

EXTERIOR POWER OPERATIONS ON RELATIVE K -THEORY

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ABSTRACT. We algebraically construct exterior power operations on higher *relative* algebraic K -groups and prove their desired properties such as the expected behaviour with respect to (tensor) products and composition. This builds on Grayson's description of relative K -groups in terms of explicit generators and relations and on work by Harris, the first author and Taelman for (absolute) K -groups. Among the new features in our approach is the observation that the product axiom in the classical notion of a λ -ring is redundant.

INTRODUCTION

Arguably most powerful and fundamental among the features of higher algebraic K -theory are long exact sequences of K -groups. The most general type of these is the long exact sequence

$$\dots \rightarrow K_{n+1}\mathcal{M} \rightarrow K_{n+1}\mathcal{N} \rightarrow K_n[F] \rightarrow K_n\mathcal{M} \rightarrow K_n\mathcal{N} \rightarrow \dots \quad (1)$$

associated with an exact functor $F: \mathcal{M} \rightarrow \mathcal{N}$ between exact categories; here, the higher relative K -group $K_n[F]$ can be thought of as the n th homotopy group of the homotopy fibre of the induced map $KF: K\mathcal{M} \rightarrow K\mathcal{N}$ between the corresponding K -theory spaces. Building on his earlier work in [Gra12] for (absolute) K -groups using so-called binary complexes, Grayson has in [Gra16] given a conjectural description of $K_n[F]$ in terms of generators and relations and established the sequence (1) purely algebraically. This conjectural description of $K_n[F]$ is confirmed in [Tu25] to be equivalent to the definition given above, and we use it as the definition of $K_n[F]$ in this paper.

Exterior power operations on K -groups provide another important tool in the (higher) K -theory of schemes. They are the core ingredients in the theory of Adams operations and, more generally, in Grothendieck's seminal Riemann-Roch theory. Their properties, such as their behaviour with respect to (tensor) products and composition, are captured in the classical, abstract notion of a λ -ring.

The object of this paper is to prove the following theorem, see Definition 5.3 and Theorem 5.4, which combines the two features mentioned above.

Let $f: X \rightarrow Y$ be a morphism between quasi-compact schemes and let $F: \mathcal{M} \rightarrow \mathcal{N}$ denote the associated pull-back functor between the exact categories $\mathcal{M} := \mathcal{P}(Y)$ and $\mathcal{N} := \mathcal{P}(X)$ of locally free modules of finite rank over \mathcal{O}_Y and \mathcal{O}_X , respectively.

Theorem. *For every $n \geq 0$, the relative K -group $K_n[F]$ can be equipped with products and exterior power operations $\lambda^k: K_n[F] \rightarrow K_n[F]$, $k \geq 1$,*

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such that $K_n[F]$ becomes a λ -ring (without unity) and such that the incoming and outgoing maps in the long exact sequence (1) become λ -ring homomorphisms. Moreover, if $n \geq 1$, the products on $K_n[F]$ are trivial and λ^k is a group homomorphism.

To be able to explain the main ideas behind this theorem, we first consider the case $n = 0$ and now roughly describe the generators of $K_0[F]$, see Definition 1.1 for more (precise) details. They are triples of the form $X = ((\begin{smallmatrix} A \\ B \end{smallmatrix}), (N, \begin{smallmatrix} d \\ d' \end{smallmatrix}), (\begin{smallmatrix} u \\ v \end{smallmatrix}))$ where A and B are chain complexes in \mathcal{M} , (N, d) and (N, d') are chain complexes in \mathcal{N} (i.e., a so-called binary complex in \mathcal{N}) and $u: FA \rightarrow (N, d)$ and $v: FB \rightarrow (N, d')$ are quasi-isomorphisms. Using the simplicial constructions from [HKT17] for (binary) complexes, we define tensor products of such triples, see Definition 2.1, and exterior powers $\Lambda^k(X)$ for $k \geq 1$, see Definition 3.1. In order to then obtain an operation $\lambda^k: K_0[F] \rightarrow K_0[F]$ which is compatible with the classical operation $\lambda^k: K_0\mathcal{M} \rightarrow K_0\mathcal{M}$ via the outgoing homomorphism $K_0[F] \rightarrow K_0\mathcal{M}$ in (1), the class $\lambda^k([X]) \in K_0[F]$ can however not be simply defined as $[\Lambda^k(X)]$, but we define it in the following a priori surprising way, see Definition 3.3:

$$\lambda^k([X]) := [\lambda^k(X - \Delta \perp X)];$$

here, $\Delta \perp X$ denotes the triple $((\begin{smallmatrix} B \\ B \end{smallmatrix}), (N, \begin{smallmatrix} d' \\ d' \end{smallmatrix}), (\begin{smallmatrix} v \\ v \end{smallmatrix}))$, the difference $X - \Delta \perp X$ is to be taken in the Grothendieck group $K_0B[F]$ of triples as above, the right-hand λ^k is computed there as well and $[\dots]$ means taking the class in the factor group $K_0[F]$. A similar trick yields the definition of products on $K_0[F]$, see Definition 2.3. This way, the outgoing map $K_0[F] \rightarrow K_0\mathcal{M}$ can be verified to be a λ -ring homomorphism, see Proposition 2.4 and Proposition 3.5. The proof of the statement that also the incoming map $K_1\mathcal{N} \rightarrow K_0[F]$ in (1) is a λ -ring homomorphism relies on the fact (proved in [HKT17]) that products in $K_1\mathcal{N}$ vanish, again see Proposition 2.4 and Proposition 3.5.

The statement that $K_0[F]$ together with the products and operations λ^k , $k \geq 1$, discussed above satisfies the properties of a λ -ring already holds on the level of $K_0B[F]$ and is proved there, see Theorem 4.2. More precisely, in order to show the expected behaviour of λ^k with respect to composition, we use the approach from [HKT17] which relies on the crucial result that the Grothendieck ring of the category $\text{Pol}_{<\infty}^0(\mathbb{Z})$ of polynomial functors over \mathbb{Z} is isomorphic to the free λ -ring in one variable. Then the proof comes down to constructing a natural functor

$$\text{Pol}_{<\infty}^0(\mathbb{Z}) \rightarrow \text{End}(B[F]).$$

Various ideas suggest themselves to be used to prove the expected behaviour of λ^k with respect to products in our context, see the paragraph before Lemma 4.1. However, we just prove the following surprising and comparatively elementary fact which seems to have not yet been observed in the literature and which will probably be convenient not just for this paper: the product axiom in the definition of a λ -ring is redundant, see Lemma 4.1.

The case when $n \geq 1$ will be treated in Section 5. This case is combinatorially more involved. The essential additional ingredient in the construction of λ^k and in the proof of our main theorem is however just the recursive

nature of the combinatorial definition of $K_n[F]$, see Definition 5.1, together with its subtle feature that the recursion can be done in any in any order, see Definition 5.3 and the proof of Theorem 5.4. For example, the statements that products on $K_n[F]$ are trivial and that $\lambda^k: K_n[F] \rightarrow K_n[F]$ is a homomorphism follow from the analogous statements for absolute K -groups proved in [HKT17].

We finally recall that, in [Bas68], Bass has also purely algebraically defined a K_1 -group and a relative K_0 -group, which we denote by $K_1^B \mathcal{N}$ and $K_0^B[F]$, respectively, and has established the corresponding exact sequence (1) for $n = 0$ when \mathcal{M} and \mathcal{N} are split exact and the functor F is cofinal. In Theorem 1.3, we provide an explicit and well-defined homomorphism

$$\Phi: K_0^B[F] \rightarrow K_0[F]$$

and prove, under the same assumptions, that Φ is an isomorphism. Furthermore, in various remarks (see 2.6, 2.7, 3.7 and 3.8), we recall/give constructions of products and exterior power operations on $K_1^B \mathcal{N}$ and $K_0^B[F]$ and relate them to the constructions described above.

1. COMBINATORIAL RELATIVE K_0

In the first part of this section we recall Grayson's combinatorial definitions of the K_1 -group of an exact category [Gra12] and of the relative K_0 -group of an exact functor between exact categories [Gra16]. We then show that Grayson's relative K_0 -group is isomorphic to Bass's relative K_0 -group assuming the usual assumptions for Bass K -theory are satisfied. We finish by proving the expected property that swapping top and bottom in the components of the generating triples of Grayson's relative K_0 -group amounts to a sign change.

Let \mathcal{M} be an exact category. We assume throughout that all exact categories support long exact sequences in the sense of [Gra12, Definition 1.4]. This ensures we have a well-behaved notion of quasi-isomorphisms for complexes, see [Gra12, Definition 2.6] and the discussion after [Gra12, Definition 1.4]. For example, by [TT90, A.9.2], every idempotent complete category supports long exact sequences.

Slightly deviating from standard notation, let $C\mathcal{M}$ denote the category of bounded chain complexes in \mathcal{M} supported in non-negative degrees. (The latter condition could be dropped in this first section but will be essential in the later sections. Anyway, this condition does not affect the associated K -theory as shown in [KZ25, Proposition A.1].) Let $C^a\mathcal{M}$ denote the full subcategory of $C\mathcal{M}$ consisting of acyclic complexes. Both $C\mathcal{M}$ and $C^a\mathcal{M}$ are exact categories in the obvious way, see [Gra12, Section 2].

We recall from [Gra12, Section 3] that a *binary complex* $M = (M., \begin{smallmatrix} d \\ \tilde{d} \end{smallmatrix})$ in \mathcal{M} is a graded object $M.$ in \mathcal{M} together with two degree -1 maps $d, \tilde{d}: M. \rightarrow M.$ such that both $d^2 = 0$ and $\tilde{d}^2 = 0$. We will write $\top M$ and $\perp M$ for the complexes $(M., d)$ and $(M., \tilde{d})$, respectively, and $\perp\!\!\!\top M$ for the pair $(\top M, \perp M)$. If $d = \tilde{d}$, the binary complex M is said to be *diagonal*. Given a complex $M = (M., d)$ in $C\mathcal{M}$, we write ΔM for the diagonal binary complex $(M., \begin{smallmatrix} d \\ d \end{smallmatrix})$. A *morphism between binary complexes* is a degree 0 map between the underlying graded objects that is a chain map with respect to both differentials.

Similarly to above, let $B\mathcal{M}$ denote the category of bounded binary chain complexes in \mathcal{M} supported in non-negative degrees. The obvious definition of short exact sequences turns $B\mathcal{M}$ into an exact category. The exact full subcategory consisting of binary complexes M in $B\mathcal{M}$ such that both $\top M$ and $\perp M$ belong to $C^q\mathcal{M}$ will be denoted $B^q\mathcal{M}$.

When applied to $n = 1$, Corollary 7.4 in [Gra12], the main outcome of [Gra12], shows that we have a canonical isomorphism

$$K_1\mathcal{M} \cong \text{coker}(\Delta: K_0(C^q\mathcal{M}) \rightarrow K_0(B^q\mathcal{M})). \quad (2)$$

Throughout this paper, we will take this as the definition of $K_1\mathcal{M}$.

Let $F: \mathcal{M} \rightarrow \mathcal{N}$ be an exact functor between exact categories. We recall the following crucial definitions from [Gra16, Section 1].

Definition 1.1.

- (a) Let $B[F]$ denote the exact category whose objects are triples of the form (M, N, u) , where M is an object in $C\mathcal{M}^2$ (i.e., a pair of objects in $C\mathcal{M}$), N is an object in $B\mathcal{N}$ and $u: FM \rightarrow \top N$ is a pair of quasi-isomorphisms of chain complexes in \mathcal{N} . A morphism

$$(M, N, u) \rightarrow (M', N', u')$$

in $B[F]$ is a pair of morphisms $\phi: M \rightarrow M'$ in $C\mathcal{M}^2$ and $\psi: N \rightarrow N'$ in $C\mathcal{N}^2$ such that the following diagram commutes:

$$\begin{array}{ccc} FM & \xrightarrow{F\phi} & FM' \\ \downarrow u & & \downarrow u' \\ \top N & \xrightarrow{\top\psi} & \top N'. \end{array}$$

The subcategory $pB[F]$ is the category containing every object of $B[F]$, and whose morphisms (ϕ, ψ) are such that ϕ is a quasi-isomorphism and ψ is an isomorphism.

- (b) Using triples (M, N, u) where both M and N are *ordinary* complexes, more precisely where $M \in C\mathcal{M}$, $N \in C\mathcal{N}$ and $u: FM \rightarrow N$ is a quasi-isomorphism, we similarly define the categories $C[F] \supset pC[F]$.
- (c) We extend the functors \top, \perp and Δ above to triples as in (a) and (b) by applying them componentwise (where $\top M$ and $\perp M$ for $M \in C\mathcal{M}^2$ are defined similarly as for $M \in B\mathcal{M}$). We also write τ for the endofunctor of $B[F]$ which swaps $\top X$ and $\perp X$ for $X \in B[F]$.
- (d) The *relative K_0 -group of F* is

$$K_0[F] := \text{coker}(\Delta: K_0pC[F] \rightarrow K_0pB[F]) \quad (3)$$

where, by abuse of notation, $K_0pC[F]$ (and similarly $K_0pB[F]$) is defined by dividing out the equivalence relation \sim on $K_0C[F]$ which is generated by $[(M, N, u)] \sim [(M', N', u')]$ whenever there is a morphism $[(M, N, u)] \rightarrow [(M', N', u')]$ in $pC[F]$.

Remark 1.2. The following two facts justify calling $K_0[F]$ the relative K_0 -group.

- (a) By [Gra16, Corollary 1.9], there is an exact sequence

$$K_1\mathcal{M} \longrightarrow K_1\mathcal{N} \longrightarrow K_0[F] \longrightarrow K_0\mathcal{M} \longrightarrow K_0\mathcal{N} \quad (4)$$

where $K_1\mathcal{M} \rightarrow K_1\mathcal{N}$ and $K_0\mathcal{M} \rightarrow K_0\mathcal{N}$ are induced by F , the homomorphism $K_1\mathcal{N} \rightarrow K_0[F]$ is induced by the fully faithful inclusion functor

$$B^q\mathcal{N} \hookrightarrow B[F], \quad N \mapsto (0, N, 0),$$

and $K_0[F] \rightarrow K_0\mathcal{M}$ is given by

$$[(M, N, u)] \mapsto [\top M] - [\perp M]$$

after identifying $K_0\mathcal{M}$ with $K_0qC\mathcal{M}$, the Grothendieck group of $C\mathcal{M}$ modulo the relations given by quasi-isomorphisms.

(b) By [Tu25, Theorem 2.1], there is a canonical isomorphism

$$K_0[F] \cong \pi_0 \text{hofib}(KF: K\mathcal{M} \rightarrow K\mathcal{N})$$

between $K_0[F]$ and π_0 of the homotopy fibre of the induced map KF between the K -theory spaces $K\mathcal{M}$ and $K\mathcal{N}$ of \mathcal{M} and \mathcal{N} .

As a further justification for calling $K_0[F]$ the relative K_0 -group, we will show over the next pages that $K_0[F]$ is isomorphic to Bass's relative K_0 -group $K_0^B[F]$ in the following standard situation for Bass K -theory: \mathcal{M} and \mathcal{N} are split exact categories (i.e., all short exact sequences in \mathcal{M} and \mathcal{N} split) and $F: \mathcal{M} \rightarrow \mathcal{N}$ is a cofinal exact functor, i.e, for every object N in \mathcal{N} , there are objects N' in \mathcal{N} and M in \mathcal{M} such that $N \oplus N' \cong FM$.

We start by recalling the definition of Bass's K_1 -group $K_1^B\mathcal{M}$ from [Bas68, Chapter VII, §1]. Let $\text{Aut}(\mathcal{M})$ denote the exact category of pairs (P, α) , where P is an object of \mathcal{M} and $\alpha: P \rightarrow P$ is an automorphism. A morphism $(P, \alpha) \rightarrow (P', \alpha')$ in $\text{Aut}(\mathcal{M})$ is a morphism $\phi: P \rightarrow P'$ such that $\phi \circ \alpha = \alpha' \circ \phi$. Bass's K_1 -group $K_1^B(\mathcal{M})$ is then defined as the quotient of the Grothendieck group $K_0 \text{Aut}(\mathcal{M})$ by the subgroup generated by elements of the form

$$[P, \beta \circ \alpha] - [P, \beta] - [P, \alpha]$$

whenever there is an object P of \mathcal{M} with automorphisms $\alpha, \beta: P \rightarrow P$. Let (P, α, id) denote the binary complex concentrated in degrees 0 and 1 whose object in degrees 0 and 1 is P , whose top bottom differential is α and whose bottom differential is id_P ; let's call such a binary complex a *binary automorphism*. It is shown in [Har15, Theorem 2.24] that mapping an object $(P, \alpha) \in \text{Aut}(\mathcal{M})$ to the binary automorphism (P, α, id) induces a well-defined isomorphism

$$K_1^B\mathcal{M} \xrightarrow{\sim} K_1\mathcal{M} \tag{5}$$

if \mathcal{M} is split exact.

Next, we recall the definition of Bass's relative K_0 -group (see [Bas68, Chapter VII, §5]). Let $\text{co}(F)$ denote the exact category whose objects are triples of the form (P, α, Q) , where $P, Q \in \mathcal{M}$ and $\alpha: FP \rightarrow FQ$ is an isomorphism in \mathcal{N} . Furthermore, a morphism $(P, \alpha, Q) \rightarrow (P', \alpha', Q')$ consists of two morphisms $f: P \rightarrow P'$ and $g: Q \rightarrow Q'$ in \mathcal{M} such that $Fg \circ \alpha = \alpha' \circ Ff$. Bass's relative K_0 -group $K_0^B[F]$ is then defined as the quotient of the Grothendieck group $K_0\text{co}(F)$ by the subgroup generated by elements of the form

$$[P, \beta\alpha, R] - [P, \alpha, Q] - [Q, \beta, R],$$

whenever there are objects P, Q, R in \mathcal{M} with isomorphisms $\alpha : FP \rightarrow FQ$ and $\beta : FQ \rightarrow FR$.

We define the functor

$$\Psi : \text{co}(F) \rightarrow B[F], \quad (P, \alpha, Q) \mapsto ((\overset{P}{Q}), FQ, (\overset{\alpha}{1})), \quad (6)$$

where $(\overset{P}{Q})$ is considered as a pair of complexes in \mathcal{M} supported in degree 0, and FQ is considered as a binary complex in \mathcal{N} supported in degree 0; the functor Ψ acts on morphisms in the obvious way.

Theorem 1.3. *The functor $\Psi : \text{co}(F) \rightarrow B[F]$ induces a well-defined homomorphism*

$$\Phi : K_0^B[F] \rightarrow K_0[F].$$

If \mathcal{M} and \mathcal{N} are split exact and F is cofinal, then Φ is an isomorphism.

In the following computations we will be writing out objects of $B[F]$ supported in degrees 0 and 1 as follows:

$$((A_1 \xrightarrow{\alpha} A_0, B_1 \xrightarrow{\beta} B_0), (N_1 \xrightarrow[d']{d} N_0), ((u_1; u_0), (v_1; v_0)))$$

where we write the quasi-isomorphisms in an order that agrees with the ordering of the objects in the complexes.

Before we begin with the proof of Theorem 1.3 we prove the following useful lemma.

Lemma 1.4. *Given $(P, \alpha, Q) \in \text{co}(F)$, we have the equality*

$$[\Psi(P, \alpha, Q)] = [(0 \xrightarrow{0} 0, P \xrightarrow{0} Q), (FP \xrightarrow[0]{\alpha} FQ), ((0; 0), (1; 1))]$$

in $K_0[F]$.

Proof. We have the following short exact sequence in $B[F]$ obtained by naïvely filtrating the middle triple:

$$\begin{array}{ccccc} ((0 \xrightarrow{0} P, 0 \xrightarrow{0} Q), (0 \xrightarrow[0]{0} FQ), ((0; \alpha), (0; 1))) & & & & \\ \downarrow 0 & \downarrow 1 & \downarrow 0 & \downarrow 1 & \downarrow 0 \\ ((P \xrightarrow{1} P, P \xrightarrow{0} Q), (FP \xrightarrow[0]{\alpha} FQ), ((1; \alpha), (1; 1))) & & & & \\ \downarrow 1 & \downarrow 0 & \downarrow 1 & \downarrow 0 & \downarrow 0 \\ ((P \xrightarrow{0} 0, P \xrightarrow{0} 0), (FP \xrightarrow[0]{0} 0), ((1; 0), (1; 0))) & & & & \end{array}$$

One easily verifies that these maps are indeed morphisms in $B[F]$. Since the bottom triple is in the image of Δ , we have that $[\Psi(P, \alpha, Q)]$ is equal to the middle triple in $K_0[F]$. To show the required identity, notice that the following arrow is in $pB[F]$:

$$\begin{array}{ccccc} ((0 \xrightarrow{0} 0, P \xrightarrow{0} Q), (FP \xrightarrow[0]{\alpha} FQ), ((0; 0), (1; 1))) & & & & \\ \downarrow 0 & \downarrow 0 & \downarrow 1 & \downarrow 1 & \downarrow 1 \\ ((P \xrightarrow{1} P, P \xrightarrow{0} Q), (FP \xrightarrow[0]{\alpha} FQ), ((1; \alpha), (1; 1))). & & & & \end{array}$$

□

Proof (of Theorem 1.3). As the functor Ψ is obviously exact, it induces a homomorphism $K_0\text{co}(F) \rightarrow K_0B[F]$. So, in order to show that Ψ induces a well-defined homomorphism $\Phi: K_0^B[F] \rightarrow K_0[F]$ it remains to check for $P, Q, R \in \mathcal{M}$ and isomorphisms $\alpha: FP \rightarrow FQ, \beta: FQ \rightarrow FR$ in \mathcal{N} that

$$[\Psi(P, \beta \circ \alpha, R)] = [\Psi(P, \alpha, Q)] + [\Psi(Q, \beta, R)] \quad (7)$$

in $K_0[F]$. To see this we first apply Lemma 1.4 to the object (Q, β, R) and notice that the following morphism is in $pB[F]$:

$$\begin{array}{ccccccc} ((0 \xrightarrow{0} 0, Q \xrightarrow{0} R), (FQ \xrightarrow[\beta]{0} FR), ((0; 0), (1; 1))) \\ \downarrow 0 \quad \downarrow 0 \downarrow 1 \quad \downarrow 1 \quad \downarrow 1 \quad \downarrow 1 \\ ((P \xrightarrow{1} P, Q \xrightarrow{0} R), (FQ \xrightarrow[\beta]{0} FR), ((\alpha; \beta \circ \alpha), (1; 1))). \end{array}$$

Now we naïvely filtrate the bottom triple above and obtain the following short exact sequence in $B[F]$:

$$\begin{array}{ccccccc} ((0 \xrightarrow{0} P, 0 \xrightarrow{0} R), (0 \xrightarrow[\beta]{0} FR), ((0; \beta \circ \alpha), (0; 1))) \\ \downarrow 0 \quad \downarrow 1 \downarrow 0 \quad \downarrow 1 \quad \downarrow 0 \quad \downarrow 1 \\ ((P \xrightarrow{1} P, Q \xrightarrow{0} R), (FQ \xrightarrow[\beta]{0} FR), ((\alpha; \beta \circ \alpha), (1; 1))) \\ \downarrow 1 \quad \downarrow 0 \downarrow 1 \quad \downarrow 0 \quad \downarrow 1 \quad \downarrow 0 \\ ((P \xrightarrow{0} 0, Q \xrightarrow{0} 0), (FQ \xrightarrow[\beta]{0} 0), ((\alpha; 0), (1; 0))) \end{array}$$

We therefore have the following equalities in $K_0[F]$:

$$\begin{aligned} [\Psi(Q, \beta, R)] &= [\Psi(P, \beta \circ \alpha, R)] + [\Psi(P, \alpha, Q)[1]] \\ &= [\Psi(P, \beta \circ \alpha, R)] - [\Psi(P, \alpha, Q)] \end{aligned}$$

where $[1]$ denotes the obvious shift functor and the second equality follows from [Gra16, Lemma 6.1]. Thus Equation (7) holds and the homomorphism Φ is defined.

We now assume that \mathcal{M} and \mathcal{N} are split exact and that F is cofinal and show that Φ is an isomorphism. To this end, we consider the diagram

$$\begin{array}{ccccccc} K_1^B \mathcal{M} & \longrightarrow & K_1^B \mathcal{N} & \longrightarrow & K_0^B[F] & \longrightarrow & K_0 \mathcal{M} \longrightarrow K_0 \mathcal{N} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \Phi & & \parallel \quad \parallel \\ K_1 \mathcal{M} & \longrightarrow & K_1 \mathcal{N} & \longrightarrow & K_0[F] & \longrightarrow & K_0 \mathcal{M} \longrightarrow K_0 \mathcal{N} \end{array}$$

which is defined as follows. The bottom row is the exact sequence (4). The top row is the exact sequence introduced in [Bas68, Theorem VII.5.3]. The two left-hand vertical isomorphisms are given by (5). In order to show that Φ is an isomorphism, by the five lemma, it suffices to show the two squares involving Φ commute. We recall the homomorphism $K_0^B[F] \rightarrow K_0 \mathcal{M}$ is given by $[P, \alpha, Q] \mapsto [P] - [Q]$, so the third square commutes by definition. Furthermore, the homomorphism $K_1^B \mathcal{N} \rightarrow K_0^B[F]$ is defined as follows: given an object (N, α) of $\text{Aut } \mathcal{N}$, choose objects N' in \mathcal{N} and M in \mathcal{M} such that $N \oplus N' \cong FM$, so that

$$[N, \alpha] = [N \oplus N', \alpha \oplus 1] = [FM, \tilde{\alpha}],$$

in $K_1^B \mathcal{N}$, where $\tilde{\alpha}$ is the automorphism of FM induced by $\alpha \oplus 1$; the image of this element in $K_0^B[F]$ is then $[M, \tilde{\alpha}, M]$. The homomorphism Φ maps this element to $[(\frac{M}{M}), FM, (\frac{\tilde{\alpha}}{1})]$ in $K_0[F]$. We now note we have the following exact sequence in $B[F]$ arising from naïvely filtrating the middle object:

$$\begin{array}{ccccccc}
((0 \xrightarrow{0} M, 0 \xrightarrow{0} M), (0 \xrightarrow[0]{0} FM), ((0; \tilde{\alpha}), (0; 1))) & & & & & & \\
\downarrow 0 & \downarrow 1 & \downarrow 0 & \downarrow 1 & \downarrow 0 & \downarrow 1 & \\
((M \xrightarrow{1} M, M \xrightarrow{1} M), (FM \xrightarrow[1]{\tilde{\alpha}} FM), ((1; \tilde{\alpha}), (1; 1))) & & & & & & \\
\downarrow 1 & \downarrow 0 & \downarrow 1 & \downarrow 0 & \downarrow 1 & \downarrow 0 & \\
((M \xrightarrow{0} 0, M \xrightarrow{0} 0), (FM \xrightarrow[0]{0} 0), ((1; 0), (1; 0))) & & & & & &
\end{array}$$

Since the bottom row is diagonal the first two rows are equal in $K_0[F]$, and since the first component of the middle row is acyclic the obvious map to the middle row from the triple

$$((0 \longrightarrow 0, 0 \longrightarrow 0), (FM \xrightarrow[1]{\tilde{\alpha}} FM), ((0; 0), (0; 0)))$$

is in $pB[F]$, and thus the class of the latter triple in $K_0[F]$ is equal to the element corresponding to the top row of the above diagram, i.e., to $[(\frac{M}{M}), FM, (\frac{\tilde{\alpha}}{1})]$. On the other hand, this element is the image of the element $[FM, \tilde{\alpha}]$ when mapped first via the isomorphism $K_1^B(\mathcal{N}) \rightarrow K_1(\mathcal{N})$ given in (5) and then by the homomorphism $K_1(\mathcal{N}) \rightarrow K_0[F]$ as in (4). Hence the second square in the diagram above commutes as well, and the proof of Theorem 1.3 is complete. \square

We end this section with the following useful fact. By [Gra16, Lemma 6.1] we know that shifting the complexes in a triple $X \in B[F]$ by 1 changes the sign of its class $[X]$ in $K_0[F]$. The following proposition shows that the same holds when swapping top and bottom in X ; this is similar to [Gra16, Corollary 8.7] where this is proved for $K_1 \mathcal{N}$.

Proposition 1.5. *For $X \in B[F]$ we have $[\tau X] = -[X]$ in $K_0[F]$.*

Proof. We write $X = ((\frac{A}{B}), (N, \frac{d}{d'}), (\frac{u}{v}))$. Let s denote the endomorphism of $N \oplus N$ which swaps the two summands and consider the two complex homomorphisms $u \oplus v$ and $s \circ (u \oplus v)$ from $F(A) \oplus F(B)$ to $\top N \oplus \perp N$ and $\perp N \oplus \top N$, respectively. The triple $X \oplus \tau X$ is isomorphic to the triple $(\Delta(A \oplus B), (N \oplus N, \frac{d \oplus d'}{d' \oplus d}), (\frac{u \oplus v}{s \circ (u \oplus v)}))$. By the second half of the proof of [Gra16, Lemma 6.3], its class $[X \oplus \tau X]$ in $K_0[F]$ is therefore equal to the image of the class $[\text{Cone}(\frac{u \oplus v}{s \circ (u \oplus v)})] \in K_1 \mathcal{N}$ under the map $K_1 \mathcal{N} \rightarrow K_0[F]$ from (4); here, $\text{Cone}(\frac{u \oplus v}{s \circ (u \oplus v)})$ is the binary complex with graded object $FA[1] \oplus FB[1] \oplus N \oplus N$ and which shares its differentials with $\text{cone}(u \oplus v)$ and $\text{cone}(s \circ (u \oplus v))$, respectively. Generalising the proof of [KKW20, Lemma 3.2], we will show in the next paragraph that this element lies in the image of the map $K_1 \mathcal{M} \rightarrow K_1 \mathcal{N}$ induced by F ; then, by the exactness of (4), the class $[X \oplus \tau X]$ in $K_0[F]$ vanishes and hence $[\tau X] = -[X]$, as

claimed.

By definition we have the binary ladder

$$\left(\text{Cone} \begin{pmatrix} u \oplus v \\ s \circ (u \oplus v) \end{pmatrix}, \text{Cone} \begin{pmatrix} u \oplus v \\ u \oplus v \end{pmatrix}, 1_{FA \oplus FB} \oplus 1_{\top N \oplus \perp N}, 1_{FA \oplus FB} \oplus s \right)$$

in the sense of [KKW20, Definition 2.2(a)], i.e., the third component of this quadruple is an isomorphism between the top parts of the binary complexes in the first two components and the fourth component is an isomorphism between the bottom parts. [KKW20, Lemma 3.3] then yields the second in the following chain of equalities in $K_1\mathcal{N}$:

$$\begin{aligned} & \left[\text{Cone} \begin{pmatrix} u \oplus v \\ s \circ (u \oplus v) \end{pmatrix} \right] = \left[\text{Cone} \begin{pmatrix} u \oplus v \\ s \circ (u \oplus v) \end{pmatrix} \right] - \left[\text{Cone} \begin{pmatrix} u \oplus v \\ u \oplus v \end{pmatrix} \right] \\ &= \sum_{i=0}^{\infty} (-1)^i \left[FA_{i-1} \oplus FB_{i-1} \oplus N_i \oplus N_i \xrightarrow[1 \oplus s]{1 \oplus 1} FA_{i-1} \oplus FB_{i-1} \oplus N_i \oplus N_i \right] \\ &= \sum_{i=0}^{\infty} (-1)^i \left[N_i \oplus N_i \xrightarrow{s} N_i \oplus N_i \right]; \end{aligned} \quad (8)$$

note that the binary complexes $FA_{i-1} \oplus FB_{i-1} \xrightarrow[1]{1} FA_{i-1} \oplus FB_{i-1}$ are diagonal, which shows the third equality above. As u is a quasi-isomorphism, the Euler characteristic of $\text{cone}(u)$ vanishes; in other words, the Euler characteristic of N is the same as of FA . Hence the same is true for their images under the homomorphism $K_0\mathcal{N} \rightarrow K_1\mathcal{N}$ induced by the exact functor $P \mapsto \left(P \oplus P \xrightarrow{s} P \oplus P \right)$. This finally proves that the alternating sum (8) is equal to

$$\sum_{i=0}^{\infty} (-1)^i \left[FA_i \oplus FA_i \xrightarrow{s} FA_i \oplus FA_i \right]$$

which is obviously in the image of the map $K_1\mathcal{M} \rightarrow K_1\mathcal{N}$, as we wanted to show. \square

2. PRODUCTS ON RELATIVE K_0

The object of this section is to turn Grayson's relative K_0 -group into a (non-unital) commutative ring. To this end, we extend the definition of so-called simplicial tensor products of complexes [HKT17] to generating triples of the relative K_0 -group and then use a certain linear combination of these tensor products to define products in the relative K_0 -group so that also the outgoing map in the associated exact sequence of K -groups becomes a ring homomorphism. We end by briefly explaining the corresponding constructions and properties for Bass K -theory.

We start by recalling the Dold-Kan correspondence. Let Δ denote the category whose objects are the totally ordered sets

$$[n] = \{0 < 1 < \dots < n-1 < n\}, \quad n \in \mathbb{N},$$

and whose morphisms are the non-decreasing functions. (We hope denoting both this category and the diagonal functor introduced in Section 1 by Δ won't lead to any confusion.) Given any category \mathcal{A} , a *simplicial object*

in \mathcal{A} is a functor $\Delta^{\text{op}} \rightarrow \mathcal{A}$. The category of such simplicial objects in \mathcal{A} is denoted by $\mathcal{A}^{\Delta^{\text{op}}}$. For $X \in \mathcal{A}^{\Delta^{\text{op}}}$, we denote the image of $[n]$ by X_n . If \mathcal{A} is an idempotent complete additive category, the Dold-Kan correspondence [Lur17, Theorem 1.2.3.7] gives us an equivalence of categories

$$\mathcal{A}^{\Delta^{\text{op}}} \xrightleftharpoons[\Gamma]{N} C^\infty \mathcal{A}$$

between $\mathcal{A}^{\Delta^{\text{op}}}$ and the category $C^\infty \mathcal{A}$ of chain complexes in \mathcal{A} supported in non-negative degrees. Here, as usual, N denotes the normalized chain complex (also called Moore complex) functor.

We now start to define products on relative K_0 -groups. So, let \mathcal{M} be an idempotent complete exact category equipped with a bi-exact functor $\otimes : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$, called a *tensor product*, which we assume to be associative and commutative in the usual sense. Furthermore, we assume we have a distinguished object I of \mathcal{M} which behaves as a “multiplicative identity” in the obvious way. We recall from [HKT17, Definition 5.4], given two chain complexes A and B in $C\mathcal{M}$, the *simplicial tensor product* of A and B is defined as follows:

$$A \otimes_\Delta B := N(\text{diag}(\Gamma A \otimes \Gamma B))$$

where $\text{diag}(\Gamma A \otimes \Gamma B) \in \mathcal{M}^{\Delta^{\text{op}}}$ is given by $\text{diag}(\Gamma A \otimes \Gamma B)_n := (\Gamma A)_n \otimes (\Gamma B)_n$ (i.e., the diagonal of the bi-simplicial object $\Gamma A \otimes \Gamma B$). Note that, by [HKT17, Lemma 5.6], $A \otimes_\Delta B$ is bounded again, i.e., an object of $C\mathcal{M}$. We extend this definition to binary chain complexes by viewing them as objects in $C\mathcal{M}^2$ and applying the simplicial tensor product componentwise. The resulting pair of chain complexes is then a binary chain complex again.

Note, by the Eilenberg-Zilber Theorem [DP61, Satz 2.15], the simplicial tensor product $A \otimes_\Delta B$ is homotopy equivalent to the total complex of the double complex $A \otimes B$. We choose to work with the former product because the former is more directly compatible with the exterior powers used in the next section.

We now assume we are given two idempotent complete exact categories \mathcal{M} and \mathcal{N} with tensor products and with identities $I_{\mathcal{M}}$ and $I_{\mathcal{N}}$, respectively, and let $F : \mathcal{M} \rightarrow \mathcal{N}$ be an exact functor which respects tensor products in the obvious way, and such that $FI_{\mathcal{M}} = I_{\mathcal{N}}$.

Definition 2.1. Given two objects $X = (M, N, u)$ and $Y = (M', N', u')$ in $C[F]$ or in $B[F]$, we define their *simplicial tensor product* as follows:

$$X \otimes_\Delta Y := (M \otimes_\Delta M', N \otimes_\Delta N', u \otimes_\Delta u').$$

Note that, by the Eilenberg-Zilber Theorem [DP61, Satz 2.15], the simplicial tensor product $u \otimes_\Delta u'$ is a quasi-isomorphism again.

Lemma 2.2. *The simplicial tensor products on $C[F]$ and $B[F]$ induce well-defined products on $K_0 C[F]$, $K_0 B[F]$, $K_0 pC[F]$ and $K_0 pB[F]$ which turn these Grothendieck groups into commutative rings with unity.*

Proof. It is obvious that the simplicial tensor product induces well-defined products on $K_0 C[F]$ and $K_0 B[F]$ and that the class of the element $[I_{\mathcal{M}}, I_{\mathcal{N}}, 1]$ is an identity element, where $I_{\mathcal{M}}$ and $I_{\mathcal{N}}$ are considered as (ordinary or binary pairs of) complexes concentrated in degree zero. The Eilenberg-Zilber

theorem [DP61, Satz 2.15] implies that these ring structures induce ring structures on $K_0pC[F]$ and $K_0pB[F]$. \square

Note that the simplicial tensor product on $K_0pB[F]$ does not directly induce a product on the quotient $K_0[F]$ because the image of the ring homomorphism $\Delta: K_0pC[F] \rightarrow K_0pB[F]$ contains 1 and is hence not an ideal. Nonetheless, we have a natural product on $K_0[F]$ which we define now. We remember that the diagonal functor Δ is split by \perp (or \top) and that therefore the obvious homomorphism

$$\ker(\perp: K_0pB[F] \rightarrow K_0pC[F]) \longrightarrow \operatorname{coker}(\Delta: K_0pC[F] \rightarrow K_0pB[F]) \quad (9)$$

is bijective. We accordingly identify $K_0[F]$, i.e., the target of this homomorphism, with its source which is an ideal in $K_0pB[F]$ and which in particular inherits a well-defined product. This is the product structure on $K_0[F]$ (without 1) we will be considering. In other words:

Definition 2.3. Given $x, y \in K_0[F]$, we choose representatives \tilde{x}, \tilde{y} in $K_0pB[F]$ and define the *product* xy as the class of $(\tilde{x} - \Delta\perp\tilde{x})(\tilde{y} - \Delta\perp\tilde{y})$ in $K_0[F]$.

If x, y are the classes of triples $X, Y \in B[F]$, then, by Proposition 1.5, their product xy in $K_0[F]$ can be written as the class of a single triple as well:

$$xy = [(X \otimes_{\Delta} Y) \oplus (\tau X \otimes_{\Delta} \Delta\perp Y) \oplus (\Delta\perp X \otimes_{\Delta} \tau Y)].$$

(Beware, although $[\Delta\perp X] = 0$ in $K_0[F]$, the class of $(\Delta\perp X) \otimes_{\Delta} \tau Y$ does not vanish in general.)

Proposition 2.4. *We equip $K_0\mathcal{M}$ with the usual product and $K_1\mathcal{N}$ with the trivial product. Then these products are compatible with the just defined product on $K_0[F]$ via the exact sequence (4).*

Proof. We first show that the outgoing homomorphism $K_0[F] \rightarrow K_0\mathcal{M}$ is compatible with products. Let $x, x' \in K_0[F]$ be represented by $X = (M, N, u)$ and $X' = (M', N', u')$ in $B[F]$. Then the product xx' which is the class of

$$X \otimes_{\Delta} X' - (\Delta\perp X) \otimes_{\Delta} X' - X \otimes_{\Delta} (\Delta\perp X') + (\Delta\perp X) \otimes_{\Delta} (\Delta\perp X')$$

in $K_0[F]$ is mapped to

$$\begin{aligned} & (\top M \otimes_{\Delta} \top M' - \perp M \otimes_{\Delta} \perp M') - (\perp M \otimes_{\Delta} \top M' - \perp M \otimes_{\Delta} \perp M') \\ & - (\top M \otimes_{\Delta} \perp M' - \perp M \otimes_{\Delta} \perp M') + (\perp M \otimes_{\Delta} \top M' - \perp M \otimes_{\Delta} \perp M') \\ & = \top M \otimes_{\Delta} \top M' - \perp M \otimes_{\Delta} \top M' - \top M \otimes_{\Delta} \perp M' + \perp M \otimes_{\Delta} \perp M' \\ & = (\top M - \perp M) \otimes_{\Delta} (\top M' - \perp M') \end{aligned}$$

which in turn is equal to the product of the images of x and y in $K_0\mathcal{M}$; note that the product on $K_0qC\mathcal{M}$ given by simplicial tensor products is compatible with the usual product on $K_0\mathcal{M}$ via the canonical isomorphism $K_0\mathcal{M} \rightarrow K_0qC\mathcal{M}$.

In order to show that the incoming homomorphism $K_1\mathcal{N} \rightarrow K_0[F]$ is compatible with products, we recall from [HKT17, Proposition 5.11] that the class $[N \otimes_{\Delta} N']$ in $K_1\mathcal{N}$ of the simplicial tensor product of any two binary complexes N, N' in $B^q\mathcal{N}$ vanishes. In particular the trivial product

on $K_1\mathcal{N}$ can be interpreted as the product induced by the simplicial tensor product or, even more complicatedly, by the product defined analogously to the product in Definition 2.3. The latter product is obviously compatible with the product in $K_0[F]$ via the incoming homomorphism. \square

Remark 2.5. Similarly to above, one easily shows that Proposition 2.4 is also true if we both replace \perp with \top in Definition 2.3 and the outgoing homomorphism $K_0[F] \rightarrow K_0\mathcal{M}$ in (4) with its negative.

Remark 2.6. We recall the following folklore statements for Bass's K_1 .

- (a) Given two objects (P, α) and (Q, β) of $\text{Aut}(\mathcal{M})$ we define their *tensor product* as follows:

$$(P, \alpha) \otimes (Q, \beta) := (P \otimes Q, \alpha \otimes \beta)$$

This turns $K_0\text{Aut}(\mathcal{M})$ into a commutative ring with unity.

- (b) Let

$$\tilde{K}_0\text{Aut}(\mathcal{M}) := \ker(K_0\perp: K_0\text{Aut}(\mathcal{M}) \rightarrow K_0\mathcal{M})$$

where $K_0\perp$ is induced by the functor $\perp: (P, \alpha) \mapsto P$ and let \tilde{I} denote the subgroup of $\tilde{K}_0\text{Aut}(\mathcal{M})$ generated by elements of the form

$$[P, \beta \circ \alpha] - [P, \alpha] - [P, \beta] + [P, 1]$$

whenever there is an object P of \mathcal{M} with automorphisms α, β . Similarly to the considerations before [Kö00, Lemma 1.2] we see that \tilde{I} is an ideal of $K_0\text{Aut}(\mathcal{M})$. Furthermore, we have the isomorphism

$$K_0\text{Aut}(\mathcal{M})/\tilde{I} \cong K_0\mathcal{M} \oplus K_1^{\text{B}}(\mathcal{M}).$$

The resulting product xy for $x, y \in K_0\mathcal{M} \oplus K_1^{\text{B}}(\mathcal{M})$ is the usual one if $x, y \in K_0\mathcal{M}$, is trivial if $x, y \in K_1^{\text{B}}(\mathcal{M})$ and defines a $K_0\mathcal{M}$ -module structure on $K_1^{\text{B}}(\mathcal{M})$ if $x \in K_0\mathcal{M}$ and $y \in K_1^{\text{B}}(\mathcal{M})$.

Remark 2.7. The following statements concerning Bass's relative K_0 -group are easy to prove (similarly to above, see also the considerations right before [Kö00, Lemma 1.3]):

- (a) Let $X := (P, \alpha, Q)$ and $Y := (R, \beta, S)$ be objects in $\text{co}(F)$. Their tensor product in $\text{co}(F)$ is defined by

$$X \otimes Y := (P \otimes R, \alpha \otimes \beta, Q \otimes S).$$

This turns $K_0\text{co}(F)$ into a commutative ring with unity.

- (b) Let

$$\tilde{K}_0\text{co}(F) := \ker(K_0\perp: K_0\text{co}(F) \rightarrow K_0\mathcal{M})$$

where $K_0\perp$ is induced by the functor $\perp: (P, \alpha, Q) \mapsto Q$ and let \tilde{J} denote the subgroup of $\tilde{K}_0\text{co}(F)$ generated by elements of the form

$$[P, \beta \circ \alpha, R] - [P, \alpha, Q] - [Q, \beta, R] + [Q, 1, Q]$$

whenever there are objects P, Q, R in \mathcal{M} and isomorphisms $\alpha: FP \rightarrow FQ$ and $\beta: FQ \rightarrow FR$ in \mathcal{N} . Then \tilde{J} is an ideal of $K_0\text{co}(F)$ and we have the isomorphism

$$K_0\text{co}(F)/\tilde{J} \cong K_0\mathcal{M} \oplus K_0^{\text{B}}[F]. \quad (10)$$

This isomorphism gives rise to a multiplication on $K_0^B[F]$ which, in contrast to the multiplication on $K_1^B\mathcal{M}$ (see Remark 2.6(b)), is in general not trivial.

- (c) This product on $K_0^B[F]$ is compatible with the trivial product on $K_1\mathcal{N}$ (see Remark 2.6(b)) and the usual product on $K_0\mathcal{M}$ via Bass's exact sequence [Bas68, Theorem VII.5.3] assuming \mathcal{M} and \mathcal{N} are split exact and F is cofinal.
- (d) The functor $\Psi: \text{co}(F) \rightarrow B[F]$ (see (6)) is compatible with tensor products. The induced homomorphism $\Phi: K_0^B[F] \rightarrow K_0[F]$ (see Theorem 1.3) is compatible with products.

3. POWER OPERATIONS ON RELATIVE K_0

In this section we equip Grayson's relative K_0 -group with exterior power operations. To this end, similarly to the previous section, we first extend the definition of exterior powers of complexes [HKT17] to generating triples of the relative K_0 -group and then use a combination of such exterior powers to define exterior power operations on the relative K_0 -group which are compatible with both the incoming and outgoing map in the associated exact sequence of K -groups. As in the previous section, we end by briefly explaining the corresponding constructions and properties for Bass K -theory.

We assume we are given an assembly of power operations as in [KZ25, Definition 1.1]. Recall this means we are given a sequence \mathcal{M}_n , $n \geq 0$, of exact categories (in most examples, all \mathcal{M}_n are equal to a single exact category \mathcal{M}), a bi-exact functor

$$\otimes: \mathcal{M}_n \times \mathcal{M}_p \rightarrow \mathcal{M}_{n+p} \quad \text{for each } n, p \geq 0,$$

and a functor

$$\text{Fil}_k(\mathcal{M}_n) \rightarrow \mathcal{M}_{nk} \quad \text{for each } k \geq 1, n \geq 0,$$

which maps an object of $\text{Fil}_k(\mathcal{M}_n)$, i.e., a sequence $V_1 \rightarrow \dots \rightarrow V_k$ of admissible monomorphisms in \mathcal{M}_n of length $k-1$, to an object $V_1 \wedge \dots \wedge V_k$ in \mathcal{M}_{nk} , and these data are subject to certain axioms. If all the $k-1$ monomorphisms are the identity on $V := V_1 = \dots = V_k$, we write $\bigwedge^k V$ for $V_1 \wedge \dots \wedge V_k$. A reader who prefers not to work in this axiomatic setup may just assume that the given assembly of power operations is the standard one where every \mathcal{M}_n is equal to the exact category $\mathcal{P}(X)$ of locally free \mathcal{O}_X -modules of finite rank on a quasi-compact scheme X , the functor \otimes is the usual tensor product and $V_1 \wedge \dots \wedge V_k$ is defined to be the image of the tensor product $V_1 \otimes \dots \otimes V_k$ in the k^{th} exterior power $\bigwedge^k V_k$.

We furthermore assume that all categories \mathcal{M}_n , $n \geq 0$, are idempotent complete. By [KZ25, Proposition 1.2], we obtain an assembly of power operations on the categories $C\mathcal{M}_n$, $n \geq 0$, as follows. Similarly to Section 2, the simplicial tensor product defines functors

$$\otimes_\Delta: C\mathcal{M}_n \times C\mathcal{M}_p \rightarrow C\mathcal{M}_{n+p} \quad n, p \geq 0;$$

in the same vein, mapping a sequence $V_1 \rightarrow \dots \rightarrow V_k$ in $\text{Fil}_k(C\mathcal{M}_n)$ to

$$V_1 \wedge \dots \wedge V_k := N(\Gamma V_1 \wedge \dots \wedge \Gamma V_k) \quad \text{in } C\mathcal{M}_{nk}$$

defines a functor

$$\mathrm{Fil}_k(\mathcal{CM}_n) \rightarrow \mathcal{CM}_{nk} \quad \text{for every } k \geq 1, n \geq 0.$$

Furthermore, according to [KZ25, Corollary 1.7], by applying the construction above to each of the two differentials in binary complexes we obtain an assembly of power operations on $B\mathcal{M}_n$, $n \geq 0$.

We now furthermore assume we are given two assemblies of power operations, on \mathcal{M}_n , $n \geq 0$, and on \mathcal{N}_n , $n \geq 0$, respectively, and exact functors $F_n: \mathcal{M}_n \rightarrow \mathcal{N}_n$, $n \geq 0$, which respect the given structures in the obvious sense. The main example we have in mind is the pull-back functor $f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ for a given morphism $f: X \rightarrow Y$ between quasi-compact schemes X and Y . We also assume that, for every quasi-isomorphism f in \mathcal{CM}_n and in \mathcal{CN}_n and for all $k \geq 1$, also $\bigwedge^k(f)$ is a quasi-isomorphism. This additional assumption is satisfied for example for the standard assembly of power operations on $\mathcal{P}(X)$, see [KZ25, Proposition 1.9].

Definition 3.1. Given $k \geq 1$ and $n \geq 0$ and a sequence $X_1 \rightarrow \dots \rightarrow X_k$ of admissible monomorphisms in $C[F_n]$ or $B[F_n]$, where $X_i = (M_i, N_i, u_i)$, we define the triple

$$X_1 \wedge \dots \wedge X_k := (M_1 \wedge \dots \wedge M_k, N_1 \wedge \dots \wedge N_k, u_1 \wedge \dots \wedge u_k)$$

in $C[F_{nk}]$ or $B[F_{nk}]$.

Note that, by [KZ25, Proposition 1.4], the morphism $u_1 \wedge \dots \wedge u_k$ is a quasi-isomorphism again.

Lemma 3.2. *The assignments*

$$(X, Y) \mapsto X \otimes_{\Delta} Y \quad \text{and} \quad (X_1 \rightarrow \dots \rightarrow X_k) \mapsto X_1 \wedge \dots \wedge X_k$$

define an assembly of power operations on the exact categories $C[F_n]$, $n \geq 0$, and $B[F_n]$, $n \geq 0$, which preserve the subcategories $pC[F_n]$ and $pB[F_n]$, respectively.

Proof. Straightforward. □

In particular we obtain power operations

$$\lambda^k: K_0 C[F_n] \rightarrow K_0 C[F_{nk}] \quad \text{and} \quad \lambda^k: K_0 B[F_n] \rightarrow K_0 B[F_{nk}]$$

and

$$\lambda^k: K_0 pC[F_n] \rightarrow K_0 pC[F_{nk}] \quad \text{and} \quad \lambda^k: K_0 pB[F_n] \rightarrow K_0 pB[F_{nk}].$$

As in Section 2, we employ the identification (9) to define power operations $\lambda^k: K_0[F_n] \rightarrow K_0[F_{nk}]$. More explicitly:

Definition 3.3. Given $x \in K_0[F_n]$, we choose a representative \tilde{x} of x in $K_0 pB[F_n]$ and define $\lambda^k(x)$ to be the class of $\lambda^k(\tilde{x} - \Delta \perp \tilde{x})$ in $K_0[F_{nk}]$.

If $x = [X]$ is the class of a triple X in $B[F_n]$, we can write $\lambda^k(x)$ as the class of a single triple in the following way.

Lemma 3.4. *Let $X = ((\begin{smallmatrix} A \\ B \end{smallmatrix}), (N, \begin{smallmatrix} d \\ d' \end{smallmatrix}), (\begin{smallmatrix} u \\ v \end{smallmatrix})) \in B[F_n]$. Then we have:*

$$\lambda^k([X]) = \left[\left(\left(\begin{smallmatrix} \Lambda_+^k(A, B) \\ \Lambda_-^k(A, B) \end{smallmatrix} \right), \left(\widehat{\Lambda}^k N, \begin{smallmatrix} \Lambda_+^k(d, d') \\ \Lambda_-^k(d, d') \end{smallmatrix} \right), \left(\begin{smallmatrix} \Lambda_+^k(u, v) \\ \Lambda_-^k(u, v) \end{smallmatrix} \right) \right) \right] \quad \text{in} \quad K_0[F_{nk}],$$

where Λ_+^k, Λ_-^k and $\widehat{\Lambda}^k$ are defined as follows:

$$\begin{aligned}\Lambda_+^k(A, B) &= \bigoplus_{\substack{a+b_1+\dots+b_u=k \\ u \text{ even} \\ a, u \geq 0, b_1, \dots, b_u \geq 1}} \Lambda^a(A) \otimes_{\Delta} \Lambda^{b_1}(B) \otimes \dots \otimes_{\Delta} \Lambda^{b_u}(B), \\ \Lambda_-^k(A, B) &= \bigoplus_{\substack{a+b_1+\dots+b_u=k \\ u \text{ odd} \\ a, u \geq 0, b_1, \dots, b_u \geq 1}} \Lambda^a(A) \otimes_{\Delta} \Lambda^{b_1}(B) \otimes \dots \otimes_{\Delta} \Lambda^{b_u}(B), \\ \widehat{\Lambda}^k N &= \bigoplus_{\substack{a+b_1+\dots+b_u=k \\ u \geq 0, a, b_1, \dots, b_u \geq 1}} \Lambda^a N \otimes_{\Delta} \Lambda^{b_1} N \otimes_{\Delta} \dots \otimes_{\Delta} \Lambda^{b_u} N.\end{aligned}$$

Note, in order to match up the index sets for $\Lambda_+^k(d, d')$ and $\widehat{\Lambda}^k N$ here, we consider any tuple (a, b_1, \dots, b_u) in $\Lambda_+^k(d, d')$ as the tuple $(\tilde{a}, \tilde{b}_1, \dots, \tilde{b}_{u-1})$ in $\widehat{\Lambda}^k N$ where $\tilde{a} = b_1, \tilde{b}_1 = b_2, \dots, \tilde{b}_{u-1} = b_u$ if $a = 0$, and as the tuple (a, b_1, \dots, b_u) if $a \geq 1$; similarly for the index sets of $\Lambda_-^k(d, d')$ and $\widehat{\Lambda}^k N$.

Proof. Using [Gra89, Equation 2.3] which computes λ^k of a difference, we obtain:

$$\begin{aligned}\lambda^k([X]) &= \lambda^k([X] - [\Delta \perp X]) \\ &= \sum_{\substack{a+b_1+\dots+b_u=k \\ a, u \geq 0, b_1, \dots, b_u \geq 1}} (-1)^u \lambda^a([X]) \lambda^{b_1}([\Delta \perp X]) \dots \lambda^{b_u}([\Delta \perp X]) \\ &= \left[\bigoplus_{\substack{a+b_1+\dots+b_u=k \\ u \geq 0, a, b_1, \dots, b_u \geq 1}} \Lambda^a(\tau^u X) \otimes_{\Delta} \Lambda^{b_1}(\Delta \perp X) \otimes_{\Delta} \dots \otimes_{\Delta} \Lambda^{b_u}(\Delta \perp X) \right]\end{aligned}$$

To see the latter equality, we use the fact that τ commutes with Λ^a and amounts to a minus sign in $K_0^B[F_{nk}]$ according to Proposition 1.5 and we notice that the terms for $a = 0$ vanish in our context. Finally, to see that the term obtained is indeed equal to $\left[\left(\left(\Lambda_+^k(A, B) \right), \left(\widehat{\Lambda}^k N, \Lambda_+^k(d, d') \right), \left(\Lambda_+^k(u, v) \right) \right), \left(\left(\Lambda_-^k(A, B) \right), \left(\widehat{\Lambda}^k N, \Lambda_-^k(d, d') \right), \left(\Lambda_-^k(u, v) \right) \right) \right]$, we split the index set into two halves according to whether u is even or odd and then, for the top components in the triples, we interpret any index where $(a \geq 1, u \text{ odd})$ as an index where $(a = 0, u \text{ even})$, and similarly for the bottom components. \square

Proposition 3.5. *Let $k \geq 1$. Via the exact sequence (4), the just defined power operation $\lambda^k: K_0[F_n] \rightarrow K_0[F_{nk}]$ is compatible with the power operation $\lambda^k: K_0\mathcal{M}_n \rightarrow K_0\mathcal{M}_{nk}$ and $\lambda^k: K_1\mathcal{N}_n \rightarrow K_1\mathcal{N}_{nk}$ resulting from the given assemblies of power operations on \mathcal{M}_n and \mathcal{N}_n .*

Proof. By Proposition 1.5, any $x \in K_0[F_n]$ can be represented by a triple $X \in B[F_n]$. We use the same notation for the components of X as in Lemma 3.4. This lemma then shows that the difference in $K_0qC\mathcal{M}_{nk}$ of the classes of the top and bottom complexes in the first component of $\lambda^k([X])$ is equal to $[\Lambda_+^k(A, B)] - [\Lambda_-^k(A, B)]$ which in turn is equal to $\lambda^k([A] - [B])$, again by [Gra89, Equation 2.3]; note λ^k here denotes the power operation resulting from the assembly of power operations on $C\mathcal{M}_n$, $n \geq 0$ (see above) which

in turn is obviously compatible with the power operation $\lambda^k: K_0\mathcal{M}_n \rightarrow K_0\mathcal{M}_{nk}$ via the canonical isomorphism $K_0\mathcal{M}_{nk} \rightarrow K_0qC\mathcal{M}_{nk}$. This shows the first statement.

To prove the second statement, let $x \in K_1\mathcal{N}_n$ be represented by $N \in B^q\mathcal{N}_n$. Then, by definition of λ^k on K_1 (see [KZ25, (3.8)]) we have in $K_1\mathcal{N}_{nk}$:

$$\begin{aligned} \lambda^k(x) &= [\Lambda^k(N)] \\ &= \sum_{\substack{a+b_1+\dots+b_u=k \\ a,u \geq 0, b_1, \dots, b_u \geq 1}} (-1)^u [\Lambda^a(N) \otimes_{\Delta} \Lambda^{b_1}(\Delta \perp N) \otimes_{\Delta} \dots \otimes_{\Delta} \Lambda^{b_u}(\Delta \perp N)] \end{aligned}$$

where all summands for $u \neq 0$ vanish by [HKT17, Proposition 5.11]. This element is mapped to the element

$$\sum_{\substack{a+b_1+\dots+b_u=k \\ a,u \geq 0, b_1, \dots, b_u \geq 1}} (-1)^u [(0, \Lambda^a(N) \otimes_{\Delta} \Lambda^{b_1}(\Delta \perp N) \otimes_{\Delta} \dots \otimes_{\Delta} \Lambda^{b_u}(\Delta \perp N), 0)]$$

in $K_0[F_{nk}]$ which in turn is equal to

$$\lambda^k\left([(0, N, 0)] - [(0, \Delta \perp N, 0)]\right)$$

again by [Gra89, Equation 2.3]. This proves the second statement and hence finishes the proof of Proposition 3.5. \square

Remark 3.6. Rather than using the somewhat combinatorially involved Lemma 3.4, one could alternatively argue as follows in the first half of the proof above.

Using [Gra89, Equation 2.3], we obtain

$$\begin{aligned} \lambda^k([X] - [\Delta \perp X]) &= \sum_{\substack{a+b_1+\dots+b_u=k \\ a,u \geq 0, b_1, \dots, b_u \geq 1}} (-1)^u \lambda^a[X] \lambda^{b_1}[\Delta \perp X] \dots \lambda^{b_u}[\Delta \perp X] \\ &= \sum_{\substack{a+b_1+\dots+b_u=k \\ a,u \geq 0, b_1, \dots, b_u \geq 1}} (-1)^u [\Lambda^a(X) \otimes_{\Delta} \Lambda^{b_1}(\Delta \perp X) \otimes_{\Delta} \dots \otimes_{\Delta} \Lambda^{b_u}(\Delta \perp X)] \end{aligned}$$

in $K_0B[F_{nk}]$. The image in $K_0qC\mathcal{M}_{nk}$ of this element's class in $K_0[F_{nk}]$ is

$$\begin{aligned} &\sum_{\substack{a+b_1+\dots+b_u=k \\ a,u \geq 0, b_1, \dots, b_u \geq 1}} (-1)^u [\Lambda^a(A) \otimes_{\Delta} \Lambda^{b_1}(B) \otimes_{\Delta} \dots \otimes_{\Delta} \Lambda^{b_u}(B)] \\ &\quad - \sum_{\substack{a+b_1+\dots+b_u=k \\ a,u \geq 0, b_1, \dots, b_u \geq 1}} (-1)^u [\Lambda^a(B) \otimes_{\Delta} \Lambda^{b_1}(B) \otimes_{\Delta} \dots \otimes_{\Delta} \Lambda^{b_u}(B)] \\ &= \lambda^k([A] - [B]) - \lambda^k([B] - [B]) = \lambda^k([A] - [B]), \end{aligned}$$

(again by [Gra89, Equation 2.3]), as was to be shown.

Remark 3.7. In this remark we look at power operations on Bass's K_1^B . First, recall from Remark 2.6 that we have an isomorphism

$$K_0\text{Aut}(\mathcal{M})/\tilde{I} \cong K_0\mathcal{M} \oplus K_1^B\mathcal{M}$$

for any exact category \mathcal{M} . Given an assembly of power operations on \mathcal{M}_n , $n \geq 0$, it is not difficult to verify (cf. [Kö00, Lemma 1.2]) that the assignment $(P, \alpha) \mapsto (\Lambda^k(P), \Lambda^k(\alpha))$ defines a map $\lambda^k: K_0\text{Aut}(\mathcal{M}_n) \rightarrow K_0\text{Aut}(\mathcal{M}_{nk})$

and then induces a map $\lambda^k: K_0\mathcal{M}_n \oplus K_1^B\mathcal{M}_n \rightarrow K_0\mathcal{M}_{nk} \oplus K_1^B\mathcal{M}_{nk}$ which becomes the usual power operation λ^k when restricted to $K_0(\mathcal{M}_n)$ and becomes a homomorphism when restricted to $K_1^B(\mathcal{M}_n)$. In the next paragraph, we sketch a proof of the fact that the canonical map $\varphi: K_1^B\mathcal{M}_n \rightarrow K_1\mathcal{M}_n$ (see (5)) is compatible with the operations λ^k .

First, beware, although we have $\lambda^k([N]) = [\Lambda^k(N)]$ in $K_1(\mathcal{M}_{nk})$ for any $N \in B^q\mathcal{N}_n$, we don't have $\lambda^k([(P, \alpha)]) = [(\Lambda^k P, \Lambda^k \alpha)]$ in $K_1^B\mathcal{M}_{nk}$ for (P, α) in $\text{Aut}(\mathcal{M}_n)$. We rather have (using [Gra89, Equation 2.3]):

$$\begin{aligned} \lambda^k([(P, \alpha)]) &= [\lambda^k((P, \alpha) - (P, \text{id}))] \\ &= \sum_{\substack{a+b_1+\dots+b_u=k \\ u \geq 0, a, b_1, \dots, b_u \geq 1}} (-1)^u [\Lambda^a(P, \alpha) \otimes \Lambda^{b_1}(P, \text{id}) \otimes \dots \otimes \Lambda^{b_u}(P, \text{id})] \end{aligned}$$

where the terms for $a = 0$ have been omitted because they vanish. The homomorphism φ sends this alternating sum to the corresponding alternating sum of the classes of the binary automorphisms

$$(\Lambda^a(P) \otimes \Lambda^{b_1}(P) \otimes \dots \otimes \Lambda^{b_u}(P), \Lambda^a(\alpha) \otimes \Lambda^{b_1}(\text{id}) \otimes \dots \otimes \Lambda^{b_u}(\text{id}), \text{id}).$$

For a fixed u , the sum of such terms is exactly the $(u+1)$ th cross-effect functor of Λ^k ; more precisely, the alternating sum arrived at above is equal to

$$\sum_{u \geq 1} (-1)^{u-1} \left[\left(\text{cr}_u(\Lambda^k)(P, \dots, P), \text{cr}_u(\Lambda^k)(\alpha, \text{id}, \dots, \text{id}), \text{id} \right) \right].$$

On the other hand, by [Kö01, Lemma 2.2] (also see [dBT15, Section 3]), the binary complex $\Lambda^k(P, \alpha, \text{id})$ of length k can be filtered by subcomplexes such that the successive quotients are the shifted binary automorphisms $(\text{cr}_u(\Lambda^k)(P, \dots, P), \text{cr}_u(\Lambda^k)(\alpha, \text{id}, \dots, \text{id}), \text{id})[u-1]$, $u \geq 1$. It remains to observe that shifting a binary complex by $u-1$ yields the sign $(-1)^{u-1}$ in K_1 by [Gra16, Corollary 8.7].

Remark 3.8. In this remark we look at power operations on Bass's relative K_0 -group. To this end, we assume the situation introduced before Definition 3.1.

- (a) For $X = (P, \alpha, Q) \in \text{co}(F_n)$ define $\Lambda^k(X) := (\Lambda^k(P), \Lambda^k(\alpha), \Lambda^k(Q))$. Then, as in Remark 3.7, see also [Kö00, Lemma 1.3], the assignment $X \mapsto \Lambda^k(X)$ defines a map $\lambda^k: K_0\text{co}(F_n) \rightarrow K_0\text{co}(F_{nk})$ and then induces a map

$$\lambda^k: K_0^B[F_n] \rightarrow K_0^B[F_{nk}]$$

via the isomorphism (10). In contrast to Remark 3.7, this operation λ^k is not a homomorphism in general.

- (b) Similarly to Lemma 3.4, for $X = (P, \alpha, Q) \in \text{co}(F_n)$, we can again describe $\lambda^k([X]) \in K_0^B[F_{nk}]$ in terms of the class of a single object of $\text{co}(F_{nk})$ in the following way. Let $\tau X := (Q, \alpha^{-1}, P)$, $\perp X := Q$ and

$\Delta \perp X := (Q, \text{id}_{FQ}, Q)$. Then we have:

$$\begin{aligned} \lambda^k([X]) &= \left[\bigoplus_{\substack{a+b_1+\dots+b_u=k \\ u \geq 0, a, b_1, \dots, b_u \geq 1}} \Lambda^a(\tau^u X) \otimes \Lambda^{b_1}(\Delta \perp X) \otimes \dots \otimes \Lambda^{b_u}(\Delta \perp X) \right] \\ &= [(\Lambda_+^k(P, Q), \hat{\Lambda}^k \alpha, \Lambda_-^k(P, Q))] \end{aligned}$$

where $\Lambda_+^k(P, Q)$ and $\Lambda_-^k(P, Q)$ are defined similarly as in Lemma 3.4 and $\hat{\Lambda}^k \alpha$ is defined as follows:

$$\hat{\Lambda}^k \alpha = \bigoplus_{\substack{a+b_1+\dots+b_u=k \\ u \geq 0, a, b_1, \dots, b_u \geq 1}} \Lambda^a(\alpha^{(-1)^u}) \otimes \Lambda^{b_1}(1) \otimes \dots \otimes \Lambda^{b_u}(1).$$

- (c) By part (b), the map $K_0^B[F_{nk}] \rightarrow K_0 \mathcal{M}_{nk}$, $[(P, \alpha, Q)] \mapsto [P] - [Q]$, sends $\lambda^k([X])$ to $[\Lambda_+^k(P, Q)] - [\Lambda_-^k(P, Q)] = \Lambda^k([P] - [Q])$ and hence commutes with λ^k . The incoming map $K_1^B \mathcal{N}_n \rightarrow K_0^B[F_n]$ in Bass's exact sequence [Bas68, Theorem VII.5.3] commutes with λ^k as well (assuming \mathcal{M}_n and \mathcal{N}_n are split exact and the F_n are cofinal); to see this we just observe that this map is the restriction of the analogously defined map $K_0 \text{Aut}(\mathcal{N}_n)/\tilde{I} \rightarrow K_0 \text{co}(F_n)/\tilde{J}$ which obviously commutes with λ^k .
- (d) Finally, the homomorphism $\Phi: K_0^B[F_n] \rightarrow K_0[F_n]$ from Theorem 1.3 also commutes with λ^k ; this follows from the very definitions.

4. LAMBDA RING AXIOMS FOR RELATIVE K_0

The main result of this section is that the products and exterior power operations introduced in the previous two sections turn Grayson's relative K_0 -group into a λ -ring (without unity). We begin by proving that the product axiom in the classical definition of a λ -ring follows from the other axioms. To prove the remaining axiom (concerning composition of exterior power operations), we follow the approach from [HKT17] which relies on the fact that the Grothendieck group of the exact category consisting of polynomial functors over \mathbb{Z} is the free λ -ring in one variable.

Recall a *pre- λ -ring* is a commutative ring K together with maps λ^k , $k \geq 1$, from K to itself such that $\lambda^1(x) = x$ for all $x \in K$ and such that

$$\lambda^k(x + y) = \sum_{i=0}^k \lambda^{k-i}(x) \lambda^i(y) \quad (11)$$

for all $x, y \in K$ and $k \geq 1$. As we want to allow our (pre-) λ -rings to be without unity, our definition does not include a λ^0 and any term such as $\lambda^0(x) \lambda^k(y)$ is to mean just $\lambda^k(y)$ here and below. The pre- λ -ring K is called a *λ -ring* if moreover

$$\lambda^k(xy) = P_k(\lambda^1(x), \lambda^2(x), \dots, \lambda^k(x); \lambda^1(y), \lambda^2(y), \dots, \lambda^k(y)), \quad (12)$$

$$\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), \lambda^2(x), \dots, \lambda^{kl}(x)) \quad (13)$$

for all $x, y \in K$ and $k, l \geq 1$ where $P_k, P_{k,l}$ are certain universal integral polynomials (see [FL85, I, §1]). We call (12) and (13) the product and composition axiom, respectively.

In the context below, the product axiom could be proved by working out a bivariate version of the proof of the composition axiom (see Theorem 4.2 below) or, similarly to the proof of [Kö00, Theorem 2.2], by using the characteristic free Cauchy decomposition as developed by Akin, Buchsbaum and Weyman in [ABW82]. We however use a shortcut provided by the following somewhat surprising lemma.

Lemma 4.1. *Let K be a pre- λ -ring which satisfies the composition axiom. Then K is a λ -ring.*

Proof. The polynomial ring $U := \mathbb{Z}[X_1, X_2, \dots; Y_1, Y_2, \dots]$ over \mathbb{Z} in the infinitely many variables X_1, X_2, \dots and Y_1, Y_2, \dots can be equipped with a unique λ -ring structure such that $\lambda^k(X_1) = X_k$ and $\lambda_k(Y_1) = Y_k$. It is called the free- λ -ring in 2 variables and can be constructed similarly to the free λ -ring in 1 variable which has been introduced in [AT69, I, §2]. By induction on k and without using the product axiom (12), we write $\lambda^k(X_1 Y_1)$ as a polynomial $Q_k \in V := \mathbb{Z}[X_1, \dots, X_{2k}; Y_1, \dots, Y_{2k}] \subset U$ as follows.

First $Q_1 = X_1 Y_1$. For the inductive step, we consider the polynomial

$$\begin{aligned} Q_{k,2} &:= \lambda^k(\lambda^2(X_1 + Y_1)) \\ &\stackrel{(13)}{=} P_{k,2}(\lambda^1(X_1 + Y_1), \dots, \lambda^{2k}(X_1 + Y_1)) \\ &\stackrel{(11)}{=} P_{k,2}\left(X_1 + Y_1, \dots, \sum_{i=0}^{2k} X_{2k-i} Y_i\right) \in V. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} Q_{k,2} &\stackrel{(11)}{=} \lambda^k(X_2 + X_1 Y_1 + Y_2) \\ &\stackrel{(11)}{=} \lambda^k(X_1 Y_1) + \sum_{i=1}^k \lambda^{k-i}(X_1 Y_1) \lambda^i(X_2 + Y_2) \\ &\stackrel{(11)}{=} \lambda^k(X_1 Y_1) + \sum_{i=1}^k \lambda^{k-i}(X_1 Y_1) \sum_{j=0}^i \lambda^{i-j}(X_2) \lambda^j(Y_2). \end{aligned}$$

Now, for $i = 1, \dots, k-1$, the inductive hypothesis yields $\lambda^{k-i}(X_1 Y_1) = Q_{k-i}$. Furthermore, for any $j \geq 1$, we have $\lambda^j(X_2) \stackrel{(13)}{=} P_{j,2}(X_1, \dots, X_{2j})$ and $\lambda^j(Y_2) \stackrel{(13)}{=} P_{j,2}(Y_1, \dots, Y_{2j})$. In other words, we have $Q_{k,2} = \lambda^k(X_1 Y_1) + R_{k,2}$ for some polynomial $R_{k,2} \in V$. Finally, defining $Q_k := Q_{k,2} - R_{k,2}$ does the desired job.

Since the calculations above only involve the axioms (11) and (13), they remain valid after replacing X_1 and Y_1 everywhere with elements x and y of K . (If K is non-unital, also observe that these calculations don't involve a constant term either.) We therefore obtain the equality

$$\lambda^k(xy) = Q_k(\lambda^1(x), \lambda^2(x), \dots, \lambda^{2k}(x); \lambda^1(y), \lambda^2(y), \dots, \lambda^{2k}(y))$$

in K . Finally, we have $Q_k = P_k(X_1, \dots, X_k; Y_1, \dots, Y_k)$ because U also satisfies the product axiom (12). This finishes the proof. \square

Now, let $f: X \rightarrow Y$ be a morphism between quasi-compact schemes and let $F := f^*: \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ denote the induced pull-back functor between

the associated categories of locally free \mathcal{O}_X - and \mathcal{O}_Y -modules of finite rank. By Lemma 3.2, we have exterior power operations $\lambda^k: K_0B[F] \rightarrow K_0B[F]$ which turn $K_0B[F]$ into a pre- λ -ring.

Theorem 4.2. *The pre- λ -ring $K_0B[F]$ is a λ -ring.*

Proof. Let $k, l \geq 1$. According to Lemma 4.1, it suffices to show that

$$\lambda^k(\lambda^l(x)) = P_{k,l}(\lambda^1(x), \lambda^2(x), \dots, \lambda^{kl}(x))$$

for all $x \in K_0B[F]$. To this end, we follow the approach used in [HKT17, Section 8]. We first note that it suffices to prove this formula when x is the class of a triple $V \in B[F]$. Due to a standard argument, the category $\text{End}(B[F])$ of endo-functors on $B[F]$ is an exact category with tensor product and exterior powers Λ^k given by $H \mapsto \Lambda^k \circ H$. We will show that the stronger identity

$$[\Lambda^k \circ \Lambda^l] = P_{k,l}([\Lambda^1], \dots, [\Lambda^{kl}])$$

holds in $K_0\text{End}(B[F])$. The desired identity in $K_0B[F]$ then follows by applying the homomorphism

$$K_0\text{End}(B[F]) \rightarrow K_0B[F] \text{ given by } [H] \mapsto [H(V)].$$

By [HKT17, Corollary 8.6], the stronger identity holds in the Grothendieck group $K_0\text{Pol}_{<\infty}^0(\mathbb{Z})$ of the category of *polynomial* endo-functors on $\mathcal{P}(\mathbb{Z}) := \mathcal{P}(\text{Spec}(\mathbb{Z}))$ of finite degree which fix 0. It therefore suffices to construct an exact functor

$$\text{Pol}_{<\infty}^0(\mathbb{Z}) \rightarrow \text{End}(B[F]) \tag{14}$$

which respects composition, tensor products and exterior powers and sends the identity functor to the identity functor. To this end, recall from [HKT17, Section 8A] that a polynomial functor H over a scheme Z assigns a $H(\mathcal{F}) \in \mathcal{P}(Z)$ to any object $\mathcal{F} \in \mathcal{P}(Z)$ and a morphism $H(\alpha): H(\mathcal{F})_T \rightarrow H(\mathcal{G})_T$ to any morphism $\alpha: \mathcal{F}_T \rightarrow \mathcal{G}_T$ for any $\mathcal{F}, \mathcal{G} \in \mathcal{P}(Z)$ and for any Z -scheme T ; the latter assignments must commute with pullback $\varphi^*: \text{Hom}(\mathcal{F}_T, \mathcal{G}_T) \rightarrow \text{Hom}(\mathcal{F}_{T'}, \mathcal{G}_{T'})$ for any morphism $\varphi: T' \rightarrow T$ of Z -schemes, but need not be linear. Furthermore, for any morphism $g: Z' \rightarrow Z$ of schemes, there is a base change functor $\text{Pol}_{<\infty}^0(Z) \rightarrow \text{Pol}_{<\infty}^0(Z')$, see the construction in *loc. cit.* The base change of any $H \in \text{Pol}_{<\infty}^0(Z)$ will be denoted H again and satisfies $H(g^*(\mathcal{F})) \cong g^*(H(\mathcal{F}))$ by construction. Now, given $H \in \text{Pol}_{<\infty}^0(\mathbb{Z})$ and $V = (M, N, u) \in B[F]$, we base-change H from $\text{Spec}(\mathbb{Z})$ to X and Y and define $H(V) := (NH\Gamma M, NH\Gamma N, NH\Gamma u)$, similarly to Section 2 and Section 3. Note that $NH\Gamma(u)$ is a quasi-isomorphism again; this is proved in [KZ25, Proposition 1.9] when H is an exterior power functor but the proof there works for any functor H which fixes 0. By construction, the resulting functor (14) satisfies the properties listed above. In particular, this finishes the proof. \square

We remark that we similarly obtain that $K_0C[F]$ is a λ -ring. Furthermore, it immediately follows that $K_0pB[F]$ and $K_0pC[F]$ are λ -rings, and we obtain:

Corollary 4.3. *The products and exterior power operations introduced in Definition 2.3 and Definition 3.3 turn the relative Grothendieck group $K_0[F]$ into a λ -ring (without unity).*

Proof. By Theorem 4.2, the axioms (11), (12) and (13) hold in the subring $\ker(\perp: K_0 pB[F] \rightarrow K_0 pC[F])$ of $K_0 B[F]$. As the isomorphism (9) is by construction compatible with products and exterior power operations, they then also hold in $K_0[F]$. \square

5. EXTERIOR POWER OPERATIONS ON HIGHER RELATIVE K -GROUPS

In this section, we first recall Grayson's combinatorial definitions of higher K -groups [Gra12] and of higher relative K -groups [Gra16]. The recursive nature of these definitions then allows us to prove the main result of this paper, namely that higher relative K -groups can be equipped with natural exterior power operations which satisfy the axioms of a λ -ring and which are compatible with the incoming and outgoing maps in the long exact sequence of K -groups.

Let \mathcal{M} be an exact category and $n \geq 0$. By [Gra12, Sections 2 and 3], the categories $C\mathcal{M}$, $B\mathcal{M}$, $C^q\mathcal{M}$ and $B^q\mathcal{M}$ are again exact categories (which support long exact sequences) and so we can iterate these constructions. As in [Gra12, Section 7], we form the n -cube $\Omega^n\mathcal{M}$ of exact categories as follows. At the corner of $\Omega^n\mathcal{M}$ corresponding to $(i_1, \dots, i_n) \in \{0, 1\}^n$ we put the exact category $G_{i_1} \dots G_{i_n}\mathcal{M}$ where $G_i := C^q$ if $i = 0$ and $G_i := B^q$ if $i = 1$. Furthermore, we put the obvious diagonal functor Δ at each edge of $\Omega^n\mathcal{M}$. For example, the 0-cube $\Omega^0\mathcal{M}$ is just \mathcal{M} , the 1-cube $\Omega^1\mathcal{M}$ is $C^q\mathcal{M} \xrightarrow{\Delta} B^q\mathcal{M}$ and the 2-cube $\Omega^2\mathcal{M}$ is the square

$$\begin{array}{ccc} C^q C^q \mathcal{M} & \xrightarrow{\Delta} & C^q B^q \mathcal{M} \\ \downarrow \Delta & & \downarrow \Delta \\ B^q C^q \mathcal{M} & \xrightarrow{\Delta} & B^q B^q \mathcal{M}. \end{array}$$

We now consider all those functors Δ in $\Omega^n\mathcal{M}$ which arrive at $(B^q)^n\mathcal{M}$ and define $\mathcal{K}_0\Omega^n\mathcal{M}$ to be the factor group of $K_0(B^q)^n\mathcal{M}$ modulo the subgroup generated by the images of all the homomorphisms $K_0\Delta$ induced by these functors Δ . In other words, $\mathcal{K}_0\Omega^n\mathcal{M}$ is obtained by taking iterated cokernels (in any order) in the cube of abelian groups obtained from $\Omega^n\mathcal{M}$ by applying K_0 . By [Gra12, Corollary 7.2] we then have a canonical isomorphism

$$K_n\mathcal{M} \cong \mathcal{K}_0\Omega^n\mathcal{M} \tag{15}$$

between Quillen's n th higher K -group $K_n\mathcal{M}$ and $\mathcal{K}_0\Omega^n\mathcal{M}$. We will take this as the definition of $K_n\mathcal{M}$.

Now, let $F: \mathcal{M} \rightarrow \mathcal{N}$ be an exact functor between exact categories. We form the $(n+1)$ -cube

$$\Omega^n[F] := \left(\Omega^n\mathcal{M} \xrightarrow{F} \Omega^n\mathcal{N} \right).$$

For example, for $n = 2$, this looks as follows:

$$\begin{array}{ccccc}
 C^q C^q \mathcal{M} & \xrightarrow{C^q C^q F} & C^q C^q \mathcal{N} & & \\
 \downarrow \Delta & \searrow \Delta & \downarrow \Delta & \searrow \Delta & \\
 & C^q B^q \mathcal{M} & \xrightarrow{C^q B^q F} & C^q B^q \mathcal{N} & \\
 & \downarrow \Delta & \downarrow \Delta & & \\
 B^q C^q \mathcal{M} & \xrightarrow{B^q C^q F} & B^q C^q \mathcal{N} & & \\
 \searrow \Delta & \downarrow \Delta & \searrow \Delta & & \\
 & B^q B^q \mathcal{M} & \xrightarrow{B^q B^q F} & B^q B^q \mathcal{N} & \\
 & & \downarrow \Delta & &
 \end{array}$$

Definition 5.1 ([Gra16, Section 2]). The n th higher relative K -group $K_n[F]$ is defined as the iterated cokernel of the n -cube $K_0 \Omega^n[F]$ of abelian groups obtained from $\Omega^n[F]$ by forming the relative K_0 -group of each of those functors in $\Omega^n[F]$ which are induced by F .

For example, $K_2[F]$ is defined as the iterated cokernel of the square

$$\begin{array}{ccc}
 K_0[C^q C^q F] & \xrightarrow{\Delta} & K_0[C^q B^q F] \\
 \downarrow \Delta & & \downarrow \Delta \\
 K_0[B^q C^q F] & \xrightarrow{\Delta} & K_0[B^q B^q F].
 \end{array}$$

Remark 5.2. The following two facts justify calling $K_n[F]$ the n th higher relative K -group.

- (a) By [Gra16, Corollary 2.3], there is a purely algebraically defined long exact sequence

$$\begin{array}{c}
 \cdots \longrightarrow K_{n+1} \mathcal{M} \longrightarrow K_{n+1} \mathcal{N} \\
 \downarrow \\
 K_n[F] \longrightarrow K_n \mathcal{M} \longrightarrow K_n \mathcal{N} \\
 \downarrow \\
 \cdots \longrightarrow K_0 \mathcal{M} \longrightarrow K_0 \mathcal{N}.
 \end{array}$$

- (b) By [Tu25, Theorem 4.3], there is a canonical isomorphism

$$K_n[F] \cong \pi_n \text{hofib}(KF: K\mathcal{M} \rightarrow K\mathcal{N})$$

between $K_n[F]$ and the n th homotopy group of the homotopy fibre of the induced map KF between the K -theory spaces $K\mathcal{M}$ and $K\mathcal{N}$ of \mathcal{M} and \mathcal{N} .

We can also describe $K_n[F]$ in the following ways. First, we turn the $(n+1)$ -cube $\Omega^n[F]$ into another $(n+1)$ -cube of exact categories by replacing every edge in $\Omega^n[F]$ of the form $G_1 \dots G_n \mathcal{M} \xrightarrow{G_1 \dots G_n F} G_1 \dots G_n \mathcal{N}$ with $C[G_1 \dots G_n F] \xrightarrow{\Delta} B[G_1 \dots G_n F]$. We tacitly assume that the notations $C[G_1 \dots G_n F]$ and $B[G_1 \dots G_n F]$ here include the subcategories $pC[G_1 \dots G_n F]$ and $pB[G_1 \dots G_n F]$, respectively, as introduced in Definition 1.1(a) and (b). We use the ad hoc notation $\text{Cube}(F, \Delta, n)$ for this cube. Then $K_n[F]$ is the iterated cokernel of the $(n+1)$ -cube $K_0 p\text{Cube}(F, \Delta, n)$ of

abelian groups obtained from $\text{Cube}(F, \Delta, n)$ by applying the functor K_0p . If we additionally replace all homomorphism $K_0p\Delta$ with $K_0p\perp$, then, similarly to (9), the iterated kernel of the resulting $(n+1)$ -cube $K_0p\text{Cube}(F, \perp, n)$ is canonically isomorphic to $K_n[F]$. For example, the iterated kernel of the cube of abelian groups obtained by applying K_0p to the cube

$$\begin{array}{ccccc}
 & & \perp & & \\
 & & \longleftarrow & & \\
 C[C^q C^q F] & & & & B[C^q C^q F] \\
 \uparrow \perp & \swarrow \perp & & \nwarrow \perp & \\
 & C[C^q B^q F] & \longleftarrow \perp & & B[C^q B^q F] \\
 & \uparrow & & \downarrow \perp & \\
 C[B^q C^q F] & \longleftarrow \perp & & & B[B^q C^q F] \\
 \swarrow \perp & & \uparrow \perp & \nwarrow \perp & \uparrow \perp \\
 & C[B^q B^q F] & \longleftarrow \perp & & B[B^q B^q F]
 \end{array}$$

is isomorphic to $K_2[F]$.

Now, as in the previous section, let $f: X \rightarrow Y$ be a morphism between quasi-compact schemes, let $\mathcal{M} := \mathcal{P}(Y)$, $\mathcal{N} := \mathcal{P}(X)$ and let $F := f^*$ denote the induced pull-back functor from \mathcal{M} to \mathcal{N} . The standard assembly of power operations on $\mathcal{P}(Y)$ and $\mathcal{P}(X)$ induce an assembly of power operations on $G_1 \dots G_n \mathcal{M}$ and $G_1 \dots G_n \mathcal{N}$ for any $G_1, \dots, G_n \in \{C^q, B^q\}$ (see [KZ25, Section 1]) and then an assembly of power operations on $C[G_1 \dots G_n F]$ and $B[G_1 \dots G_n]$ (see Lemma 3.2) and finally products and exterior power operations λ^k , $k \geq 1$, on $K_0[G_1 \dots G_n F]$ and $K_0p[G_1 \dots G_n F]$ which turn these Grothendieck groups into pre- λ -rings. As these pre- λ -ring structures are compatible with all homomorphisms in the cube $K_0p\text{Cube}(F, \perp, n)$, we also obtain a pre- λ -ring structure on the iterated kernel of this cube.

The following definition extends Definition 2.3 and Definition 3.3 from $n = 0$ to any $n \geq 0$.

Definition 5.3. The pre- λ -ring structure on $K_n[F]$ is defined to be the one obtained by transporting the pre- λ -ring structure on the iterated kernel of $K_0p\text{Cube}(F, \perp, n)$ to $K_n[F]$ via the canonical isomorphism described above.

Theorem 5.4.

- (a) The pre- λ -ring $K_n[F]$ is a λ -ring.
- (b) If $n \geq 1$, all products on $K_n[F]$ vanish and the exterior power operations λ^k , $k \geq 1$, on $K_n[F]$ are homomorphisms.
- (c) Products and exterior power operations on $K_n[F]$ are compatible with products and exterior power operations on the higher K -groups of X and Y via the long exact sequence mentioned in Remark 5.2(a).

Proof.

(a) As in Corollary 4.3, the first statement follows from the stronger statement that $K_0B[(B^q)^n F]$ is a λ -ring. To prove the latter statement we proceed as in the proof of Theorem 4.2. Similarly to there, this comes down to constructing an exact functor

$$\text{Pol}_{<\infty}^0(\mathbb{Z}) \rightarrow \text{End}(B[(B^q)^n F]) \quad (16)$$

which respects composition, tensor products and exterior powers and sends the identity functor to the identity functor. This functor in turn is obtained as in the proof of Theorem 4.2 after first inductively defining the functors $\text{Pol}_{<\infty}^0(\mathbb{Z}) \rightarrow \text{End}((B^q)^n \mathcal{P}(Y))$ and $\text{Pol}_{<\infty}^0(\mathbb{Z}) \rightarrow \text{End}((B^q)^n \mathcal{P}(X))$ as in the proof of [HKT17, Theorem 8.18].

(b) Due to axiom (11), we only need to show the first statement. By definition this means we need to show that products on the iterated kernel of the cube $K_0 p\text{Cube}(F, \perp, n)$ vanish. This iterated kernel in turn is a quotient of the iterated kernel of $K_0 \text{Cube}(F, \perp, n)$, and the latter kernel carries products similar to and hence compatible with the ones on the former. Now, products on higher K -groups such as $K_n B[F]$ vanish by [HKT17, Proposition 5.11], hence also the induced products on $\ker(\perp : K_n B[F] \rightarrow K_n C[F])$. The latter is the same as the iterated kernel of $K_0 \text{Cube}(F, \perp, n)$ because we may think of each category $G[G_1 \dots, G_n F]$ in $\text{Cube}(F, \perp, n)$ as $G_1 \dots G_n(G[F])$ and because forming iterated kernels does not depend on the order of iterations. Overall, we conclude that products on $K_n[F]$ vanish as claimed.

(c) We associate the exact sequence (4) with each of those arrows in the $(n+1)$ -cube $\Omega^n[F]$ which are induced by F . This way we obtain the exact sequence of n -cubes of abelian groups

$$K_1 \Omega^n \mathcal{M} \rightarrow K_1 \Omega^n \mathcal{N} \rightarrow K_0 \Omega^n[F] \rightarrow K_0 \Omega^n \mathcal{M} \rightarrow K_0 \Omega^n \mathcal{N}.$$

Now we replace every homomorphism induced by Δ in every cube of this sequence with the homomorphism induced by \perp in the opposite direction. Similar to the proof of part (b), and as explained in [Gra16, Lemma 2.2 and proof of Corollary 2.3], the sequence obtained by taking iterated kernels of each cube in the resulting exact sequence of n -cubes is then the same as the part

$$K_{n+1} \mathcal{M} \rightarrow K_{n+1} \mathcal{N} \rightarrow K_n[F] \rightarrow K_n \mathcal{M} \rightarrow K_n \mathcal{N} \quad (17)$$

of the sequence in Remark 5.2(a). In particular, the exact sequence (17) is a subsequence of the sequence

$$K_1 (B^q)^n \mathcal{M} \rightarrow K_1 (B^q)^n \mathcal{N} \rightarrow K_0 [(B^q)^n F] \rightarrow K_0 (B^q)^n \mathcal{M} \rightarrow K_0 (B^q)^n \mathcal{N}$$

and, by construction, all the products and exterior power operations are compatible with this inclusion. Now part (b) follows from Proposition 2.4 and Proposition 3.5. \square

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