable and $f^{(1)} \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$, then

$$f^{(1)} = \widetilde{f}\frac{2}{\tau_b - \tau_a}\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\mathfrak{C} =: \widetilde{f}\mathcal{D}\mathfrak{C} \ ; \tag{9}$$

we call $\mathcal{D}$ the *differentiation matrix* whose entries are computed according to standard formulas, see Example 4 p. 4 in Rapisarda et al. (2024). The derivative of any $\mathcal{L}_2(\mathbb{I}, \mathbb{R})$ function can be computed with CPOB and the matrix $\mathcal{D}$.

Let $f \in \mathcal{L}_2(\mathbb{I}, \mathbb{R})$ be represented by (8), we define the bijective projection of f on the space of Chebyshev coefficient sequences $\Pi : \mathcal{L}_2(\mathbb{I}, R) \to \ell_2(\mathbb{N}, \mathbb{R})$ by

$$\Pi(f) := \widetilde{f}, \tag{10}$$

and its N^{th} degree *truncation* by

$$\Pi_N(f) := \left[\widetilde{f}_0\ \widetilde{f}_1\ \cdots\ \widetilde{f}_{N-1}\right] . \tag{11}$$

We call $f - \Pi_N(f)\left[C_0\ \cdots\ C_{N-1}\right]^\top$ the *approximation error.* It can be shown that the approximation error decays with N. The truncation of the derivative of the Chebyshev series for f can be computed by using finite submatrices of $\mathcal{D}$, see Section II.D p. 4 in Rapisarda et al. (2024).

2.3 The continuous-time 'fundamental lemma's

We state two results that enable the CPOB representation of *all* system trajectories from one 'sufficiently informative' input-output trajectory produced by a continuous-time LTI system.

We first state some definitions. Associate to (2) its *input-output* behaviour on $\mathbb{I} = [\tau_a, \tau_b]$, defined by

$$\mathfrak{B} := \{\text{col}(u,y) \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^{m+p}) \mid \exists\, x \in \mathcal{C}^\infty(\mathbb{I}, \mathbb{R}^n) \text{ s.t. } \text{col}(u,x,y) \text{ satisfies (2)}\}. \tag{12}$$

Define the *system lag* $\ell(\mathfrak{B})$ as in Section 3 p. 3 in Rapisarda et al. (2023b). We denote the *system order* by $n(\mathfrak{B})$.

2.4 The 'CPOB approach'

The following is Definition 1 p. 589 in Rapisarda et al. (2023a).

Definition 1. *Let* $\mathbb{I} = [\tau_a, \tau_b]$. $f : \mathbb{I} \to \mathbb{R}^m$ *is* persistently exciting of order L *on* $\mathbb{I}$ *if:*

a) f *is* $(L-1)$*-times continuously differentiable in* $\mathbb{I}$*;*
b) For every $v := [v_0 \ \dots \ v_{L-1}] \in \mathbb{R}^{1\times Lm}$ *it holds that*

$$v\Lambda_L(f)(t) = 0 \ \forall\, t \in \mathbb{I} \Longrightarrow v_0, \dots, v_{L-1} = 0. \tag{13}$$

The projection of $\mathfrak{B}$ on the space of Chebyshev coefficient sequences is defined by

$$\Pi(\mathfrak{B}) := \{\text{col}(\widetilde{u}, \widetilde{y}) \in \ell_2(\mathbb{N}, \mathbb{R}^{m+p}) \mid \exists\ \text{col}(u,y) \in \mathfrak{B} \ \text{ s.t. } \text{col}(\widetilde{u}, \widetilde{y}) = \Pi(\text{col}(u,y))\}. \tag{14}$$

Define the following *jets* of Chebyshev coefficients

$$\mathcal{W}_L(\widetilde{u}) := \Pi(\Lambda_L(u));\ \mathcal{W}_L(\widetilde{y}) := \Pi(\Lambda_L(y)) \tag{15}$$

The following is Theorem 4 p. 6 in Rapisarda et al. (2023b).

Theorem 1. *Define* $\mathfrak{B}$ *by* (12) *and let* $\text{col}\,(u,y) \in \mathfrak{B}$. *Assume that* (A,B) *is controllable and that* u *is persistently exciting of order* $L > \ell(\mathfrak{B}) + n(\mathfrak{B})$. *Define* $\mathcal{W}_L(\widetilde{u})$, $\mathcal{W}_L(\widetilde{y})$ *by* (15)*, then* $\dim \mathcal{R}\,[\text{col}(\mathcal{W}_L(\widetilde{u}), \mathcal{W}_L(\widetilde{y}))] = Lm + n(\mathfrak{B}) =: d$.

Let $V_u \in \mathbb{R}^{Lm\times d}$ *and* $V_y \in \mathbb{R}^{Lp\times d}$ *be such that* $\text{col}(V_u, V_y)$ *is a basis matrix for* $\mathcal{R}\,[\text{col}(\mathcal{W}_L(\widetilde{u}), \mathcal{W}_L(\widetilde{y}))]$. *Define* $\Pi(\mathfrak{B})$ *by* (14)*. The following are equivalent:*

(1) $\text{col}\,(\widetilde{u}', \widetilde{y}') \in \Pi(\mathfrak{B})$*;*
(2) There exists $G \in \mathbb{R}^{d\times\infty}$ *such that*

$$\begin{bmatrix} \mathcal{W}(\widetilde{u}') \\ \mathcal{W}(\widetilde{y}') \end{bmatrix} = \begin{bmatrix} V_u \\ V_y \end{bmatrix} G\,, \tag{16}$$

(3) There exists $G \in \mathbb{R}^{d\times\infty}$ *such that*

$$\begin{aligned} (V_uG)_{1,:} &= \widetilde{u}' \\ (V_yG)_{1,:} &= \widetilde{y}' \\ (V_uG)_{1,:}\mathcal{D}^i - (V_uG)_{i+1,:} &= 0 \\ (V_yG)_{1,:}\mathcal{D}^i - (V_yG)_{i+1,:} &= 0\,, \end{aligned} \tag{17}$$

$i = 1, \dots, L-1$.

where $\mathcal{D}$ *is defined as in* (9).

2.5 The 'Gramian approach'

Given $L \in \mathbb{N}\backslash\{0\}$ and $f \in H^{L-1}(\mathbb{I}, \mathbb{R}^d)$, we define the Gramian

$$\Gamma_L(f) := \int_{\mathbb{I}} \Lambda_L(f)\Lambda_L(f)^\top dt\,. \tag{18}$$

The following is Definition 16 p. 6 in Schmitz et al. (2024).

Definition 2. *Let* $L \in \mathbb{N}\backslash\{0\}$. *A function* $f : \mathbb{I} \to \mathbb{R}^d$ *is* persistently exciting of order L *if* $f \in H^{L-1}(\mathbb{I}, \mathbb{R}^d)$ *and the Gramian* $\Gamma_L(f)$ *in* (18) *is positive definite.*

We extend the definitions of the *jet* and Gramian to

$$\Lambda_{L,K}(u,y) := \begin{bmatrix} \Lambda_L(u) \\ \Lambda_K(y) \end{bmatrix}; \tag{19}$$

$$\Gamma_{L,K}(u,y) := \int_{\mathbb{I}} \Lambda_{L,K}(u,y)\Lambda_{L,K}(u,y)^\top dt\,. \tag{20}$$

The following is Theorem 22 p. 7 in Schmitz et al. (2024).

Theorem 2. *Suppose that* $\mathfrak{B}$ *is controllable. Let* $\text{col}(u,y) \in \mathfrak{B}$ *be such that* u *is persistently exciting of order* $L > \ell(\mathfrak{B}) + n(\mathfrak{B})$. *For* $\text{col}(u',y') \in H^{L-1}(\mathbb{I}, \mathbb{R}^{m+p})$ *and* $K \in \mathbb{N}$, $\ell(\mathfrak{B}) + 1 \le K \le L$*, the following statements are equivalent:*

(1) $\text{col}(u',y') \in \mathfrak{B}$*;*
(2) There exists $g \in \mathcal{L}_2(\mathbb{I}, \mathbb{R}^{Lm+Kp})$ *such that*

$$\Lambda_{L,K}(u',y') = \Gamma_{L,K}(u,y)g. \tag{21}$$

Moreover, rank $\Gamma_{L,K}(u,y) = Lm + n(\mathfrak{B})$.

The following is Corollary 25 p. 8 in Schmitz et al. (2024) for Chebyshev polynomial bases.

Corollary 1. *Let the assumption of Theorem 2 hold. Consider the partition*

$$\left[\Gamma_u^\top\ \Gamma_{u^{(1)}}^\top\ \cdots\ \Gamma_{u^{L-1}}^\top\ \Gamma_y^\top\ \Gamma_{y^{(1)}}^\top\ \cdots\ \Gamma_{y^{L-1}}^\top\right]^\top = \Gamma_{L,K}(u,y) \tag{22}$$

where $\Gamma_{u^{(j)}} \in \mathbb{R}^{m\times(Lm+Kp)}$, $j = 0, \dots, L-1$, *and* $\Gamma_{y^{(i)}} \in \mathbb{R}^{p\times(Lm+Kp)}$, $i = 0, \dots, K-1$. *For* $\text{col}(u',y') \in H^{L-1}(\mathbb{I}, \mathbb{R}^{m+p})$ *with* $\widetilde{u}' = \Pi(u')$, $\widetilde{y}' = \Pi(y')$*, the following are equivalent:*

(1) $\text{col}(u',y') \in \mathfrak{B}$*;*
(2) There exists $G \in \ell_2(\mathbb{N}, \mathbb{R}^{Lm+Kp})$ *such that*

$$\begin{aligned} \widetilde{u}' &= \Gamma_{u^{(0)}}G, \\ \widetilde{y}' &= \Gamma_{y^{(0)}}G, \\ \Gamma_{u^{(j-1)}}G\mathcal{D} &= \Gamma_{u^{(j)}}G, j = 1, \dots, L-1, \\ \Gamma_{y^{(i-1)}}G\mathcal{D} &= \Gamma_{y^{(i)}}G, i = 1, \dots, K-1, \end{aligned} \tag{23}$$

where $\mathcal{D}$ *is defined as in* (9).

Remark 1. *Instead of using the data matrix* $\Gamma_{L,K}(u,y)$*, one can compute basis matrices* V_u *and* V_y *as in Section 2.4 such that* $\text{col}(V_u,V_y)$ *is a basis matrix for* $\mathcal{R}\,[\Gamma_{L,K}(u,y)]$. *One such method of performing this computation is to utilise the reduced singular value decomposition of* $\Gamma_{L,K}(u,y)$.

3. DATA-DRIVEN NOILC FOR CT SYSTEMS

In this section, we present a data-driven NOILC for continuous-time systems framework that utilises the 'Gramian approach' (of Section 2.5) to the continuous-time 'fundamental lemma'. The following framework inherits all convergence properties of the model-based NOILC algorithm including monotonic tracking error norm convergence - see e.g. Proposition 2 p. 4629 in Chu and Rapisarda (2023).

Proposition 1. *Assume that* (A,B) *is controllable and let* $\text{col}(u,y)$ *be an input-output trajectory of* (2)*. Assume that* u *is persistently exciting of order* $L > \ell(\mathfrak{B}) + n(\mathfrak{B})$ *according to Definition 2. Define* K *as in Theorem 2; define* $\Gamma_{L,K}(u,y)$ *as* (20) *and consider the partition* (22).

The optimal input

$$u_{k+1} = \Gamma_u G_{k+1} \mathfrak{C} \tag{24}$$

solves the NOILC design problem (4) *on trial* $k+1$ *if and only if there exists* $G_{k+1} \in \mathbb{R}^{Lm+Kp}$ *that solves the following optimisation problem*

$$\begin{aligned} &\min_{G_{k+1}} \{\|r - \Gamma_y G_{k+1}\mathfrak{C}\|_Q^2 + \|\Gamma_u G_{k+1}\mathfrak{C} - u_k\|_R^2\} \\ &\text{s.t. } \Gamma_{u^{(i-1)}} G_{k+1}\mathcal{D} - \Gamma_{u^{(i)}} G_{k+1} = 0 \\ &\quad i = 1, \ldots, L-1. \\ &\quad \Gamma_{y^{(i-1)}} G_{k+1}\mathcal{D} - \Gamma_{y^{(i)}} G_{k+1} = 0 \\ &\quad i = 1, \ldots, K-1. \\ &\quad \Gamma_{y^{(i)}} G_{k+1}\mathcal{D}\mathfrak{C}(\tau_a) = 0,\ i = 0, \ldots, q-1\ . \end{aligned} \tag{25}$$

Proof. The first $L + K - 2$ constraints in (25) are equivalent to requiring that $\text{col}(u_{k+1}, y_{k+1})$ is an input-output trajectory of (2). Using Theorem 2, Corollary 1 and the assumption of persistent excitation of u (according to Definition 2), $\text{col}(u_{k+1}, y_{k+1})$ is a system trajectory if and only if it satisfies (23). To complete the proof, recall that the system output is required to satisfy the zero initial condition, equivalently the last q constraints in (25). □

Remark 2. *The proposed design framework involves the use of (high-order) derivatives of the input and output signals. However, there is no need to measure them (which can be difficult in practice) as they can be directly computed using CPOB representations and* (9).

Remark 3. *In practice, for computational purposes, one would truncate the framework* (25) *to use a finite number* N *of Chebyshev coefficients, see e.g. Remark 2 p. 372 in Wolski et al. (2024).*

Remark 4. *An analogous approach to Proposition 1 can be defined for the 'CPOB approach', see Proposition 1 p. 4629 in Chu and Rapisarda (2023).*

4. NUMERICAL INVESTIGATIONS

Consider the dynamical system (from Oomen et al. (2009)),

$$x^{(2)} = u,\ y = x. \tag{26}$$

The system has a lag of $\ell(\mathfrak{B}) = 2$ and order $n(\mathfrak{B}) = 2$. In the following examples, we use the `Chebfun` toolbox within `MATLAB` to compute CPOB representations and the command `chebop` to simulate the system (26). The Gramian matrices $\Gamma_L(u)$ and $\Gamma_{L,K}(u, y)$ are computed by using CPOB representations of (u, y), the `chebcoeffs` command within `Chebfun` and the matrix $\mathcal{D}$ in Section 2.2. The rank of a matrix is computed with the `rank` command in `MATLAB` with the default tolerance. We use the `mosek` solver within `yalmip` to solve (25).

Example 1 (Model-driven). *The system* (26) *is operating over* $\mathbb{I} = [0, 0.15]$ *with initial condition* $x_k(0) = 0$. *It can be shown that the following is a basis of* $\mathcal{R}[\text{col}(\mathcal{W}_L(\tilde{u}), \mathcal{W}_L(\tilde{y}))]$ *(as defined in Theorem 1) and can be used to characterise all system trajectories, see* (16),

$$\begin{bmatrix} V_u \\ V_y \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&1&0&0 \\ 0&1&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \end{bmatrix}^\top . \tag{27}$$

We consider this a 'numerically ideal' basis of the system (26) *since it was derived with knowledge of the system and numerically exactly characterises the trajectories of the system - it is therefore effectively a model.*

Figure 1 shows the tracking error $e(t)$ *with* $r(t) = H(t - 0.02)$, *the shifted Heaviside step, after 10 iterations with the values of the Chebyshev truncation index* $N = 15, 30, 60$ *in our framework using the basis* (27) *- labelled as 'CT-ILC'. We compare this to the multirate ILC scheme in Oomen et al. (2009) (labelled as 'MR-ILC' in Figure 1) in which the tracking error is sampled at* $f^h = 400Hz$ *(shown by the solid line in Figure 1) and the controller operates at* $f^l = 100Hz$ *(shown by the dots on the line in Figure 1). In both frameworks, we use the weights* $Q = 1$, $R = 10^{-12}$.

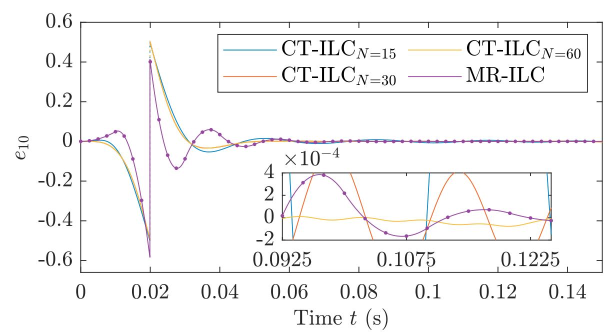

Fig. 1. Tracking error e on the 10th iteration for Example 1 with different values of N in our framework and that of the multirate ILC scheme in Oomen et al. (2009).

Example 2 (Data-driven: data matrices). *In this example, we investigate numerically the data-driven approaches in Section 3 in terms of their persistency of excitation definitions and data matrices. Consider the system* (26), *let* $L = 5 > \ell(\mathfrak{B}) + n(\mathfrak{B})$. *It can be shown that the input signals whose values at* t *are* $u(t) = t^4$; $u(t) = -5e^{-6t} - 4e^{-5t} - 3e^{-4t} - 2e^{-3t} - e^{-2t}$ *are persistently exciting of order* 5 *in the 'CPOB approach' (according to Definition 1) and belong to* $H^{L-1}(\mathbb{I}, \mathbb{R}^m)$ *for any finite* $\mathbb{I} = [\tau_a, \tau_b]$. *Definition 2 states that in order for* u *to be persistently exciting in the 'Gramian approach', the Gramian matrix* $\Gamma_L(u)$ *must also be positive definite. By definition,* $\Gamma_L(u)$ *depends on the time interval* $\mathbb{I} = [\tau_a, \tau_b]$. *Figure 2 shows the values of* N *and* τ_b, *on a finite grid, for which* $\Gamma_L(u)$ *has* 5 *positive eigenvalues for the two aforementioned input signals thus ascertaining the persistency of excitation condition in the 'Gramian Approach'. Red points in Figure 2 indicate that* $\Gamma_L(u)$ *is not positive definite for the corresponding values of* N *and* τ_b.

Theorems 1 and 2 state that the data matrices $\mathcal{W}_L(\tilde{u}, \tilde{y})$ *and* $\Gamma_{L,K}(u, y)$ *have rank* $d := Lm + n(\mathfrak{B}) = 7$. *Figure 3 shows the values of* N *and* τ_b *for which this rank result is true in simulation for the two data-driven approaches given the input signal* $u(t) = t^4$.

Example 3 (Data-driven: ILC). *Figure 3 from Example 2 shows that the data matrices in both the 'CPOB' and 'Gramian' approaches have rank* 7 *when* $u(t) = t^4$, $N = 10$ *and* $\tau_b = 3s$. *In this example, we apply both data-driven approaches to the ILC design problem for the system* (26). *We compare these results to our model-driven approach,*

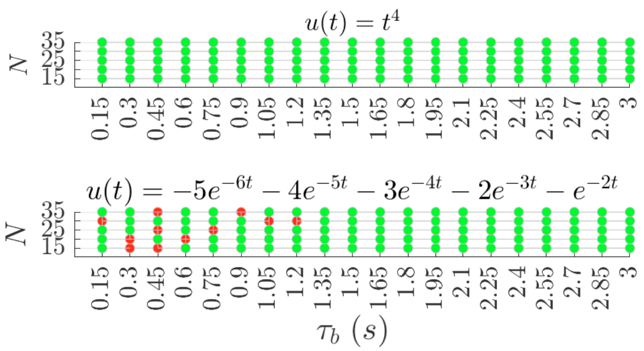

Fig. 2. The values of N and τ_b in our framework for which the Gramian matrices $\Gamma_L(u)$ with the input signals in Example 2 are positive definite (indicated in green).

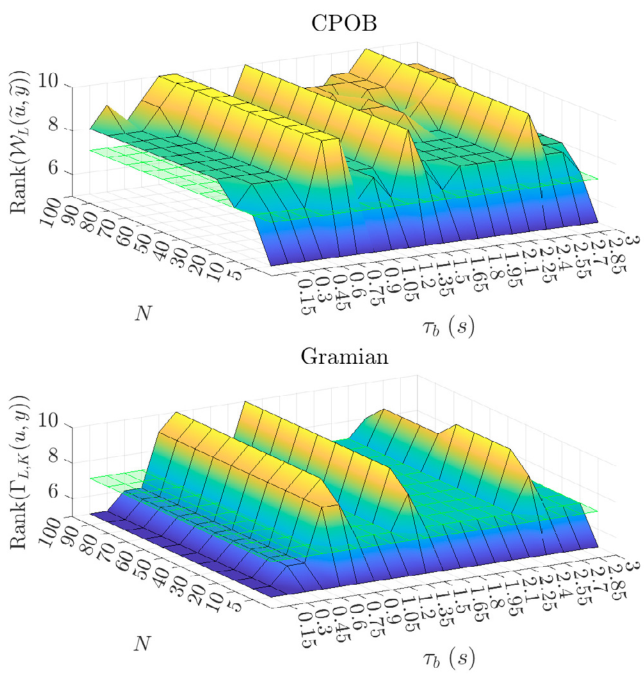

Fig. 3. The ranks of the CPOB matrix $\mathcal{W}_L(\widetilde{u}, \widetilde{y})$ and the Gramian matrix $\Gamma_{L,K}(u, y)$ with different values of N and τ_b in Example 2.

i.e. when we use (27) *as the basis, as well as the multirate approach of Oomen et al. (2009).*

The system (26) *is operating over* $\mathbb{I} = [0, 3]$ *with initial condition* $x_k(0) = 0$ *and whose output is required to track the reference* $r(t) = t^3$*. We use* 10 *samples of the tracking error, i.e.* $N = 10$ *in our framework and retain* $f^h = 400Hz$*;* $f^l = 100Hz$ *in the multirate scheme.*

Figure 4 shows the tracking error on the 100^{th} *iteration for our model-driven and data-driven approaches and the multirate scheme in Oomen et al. (2009), and the 2-norm of the tracking error sampled at 10kHz to capture the intersample behaviour over* 100 *iterations for each framework.*

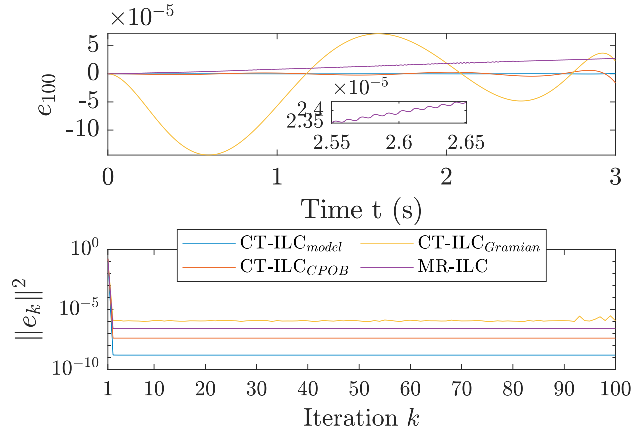

Fig. 4. Tracking error on the 100^{th} iteration (Top) and 2-norm of the tracking error over 100 iterations (Bottom) for Example 3 with our framework (model-driven, 'CPOB approach' and 'Gramian approach') and that of the multirate ILC scheme in Oomen et al. (2009).

5. DISCUSSION

From the examples in the previous section, we can draw some indications about the numerical differences between the two data-driven approaches in Section 3 and model-driven schemes.

Comment 1 (Model-driven comparison)**.** *Figure 1 shows the performance of our model-driven framework with* 15*,* 30 *and* 60 *samples of the tracking error against the scheme in Oomen et al. (2009) with* 60 *samples. It is clear that with less samples* (15 *or* 30) *than the multirate scheme, our framework performs well in terms of settling time in comparison to that of Oomen et al. (2009). With the same number of samples* (60)*, Figure 1 shows that our model-driven framework outperforms that of the discrete-time multirate scheme with regards to tracking performance. This is despite of the fact that the reference function is infeasible for the continuous-time system* (26)*.*

Comment 2 (Persistency of excitation conditions)**.** *It is a simple task to generate continuous functions that are persistently exciting of a given order in the 'CPOB approach' (Definition 1). The processes for finding functions that are persistently exciting according to the 'Gramian approach' (Definition 2) are less straightforward. In Example 2, we investigate the roles of the parameters* N *and* τ_b *on this notion. Figure 2 shows that even when a function is considered persistently exciting according to the 'CPOB approach', this only also holds according to the 'Gramian approach' for a limited set of values of* N *and* τ_b*. Several factors may influence this such as the number of computational operations involved and the algorithms used for such operations. It is not clear at this stage why it is more difficult numerically to achieve the persistency of excitation condition for the 'Gramian approach'. An examination of more and different cases needs to be performed before drawing any firm conclusions on the matter.*

Comment 3 (Ranks of data matrices)**.** *Theorems 1 and 2 state that the data matrices* $\mathcal{W}_L(\widetilde{u}, \widetilde{y})$ *and* $\Gamma_{L,K}(u, y)$ *have rank* $d := Lm + n(\mathfrak{B})$*. The verification of the rank condition is essential in establishing sufficient informativity. In Example 2, we analyse how accurately such conditions can* *be ascertained with the 'CPOB' and 'Gramian' approaches. We use an input that has been verified to be persistently exciting in both approaches for the relevant values of N and τ_b in Example 2. Figure 3 shows that with different values of N and τ_b, the rank of the data matrix in the 'CPOB approach' $\mathcal{W}_L(\widetilde{u}, \widetilde{y})$ is inconsistent and moreover is rarely equal to $d = 7$ (shown by the flat intersecting plane). The rank of the 'Gramian approach' data matrix is also shown to be similarly inconsistent in Figure 3 but does provide some flat regions that mirror the rank result of Theorem 2. In more experiments involving different input signals (omitted here for reasons of space), this flat plane is consistently present in the rank of $\Gamma_{L,K}(u, y)$. We infer that, in practice, the truncation index N as well as the length of the time interval $[\tau_a, \tau_b]$ greatly affect both the persistent excitation of CPOB representations of continuous functions and the rank condition of the data matrices in both data-driven approaches. There is however some indication that the 'Gramian approach' provides a larger set of values of N and τ_b for which the rank condition is consistently verified. We recognise that numerical issues may affect implementations with Chebyshev coefficients. Such important issue is a matter for future research.*

Comment 4 (Continuous-time vs. discrete-time ILC)**.** *In Example 3, we compare the tracking error, including the intersample behaviour, over* 100 *iterations with our framework when using a model (see* (27)*) as well as two data-driven approaches against the discrete-time multirate ILC scheme of Oomen et al. (2009). Despite the fact that we are using only* 10 *samples of the tracking error in our approaches and* 1200 *in that of the multirate scheme of Oomen et al. (2009), Figure 4 clearly shows that our framework equipped with a model or the 'CPOB' approach outperforms that of Oomen et al. (2009) on all iterations. It is interesting and unclear at this stage why the 'Gramian' approach in Example 3 not only performs worse than the 'CPOB' approach but also the multirate scheme.*

6. CONCLUSION

We presented a data-driven ILC framework for continuous-time systems that uses the continuous-time version of the 'fundamental lemma' presented in Schmitz et al. (2024). We compared this 'fundamental lemma' against that in Rapisarda et al. (2023b) numerically using experiments with ILC. We also compared a model-driven discrete-time approach that considers 'intersample behaviour' to our framework equipped with a system model and separately with just data.

Admittedly, the results presented here are based on a few simulations on a specific example in which standard `MATLAB` operations are used without special attention being paid to the numerical accuracy of the algorithms used. Future work includes more rigorous numerical experimentation and analysis from which we can detail the advantages of the entire set of continuous-time 'fundamental lemma's.

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