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Teaching Introductory Calculus: approaching key ideas with dynamic software

Theodosios Zachariades (University of Athens, Greece), Paris Pamfilos (University of Crete, Greece), Constantinos Christou (University of Cyprus), Rumen Maleev (University of Sofia, Bulgaria) and Keith Jones (University of Southampton, UK)

While, commonly across the world, selected key ideas of the Calculus are introduced to students in the final years of schooling, and are thence built upon as students take a full course in Analysis at University, there remains much to learn about how best to introduce such ideas and how to develop and expand the ideas at University level. This paper reports on the work of a European-funded project involving four countries in which the potential of dynamic software was exploited in the teaching of topics such as infinite processes, limits, continuity, differentiation and integration. Amongst the approaches adopted in the project, problem-solving situations were developed through which students, while their knowledge may initially be inadequate, could approach intuitively the central mathematical notion in ways that are consistent with formal mathematical definitions. Amongst the implications of the project, in terms of the debate about what is suitable preparation for students embarking on a course of analysis at University level, are that it might be useful to think in terms of two categories of learning activity – the first is introducing student to relevant concepts and the second focuses on the teaching of theorems. These two categories entail a different design of learning activity.

Introduction

In most countries, the Calculus is a key mathematical topic in which ideas introduced in the final years of schooling are developed and expanded as students take a full course in Analysis at University. As such, there is likely to be, across countries, some variation in what is emphasised at the school level and how this is taken up at the University level, especially in relation to relevant significant mathematical ideas such as function, continuity, and limits.

The use of geometry, and specifically the use of dynamic geometry software tools, is an appropriate, and also a necessary, vehicle in teaching calculus concepts. The fact that calculus concepts are abstract and complicated makes the teaching and learning of calculus concepts demanding and even frustrating for teachers and students. Additionally the dynamic nature of calculus concepts (for example, the ε-δ definition of the limit) makes the use of dynamic geometry tools an appropriate way of teaching and learning calculus. The historical origin of many calculus concepts derives from the necessity to solve geometric problems, such as the problems of calculating the area between curves, the tangent notion, and so on.

This paper reports on a European-funded project involving four countries in which the central aim was to exploit the potential of dynamic software in developing not only teaching material but also a resource pack for teacher/lecturer professional development, covering topics such as an introduction to infinite processes, limits, continuity, differentiation and integration. The aim of the paper is to examine the implications of the project in terms of the debate about what is suitable preparation for students embarking on a course of Analysis at University level.

Background

Research shows that many students face serious problems in understanding some of the key concepts of Calculus [1] [2]. As such, there is a need for more detailed investigations into students' conceptualisations of calculus concepts, and, more specifically, on understanding how students learn and come to know calculus concepts and how the Calculus can be taught most effectively.

The work presented in this paper originates in a three-year project entitles *CalGeo* (Teaching Calculus Using Dynamic Geometric Tools), funded by the European Union and involving five Universities from four countries. More details of the project can be found at: http://www.math.uoa.gr/calgeo/

The idea behind the project is to use ideas from international research that suggest that the abstract nature of the concepts of Calculus requires a pedagogical approach that begins with the informal-spontaneous perceptions of students; in particular, that appropriate intuitions and images of calculus can be created and developed through geometrical concepts, with the use of computer technology (ICT) supporting this development.

The main objectives of the project were to develop research-based ideas that can be incorporated into the teaching of Calculus, and design a programme of in-service training for mathematics teachers on these ideas.

The *CalGeo* Project

A central theme of the *CalGeo* project was to exploit the potential of dynamic software in the teaching of calculus concepts. Software was chosen that provided an environment in which students could explore relationships and make and test conjectures (Example included *Autograph*, *Cabri*, *EucliDraw*, *Sketchpad*).

One teaching approach utilised in the project was to offer problem-solving situations through which, while students' knowledge may initially be inadequate, students can approach intuitively the corresponding mathematical notion in ways that are consistent with formal mathematical definitions. For example, visual representation of the ε-δ definition of the limit, the ε-δ definition of continuity, definite integral, and so on) were used to lead students to formal definitions. Other teaching activities developed in the project offer problem-solving situations as a starting point in order to lead students to the statement of a theorem, to examine the necessity of its hypothesis, and to uncover if the reverse of the theorem is correct.

More details of the project material can be found in [3] and [4] and on the project website.

Outcomes of the Project

In trialling the project material with teachers (through the training course that was designed as part of the project and that was trialled in four countries), and through follow-up research in classrooms, the teaching activities developed through the project were mostly well-received. In particular, the uses of graphical representations, especially via the use of dynamic computer environments were received most positively.

Naturally, not everything was well-received. For example, for teachers in a number of the project countries, the specified curriculum (say, at the pre-University level) seems to them to focus mainly on procedures, and, as such, seems to limit attention given to students *understanding* the concepts of calculus. What is more, teachers at this level report facing time-pressures because of the curriculum they must teach and this can restrict their ability to experiment with new environments and technologies. It also seems that some institutions (at the pre-University level) in some countries can lack suitable ICT equipment and some teachers who work in these institutions can be unfamiliar with the use of computers in teaching. A particular concern for some teachers related to the fact that some of the activities referred to themes which are not examined in school.

Both the positive outcomes, and the teacher reservations about some of the project material, helped to inform the implications of the project. The results of the project indicate that teachers can be willing to use the provided software tools as a means for their students to explore and solve the real-world problems presented in the project activities. Additionally, students that participated used the available software tools for examining the setting of the problem, formulating hypotheses, examining the hypotheses and finally reaching a solution to the problem, through their introduction to the related concept. An interesting dimension of students' work was their willingness to try different strategies, supported by the provided software tools.

Students used the software tools sometimes as an investigative or exploratory tool, and sometimes to verify their hypotheses by using the tools as a confirmatory method. As a result, they engaged in a fruitful problem-solving process. This result is not common in the traditional calculus curriculum in which students can appear to lack the know-how to use knowledge in a flexible manner in unfamiliar problems [5].

One of the implications of the project is related to the use of representations. Among the aims of the activities was the introduction of concepts using a multiple-representational approach. This aimed to helped students making the necessary connections and translations between the graphical, iconic, symbolic representation of the related concepts. As such, concepts not only need to be presented in a multiple representational view, but links between the representations need to be shown. The presentation of problems in different representations, therefore, compels students to build and understanding of the ties among representations. This also had implications for teachers. It assisted them in being aware of the differences in the variety of concepts' representations and in affording students the opportunity to make translations and ties among representations.

Nevertheless, the results of the project indicate that access to software tools alone is not sufficient to develop better understanding of graphical, symbolic and verbal relationships or calculus concepts such as limits, derivatives and integrals. The work of the project suggests that by providing teachers with appropriate training and teaching materials, it is possible to enhance students' understanding of calculus concepts.

Implications for the Teaching of Calculus

Given that Calculus topics are important at the University level, and that the teaching approaches and training material developed by the *CalGeo* project are applicable across the school-University transition, there are implications of the project for the teaching of Calculus for university and college instructors.

First, the use of technology in the teaching of college calculus can be beneficial, as long as this use is accompanied by with appropriately designed activities. This introduction of technology at the university level is likely not only to assist university instructors in their courses but it also university students (including prospective mathematics teachers) in their future work by using technology not as a replacement to conventional learning of concepts in calculus, but more as an exploratory tool in attempting unfamiliar problems in novel problem situations and in theorem proving.

The project also shed light on showing how the use of technological tools, in conjunction with appropriately designed activities has implications for the approach to the teaching of theorems at the first year level of University mathematics. The findings of the project point to two categories of learning activity – the first is introducing student to relevant concepts and the second focuses on the teaching of theorems.

Learning activities in the first category are designed as follows: introduce a problem which cannot be solved by students with their current knowledge; discuss with students; through the discussion, students realise the need to introduce new concept; students discuss the concepts through different kinds of representations such as graphical, symbolic and verbal representations. An example – "Introduction to continuity of a function at a point" – can be found in appendix 1.

Learning activities in the second category are designed as follows: start with a problem and a discussion of a problem with the students; aim to reach a conjecture that students can study and investigate; students can define the theorem and then move to the proof of the theorem; students at University level can be led through appropriate activities to realise the set of conjectures that are necessary for the proving of the theorem and also to appreciate applications of the theorem; through all these activities, students return to the initial problem and should be able to solve. An example – "Mean Value Theorem" - can be found in appendix 2.

As such, these learning activities may be useful in developing students' *conceptual understanding* in conjunction with their *procedural capability*, something that is known to be difficult.

Concluding Comments

This paper reports on a project in which the potential of dynamic software was exploited in the teaching of key Calculus topics such as infinite processes, limits, continuity, differentiation and integration. The contribution of the paper concerns the implications of the project in terms of the debate about what is suitable preparation for students embarking on a course of Analysis at University level and how the ideas that students bring to their University-level course can be successfully developed.

A key implication of the project is the possibility of developing a new theoretical model for the teaching of calculus at the upper secondary school and university level. Such a model for the teaching of calculus would consider the following two dimensions: (a) Teaching Concepts and (b) Teaching Theorems. The use of novel real-world problems might serve as a starting point for mathematical discussions, setting conjectures, exploring and testing these conjectures. Through this process students are expected to perceive the necessity of related concepts and theorems, using different representations and counter examples.

The classroom use of such an approach in the CalGeo project appeared to assist teachers in building fruitful mathematical discourse in their classes which resulted in the demonstration of students' ideas. It also assisted students to reason mathematically and consequently to be moved towards the formal definition and proving of theorems.

Authors' note

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Appendix 1

ACTIVITY 3.1 Introduction to continuity of a function at a point

WORKSHEET (Student Version, using *EucliDraw* software)

Step 1:

A chemical and health care corporation is about to produce a new antibiotic pill which will be able to cure some diseases.
It is known that the pill should be of 3gr in order to provide the patient with the right dose of medicine.
The function $f(x)=\sqrt{x+1}-1$ gives the amount $f(x)$ of the antibiotic which is detected in the blood, when a patient gets a x gr pill.
According to current research results, if the antibiotic detected in blood is equal or less than 0.8gr, there will be no effect on the patient's health and if it is equal or more than 1.2gr the patient is in danger due to overdose.

Q1 Which amount of medicine that should ideally be detected in the patient's blood?

Q2 Which is the allowed error ε from which could the detected amount of medicine be far from the ideal value, in a way that the pills are safe and effective?

The machine currently available to the corporation can produce pills of t=3gr, but it also has an accuracy level adjusted to δ=1.1
This means that although the machine is programmed to produce 3gr pills, the pills are not always 3gr but their weights vary between 3 – 1.1gr and 3 + 1.1gr.

Q3: Is the machine capable of producing pills within the allowed error?

- In a new *EucliDraw* file, sketch the graph of the function and set the axes active.
- Show (3,0) , (3,f(3)) and (0,f(3)) on the graph.
- Display 0.8 and 1.2 on y-axis and 3.9 and 6.1 on x-axis.

The machine can be adjusted to a lower accuracy level. However, this change will make the pill production more expensive.

Q4: Has the machine to be adjusted in order to produce pills within the allowed error?

The results of a new research indicated that the error level should be reduced to ε=0.1.

**Q5: Has the machine any problem with this change?
Does the accuracy level have to be decreased again?**

Step 2:

Q1: If the results of another research suggest that ε should be less and less, will the corporation be always able to adjust the machine?

Hints:
1. Open *activity3.1.1.euc* file and check whether we can always find an adequate δ>0, as ε gets smaller and smaller. Experiment graphically.
2. Display the Red/Green region and describe what it means for the function graph to lay in the green, red or white region.
3. If necessary, use the magnification window.

Q2: In each of the following sentences, fill in the blanks with the correct colours:

a. Whenever we are given an ε, we can find a δ,
such that the function does not lie in the .... region.

b. For every ε we can find a δ such that for every x in the accuracy of the machine, f(x) lies in the .... region.

Q3: Write sentence b, replacing the colours with algebraic relations.

Your answer to Q3 is called the "continuity property".

Step 3:
Another research shows that the formula used to find the detected antibiotic in blood works well for values less than 3gr, but it shows 0.06gr less than the real amount for values greater or equal to 3.

Q1. Complete the formula, taking into account the results of the research.

$$f(x)=\begin{cases}\sqrt{x+1}-1 & , \ x<3 \\ \text{.................} & , \ x\geq 3\end{cases}$$

Q2: Can the machine be adjusted properly to produce pills?
Which δ should work for ε=0.1?

- Open *activity3.1.2.euc* file
- Give your answer by giving suitable values δ.
- If needed use the magnification window.

Q3: What will happen if ε is reduced to 0.06?
Can you find an adequate δ?

Hint:

- Observe that if ε=0.06, every δ>0 fails to prevent the function from remaining in the red area.

Q4: Does the function have the continuity property?

Q5. What causes this failure?

Q6. In each of the following sentences, fill in the blanks with the correct colors:

a. For a given an ε, no δ could prevent the function from lying in the .... region.

b. There is an ε, such that for every δ, some x in the accuracy of the machine has an f(x) in the .... region.

Q7. Write sentence b using algebraic relationships instead of colours.

Your answer to question 6 is called the "discontinuity property"

Appendix 2

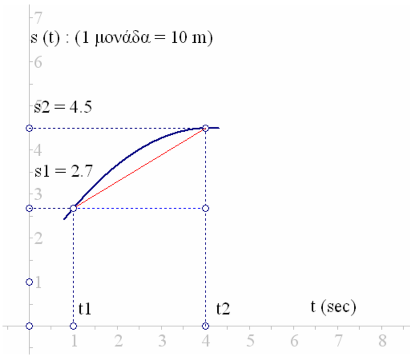
ACTIVITY 4.2.1 The Mean Value Theorem of Differential Calculus

WORKSHEET (Student Version, using *EucliDraw* software)

Step 1: Geometric representation of the Mean Value Theorem

The situation:

A train is moving in a straight line under an equation $s(t)$ which is continuous and smooth function of time, where t represents the time (h) and $s(t)$ the distance positive or negative (km) of the train from some origin.

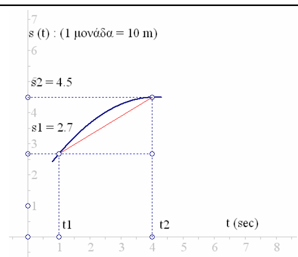


Q1: Can you find the average velocity of the train moving from s_1 to s_2 during the time interval [t_1 , t_2], where t_1 = 1 *sec* and t_2 = 4 *sec*?
How could you represent graphically the average velocity that you found?

Q2: Do you think that during the train movement from t_1 = 1 *sec* to t_2 = 4 *sec*, there is some instant t_0 , where the measure of the instant speed equals to the average speed you found before for the time interval t_1 , t_2]?

Q4: What is the principal mathematical notion underlying these expressions : *instantaneous velocity (or rate of change), the tangent of the graph of a function at a point?*

Q4: How can the value of the instant speed at an instant t_0 = 3 *sec* between t_1 and t_2 be represented graphically?
Could you try to sketch it in the graph given?

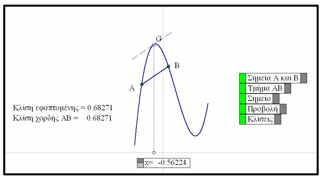


Q5: Could you give a geometrical interpretation for question Q2;

Q6: How could you express the conjecture above for a function f defined on an interval [x_1 , x_2];
What kind of attributes do you think that such a function must dispose?

.

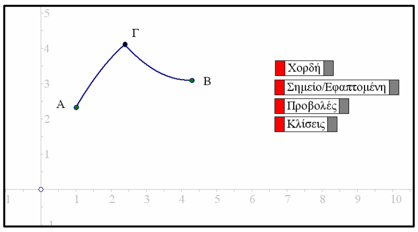
❖Open the *4.2.1 MVT Control.euc* EucliDraw file and press the appropriate buttons to see the environment:



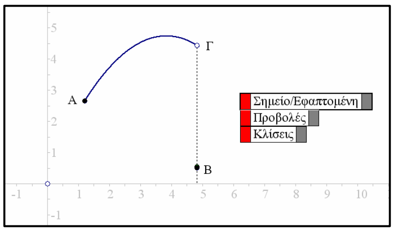
Q7: By moving point G between points *A* and *B*, could you check if there is some point x_0 of the open interval (x_1 , x_2), satisfying the conjecture of Q6?

Q8: For the following graphs is there any real number ξ, satisfying Q6's conjecture in the appropriate interval?

A)



B)



❖You can open the corresponding *4.2.2 ExampleA.euc* and *4.2.3 ExampleB.euc* EucliDraw files to verify with the help of counters if the conjecture formulated in Q6 is true.

Q9: What is the reason that makes Q6's conjecture not true for each of the previous cases?

Q10: How could you express the conjecture formulated in Q6 and Q9 by using mathematical symbols and terms?