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THE ECONOMIC AND ENVIRONMENTAL EFFECT OF REMANUFACTURING AUTHORIZATION IN THE FUZZY ENVIRONMENT

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Abstract. Remanufacturing has attracted considerable attention because it significantly contributes to a sustainable supply chain where used parts/products can be recycled to reduce waste. Faced with the rise of the remanufacturing business, many original equipment manufacturers (OEMs) choose to authorize independent remanufacturers (IRs) for profit. In this paper, we consider a sustainable supply chain consisting of an OEM and an IR in a fuzzy environment where production cost and market demand are uncertain to study whether such an authorization decision would benefit both the economy and the environment. An unauthorized model (Nash game model) and two authorized models (Cooperative game and Stackelberg game models) are developed. Our results show that the cooperation between the OEM and the IR increases the selling price of the product, and therefore encourages authorization financially. However, a leader position of the OEM (*i.e.*, in the Stackelberg game) helps achieve a win–win situation where profit maximization also improves the environment. Furthermore, our analysis illustrates that considering a fuzzy environment will boost the chances of authorization and help achieve a win–win situation.

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1. Introduction

Sustainable supply chains have gained significant attention due to the emerging need for Net Carbon Zero. According to the Paris Agreement reached in 2015, 194 states and the EU, representing over 98% of global greenhouse gas emissions (GHGs), have aimed to reduce global emissions to prevent the planet from warming by more than two degrees Celsius [41]. The UK government, for example, has developed policies and proposals for decarbonizing all sectors of the UK economy to meet the target by 2050. Driven by these goals, Accenture [1] reported that many CEOs plan to adopt circular business models, among which remanufacturing becomes popular in practice so that recycled (parts of) products can be used in production to minimize waste while generating profits. The higher the proportion of frequently recycled products utilized, the more significant the

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positive impact on the environment. Consequently, in alignment with the method presented in [43], our study adopts the demand for remanufactured products as a metric to gauge environmental commitment.

Xu [48] examined how remanufacturing can reduce energy consumption and save production costs compared to the classic model, resulting in significant ecological and economic benefits. The price of remanufactured products, due to the cost advantages, is often lower than that of new products, which attracts a good percentage of the population in the consumer market. In practice, original equipment manufacturers (OEMs) may decide to perform remanufacturing themselves either in-house or outsourced [43]. However, OEMs may lack remanufacturing capabilities [13] or face high remanufacturing costs due to the poor quality of used products [47], deterring their participation in remanufacturing initiatives.

The rapid expansion of the remanufacturing business has nevertheless forced OEMs to embrace the remanufacturing market. Taking electronics as an example, one may find both new and remanufactured smartphones in the market at the same time, and many of those remanufactured products (for example, those sold on eBay) may not be produced by the OEMs. Indeed, attracted by the high customer demand for remanufactured products, many independent remanufacturers (IRs) have entered the business. This undoubtedly imposes greater competitive pressure on new products produced by OEMs: price-sensitive customers tend to buy relatively cheap refurbished products, resulting in the market share of new products will decrease (see [2]). Remanufacturing authorization is therefore a better option for OEMs, as it shifts responsibility for production and sales to the IR [14, 59, 61]. Through authorization, OEMs can directly profit from remanufacturing by charging authorization fees and improving customers' willingness-to-pay (WTP) for remanufactured products [26, 58]. In addition, OEMs can opt for dual dedicated channels to separately manage the distribution of new and remanufactured products [37]. Consequently, OEMs are willing to authorize IRs to manage remanufactured products if doing so enhances their profitability. How to optimize the coexistence of both products or business parties to achieve a win–win situation that boosts the performance of such a supply chain financially and environmentally thus becomes a hot topic under investigation.

Significant research on remanufacturing authorization can be found in the literature review (see Sect. 2). It is worth noting that all the works mentioned above consider deterministic environments where all parameters are assumed to be known in advance. In the real world, however, incomplete and inaccurate information often leads to uncertainty in demand, costs, and other factors. In fact, uncertainty in supply chains is inevitable and cannot be disregarded. Some studies on remanufacturing have acknowledged this inherent uncertain characteristic (see [3, 9, 17, 20, 23, 38, 40, 50, 52, 53]). It is noted that they employ either probability theory or fuzzy set theory to characterize the uncertainty involved. Unlike those assumptions made in probabilistic models, experts and managers may not have accurate information due to the constantly evolving nature of data resulting from product updates and/or the absence of historical data in some cases, which precludes the derivation of a probability distribution function for the parameters [30,53]. Instead, industry experience and expertise can easily provide fuzzy information such as a range (*e.g.*, the maximum and minimum value it could take). Consequently, fuzzy set theory is favored as it utilizes a fuzzy variable representation based primarily on expert judgment and experience. However, after a thorough review of the literature, the authors have found no studies that specifically address the authorization problem in a fuzzy environment.

To fill this gap, this paper considers a sustainable supply chain consisting of an OEM and an IR, where production costs and demand for new and remanufactured products are treated as fuzzy variables. We examine the authorization decision and analyze the impact of authorization both economically and environmentally. To the best of the authors' knowledge, we are the first to explore remanufacturing authorization in a fuzzy environment (see Tab. 1). Moreover, in contrast to the literature [26, 55, 57, 58], we adopt a distinct approach to determining the authorization fee. Specifically, we distribute the profits generated from remanufactured products using the Shapley value, which is derived from a fuzzy cooperative game. The share allocated to OEMs is then utilized as the authorization fee. Note that authorization is a form of cooperation: the IR would need such authorization to improve customers' preferences. Meanwhile, the OEM gets profits in return (*via* authorization fees). Therefore, the authorization fee should be decided based on the surplus generated by such a coalition. We remark that solving the Shapley value in a fuzzy environment is technically challenging as the four basic axioms of the Shapley value are no longer satisfied. Hukuhara-Shapley value is therefore used to bypass such difficulty [51].

In addition, profitability and environmental goals are often known to conflict [43]. We therefore investigate the impact of factors like market demand, cost advantage of remanufactured products, quality of unlicensed remanufactured products, and customers' WTP on the authorization decision and environmental effect in the fuzzy setting. The ultimate goal is to understand if a win–win situation can be achieved (and when if possible). To this aim, we analyze two scenarios (unauthorized and authorized situations) and propose three models. In the unauthorized scenario, we establish a general Nash game (Model U) assuming OEM would compete against IR in the market. In the authorized scenario, they collaborate and the OEM's profit derives from two sources: original product sales and remanufactured product operations. We thus develop a cooperative game model (Model C) and a Stackelberg game model (Model S) separately. The former assumes that the OEM attaches importance to the remanufacturing of the used products and makes decisions simultaneously with the IR to maximize the profit of the cooperative game, while the latter means that the OEM places greater emphasis on maximizing its own profit and makes pricing decisions sequentially, prior to the IR's decisions. Additionally, a comparative numerical analysis is performed to evaluate differences between optimal solutions in fuzzy and deterministic settings. We find that authorization in the fuzzy environment yields higher profits and can help achieve economic and ecological benefits at once. Interestingly, our results indicate that it is easier to achieve such a win–win situation for Model S compared to Model C.

There are several contributions in our work that are enlisted as follows:

(i) The remanufacturing authorization problem in a fuzzy environment is considered for the first time;
(ii) The impact of integrating fuzzy variables into the model on authorization decisions, product pricing, member profits is examined in our study. Compared to the results in a deterministic environment, both the OEM and the IR are more likely to engage in authorized remanufacturing and more profitable in the fuzzy environment;
(iii) Given that many firms have increasingly prioritized energy conservation and environmental protection, both the economic and environmental impacts of authorization are discussed. It is observed that, in the fuzzy environment, a win–win situation is more likely to occur.

The remainder of this paper is organized as follows: Section 2 reviews the related research and outlines our motivation. Section 3 describes the preparations for constructing the model, including parameter settings, model assumptions, and notation. Section 4 develops an unauthorized game model and two authorized game models with theoretical analysis. Section 5 implements numerical analysis to verify the analytical results and provides additional insights. Section 6 offers managerial guidance for decision makers. Finally, Section 7 concludes our research and points out directions for future improvement.

2. Literature review

This section presents a literature survey on sustainable supply chain and remanufacturing, authorization, and fuzzy environment. Further, we identify the research gap and explain our motivation for the present study.

2.1. Sustainable supply chain and remanufacturing

In recent years, extensive research has been conducted on various aspects of supply chain operation and management, including sustainable supply chains [5, 29], and inventory management [6–8, 24, 28, 31–33]. It is noteworthy that remanufacturing is a significant topic in sustainable supply chain research, having attracted considerable attention from scholars. Tao *et al.* [40] explored the planning and operational decisions of a sustainable closed-loop supply chain, with particular emphasis on retail network configuration. Wei and Huang [45] modeled two different channel choices in a sustainable reverse supply chain to study the operational strategy of remanufactured products. Jin *et al.* [13] examined the effect of third-party remanufacturing on reverse supply chains. Wang *et al.* [43] explored whether profit maximization inevitably results in environmental harm by designing a reverse supply chain for remanufacturing from the retailer's perspective, where the remanufacturing process is either conducted internally or outsourced to a third party. The results suggested that, under certain conditions, both economic and environmental objectives can be achieved simultaneously. Wang *et al.* [44] investigated the two strategic choices available to the OEM when confronted with the impact of remanufactured products on new product sales: abandoning the remanufacturing business or collaborating with a third party. Through the development of two game theory models, the study found that while the OEM can consistently achieve profitability through cooperation, such collaboration may have adverse environmental implications. Haque *et al.* [10] studied a sustainable inventory model with a two-stage production process, where the first stage is responsible for manufacturing and the second stage is responsible for manufacturing and remanufacturing. Pervin *et al.* [34] developed an inventory model that considers investments in green technologies, enabling the production of new products as well as the remanufacturing of used products.

2.2. Authorization

Authorization is an important component of the product remanufacturing problem, it is supported by many OEMs as a strategy to enter secondary markets. Research on it can be developed in two ways. Firstly, authorization serves as a cooperative mechanism between OEMs and IRs to enable OEMs to access secondary market profits and achieve sustainability objectives. Hence, some studies have compared various remanufacturing autorization methods. [14,55,57]. Jin *et al.* [14] considered two modes of authorization, *i.e.*, dealer authorisation and remanufacturing authorisation, and derived the conditions for selecting which of them *via* theoretical analysis. Zheng and Jin [55] discussed and compared the equilibrium production quantity, price, consumer surplus, environmental performance, and social welfare under two remanufacturing licensing strategies: licensed retailers or third parties. Secondly, some scholars have also examined how the presence of authorized remanufactured products affects market demand for new products, the profits of OEMs, and supply chain decisions [12,21,44,59]. For example, Huang and Wang [12] investigated the equilibrium price decision when the remanufacturing behavior was performed by each member of the supply chain separately, and a random variable describing the existence of market demand information sharing between the manufacturer and the distributor was added to the model. Ma *et al.* [21] explored the conditions for cooperation or competition between an OEM and an unauthorized remanufacturer when they choose to license the latter to perform remanufacturing in a market environment where new and remanufactured products coexist. Furthermore, it is also worthwhile to study the rational formulation of authorization fees to ensure that the profits of OEMs and IRs can be improved. Based on existing relevant studies, three main forms of authorization fees can be identified [11, 26, 54, 55, 58–60]: OEM-determined fees, contractual payment mechanisms, and power-based negotiations among stakeholders. Zhou *et al.* [60] studied the conditions under which the IR chooses to accept technology licensing with a licensing fee determined by the OEM or to self-develop it in-house. Oraiopoulos *et al.* [26] considered a licensing fee mechanism where the OEM gains fees from customers who purchase refurbished products. Zhou *et al.* [59] constructed a Nash bargaining game model, namely, the optimal authorization fee is determined according to the bargaining power of the two parties, which is more practical than the assumption that it is completely determined by the OEM, as seen in other studies. It is important to highlight that the aforementioned studies on remanufacturing authorization were conducted under deterministic conditions. However, uncertainty is an inherent characteristic of real-world supply chain operations, and numerous studies have considered this dimension, yielding promising outcomes.

2.3. Fuzzy environment

In the existing literature, Zhao *et al.* [53] investigated pricing strategies for substitutable products under fuzzy conditions, modeling both manufacturing costs and customer demand as fuzzy variables. Building upon this work, Wei *et al.* [46] extended the analysis by considering remanufacturing cost as a fuzzy variable, and examined the pricing decision problem in a closed-loop supply chain. Furthermore, Alamdar *et al.* [3], Liu *et al.* [20], and Zhang *et al.* [52] explored the pricing and collection decisions in a fuzzy closed-loop supply chain by using game theory and fuzzy sets theory. Moreover, the research [52] compared the experimental results between

Table 1. Overview of the most relevant literature.

	Year	SSC	Remanufacturing	Authorization	Fuzzy environment
Zhao *et al.* [54]	2019		√	√ (A relicensing fee machine)	
Tao *et al.* [40]	2020	√			
Yan *et al.* [49]	2021		√		√
Zhou *et al.* [58]	2021		√	√ (GNBG)	
Zheng and Jin [55]	2022		√	√ (The OEM determines)	
Barman *et al.* [5]	2023	√			√
Haque *et al.* [10]	2024		√		
Wei and Huang [45]	2024	√	√		
Our paper		√	√	√ (CG)	√

Notes. The SSC denotes a sustainable supply chain. The GNBG denotes a generalized nash bargaining game. The CG denotes a cooperative game.

fuzzy and deterministic environments and discovered that in the fuzzy environment, collection firms gather a greater quantity of used products and achieve higher total profits for both individual members and the entire supply chain. And Karimabadi [15] studied the pricing and remanufacturing decisions in a fuzzy dual-channel supply chain. Additionally, Kumar *et al.* [17] modeled production and ordering costs as fuzzy numbers and established a reverse logistics system to support customers in addressing issues related to defective products and carbon emissions. Their results showed that the fuzzy model is economically superior to the deterministic model. Sugapriya *et al.* [38] presented two models with intuitionistic fuzzy parameters to analyze the impact of manufacturing and remanufacturing processes in the context of changing supply chains. Barman *et al.* [4] developed an economic production model incorporating cloudy fuzzy parameters to optimize inventory costs, with key variables including demand rate and inflation rate. Mondal *et al.* [22] designed a three-dimensional decision model in which the Fermatean fuzzy set is utilized to assess the ambiguity of information for effective supply chain management. These articles predominantly examined pricing, collection, and remanufacturing decisions, as well as production and inventory management within supply chains. To a certain extent, they demonstrated that incorporating fuzzy factors can be advantageous for supply chain operations.

After synthesizing and analyzing the existing literature, it is crucial to integrate fuzzy variables, including production costs and market demand, when examining the remanufacturing authorization issue. This approach enhances the analysis from both practical market perspectives and research findings. In fact, our results demonstrate that remanufacturing authorization and member profitability are more conducive in a fuzzy environment, and that there is a higher probability of achieving an economic and environmental win–win situation. The differences between the existing literature and the current study are summarized in Table 1.

3. Problem description

In this section, we first introduce the authorization problem under investigation with assumptions for each parameter. Subsequently, we briefly introduce the fuzzy game theory and fuzzy operators needed for solving the model.

3.1. Model assumption

Consider a supply chain consisting of an original equipment manufacturer (OEM) and an independent remanufacturer (IR), where the OEM and the IR are responsible for the production and sale of mutually substitutable new and remanufactured products in the same market, respectively. Confronted with the detrimental impact of unauthorized remanufactured products on the market share of its new products, the OEM chooses to authorize the IR, thereby generating revenues from the remanufacturing market to offset the lost profits from the new product. The IR can decide whether or not to accept the authorization. If IR accepts it, it will produce authorized remanufactured products for which customers have a higher WTP. However, this decision also requires the IR to share a portion of the profits from the remanufactured products with the OEM as a fee for obtaining the authorization.

Without loss of generality, let $\tilde{c}_n$ and $\tilde{c}_r$ denote the unit production costs of new and remanufactured products, respectively. We use the symbol "$\sim$" to represent that they are fuzzy (we will explain this further in Sect. 3.2 below), and the subscripts n and r indicate new and remanufactured products, respectively. The same symbols appearing thereafter are consistent with this definition. Normally, the remanufactured product is made by utilizing a used product, which implies that $\tilde{c}_n > \tilde{c}_r$. Hence, we denote the cost advantage of remanufacturing as $\alpha(\alpha \in (0,1))$, indicating the proportion of the cost of remanufacturing relative to the new product. Based on the view that the cost of a remanufactured product is a quadratic function of its quality [27], we derive the production cost of a remanufactured product in the unauthorized scenario as $\tilde{c}_r = \alpha\lambda^2\tilde{c}_n$. And $\lambda(\lambda \in (0,1))$ denotes the unauthorized remanufactured product's quality. For simplicity, the quality level of new products is normalized to 1. Nevertheless, IR normally lacks quality control and technical support compared to OEM. For authorized remanufactured products, we consider the situation "as good as new", that their quality is the same as 1. This is because the remanufactured product is inspected, cleaned, and repaired to meet the OEM's quality requirements in an authorized situation, and its quality can be guaranteed to be close to the new product. Thus, the production cost of the remanufactured product in the authorization scenario is $\tilde{c}_r = \alpha\tilde{c}_n$.

The demand for two substitutable products can be represented as a linear function of their selling prices [9,20,53]. This function is characterized by two critical parameters: the self-price coefficients and the cross-price coefficients. The former measures the sensitivity of demand for a product in response to changes in its own price, while the latter captures the sensitivity of demand for a product to changes in the price of its substitute counterpart. In addition, demand is also related to the product's quality level (λ) and the customers' WTP (δ), and the WTP for remanufactured products is often believed to be inferior to the new product [58,60]. From the literature [54,57,58], the demand functions in a deterministic environment are obtained. We maintain the format in the paper. Instead of standardizing the market size to 1, we assume that new products and remanufactured products have different market potentials ($\tilde{a}$ and $\tilde{b}$, respectively). Indeed, this is consistent with the literature that authorization can improve the quality of products and increase customers' WTP at the cost of the market share for new products [58]. Consequently, the demand functions for the two products in the unauthorized scenario are derived:

$$\widetilde{D}_n^U = \tilde{a}^U - \frac{1}{1-\delta\lambda}p_n^U + \frac{1}{1-\delta\lambda}p_r^U, \tag{1}$$

$$\widetilde{D}_r^U = \tilde{b}^U - \frac{1}{\delta\lambda(1-\delta\lambda)}p_r^U + \frac{1}{1-\delta\lambda}p_n^U, \tag{2}$$

where the cross-price coefficients (effects) for new products and remanufactured products are asymmetric. Particularly, the demand for new products is less affected by the price of remanufactured products, and we ensure that there is no demand for remanufactured products when the quality is too low (*i.e.*, $\lambda \to 0$). The definitions of the symbols used in the formulas are provided in Table 2.

Similarly, for authorized scenario, we obtain the demand functions as follows:

$$\widetilde{D}_n^A = \tilde{a}^A - \frac{1}{1-\delta}p_n^A + \frac{1}{1-\delta}p_r^A, \tag{3}$$

$$\widetilde{D}_r^A = \tilde{b}^A - \frac{1}{\delta(1-\delta)}p_r^A + \frac{1}{1-\delta}p_n^A. \tag{4}$$

Following the argument above, we further assume that the market potential satisfies: $E(\tilde{a}^U) > E(\tilde{a}^A) > E(\tilde{b}^A) > E(\tilde{b}^U)$ and $E(\tilde{a}^A) + E(\tilde{b}^A) > E(\tilde{a}^U) + E(\tilde{b}^U)$. Namely, new products have a higher market potential in the unauthorized scenario, and the market potential of remanufactured products should not exceed the one for new products. Moreover, we assume authorization increases the total market potential; otherwise, the coalition

Table 2. The definition of the symbols.

Symbol	Definition
Model parameters	
α	An indicator of remanufacturing cost advantage: the smaller the value, the larger the advantage
δ	The customers' willingness-to-pay for the remanufactured product, and $\delta \in (0,1)$
λ	An indicator of the remanufactured product quality relative to the new product
$\tilde{c}_n$	Cost of producing a new product
$\tilde{c}_r^i$	Cost of producing a remanufactured product in scenario i. Particularly, $\tilde{c}_r^U = \alpha\lambda^2\tilde{c}_n$, $\tilde{c}_r^A = \alpha\tilde{c}_n$
$\tilde{a}^i$	Potential market demand of new products in scenario i
$\tilde{b}^i$	Potential market demand of remanufactured products in scenario i
Decision variables	
$p_n^{ij^*}$	Optimal sale price of a new product in scenario i and model j
$p_r^{ij^*}$	Optimal sale price of a remanufactured product in scenario i and model j
Other notations	
i	$i \in \{U, A\}$, U denotes unauthorization scenario; A denotes authorization scenario
j	$j \in \{U, C, S\}$ represent three different models respectively
$E^{*j}(\pi^i_{\text{OEM}})$	Optimal expected profit of the OEM in scenario i and model j
$E^{*j}(\pi^i_{\text{IR}})$	Optimal expected profit of the IR in scenario i and model j
$E(\pi^c)$	Expected profit of cooperation game

would be economically unsustainable. In addition, the expected demand for new and remanufactured products in both authorized and unauthorized scenarios meets $E(\widetilde{D}_n) \geq E(\widetilde{D}_r)$, which ensures that the used products available for remanufacturing are adequate. According to [43], we assess the environmental benefits in this paper by examining the demand for remanufactured products, *i.e.*, the higher the demand for remanufactured products, the greater the environmental improvements (due to recycled products).

Finally, the premise of this study being situated in a mature market implies that we assume all activities within the supply chain occur within a single period. This approach is consistent with the main body of existing research on remanufacturing [14, 27, 36, 53, 61].

3.2. Precursor fuzzy set theory

To account for the incompletion of information, we employ the common triangle fuzzy numbers in this paper. That is, for a triangle fuzzy number $\tilde{z} = (x_1, x_2, x_3)$, one can view x_1 as the smallest possible value, x_2 the most likely value, and x_3 the biggest possible value for any parameter. For fuzzy numbers, the basic operations need to be re-defined. Let two independent triangular fuzzy numbers $\tilde{x} = (x_1, x_2, x_3)$ and $\tilde{y} = (y_1, y_2, y_3)$. Following [8], we define:

(i) $\tilde{x} + \tilde{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$;
(ii) $\tilde{x} - \tilde{y} = (x_1 - y_3, x_2 - y_2, x_3 + y_1)$;
(iii) $\eta\tilde{x} = (\eta x_1, \eta x_2, \eta x_3)$, if η is a nonnegative number; $\mu\tilde{x} = (\mu x_3, \mu x_2, \mu x_1)$, if μ is negative number;
(iv) $\tilde{x} \times \tilde{y} = (x_1 y_1, x_2 y_2, x_3 y_3)$.

Consequently, the solution of a fuzzy game is different from that of any conventional model. Shapley value, for example, which has been widely used in the literature [19, 56], is no longer available in a fuzzy environment as its four axioms are not fulfilled [51]. To bypass this difficulty, we follow the research results in [51] and use the Hukuhara-Shapley value instead to assign the surplus as shown in equation (5).

$$Sh_i(N,\widetilde{\omega}) = \sum_{T\in P(N\backslash\{i\})} \frac{|T|!\cdot(|N|!-|T|!-1)}{|N|!}\cdot[\widetilde{\omega}(T\cup\{i\})\ominus\widetilde{\omega}(T)], \qquad \forall i\in N, \tag{5}$$

where the symbol $\ominus$ denotes the Hukuhara difference, and its definition is provided in Definition 3.1. That is, let $\tilde{x}=(x_1,x_2,x_3)$ and $\tilde{y}=(y_1,y_2,y_3)$ be triangular fuzzy numbers in the real number domain if a Hukuhara difference exists, then $\tilde{x}\ominus\tilde{y}=(x_1-y_1,x_2-y_2,x_3-y_3)$.

Definition 3.1 (See [51])**.** If there exists $\tilde{z}\in\widetilde{R}$ such that $\tilde{x}=\tilde{y}+\tilde{z}$, $\tilde{x},\tilde{y}\in\widetilde{R}$, then $\tilde{z}$ is called the Hukuhara difference of $\tilde{x}$ and $\tilde{y}$.

Like probabilistic models, we then need to solve the expectations of the objective function containing fuzzy variables. However, the introduction of the Hukuhara difference means the theory of expectation of fuzzy variables proposed in [42] cannot be applied here. Therefore, we follow [18] to alleviate the issue *via* defuzzification. The calculation of the expectation of a triangular fuzzy number is given in equation (6).

$$E(\tilde{z}) = \frac{z_1+2z_2+z_3}{4}\cdot \tag{6}$$

4. Model formulation and analytical solution

In this section, we first consider the situation where no authorization has been agreed upon and formulate the Nash game (Model U) as the benchmark. In the authorized scenarios, we formulate a cooperative game (Model C) and a Stackelberg game (Model S) individually. Finally, we compare the analytical solutions of Model C and Model S to derive theoretical insights into their profitability. Remark that the analytical results are in a very complex form. We will present a detailed comparison of these three models in the next section *via* numerical examples.

4.1. Unauthorized scenario (Model U)

We consider the scenario where there is no authorization contract between the OEM and the IR, that is, the OEM does not enter the remanufacturing market but is only involved in the production and sale of new products, while the IR is engaged in the production and sale of unauthorized remanufactured products. Given the quality of the unauthorized remanufactured product, we construct a Nash game process. That is, the OEM and the IR simultaneously decide the prices of new and unauthorized remanufactured products, respectively, with profits derived from their respective product sales. Consequently, their profits are expressed as follows:

$$\pi_{\mathrm{OEM}}^U = \left(p_n^U-\tilde{c}_n\right)\left(\tilde{a}^U - \frac{1}{1-\delta\lambda}p_n^U + \frac{1}{1-\delta\lambda}p_r^U\right), \tag{7}$$

$$\pi_{\mathrm{IR}}^U = \left(p_r^U-\tilde{c}_r^U\right)\left(\tilde{b}^U - \frac{1}{\delta\lambda(1-\delta\lambda)}p_r^U + \frac{1}{1-\delta\lambda}p_n^U\right). \tag{8}$$

According to the arithmetic rules of the triangular fuzzy numbers defined in Section 3, we calculate the triangular fuzzy number form of the above profit functions and then defuzzify them using equation (6). Thus, the expected profits and optimization problems of the members can be obtained, respectively:

$$\max E\left(\pi_{\mathrm{OEM}}^U\right) = -\frac{1}{1-\delta\lambda}\left(p_n^U\right)^2 + \frac{1}{1-\delta\lambda}p_n^Up_r^U + \left(E\left(\tilde{a}^U\right)+\frac{E(\tilde{c}_n)}{1-\delta\lambda}\right)p_n^U - \frac{E(\tilde{c}_n)}{1-\delta\lambda}p_r^U + E\left(-\tilde{c}_n\tilde{a}^U\right), \tag{9}$$

$$\max E\left(\pi_{\mathrm{IR}}^U\right) = -\frac{1}{\delta\lambda(1-\delta\lambda)}\left(p_r^U\right)^2 + \frac{1}{1-\delta\lambda}p_n^Up_r^U + \left(E\left(\tilde{b}^U\right)+\frac{\alpha\lambda E(\tilde{c}_n)}{\delta(1-\delta\lambda)}\right)p_r^U$$

$$-\frac{\alpha\lambda^2 E(\tilde{c}_n)}{1-\delta\lambda}p_n^U + \alpha\lambda^2 E\Big(-\tilde{c}_n\tilde{b}^U\Big). \tag{10}$$

Given the quality of the remanufactured product, we employ the Karush–Kuhn–Tucker (KKT) conditions to solve the decision problems. The optimal prices for both the new and the remanufactured product can be found in Proposition 4.1, with the corresponding optimal profits provided in Appendix A. To save space, we defer the derivation to Appendix C.

Proposition 4.1. *In Model U, there exists thresholds:* $\alpha_1^U = \frac{\delta(1-\delta\lambda)((2-\delta\lambda)\mathrm{E}(\tilde{b}^U)-\mathrm{E}(\tilde{a}^U))+\delta(3-\delta\lambda)\mathrm{E}(\tilde{c}_n)}{2\lambda\mathrm{E}(\tilde{c}_n)}$ *and* $\alpha_2^U = \frac{\delta(1-\delta\lambda)(\mathrm{E}(\tilde{a}^U)+2\mathrm{E}(\tilde{b}^U))+\delta\mathrm{E}(\tilde{c}_n)}{\lambda(2-\delta\lambda)\mathrm{E}(\tilde{c}_n)}$*, such that*

(i) *if* $\alpha < \alpha_1^U$*, the IR will collect and remanufacture all used products. And the optimal pricing decisions for the OEM and the IR are respectively:* $p_n^{UU^*} = \frac{\delta(1-\delta\lambda)(2\mathrm{E}(\tilde{a}^U)-(1-\delta\lambda)\mathrm{E}(\tilde{b}^U))+\alpha\lambda(1+\delta\lambda)\mathrm{E}(\tilde{c}_n)}{\delta(3-\delta\lambda)}$ *and* $p_r^{UU^*} = \frac{\delta\lambda(1-\delta\lambda)(\mathrm{E}(\tilde{a}^U)+\mathrm{E}(\tilde{b}^U))+2\alpha\lambda^2\mathrm{E}(\tilde{c}_n)}{3-\delta\lambda}$;

(ii) *if* $\alpha_1^U \le \alpha \le \alpha_2^U$*, the IR will collect and remanufacture some of used products. And the optimal pricing decisions for the OEM and the IR are respectively:* $p_n^{UU^*} = \frac{(1-\delta\lambda)(2\mathrm{E}(\tilde{a}^U)+\delta\lambda\mathrm{E}(\tilde{b}^U))+(2+\alpha\lambda^2)\mathrm{E}(\tilde{c}_n)}{4-\delta\lambda}$ *and* $p_r^{UU^*} = \frac{\delta\lambda(1-\delta\lambda)(\mathrm{E}(\tilde{a}^U)+2\mathrm{E}(\tilde{b}^U))+\lambda(2\alpha\lambda+\delta)\mathrm{E}(\tilde{c}_n)}{4-\delta\lambda}$;

(iii) *if* $\alpha > \alpha_2^U$*, then the IR will not implement remanufactured activities due to high production cost of remanufactured products.*

To guarantee that the thresholds α_1^U and α_2^U satisfy $\alpha_1^U < \alpha_2^U$, we require $E(\tilde{a}^U) + \delta\lambda E(\tilde{b}^U) > E(\tilde{c}_n)$. Specifically, the constraint is in agreement with [59] when both $E(\tilde{a}^U) = 1$ and $E(\tilde{b}^U) = 0$ hold. The optimal price decisions reveal that the quantity of used products collected by the IR for remanufacturing is related to remanufacturing costs. Thus, the equilibrium pricing solutions diverge across alternative remanufacturing strategy scenarios.

When the remanufacturing cost is relatively low (*i.e.*, $\alpha < \alpha_1^U$), the expected demand for the remanufactured product equals that of the new product. With the increase in remanufacturing cost (*i.e.*, $\alpha_1^U \le \alpha \le \alpha_2^U$), the IR will only collect and remanufacture partially used products. Further, the IR does not enter the remanufacturing market if the cost is high (*i.e.*, $\alpha > \alpha_2^U$). Moreover, the prices of the new and remanufactured products are related to the production cost and quality of the remanufactured product. It is plausible that the prices of both remanufactured and new products increase as the cost advantage of remanufacturing decreases. Note that with a price change, the demand for both new and remanufactured products will adjust accordingly.

4.2. Authorized scenario

If the authorization can be agreed upon between the OEM and the IR, the OEM will guarantee the quality of remanufactured products, thereby enhancing customers' WTP for these products. Consequently, such collaboration would enlarge the total profit potential of new and remanufactured products, as otherwise at least one of them would bear a loss and reject the authorization. For this reason, we view authorization as a form of cooperation and propose calculating the authorization fee as the share of the surplus (of remanufactured products) allocated *via* the Shapley value. To this end, we first derive the total profitability of remanufactured products in the authorization scenario.

$$\pi^c = \big(p_r^A - \tilde{c}_r^A\big)\bigg(\tilde{b}^A - \frac{1}{\delta(1-\delta)}p_r^A + \frac{1}{1-\delta}p_n^A\bigg). \tag{11}$$

Following equation (5), the benefits accrued by the OEM and IR from the collaboration are, respectively, $\frac{1}{2}\pi^c \ominus \frac{1}{2}(\pi_{\mathrm{IR}}^U)^*$ and $\frac{1}{2}\pi^c + \frac{1}{2}(\pi_{\mathrm{IR}}^U)^*$, where the $(\pi_{\mathrm{IR}}^U)^*$ is IR's optimal profit in the unauthorized scenario. The profit functions of both parties can then be formulated.

$$\pi_{\mathrm{OEM}}^A = \big(p_n^A - \tilde{c}_n\big)\bigg(\tilde{a}^A - \frac{1}{1-\delta}p_n^A + \frac{1}{1-\delta}p_r^A\bigg) + \frac{1}{2}\pi^c \ominus \frac{1}{2}\big(\pi_{\mathrm{IR}}^U\big)^*, \tag{12}$$

$$\pi_{\mathrm{IR}}^{A} = \frac{1}{2}\pi^{c} + \frac{1}{2}\left(\pi_{\mathrm{IR}}^{U}\right)^{*}, \tag{13}$$

where the OEM makes profits from both new products and the coalition of remanufactured products, while the IR only receives profits from the latter. Note here the shares of profits of remanufactured products for OEM and IR could be different, depending on their original market conditions and other factors.

Applying the Hukuhara difference and operations defined in Section 3.2, we obtain the expected profitability for OEM and IR, respectively.

$$\begin{aligned} E(\pi_{\mathrm{OEM}}^{A}) = & -\frac{1}{1-\delta}\left(\left(p_n^A\right)^2 + \frac{1}{2\delta}\left(p_r^A\right)^2\right) + \frac{3}{2(1-\delta)}p_n^A p_r^A + \left(E\left(\tilde{a}^A\right) + \frac{E(\tilde{c}_n)}{1-\delta}\right. \\ & \left. - \frac{\alpha E(\tilde{c}_n)}{2(1-\delta)}\right)p_n^A + \left(\frac{1}{2}\left(E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)}\right) - \frac{E(\tilde{c}_n)}{1-\delta}\right)p_r^A + E\left(-\tilde{c}_n\tilde{a}^A\right) \\ & + \frac{1}{2}\alpha E\left(-\tilde{c}_n\tilde{b}^A\right) - \frac{1}{2}E^{*U}\left(\pi_{\mathrm{IR}}^{U}\right), \end{aligned} \tag{14}$$

$$\begin{aligned} E(\pi_{\mathrm{IR}}^{A}) = & -\frac{1}{2\delta(1-\delta)}\left(p_r^A\right)^2 + \frac{1}{2(1-\delta)}p_n^A p_r^A + \frac{1}{2}\left(E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)}\right)p_r^A - \frac{\alpha E(\tilde{c}_n)}{2(1-\delta)}p_n^A \\ & + \frac{1}{2}\alpha E\left(-\tilde{c}_n\tilde{b}^A\right) + \frac{1}{2}E^{*U}\left(\pi_{\mathrm{IR}}^{U}\right). \end{aligned} \tag{15}$$

Due to the nature of this authorization problem, we assume that the OEM has greater bargaining power (in terms of pricing) in this coalition. Therefore, the OEM can decide to cooperate with the IR and make joint decisions on pricing. This would result in a cooperative game, which we denote as Model C. Alternatively, the OEM can take advantage of its bargaining power in the collaboration, where it acts as the leader who only maximizes its own profits and forces the IR to follow its pricing decision. This would then form a Stackelberg game, which we denote as Model S. We will explain how these two models can be formulated and solved in the following sections.

4.2.1. Cooperative game (Model C)

We now consider the situation where OEM and IR would make the joint decision on pricing. In this scenario, the prices of new and remanufactured products will be made simultaneously to maximize the profitability of remanufactured products. They then share the profits according to the Hukuhara-Shapley value given in the previous section. The expected profit from cooperation can be formulated as follows.

$$\begin{aligned} & \max_{p_n^A, p_r^A} E(\pi^c) = -\frac{1}{\delta(1-\delta)}\left(p_r^A\right)^2 + \frac{1}{1-\delta}p_n^A p_r^A + \left(E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)}\right)p_r^A - \frac{\alpha E(\tilde{c}_n)}{1-\delta}p_n^A + \alpha E\left(-\tilde{c}_n\tilde{b}^A\right), \\ & \text{subject to } E\left(\widetilde{D}_n^A\right) \geq E\left(\widetilde{D}_r^A\right) > 0. \end{aligned} \tag{16}$$

Although the objective function is not concave, optimal decisions can also be obtained by the KKT necessary conditions (one can check the sufficient conditions in [25], page 345, Thm. 12.6). The corresponding results are summarized in Proposition 4.2.

Proposition 4.2. *In Model C, there exists a threshold* $\alpha^C = \frac{\delta(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))}{\mathrm{E}(\tilde{c}_n)}$*, such that*

(i) *if* $\alpha < \alpha^C$*, then the IR will collect and remanufacture all used products. And the optimal pricing for the OEM and IR are respectively:* $p_n^{AC^*} = \frac{\delta((3-\delta)\mathrm{E}(\tilde{a}^A)+(3\delta-1)\mathrm{E}(\tilde{b}^A))+\alpha(1+\delta)\mathrm{E}(\tilde{c}_n)}{4\delta}$ *and* $p_r^{AC^*} = \frac{\delta(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))+\alpha\mathrm{E}(\tilde{c}_n)}{2}$*;*
(ii) *if* $\alpha \geq \alpha^C$*, then the IR will not remanufacture used products.*

Proposition 4.2 reveals an interesting observation that IR has only two options in Model C: (1) IR should recycle and remanufacture all used products; (2) IR should not produce anything at all. In other words, if OEM and IR decide to collaborate and exploit the remanufactured market, they will only do so if the remanufacturing advantage is large enough (*i.e.*, α is small), and it is never optimal for them to remanufacture only part of the used products.

4.2.2. Stackelberg game (Model S)

If the OEM decides to maximize its own profits rather than those of the coalition, this would lead to a Stackelberg game where the OEM is the leader and the IR is the follower. In this case, the OEM has the power to set the price for new products after observing the price for remanufactured products. Therefore, we model it as a bi-level problem, where the IR first decides the price for remanufactured products, and then the OEM decides the price for new products.

$$\begin{cases} \max_{p_n^A} & E(\pi_{\text{OEM}}^A), \\ \text{subject to} & E\left(\widetilde{D}_n^A\right) \geq E\left(\widetilde{D}_r^A\right), \\ & \begin{cases} \max_{p_r^A} & E(\pi^c), \\ \text{subject to} & E\left(\widetilde{D}_n^A\right) \geq E\left(\widetilde{D}_r^A\right) > 0. \end{cases} \end{cases} \tag{17}$$

Again, solving the model above *via* the KKT condition would lead to Proposition 4.3 below.

Proposition 4.3. *In Model S, there exists two thresholds* $\alpha_1^S = \frac{\delta(1-\delta)((3\delta-2)\mathrm{E}(\tilde{a}^A)+(\delta^2-2\delta+4)\mathrm{E}(\tilde{b}^A))+\delta(2-\delta)(3-\delta)\mathrm{E}(\tilde{c}_n)}{2(2-\delta^2)\mathrm{E}(\tilde{c}_n)}$ *and* $\alpha_2^S = \frac{\delta(1-\delta)(2\mathrm{E}(\tilde{a}^A)+(4-\delta)\mathrm{E}(\tilde{b}^A))+\delta(2-\delta)\mathrm{E}(\tilde{c}_n)}{(4-3\delta)\mathrm{E}(\tilde{c}_n)}$ *, such that*

(i) *if* $\alpha < \alpha_1^S$*, the IR will collect and remanufacture all used products. And the optimal pricing for the OEM and IR are respectively:* $p_n^{AS^*} = \frac{\delta(1-\delta)(2\mathrm{E}(\tilde{a}^A)-(1-\delta)\mathrm{E}(\tilde{b}^A))+\alpha(1+\delta)\mathrm{E}(\tilde{c}_n)}{\delta(3-\delta)}$ *and* $p_r^{AS^*} = \frac{\delta(1-\delta)(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))+2\alpha\mathrm{E}(\tilde{c}_n)}{3-\delta}$*;*

(ii) *if* $\alpha_1^S \leq \alpha < \alpha_2^S$*, the IR will collect and remanufacture some of used products. And the optimal pricing decisions for the OEM and the IR are respectively:* $p_n^{AS^*} = \frac{(1-\delta)(4\mathrm{E}(\tilde{a}^A)+3\delta\mathrm{E}(\tilde{b}^A))+(4-2\delta+\alpha)\mathrm{E}(\tilde{c}_n)}{8-5\delta}$ *and* $p_r^{AS^*} = \frac{\delta(1-\delta)(2\mathrm{E}(\tilde{a}^A)+(4-\delta)\mathrm{E}(\tilde{b}^A))+(2-\delta)(2\alpha+\delta)\mathrm{E}(\tilde{c}_n)}{8-5\delta}$*;*

(iii) *if* $\alpha \geq \alpha_2^S$*, then the IR will not conduct remanufacture.*

Note here we need $E(\tilde{a}^A) + \delta E(\tilde{b}^A) > E(\tilde{c}_n)$ to ensure that $\alpha_1^S \leq \alpha < \alpha_2^S$, which implies that the remanufacturing cost should be smaller than the weighted market base of new and remanufactured products. Then, we can see that Proposition 4.3 is similar to Proposition 4.1, where IR's optimal decision will be subjected to the market condition and the remanufacturing cost.

4.2.3. Comparative analysis of profits

In this section, we will compare the profits of OEM and IR in Models C, S, and U to identify the conditions for cooperation in the authorized model. We will explore which model would be preferred (and under what conditions) if, for example, OEM has such power and freedom.

By examining Propositions 4.2 and 4.3, we first derive the following Corollary 4.1. It is evident that the threshold conditions in Models U and C align consistently with Lemma 2 in the literature [60] when the production cost of the new product and the quality of the unauthorized remanufactured product fall within a specified range. Specifically, the thresholds in the unauthorized case exceed those in the authorized case, meaning that it is easier to remanufacture full demand without authorization. This is because, in the authorization scenario, the partnership between the OEM and the IR results in the IR sharing a portion of its profits with the OEM for authorization, which directly affects the IR's profitability.

Corollary 4.1. *Define the thresholds* $\mathrm{E}(\tilde{c}_n)_1$*,* $\mathrm{E}(\tilde{c}_n)_2$*,* $\mathrm{E}(\tilde{c}_n)_3$*,* λ_1*, and* λ_2 *given in Appendix B. We have:*

(i) *The boundary conditions for Models U and S are related as follows:* $\alpha_2^U > \alpha_2^S > \alpha_1^U > \alpha_1^S$ *when* $\mathrm{E}(\tilde{c}_n) \in (\max\{\mathrm{E}(\tilde{c}_n)_1, \mathrm{E}(\tilde{c}_n)_2\}, \mathrm{E}(\tilde{c}_n)_3)$ *and* $\lambda \in (0, \min\{\lambda_1, \lambda_2\})$*;*

(ii) *The boundary conditions for Models C and S are related as follows:* $\alpha^C > \alpha_2^S > \alpha_1^S$ *when* $\mathrm{E}(\tilde{a}^A) + \delta\mathrm{E}(\tilde{b}^A) > \mathrm{E}(\tilde{c}_n)$*.*

However, the threshold values above do not necessarily reflect the profitability of the two parties as the demand for remanufactured products, which is bounded by the demand for new products, would change dynamically.

Rationally, the two parties would collaborate only if the profits of both will increase compared to those in the unauthorized scenario. Based on Corollary 4.1, we discuss the profitability of the IR and the OEM individually subjected to different ranges of the remanufacturing cost advantage (*i.e.*, α). Technically, we require $E(\tilde{c}_n) > \frac{(1-\delta)((2-3\delta)E(\tilde{a}^A)-(4-2\delta+\delta^2)E(\tilde{b}^A))}{(2-\delta)(3-\delta)}$ to ensure that $\alpha_1^S > 0$. All propositions are proved in Appendix C.

Proposition 4.4. *The optimal profits of the OEM for* $0 < \alpha < \alpha_1^S$ *in Models C and S are related as follows:*

(i) *if* $0 < \delta < \frac{\sqrt{57}-7}{2}$ *and* $\mathrm{E}(\tilde{c}_n) < \frac{(5-11\delta)\mathrm{E}(\tilde{a}^A)-(7-5\delta+4\delta^2)\mathrm{E}(\tilde{b}^A)}{4(3-\delta)}$, *then* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) < \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for any* $\alpha < \alpha_1^S$*;*

(ii) *if* $0 < \delta < \frac{\sqrt{57}-7}{2}$ *and* $\mathrm{E}(\tilde{c}_n) > \frac{(5-11\delta)\mathrm{E}(\tilde{a}^A)-(7-5\delta+4\delta^2)\mathrm{E}(\tilde{b}^A)}{4(3-\delta)}$, *then there is a point* $\bar{\alpha}$ *that makes* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) > \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for* $0 < \alpha < \bar{\alpha}$ *and* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) < \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for* $\bar{\alpha} < \alpha < \alpha_1^S$*;*

(iii) *if* $\frac{\sqrt{57}-7}{2} < \delta < 1$, *then* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) > \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for any* $\alpha < \alpha_1^S$.

Proposition 4.5. *The optimal profits of the OEM for* $\alpha_1^S \le \alpha < \alpha_2^S$ *in Models C and S are related as follows:*

(i) *if* $0 < \delta < \frac{\sqrt{57}-7}{2}$, *then* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) < \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for any* $\alpha_1^S \le \alpha < \alpha_2^S$*;*

(ii) *if* $\frac{\sqrt{57}-7}{2} < \delta < \frac{6}{7}$, *then there is a point* $\bar{\bar{\alpha}}$ *that makes* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) > \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for* $\alpha_1^S \le \alpha < \bar{\bar{\alpha}}$ *and* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) < \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for* $\bar{\bar{\alpha}} < \alpha < \alpha_2^S$*;*

(iii) *if* $\frac{6}{7} < \delta < 1$, *then* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}}) > \mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})$ *for any* $\alpha_1^S \le \alpha < \alpha_2^S$.

Propositions 4.4 and 4.5 describe the situation in which the OEM would find it profitable in Model S compared to in Model C. Intuitively, and as discussed above, both parties would always remanufacture the "full demand" (subjected to the demand for new products) in Model C, but not always in Model S. This suggests that the OEM will produce more new products in Model S if the cost of new products is not high (*i.e.*, $E(\tilde{c}_n) < \frac{(5-11\delta)E(\tilde{a}^A)-(7-5\delta+4\delta^2)E(\tilde{b}^A)}{4(3-\delta)}$) and/or the customers' WTP for remanufactured products is not high (*i.e.*, $\delta < \frac{\sqrt{57}-7}{2}$) when the remanufacturing cost advantage is large enough (*i.e.*, $\alpha < \alpha_2^S$).

Furthermore, when the cost advantage is very large (*i.e.*, $\bar{\alpha} < \alpha < \alpha_1^S$), Model S would be preferred if the cost of new products is high (*i.e.*, $E(\tilde{c}_n) > \frac{(5-11\delta)E(\tilde{a}^A)-(7-5\delta+4\delta^2)E(\tilde{b}^A)}{4(3-\delta)}$) and the customers' WTP is low (*i.e.*, $\delta < \frac{\sqrt{57}-7}{2}$). For a smaller cost advantage (*i.e.*, $\bar{\bar{\alpha}} < \alpha < \alpha_2^S$), we require a moderate customers' WTP (*i.e.*, $\frac{\sqrt{57}-7}{2} < \delta < \frac{6}{7}$) for Model S to be preferred.

Conversely, if customers like remanufactured products (*i.e.*, $\delta > \frac{\sqrt{57}-7}{2}$) and the cost advantage is large enough (*i.e.*, $\alpha < \bar{\bar{\alpha}}$), then remanufacturing becomes attractive to the OEM and it is more profitable in Model C. Even if the customers' WTP is low (*i.e.*, $\delta < \frac{\sqrt{57}-7}{2}$), the OEM still profits more in Model C if the cost of the new product is not low (*i.e.*, $E(\tilde{c}_n) > \frac{(5-11\delta)E(\tilde{a}^A)-(7-5\delta+4\delta^2)E(\tilde{b}^A)}{4(3-\delta)}$) and the cost advantage is relatively high (*i.e.*, $\alpha < \bar{\alpha}$).

Simply put, OEM will choose Model C if remanufactured products are more profitable and Model S otherwise. However, for IR, since it only generates profits from remanufactured products, Model C is always preferred, as shown in Proposition 4.6 below.

Proposition 4.6. *The optimal profits of the IR in Models C and S always satisfy* $\mathrm{E}^{*C}(\pi^A_{\mathrm{IR}}) > \mathrm{E}^{*S}(\pi^A_{\mathrm{IR}})$ *for* $0 < \alpha < \alpha_2^S$.

The rationale is straightforward as both the OEM and the IR would prioritize the profits from remanufactured products in Model C. Intuitively, Model C is a better deal for both the OEM and the IR, as long as the OEM achieves a higher profit in this model. However, this does not mean that Model S is unlikely to be chosen, as the profit for the OEM in Model C could be lower than those in Model S (according to Props. 4.4(ii) and 4.5(ii)) or even Model U (as observed in Sect. 5.2).

The comparison between Model C (or Model S) and Model U is however much more complicated. For Model C, a relatively large remanufacturing cost advantage (*i.e.*, $\alpha < \alpha_1^U$) may render the OEM more profits in the authorized scenario than in the unauthorized one, particularly when the quality level of the unauthorized remanufactured product, the customers' WTP for the remanufactured product, and the relative difference in potential demands for the two products are not large, while the cost of the new product remains relatively high. Given such a cost advantage, however, the conditions under which IR accepts authorization are different and makes it extremely difficult to analytically present when both parties would accept the authorization.

As the cost of remanufacturing increases (*i.e.*, $\alpha_1^U < \alpha < \alpha_2^S$), the moderate remanufacturing cost advantage promotes the IR to achieve higher profits in authorization compared to the unauthorized scenario given that the quality level of the unauthorized remanufactured product is not high, the customers' WTP is sufficiently high, and the cost of the new product is not low.

For Model S, a large remanufacturing cost advantage (*i.e.*, $\alpha < \alpha_1^S$) is favorable for the OEM to authorize under the condition that the quality of the unauthorized remanufactured product and the cost of the new product are not high. In contrast, a moderate remanufacturing cost (*i.e.*, $\alpha_1^S < \alpha < \alpha_1^U$) makes it more likely that the IR will accept the authorization when the cost of new products is not high.

Again, the conditions under which the two can cooperate are complicated in analysis. Therefore, we leave the details in Appendix B and use numerical experiments in Section 5.2 to visualize the effect of different parameters on the authorization decisions.

4.2.4. Comparative analysis of environmental impact

In this section, we further discuss the environmental impact of authorization for Models C and S. Following [43], we use the demand for remanufactured products to measure environmental friendliness as it implies more products to be recycled.

Proposition 4.7. *When $0 < \alpha < \alpha_1^U$, the remanufactured product's demands in Models C and U are related as follows:*

(i) *if* $0 < \delta < \min\{\frac{\alpha \mathrm{E}(\tilde{c}_n)+\sqrt{\alpha \mathrm{E}(\tilde{c}_n)(\alpha \mathrm{E}(\tilde{c}_n)+16(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A)))}}{2(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))}, 1\}$, *the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $\max\{0, \bar{\lambda}\} < \lambda < \max\{\bar{\lambda}, 1\}$ *and the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $0 < \lambda < \max\{0, \bar{\lambda}\}$*;*

(ii) *if* $\min\{\frac{\alpha \mathrm{E}(\tilde{c}_n)+\sqrt{\alpha \mathrm{E}(\tilde{c}_n)(\alpha \mathrm{E}(\tilde{c}_n)+16(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A)))}}{2(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))}, 1\} < \delta < 1$, *the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $0 < \lambda < \max\{0, \bar{\lambda}\}$ *and the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $\max\{0, \bar{\lambda}\} < \lambda < \max\{\bar{\lambda}, 1\}$*,*

where the $\bar{\lambda} = \frac{\delta(4(\mathrm{E}(\tilde{a}^U)+\mathrm{E}(\tilde{b}^U))-3(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A)))+3\alpha \mathrm{E}(\tilde{c}_n)}{(4+\delta)\alpha \mathrm{E}(\tilde{c}_n)-\delta^2(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))}$.

Proposition 4.7 posits that the demand for remanufactured products in Model C, relative to the unauthorized scenario, is influenced by customers' WTP for remanufactured products and the quality level of the unauthorized remanufactured products. The IR collects a greater number of products for remanufacturing in Model C only under the conditions of low WTP and high quality levels (*i.e.*, the conclusion (i)), or high WTP and low quality levels (*i.e.*, the conclusion (ii)). Intuitively, the higher quality of a remanufactured product incurs a higher production cost. Therefore, when customers exhibit lower enthusiasm for remanufactured products, the absence of quality advantages in unauthorized contexts leads to a further decline in demand for these products. However, when the remanufactured product is attractive enough to customers, its low quality in the unauthorized scenario exacerbates the quality difference between it and that in the authorized scenario, indicating that demand for the remanufactured product in the authorized scenario will be higher compared to the unauthorized one.

Proposition 4.8. *The demands for remanufactured products are related in Models C and U for $\alpha_1^U < \alpha < \alpha_2^S$ as follows:*

(i) *if* $0 < \lambda < \frac{8+5\delta-\sqrt{9\delta^2+16\delta+64}}{2\delta(4+\delta)}$, *then the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\alpha_1^U, \max\{\alpha_1^U, \min\{\alpha_2^S, \alpha^*\}\})$ *and the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\max\{\alpha_1^U, \alpha^*\}, \max\{\alpha_2^S, \alpha^*\})$;

(ii) *if* $\frac{8+5\delta-\sqrt{9\delta^2+16\delta+64}}{2\delta(4+\delta)} < \lambda < 1$, *then the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\max\{\alpha_1^U, \alpha^*\}, \max\{\alpha_2^S, \alpha^*\})$ *and the* $\mathrm{E}^{*C}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\alpha_1^U, \max\{\alpha_1^U, \min\{\alpha_2^S, \alpha^*\}\})$,

where the $\alpha^* = \frac{\delta(1-\delta\lambda)((4-\delta\lambda)(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))-4(\mathrm{E}(\tilde{a}^U)+2\mathrm{E}(\tilde{b}^U)))-4\delta\mathrm{E}(\tilde{c}_n)}{((1-\delta\lambda)(4-\delta\lambda)-4\lambda(2-\delta\lambda))\mathrm{E}(\tilde{c}_n)}$.

When the remanufacturing cost advantage is smaller, Proposition 4.8 implies the market demand for low-quality remanufactured products remains limited, even under relatively low remanufacturing costs (*i.e.*, $\alpha < \max\{\alpha_1^U, \min\{\alpha_2^S, \alpha^*\}\}$). In other words, it is more environmentally beneficial to authorize (*i.e.*, authorization would guarantee quality and thus increase demand). As the quality of unauthorized remanufactured products improves, their production costs and selling prices would rise. Nevertheless, high-quality authorized remanufactured products become more appealing to customers even at a higher cost.

Proposition 4.9. *When* $0 < \alpha < \alpha_1^S$, *the remanufactured product's demands in Models S and U are related as follows:*

(i) *if* $0 < \delta < \min\{\sqrt{\frac{3\alpha\mathrm{E}(\tilde{c}_n)}{\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A)}}, 1\}$, *the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $\max\{0, \lambda^*\} < \lambda < \max\{\lambda^*, 1\}$ *and the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $0 < \lambda < \max\{0, \lambda^*\}$;

(ii) *if* $\min\{\sqrt{\frac{3\alpha\mathrm{E}(\tilde{c}_n)}{\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A)}}, 1\} < \delta < 1$, *the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $0 < \lambda < \max\{0, \lambda^*\}$ *and the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $\max\{0, \lambda^*\} < \lambda < \max\{\lambda^*, 1\}$,

where the $\lambda^* = \frac{\delta((3-\delta)(\mathrm{E}(\tilde{a}^U)+\mathrm{E}(\tilde{b}^U))-3(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A)))+3\alpha\mathrm{E}(\tilde{c}_n)}{3\alpha\mathrm{E}(\tilde{c}_n)-\delta^2(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))}$.

Proposition 4.10. *The remanufactured product's demands for* $\alpha_1^S < \alpha < \alpha_1^U$ *in Models S and U are related as follows:*

(i) *if* $0 < \lambda < \max\{0, \lambda^{**}\}$, *the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\alpha_1^S, \max\{\alpha_1^S, \min\{\alpha_1^U, \alpha^{**}\}\})$ *and the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\max\{\alpha_1^S, \alpha^{**}\}, \max\{\alpha_1^U, \alpha^{**}\})$;

(ii) *if* $\max\{0, \lambda^{**}\} < \lambda < \max\{\lambda^{**}, 1\}$, *the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\max\{\alpha_1^S, \alpha^{**}\}, \max\{\alpha_1^U, \alpha^{**}\})$ *and the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds for* $\alpha \in (\alpha_1^S, \max\{\alpha_1^S, \min\{\alpha_1^U, \alpha^{**}\}\})$,

where the $\alpha^{**} = \frac{\delta^2(1-\delta)(2\mathrm{E}(\tilde{a}^A)+(4-\delta)\mathrm{E}(\tilde{b}^A))+\delta^2(2-\delta)\mathrm{E}(\tilde{c}_n)}{(2\delta^2-9\delta+8)\mathrm{E}(\tilde{c}_n)}$ *and the* λ^{**} *is presented in Appendix B.*

Now we derive Propositions 4.9 and 4.10 for Model S. Similarly, customers' low (high) WTP and high (low) quality level of remanufactured products would motivate the IR to collect more used products in the Model S when the cost advantage of remanufacturing is large (*i.e.*, $\alpha < \alpha_1^S$). With the increase in remanufacturing costs (*i.e.*, $\alpha_1^S < \alpha < \alpha_1^U$), however, the condition becomes different. When the quality level of unauthorized remanufactured products is not high, the cost of remanufacturing must fall below a certain threshold (*i.e.*, $\alpha < \max\{\alpha_1^S, \min\{\alpha_1^U, \alpha^{**}\}\}$) for authorization to be environmentally beneficial. If both the quality level and the remanufacturing cost are relatively high, the market demand for remanufactured products in the authorized scenario will also be higher. This is because authorization can increase the customers' WTP and ensure that the remanufactured product matches the quality of a new one, thereby mitigating the negative impact of high remanufacturing costs on demand.

Proposition 4.11. *The demands for remanufactured products in Models S and U for* $\alpha_1^U < \alpha < \alpha_2^S$ *are related as follow:*

(i) *The* $\mathrm{E}^{*S}(\widetilde{D}_r^A) > \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $\alpha \in (\alpha_1^U, \max\{\alpha_1^U, \min\{\alpha_2^S, \alpha^{***}\})$;

(ii) *The* $\mathrm{E}^{*S}(\widetilde{D}_r^A) < \mathrm{E}^{*U}(\widetilde{D}_r^U)$ *holds when* $\alpha \in (\max\{\alpha_1^U, \alpha^{***}\}, \max\{\alpha_2^S, \alpha^{***}\})$, *where the* α^{***} *is provided in Appendix B.*

Given a smaller remanufacturing cost advantage (*i.e.*, $\alpha_1^U < \alpha < \alpha_2^S$), IR would collect and remanufacture only partial products in Models S and U. Proposition 4.11 indicates that a lower remanufacturing cost would lead to a higher demand for remanufactured products in the authorized scenario, making it more environmentally friendly.

Recall Corollary 4.1, it would be easier to have full demand for remanufactured products in Model C than in Model S. That is, due to the contribution of the OEM in Model C, the IR can get full demand with a smaller remanufacturing cost advantage (*i.e.*, $\alpha_1^S < \alpha^C$). This directly leads to the observation that when the remanufacturing advantage is relatively small (*i.e.*, $\alpha_2^S < \alpha < \alpha^C$), IR will remanufacture all used products in Model C and produce nothing in Model S, which means the authorization only works in Model C. When the remanufacturing cost advantage is moderate (*i.e.*, $\alpha < \alpha_2^S$), however, we need to check the optimal prices for the new and remanufactured products.

Corollary 4.2. *The optimal prices of two products in the authorized scenario satisfy* $p_n^{AC^*} > p_n^{AS^*}$, $p_r^{AC^*} > p_r^{AS^*}$ *for any* $\alpha < \alpha_2^S$ *when* $\mathrm{E}(\tilde{a}^A) + \delta\mathrm{E}(\tilde{b}^A) > \mathrm{E}(\tilde{c}_n)$.

Corollary 4.2 shows that the optimal selling price of new and remanufactured products in Model C will always be higher than in Model S. In Model S, the OEM would reduce the price of new products to increase its market share. Consequently, IR would lower the price of the remanufactured products to maintain demand. When the remanufacturing cost advantage decreases (*i.e.*, remanufacturing cost becomes closer to the cost of producing new products), the remanufacturing market will shrink in Model S (because the price of remanufactured products is less attractive due to the low price of new products) and result in less profitability for remanufactured products. However, the OEM could potentially compensate by selling new products.

Proposition 4.12. *When* $\mathrm{E}(\tilde{a}^A) + \delta\mathrm{E}(\tilde{b}^A) > \mathrm{E}(\tilde{c}_n)$, *the demands for remanufactured products in Models C and S for* $0 < \alpha < \alpha_2^S$ *are related as follow:*

(i) *The* $\mathrm{E}^{*S}(\widetilde{D}_r^A) > \mathrm{E}^{*C}(\widetilde{D}_r^A)$ *is always satisfied for* $\alpha \in (0, \alpha_1^S)$;

(ii) *The* $\mathrm{E}^{*S}(\widetilde{D}_r^A) > \mathrm{E}^{*C}(\widetilde{D}_r^A)$ *is satisfied when* $\alpha \in \left(\alpha_1^S, \frac{\delta(1-\delta)(5\delta\mathrm{E}(\tilde{a}^A)+(8+\delta)\mathrm{E}(\tilde{b}^A))+4\delta(2-\delta)\mathrm{E}(\tilde{c}_n)}{(8+\delta-5\delta^2)\mathrm{E}(\tilde{c}_n)}\right)$ *and the* $\mathrm{E}^{*S}(\widetilde{D}_r^A) < \mathrm{E}^{*C}(\widetilde{D}_r^A)$ *is satisfied when* $\alpha \in \left(\frac{\delta(1-\delta)(5\delta\mathrm{E}(\tilde{a}^A)+(8+\delta)\mathrm{E}(\tilde{b}^A))+4\delta(2-\delta)\mathrm{E}(\tilde{c}_n)}{(8+\delta-5\delta^2)\mathrm{E}(\tilde{c}_n)}, \alpha_2^S\right)$.

Proposition 4.12 shows that the demand for remanufactured products in Model S is always higher compared to Model C when IR will do full remanufacturing in both scenarios. That is, Model S is more environmentally efficient. Indeed, this can be derived directly from the conclusion of Corollary 4.2 as a lower unit price for new products always attracts more customers, thus creating a higher demand for remanufactured products.

With the rising cost of remanufacturing, although IR will only collect and remanufacture partially used products in Model S, the demand for remanufactured products can be higher when such cost is below a certain threshold (*i.e.*, $\alpha < \frac{\delta(1-\delta)(5\delta E(\tilde{a}^A)+(8+\delta)E(\tilde{b}^A))+4\delta(2-\delta)E(\tilde{c}_n)}{(8+\delta-5\delta^2)E(\tilde{c}_n)}$). We notice that the demand for remanufactured products is higher in Model C when remanufacturing costs are relatively high, even though the selling price of remanufactured products is higher (recall Cor. 4.2). This is because the OEM values the profit from the new product more in Model S, causing it to lower its expectation of profit from the remanufacturing market in situations where remanufacturing is more costly.

To summarize, though it appears that Model C could be more environmentally friendly at first glance, it turns out that Model S has a better environmental impact than Model C in many situations by creating a larger market for remanufactured products.

5. Numerical results

For better illustration, in this section, we use numerical experiments to compare the performances of Models U, C, and S introduced above to derive further insights. We use Model U as the benchmark, and authorization

TABLE 3. Model parameter values.

Parameter	Unauthorized scenario	Authorized scenario
Manufacturing cost $\tilde{c}_n$	$(0.1, 0.2, 0.25)$	$(0.1, 0.2, 0.25)$
Market base $\tilde{a}^i$	$(0.8, 0.85, 0.9)$	$(0.75, 0.8, 0.9)$
Market base $\tilde{b}^i$	$(0.2, 0.3, 0.4)$	$(0.4, 0.45, 0.5)$
Customers' WTP δ	0.3	0.3
Quality level λ	0.4	1

TABLE 4. Optimal expected profit for the members in the deterministic environment with different cost advantages.

		$\alpha < 0.5503$ ($\alpha = 0.2$)	$0.5503 \leq \alpha < 0.6081$ ($\alpha = 0.56$)	$0.6081 \leq \alpha < 1$ ($\alpha = 0.8$)
Model U	$E^{*U}(\pi^U_{\text{OEM}})$	0.0988	0.1031	0.1046
	$E^{*U}(\pi^U_{\text{IR}})$	0.0153	0.0127	0.0105
Model C	$E^{*C}(\pi^A_{\text{OEM}})$	**0.1196**	**0.1067**	0.0944
	$E^{*C}(\pi^A_{\text{IR}})$	**0.031**	**0.0208**	0.0149
Model S	$E^{*S}(\pi^A_{\text{OEM}})$	0.0919	**0.1063**	0.1118
	$E^{*S}(\pi^A_{\text{IR}})$	0.0256	**0.0173**	0.01

Notes. The boundary values of α are obtained from Propositions 4.1–4.3, where the values in brackets are used for calculation. Authorization can be achieved for highlighted data.

will be accepted/feasible only if the profits obtained in Model C/S for both parties can be higher than in Model U. Furthermore, we compare the model results in the fuzzy environment with those in the deterministic setting. Sensitivities of parameters on authorization decisions and environmental effects are then analyzed. The experimental data satisfying the model assumptions are summarized in Table 3.

5.1. Fuzzy environment *vs.* deterministic environment

Before presenting the results of our fuzzy models, we first summarize the results of our model as if all parameters are deterministic and known. To this aim, we follow the literature and use the most probable value of the triangular fuzzy number for the deterministic setting [16, 52]. The optimal results are summarized in Tables 4–6.

The boundary conditions (of α) in Tables 4–11 for IR to implement different remanufacturing strategies across various models are obtained from Propositions 4.1–4.3. That is, for a remanufacturing cost advantage of $\alpha < 0.5503$, IR will collect and remanufacture all used products in both authorized and unauthorized scenarios. With the decrease in the cost advantage of remanufacturing (*i.e.*, $0.5503 \leq \alpha < 0.6081$), IR will remanufacture all used products in Models U and C, and part of used products in Model S. IR will perform full remanufacturing only in Model C, and partial remanufacturing in Models U and S when the cost advantage is small, *i.e.*, $0.6081 \leq \alpha < 1$. For simplicity, we take $\alpha = 0.2$, 0.56, and 0.8 for each remanufacturing strategy, respectively.

Recall that authorization can be agreed upon only if the OEM and IR make more profits from the partnership (*i.e.*, the highlighted data in Tab. 4). When IR performs full remanufacturing in the authorized and unauthorized scenarios (*i.e.*, $\alpha < 0.5503$), an authorized partnership can be established in Model C. With a smaller remanufacturing cost advantage (*i.e.*, $0.5503 \leq \alpha < 0.6081$), authorization can be achieved in both Models C and S.

TABLE 5. Impact of authorization cooperation on optimal sale prices (compared to Model U) in the deterministic environment.

		$\alpha < 0.5503$ ($\alpha = 0.2$)	$0.5503 \leq \alpha < 0.6081$ ($\alpha = 0.56$)	$0.6081 \leq \alpha < 1$ ($\alpha = 0.8$)
Model C	Δp_n^{*C}	+0.1126	+0.1532	/
	Δp_r^{*C}	+0.1609	+0.1889	/
Model S	Δp_n^{*S}	/	+0.0132	/
	Δp_r^{*S}	/	+0.1252	/

Notes. The boundary values of α are obtained from Propositions 4.1–4.3, where the values in brackets are used for calculation.

TABLE 6. Impact of authorization cooperation on customer demand (compared to Model U) in the deterministic environment.

		$\alpha < 0.5503$ ($\alpha = 0.2$)	$0.5503 \leq \alpha < 0.6081$ ($\alpha = 0.56$)	$0.6081 \leq \alpha < 1$ ($\alpha = 0.8$)
Model C	$\Delta E^{*C}(\widetilde{D}_n)$	−0.1016	−0.1283	/
	$\Delta E^{*C}(\widetilde{D}_r)$	−0.1016	−0.1283	/
Model S	$\Delta E^{*S}(\widetilde{D}_n)$	/	−0.0194	/
	$\Delta E^{*S}(\widetilde{D}_r)$	/	−0.0248	/

Notes. The boundary values of α are obtained from Propositions 4.1–4.3, where the values in brackets are used for calculation.

On the premise that the authorization relationship is established, Table 5 gives the variation in the sales price of the product concerning the authorization decision, where $\Delta p_n^{*C(S)}$ and $\Delta p_r^{*C(S)}$ denote the price difference between the new and remanufactured products in Model C(S) and the corresponding product in Model U, respectively. The results indicate that authorization cooperation induces the OEM and IR to increase the sales prices of new and remanufactured products, respectively, compared to unauthorized ones, which is consistent with the findings in literature [60]. Notably, the price of remanufactured products rises more than that of new products. Indeed, increasing the price gap between new and remanufactured products could help mitigate the negative impact of the remanufacturing market on the market for new product.

Rising prices inevitably affect customer demand, as illustrated in Table 6. In this table, $\Delta E^{*C(S)}(\widetilde{D}_n)$ and $\Delta E^{*C(S)}(\widetilde{D}_r)$ represent the expected demand for new and remanufactured products in Model C(S), respectively, subtracting the expected demand for the corresponding products in Model U. In this deterministic setting, it is plausible that the market demand for both new and remanufactured products is reduced under the authorized scenario compared to the unauthorized scenario. Combined with Table 4, the reduced market demand in Models C and S implies that the authorization would hurt the environment. This conflict between the economic and environmental benefits in Models C and S arises because the increase in the price of remanufactured products in the authorized scenario suppresses customer demand.

In comparison with results in the deterministic context, we observe that such a conflict would ease, which means one could actually achieve a win–win situation with both economic and environmental benefits in the fuzzy setting (see Tabs. 7 and 9). Particularly, we use the same α values as in the deterministic environment to facilitate the comparison. From Table 7, it is observed that the IR is consistently more profitable in Model C than in Model S, regardless of whether the remanufacturing cost advantage is high or low, which is consistent with Proposition 4.6. Indeed, authorization can be achieved in Model C for high and moderate cost advantages,

TABLE 7. Optimal expected profit for the members in the fuzzy environment with different cost advantages.

		$\alpha < 0.5589$ ($\alpha = 0.2$)	$0.5589 \leq \alpha < 0.5766$ ($\alpha = 0.56$)	$0.5766 \leq \alpha < 1$ ($\alpha = 0.8$)
Model U	$E^{*U}(\pi^U_{\text{OEM}})$	0.1053	0.1091	0.1104
	$E^{*U}(\pi^U_{\text{IR}})$	0.0155	0.0133	0.0112
Model C	$E^{*C}(\pi^A_{\text{OEM}})$	**0.1306**	**0.1182**	0.1067
	$E^{*C}(\pi^A_{\text{IR}})$	**0.0322**	**0.0228**	0.0172
Model S	$E^{*S}(\pi^A_{\text{OEM}})$	0.1035	**0.1172**	**0.1224**
	$E^{*S}(\pi^A_{\text{IR}})$	0.0266	**0.0192**	**0.0122**

Notes. The boundary values of α are obtained from Propositions 4.1–4.3, where the values in brackets are used for calculation. Authorization can be achieved for highlighted data.

TABLE 8. Optimal selling prices in the deterministic (fuzzy) environments with different cost advantages.

		$\alpha = 0.2$		$\alpha = 0.56$		$\alpha = 0.8$	
		p^*_n	p^*_r	p^*_n	p^*_r	p^*_n	p^*_r
Deterministic environment	Model U	**0.4595**	**0.0466**	**0.4969**	**0.0546**	**0.5034**	**0.0588**
	Model C	0.5721	0.2075	0.6501	0.2435	0.7021	0.2675
	Model S	0.3973	0.1269	0.5101	0.1798	0.5175	0.2049
Fuzzy environment	Model U	**0.4582**	**0.0463**	**0.4932**	**0.0538**	**0.4966**	**0.0576**
	Model C	0.5778	0.2081	0.6509	0.2419	0.6997	0.2644
	Model S	0.3998	0.126	0.5078	0.1759	0.5148	0.1995

Notes. Price decisions in the fuzzy environment are more consumer-friendly for the highlighted data.

TABLE 9. Optimal expected demand for two product types in the deterministic (fuzzy) environments with different cost advantages.

		$\alpha = 0.2$		$\alpha = 0.56$		$\alpha = 0.8$	
		$E^*(\widetilde{D}_n)$	$E^*(\widetilde{D}_r)$	$E^*(\widetilde{D}_n)$	$E^*(\widetilde{D}_r)$	$E^*(\widetilde{D}_n)$	$E^*(\widetilde{D}_r)$
Deterministic environment	Model U	0.3808	0.3808	0.3475	0.3475	0.3448	0.3148
	Model C	0.2792	0.2792	0.2192	0.2192	0.1792	0.1792
	Model S	0.4136	0.4136	0.3281	0.3227	0.3534	0.2137
Fuzzy environment	Model U	0.3819	0.3819	0.3507	0.3507	0.3512	0.3185
	Model C	0.2844	0.2844	0.2281	0.2281	0.1906	0.1906
	Model S	0.4213	0.4213	0.3383	0.3377	0.3621	0.2355

Notes. Authorization can be achieved in both fuzzy and deterministic environment for highlighted data.

TABLE 10. Impact of authorization cooperation on optimal sale prices (compared to Model U) in the fuzzy environment.

		$\alpha < 0.5589$ ($\alpha = 0.2$)	$0.5589 \leq \alpha < 0.5766$ ($\alpha = 0.56$)	$0.5766 \leq \alpha < 1$ ($\alpha = 0.8$)
Model C	Δp_n^{*C}	+0.1196	+0.1577	/
	Δp_r^{*C}	+0.1618	+0.1881	/
Model S	Δp_n^{*S}	/	+0.0146	+0.0182
	Δp_r^{*S}	/	+0.1221	+0.1419

Notes. The boundary values of α are obtained from Propositions 4.1–4.3, where the values in brackets are used for calculation.

TABLE 11. Impact of authorization cooperation on customer demand (compared to Model U) in the fuzzy environment.

		$\alpha < 0.5589$ ($\alpha = 0.2$)	$0.5589 \leq \alpha < 0.5766$ ($\alpha = 0.56$)	$0.5766 \leq \alpha < 1$ ($\alpha = 0.8$)
Model C	$\Delta E^{*C}(\widetilde{D}_n)$	−0.0975	−0.1226	/
	$\Delta E^{*C}(\widetilde{D}_r)$	−0.0975	−0.1226	/
Model S	$\Delta E^{*S}(\widetilde{D}_n)$	/	−0.0124	+0.0109
	$\Delta E^{*S}(\widetilde{D}_r)$	/	−0.013	−0.083

Notes. The boundary values of α are obtained from Propositions 4.1–4.3, where the values in brackets are used for calculation.

whilst Model S would be feasible for moderate and low cost advantage; recall Propositions 4.4 and 4.5. Compared to the optimal results in Table 4, we find that in the fuzzy environment, with the same remanufacturing cost advantage and game model, the supply chain members are more profitable due to the smaller effect of price on demand.

Tables 8 and 9 characterize the optimal price decisions and expected demand for new and remanufactured products in both the fuzzy and deterministic environments with different models and remanufacturing cost advantages, respectively. Table 8 clearly shows that, for the same cost advantage, both new and remanufactured products are consistently priced higher in Model C compared to Model S, regardless of whether market uncertainty is considered. This observation implies that the conclusions drawn in Corollary 4.2 remain valid even in a deterministic setting. In addition, according to the highlighted data in the table, price decisions made in a fuzzy environment tend to be more consumer-friendly in the unauthorized scenario. Table 9 indicates that for the same model, there is a greater market demand for both products in the fuzzy environment compared to the deterministic environment. This (partly) explains why the conflicts between economic and environmental benefits can be eased.

Similar to Tables 5 and 6, we further discuss the effect of authorization on the price and demand of the two types of products in the fuzzy environment and obtain Tables 10 and 11. Since the influence of authorization on price decisions in the fuzzy environment is similar to that in the deterministic environment, as shown in Table 10, we will not reiterate it here. Table 11 concludes that, in Model C, customer demand is lower than in the unauthorized case because authorization increases the product's price, and the higher the remanufacturing cost, the more significant the impact of price on demand. Interestingly, the market demand for the new product increases in Model S when the remanufacturing cost is high (*i.e.*, $\alpha = 0.8$), even though the price of the new product increases. The reason for this is that the price gap between the new product and the authorized remanufactured product is narrowed, thus making the new product in Model S more attractive.

5.2. The economic and environmental impact of parameters

The comparative analysis of OEM's/IR's profits is complex; recall Section 4.2.3. Specifically, the authorization decisions of the OEM and IR are constrained by multiple parameters. To better comprehend the impact of these parameters, the changes in the authorization decisions in Models C and S with respect to the cost advantage of remanufacturing α, the customers' WTP for the remanufactured products δ, and the quality level of unauthorized remanufactured products λ are considered below, holding constant the values of the new product's production cost and the market conditions (introduced in Tab. 4). Again, we compare the results in fuzzy and deterministic contexts.

Figure 1 depicts the regions where authorized cooperation can be realized in different contexts and the environmental impacts of the corresponding authorization models when the cost advantage of remanufacturing is high ($i.e.$, $\alpha = 0.3$) and low ($i.e.$, $\alpha = 0.8$). The red lines draw the border that the profit of OEM/IR in Model S equals the profit in Model U. The magenta (or blue) line marks the edge where the profit of OEM/IR in Model C is equal to that in Model U (or S). The green line denotes that the total profits of the OEM and IR are equal in Models C and S. And the black dotted and dashed lines represent the demand for remanufactured products in Models C and S, respectively, being equivalent to the demand in Model U (and thus gives the indicator of environmental effect). In Figure 1(Environmental efforts), to the right of the black dotted line, the demand for remanufactured products is lower in Model C compared to Model U. Similarly, within the region bounded by the black dashed line, the demand is also lower in Model S than in Model U. In Figure 1(Authorization models in area III), the green line separates region III into two areas, with the left side indicating that the total profit is higher for the OEM and IR in Model S, and the opposite on the right side.

According to Figure 1(Authorization regions), it is shown that high remanufacturing costs also render the remanufacturing market less appealing to IR, which leads to an expansion of the unauthorized region in both the deterministic and fuzzy settings. However, authorization is always easier to achieve in the fuzzy environment, regardless of whether the remanufacturing cost is high or low.

On the economic side, for Model C, from Figure 1(Authorization regions), the OEM and IR opt to cooperate in Model C when the customers' WTP is not low, and the likelihood of establishing a partnership in this model is greater compared to Model S (according to the authorization region). Especially, Model C is more likely to be chosen when the cost advantage is high, as seen by comparing Figures 1A with 1G, and 1D with 1J, respectively. Because the elevated selling price in Model C diminishes demand for remanufactured products when the customers' WTP is particularly low and the remanufacturing cost is high. Moreover, a comparison of Figures 1A with 1D and 1G with 1J reveals that the OEM and IR are more inclined to cooperate on authorization in a fuzzy environment. Additionally, the total profit of OEM and IR is higher in Model C than that in Model S for relatively high WTP based on Figure 1(Authorization models in area III).

For Model S, when the cost advantage is high, it is chosen in situations where the customers' WTP and quality of unauthorized remanufactured products are not high, or the WTP is low but the quality level is high. With the decrease of cost advantage, the possibility of opting for the authorized Model S to improve profitability becomes more significant in both deterministic and fuzzy environments by comparing Figures 1A and 1G, and 1D and 1J, respectively. This is because an increase in the cost of remanufacturing results in a higher price for both new and remanufactured products, with the price increase being more significant for remanufactured products. Consequently, customers are more inclined to purchase new products. Moreover, while IR causes a reduction in demand due to higher prices, the increased unit price of remanufactured products can offset this loss and potentially generate additional revenue (based on Tabs. 8 and 9). When the customers' WTP is relatively low based on Figure 1(Authorization models in area III), both parties will choose to authorize cooperation in Model S (based on total profits). In other words, a low willingness to pay suggests that customers are less likely to opt for remanufactured products. Consequently, the OEM will focus more on the new product market, making Model S a more favorable option.

On the environmental side, the economic and environmental benefits in Model C always conflict when remanufacturing costs are low (based on Figs. 1B and 1E). However, Figures 1H and 1K show that moderate WTP

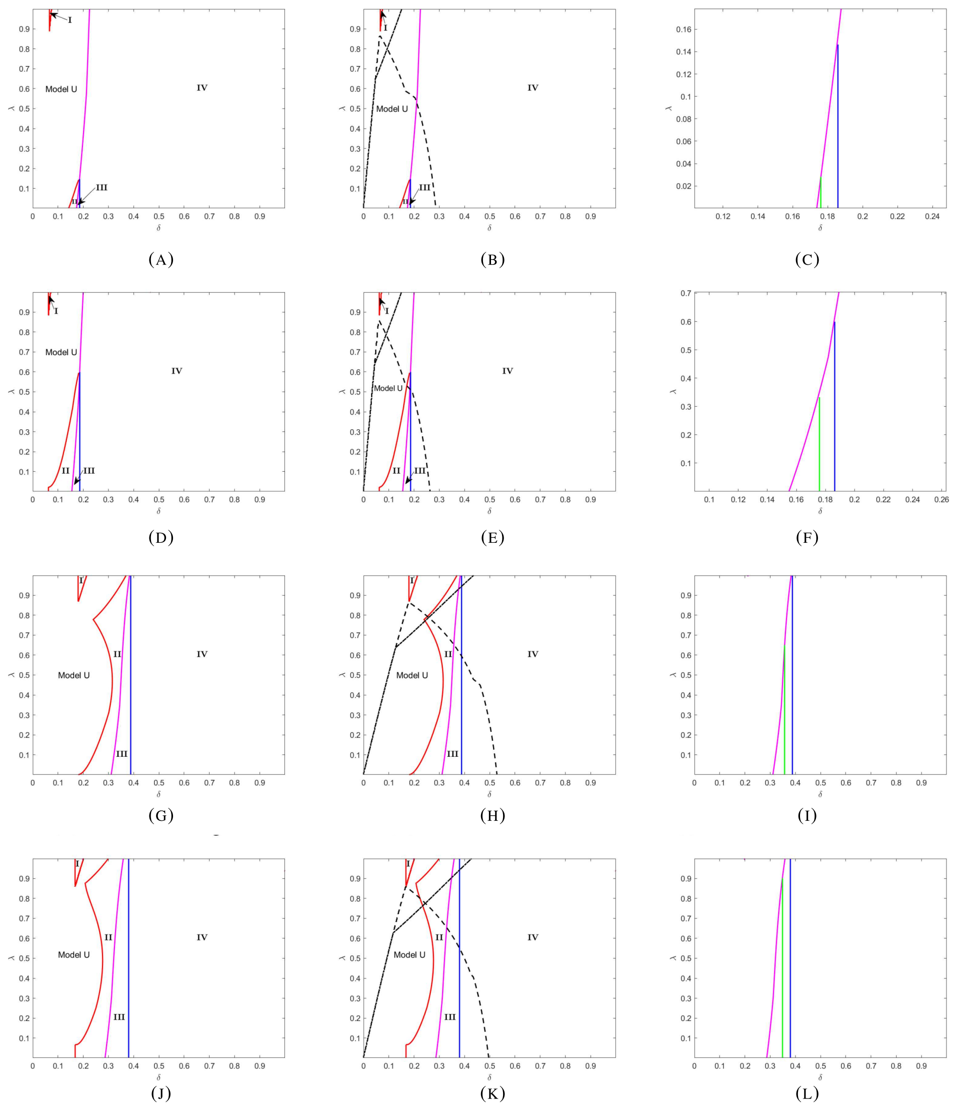

FIGURE 1. Authorization regions and environment efforts for Models C and S under the deterministic environment with $\alpha = 0.3$ (*first row*), fuzzy environment with $\alpha = 0.3$ (*second row*), deterministic environment with $\alpha = 0.8$ (*third row*), and fuzzy environment with $\alpha = 0.8$ (*fourth row*), respectively. Region I indicates that the profits of both the OEM and the IR are higher in Model S compared to the unauthorized scenario, where the model chosen in region II is the same as in region I. Region III demonstrates that both Models C and S are more profitable compared to the unauthorized scenario, with the OEM finding Model S more advantageous, and the IR preferring Model C. Region IV means that the profits of both entities meet the authorization conditions only in Model C. (A) Authorization regions. (B) Environment efforts. (C) Authorization models in area III. (D) Authorization regions. (E) Environment efforts. (F) Authorization models in area III. (G) Authorization regions. (H) Environment efforts. (I) Authorization models in area III. (J) Authorization regions. (K) Environment efforts. (L) Authorization models in area III.

and the high quality level of unauthorized remanufactured products allow the supply chain to achieve a win–win situation for both the economy and the environment in Model C when the cost of remanufactured products is high. In Model C, the possibility of reaching a win–win situation is greater in the fuzzy environment by comparing Figures 1H with 1K (according to the left side of the dotted line in regions III and IV).

For Model S, by comparing Regions I and II of Figures 1B and 1E, as well as Figures 1H and 1K, it is found that win–win cooperation can be achieved when the quality level of the unauthorized remanufactured product is high and the customers' WTP is low (*i.e.*, region I in Figure 1(Authorization regions)). Intuitively, the high quality level of the unauthorized remanufactured product increases its selling price, which encourages customers to be more willing to purchase an authorized remanufactured product of comparable quality to the new product, thereby improving environmental benefits. When the remanufacturing cost is also high, the moderate WTP for remanufacturing products positively impacts the environment in this model (*i.e.*, regions II and III in Figs. 1H and 1K). Obviously, there is a greater likelihood that the supply chain will achieve a win–win situation in Model S than in Model C. The other two parameters, δ and λ, can also be analyzed, yielding similar conclusions as above, and we leave the details to Appendix D for interested readers.

6. Managerial insights

To facilitate remanufacturing cooperation between the OEM and IR and the sustainable development of a supply chain, there are some management implications provided for decision makers based on the analytical studies and numerical results.

(i) From the perspective of economic benefits, the OEM and IR should actively establish strategic partnerships in most cases. In the sensitivity analysis section, we clearly observe that the authorized area exceeds the unauthorized area, suggesting that collaboration between the OEM and the IR is generally advantageous for both parties. Particularly, Model C demonstrates robust superiority when customers exhibit elevated WTP for remanufactured products, maintaining optimal performance across variations in both remanufacturing cost advantages and product quality levels. This indicates that, at this juncture, WTP is the primary driving force in selecting between Models C and S. However, in the fuzzy environment, when the values of these influences are not extreme (*i.e.*, at a moderate level), authorized cooperation remains feasible. The main issue then becomes how to allocate the profits within the supply chain (*i.e.*, whether to adopt Model C or S). Consequently, it is crucial for the OEM and IR to reach a consensus on this matter during the collaboration process;

(ii) From the perspective of ecological benefits, in the fuzzy environment, although both Models C and S can enhance firm profitability for moderate customers WTP, environmental factors should also be considered to ensure sustainable development in this case, aiming to achieve both economic and ecological benefits. Based on in-depth discussion and analysis, Model S is more likely to be selected to create a win–win situation. Specifically, when the quality of the unauthorized remanufactured product is very high (even approaching that of a new product), authorizing cooperation under Model C also proves to be environmentally sustainable;

(iii) Managers should focus on the ambiguities in the market and apply appropriate fuzzy parameters while making decisions to enhance profitability. In fact, market situations are constantly changing. If the decision maker is modeling in a fuzzy environment, more market information can be exploited to obtain more accurate judgments. The findings from the numerical analysis also illustrate that incorporating uncertainties into models can facilitate the formation of cooperative relationships and win–win situations. This also suggests a new direction for future research in similar studies.

7. Conclusion and future work

This paper investigates the authorization decision in the remanufacturing business for a supply chain consisting of an original equipment manufacturer (OEM) and a third-party independent remanufacturer (IR) in the fuzzy setting. Under the premise that the profit assigned by the cooperative game is regarded as the authorization fee, we utilize the expectation value of fuzzy variables to bypass the difficulty of solving the fuzzy optimization problem. By incorporating the KKT conditions, the optimal pricing decisions are obtained for three models (one licensed and two unlicensed) under various remanufacturing strategies. Further, we compare the profits and remanufactured product demands of OEM/IR in the three models to discuss their economic and environmental effect of the authorization. Particularly, a numerical analysis is presented to visualize the effects of market fuzziness on pricing, authorization decisions, and environmental impacts.

The superiority of the proposed model is that it incorporates demand and cost uncertainties into the remanufacturing authorization problem and considers the ecological impact of authorization decisions. We hope to demonstrate that our results contribute directly to the understanding of remanufacturing business, and could shed light on similar kinds of collaboration in other problem settings. Both analytical and numerical results suggest that (1) considering a fuzzy environment would help ease the economic and environmental conflict for the supply chain by encouraging the authorization; (2) a dominant position of the OEM (*i.e.*, Model S) would make it easier to boost the demand of remanufactured products, and thus benefit the environment through the recycling activities. Indeed, fuzzy variables inherently incorporate richer informational content compared to deterministic values, enabling enhanced model representation. In the fuzzy setting, both parties would expect a higher profit (if they collaborate), and thus motivate them to make the authorization decision.

However, there are some limitations to our study. Currently, we consider only one product, one OEM, and one IR, which could be further extended in the future. Moreover, the costs associated with remanufacturing activities include not only the costs of manufacturing the recovered products but also the expenses incurred in collecting, transporting, screening. A more comprehensive cost structure can be developed in subsequent studies to provide more accurate decision-making recommendations for managing the supply chain. Additionally, it is also possible to consider carbon emissions and carbon policies in supply chain management in future research to help companies fulfill their social responsibility goals while maintaining profits. Lastly, the analysis of real data also needs to be further refined in subsequent work.

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Data availability statement

The research data associated with this article are included in the article.

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Appendix A. Profit function

See Tables A.1 and A.2.

Table A.1. Optimal profits of the OEM and IR in unauthorized scenario.

	$\alpha < \alpha_1^U$	$\alpha_1^U \leq \alpha \leq \alpha_2^U$
$E^{*U}(\pi_{0EM}^U)$	$\frac{-\lambda(\lambda(1+\delta\lambda)\alpha^2-\delta(3-\delta\lambda)\alpha)E^2(\tilde{c}_n)+\delta^2(3-\delta\lambda)((2-\delta\lambda)E(\tilde{a}^U)-E(\tilde{b}^U))E(\tilde{c}_n)+\delta\lambda\alpha((3\delta\lambda-1)E(\tilde{a}^U)+(\delta^2\lambda^2-\delta\lambda+2)E(\tilde{b}^U))E(\tilde{c}_n)+\delta^2(1-\delta\lambda)(E(\tilde{a}^U)+E(\tilde{b}^U))(2E(\tilde{a}^U)-(1-\delta\lambda)E(\tilde{b}^U))+\delta^2(3-\delta\lambda)^2E(-\tilde{c}_n\tilde{a}^U)}{\delta^2(3-\delta\lambda)^2}$	$\frac{(\alpha\lambda^2-(2-\delta\lambda))^2E^2(\tilde{c}_n)+(1-\delta\lambda)((\delta^2\lambda^2-4\delta\lambda+8)E(\tilde{a}^U)-2\delta\lambda(2-\delta\lambda)E(\tilde{b}^U))E(\tilde{c}_n)+2\alpha\lambda^2(1-\delta\lambda)(2E(\tilde{a}^U)+\delta\lambda E(\tilde{b}^U))E(\tilde{c}_n)+(1-\delta\lambda)^2(2E(\tilde{a}^U)+\delta\lambda E(\tilde{b}^U))^2+(1-\delta\lambda)(4-\delta\lambda)^2E(-\tilde{c}_n\tilde{a}^U)}{(1-\delta\lambda)(4-\delta\lambda)^2}$
$E^{*U}(\pi_{\mathrm{IR}}^U)$	$\frac{\alpha^2\lambda^3(1-\delta\lambda)E^2(\tilde{c}_n)+\delta\lambda^2\alpha((\delta^2\lambda^2-4\delta\lambda+7)E(\tilde{b}^U)-2(1-\delta\lambda)E(\tilde{a}^U))E(\tilde{c}_n)+\delta^2\lambda(1-\delta\lambda)(E(\tilde{a}^U)+E(\tilde{b}^U))^2+\alpha\delta\lambda^2(3-\delta\lambda)^2E(-\tilde{c}_n\tilde{b}^U)}{\delta(3-\delta\lambda)^2}$	$\frac{\lambda^2(\alpha(2-\delta\lambda)\lambda-\delta)^2E^2(\tilde{c}_n)+2\delta^2\lambda^2(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))E(\tilde{c}_n)+\alpha\delta\lambda^3(1-\delta\lambda)((\delta^2\lambda^2-4\delta\lambda+8)E(\tilde{b}^U)-2(2-\delta\lambda)E(\tilde{a}^U))E(\tilde{c}_n)+\delta^2\lambda^2(1-\delta\lambda)^2(E(\tilde{a}^U)+2E(\tilde{b}^U))^2+\alpha\delta\lambda^3(1-\delta\lambda)(4-\delta\lambda)^2E(-\tilde{c}_n\tilde{b}^U)}{\delta\lambda(1-\delta\lambda)(4-\delta\lambda)^2}$

TABLE A.2. Optimal profits of the OEM and IR in authorized scenario.

	$E^{*}(\pi_{0EM}^{A})$	$E^{*}(\pi_{\mathrm{IR}}^{A})$
$\alpha < \alpha_1^C$	$\frac{\alpha(4\delta-\alpha)E^2(\tilde{c}_n)+4\delta^2(3E(\tilde{a}^A)-E(\tilde{b}^A))E(\tilde{c}_n) + 2\alpha\delta((1+2\delta)E(\tilde{b}^A)-E(\tilde{a}^A))E(\tilde{c}_n)+\delta^2 (E(\tilde{a}^A)+E(\tilde{b}^A))(3E(\tilde{a}^A)+(4\delta-1)E(\tilde{b}^A)) + 8\delta^2(2E(-\tilde{c}_n\tilde{a}^A)+\alpha E(-\tilde{c}_n\tilde{b}^A)-E^{*}(\pi_{\mathrm{IR}}^{U}))}{16\delta^2}$	$\frac{\alpha^2E^2(\tilde{c}_n)+2\alpha\delta(3E(\tilde{b}^A)-E(\tilde{a}^A))E(\tilde{c}_n)+\delta^2(E(\tilde{a}^A) + E(\tilde{b}^A))^2+8\delta(\alpha E(-\tilde{c}_n\tilde{b}^A)+E^{*}(\pi_{\mathrm{IR}}^{U}))}{16\delta}$
$\alpha < \alpha_1^S$	$\frac{\alpha(2\delta(3-\delta)-(\delta^2+\delta+2)\alpha)E^2(\tilde{c}_n)+2\delta^2(3 -\delta)((2-\delta)E(\tilde{a}^A)-E(\tilde{b}^A))E(\tilde{c}_n)+\alpha\delta(2(\delta^2 +2\delta-1)E(\tilde{a}^A)+(\delta^3-2\delta^2+5\delta+4)E(\tilde{b}^A)) E(\tilde{c}_n)+\delta^2(1-\delta)(E(\tilde{a}^A)+E(\tilde{b}^A))((4+\delta) E(\tilde{a}^A)+(3\delta-2)E(\tilde{b}^A))+\delta^2(3-\delta)^2 (2E(-\tilde{c}_n\tilde{a}^A)+\alpha E(-\tilde{c}_n\tilde{b}^A)-E^{*}(\pi_{\mathrm{IR}}^{U}))}{2\delta^2(3-\delta)^2}$	$\frac{(1-\delta)\alpha^2E^2(\tilde{c}_n)+\alpha\delta((\delta^2-4\delta+7)E(\tilde{b}^A)-2(1 -\delta)E(\tilde{a}^A))E(\tilde{c}_n)+\alpha\delta\lambda^3(1-\delta\lambda)((\delta^2\lambda^2-4\delta\lambda+8) E(\tilde{b}^U)-2(2-\delta\lambda)E(\tilde{a}^U))E(\tilde{c}_n)+\delta^2(1-\delta)(E(\tilde{a}^A) +E(\tilde{b}^A))^2+\delta(3-\delta)^2(\alpha E(-\tilde{c}_n\tilde{b}^A)+E^{*}(\pi_{\mathrm{IR}}^{U}))}{2\delta(3-\delta)^2}$
$\alpha_1^S \le \alpha < \alpha_2^S$	$\frac{((2-\delta)\alpha^2+2\delta(2\delta-3)\alpha+\delta(2-\delta)^2)E^2(\tilde{c}_n) +(2\delta(2(2-\delta)E(\tilde{a}^A)-\delta(1-\delta)E(\tilde{b}^A)))E(\tilde{c}_n) +\alpha\delta(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))E(\tilde{c}_n) +\delta(1-\delta)^2(4E^2(\tilde{a}^A)+6\delta E(\tilde{a}^A)E(\tilde{b}^A)+\delta(2+\delta) E^2(\tilde{b}^A))+\delta(1-\delta)(8-5\delta)(2E(-\tilde{c}_n\tilde{a}^A)+\alpha E(-\tilde{c}_n\tilde{b}^A)-E^{*}(\pi_{\mathrm{IR}}^{U}))}{2\delta(1-\delta)(8-5\delta)}$	$\frac{((4-3\delta)\alpha-\delta(2-\delta))^2E^2(\tilde{c}_n)+2\delta^2(1-\delta)(2-\delta) (2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))E(\tilde{c}_n)+\alpha\delta(1-\delta)((19\delta^2 -48\delta+32)E(\tilde{b}^A)-4(4-3\delta)E(\tilde{a}^A))E(\tilde{c}_n)+\delta^2 (1-\delta)^2(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))^2+\delta(1-\delta) (8-5\delta)^2(\alpha E(-\tilde{c}_n\tilde{b}^A)+E^{*}(\pi_{\mathrm{IR}}^{U}))}{2\delta(1-\delta)(8-5\delta)^2}$

APPENDIX B. COMPARING THE PROFITS IN MODELS C/S AND U

(1) Model C *vs.* Model U

Proposition B.1. *When* $\lambda \in (0, \frac{3}{4+\delta})$, $\delta \in (0, \frac{4\lambda\sqrt{7(1-\lambda^2)}-(3+4\lambda^2)}{\lambda(8\lambda^2-1)})$, *and* $\frac{2\mathrm{E}(\tilde{a}^U)-\mathrm{E}(\tilde{b}^U)}{3\mathrm{E}(\tilde{a}^A)-\mathrm{E}(\tilde{b}^A)} < \frac{3}{4}$, *the authorization decisions of the OEM in Model C for any* $\alpha < \alpha_1^U$ *are characterized as follows:*

- (i) *if* $\mathrm{E}(\tilde{c}_n) \in (\max\{0, \mathrm{E}(\tilde{c}_n)_4\}, +\infty)$, *there exists a threshold* α_1^* *that makes* $E^{*C}(\pi_{\mathrm{OEM}}^A) > E^{*U}(\pi_{\mathrm{OEM}}^U)$ *for* $\alpha \in (0, \min\{\alpha_1^U, \alpha_1^*\})$;
- (ii) *if* $\mathrm{E}(\tilde{c}_n) \in (\mathrm{E}^{*1}(\tilde{c}_n), \max\{\mathrm{E}^{*1}(\tilde{c}_n), \mathrm{E}(\tilde{c}_n)_4\})$, $(\mathrm{E}^{*C}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=\alpha_2^*} > 0$, *then there exists two thresholds* α_1^* *and* α_3^* *such that* $E^{*C}(\pi_{\mathrm{OEM}}^A) > E^{*U}(\pi_{\mathrm{OEM}}^U)$ *for* $\alpha \in (\alpha_3^*, \max\{\alpha_3^*, \min\{\alpha_1^*, \alpha_1^U\}\})$,

where the positive zero of a quadratic function $(\frac{\partial(\mathrm{E}^{*C}(\pi_{\mathrm{OEM}}^A)-\mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha}) \mid_{\alpha=0}$ *about* $\mathrm{E}(\tilde{c}_n)$ *is denoted by* $\mathrm{E}^{*1}(\tilde{c}_n)$, *the* α_2^* *is the maximum point of* $\mathrm{E}^{*C}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)$, *and the* $\mathrm{E}(\tilde{c}_n)_4$ *is defined in Table B.1.*

Proposition B.2. *The conditions under which the IR accepts authorization in Model C for any* $\alpha < \alpha_1^S$ *are as follows:*

- (i) *if* $\mathrm{E}(\tilde{c}_n) \in (0, \mathrm{E}^{*2}(\tilde{c}_n))$ *and* $(\mathrm{E}^{*C}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=\alpha_4^*} < 0$, *there is a threshold* α_5^* *that makes* $E^{*C}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for any* $\alpha \in (0, \min\{\alpha_1^S, \alpha_5^*\})$;
- (ii) *if* $\mathrm{E}(\tilde{c}_n) \in (\mathrm{E}^{*2}(\tilde{c}_n), +\infty)$, $(\mathrm{E}^{*C}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=\alpha_4^*} < 0$, *and* $\alpha_4^* > 0$, *then there is a threshold* α_6^* *that makes* $E^{*C}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for* $\alpha \in (0, \alpha_5^*) \cup (\alpha_6^*, \max\{\alpha_6^*, \alpha_1^S\})$;
- (iii) *if* $(\mathrm{E}^{*C}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=\alpha_4^*} > 0$ *or* $\alpha_4^* < 0$, *the IR's profit always meets* $E^{*C}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for any* $\alpha \in (0, \alpha_1^S)$,

where the zero point greater than 0 of $\alpha_1^S = \alpha_4^*$ *is denoted by* $\mathrm{E}^{*2}(\tilde{c}_n)$ *and the* α_4^* *is the minimum point of* $\mathrm{E}^{*C}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)$.

Proposition B.3. *The authorization decisions of the OEM in Model C for any* $\alpha_1^U < \alpha < \alpha_2^S$ *are derived as follows:*

- (i) *if* $(\mathrm{E}^{*C}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=0} > 0$, *there exists a threshold* α_7^* *that makes* $E^{*C}(\pi_{\mathrm{OEM}}^A) > E^{*U}(\pi_{\mathrm{OEM}}^U)$ *for* $\alpha \in (\alpha_1^U, \max\{\alpha_1^U, \min\{\alpha_7^*, \alpha_2^S\}\})$;

(ii) *if* $(\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=0}<0$, $(\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=\alpha_8^*}>0$, *and* $\mathrm{E}(\tilde{c}_n)\in(\mathrm{E}^{*3}(\tilde{c}_n),+\infty)$, *there exists two thresholds* α_7^* *and* α_9^* *that make* $E^{*C}(\pi^A_{\mathrm{OEM}})>E^{*U}(\pi^U_{\mathrm{OEM}})$ *for* $\alpha\in(\max\{\alpha_1^U,\alpha_9^*\},\max\{\alpha_1^U,\alpha_9^*,\min\{\alpha_7^*,\alpha_2^S\}\})$,

where the positive zero of a quadratic function $(\frac{\partial(\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))}{\partial\alpha})\mid_{\alpha=0}$ *about* $\mathrm{E}(\tilde{c}_n)$ *is denoted by* $\mathrm{E}^{*3}(\tilde{c}_n)$) *and the* α_8^* *is the maximum point of* $\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}})$.

Proposition B.4. *When* $\lambda\in(0,\sqrt[3]{\frac{1-\delta}{4}})$ *and* $\delta>\frac{(\mathrm{E}(\tilde{a}^U)+2\mathrm{E}(\tilde{b}^U))^2}{6(\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))^2}$, *the conditions under which the IR accepts authorization in Model C for any* $\alpha_1^U<\alpha<\alpha_2^S$ *are provided as follows:*

(i) *if* $\mathrm{E}(\tilde{c}_n)\in(0,\min\{\mathrm{E}^{*4}(\tilde{c}_n),\mathrm{E}(\tilde{c}_n)_5\})$ *and* $(\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=\alpha_{10}^*}<0$, *there exists two thresholds* α_{11}^* *and* α_{12}^* *that make* $E^{*C}(\pi^A_{\mathrm{IR}})>E^{*U}(\pi^U_{\mathrm{IR}})$ *for* $\alpha\in((0,\alpha_{11}^*)\cup(\alpha_{12}^*,\min\{\alpha_{12}^*,1\}))\cap(\alpha_1^U,\alpha_2^S)$;
(ii) *if* $(\mathrm{E}^{*C}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=\alpha_{10}^*}>0$ *or* $\mathrm{E}(\tilde{c}_n)\in(\mathrm{E}^{*4}(\tilde{c}_n),\max\{\mathrm{E}^{*4}(\tilde{c}_n),\mathrm{E}(\tilde{c}_n)_5\})$, *the IR's profit meets* $E^{*C}(\pi^A_{\mathrm{IR}})>E^{*U}(\pi^U_{\mathrm{IR}})$ *for any* $\alpha\in(\alpha_1^U,\alpha_2^S)$;
(iii) *if* $\mathrm{E}(\tilde{c}_n)\in(\max\{0,\mathrm{E}(\tilde{c}_n)_5\},+\infty)$, *then there is a threshold* α_{12}^* *that makes* $E^{*C}(\pi^A_{\mathrm{IR}})>E^{*U}(\pi^U_{\mathrm{IR}})$ *for* $\alpha\in(\max\{\alpha_1^U,\alpha_{12}^*\},\max\{\alpha_2^S,\alpha_{12}^*\})$,

where the positive zero of a quadratic function $(\frac{\partial(\mathrm{E}^{*C}(\pi^A_{\mathrm{IR}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{IR}}))}{\partial\alpha})\mid_{\alpha=0}$ *about* $\mathrm{E}(\tilde{c}_n)$ *is denoted by* $\mathrm{E}^{*4}(\tilde{c}_n)$), *the* α_{10}^* *is the minimum point of* $\mathrm{E}^{*C}(\pi^A_{\mathrm{IR}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{IR}})$, *and the* $\mathrm{E}(\tilde{c}_n)_5$ *is defined in Table B.1.*

(2) Model S *vs.* Model U

Proposition B.5. *When* $\lambda\in(0,\frac{(3-\delta)(2\mathrm{E}(\tilde{a}^U)-\mathrm{E}(\tilde{b}^U))-3((2-\delta)\mathrm{E}(\tilde{a}^A)-\mathrm{E}(\tilde{b}^A))}{((3-\delta)\mathrm{E}(\tilde{a}^U)-(2-\delta)\mathrm{E}(\tilde{a}^A)+\mathrm{E}(\tilde{b}^A))\delta})$, *the authorization decisions of the OEM in Model S for any* $\alpha<\alpha_1^S$ *are characterized as follows:*

(i) *if* $\mathrm{E}(\tilde{c}_n)\in(0,\max\{0,\mathrm{E}(\tilde{c}_n)_6\})$, *there is a threshold* α_{13}^* *such that* $E^{*S}(\pi^A_{\mathrm{OEM}})>E^{*U}(\pi^U_{\mathrm{OEM}})$ *for* $\alpha\in(0,\min\{\alpha_1^S,\alpha_{13}^*\})$;
(ii) *if* $\mathrm{E}(\tilde{c}_n)\in(\max\{\mathrm{E}^{*5}(\tilde{c}_n),\mathrm{E}(\tilde{c}_n)_6\},+\infty)$, $(\mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=\alpha_{14}^*}>0$, *there are two thresholds* α_{13}^* *and* α_{15}^* *that make* $E^{*S}(\pi^A_{\mathrm{OEM}})>E^{*U}(\pi^U_{\mathrm{OEM}})$ *for* $\alpha\in(\alpha_{15}^*,\max\{\alpha_{15}^*,\min\{\alpha_{13}^*,\alpha_1^S\}\})$,

where the positive zero of a quadratic function $(\frac{\partial(\mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))}{\partial\alpha})\mid_{\alpha=0}$ *about* $\mathrm{E}(\tilde{c}_n)$ *is denoted by* $\mathrm{E}^{*5}(\tilde{c}_n)$), *the* α_{14}^* *is the maximum point of* $E^{*S}(\pi^A_{\mathrm{OEM}})-E^{*U}(\pi^U_{\mathrm{OEM}})$, *and the* $\mathrm{E}(\tilde{c}_n)_6$ *is defined in Table B.1.*

Proposition B.6. *When* $\lambda\in(0,\lambda_3)$ *and* $\delta\in(0,\min\{\delta_1,\delta_2\})$, *the conditions under which the IR accepts authorization in Model S for any* $\alpha<\alpha_1^S$ *are as follows:*

(i) *if* $\mathrm{E}(\tilde{c}_n)\in(0,\mathrm{E}^{*6}(\tilde{c}_n))$ *and* $(\mathrm{E}^{*S}(\pi^A_{\mathrm{IR}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{IR}}))\mid_{\alpha=\alpha_{16}^*}<0$, *there is a threshold* α_{17}^* *that makes the IR's profit meets* $E^{*S}(\pi^A_{\mathrm{IR}})>E^{*U}(\pi^U_{\mathrm{IR}})$ *for* $\alpha\in(0,\min\{\alpha_1^S,\alpha_{17}^*\})$;
(ii) *if* $\mathrm{E}(\tilde{c}_n)\in(\mathrm{E}^{*6}(\tilde{c}_n),+\infty)$, $(\mathrm{E}^{*S}(\pi^A_{\mathrm{IR}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{IR}}))\mid_{\alpha=\alpha_{16}^*}<0$, *and* $\alpha_{16}^*>0$, *there exists another threshold* α_{18}^* *that makes* $E^{*S}(\pi^A_{\mathrm{IR}})>E^{*U}(\pi^U_{\mathrm{IR}})$ *for* $\alpha\in(0,\alpha_{17}^*)\cup(\alpha_{18}^*,\max\{\alpha_{18}^*,\alpha_1^S\})$;
(iii) *if* $(\mathrm{E}^{*S}(\pi^A_{\mathrm{IR}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{IR}}))\mid_{\alpha=\alpha_{16}^*}>0$ *or* $\alpha_{16}^*<0$, *the IR's profit always meets* $E^{*S}(\pi^A_{\mathrm{IR}})>E^{*U}(\pi^U_{\mathrm{IR}})$ *for any* $\alpha\in(0,\alpha_1^S)$,

where the δ_1, δ_2, *and* λ_2 *are defined in Tables B.2 and B.3, the zero point greater than 0 of* $\alpha_1^S=\alpha_{16}^*$ *is denoted by* $\mathrm{E}^{*6}(\tilde{c}_n)$, *and the* α_{16}^* *is the minimum point of* $\mathrm{E}^{*S}(\pi^A_{\mathrm{IR}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{IR}})$.

Proposition B.7. *The OEM's authorization decisions in Model S for any* $\alpha_1^S<\alpha<\alpha_1^U$ *are given as follows:*

(i) *if* $(\mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=\alpha_{19}^*}>0$, *then the OEM's profit always meets* $E^{*S}(\pi^A_{\mathrm{OEM}})>E^{*U}(\pi^U_{\mathrm{OEM}})$ *for* $\alpha\in(\alpha_1^S,\alpha_1^U)$;
(ii) *if* $(\mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=\alpha_{19}^*}<0$, *the* $\alpha_{19}^*>0$, *and* $(\mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=0}>0$, *there exists two positive zero points* α_{20}^* *and* α_{21}^* *that make* $E^{*S}(\pi^A_{\mathrm{OEM}})>E^{*U}(\pi^U_{\mathrm{OEM}})$ *for* $\alpha\in((0,\alpha_{20}^*)\cup(\alpha_{21}^*,\min\{\alpha_{21}^*,1\}))\cap(\alpha_1^S,\alpha_1^U)$;
(iii) *if* $(\mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=0}>0$ *and the* $\alpha_{19}^*<0$, *then the the OEM's profit always satisfies* $E^{*S}(\pi^A_{\mathrm{OEM}})>E^{*U}(\pi^U_{\mathrm{OEM}})$ *for* $\alpha\in(\alpha_1^S,\alpha_1^U)$;
(iv) *if* $(\mathrm{E}^{*S}(\pi^A_{\mathrm{OEM}})-\mathrm{E}^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=0}<0$, *there is a threshold* α_{21}^* *that makes* $E^{*S}(\pi^A_{\mathrm{OEM}})>E^{*U}(\pi^U_{\mathrm{OEM}})$ *for* $\alpha\in(\max\{\alpha_1^S,\alpha_{21}^*\},\max\{\alpha_1^U,\alpha_{21}^*\})$,

where the α_{19}^* *is the minimum point of* $\mathrm{E}^{*S}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)$.

Proposition B.8. *The conditions under which the IR accepts authorization in Model S for any* $\alpha_1^S < \alpha < \alpha_1^U$ *are as follows:*

(i) *if* $\mathrm{E}(\tilde{c}_n) \in (0, \max\{0, \mathrm{E}(\tilde{c}_n)_7\})$, *then there is a threshold* α_{22}^* *that makes* $E^{*S}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for* $\alpha \in (\max\{\alpha_1^S, \alpha_{22}^*\}, \max\{\alpha_1^U, \alpha_{22}^*\})$;
(ii) *if* $\mathrm{E}(\tilde{c}_n) \in (\max\{0, \mathrm{E}(\tilde{c}_n)_7\}, +\infty)$, $(\mathrm{E}^{*S}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=\alpha_{23}^*} < 0$ *and* $\alpha_{23}^* > 0$, *there are two thresholds* α_{22}^* *and* α_{24}^* *that make* $E^{*S}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for* $\alpha \in ((0, \alpha_{24}^*) \cup (\alpha_{22}^*, \min\{\alpha_{22}^*, 1\})) \cap (\alpha_1^S, \alpha_1^U)$;
(iii) *if* $\mathrm{E}(\tilde{c}_n) \in (\max\{0, \mathrm{E}(\tilde{c}_n)_7\}, +\infty)$ *and* $\alpha_{23}^* < 0$, *the IR's profit meets* $E^{*S}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for any* $\alpha \in (\alpha_1^S, \alpha_1^U)$;
(iv) *if* $(\mathrm{E}^{*S}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=\alpha_{23}^*} > 0$, *the IR's profit meets* $E^{*S}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for any* $\alpha \in (\alpha_1^S, \alpha_1^U)$,

where the $\mathrm{E}(\tilde{c}_n)_7$ *is defined in Table B.1.*

Proposition B.9. *The authorization decisions of the OEM in Model S for any* $\alpha_1^U < \alpha < \alpha_2^S$ *are derived as follows:*

(i) *if* $(\mathrm{E}^{*S}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=0} > 0$, $(\mathrm{E}^{*S}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=\alpha_{25}^*} < 0$, *and* $\alpha_{25}^* > 0$, *there are two thresholds* α_{26}^* *and* α_{27}^* *that make* $E^{*S}(\pi_{\mathrm{OEM}}^A) > E^{*U}(\pi_{\mathrm{OEM}}^U)$ *for* $\alpha \in ((0, \alpha_{26}^*) \cup (\alpha_{27}^*, \min\{\alpha_{27}^*, 1\})) \cap (\alpha_1^U, \alpha_2^S)$;
(ii) *if* $(\mathrm{E}^{*S}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=0} > 0$ *and* $\alpha_{25}^* < 0$, *the OEM's profit always meets* $E^{*S}(\pi_{\mathrm{OEM}}^A) > E^{*U}(\pi_{\mathrm{OEM}}^U)$ *for any* $\alpha \in (\alpha_1^U, \alpha_2^S)$;
(iii) *if* $(\mathrm{E}^{*S}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=0} < 0$, *there is a threshold* α_{27}^* *that makes* $E^{*S}(\pi_{\mathrm{OEM}}^A) > E^{*U}(\pi_{\mathrm{OEM}}^U)$ *for* $\alpha \in (\max\{\alpha_1^U, \alpha_{27}^*\}, \max\{\alpha_2^S, \alpha_{27}^*\})$;
(iv) *if* $(\mathrm{E}^{*S}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=\alpha_{25}^*} > 0$, *the OEM's profit always satisfies* $E^{*S}(\pi_{\mathrm{OEM}}^A) > E^{*U}(\pi_{\mathrm{OEM}}^U)$ *for any* $\alpha \in (\alpha_1^U, \alpha_2^S)$,

where the α_{25}^* *is the minimum point of* $\mathrm{E}^{*S}(\pi_{\mathrm{OEM}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{OEM}}^U)$.

Proposition B.10. *When* $\delta \in (\delta_3, 1)$ *and* $\lambda \in (0, \lambda_4)$, *the conditions under which the IR accepts authorization in Model S for any* $\alpha_1^U < \alpha < \alpha_2^S$ *are provided as follows:*

(i) *if* $(\mathrm{E}^{*S}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=\alpha_{28}^*} < 0$ *and* $\alpha_{28}^* > 0$, *there exists two thresholds* α_{29}^* *and* α_{30}^* *such that* $E^{*S}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for* $\alpha \in ((0, \alpha_{29}^*) \cup (\alpha_{30}^*, \min\{\alpha_{30}^*, 1\})) \cap (\alpha_1^U, \alpha_2^S)$;
(ii) *if* $(\mathrm{E}^{*S}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=\alpha_{28}^*} > 0$ *or* $\alpha_{28}^* < 0$, *the IR's profit meets* $E^{*S}(\pi_{\mathrm{IR}}^A) > E^{*U}(\pi_{\mathrm{IR}}^U)$ *for any* $\alpha \in (\alpha_1^U, \alpha_2^S)$,

where the δ_3 *and* λ_3 *are provided in Tables B.2 and B.3, and the* α_{28}^* *is the minimum point of* $\mathrm{E}^{*S}(\pi_{\mathrm{IR}}^A) - \mathrm{E}^{*U}(\pi_{\mathrm{IR}}^U)$.

TABLE B.1. Zero points related to $E(\tilde{c}_n)$.

Symbol	Expression
$E(\tilde{c}_n)_1$	$\dfrac{\lambda(1-\delta)(2-\delta\lambda)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A)) - (1-\delta\lambda)(4-3\delta)(E(\tilde{a}^U)+2E(\tilde{b}^U))}{4-3\delta-\lambda(2-\delta)(2-\delta\lambda)}$
$E(\tilde{c}_n)_2$	$\dfrac{\lambda(1-\delta)((3\delta-2)E(\tilde{a}^A)+(\delta^2-2\delta+4)E(\tilde{b}^A)) - (2-\delta^2)(1-\delta\lambda)((2-\delta\lambda)E(\tilde{b}^U)-E(\tilde{a}^U))}{(2-\delta^2)(3-\delta\lambda)-\lambda(2-\delta)(3-\delta)}$
$E(\tilde{c}_n)_3$	$\dfrac{2\lambda(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))-(1-\delta\lambda)(4-3\delta)((2-\delta\lambda)E(\tilde{b}^U)-E(\tilde{a}^U))}{(4-3\delta)(3-\delta\lambda)-2\lambda(2-\delta)}$
$E(\tilde{c}_n)_4$	$\dfrac{8(1-\delta\lambda)(E(\tilde{a}^U)+E(\tilde{b}^U))((4+\delta\lambda)E(\tilde{a}^U)+(3\delta\lambda-2)E(\tilde{b}^U)) - (3-\delta\lambda)^2(E(\tilde{a}^A)+E(\tilde{b}^A))(3E(\tilde{a}^A)+(4\delta-1)E(\tilde{b}^A))-16\delta(3-\delta\lambda)^2(E(-\tilde{c}_n\tilde{a}^A)-E(-\tilde{c}_n\tilde{a}^U))}{4(3-\delta\lambda)((3-\delta\lambda)(3E(\tilde{a}^A)-E(\tilde{b}^A))-4((2-\delta\lambda)E(\tilde{a}^U)-E(\tilde{b}^U)))}$
$E(\tilde{c}_n)_5$	$\dfrac{(4-\delta\lambda)(E(\tilde{a}^A)+E(\tilde{b}^A))\sqrt{2\lambda(1-\delta\lambda)}-4\lambda(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))}{4\lambda}$
$E(\tilde{c}_n)_6$	$\dfrac{(3-\delta)^2(1-\delta\lambda)(E(\tilde{a}^U)+E(\tilde{b}^U))((4+\delta\lambda)E(\tilde{a}^U)+(3\delta\lambda-2)E(\tilde{b}^U)) - (3-\delta\lambda)^2(1-\delta)(E(\tilde{a}^A)+E(\tilde{b}^A))((4+\delta)E(\tilde{a}^A)+(3\delta-2)E(\tilde{b}^A)) - 2(3-\delta)^2(3-\delta\lambda)^2(E(-\tilde{c}_n\tilde{a}^A)-E(-\tilde{c}_n\tilde{a}^U))}{2(3-\delta)(3-\delta\lambda)((3-\delta\lambda)((2-\delta)E(\tilde{a}^A)-E(\tilde{b}^A))-(3-\delta)((2-\delta\lambda)E(\tilde{a}^U)-E(\tilde{b}^U)))}$
$E(\tilde{c}_n)_7$	$\dfrac{\lambda(1-\delta\lambda)(8-5\delta)^2(E(\tilde{a}^U)+E(\tilde{b}^U))^2-(1-\delta)(3-\delta\lambda)^2(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))^2}{2(2-\delta)(3-\delta\lambda)^2(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))}$

TABLE B.2. Symbols related to δ and α.

Symbol	Expression
δ_1	$\dfrac{(1+\lambda)(3(1+\lambda^2)+\lambda) - \sqrt{(1+\lambda)^2(3(1+\lambda^2)+\lambda)^2 - 12\lambda(1+\lambda+\lambda^2)^2}}{2\lambda(1+\lambda+\lambda^2)}$
δ_2	$\dfrac{3((E(\tilde{a}^U)+E(\tilde{b}^U))^2 + 4(E(\tilde{a}^A)+E(\tilde{b}^A))^2) - 12(E(\tilde{a}^A) + E(\tilde{b}^A))\sqrt{(E(\tilde{a}^A)+E(\tilde{b}^A))^2 - (E(\tilde{a}^U)+E(\tilde{b}^U))^2}}{(E(\tilde{a}^U)+E(\tilde{b}^U))^2 + 24(E(\tilde{a}^A)+E(\tilde{b}^A))^2}$
δ_3	$\dfrac{40(E(\tilde{a}^U)+2E(\tilde{b}^U))^2 + 24(2E(\tilde{a}^A)+3E(\tilde{b}^A))^2 - 24(2E(\tilde{a}^A) + 3E(\tilde{b}^A))\sqrt{(2E(\tilde{a}^A)+3E(\tilde{b}^A))^2 - 2(E(\tilde{a}^U)+2E(\tilde{b}^U))^2}}{25(E(\tilde{a}^U)+2E(\tilde{b}^U))^2 + 48(2E(\tilde{a}^A)+3E(\tilde{b}^A))^2}$
α^{***}	$\dfrac{\delta(1-\delta)(1-\delta\lambda)((4-\delta\lambda)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A)) - (8-5\delta)(E(\tilde{a}^U)+2E(\tilde{b}^U))) + \delta^2(9-5\delta-(2-\delta)(5-\delta\lambda)\lambda)E(\tilde{c}_n)}{(\delta(2\delta^2-9\delta+8)\lambda^2 + (5\delta^2+6\delta-16)\lambda + 4(4-3\delta))E(\tilde{c}_n)}$

TABLE B.3. Symbols related to λ.

Symbol	Expression
λ_1	$\dfrac{(2-\delta)E(\tilde{a}^U)+(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A)) - \sqrt{((2-\delta)E(\tilde{a}^U)+(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A)))^2 + 4\delta(2-\delta)(4-3\delta)E(\tilde{b}^U)(E(\tilde{a}^U)+E(\tilde{b}^U))}}{-2\delta(2-\delta)E(\tilde{b}^U)}$
λ_2	$\dfrac{(2+\delta)E(\tilde{a}^A)+(2+\delta^2)E(\tilde{b}^A)-\delta(E(\tilde{a}^U)-3E(\tilde{b}^U)) - \sqrt{((2+\delta)E(\tilde{a}^A)+(2+\delta^2)E(\tilde{b}^A)-\delta(E(\tilde{a}^U)-3E(\tilde{b}^U)))^2 - 4\delta^2E(\tilde{b}^U)(3(E(\tilde{a}^A)+\delta E(\tilde{b}^A)) - E(\tilde{a}^U)+2E(\tilde{b}^U))}}{2\delta^2E(\tilde{b}^U)}$
λ_3	$\dfrac{6\delta(1-\delta)(E(\tilde{a}^A)+E(\tilde{b}^A))^2 + (3-\delta)^2(E(\tilde{a}^U)+E(\tilde{b}^U))^2 - (3-\delta)(E(\tilde{a}^U)+E(\tilde{b}^U))\sqrt{(3-\delta)^2(E(\tilde{a}^U)+E(\tilde{b}^U))^2 - 24\delta(1-\delta)(E(\tilde{a}^A)+E(\tilde{b}^A))^2}}{2\delta(\delta(1-\delta)(E(\tilde{a}^A)+E(\tilde{b}^A))^2 + (3-\delta)^2(E(\tilde{a}^U)+E(\tilde{b}^U))^2)}$

TABLE B.3. continued.

Symbol	Expression
λ_4	$\dfrac{8\delta(2-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))+(1-\delta)(8-5\delta)^2(E(\tilde{a}^U)+2E(\tilde{b}^U))-(8-5\delta)\sqrt{(1-\delta)(E(\tilde{a}^U)+2E(\tilde{b}^U))((1-\delta)(8-5\delta)^2(E(\tilde{a}^U)+2E(\tilde{b}^U))+16\delta(2-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A)))}}{2\delta^2(2-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))}$
λ^{**}	$\dfrac{\delta(1-\delta)(3(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))-(8-5\delta)(E(\tilde{a}^U)+E(\tilde{b}^U)))+3(\delta(2-\delta)-(4-3\delta)\alpha)E(\tilde{c}_n)}{\delta^2(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))+(\delta^2(2-\delta)-(2\delta^2-9\delta+8)\alpha)E(\tilde{c}_n)}$

APPENDIX C. PROOFS

Proof of Proposition 4.1. Rewrite constraint $E(\widetilde{D}_n^U)\geq E(\widetilde{D}_r^U)\geq 0$ as $\frac{1}{\delta\lambda}p_r^U-(1-\delta\lambda)E(\tilde{b}^U)\leq p_n^U\leq\frac{(1-\delta\lambda)(E(\tilde{a}^U)-E(\tilde{b}^U))}{2}+\frac{1+\delta\lambda}{2\delta\lambda}p_r^U$. Thus, the Lagrangian and the KKT optimality conditions for the OEM's and the IR's optimization problems in unauthorized scenario are as follows:

$$\begin{aligned}L_1\left(p_n^U,\mu_1\right)=&-\frac{1}{1-\delta\lambda}\left(p_n^U\right)^2+\frac{1}{1-\delta\lambda}p_n^Up_r^U+\left(E\left(\tilde{a}^U\right)+\frac{E(\tilde{c}_n)}{1-\delta\lambda}\right)p_n^U-\frac{E(\tilde{c}_n)}{1-\delta\lambda}p_r^U\\&+E\left(-\tilde{c}_n\tilde{a}^U\right)+\mu_1\left(\frac{(1-\delta\lambda)\left(E(\tilde{a}^U)-E\left(\tilde{b}^U\right)\right)}{2}+\frac{1+\delta\lambda}{2\delta\lambda}p_r^U-p_n^U\right),\end{aligned}\tag{C.1}$$

$$\begin{aligned}L_2\left(p_r^U,\mu_2,\mu_3\right)=&-\frac{1}{\delta\lambda(1-\delta\lambda)}\left(p_r^U\right)^2+\frac{1}{1-\delta\lambda}p_n^Up_r^U+\left(\frac{\alpha\lambda^2E(\tilde{c}_n)}{\delta\lambda(1-\delta\lambda)}+E\left(\tilde{b}^U\right)\right)p_r^U\\&-\frac{\alpha\lambda^2E(\tilde{c}_n)}{1-\delta\lambda}p_n^U+\alpha\lambda^2E\left(-\tilde{c}_n\tilde{b}^U\right)+\mu_2\left(\frac{(1-\delta\lambda)\left(E(\tilde{a}^U)-E\left(\tilde{b}^U\right)\right)}{2}\right.\\&\left.+\frac{1+\delta\lambda}{2\delta\lambda}p_r^U-p_n^U\right)+\mu_3\left((1-\delta\lambda)E\left(\tilde{b}^U\right)+p_n^U-\frac{1}{\delta\lambda}p_r^U\right),\end{aligned}\tag{C.2}$$

$$\frac{\partial L_1\left(p_n^U,\mu_1\right)}{\partial p_n^U}=-\frac{2}{1-\delta\lambda}p_n^U+\frac{1}{1-\delta\lambda}p_r^U+E\left(\tilde{a}^U\right)+\frac{E(\tilde{c}_n)}{1-\delta\lambda}-\mu_1=0,\tag{C.3}$$

$$\begin{aligned}\frac{\partial L_2\left(p_r^U,\mu_2,\mu_3\right)}{\partial p_r^U}=&-\frac{2}{\delta\lambda(1-\delta\lambda)}p_r^U+\frac{1}{1-\delta\lambda}p_n^U+E\left(\tilde{b}^U\right)+\frac{\alpha\lambda^2E(\tilde{c}_n)}{\delta\lambda(1-\delta\lambda)}\\&+\frac{1+\delta\lambda}{2\delta\lambda}\mu_2-\frac{1}{\delta\lambda}\mu_3=0,\end{aligned}\tag{C.4}$$

$$\frac{1}{\delta\lambda}p_r^U-(1-\delta\lambda)E\left(\tilde{b}^U\right)\leq p_n^U\leq\frac{(1-\delta\lambda)\left(E(\tilde{a}^U)-E\left(\tilde{b}^U\right)\right)}{2}+\frac{1+\delta\lambda}{2\delta\lambda}p_r^U,\tag{C.5}$$

$$\begin{aligned}\mu_1\left(\frac{(1-\delta\lambda)\left(E(\tilde{a}^U)-E\left(\tilde{b}^U\right)\right)}{2}+\frac{1+\delta\lambda}{2\delta\lambda}p_r^U-p_n^U\right)=\mu_2\left(\frac{(1-\delta\lambda)\left(E(\tilde{a}^U)-E\left(\tilde{b}^U\right)\right)}{2}\right.\\\left.+\frac{1+\delta\lambda}{2\delta\lambda}p_r^U-p_n^U\right)=\mu_3\left(p_n^U-\frac{1}{\delta\lambda}p_r^U+(1-\delta\lambda)E\left(\tilde{b}^U\right)\right)=0,\end{aligned}\tag{C.6}$$

$$\mu_1\geq 0,\mu_2\geq 0,\mu_3\geq 0.\tag{C.7}$$

Different values of the Lagrange multipliers are discussed, corresponding to the following optimal results:

(i) $\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 = 0$: according to equation (C.6), we have $p_n^U = \frac{(1-\delta\lambda)(E(\tilde{a}^U)-E(\tilde{b}^U))}{2} + \frac{1+\delta\lambda}{2\delta\lambda}p_r^U$, combining with equation (C.4), the optimal results are derived $p_n^{UU^*} = \frac{\delta(1-\delta\lambda)(2E(\tilde{a}^U)-(1-\delta\lambda)E(\tilde{b}^U))+\alpha\lambda(1+\delta\lambda)E(\tilde{c}_n)}{\delta(3-\delta\lambda)}$ and $p_r^{UU^*} = \frac{\delta\lambda(1-\delta\lambda)(E(\tilde{a}^U)+E(\tilde{b}^U))+2\alpha\lambda^2E(\tilde{c}_n)}{3-\delta\lambda}$. Furthermore, the multiplier $\mu_1 > 0$ and equation (C.5) require $\alpha < \frac{\delta(1-\delta\lambda)((2-\delta\lambda)E(\tilde{b}^U)-E(\tilde{a}^U))+\delta(3-\delta\lambda)E(\tilde{c}_n)}{2\lambda E(\tilde{c}_n)}$;

(ii) $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 = 0$: according to equation (C.6), we have $p_n^U = \frac{(1-\delta\lambda)(E(\tilde{a}^U)-E(\tilde{b}^U))}{2} + \frac{1+\delta\lambda}{2\delta\lambda}p_r^U$, combining with equation (C.3), we have $p_n^{UU^*} = \frac{(1-\delta\lambda)(E(\tilde{a}^U)+\delta\lambda E(\tilde{b}^U))+(1+\delta\lambda)E(\tilde{c}_n)}{2}$ and $p_r^{UU^*} = \delta\lambda((1-\delta\lambda)E(\tilde{b}^U)+E(\tilde{c}_n))$. The multiplier $\mu_2 > 0$ and equation (C.5) require $\alpha < \frac{\delta(1-\delta\lambda)((2-\delta\lambda)E(\tilde{b}^U)-E(\tilde{a}^U))+\delta(3-\delta\lambda)E(\tilde{c}_n)}{2\lambda E(\tilde{c}_n)}$. However, in this scenario, corresponding the IR's optimal profit is not equal 0 when $\lambda = 0$, which doesn't fit the actual context. Thus, we eliminate it;

(iii) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 > 0$: according to equation (C.6), we have $p_n^U = \frac{1}{\delta\lambda}p_r^U - (1-\delta\lambda)E(\tilde{b}^U)$, combining with equation (C.3), we obtain $p_n^{UU^*} = \frac{(1-\delta\lambda)(E(\tilde{a}^U)+\delta\lambda E(\tilde{b}^U))+E(\tilde{c}_n)}{2-\delta\lambda}$ and $p_r^{UU^*} = \frac{\delta\lambda(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))+\delta\lambda E(\tilde{c}_n)}{2-\delta\lambda}$. Furthermore, to satisfy the multiplier $\mu_3 > 0$ and equation (C.5), we need $\alpha > \frac{\delta(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))+\delta E(\tilde{c}_n)}{\lambda(2-\delta\lambda)E(\tilde{c}_n)}$. In this scenario, the IR does not enter remanufacturing market because of high production cost;

(iv) $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 = 0$: according to equation (C.6), we only have $p_n^U = \frac{(1-\delta\lambda)(E(\tilde{a}^U)-E(\tilde{b}^U))}{2} + \frac{1+\delta\lambda}{2\delta\lambda}p_r^U$, that is the optimal results are unavailable. Similarly, in the scenarios $\mu_1 > 0$, $\mu_2 = 0$, $\mu_3 > 0$ and $\mu_1 = 0$, $\mu_2 > 0$, $\mu_3 > 0$, we can't find the optimal solution;

(v) $\mu_1 = 0$, $\mu_2 = 0$, $\mu_3 = 0$: according to equations (C.3) and (C.4), the optimal results are obtained: $p_n^{UU^*} = \frac{(1-\delta\lambda)(2E(\tilde{a}^U)+\delta\lambda E(\tilde{b}^U))+(2+\alpha\lambda^2)E(\tilde{c}_n)}{4-\delta\lambda}$ and $p_r^{UU^*} = \frac{\delta\lambda(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))+\lambda(2\alpha\lambda+\delta)E(\tilde{c}_n)}{4-\delta\lambda}$. Additionally, in order to satisfy equation (C.5), we need $\frac{\delta(1-\delta\lambda)((2-\delta\lambda)E(\tilde{b}^U)-E(\tilde{a}^U))+\delta(3-\delta\lambda)E(\tilde{c}_n)}{2\lambda E(\tilde{c}_n)} \leq \alpha \leq \frac{\delta(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))+\delta E(\tilde{c}_n)}{\lambda(2-\delta\lambda)E(\tilde{c}_n)}$;

(vi) $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$: according to equation (C.6), we have $E(\widetilde{D}_n) = E(\widetilde{D}_r) = 0$, which is meaningless.

We denote $\alpha_1^U = \frac{\delta(1-\delta\lambda)((2-\delta\lambda)E(\tilde{b}^U)-E(\tilde{a}^U))+\delta(3-\delta\lambda)E(\tilde{c}_n)}{2\lambda E(\tilde{c}_n)}$ and $\alpha_2^U = \frac{\delta(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))+\delta E(\tilde{c}_n)}{\lambda(2-\delta\lambda)E(\tilde{c}_n)}$, and the $\alpha_2^U > \alpha_1^U$ when $E(\tilde{a}^U) + \delta\lambda E(\tilde{b}^U) > E(\tilde{c}_n)$. Therefore, the Proposition 4.1 is proved. □

Proof of Proposition 4.2. Rewrite constraint $E(\widetilde{D}_n^A) \geq E(\widetilde{D}_r^A) > 0$ as $\frac{1}{\delta}p_r^A - (1-\delta)E(\tilde{b}^A) < p_n^A \leq \frac{(1-\delta)(E(\tilde{a}^A)-E(\tilde{b}^A))}{2} + \frac{1+\delta}{2\delta}p_r^A$. Thus, the Lagrangian and the KKT optimality conditions for cooperation problems in the C model are as follows:

$$\begin{aligned} L\left(p_n^A, p_r^A, \mu_1, \mu_2\right) = &- \frac{1}{\delta(1-\delta)}\left(p_r^A\right)^2 + \frac{1}{1-\delta}p_n^A p_r^A + \left(E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)}\right)p_r^A \\ &- \frac{\alpha E(\tilde{c}_n)}{1-\delta}p_n^A + \alpha E\left(-\tilde{c}_n\tilde{b}^A\right) + \mu_1\left(p_n^A - \frac{1}{\delta}p_r^A + (1-\delta)E\left(\tilde{b}^A\right)\right) \\ &+ \mu_2\left(\frac{(1-\delta)\left(E(\tilde{a}^A) - E\left(\tilde{b}^A\right)\right)}{2} + \frac{1+\delta}{2\delta}p_r^A - p_n^A\right), \end{aligned} \tag{C.8}$$

$$\frac{\partial L\left(p_n^A, p_r^A, \mu_1, \mu_2\right)}{\partial p_n^A} = \frac{1}{1-\delta}p_r^A - \frac{\alpha E(\tilde{c}_n)}{1-\delta} + \mu_1 - \mu_2 = 0, \tag{C.9}$$

$$\frac{\partial L\left(p_n^A, p_r^A, \mu_1, \mu_2\right)}{\partial p_r^A} = -\frac{2}{\delta(1-\delta)}p_r^A + \frac{1}{1-\delta}p_n^A + E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)} - \frac{1}{\delta}\mu_1 + \frac{1+\delta}{2\delta}\mu_2 = 0, \tag{C.10}$$

$$\frac{1}{\delta}p_r^A - (1-\delta)E\left(\tilde{b}^A\right) < p_n^A \leq \frac{(1-\delta)\left(E(\tilde{a}^A) - E\left(\tilde{b}^A\right)\right)}{2} + \frac{1+\delta}{2\delta}p_r^A, \tag{C.11}$$

$$\mu_1\left(p_n^A - \frac{1}{\delta}p_r^A + (1-\delta)E\left(\tilde{b}^A\right)\right) = \mu_2\left(\frac{(1-\delta)\left(E(\tilde{a}^A) - E\left(\tilde{b}^A\right)\right)}{2} + \frac{1+\delta}{2\delta}p_r^A - p_n^A\right) = 0, \tag{C.12}$$

$$\mu_1 \geq 0, \mu_2 \geq 0. \tag{C.13}$$

Solving the optimality conditions simultaneously, the solutions are concluded:

(i) $\mu_1 > 0$, $\mu_2 = 0$: according to equation (C.12), we have $p_n^A = \frac{1}{\delta}p_r^A - (1-\delta)E(\tilde{b}^A)$ (*i.e.*, $E(\widetilde{D}_r^A) = 0$), which is discussed unnecessarily in authorized case. Similar to it, the results in the scenario $\mu_1 > 0$, $\mu_2 > 0$ are unavailable;

(ii) $\mu_1 = 0$, $\mu_2 > 0$: according to equation (C.12), we know $p_n^A = \frac{(1-\delta)(E(\tilde{a}^A)-E(\tilde{b}^A))}{2} + \frac{1+\delta}{2\delta}p_r^A$, combing with equations (C.9) and (C.10), we have $p_n^{AC^*} = \frac{\delta((3-\delta)E(\tilde{a}^A)+(3\delta-1)E(\tilde{b}^A))+\alpha(1+\delta)E(\tilde{c}_n)}{4\delta}$ and $p_r^{AC^*} = \frac{\delta(E(\tilde{a}^A)+E(\tilde{b}^A))+\alpha E(\tilde{c}_n)}{2}$. The multiplier $\mu_2 > 0$ and equation (C.11) require $\alpha < \frac{\delta(E(\tilde{a}^A)+E(\tilde{b}^A))}{E(\tilde{c}_n)}$;

(iii) $\mu_1 = 0$, $\mu_2 = 0$: according to equations (C.9) and (C.10), we can know that $p_n^{AC^*} = \frac{\alpha E(\tilde{c}_n)}{\delta} - (1-\delta)E(\tilde{b}^A)$ and $p_r^{AC^*} = \alpha E(\tilde{c}_n)$, however, this solution corresponds to $E(\widetilde{D}_r^A) = 0$. Therefore, we eliminate it.

In this non-concave problem, we can obtain the critical cone at the point $(p_n^{AC^*}, p_r^{AC^*})$ (the result from (ii)), which is given by $F_2 = \left\{(d_1, d_2) \mid d_1 = \frac{1+\delta}{2\delta}d_2, d_2 \neq 0\right\}$. Therefore, we can verify the second-order sufficient condition holds based on Theorem 12.6 in the literature [25]. That is,

$$\begin{pmatrix} \frac{1+\delta}{2\delta}d_2 & d_2 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{1-\delta} \\ -\frac{1}{1-\delta} & \frac{2}{\delta(1-\delta)} \end{pmatrix} \begin{pmatrix} \frac{1+\delta}{2\delta}d_2 \\ d_2 \end{pmatrix} = \frac{d_2^2}{\delta} > 0.$$

Letting $\alpha^C = \frac{\delta(E(\tilde{a}^A)+E(\tilde{b}^A))}{E(\tilde{c}_n)}$, the Proposition 4.2 is proved. □

Proof of Proposition 4.3. In Model S, the OEM decides new product's selling price and next the IR decides remanufactured product's price. Hence, the Lagrangian and the KKT optimality conditions for the IR's optimization problem are formulated:

$$\begin{aligned} L_1\left(p_r^A, \mu_1\right) = &-\frac{1}{\delta(1-\delta)}\left(p_r^A\right)^2 + \frac{1}{1-\delta}p_n^A p_r^A + \left(E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)}\right)p_r^A - \frac{\alpha E(\tilde{c}_n)}{1-\delta}p_n^A \\ &+ \alpha E\left(-\tilde{c}_n\tilde{b}^A\right) + \mu_1\left(p_n^A - \frac{1}{\delta}p_r^A + (1-\delta)E\left(\tilde{b}^A\right)\right), \end{aligned} \tag{C.14}$$

$$\frac{\partial L_1(p_r^A, \mu_1)}{\partial p_r^A} = -\frac{2}{\delta(1-\delta)}p_r^A + \frac{1}{1-\delta}p_n^A + E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)} - \frac{1}{\delta}\mu_1 = 0, \tag{C.15}$$

$$p_n^A > \frac{1}{\delta}p_r^A - (1-\delta)E\left(\tilde{b}^A\right), \tag{C.16}$$

$$\mu_1\left(p_n^A - \frac{1}{\delta}p_r^A + (1-\delta)E\left(\tilde{b}^A\right)\right) = 0, \tag{C.17}$$

$$\mu_1 \geq 0. \tag{C.18}$$

To meet the original feasibility condition, we just need to analyze the scenario $\mu_1 = 0$. According to equation (C.17), the two products' prices satisfy the relationship: $p_r^A = \frac{\delta p_n^A + \delta(1-\delta)E(\tilde{b}^A)+\alpha E(\tilde{c}_n)}{2}$. Examining the equation (C.16), we have $p_n^A > \frac{\alpha E(\tilde{c}_n)-\delta(1-\delta)E(\tilde{b}^A)}{\delta}$. Further, the constraint $E(\widetilde{D}_n^A) \geq E(\widetilde{D}_r^A)$ in the OEM's optimization problem is rewritten as $p_n^A \leq \frac{\delta(1-\delta)(2E(\tilde{a}^A)-(1-\delta)E(\tilde{b}^A))+\alpha(1+\delta)E(\tilde{c}_n)}{\delta(3-\delta)}$, corresponding the problem is given:

$$\begin{aligned} L_2\left(p_n^A, \mu_2, \mu_3\right) = &-\frac{1}{1-\delta}\left(\left(p_n^A\right)^2 + \frac{1}{2\delta}\left(p_r^A\right)^2\right) + \frac{3}{2(1-\delta)}p_n^A p_r^A + \left(E\left(\tilde{a}^A\right) + \frac{E(\tilde{c}_n)}{1-\delta}\right. \\ &\left.- \frac{\alpha E(\tilde{c}_n)}{2(1-\delta)}\right)p_n^A + \left(\frac{1}{2}\left(E\left(\tilde{b}^A\right) + \frac{\alpha E(\tilde{c}_n)}{\delta(1-\delta)}\right) - \frac{E(\tilde{c}_n)}{1-\delta}\right)p_r^A + E\left(-\tilde{c}_n\tilde{a}^A\right) \\ &+ \frac{1}{2}\alpha E\left(-\tilde{c}_n\tilde{b}^A\right) - \frac{1}{2}E^{*U}\left(\pi_{\mathrm{IR}}^U\right) + \mu_2\left(p_n^A - \frac{\alpha E(\tilde{c}_n) - \delta(1-\delta)E\left(\tilde{b}^A\right)}{\delta}\right) \\ &+ \mu_3\left(\frac{\delta(1-\delta)\left(2E(\tilde{a}^A) - (1-\delta)E\left(\tilde{b}^A\right)\right) + \alpha(1+\delta)E(\tilde{c}_n)}{\delta(3-\delta)} - p_n^A\right), \end{aligned} \tag{C.19}$$

$$\begin{aligned} \frac{\partial L_2(p_n^A, \mu_2, \mu_3)}{\partial p_n^A} = &\frac{3\delta-8}{4(1-\delta)}p_n^A + \frac{1}{1-\delta}p_r^A + \frac{(2-\delta)E(\tilde{c}_n)}{2(1-\delta)} - \frac{\alpha E(\tilde{c}_n)}{4(1-\delta)} \\ &+ E\left(\tilde{a}^A\right) + \frac{\delta}{4}E\left(\tilde{b}^A\right) + \mu_2 - \mu_3 = 0, \end{aligned} \tag{C.20}$$

$$\frac{\alpha E(\tilde{c}_n) - \delta(1-\delta)E\left(\tilde{b}^A\right)}{\delta} < p_n^A \leq \frac{\delta(1-\delta)\left(2E(\tilde{a}^A) - (1-\delta)E\left(\tilde{b}^A\right)\right) + \alpha(1+\delta)E(\tilde{c}_n)}{\delta(3-\delta)}, \tag{C.21}$$

$$\mu_2\left(p_n^A - \frac{\alpha E(\tilde{c}_n) - \delta(1-\delta)E\left(\tilde{b}^A\right)}{\delta}\right) = \mu_3\left(\frac{\delta(1-\delta)\left(2E(\tilde{a}^A) - (1-\delta)E\left(\tilde{b}^A\right)\right) + \alpha(1+\delta)E(\tilde{c}_n)}{\delta(3-\delta)} - p_n^A\right) = 0, \tag{C.22}$$

$$\mu_2 \geq 0, \mu_3 \geq 0. \tag{C.23}$$

Based on equation (C.21), we only need to discuss two scenarios, *i.e.*, $\mu_2 = 0$, $\mu_3 > 0$ and $\mu_2 = 0$, $\mu_3 = 0$

(i) $\mu_2 = 0$, $\mu_3 > 0$: according to equation (C.22), we know $p_n^{AS^*} = \frac{\delta(1-\delta)(2E(\tilde{a}^A)-(1-\delta)E(\tilde{b}^A))+\alpha(1+\delta)E(\tilde{c}_n)}{\delta(3-\delta)}$, and $p_r^A = \frac{\delta p_n^A+\delta(1-\delta)E(\tilde{b}^A)+\alpha E(\tilde{c}_n)}{2}$. Thus, $p_r^{AS^*} = \frac{\delta(1-\delta)(E(\tilde{a}^A)+E(\tilde{b}^A))+2\alpha E(\tilde{c}_n)}{3-\delta}$. In order to meet $\mu_3 > 0$ and equation (C.21), we need $\alpha < \frac{\delta(1-\delta)((3\delta-2)E(\tilde{a}^A)+(\delta^2-2\delta+4)E(\tilde{b}^A))+\delta(2-\delta)(3-\delta)E(\tilde{c}_n)}{2(2-\delta^2)E(\tilde{c}_n)}$;

(ii) $\mu_2 = 0$, $\mu_3 = 0$: according to equation (C.20), we have $p_n^A = \frac{(1-\delta)(4E(\tilde{a}^A)+\delta E(\tilde{b}^A))+(4-2\delta-\alpha)E(\tilde{c}_n)+4p_r^A}{8-3\delta}$, substituting it into $p_r^A = \frac{\delta p_n^A+\delta(1-\delta)E(\tilde{b}^A)+\alpha E(\tilde{c}_n)}{2}$. Hence, we obtain $p_n^{AS^*} = \frac{(1-\delta)(4E(\tilde{a}^A)+3\delta E(\tilde{b}^A))+(4-2\delta+\alpha)E(\tilde{c}_n)}{8-5\delta}$ and $p_r^{AS^*} = \frac{\delta(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))+(2-\delta)(2\alpha+\delta)E(\tilde{c}_n)}{8-5\delta}$. The constraint condition equation (C.21) requires that the relationship $\frac{\delta(1-\delta)((3\delta-2)E(\tilde{a}^A)+(\delta^2-2\delta+4)E(\tilde{b}^A))+\delta(2-\delta)(3-\delta)E(\tilde{c}_n)}{2(2-\delta^2)E(\tilde{c}_n)} \leq \alpha < \frac{\delta(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))+\delta(2-\delta)E(\tilde{c}_n)}{(4-3\delta)E(\tilde{c}_n)}$ holds.

Letting $\alpha_1^S = \frac{\delta(1-\delta)((3\delta-2)E(\tilde{a}^A)+(\delta^2-2\delta+4)E(\tilde{b}^A))+\delta(2-\delta)(3-\delta)E(\tilde{c}_n)}{2(2-\delta^2)E(\tilde{c}_n)}$ and $\alpha_2^S = \frac{\delta(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))+\delta(2-\delta)E(\tilde{c}_n)}{(4-3\delta)E(\tilde{c}_n)}$, and the $\alpha_2^S > \alpha_1^S$ when $E(\tilde{a}^A) + \delta E(\tilde{b}^A) > E(\tilde{c}_n)$. Thus, the Proposition 4.3 is proved. □

Proof of Proposition 4.4. To examine which of the two authorization models is better for the OEM under different remanufacturing cost advantages, it is essential to analyze the fluctuation in profit margins with respect to the cost advantage. When $\alpha < \alpha_1^S$, the first and second order derivatives of the profit differential $E^{*S}(\pi_{\text{OEM}}^A) - E^{*C}(\pi_{\text{OEM}}^A)$ relative to α are given below:

$$\frac{\partial\left(E^{*S}\left(\pi_{\text{OEM}}^A\right) - E^{*C}\left(\pi_{\text{OEM}}^A\right)\right)}{\partial\alpha} = \frac{(1+\delta)\Big((2\delta(3-\delta) - 7(1+\delta)\alpha)E^2(\tilde{c}_n) + \delta\Big((1+9\delta)E\left(\tilde{a}^A\right) + \left(2\delta^2+\delta+7\right)E\left(\tilde{b}^A\right)\Big)E(\tilde{c}_n)\Big)}{8\delta^2(3-\delta)^2},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\text{OEM}}^A\right) - E^{*C}\left(\pi_{\text{OEM}}^A\right)\right)}{\partial\alpha^2} = -\frac{7(1+\delta)^2E^2(\tilde{c}_n)}{8\delta^2(3-\delta)^2},$$

$$\left(\frac{\partial\left(E^{*S}\left(\pi_{\text{OEM}}^A\right) - E^{*C}\left(\pi_{\text{OEM}}^A\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0} = \frac{2\delta(1+\delta)(3-\delta)E^2(\tilde{c}_n) + \delta(1+\delta)\Big((1+9\delta)E\left(\tilde{a}^A\right) + \left(2\delta^2+\delta+7\right)E\left(\tilde{b}^A\right)\Big)E(\tilde{c}_n)}{8\delta^2(3-\delta)^2}.$$

Obviously, we have $\frac{\partial^2(E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A))}{\partial\alpha^2} < 0$ and $\left(\frac{\partial(E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A))}{\partial\alpha}\right)\Big|_{\alpha=0} > 0$ hold, meaning that the function $E^{*S}(\pi_{\text{OEM}}^A) - E^{*C}(\pi_{\text{OEM}}^A)$ is increasing for $\alpha \in (0, \frac{2\delta(3-\delta)E(\tilde{c}_n)+((1+9\delta)E(\tilde{a}^A)+(2\delta^2+\delta+7)E(\tilde{b}^A))}{7(1+\delta)E(\tilde{c}_n)})$. Due to $\frac{2\delta(3-\delta)E(\tilde{c}_n)+((1+9\delta)E(\tilde{a}^A)+(2\delta^2+\delta+7)E(\tilde{b}^A))}{7(1+\delta)E(\tilde{c}_n)} > \alpha_1^S$, the $E^{*S}(\pi_{\text{OEM}}^A) - E^{*C}(\pi_{\text{OEM}}^A)$ is increasing for $\alpha \in (0, \alpha_1^S)$. And the interval holds (*i.e.*, $\alpha_1^S > 0$) when $E(\tilde{c}_n) > \frac{(1-\delta)((2-3\delta)E(\tilde{a}^A)-(4-2\delta+\delta^2)E(\tilde{b}^A))}{(2-\delta)(3-\delta)}$. What's more, the function value of $E^{*S}(\pi_{\text{OEM}}^A) - E^{*C}(\pi_{\text{OEM}}^A)$ at the point α_1^S is equal to $\frac{-(1+\delta)(2-\delta)(\delta^2+7\delta-2)(E(\tilde{a}^A)+\delta E(\tilde{b}^A)-E(\tilde{c}_n))^2}{64(2-\delta^2)^2}$. As such, the function value is more than 0 when $0 < \delta < \frac{\sqrt{57}-7}{2}$ and less than 0 otherwise. And it is easy to prove Proposition 4.4. □

Proof of Proposition 4.5. Similarly, when $\alpha_1^S \le \alpha < \alpha_2^S$, the first and second order derivatives of the profit differential $E^{*S}(\pi_{\text{OEM}}^A) - E^{*C}(\pi_{\text{OEM}}^A)$ with respect to α are derived:

$$\frac{\partial\left(E^{*S}\left(\pi_{\text{OEM}}^A\right) - E^{*C}\left(\pi_{\text{OEM}}^A\right)\right)}{\partial\alpha} = \frac{(2\delta(2\delta-3)+2(2-\delta)\alpha)E^2(\tilde{c}_n) + \delta(1-\delta)\left(2E\left(\tilde{a}^A\right) + (4-\delta)E\left(\tilde{b}^A\right)\right)E(\tilde{c}_n)}{2\delta(1-\delta)(8-5\delta)},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\text{OEM}}^A\right) - E^{*C}\left(\pi_{\text{OEM}}^A\right)\right)}{\partial\alpha^2} = \frac{\left(8+3\delta-3\delta^2\right)E^2(\tilde{c}_n)}{8\delta^2(1-\delta)(8-5\delta)},$$

$$\left(\frac{\partial\left(E^{*S}\left(\pi_{\text{OEM}}^A\right) - E^{*C}\left(\pi_{\text{OEM}}^A\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0} = \frac{2\left(3\delta^2+\delta-8\right)E^2(\tilde{c}_n) + (1-\delta)\left((8+3\delta)E\left(\tilde{a}^A\right) + \left(6\delta^2+5\delta-8\right)E\left(\tilde{b}^A\right)\right)E(\tilde{c}_n)}{8\delta(1-\delta)(8-5\delta)}.$$

Obviously, the $\frac{\partial^2(E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A))}{\partial\alpha^2} > 0$ holds. When the new product's cost satisfies $0 < E(\tilde{c}_n) < \frac{(1-\delta)((8+3\delta)E(\tilde{a}_A)+(6\delta^2+5\delta-8)E(\tilde{b}_A))}{2(8-3\delta^2-\delta)}$, the expression $\left(\frac{\partial(E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A))}{\partial\alpha}\right)\Big|_{\alpha=0}$ is more than 0. Otherwise, it is less than 0. (i) If $\left(\frac{\partial(E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A))}{\partial\alpha}\right)\Big|_{\alpha=0} > 0$, then the function $E^{*S}(\pi_{\text{OEM}}^A) - E^{*C}(\pi_{\text{OEM}}^A)$ is increasing for $\alpha \in (0,1)$. Due to $\frac{(1-\delta)((2-3\delta)E(\tilde{a}^A)-(4-2\delta+\delta^2)E(\tilde{b}^A))}{(2-\delta)(3-\delta)} < \frac{(1-\delta)((8+3\delta)E(\tilde{a}^A)+(6\delta^2+5\delta-8)E(\tilde{b}^A))}{2(8-3\delta^2-\delta)}$, we known the $E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A)$ is increasing for $\alpha \in (\alpha_1^S, \alpha_2^S)$. Furthermore, the function value of the $E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A)$ at α_1^S is equal to $\frac{-(1+\delta)(2-\delta)(\delta^2+7\delta-2)(E(\tilde{a}^A)+\delta E(\tilde{b}^A)-E(\tilde{c}_n))^2}{64(2-\delta^2)^2}$ and at α_2^S is equal to $\frac{-3(2-\delta)(7\delta-6)(E(\tilde{a}^A)+\delta E(\tilde{b}^A)-E(\tilde{c}_n))^2}{16(4-3\delta)^2}$. Thus, the function at α_1^S is greater than 0 if $0 < \delta < \frac{\sqrt{57}-7}{2}$ and *vice versa*. Meanwhile, the function at α_2^S is more than 0 if $0 < \delta < \frac{6}{7}$ and *vice versa*. (ii) If $\left(\frac{\partial(E^{*S}(\pi_{\text{OEM}}^A)-E^{*C}(\pi_{\text{OEM}}^A))}{\partial\alpha}\right)\Big|_{\alpha=0} < 0$, then $E^{*S}(\pi_{\text{OEM}}^A) - E^{*C}(\pi_{\text{OEM}}^A)$ is growing for $\alpha > \frac{\delta\left(2(8-3\delta^2-\delta)E(\tilde{c}_n)-(1-\delta)((8+3\delta)E(\tilde{a}^A)+(6\delta^2+5\delta-8)E(\tilde{b}^A))\right)}{8+3\delta-3\delta^2}$ and dropping for $0 < \alpha < \frac{\delta\left(2(8-3\delta^2-\delta)E(\tilde{c}_n)-(1-\delta)((8+3\delta)E(\tilde{a}^A)+(6\delta^2+5\delta-8)E(\tilde{b}^A))\right)}{8+3\delta-3\delta^2}$. In addition, we can derive $\frac{\delta\left(2(8-3\delta^2-\delta)E(\tilde{c}_n)-(1-\delta)((8+3\delta)E(\tilde{a}^A)+(6\delta^2+5\delta-8)E(\tilde{b}^A))\right)}{8+3\delta-3\delta^2} < \alpha_1^S < \alpha_2^S$ under the condition $E(\tilde{a}^A) + \delta E(\tilde{b}^A) > E(\tilde{c}_n)$, combing with the discussion in (i). It is easy to prove Proposition 4.5. □

Proof of Proposition 4.6. In order to investigate which model is more favorable to the IR under different remanufacturing cost advantages, we need to consider the amount of the IR's profits in Models C and S. When $\alpha < \alpha_1^S$, the first and second order derivatives of the profit differential $E^{*S}(\pi_{\text{IR}}^A) - E^{*C}(\pi_{\text{IR}}^A)$ with respect to α are as follows:

$$\frac{\partial\left(E^{*S}\left(\pi_{\text{IR}}^A\right) - E^{*C}\left(\pi_{\text{IR}}^A\right)\right)}{\partial\alpha} = \frac{\delta(1+\delta)^2\left(E(\tilde{a}^A) + E\left(\tilde{b}^A\right)\right)E(\tilde{c}_n) - \alpha(1+\delta)^2E^2(\tilde{c}_n)}{8\delta(3-\delta)^2},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\text{IR}}^A\right) - E^{*C}\left(\pi_{\text{IR}}^A\right)\right)}{\partial\alpha^2} = -\frac{(1+\delta)^2E^2(\tilde{c}_n)}{8\delta(3-\delta)^2},$$

$$\left(\frac{\partial\left(E^{*S}\left(\pi_{\text{IR}}^A\right) - E^{*C}\left(\pi_{\text{IR}}^A\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0} = \frac{(1+\delta)^2\left(E(\tilde{a}^A) + E\left(\tilde{b}^A\right)\right)E(\tilde{c}_n)}{8(3-\delta)^2}.$$

Clearly, the $\frac{\partial^2(E^{*S}(\pi_{\text{IR}}^A)-E^{*C}(\pi_{\text{IR}}^A))}{\partial\alpha^2} < 0$ and $\left(\frac{\partial(E^{*S}(\pi_{\text{IR}}^A)-E^{*C}(\pi_{\text{IR}}^A))}{\partial\alpha}\right)\Big|_{\alpha=0} > 0$, which imply that the $E^{*S}(\pi_{\text{IR}}^A)-E^{*C}(\pi_{\text{IR}}^A)$ is increasing for $0 < \alpha < \frac{\delta(E(\tilde{a}^A)+E(\tilde{b}^A))}{E(\tilde{c}_n)}$ and decreasing for $\alpha > \frac{\delta(E(\tilde{a}^A)+E(\tilde{b}^A))}{E(\tilde{c}_n)}$. Owing to the maximum of $E^{*S}(\pi_{\text{IR}}^A) - E^{*C}(\pi_{\text{IR}}^A)$ is equal to $-\frac{\delta(1+\delta)^2(E(\tilde{a}^A)+E(\tilde{b}^A))^2}{16(3-\delta)^2}$, it is less than 0. As such, the relationship $E^{*S}(\pi_{\text{IR}}^A) < E^{*C}(\pi_{\text{IR}}^A)$ holds for any $\alpha \in (0,1)$. Furthermore, when $\alpha_1^S \le \alpha < \alpha_2^S$, the first and second order derivatives of the profit differential $E^{*S}(\pi_{\text{IR}}^A) - E^{*C}(\pi_{\text{IR}}^A)$ with respect to α are also provided:

$$\frac{\partial\left(E^{*S}\left(\pi_{\text{IR}}^A\right) - E^{*C}\left(\pi_{\text{IR}}^A\right)\right)}{\partial\alpha} = \frac{\begin{array}{c}8\delta(2-\delta)(3\delta-4)E^2(\tilde{c}_n) + \alpha\left(25\delta^3 - 33\delta^2 - 48\delta + 64\right)\\ E^2(\tilde{c}_n) + \delta(1-\delta)\left(\delta(25\delta-32)E\left(\tilde{a}^A\right) + \left(\delta^2 + 48\delta - 64\right)\right.\\ \left.E\left(\tilde{b}^A\right)\right)E(\tilde{c}_n)\end{array}}{8\delta(1-\delta)(8-5\delta)^2},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\text{IR}}^A\right) - E^{*C}\left(\pi_{\text{IR}}^A\right)\right)}{\partial\alpha^2} = \frac{\left(25\delta^3 - 33\delta^2 - 48\delta + 64\right)E^2(\tilde{c}_n)}{8\delta(1-\delta)(8-5\delta)^2},$$

$$\left(\frac{\partial\left(E^{*S}\left(\pi_{\text{IR}}^A\right) - E^{*C}\left(\pi_{\text{IR}}^A\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0} = \frac{\begin{array}{c}8(2-\delta)(3\delta-4)E^2(\tilde{c}_n) + (1-\delta)\left(\delta(25\delta-32)\right.\\ \left.E\left(\tilde{a}^A\right) + \left(\delta^2+48\delta-64\right)E\left(\tilde{b}^A\right)\right)E(\tilde{c}_n)\end{array}}{8(1-\delta)(8-5\delta)^2}.$$

Similarly, the $\frac{\partial^2(E^{*S}(\pi_{\text{IR}}^A) - E^{*C}(\pi_{\text{IR}}^A))}{\partial\alpha^2} > 0$ and $\left(\frac{\partial(E^{*S}(\pi_{\text{IR}}^A) - E^{*C}(\pi_{\text{IR}}^A))}{\partial\alpha}\right)\Big|_{\alpha=0} < 0$ be observed. And when $E(\tilde{a}^A) + \delta E(\tilde{b}^A) > E(\tilde{c}_n)$, the $\alpha_1^S < \frac{8\delta(2-\delta)(3\delta-4)E(\tilde{c}_n)+\delta(1-\delta)(\delta(25\delta-32)E(\tilde{a}^A)+(\delta^2+48\delta-64)E(\tilde{b}^A))}{(25\delta^3-33\delta^2-48\delta+64)E(\tilde{c}_n)} < \alpha_2^S$ holds. Therefore, the monotonicity of the profit difference function $E^{*S}(\pi_{\text{IR}}^A) - E^{*C}(\pi_{\text{IR}}^A)$ is derived, *i.e.*, it is dropping within the range $\alpha \in (\alpha_1^S, \frac{8\delta(2-\delta)(3\delta-4)E(\tilde{c}_n)+\delta(1-\delta)(\delta(25\delta-32)E(\tilde{a}^A)+(\delta^2+48\delta-64)E(\tilde{b}^A))}{(25\delta^3-33\delta^2-48\delta+64)E(\tilde{c}_n)})$ and growing within the range $\alpha \in (\frac{8\delta(2-\delta)(3\delta-4)E(\tilde{c}_n)+\delta(1-\delta)(\delta(25\delta-32)E(\tilde{a}^A)+(\delta^2+48\delta-64)E(\tilde{b}^A))}{(25\delta^3-33\delta^2-48\delta+64)E(\tilde{c}_n)}, \alpha_2^S)$. Moreover, its function values at above three endpoints are calculated respectively: $\frac{-\delta(1+\delta)^2(2-\delta)^2(E(\tilde{a}^A)+\delta E(\tilde{b}^A)-E(\tilde{c}_n))^2}{64(2-\delta^2)^2}$, $\frac{-\delta(2-\delta)^2(E(\tilde{a}^A)+\delta E(\tilde{b}^A)-E(\tilde{c}_n))^2}{2(25\delta^3-33\delta^2-48\delta+64)}$ and $\frac{-\delta(2-\delta)^2(E(\tilde{a}^A)+\delta E(\tilde{b}^A)-E(\tilde{c}_n))^2}{16(4-3\delta)^2}$. Evidently, these values are all less than 0. Hence, it is easy to prove Proposition 4.6. □

Proof of Proposition 4.7. To examine the environmental impact of authorization decision. We need to compare the demand for remanufactured products in the three models. And when $0 < \alpha < \alpha_1^U$, the relationship between the demand in Models C and U is derived:

$$E^{*C}\left(\widetilde{D}_r^A\right) - E^{*U}\left(\widetilde{D}_r^U\right) = \frac{\begin{array}{c}\delta\left((3-\delta\lambda)\left(E\left(\tilde{a}^A\right) + E\left(\tilde{b}^A\right)\right) - 4\left(E\left(\tilde{a}^U\right) + E\left(\tilde{b}^U\right)\right)\right)\\ + ((4+\delta)\lambda - 3)\alpha E(\tilde{c}_n)\end{array}}{4\delta(3-\delta\lambda)},$$

$$= \frac{\begin{array}{c}\left((4+\delta)\alpha E(\tilde{c}_n) - \delta^2\left(E\left(\tilde{a}^A\right) + E\left(\tilde{b}^A\right)\right)\right)\lambda + \delta\left(3\left(E\left(\tilde{a}^A\right)\right.\right.\\ \left.\left. + E\left(\tilde{b}^A\right)\right) - 4\left(E\left(\tilde{a}^U\right) + E\left(\tilde{b}^U\right)\right)\right) - 3\alpha E(\tilde{c}_n)\end{array}}{4\delta(3-\delta\lambda)}.$$

The sign of the coefficient of λ can be determined by controlling the range of δ. And the coefficient is more than 0 when $\delta < \frac{\alpha E(\tilde{c}_n)+\sqrt{\alpha E(\tilde{c}_n)(\alpha E(\tilde{c}_n)+16(E(\tilde{a}^A)+E(\tilde{b}^A)))}}{2(E(\tilde{a}^A)+E(\tilde{b}^A))}$ and less than 0 *vice versa*. Therefore, Proposition 4.7 is proved based on above discussion. □

Proof of Proposition 4.8. When $\alpha_1^U < \alpha < \alpha_2^S$, the relationship between the remanufactured product's demand in Models C and U is characterized:

$$E^{*C}\left(\widetilde{D}_r^A\right) - E^{*U}\left(\widetilde{D}_r^U\right) = \frac{\begin{array}{c}\delta(1-\delta\lambda)\left((4-\delta\lambda)\left(E\left(\tilde{a}^A\right) + E\left(\tilde{b}^A\right)\right) - 4\left(E\left(\tilde{a}^U\right) + 2E\left(\tilde{b}^U\right)\right)\right)\\ + ((4\lambda(2-\delta\lambda) - (1-\delta\lambda)(4-\delta\lambda))\alpha - 4\delta)E(\tilde{c}_n)\end{array}}{4\delta(1-\delta\lambda)(4-\delta\lambda)}.$$

Although we cannot directly control the ranges of δ and λ to identify the sign of $E^{*C}(\widetilde{D}_r^A) - E^{*U}(\widetilde{D}_r^U)$, this can be accomplished by taking values of α. And the coefficient of α is greater than 0 when $\lambda < \frac{8+5\delta-\sqrt{9\delta^2+16\delta+64}}{2\delta(4+\delta)}$ and less than 0 *vice versa*. Thus, Proposition 4.8 is proved. □

Proof of Proposition 4.9. When $0 < \alpha < \alpha_1^S$, the relationship between the remanufactured product's demand in Models S and U is provided:

$$E^{*S}\left(\widetilde{D}_r^A\right) - E^{*U}\left(\widetilde{D}_r^U\right) = \frac{\delta\Big((3-\delta\lambda)\Big(E\Big(\tilde{a}^A\Big) + E\Big(\tilde{b}^A\Big)\Big) - (3-\delta)\Big(E\Big(\tilde{a}^U\Big) + E\Big(\tilde{b}^U\Big)\Big)\Big) - 3(1-\lambda)\alpha E(\tilde{c}_n)}{\delta(3-\delta)(3-\delta\lambda)},$$

$$= \frac{\Big(3\alpha E(\tilde{c}_n) - \delta^2\Big(E\Big(\tilde{a}^A\Big) + E\Big(\tilde{b}^A\Big)\Big)\Big)\lambda + \delta\Big(3\Big(E\Big(\tilde{a}^A\Big) + E\Big(\tilde{b}^A\Big)\Big) - (3-\delta)\Big(E\Big(\tilde{a}^U\Big) + E\Big(\tilde{b}^U\Big)\Big)\Big) - 3\alpha E(\tilde{c}_n)}{\delta(3-\delta)(3-\delta\lambda)}.$$

The analytical process is similar to Proposition 4.7, *i.e.*, the coefficient of λ is more than 0 for $\delta < \sqrt{\frac{3\alpha E(\tilde{c}_n)}{E(\tilde{a}^A)+E(\tilde{b}^A)}}$ and *vice versa*. Hence, Proposition 4.9 is proved. □

Proof of Proposition 4.10. When $\alpha_1^S < \alpha < \alpha_1^U$, the relationship between the demand for remanufactured product in Models S and U is derived:

$$E^{*S}\left(\widetilde{D}_r^A\right) - E^{*U}\left(\widetilde{D}_r^U\right) = \frac{\delta(1-\delta)\Big((3-\delta\lambda)\Big(2E\Big(\tilde{a}^A\Big) + (4-\delta)E\Big(\tilde{b}^A\Big)\Big) - (8-5\delta)\Big(E\Big(\tilde{a}^U\Big) + E\Big(\tilde{b}^U\Big)\Big)\Big) + \delta(2-\delta)(3-\delta\lambda)E(\tilde{c}_n) + \big((2\delta^2-9\delta+8)\lambda - 3(4-3\delta)\big)\alpha E(\tilde{c}_n)}{\delta(1-\delta)(8-5\delta)(3-\delta\lambda)},$$

$$= \frac{\Big(\big((2\delta^2-9\delta+8)\alpha - \delta^2(2-\delta)\big)E(\tilde{c}_n) - \delta^2(1-\delta)\Big(2E\Big(\tilde{a}^A\Big) + (4-\delta)E\Big(\tilde{b}^A\Big)\Big)\Big)\lambda + \delta(1-\delta)\Big(3\Big(2E\Big(\tilde{a}^A\Big) + (4-\delta)E\Big(\tilde{b}^A\Big)\Big) - (8-5\delta)\Big(E\Big(\tilde{a}^U\Big) + E\Big(\tilde{b}^U\Big)\Big)\Big) + 3(\delta(2-\delta) - (4-3\delta)\alpha)E(\tilde{c}_n)}{\delta(1-\delta)(8-5\delta)(3-\delta\lambda)}.$$

Likewise, it is necessary to identify the sign of the coefficient of parameter λ. And it is difficult to accomplish it through judging the range of δ. Instead, the range of α is obtained, *i.e.*, the coefficient is more than 0 when $\alpha > \frac{\delta^2(1-\delta)(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))+\delta^2(2-\delta)E(\tilde{c}_n)}{(2\delta^2-9\delta+8)E(\tilde{c}_n)}$ and *vice versa*. Then, it is easy to prove Proposition 4.10. □

Proof of Proposition 4.11. When $\alpha_1^U < \alpha < \alpha_2^S$, the relationship between the demand for remanufactured product in Models S and U is given:

$$E^{*S}\left(\widetilde{D}_r^A\right) - E^{*U}\left(\widetilde{D}_r^U\right) = \frac{\delta(1-\delta)(1-\delta\lambda)\Big((4-\delta\lambda)\Big(2E\Big(\tilde{a}^A\Big) + (4-\delta)E\Big(\tilde{b}^A\Big)\Big) - (8-5\delta)\Big(E\Big(\tilde{a}^U\Big) + 2E\Big(\tilde{b}^U\Big)\Big)\Big) + \delta^2\big(\delta(2-\delta)\lambda^2 - 5(2-\delta)\lambda + 9 - 5\delta\big)E(\tilde{c}_n) - \big(\delta\big(2\delta^2-9\delta+8\big)\lambda^2 + (5\delta^2 + 6\delta - 16)\lambda + 4(4-3\delta)\big)\alpha E(\tilde{c}_n)}{\delta(1-\delta)(8-5\delta)(1-\delta\lambda)(4-\delta\lambda)}.$$

Similar to Proposition 4.8, the $E^{*S}(\widetilde{D}_r^A) - E^{*U}(\widetilde{D}_r^U)$ is regarded as a function with respect to α. And Proposition 4.11 is proved by solving the above inequality. □

Proof of Proposition 4.12. Furthermore, our research also discuss which authorization model is more environmentally friendly. When $0 < \alpha < \alpha_2^S$, the relationship between the demand for remanufactured product in Models C and S is derived:

$$E^{*C}\left(\widetilde{D}_r^A\right) - E^{*S}\left(\widetilde{D}_r^A\right) = \frac{\delta(1+\delta)\left(\alpha E(\tilde{c}_n) - \left(E\left(\tilde{a}^A\right) + E\left(\tilde{b}^A\right)\right)\right)}{4\delta(3-\delta)},$$

$$E^{*C}\left(\widetilde{D}_r^A\right) - E^{*S}\left(\widetilde{D}_r^A\right) = \frac{-\delta(1-\delta)\left(5\delta E\left(\tilde{a}^A\right) + (8+\delta)E\left(\tilde{b}^A\right)\right) + \left((8+\delta-5\delta^2)\alpha - 4\delta(2-\delta)\right)E(\tilde{c}_n)}{4\delta(1-\delta)(8-5\delta)}.$$

Obviously, the $E^{*C}(\widetilde{D}_r^A) - E^{*S}(\widetilde{D}_r^A)$ is less than 0 when $0 < \alpha < \alpha_1^S$. Furthermore, when $\alpha_1^S \le \alpha < \alpha_2^S$, the $E^{*C}(\widetilde{D}_r^A) = E^{*S}(\widetilde{D}_r^A)$ is satisfied by $\alpha = \frac{\delta(1-\delta)(5\delta E(\tilde{a}^A)+(8+\delta)E(\tilde{b}^A))+4\delta(2-\delta)E(\tilde{c}_n)}{(8+\delta-5\delta^2)E(\tilde{c}_n)}$. And, it falls within the interval (α_1^S, α_2^S) under the condition $E(\tilde{a}^A) + \delta E(\tilde{b}^A) > E(\tilde{c}_n)$. Then, it is easy to prove Proposition 4.12. □

Proof of Proposition B.1. To investigate the conditions of the OEM authorization in Model C, we need to discuss the parameters value ranges under different strategies. When the remanufacturing cost advantage satisfies $\alpha < \alpha_1^S$, the variation of the profit margin $E^{*C}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U)$ with regard to α is analyzed as follows:

$$\frac{\partial\left(E^{*C}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha} = \frac{\begin{gathered}2\delta(3-\delta\lambda)(3-\lambda(4+\delta))E^2(\tilde{c}_n) + \alpha\Big(8\lambda^2\left(\delta^2\lambda^2+\delta\lambda+2\right) \\ - (3-\delta\lambda)^2\Big)E^2(\tilde{c}_n) + \delta\Big((3-\delta\lambda)^2\left((1+2\delta)E\left(\tilde{b}^A\right) - E\left(\tilde{a}^A\right)\right) \\ - 4\lambda\Big(2\left(\delta^2\lambda^2+2\delta\lambda-1\right)E\left(\tilde{a}^U\right) + \left(\delta^3\lambda^3-2\delta^2\lambda^2+5\delta\lambda+4\right) \\ E\left(\tilde{b}^U\right)\Big)\Big)E(\tilde{c}_n) + 4\delta^2(3-\delta\lambda)^2\left(E\left(-\tilde{c}_n\tilde{b}^A\right) - \lambda^2 E\left(-\tilde{c}_n\tilde{b}^U\right)\right)\end{gathered}}{8\delta^2(3-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*C}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha^2} = \frac{\left(8\lambda^2\left(\delta^2\lambda^2+\delta\lambda+2\right) - (3-\delta\lambda)^2\right)E^2(\tilde{c}_n)}{8\delta^2(3-\delta\lambda)^2},$$

$$\left(\frac{\partial\left(E^{*C}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0} = \frac{\begin{gathered}2(3-\delta\lambda)(3-\lambda(4+\delta))E^2(\tilde{c}_n) + \Big((3-\delta\lambda)^2 \\ \left((1+2\delta)E\left(\tilde{b}^A\right) - E\left(\tilde{a}^A\right)\right) - 4\lambda\Big(2\left(\delta^2\lambda^2+2\delta\lambda-1\right)E\left(\tilde{a}^U\right) \\ + \left(\delta^3\lambda^3-2\delta^2\lambda^2+5\delta\lambda+4\right)E\left(\tilde{b}^U\right)\Big)\Big)E(\tilde{c}_n) + 4\delta(3 \\ -\delta\lambda)^2\left(E\left(-\tilde{c}_n\tilde{b}^A\right) - \lambda^2 E\left(-\tilde{c}_n\tilde{b}^U\right)\right)\end{gathered}}{8\delta(3-\delta\lambda)^2},$$

$$\left(E^{*C}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)\Big|_{\alpha=0} = \frac{\begin{gathered}4(3-\delta\lambda)\Big((3-\delta\lambda)\left(3E\left(\tilde{a}^A\right) - E\left(\tilde{b}^A\right)\right) - 4\Big((2-\delta\lambda)E\left(\tilde{a}^U\right) \\ - E\left(\tilde{b}^U\right)\Big)\Big)E(\tilde{c}_n) + (3-\delta\lambda)^2\left(E\left(\tilde{a}^A\right) + E\left(\tilde{b}^A\right)\right)\Big(3E\left(\tilde{a}^A\right) + (4\delta \\ -1)E\left(\tilde{b}^A\right)\Big) - 8(1-\delta\lambda)\left(E\left(\tilde{a}^U\right) + E\left(\tilde{b}^U\right)\right)\Big((4+\delta\lambda)E\left(\tilde{a}^U\right) \\ + (3\delta\lambda-2)E\left(\tilde{b}^U\right)\Big) + 16\delta(3-\delta\lambda)^2\left(E\left(-\tilde{c}_n\tilde{a}^A\right) - E\left(-\tilde{c}_n\tilde{a}^U\right)\right)\end{gathered}}{16\delta(3-\delta\lambda)^2}.$$

To determine the sign of $E^{*C}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U)$, it is necessary to derive the first and second order derivatives of it. The $\frac{\partial^2(E^{*C}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha^2} < 0$ when $\lambda < \frac{3}{4}$ and $0 < \delta < \frac{4\lambda\sqrt{7(1-\lambda^2)}-(3+4\lambda^2)}{\lambda(8\lambda^2-1)}$. Furthermore, if the expression $\left(\frac{\partial(E^{*C}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}$ is considered as a quadratic function of $E(\tilde{c}_n)$, we know that the quadratic coefficient is more than 0 for $\lambda < \frac{3}{4+\delta}$. Obviously, the $E(-\tilde{c}_n\tilde{b}^A) < \lambda^2 E(-\tilde{c}_n\tilde{b}^U)$. Thus, there exists a root greater than 0 denoted by $E^{*1}(\tilde{c}_n)$ that makes (i) $\left(\frac{\partial(E^{*C}(\pi^A_{\mathrm{OEM}})-E^{*U}(\pi^U_{\mathrm{OEM}}))}{\partial\alpha}\right)\Big|_{\alpha=0} > 0$ within the range $E(\tilde{c}_n) > E^{*1}(\tilde{c}_n)$ and (ii) $\left(\frac{\partial(E^{*C}(\pi^A_{\mathrm{OEM}})-E^{*U}(\pi^U_{\mathrm{OEM}}))}{\partial\alpha}\right)\Big|_{\alpha=0} < 0$ for $0 < E(\tilde{c}_n) < E^{*1}(\tilde{c}_n)$. In scenario (i), the profit function $E^{*C}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}})$ is increasing and then decreasing when $\alpha > 0$. Moreover, the $(E^{*C}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}}))\ |_{\alpha=0} > 0$ when the new product's production cost $E(\tilde{c}_n) > E(\tilde{c}_n)_4$ is met by $\lambda > \frac{4(2E(\tilde{a}^U)-E(\tilde{b}^U))-3(3E(\tilde{a}^A)-E(\tilde{b}^A))}{(4E(\tilde{a}^U)-3E(\tilde{a}^A)+E(\tilde{b}^A))\delta}$ and $\frac{2E(\tilde{a}^U)-E(\tilde{b}^U)}{3E(\tilde{a}^A)-E(\tilde{b}^A)} < \frac{3}{4}$. In scenario (ii), the profit function $E^{*C}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}})$ is dropping on $\alpha \in (0,1)$. And the $(E^{*C}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}}))\ |_{\alpha=0} < 0$ in case $E(\tilde{c}_n) < E(\tilde{c}_n)_4$. Hence, it is easy to prove Proposition B.1 by analyzing above cases. □

Proof of Proposition B.2. To examine the conditions of the IR accepts authorization, we need to identify the sign of the difference between the IR's profit in the unauthorized and authorized scenarios. In Model C, it is analyzed across three different intervals. When $\alpha < \alpha_1^S$, the derivations of the function $E^{*C}(\pi^A_{\mathrm{IR}}) - E^{*U}(\pi^U_{\mathrm{IR}})$ are given:

$$\frac{\partial\left(E^{*C}\left(\pi^A_{\mathrm{IR}}\right) - E^{*U}\left(\pi^U_{\mathrm{IR}}\right)\right)}{\partial\alpha} = \frac{\begin{aligned}&\alpha\left((3-\delta\lambda)^2 - 8\lambda^3(1-\delta\lambda)\right)E^2(\tilde{c}_n) + \delta\Big((3-\delta\lambda)^2\Big(3E\Big(\tilde{b}^A\Big)\\ &- E\Big(\tilde{a}^A\Big)\Big) - 4\lambda^2\Big(\left(\delta^2\lambda^2 - 4\delta\lambda + 7\right)E\Big(\tilde{b}^U\Big) - 2(1-\delta\lambda)E\Big(\tilde{a}^U\Big)\Big)\Big)\\ &E(\tilde{c}_n) + 4\delta(3-\delta\lambda)^2\Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2 E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{aligned}}{8\delta(3-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*C}\left(\pi^A_{\mathrm{IR}}\right) - E^{*U}\left(\pi^U_{\mathrm{IR}}\right)\right)}{\partial\alpha^2} = \frac{\left((3-\delta\lambda)^2 - 8\lambda^3(1-\delta\lambda)\right)E^2(\tilde{c}_n)}{8\delta(3-\delta\lambda)^2},$$

$$\left(E^{*C}\left(\pi^A_{\mathrm{IR}}\right) - E^{*U}\left(\pi^U_{\mathrm{IR}}\right)\right)\ |_{\alpha=0} = \frac{\delta\left((3-\delta\lambda)^2\left(E\left(\tilde{a}^A\right) + E\left(\tilde{b}^A\right)\right)^2 - 8\lambda(1-\delta\lambda)\left(E\left(\tilde{a}^U\right) + E\left(\tilde{b}^U\right)\right)^2\right)}{16(3-\delta\lambda)^2}.$$

Obviously, the second derivative meets $\frac{\partial^2(E^{*C}(\pi^A_{\mathrm{IR}})-E^{*U}(\pi^U_{\mathrm{IR}}))}{\partial\alpha^2} > 0$. According to $\delta, \lambda \in (0,1)$ and $E(\tilde{a}^A) + E(\tilde{b}^A) > E(\tilde{a}^U) + E(\tilde{b}^U)$, we know the $(E^{*C}(\pi^A_{\mathrm{IR}}) - E^{*U}(\pi^U_{\mathrm{IR}}))\ |_{\alpha=0} > 0$. Moreover, the relationship of the boundary condition α_1^S and minimum point α_4^* by controlling the range of $E(\tilde{c}_n)$, which means that the situation (i) $\alpha_1^S > \alpha_4^*$ for $E(\tilde{c}_n) > E^{*2}(\tilde{c}_n)$ and situation (ii) $\alpha_1^S < \alpha_4^*$ for $0 < E(\tilde{c}_n) < E^{*2}(\tilde{c}_n)$. Likewise, a root of the equation $\alpha_1^S = \alpha_4^*$ is denoted by $E^{*2}(\tilde{c}_n)$. Thus, we can prove Proposition B.2 by discussing the signs of the profit margin's minimum value in situations (i) and (ii). □

Proof of Proposition B.3. When the remanufacturing cost advantage satisfies $\alpha_1^U < \alpha < \alpha_2^S$, the first and second order derivatives of the profit margin $E^{*C}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}})$ are as follows:

$$\frac{\partial\left(E^{*C}\left(\pi^A_{\mathrm{OEM}}\right) - E^{*U}\left(\pi^U_{\mathrm{OEM}}\right)\right)}{\partial\alpha} = \frac{\begin{aligned}&2\delta\left((1-\delta\lambda)(4-\delta\lambda)^2 + 12\delta\lambda^2(2-\delta\lambda)\right)E^2(\tilde{c}_n) - \alpha\Big((1-\delta\lambda)\\ &(4-\delta\lambda)^2 + 8\delta\lambda^3\left(\delta^2\lambda^2 - 2\delta\lambda + 4\right)\Big)E^2(\tilde{c}_n) + \delta(1-\delta\lambda)\\ &\Big((4-\delta\lambda)^2\Big((1+2\delta)E\Big(\tilde{b}^A\Big) - E\Big(\tilde{a}^A\Big)\Big) - 4\delta\lambda^2\Big(2(2+\delta\lambda)E\Big(\tilde{a}^U\Big)\\ &+ \left(8+\delta^2\lambda^2\right)E\Big(\tilde{b}^U\Big)\Big)\Big)E(\tilde{c}_n) + 4\delta^2(1-\delta\lambda)(4-\delta\lambda)^2\Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2\\ &E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{aligned}}{8\delta^2(1-\delta\lambda)(4-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*C}\left(\pi^A_{\mathrm{OEM}}\right) - E^{*U}\left(\pi^U_{\mathrm{OEM}}\right)\right)}{\partial\alpha^2} = -\frac{\left((1-\delta\lambda)(4-\delta\lambda)^2 + 8\delta\lambda^3\left(\delta^2\lambda^2 - 2\delta\lambda + 4\right)\right)E^2(\tilde{c}_n)}{8\delta^2(1-\delta\lambda)(4-\delta\lambda)^2},$$

$$\left(\frac{\partial\left(E^{*C}\left(\pi_{\text{OEM}}^A\right)-E^{*U}\left(\pi_{\text{OEM}}^U\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0}=\frac{\begin{array}{l}2\left((1-\delta\lambda)(4-\delta\lambda)^2+12\delta\lambda^2(2-\delta\lambda)\right)E^2(\tilde{c}_n)\\+(1-\delta\lambda)\left((4-\delta\lambda)^2\left((1+2\delta)E\left(\tilde{b}^A\right)-E\left(\tilde{a}^A\right)\right)-4\delta\lambda^2\right.\\\left.\left(2(2+\delta\lambda)E\left(\tilde{a}^U\right)+\left(8+\delta^2\lambda^2\right)E\left(\tilde{b}^U\right)\right)\right)E(\tilde{c}_n)+4\delta(1-\delta\lambda)\\(4-\delta\lambda)^2\left(E\left(-\tilde{c}_n\tilde{b}^A\right)-\lambda^2E\left(-\tilde{c}_n\tilde{b}^U\right)\right)\end{array}}{8\delta(1-\delta\lambda)(4-\delta\lambda)^2}.$$

Obviously, the second derivative satisfies $\frac{\partial^2(E^{*C}(\pi_{\text{OEM}}^A)-E^{*U}(\pi_{\text{OEM}}^U))}{\partial\alpha^2}<0$, the quadratic term's coefficient is greater than 0 and constant term is less than 0 for the function $\left(\frac{\partial(E^{*C}(\pi_{\text{OEM}}^A)-E^{*U}(\pi_{\text{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}$ of $E(\tilde{c}_n)$. Thus, there is a root more than 0 marked by $E^{*3}(\tilde{c}_n)$ that makes (i) $\left(\frac{\partial(E^{*C}(\pi_{\text{OEM}}^A)-E^{*U}(\pi_{\text{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}>0$ for $E(\tilde{c}_n)>E^{*3}(\tilde{c}_n)$ and (ii) $\left(\frac{\partial(E^{*C}(\pi_{\text{OEM}}^A)-E^{*U}(\pi_{\text{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}<0$ for $0<E(\tilde{c}_n)<E^{*3}(\tilde{c}_n)$. Owing to the complaint form of $(E^{*C}(\pi_{\text{OEM}}^A)-E^{*U}(\pi_{\text{OEM}}^U))\,|_{\alpha=0}$, resulting in we do not determine its sign. Therefore, the Proposition B.3 is proved by examining the above cases in categories. □

Proof of Proposition B.4. When the remanufacturing cost advantage satisfies $\alpha_1^U<\alpha<\alpha_2^S$, the profit difference function $E^{*C}(\pi_{\text{IR}}^A)-E^{*U}(\pi_{\text{IR}}^U)$ in Model C is investigated:

$$\frac{\partial\left(E^{*C}\left(\pi_{\text{IR}}^A\right)-E^{*U}\left(\pi_{\text{IR}}^U\right)\right)}{\partial\alpha}=\frac{\begin{array}{l}8\delta\lambda^2(2-\delta\lambda)E^2(\tilde{c}_n)+\alpha\left((1-\delta\lambda)(4-\delta\lambda)^2-8\lambda^3(2-\delta\lambda)^2\right)\\E^2(\tilde{c}_n)+\delta(1-\delta\lambda)\left((4-\delta\lambda)^2\left(3E\left(\tilde{b}^A\right)-E\left(\tilde{a}^A\right)\right)-4\lambda^2\right.\\\left.\left(\left(\delta^2\lambda^2-4\delta\lambda+8\right)E\left(\tilde{b}^U\right)-2(2-\delta\lambda)E\left(\tilde{a}^U\right)\right)\right)E(\tilde{c}_n)\\+4\delta(1-\delta\lambda)(4-\delta\lambda)^2\left(E\left(-\tilde{c}_n\tilde{b}^A\right)-\lambda^2E\left(-\tilde{c}_n\tilde{b}^U\right)\right)\end{array}}{8\delta(1-\delta\lambda)(4-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*C}\left(\pi_{\text{IR}}^A\right)-E^{*U}\left(\pi_{\text{IR}}^U\right)\right)}{\partial\alpha^2}=\frac{\left((1-\delta\lambda)(4-\delta\lambda)^2-8\lambda^3(2-\delta\lambda)^2\right)E^2(\tilde{c}_n)}{8\delta(1-\delta\lambda)(4-\delta\lambda)^2},$$

$$\left(\frac{\partial\left(E^{*C}\left(\pi_{\text{IR}}^A\right)-E^{*U}\left(\pi_{\text{IR}}^U\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0}=\frac{\begin{array}{l}8\lambda^2(2-\delta\lambda)E^2(\tilde{c}_n)+(1-\delta\lambda)\left((4-\delta\lambda)^2\left(3E\left(\tilde{b}^A\right)-E\left(\tilde{a}^A\right)\right)\right.\\\left.-4\lambda^2\left(\left(\delta^2\lambda^2-4\delta\lambda+8\right)E\left(\tilde{b}^U\right)-2(2-\delta\lambda)E\left(\tilde{a}^U\right)\right)\right)\\E(\tilde{c}_n)+4(1-\delta\lambda)(4-\delta\lambda)^2\left(E\left(-\tilde{c}_n\tilde{b}^A\right)-\lambda^2E\left(-\tilde{c}_n\tilde{b}^U\right)\right)\end{array}}{8(1-\delta\lambda)(4-\delta\lambda)^2},$$

$$\left(E^{*C}\left(\pi_{\text{IR}}^A\right)-E^{*U}\left(\pi_{\text{IR}}^U\right)\right)|_{\alpha=0}=\frac{\begin{array}{l}-8\delta\lambda E^2(\tilde{c}_n)-16\delta\lambda(1-\delta\lambda)\left(E\left(\tilde{a}^U\right)+2E\left(\tilde{b}^U\right)\right)E(\tilde{c}_n)+\delta(1-\delta\lambda)\\\left((4-\delta\lambda)^2\left(E\left(\tilde{a}^A\right)+E\left(\tilde{b}^A\right)\right)^2-8\lambda(1-\delta\lambda)\left(E\left(\tilde{a}^U\right)+2E\left(\tilde{b}^U\right)\right)^2\right)\end{array}}{16(1-\delta\lambda)(4-\delta\lambda)^2}.$$

The second derivative meets $\frac{\partial^2(E^{*C}(\pi_{\text{IR}}^A)-E^{*U}(\pi_{\text{IR}}^U))}{\partial\alpha^2}>0$ when $0<\lambda<\sqrt[3]{\frac{1-\delta}{4}}$ and $\lambda<\frac{4((1-\delta-8\lambda^3)-\lambda\sqrt{2\lambda(1-\delta)})}{\delta(1-\delta-8\lambda^3)}$. Yet $\frac{4((1-\delta-8\lambda^3)-\lambda\sqrt{2\lambda(1-\delta)})}{\delta(1-\delta-8\lambda^3)}>\sqrt[3]{\frac{1-\delta}{4}}$. Obviously, the quadratic term of a function $\left(\frac{\partial(E^{*C}(\pi_{\text{IR}}^A)-E^{*U}(\pi_{\text{IR}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}$ has a coefficient that is greater than 0 and a constant term that is less than 0, which implies that the scenario (i) $\left(\frac{\partial(E^{*C}(\pi_{\text{IR}}^A)-E^{*U}(\pi_{\text{IR}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}>0$ for $E(\tilde{c}_n)>E^{*4}(\tilde{c}_n)$ and scenario (ii) $\left(\frac{\partial(E^{*C}(\pi_{\text{IR}}^A)-E^{*U}(\pi_{\text{IR}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}<0$ for $0<E(\tilde{c}_n)<$

$E^{*4}(\tilde{c}_n)$. We note that the root of this function larger than 0 is $E^{*4}(\tilde{c}_n)$. What's more, when $\delta > \frac{(E(\tilde{a}^U)+2E(\tilde{b}^U))^2}{6(E(\tilde{a}^A)+E(\tilde{b}^A))^2}$, the term in the expression $(E^{*C}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=0}$ that does not include $E(\tilde{c}_n)$ is greater than 0. Then, the $(E^{*C}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=0} > 0$ if $0 < E(\tilde{c}_n) < \frac{(4-\delta\lambda)(E(\tilde{a}^A)+E(\tilde{b}^A))\sqrt{2\lambda(1-\delta\lambda)}-4\lambda(1-\delta\lambda)(E(\tilde{a}^U)+2E(\tilde{b}^U))}{4\lambda}$ (denoted by $E(\tilde{c}_n)_5$) and $(E^{*C}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U)) \mid_{\alpha=0} < 0$ otherwise. Therefore, it is easy to prove Proposition B.4 by discussing scenarios (i) and (ii). □

Proof of Proposition B.5. Similarly, the following analyzes the OEM's authorization choice in Model S under three different intervals. When the remanufacturing cost advantage has $\alpha < \alpha_1^S$, our discussion process is provided:

$$\frac{\partial\left(E^{*S}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha} = \frac{\begin{array}{l}6\delta(1-\lambda)(3-\delta)(3-\delta\lambda)E^2(\tilde{c}_n) + 2\alpha\Big(\lambda^2(3-\delta)^2\big(\delta^2\lambda^2+\delta\lambda+2\big)\\ -(3-\delta\lambda)^2\big(\delta^2+\delta+2\big)\Big)E^2(\tilde{c}_n) + \delta\Big((3-\delta\lambda)^2\Big(2\big(\delta^2+2\delta-1\big)\\ E\Big(\tilde{a}^A\Big) + \big(\delta^3-2\delta^2+5\delta+4\big)E\Big(\tilde{b}^A\Big)\Big) - \lambda(3-\delta)^2\Big(2\big(\delta^2\lambda^2+2\delta\lambda-1\big)\\ E\Big(\tilde{a}^U\Big) + \big(\delta^3\lambda^3-2\delta^2\lambda^2+5\delta\lambda+4\big)E\Big(\tilde{b}^U\Big)\Big)\Big)E(\tilde{c}_n) + \delta^2(3-\delta)^2\\ (3-\delta\lambda)^2\Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2 E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{array}}{2\delta^2(3-\delta)^2(3-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha^2} = \frac{\big(\lambda^2(3-\delta)^2\big(\delta^2\lambda^2+\delta\lambda+2\big) - (3-\delta\lambda)^2\big(\delta^2+\delta+2\big)\big)E^2(\tilde{c}_n)}{\delta^2(3-\delta)^2(3-\delta\lambda)^2},$$

$$\left(\frac{\partial\left(E^{*S}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha}\right)\Bigg|_{\alpha=0} = \frac{\begin{array}{l}6(1-\lambda)(3-\delta)(3-\delta\lambda)E^2(\tilde{c}_n) + \Big((3-\delta\lambda)^2\Big(2\big(\delta^2+2\delta-1\big)E\Big(\tilde{a}^A\Big)\\ +\big(\delta^3-2\delta^2+5\delta+4\big)E\Big(\tilde{a}^A\Big)\Big) - \lambda(3-\delta)^2\Big(2\big(\delta^2\lambda^2+2\delta\lambda-1\big)E\Big(\tilde{a}^U\Big)\\ +\big(\delta^3\lambda^3-2\delta^2\lambda^2+5\delta\lambda+4\big)E\Big(\tilde{b}^U\Big)\Big)E(\tilde{c}_n)\Big) + \delta(3-\delta)^2(3-\delta\lambda)^2\\ \Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2 E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{array}}{2\delta(3-\delta)^2(3-\delta\lambda)^2},$$

$$\left(E^{*S}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right) \mid_{\alpha=0} = \frac{\begin{array}{l}2(3-\delta)(3-\delta\lambda)\Big((3-\delta\lambda)\Big((2-\delta)E\Big(\tilde{a}^A\Big) - E\Big(\tilde{b}^A\Big)\Big) - (3-\delta)\\ \Big((2-\delta\lambda)E\Big(\tilde{a}^U\Big) - E\Big(\tilde{b}^U\Big)\Big)\Big)E(\tilde{c}_n) + (1-\delta)(3-\delta\lambda)^2\\ \Big(E\Big(\tilde{a}^A\Big)+E\Big(\tilde{b}^A\Big)\Big)\Big((4+\delta)E\Big(\tilde{a}^A\Big) + (3\delta-2)E\Big(\tilde{b}^A\Big)\Big) - (1-\delta\lambda)(3-\delta)^2\\ \Big(E\Big(\tilde{a}^U\Big)+E\Big(\tilde{b}^U\Big)\Big)\Big((4+\delta\lambda)E\Big(\tilde{a}^U\Big) + (3\delta\lambda-2)E\Big(\tilde{b}^U\Big)\Big)\\ +2(3-\delta)^2(3-\delta\lambda)^2\Big(E\Big(-\tilde{c}_n\tilde{a}^A\Big) - E\Big(-\tilde{c}_n\tilde{a}^U\Big)\Big)\end{array}}{2(3-\delta)^2(3-\delta\lambda)^2}.$$

Based on the ranges of the δ and λ, we know $\frac{\partial^2(E^{*S}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha^2} < 0$. And, the quadratic coefficient and constant term have opposite signs for the function $\left(\frac{\partial(E^{*S}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}$. Hence, the situation (i) $\left(\frac{\partial(E^{*S}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0} > 0$ for $E(\tilde{c}_n) > E^{*5}(\tilde{c}_n)$ and situation (ii) $\left(\frac{\partial(E^{*S}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0} < 0$ for $0 < E(\tilde{c}_n) < E^{*5}(\tilde{c}_n)$, where the $E^{*5}(\tilde{c}_n)$ is a root more than 0 of $\left(\frac{\partial(E^{*S}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0} = 0$. Further, the coefficient of $E(\tilde{c}_n)$ in the function $(E^{*S}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=0}$ is less than 0 when the unauthorized remanufactured product's quality meets $\lambda < \frac{(3-\delta)(2E(\tilde{a}^U)-E(\tilde{b}^U))-3((2-\delta)E(\tilde{a}^A)-E(\tilde{b}^A))}{((3-\delta)E(\tilde{a}^U)-(2-\delta)E(\tilde{a}^A)+E(\tilde{b}^A))\delta}$. Thus, the $(E^{*S}(\pi_{\mathrm{OEM}}^A)-E^{*U}(\pi_{\mathrm{OEM}}^U)) \mid_{\alpha=0} > 0$

holds for $E(\tilde{c}_n) < E(\tilde{c}_n)_6$ and $(E^{*S}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=0} < 0$ holds for $E(\tilde{c}_n) > E(\tilde{c}_n)_6$, where the root of equation $(E^{*S}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=0} = 0$ is denoted by $E(\tilde{c}_n)_6$. Likewise, we discuss situation (i) and (ii) to prove Proposition B.5. □

Proof of Proposition B.6. Similarly, to study the conditions of the IR accepts authorization in Model S. We analyze its profit difference values across three different intervals. When the cost advantage has $\alpha < \alpha^S_1$, the research process is as follows:

$$\frac{\partial\left(E^{*S}(\pi^A_{\mathrm{IR}}) - E^{*U}(\pi^U_{\mathrm{IR}})\right)}{\partial\alpha} = \frac{\begin{aligned}&2\alpha\big((1-\delta)(3-\delta\lambda)^2 - \lambda^3(1-\delta\lambda)(3-\delta)^2\big)E^2(\tilde{c}_n) + \delta\Big((3-\delta\lambda)^2\Big((\delta^2-4\delta+7)E\Big(\tilde{b}^A\Big)\\&-2(1-\delta)E\Big(\tilde{a}^A\Big)\Big) - \lambda^2(3-\delta)^2\Big((\delta^2\lambda^2-4\delta\lambda+7)E\Big(\tilde{b}^U\Big) - 2(1-\delta\lambda)E\Big(\tilde{a}^U\Big)\Big)\Big)\\&E(\tilde{c}_n) + \delta(3-\delta)^2(3-\delta\lambda)^2\Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{aligned}}{2\delta(3-\delta)^2(3-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*S}(\pi^A_{\mathrm{IR}}) - E^{*U}(\pi^U_{\mathrm{IR}})\right)}{\partial\alpha^2} = \frac{\big((1-\delta)(3-\delta\lambda)^2 - \lambda^3(1-\delta\lambda)(3-\delta)^2\big)E^2(\tilde{c}_n)}{\delta(3-\delta)^2(3-\delta\lambda)^2},$$

$$\left(E^{*S}\left(\pi^A_{\mathrm{IR}}\right) - E^{*U}\left(\pi^U_{\mathrm{IR}}\right)\right)\mid_{\alpha=0} = \frac{\delta\left((1-\delta)(3-\delta\lambda)^2\left(E\left(\tilde{a}^A\right)+E\left(\tilde{b}^A\right)\right)^2 - \lambda(1-\delta\lambda)(3-\delta)^2\left(E\left(\tilde{a}^U\right)+E\left(\tilde{b}^U\right)\right)^2\right)}{2(3-\delta)^2(3-\delta\lambda)^2}.$$

We can know that the second derivative satisfies $\frac{\partial^2(E^{*S}(\pi^A_{\mathrm{IR}})-E^{*U}(\pi^U_{\mathrm{IR}}))}{\partial\alpha^2} > 0$ when the customers' WTP meets $0 < \delta < \frac{(1+\lambda)(3(1+\lambda^2)+\lambda)-\sqrt{(1+\lambda)^2(3(1+\lambda^2)+\lambda)^2-12\lambda(1+\lambda+\lambda^2)^2}}{2\lambda(1+\lambda+\lambda^2)}$ (denoted by δ_1). Also, the $(E^{*S}(\pi^A_{\mathrm{IR}}) - E^{*U}(\pi^U_{\mathrm{IR}}))\mid_{\alpha=0}$ is more than 0 for $0 < \delta < \frac{3((E(\tilde{a}^U)+E(\tilde{b}^U))^2+4(E(\tilde{a}^A)+E(\tilde{b}^A))^2)-12(E(\tilde{a}^A)+E(\tilde{b}^A))\sqrt{(E(\tilde{a}^A)+E(\tilde{b}^A))^2-(E(\tilde{a}^U)+E(\tilde{b}^U))^2}}{(E(\tilde{a}^U)+E(\tilde{b}^U))^2+24(E(\tilde{a}^A)+E(\tilde{b}^A))^2}$ (noted by δ_2) and $0 < \lambda < \lambda_3$. Although it is difficult to identify the sign of the $\left(\frac{\partial(E^{*S}(\pi^A_{\mathrm{IR}})-E^{*U}(\pi^U_{\mathrm{IR}}))}{\partial\alpha}\right)\Big|_{\alpha=0}$, we know the relationship of the threshold α^S_1 and minimum point α^*_{16} of the function $E^{*S}(\pi^A_{\mathrm{IR}}) - E^{*U}(\pi^U_{\mathrm{IR}})$. That is, the correlativity (i) $\alpha^S_1 > \alpha^*_{16}$ holds for $E(\tilde{c}_n) > E^{*6}(\tilde{c}_n)$ and (ii) $\alpha^S_1 < \alpha^*_{16}$ otherwise. In cases (i) and (ii), Proposition B.6 is proved combing with above discussion. □

Proof of Proposition B.7. When the remanufacturing cost advantage meets $\alpha^S_1 < \alpha < \alpha^U_1$, the derivations of function $E^{*S}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}})$ are derived below:

$$\frac{\partial\left(E^{*S}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}})\right)}{\partial\alpha} = \frac{\begin{aligned}&2\delta(3-\delta\lambda)(\delta(3-\delta\lambda)(2\delta-3) - \lambda(1-\delta)(8-5\delta))E^2(\tilde{c}_n) + 2\alpha\big(\delta(2-\delta)(3-\delta\lambda)^2\\&+\lambda^2(1-\delta)(8-5\delta)\big(\delta^2\lambda^2+\delta\lambda+2\big)\big)E^2(\tilde{c}_n) + \delta(1-\delta)\Big(\delta(3-\delta\lambda)^2\Big(2E\Big(\tilde{a}^A\Big)\\&+(4-\delta)E\Big(\tilde{b}^A\Big)\Big) - \lambda(8-5\delta)\Big(2\big(\delta^2\lambda^2+2\delta\lambda-1\big)E\Big(\tilde{a}^U\Big) + (\delta^3\lambda^3-2\delta^2\lambda^2+5\delta\lambda\\&+4)E\Big(\tilde{b}^U\Big)\Big)\Big)E(\tilde{c}_n) + \delta^2(1-\delta)(8-5\delta)(3-\delta\lambda)^2\Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{aligned}}{2\delta^2(1-\delta)(8-5\delta)(3-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*S}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}})\right)}{\partial\alpha^2} = \frac{\big(\delta(2-\delta)(3-\delta\lambda)^2 + \lambda^2(1-\delta)(8-5\delta)\big(\delta^2\lambda^2+\delta\lambda+2\big)\big)E^2(\tilde{c}_n)}{\delta^2(1-\delta)(8-5\delta)(3-\delta\lambda)^2}.$$

Obviously, the second derivative meets $\frac{\partial^2(E^{*S}(\pi^A_{\mathrm{OEM}})-E^{*U}(\pi^U_{\mathrm{OEM}}))}{\partial\alpha^2} > 0$. However, it is uneasy to identify the signs of $\left(\frac{\partial(E^{*S}(\pi^A_{\mathrm{OEM}})-E^{*U}(\pi^U_{\mathrm{OEM}}))}{\partial\alpha}\right)\Big|_{\alpha=0}$ and $(E^{*S}(\pi^A_{\mathrm{OEM}}) - E^{*U}(\pi^U_{\mathrm{OEM}}))\mid_{\alpha=0}$. Thus, we only assume their signs and then discuss. □

Proof of Proposition B.8. When the remanufacturing cost advantage meets $\alpha_1^S < \alpha < \alpha_1^U$, the first and second order derivations of profit margin $E^{*S}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U)$ in Model S are derived below:

$$\frac{\partial\left(E^{*S}\left(\pi_{\mathrm{IR}}^A\right) - E^{*U}\left(\pi_{\mathrm{IR}}^U\right)\right)}{\partial\alpha} = \frac{\begin{array}{l}2\delta(2-\delta)(3\delta-4)(3-\delta\lambda)^2E^2(\tilde{c}_n) + 2\alpha((3\delta-4)^2(3-\delta\lambda)^2 - \lambda^3(1-\delta)(1-\delta\lambda)\\ (8-5\delta)^2)E^2(\tilde{c}_n) + \delta(1-\delta)\Big((3-\delta\lambda)^2\Big(4(3\delta-4)E\Big(\tilde{a}^A\Big) + \big(19\delta^2-48\delta+32\big)E\Big(\tilde{b}^A\Big)\Big)\\ -\lambda^2(8-5\delta)^2\Big(\big(\delta^2\lambda^2-4\delta\lambda+7\big)E\Big(\tilde{b}^U\Big) - 2(1-\delta\lambda)E\Big(\tilde{a}^U\Big)\Big)\Big)E(\tilde{c}_n) + \delta(1-\delta)\\ (3-\delta\lambda)^2(8-5\delta)^2\Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{array}}{2\delta(1-\delta)(3-\delta\lambda)^2(8-5\delta)^2},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\mathrm{IR}}^A\right) - E^{*U}\left(\pi_{\mathrm{IR}}^U\right)\right)}{\partial\alpha^2} = \frac{\big((3\delta-4)^2(3-\delta\lambda)^2 - \lambda^3(1-\delta)(1-\delta\lambda)(8-5\delta)^2\big)E^2(\tilde{c}_n)}{\delta(1-\delta)(3-\delta\lambda)^2(8-5\delta)^2},$$

$$\Big(E^{*S}\Big(\pi_{\mathrm{IR}}^A\Big) - E^{*U}\Big(\pi_{\mathrm{IR}}^U\Big)\Big)\ |_{\alpha=0} = \frac{\begin{array}{l}2\delta(2-\delta)(3-\delta\lambda)^2\Big(2E\Big(\tilde{a}^A\Big) + (4-\delta)E\Big(\tilde{b}^A\Big)\Big)E(\tilde{c}_n) + \delta\Big((1-\delta)\\ (3-\delta\lambda)^2\Big(2E\Big(\tilde{a}^A\Big) + (4-\delta)E\Big(\tilde{b}^A\Big)\Big)^2 - \lambda(1-\delta\lambda)(8-5\delta)^2\Big(E\Big(\tilde{a}^U\Big) + E\Big(\tilde{b}^U\Big)\Big)^2\Big)\end{array}}{2(3-\delta\lambda)^2(8-5\delta)^2}.$$

We can know that the second derivative meets $\frac{\partial^2(E^{*S}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U))}{\partial\alpha^2} > 0$. Moreover, the relationship $(E^{*S}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U))\ |_{\alpha=0} > 0$ holds when $E(\tilde{c}_n) > \frac{\lambda(1-\delta\lambda)(8-5\delta)^2(E(\tilde{a}^U)+E(\tilde{b}^U))^2 - (1-\delta)(3-\delta\lambda)^2(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))^2}{2(2-\delta)(3-\delta\lambda)^2(2E(\tilde{a}^A)+(4-\delta)E(\tilde{b}^A))}$ (noted by $E(\tilde{c}_n)_7$). Likewise, we do not recognize the sign of the expression $\left(\frac{\partial(E^{*S}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}$. Hence, Proposition B.8 is proved by classification discussion. □

Proof of Proposition B.9. When the remanufacturing cost advantage has $\alpha_1^U < \alpha < \alpha_2^S$, the difference $E^{*S}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U)$ is analyzed:

$$\frac{\partial\left(E^{*S}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha} = \frac{\begin{array}{l}2\delta\big((1-\delta\lambda)(2\delta-3)(4-\delta\lambda)^2 + 3\lambda^2(1-\delta)(2-\delta\lambda)(8-5\delta)\big)\\ E^2(\tilde{c}_n) + 2\alpha\big((1-\delta\lambda)(2-\delta)(4-\delta\lambda)^2 - \lambda^3(1-\delta)(8-5\delta)\\ \big(\delta^2\lambda^2-2\delta\lambda+4\big)\big)E^2(\tilde{c}_n) + \delta(1-\delta)(1-\delta\lambda)\Big((4-\delta\lambda)^2\\ \Big(2E\Big(\tilde{a}^A\Big) + (4-\delta)E\Big(\tilde{b}^A\Big)\Big) - \lambda^2(8-5\delta)\big(2(2+\delta\lambda)E\Big(\tilde{a}^U\Big)\\ +\big(8+\delta^2\lambda^2\big)E\Big(\tilde{b}^U\Big)\big)\Big)E(\tilde{c}_n) + \delta(1-\delta)(1-\delta\lambda)(8-5\delta)(4-\delta\lambda)^2\\ \Big(E\Big(-\tilde{c}_n\tilde{b}^A\Big) - \lambda^2E\Big(-\tilde{c}_n\tilde{b}^U\Big)\Big)\end{array}}{2\delta(1-\delta)(1-\delta\lambda)(8-5\delta)(4-\delta\lambda)^2},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\mathrm{OEM}}^A\right) - E^{*U}\left(\pi_{\mathrm{OEM}}^U\right)\right)}{\partial\alpha^2} = \frac{\big((1-\delta\lambda)(2-\delta)(4-\delta\lambda)^2 - \lambda^3(1-\delta)(8-5\delta)\big(\delta^2\lambda^2-2\delta\lambda+4\big)\big)E^2(\tilde{c}_n)}{\delta(1-\delta)(1-\delta\lambda)(8-5\delta)(4-\delta\lambda)^2}.$$

Based on the ranges of the δ and λ, we know that the $\frac{\partial^2(E^{*S}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha^2} > 0$. In addition, it is difficult to recognize the signs of $\left(\frac{\partial(E^{*S}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U))}{\partial\alpha}\right)\Big|_{\alpha=0}$ and $(E^{*S}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U))\ |_{\alpha=0}$ due to the lack of information of the difference between $E(\tilde{a}^U)$, $E(\tilde{b}^U)$, $E(\tilde{a}^A)$ and $E(\tilde{b}^A)$ as well as the frequency of the parameters. Thus, Proposition B.9 is proved by discussing the signs of $(E^{*S}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U))\ |_{\alpha=0}$ and $(E^{*S}(\pi_{\mathrm{OEM}}^A) - E^{*U}(\pi_{\mathrm{OEM}}^U))\ |_{\alpha=\alpha_{25}^*}$. □

Proof of Proposition B.10. Similarly, when the remanufacturing cost advantage satisfies $\alpha_1^U < \alpha < \alpha_2^S$, the analysis process of profit margin function $E^{*S}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U)$ is given:

$$\frac{\partial\left(E^{*S}\left(\pi_{\mathrm{IR}}^A\right) - E^{*U}\left(\pi_{\mathrm{IR}}^U\right)\right)}{\partial\alpha} = \frac{\begin{array}{l}2\delta\left((1-\delta\lambda)(2-\delta)(3\delta-4)(4-\delta\lambda)^2 + \lambda^2(1-\delta)(2-\delta\lambda)(8-5\delta)^2\right)E^2(\tilde{c}_n)\\ +\, 2\alpha\left((1-\delta\lambda)(3\delta-4)^2(4-\delta\lambda)^2 - \lambda^3(1-\delta)(2-\delta\lambda)^2(8-5\delta)^2\right)E^2(\tilde{c}_n)\\ +\, \delta(1-\delta)(1-\delta\lambda)\left((4-\delta\lambda)^2\left(4(3\delta-4)E\left(\tilde{a}^A\right) + \left(19\delta^2 - 48\delta + 32\right)E\left(\tilde{b}^A\right)\right)\right.\\ \left. -\, \lambda^2(8-5\delta)^2\left(\left(\delta^2\lambda^2 - 4\delta\lambda + 8\right)E\left(\tilde{b}^U\right) - 2(2-\delta\lambda)E\left(\tilde{a}^U\right)\right)\right)E(\tilde{c}_n) + \delta(1-\delta)\\ (1-\delta\lambda)(4-\delta\lambda)^2(8-5\delta)^2\left(E\left(-\tilde{c}_n\tilde{b}^A\right) - \lambda^2 E\left(-\tilde{c}_n\tilde{b}^U\right)\right)\end{array}}{2\delta(1-\delta)(1-\delta\lambda)(4-\delta\lambda)^2(8-5\delta)^2},$$

$$\frac{\partial^2\left(E^{*S}\left(\pi_{\mathrm{IR}}^A\right) - E^{*U}\left(\pi_{\mathrm{IR}}^U\right)\right)}{\partial\alpha^2} = \frac{\left((1-\delta\lambda)(3\delta-4)^2(4-\delta\lambda)^2 - \lambda^3(1-\delta)(2-\delta\lambda)^2(8-5\delta)^2\right)E^2(\tilde{c}_n)}{\delta(1-\delta)(1-\delta\lambda)(4-\delta\lambda)^2(8-5\delta)^2},$$

$$\left(E^{*S}\left(\pi_{\mathrm{IR}}^A\right) - E^{*U}\left(\pi_{\mathrm{IR}}^U\right)\right)|_{\alpha=0} = \frac{\begin{array}{l}\delta\left((1-\delta\lambda)(2-\delta)^2(4-\delta\lambda)^2 - \lambda(1-\delta)(8-5\delta)^2\right)E^2(\tilde{c}_n) + 2\delta(1-\delta\lambda)\\ \left((2-\delta)(4-\delta\lambda)^2\left(2E\left(\tilde{a}^A\right) + (4-\delta)E\left(\tilde{b}^A\right)\right) - \lambda(1-\delta)(8-5\delta)^2\right.\\ \left.\left(E\left(\tilde{a}^U\right) + 2E\left(\tilde{b}^U\right)\right)\right)E(\tilde{c}_n) + \delta(1-\delta)(1-\delta\lambda)\left((1-\delta)(4-\delta\lambda)^2\right.\\ \left.\left(2E\left(\tilde{a}^A\right) + (4-\delta)E\left(\tilde{b}^A\right)\right)^2 - \lambda(1-\delta\lambda)(8-5\delta)^2\left(E\left(\tilde{a}^U\right) + 2E\left(\tilde{b}^U\right)\right)^2\right)\end{array}}{2(1-\delta)(1-\delta\lambda)(4-\delta\lambda)^2(8-5\delta)^2}.$$

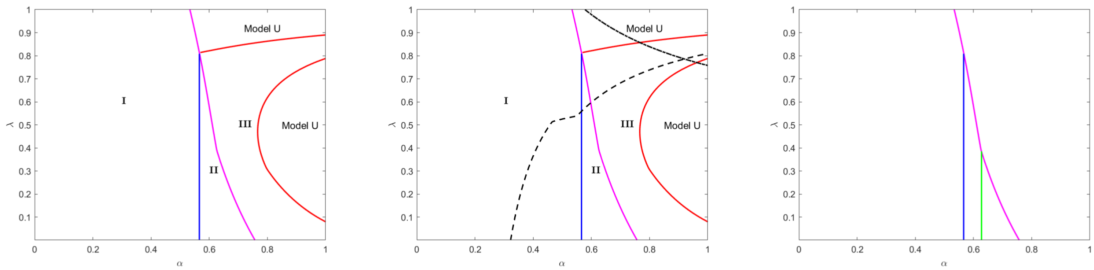
According to the values of customers' WTP δ and quality level λ, we can determine the sign of the second order derivation, *i.e.*, $\frac{\partial^2(E^{*S}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U))}{\partial\alpha^2} > 0$. Obviously, the quadratic coefficient of the function $(E^{*S}(\pi_{\mathrm{IR}}^A) - E^{*U}(\pi_{\mathrm{IR}}^U))\,|_{\alpha=0}$ about $E(\tilde{c}_n)$ is greater than 0. And its coefficient of primary term and constant term are also more than 0 when $0 < \lambda < \lambda_4$ and $\frac{40(E(\tilde{a}^U)+2E(\tilde{b}^U))^2 + 24(2E(\tilde{a}^A)+3E(\tilde{b}^A))^2 - 24(2E(\tilde{a}^A)+3E(\tilde{b}^A))\sqrt{(2E(\tilde{a}^A)+3E(\tilde{b}^A))^2 - 2(E(\tilde{a}^U)+2E(\tilde{b}^U))^2}}{25(E(\tilde{a}^U)+2E(\tilde{b}^U))^2 + 48(2E(\tilde{a}^A)+3E(\tilde{b}^A))^2} < \delta < 1$. Therefore, it is easy to prove Proposition B.10 based on above discussion. □

Proof of Corollary 4.1. Based on Propositions 4.1 and 4.2, we compare the boundary conditions for IR implements different remanufacturing strategies. When $E(\tilde{c}_n) > E(\tilde{c}_n)_1$, the relationship $\alpha_2^U > \alpha_2^S$ holds. And the $\alpha_1^U > \alpha_1^S$ is derived for $E(\tilde{c}_n) > E(\tilde{c}_n)_2$ and the $\alpha_2^S > \alpha_1^U$ is derived for $E(\tilde{c}_n) < E(\tilde{c}_n)_3$. To ensure that the three ranges intersect, we require $0 < \lambda < \lambda_1$ to make $E(\tilde{c}_n)_3 > E(\tilde{c}_n)_1$ and $0 < \lambda < \lambda_2$ to make $E(\tilde{c}_n)_3 > E(\tilde{c}_n)_2$. □

Proof of Corollary 4.2. When $E(\tilde{a}^A) + \delta E(\tilde{b}^A) > E(\tilde{c}_n)$, it is easy to prove this corollary by subtracting the price of the new product (or remanufactured product) in Model C from that in Model S. □

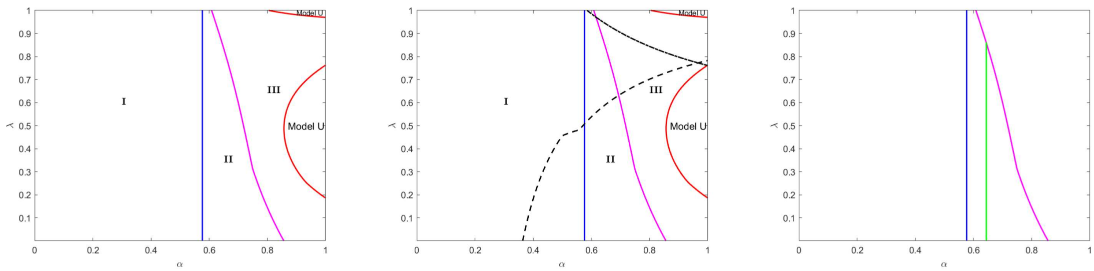
Appendix D. The impact of parameters δ and λ on authorization decisions

In Figures C.1A and C.2A, region I represents the OEM and IR select authorization Model C, while region III denotes the Model S is chosen. For region II, choosing either Model C or S is more profitable, but the latter is the better choice for the OEM. Conversely, the IR is more willing to adopt Model C. The Model U then indicates an unauthorized area. Furthermore, the total profit analysis shows that the Model C is preferable for both the OEM and IR to the left of the green line (in Figs. C.1C and C.2C), whereas the Model S is preferable to the right. Figures C.1B and C.2B show that in the lower right region of the black dashed line, demand for remanufactured products is lower in Model S compared to Model U, while it is higher in the upper left region. And in the upper right region of the black dotted line, the demand for remanufactured products is higher in Model C than in Model U, while it is lower in the lower left region. When customers exhibit a high WTP for remanufactured products (*i.e.*, $\delta = 0.8$), both the OEM and the IR are consistently inclined to establish an authorization relationship for any $\alpha, \lambda \in (0, 1)$ and they achieve higher profits under Model C



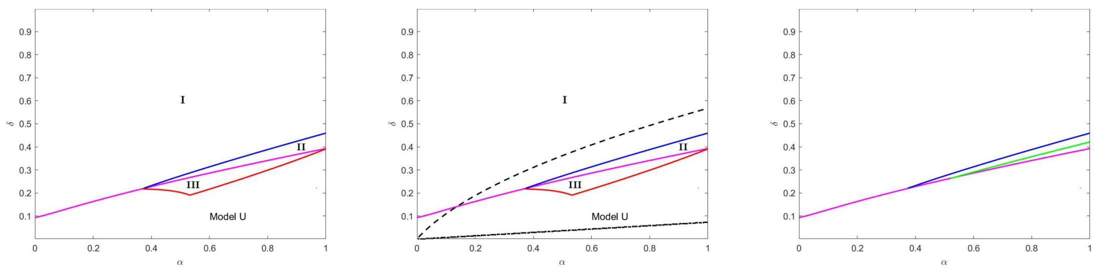
(A) (B) (C)

FIGURE C.1. Authorization regions and environment efforts for Models C and S under the deterministic environment with $\delta = 0.3$. (A) Authorization regions. (B) Environment efforts. (C) Authorization models in area II.



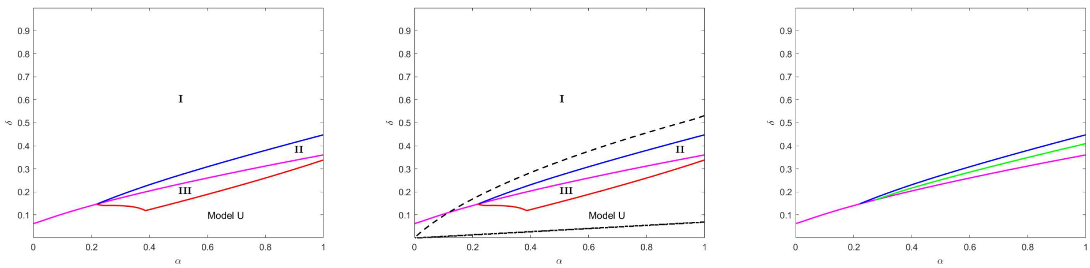
(A) (B) (C)

FIGURE C.2. Authorization regions and environment efforts for Models C and S under the fuzzy environment with $\delta = 0.3$. (A) Authorization regions. (B) Environment efforts. (C) Authorization models in area II.



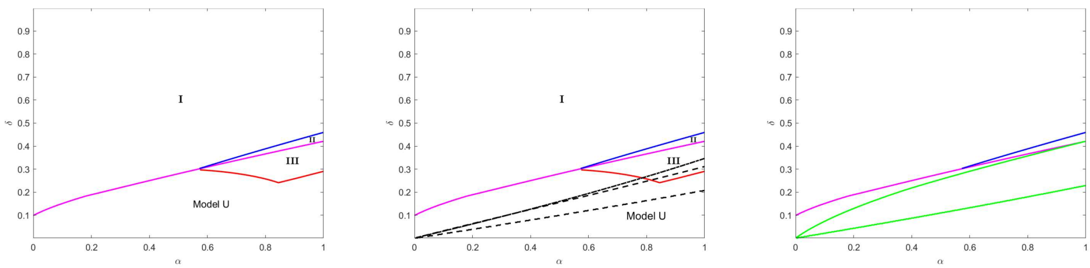
(A) (B) (C)

FIGURE D.1. Authorization regions and environment efforts for Models C and S under the deterministic environment with $\lambda = 0.3$. (A) Authorization regions. (B) Environment efforts. (C) Authorization models in area II.



(A) (B) (C)

FIGURE D.2. Authorization regions and environment efforts for Models C and S under the fuzzy environment with $\lambda = 0.3$. (A) Authorization regions. (B) Environment efforts. (C) Authorization models in area II.



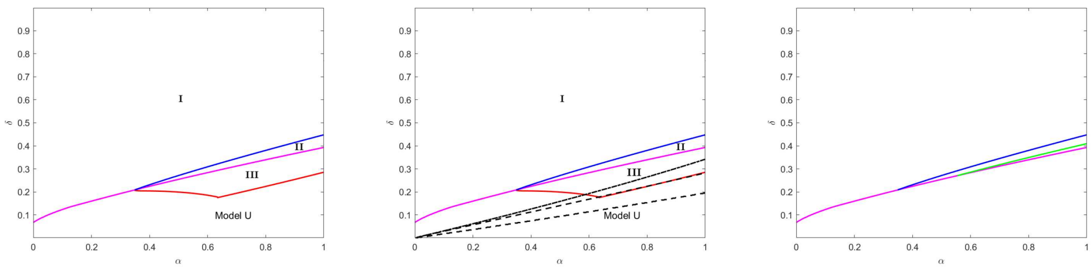
(A) (B) (C)

FIGURE D.3. Authorization regions and environment efforts for Models C and S under the deterministic environment with $\lambda = 0.8$. (A) Authorization regions. (B) Environment efforts. (C) Authorization models in area II.

compared to Model S. Nonetheless, the demand for remanufactured products in authorization Model C is lower than that in the unauthorized scenario.

In Figures D.1A and D.2A, region I indicates that both OEM and IR make more profits with Model C, while region III denotes that Model S is superior. And in region II, both OEM and IR improve their returns by choosing either Model C or S, but Model S is more profitable for the OEM and the opposite is true for the IR. Nevertheless, Figures D.1C and D.2C illustrate that their total profits are higher for the Model C above the green line and higher for the Model S below the green line. With regard to the environmental impact of the authorization, in Figures D.1B and D.2B, the demand for remanufactured products is lower than in the unauthorized scenario for Model C above the black dotted line and for Model S inside the black dotted line.

In Figures D.3A and D.4A, region I represents that Model C is superior for both OEM and IR, while region III indicates that Model S is better selection. And the meanings of region II is the same as that of region II in Figures D.1 and D.2. Additionally, based on the total profits of OEM and IR, the Model C is consistently chosen in region II of Figure D.3B and above the green line in Figure D.4C, while the Model S is preferred below the green line in Figure D.4C. As for environmental impacts, Figures D.3B and D.4B show that the remanufactured product's demand is higher in Model S above the black dashed line compared to unauthorized Model U. Conversely, the demand is lower within the black dashed line and in Model C above the black dotted line.



(A) (B) (C)

FIGURE D.4. Authorization regions and environment efforts for Models C and S under the fuzzy environment with $\lambda = 0.8$. (A) Authorization regions. (B) Environment efforts. (C) Authorization models in area II.