UNIVERSITY OF SOUTHAMPTON

Faculty of Social Sciences
School of Mathematical Sciences

Automorphisms, Free Products, and Properties of Groups through Spaces of Trees

by

Harry M. J. Iveson

ORCiD: 0009-0009-3564-5873

A thesis for the degree of
Doctor of Philosophy

9th February 2026

University of Southampton

Abstract

Faculty of Social Sciences
School of Mathematical Sciences

Doctor of Philosophy

Automorphisms, Free Products, and Properties of Groups through Spaces of Trees

by Harry M. J. Iveson

Given a group acting on a tree, one may obtain a space of such trees, which is acted on by automorphisms of the group. In this thesis, we study these actions to derive various results regarding automorphisms of free products (which act on particularly nice trees).

This is a three paper thesis; the main body of the work is contained in the following papers:

[1] Harry Iveson. A presentation for the group of pure symmetric outer automorphisms of a given splitting of a free product. Preprint, December 2024, available at `arXiv:2412.04250`.

[2] Harry Iveson. Generators for the pure symmetric outer automorphisms of a free product. Preprint, July 2025, available at `arXiv:2507.16662`.

[3] Harry Iveson, Armando Martino, Wagner Sgobbi, and Peter Wong with an appendix by Francesco Fournier-Facio. Property R_∞ for groups with infinitely many ends. Preprint, April 2025, available at `arXiv:2504.12002`.

In [1] (Paper 1), we give a concise presentation for the group of pure symmetric outer automorphisms, $\mathrm{Out}_{\mathfrak{S}}(G)$, of a given splitting of a free product $G = G_1 * \cdots * G_n$. This is achieved by applying a theorem of K. S. Brown to a particular subcomplex, $\mathcal{C}_n$, of Guirardel and Levitt's Outer Space for a free product — a space of trees, viewed through the language of graphs of groups, via Bass–Serre theory. The application of Brown's theorem occupies the first half of the paper, while the second half of the paper develops a rigorous argument as to why the subcomplex $\mathcal{C}_n$ is simply connected (a requirement of Brown's theorem).

In [2] (Paper 2), we give a geometric proof that $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by Whitehead (and factor) automorphisms. The motivation behind this was to provide a direct proof for Proposition 3.1.3 (and hence also for Corollary 3.1.4) of Paper 1 [1]. That is, to show that a particular subcomplex ($\mathcal{C}_n$, or in this case, its subcomplex $\mathcal{S}_n$) of the spine of the Outer Space for a free product is path connected.

In [3] (Paper 3), we use actions on $\mathbb{R}$-trees to show that any group with infinitely many ends (and in fact, a larger class of groups; those which act 'sufficiently canonically' on a 'sufficiently nice' $\mathbb{R}$-tree) has the property R_∞. That is, that every automorphism of the group has infinitely many twisted conjugacy classes. As a corollary, we see that the problem of if a group has property R_∞ or not is undecidable amongst the class of finitely presented groups.

Contents

3 Property R_∞ for groups with infinitely many ends 181

List of Figures

List of Tables

Declaration of Authorship

I declare that this thesis and the work presented in it is my own and has been generated by me as the result of my own original research.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Parts of this work have been published as:

   [1] Harry Iveson. A presentation for the group of pure symmetric outer automorphisms of a given splitting of a free product. Preprint, December 2024, available at `arXiv:2412.04250`.

   [2] Harry Iveson. Generators for the pure symmetric outer automorphisms of a free product. Preprint, July 2025, available at `arXiv:2507.16662`.

   [3] Harry Iveson, Armando Martino, Wagner Sgobbi, and Peter Wong, with an appendix by Francesco Fournier-Facio. Property R_∞ for groups with infinitely many ends. Preprint, April 2025, available at `arXiv:2504.12002`.

Signed:.. Date:..................

Acknowledgements

I would like to begin by thanking Armando — there is nobody I would rather have had to guide me through this period. You have helped me grow, mathematically and as a person, and have always been supportive (even when the problems I've encountered were a direct result of not listening to your previous advice). Thank you for being so patient with me, and for always being happy to talk when I've needed it.

Thank you Charles for being an amazing and wonderful partner and person. You have supported me, emotionally and physically, throughout my PhD (and before), and you have made things bearable that otherwise would not have been, and made things enjoyable that would otherwise have simply been bearable. I am grateful for all that you have done for and with me.

I would also like to thank Naomi; my unofficial "second supervisor". You were an integral part of the first year of my PhD, and I will always be grateful for the time you have spent helping me, even after you left Southampton. You have always made me feel welcome, and tried to ensure that things are as accessible as possible for me. Thank you for your continued support, and for your friendship.

It's strange to think that four years ago I had no idea how to go about solving a cryptic crossword. Thank you to all of the PhD students and PostDocs who introduced me to this pastime, and to all who have subsequently joined and shared in this lunchtime activity with me. I would also like to thank Alexis and Guy for inviting me to play Dungeons and Dragons, and giving me something to look forward to every couple of weeks (or months) that is (mostly) unrelated to my work.

Thank you to all in the mathematical community, in Southampton and beyond, who have taught me, welcomed me, kept me company, and made my time here more fulfilling and enjoyable.

Thank you as well to those outside of Southampton, friends and family, who don't mind that I never message, and always offer me somewhere to stay when I need to escape for a bit.

Definitions and Abbreviations

$\mathbb{N}$	The set of natural numbers, including 0
$\mathbb{Z}$	The set of integers, often viewed as a group under addition, occasionally viewed as a ring
$\mathbb{R}$	The set of real numbers, often viewed as 'the real line', a metric space with $d(x, y) = \lvert x - y \rvert$
S_n	The symmetric group on n points
$F_n, F(X)$	The free group of rank n and the free group with generating set X, respectively
$X \subseteq Y, X \subset Y$	X is a subset/proper subset (respectively) of Y — we may write $X \subsetneq Y$ when we wish to emphasise the properness of a subset
X-Y	The set of elements which belong to X and do not belong to Y
$\bigcup X$	The union $\{y \in x \mid x \in X\}$
$H \leqslant G, H \trianglelefteqslant G$	H is a subgroup/normal subgroup (respectively) of G
$G * H$	The free product of G and H
$G \times H$	The direct product of G and H
$N \rtimes H$	The semidirect product with kernel N and quotient H
$G \underline{\diamond} G \times \cdots \times G$	G is the diagonal subgroup $\{(g, \ldots, g) \mid g \in G\}$ of $G \times \cdots \times G$
$Z(G)$	The centre of G
$\mathrm{Stab}_G(x)$	The subgroup of G stabilising the point x
$\langle\langle S \rangle\rangle_G$	The normal closure of S in G
${}^{G}\!\big/_{N}$	The quotient of G by a normal subgroup N
${}^{G}\!\big/_{\simeq}$	The quotient of G by $\langle\langle \{g \in G \mid g \simeq 1\} \rangle\rangle_G$ where $\simeq$ is an equivalence relation on G
$\langle X \mid R \rangle$	The quotient ${}^{F(X)}\!\big/_{\langle\langle R \rangle\rangle_{F(X)}}$
$\mathrm{Aut}(G)$	The automorphism group of G
$\mathrm{Inn}(G)$	The normal subgroup of $\mathrm{Aut}(G)$ comprising inner automorphisms of G

$\mathrm{Out}(G)$	The outer automorphism group of G; the quotient ${}^{\mathrm{Aut}(G)}\!\big/_{\mathrm{Inn}(G)}$
$\mathrm{Aut}_{\mathfrak{S}}(G)$	The pure symmetric automorphisms relative to a splitting $\mathfrak{S}$ of a free product G
$\mathrm{Out}_{\mathfrak{S}}(G)$	The image of $\mathrm{Aut}_{\mathfrak{S}}(G)$ in $\mathrm{Out}(G)$
$[v_0, \ldots, v_n]$	The n-simplex on vertices $v_0, \ldots, v_n$
$u \mathbin{-\!\!-} v$	The edge/1-simplex with endpoints u and v
$o(e), t(e)$	The origin/terminus (respectively) of an edge e
ι_e	The inclusion $G_e \hookrightarrow G_{o(e)}$ of an edge group into its (origin) vertex group
$\varphi : G \hookrightarrow H$	φ is an injective homomorphism from G to H
$\varphi : G \twoheadrightarrow H$	φ is a surjective homomorphism from G to H
$\mathcal{O}, \mathcal{PO}, \mathcal{SO}$	The outer space/projectivised outer space/spine of outer space (respectively) introduced by V. Guirardel and G. Levitt in [15]
$\mathcal{C}_n, \mathcal{S}_n$	The subcomplexes of $\mathcal{SO}$ studied in Papers 1 and 2, respectively

Introduction

In this introduction we provide background material and context for the three papers that form the main body of this thesis. Papers 1 and 2 are single author papers; Paper 3 is a joint paper with Armando Martino (my supervisor), Wagner Sgobbi, and Peter Wong (with an appendix by Francesco Fournier-Facio).

The (non-background) statements and proofs in the main body of Paper 3 were jointly formulated for the most part, mostly by Armando and myself, and were developed over the course of several meetings. I wrote the majority of the first draft, at that point roughly comprising material for Sections 1–5 and Section 7 (Appendix A). The four authors then reviewed and revised the manuscript, both independently and jointly during (online) meetings, before a final version was produced. Appendix B was written solely by Francesco Fournier-Facio, who contacted us after having read an earlier version of the paper.

This introduction is split into 10 sections. Sections 1 and 2 give general background for all papers, with an overview of (mostly undergraduate level) group theory in Section 1, and a crash course on Bass–Serre theory in Section 2.

Papers 1 and 2 are closely related and therefore share a lot of background; Sections 3–6 provide this. The objects of study in Papers 1 and 2 are subcomplexes ($\mathcal{C}_n$ and $\mathcal{S}_n$, respectively) of (the spine of) Guirardel and Levitt's 'Outer Space for a free product' [15]. These objects encode information on (pure symmetric outer) automorphisms of free products.

Section 3 briefly extends Subsections 1.3 and 1.5 to describe the automorphisms specific to a free product. Section 4 introduces simplicial complexes (which $\mathcal{C}_n$ and $\mathcal{S}_n$ both are), giving examples of specific constructions which will be referenced in Papers 1 and 2, as well as describing properties they may (or may not) have. Note that Subsection 4.3 (which discusses a theorem of K. S. Brown) is not relevant for Paper 2 (and does give more detail than is needed for Paper 1 [1]). Section 5 summarises [15] to give descriptions of the spaces $\mathcal{O}$, $\mathcal{PO}$, and $\mathcal{SO}$. This is not strictly necessary to understand the spaces

---

[1] It is useful context though, and **is** necessary for extending the results of Paper 1 to include automorphisms of free generators.

$\mathcal{C}_n$ and $\mathcal{S}_n$, but does provide context for their existence. Section 6 briefly describes the complexes $\mathcal{C}_n$ and $\mathcal{S}_n$, as well as introducing a complex $\mathcal{C}_{n,k,d}$ in Subsection 6.1 which would allow the results of Paper 1 to be extended to groups of the form $G_1 * \cdots * G_n * F_k$ for $k > 0$. However, due to the length of this thesis, Subsection 6.1 is not as detailed as it could be, and $\mathcal{C}_{n,k,d}$ does not appear in any of the papers in this thesis.

Paper 3 is relatively self-contained, and contains plenty of references for further background reading on topics encountered. As such, we give only a brief background in this introduction, with Section 7 providing a brief overview of the R_∞ property, and Section 9 providing a brief introduction to $\mathbb{R}$-trees. We then choose a couple of topics referenced in Paper 3, namely ‘ends’ and ‘reduced trees’, to review in more depth, explored in Sections 8 and 10, respectively.

1 Groups and Automorphisms

We begin with a review (mostly for notational purposes) of some standard undergraduate definitions.

Definition 1.0.1. A *group* $(G, \cdot)$ is a non-empty set G together with a binary operation $\cdot$ which satisfies:

Associativity: $\forall x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Identity: $\exists 1 \in G$ such that $\forall x \in G$, $x \cdot 1 = 1 \cdot x = x$

Inverses: $\forall x \in G$, $\exists y \in G$ such that $x \cdot y = y \cdot x = 1$

We will usually write our groups ‘multiplicatively’, writing G for a group $(G, \cdot)$ and xy for the element $x \cdot y$ of G. Unless otherwise specified, we will denote the identity element of a group by 1 and the inverse of an element x by x^{-1}.

Notation 1.0.2. Given two sets X and Y we write X-Y for the set $\{x \in X | x \notin Y\}$. While Y will often be a subset of X, this is not a requirement of the definition.

Example 1.0.3 (Symmetric Group)**.** The *symmetric group* on n letters, denoted S_n, is the group comprising all bijections of the set $\{1, \ldots, n\}$ for some fixed $n \in \mathbb{N}$, where the group operation is composition. That is, given $\sigma, \tau \in S_n$, $\sigma \circ \tau : j \mapsto \sigma(\tau(j))$ for all $j \in \{1, \ldots, n\}$.

Definition 1.0.4. A subset H of a group G is called a *subgroup* of G (denoted $H \leqslant G$) if $1 \in H$ and for all $g, h \in H$ we have $g^{-1}h \in H$.

We say a subgroup N of G is *normal* (denoted $N \trianglelefteq G$) if for all $n \in N$ and $g \in G$ we have $g^{-1}ng \in N$. We say a subgroup M of a group G is *malnormal* if for all $g \in G$-M we have $g^{-1}Mg \cap M = \{1\}$.

Remark. Note that a subgroup $H \leqslant G$ is normal if and only if $g^{-1}Hg = H$ for all $g \in G$. For any subgroup $H \leqslant G$ we have that $h^{-1}Hh = H$ for all $h \in H$. So a subgroup $H \leqslant G$ being malnormal means it is as far from being normal as possible, in the sense that $g^{-1}hg \notin H$ for all $g \in G\text{-}H$ and all $h \in H$.

Notation 1.0.5. Let G be a group and S a subset of G.

1. $Z(G) := \{g \in G \mid g^{-1}xg = x \ \forall x \in G\}$ is the *center* of G; it is a normal subgroup of G.
2. $\langle\langle S \rangle\rangle_G := \bigcap\limits_{S \subseteq N \trianglelefteqslant G} N$ is the *normal closure* of S in G; it is the smallest normal subgroup of G containing S.

Proof. We justify the above claims:

1. Since $1^{-1}x1 = x$ for all $x \in G$ then $1 \in Z(G)$. Let $g, h \in Z(G)$. Then for all $x \in G$ we have $g^{-1}xg = x = h^{-1}xh$. So for any $x \in G$, we have $(g^{-1}h)^{-1}x(g^{-1}h) = (h^{-1}g)g^{-1}xg(g^{-1}h) = h^{-1}xh = x$. Thus $g^{-1}h \in Z(G)$ and hence $Z(G) \leqslant G$.

   Now let $g \in Z(G)$ and $h \in G$. Then for any $x \in G$ we have $(h^{-1}gh)^{-1}x(h^{-1}gh) = h^{-1}\left(g^{-1}(hxh^{-1})g\right)h = h^{-1}(hxh^{-1})h = x$, hence $h^{-1}gh \in Z(G)$. Thus $Z(G) \trianglelefteqslant G$.
2. Since $\langle\langle S \rangle\rangle_G$ is an intersection of normal subgroups it must itself be a normal subgroup. Moreover, $\left|\bigcap\limits_{S \subseteq N \trianglelefteqslant G} N\right| \leqslant \min\limits_{S \subseteq N \trianglelefteqslant G} |N|$, thus there is no subgroup $H \trianglelefteqslant G$ containing S with cardinality less than that of $\langle\langle S \rangle\rangle_G$.

□

Example 1.0.6 (Quotient Group)**.** Given a group G and a normal subgroup N of G, we can form the *quotient group* ${}^{G}\!/\!{}_{N}$ whose elements are (left) *cosets* $gN := \{gn | n \in N\}$ where the group operation is given by $gN \cdot hN = ghN$.

1.1 Free Groups and Group Presentations

'Free groups' are a fundamental concept in the theory of infinite groups. As such, there are many expository works on the subject, leading to several different (but equivalent) definitions, depending on the author consulted.

R. Lyndon and P. Schupp, in their textbook 'Combinatorial Group Theory' [19], take the approach of a 'universal property' for functions with domain X a subset of the free group F.

Definition 1.1.1 (Lyndon and Schupp [19]). Let F be a group and $X \subset F$ a subset of F. We say that F is *free* with basis X if every function $f : X \to G$ from X to some group G extends to a unique homomorphism $\varphi : F \to G$. That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & F \arrow[d, dashed, "\exists!\varphi"] \\ & G \end{tikzcd}$$

where $i : X \hookrightarrow F$ is the inclusion map.

D. Cohen in 'Combinatorial Group Theory: A Topological Approach' [6] (or [5] for the earlier edition) takes a similar view, but does not initially require that X be a subset of F. However he later shows there must be a monomorphism from X to F, which may be regarded as inclusion. Cohen also gives an explicit construction of a free group based on the idea of 'reduced words', which can be combined by concatenation and reduction.

Example 1.1.2 (Cohen [6]). Let X be a set and let

$$F(X) := \left\{ x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_n}^{\varepsilon_n} \mid n \in \mathbb{N}, \text{ and } \forall j, x_{i_j} \in X, \varepsilon_j \in \{\pm 1\}, \text{ and } x_{i_{j+1}} = x_{i_j} \Rightarrow \varepsilon_{j+1} = \varepsilon_j \right\}$$

be the set of reduced words on $X \cup X^{-1}$. Then $F(X)$ is a group under the operation "concatenate and reduce".

Explicitly, given two elements $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_n}^{\varepsilon_n}$ and $x_{j_1}^{\varepsilon'_1} x_{j_2}^{\varepsilon'_2} \dots x_{j_m}^{\varepsilon'_m}$, we define $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_n}^{\varepsilon_n} \cdot x_{j_1}^{\varepsilon'_1} x_{j_2}^{\varepsilon'_2} \dots x_{j_m}^{\varepsilon'_m}$ to be the element of $F(X)$ obtained by recursively deleting pairs of the form $x_{i_k}^{\varepsilon_k} x_{i_k}^{-\varepsilon_k}$ (i.e. pairs $x_{i_k}^{\varepsilon_k} x_{i_{k+1}}^{\varepsilon_{k+1}}$ where $x_{i_{k+1}} = x_{i_k}$ and $\varepsilon_{k+1} \neq \varepsilon_k$) from the word $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_n}^{\varepsilon_n} x_{i_{n+1}}^{\varepsilon_{n+1}} \dots x_{i_{n+m}}^{\varepsilon_{n+m}} := x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_n}^{\varepsilon_n} x_{j_1}^{\varepsilon'_1} x_{j_2}^{\varepsilon'_2} \dots x_{j_m}^{\varepsilon'_m}$.

The identity element is the empty word, denoted '1', and the inverse of an element $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_n}^{\varepsilon_n}$ is the word $x_{i_n}^{-\varepsilon_n} \dots x_{i_2}^{-\varepsilon_2} x_{i_1}^{-\varepsilon_1}$.

Lemma 1.1.3 ([6, Theorems 3 and 4, Proposition 6]). *Let X be any set. Then $F(X)$ is free with basis X. Moreover, if Y is another set, then $F(X) \cong F(Y)$ if and only if the sets X and Y have the same cardinality.*

Given a set X, we call the cardinality of X the *rank* of the free group it generates. When $X = \{x_1, \dots, x_k\}$ is a set of cardinality k, we may denote the free group $F(X)$ by F_k.

Somewhat contrarily, A. Hatcher in 'Algebraic Topology' [17] simply defines a free group as a free product (see Definition 1.5.1) of infinite cyclic groups, with F_k being the free product of k copies of $\mathbb{Z}$.

Notation 1.1.4 (Group Presentation)**.** Let X be a set, $F(X)$ be the free group with generating set X, and let R be a subset of $F(X)$. We denote by $\langle\, X \mid R\rangle$ the quotient group $G = {}^{F(X)}\!/_{\langle\langle R\rangle\rangle_{F(X)}}$. We call $\langle\, X \mid R\rangle$ a *presentation* for G and we call elements $x \in X$ *generators* of G and elements $r \in R$ *relators* for G.

We will usually omit set brackets when writing out presentations (i.e. we write $\langle x, y, z \mid a, b, c\rangle$ rather than $\langle \{x, y, z\} \mid \{a, b, c\}\rangle$). We sometimes prefer to use the notation of *relations* over that of relators, writing $\langle X \mid a = b\rangle$ to represent the presentation $\langle X \mid ab^{-1}\rangle$ (where a and b are both words on $X \cup X^{-1}$). If H is a group with presentation $\langle y_1, \ldots, y_m \mid s_1, \ldots, s_l\rangle$, we may write $\langle H, x_1, \ldots, x_n \mid r_1, \ldots, r_k\rangle$ as shorthand for the presentation $\langle x_1, \ldots, x_n, y_1, \ldots, y_m \mid r_1, \ldots, r_k, s_1, \ldots, s_l\rangle$.

Every group can be expressed as a quotient of a free group in this way (see, for example, [6, Proposition 13]), though such a presentation is far from unique.

Definition 1.1.5. We say a group G is *finitely generated* if there exists a presentation $\langle X|R\rangle$ for G such that the cardinality $|X|$ of X is finite (that is, there exists a surjective homomorphism from a finite rank free group to G). We say a group G is *finitely presented* if there exists a presentation $\langle X|R\rangle$ for G such that the cardinalities $|X|$ and $|R|$ of X and R are both finite.

1.2 Group Actions

Definition 1.2.1. Given a group G and a set X, a left *action* of G on X is a function $\rho : G \times X \to X$ satisfying $\rho((1, x)) = x\ \forall x \in X$ and $\rho((g, \rho((h, x)))) = \rho((gh, x))\ \forall g, h \in G$ and $\forall x \in X$. We call such a set X equipped with an action of G a *G-set*. We will write actions multiplicatively, denoting $g \cdot x := \rho(g, x)$. The above conditions then become $1 \cdot x = x$ and $g \cdot (h \cdot x) = gh \cdot x\ \forall g, h \in G$ and $\forall x \in X$. Right actions of G on X are defined analogously.

Remark. Note that for simplicity, in this introduction, all actions will be left actions. However, many actions in Papers 1, 2, and 3 are right actions (though not necessarily consistently across different papers), so care ought to be taken here. These differences are summarised in Table 2, where G is a group, $\mathrm{Aut}(G)$ is its automorphism group (see Section 1.3), T is a G-tree, and $\mathcal{D}(T)$ is a deformation space (e.g. Outer Space, as in Section 5) containing T.

Oftentimes our set X will have some additional structure, such as that of a group, or a tree, or a CW-complex. In these cases we will often care about actions that respect the structure of X.

Definition 1.2.2. Let G be a group acting on a G-set X.

1. The action of G on X is *free* if for all $g \in G - \{1\}$ and $x \in X$, we have $g \cdot x \neq x$

	Paper 1	**Paper 2**	**Paper 3**
Conjugation:	$a^b := bab^{-1}$ $(a^b)^c = a^{cb}$	$a^b := b^{-1}ab$ $(a^b)^c = a^{bc}$	$a^b := b^{-1}ab$ $(a^b)^c = a^{bc}$
Action of G on T:	Left: $g \cdot x$ $\mathrm{Stab}_T(g \cdot x)$ $= g(\mathrm{Stab}_T(x))g^{-1}$ $= (\mathrm{Stab}_T(x))^g$	Right: $x \cdot g$ $\mathrm{Stab}_T(x \cdot g)$ $= g^{-1}(\mathrm{Stab}_T(x))g$ $= (\mathrm{Stab}_T(x))^g$	Right: $x \cdot g$ $\mathrm{Stab}_T(x \cdot g)$ $= g^{-1}(\mathrm{Stab}_T(x))g$ $= (\mathrm{Stab}_T(x))^g$
Action of $\mathrm{Aut}(G)$ on G:	Right: $(g)\varphi$ $\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[rr, bend right, "\varphi\psi"'] & G \arrow[r, "\psi"] & G \end{tikzcd}$	Left: $\varphi(g)$ $\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\varphi"'] & G \arrow[r, "\psi"] & G \end{tikzcd}$	Right: $\varphi(g)$ $\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[rr, bend right, "\varphi\psi"'] & G \arrow[r, "\psi"] & G \end{tikzcd}$
Action of $\mathrm{Aut}(G)$ on $\mathcal{D}(T)$:	Right: $T\varphi$ $g \cdot_{T\varphi} x := (g)\varphi^{-1} \cdot_T x$ $\mathrm{Stab}_{T\varphi}(x)$ $= (\mathrm{Stab}_T(x))\varphi$	Left: φT $g \cdot_{\varphi T} x := \varphi^{-1}(g) \cdot_T x$ $\mathrm{Stab}_{\varphi T}(x)$ $= \varphi(\mathrm{Stab}_T(x))$	Left: φT $x \cdot_{\varphi T} g := x \cdot_T (g)\varphi$ $\mathrm{Stab}_{\varphi T}(x)$ $= (\mathrm{Stab}_T(x))\varphi^{-1}$

TABLE 2: Summary of Action Handedness in Thesis

2. The action of G on X is *transitive* if there exists some $x \in X$ such that for all $y \in X$, there exists $g_y \in G$ such that $y = g_y \cdot x$.
3. If X is a metric space with metric d, then the action of G on X is *isometric* if for any $g \in G$ and any $x, y \in X$, $d(g \cdot x, g \cdot y) = d(x, y)$.
4. If X is a CW-complex, then the action of G on X is *cellular* if for any $g \in G$ and any cell σ of X, $g \cdot \sigma$ is a cell of X, and moreover, if $g \cdot \sigma = \sigma$ as sets, then g pointwise fixes σ.

If the action of G on X is [adjective], then we say that G *acts [adjective-ly]* on X. If G acts isometrically, we may say that G 'acts by isometries'.

Definition 1.2.3. Let G be a group acting on a set X. The *G-stabiliser* of a point x in X, denoted $\mathrm{Stab}_G(x)$ is the set $\{g \in G | g \cdot x = x\}$. This is a subgroup of G. When G is assumed and there is no ambiguity, we may simply refer to the 'stabiliser' of a point, and denote it $\mathrm{Stab}(x)$. We will sometimes write G_x for $\mathrm{Stab}_G(x)$ when space is in short supply.

Definition 1.2.4. Let G be a group acting on a set X, let $g \in G$ and $x \in X$, and let $Y \subseteq X$ be a subset of X.

- We denote by $G \cdot x$ the set $\{h \cdot x | h \in G\}$; this is the *orbit* of x under G.
- We denote by $g \cdot Y$ the set $\{g \cdot y | y \in Y\}$; this is the *image* of Y under g.
- We denote by $G \cdot Y$ the set $\{h \cdot y | h \in G, y \in Y\}$.

If $X \subseteq G \cdot Y$, we call Y a *fundamental domain* for the action of G on X. If in addition we have that for any $y_1, y_2 \in Y$ and $g \in G$, $y_2 = g \cdot y_1 \Rightarrow y_1 = y_2$ we say that the fundamental domain Y is *strict*.

In plainer language, a subset Y of X is a fundamental domain for the action of G on X if every element of X has some orbit representative in Y, and a fundamental domain Y of X is strict if such orbit representatives are always unique.

1.3 Group Automorphisms

Definition 1.3.1. Given two groups $(G, \bullet)$ and $(H, *)$, a *homomorphism* from G to H is a map $\varphi : G \to H$ such that for all $x, y \in G$ we have $\varphi(x \bullet y) = \varphi(x) * \varphi(y)$. A homomorphism $\varphi : G \to H$ is called an:

- *Isomorphism* if φ is a bijection.
- *Endomorphism* if $H = G$.
- *Automorphism* if φ is a both an isomorphism and an endomorphism.

We denote the set of automorphisms of G by $\mathrm{Aut}(G)$. We say an automorphism φ of G is *inner* if there exists some $x \in G$ such that for all $g \in G$, $\varphi(g) = g^x$, where g^x denotes conjugation of G by x. We will sometimes write Ad_x or ι_x for the inner automorphism $g \mapsto g^x$. We denote by $\mathrm{Inn}(G)$ the subset of $\mathrm{Aut}(G)$ comprising all inner automorphisms of G.

There are several key theorems regarding group isomorphisms, commonly referred to as the 'Isomorphism Theorems'. There is no consistent labelling of these in the literature (the Wikipedia page 'Isomorphism Theorems' [23] has a nice breakdown of the variations). We adopt the numbering of Dummit and Foote's 'Abstract Algebra' [12, pp.97-98].

Theorem 1.3.2 (Isomorphism Theorems)**.**

1. *Let G and H be groups and $\varphi : G \to H$ a homomorphism.*
   Then $\ker(\varphi) \trianglelefteq G$, $\mathrm{im}(\varphi) \leqslant H$, *and* $\mathrm{im}(\varphi) \cong {}^{G}\!\diagup_{\ker(\varphi)}$.

2. *Let G be a group, $S \leqslant G$ a subgroup of G, and $N \trianglelefteq G$ a normal subgroup of G. Then $SN \leqslant G$, $N \trianglelefteq SN$, $S \cap N \trianglelefteq S$, and ${}^{SN}\!/_{N} \cong {}^{S}\!/_{S \cap N}$.*

3. *Let G be a group, and $N, K \trianglelefteq G$ normal subgroups of G such that $N \subseteq K \subseteq G$. Then ${}^{\left({}^{G}\!/_{N}\right)}\!/_{\left({}^{K}\!/_{N}\right)} \cong {}^{G}\!/_{K}$.*

Definition 1.3.3. We say a sequence of homomorphisms

$$\ldots \xrightarrow{\varphi_{n-1}} G_{n-1} \xrightarrow{\varphi_n} G_n \xrightarrow{\varphi_{n+1}} G_{n+1} \xrightarrow{\varphi_{n+2}} \ldots$$

is *exact* if for all i we have $\operatorname{im}(\varphi_i) = \ker(\varphi_{i+1})$. An exact sequence of the form

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1$$

is called a *short exact sequence*. In this case, we see that ι is injective and π is surjective, so by the First Isomorphism Theorem, $Q \cong {}^{G}\!/_{N}$. We may thus often consider ι to be inclusion, and π to be the quotient map.

We say a short exact sequence *splits* if there exists some homomorphism $\psi : Q \to G$ such that $\pi \circ \psi$ is the identity map on Q. We will later (Definition 1.4.1 and Lemma 1.4.5) see that in this case we have $G \cong N \rtimes Q$.

Lemma 1.3.4. *Let G be a group. Then:*

1. $\operatorname{Aut}(G)$ *is a group under composition.*

2. $\operatorname{Inn}(G)$ *is a normal subgroup of* $\operatorname{Aut}(G)$ *and is isomorphic to* ${}^{G}\!/_{Z(G)}$.

3. *There is an exact sequence:*

$$1 \longrightarrow Z(G) \longrightarrow G \longrightarrow \operatorname{Aut}(G) \longrightarrow \operatorname{Out}(G) \longrightarrow 1$$

where $\operatorname{Out}(G) := {}^{\operatorname{Aut}(G)}\!/_{\operatorname{Inn}(G)}$ *is the* outer automorphism group *of G.*

Proof. Since composition of functions is always associative, it is easy to see that $\operatorname{Aut}(G)$ is a group under composition; indeed since the inverse of a bijective homomorphism is itself a bijective homomorphism then inverses exist, and the identity map $\operatorname{Id}_G : x \mapsto x$ on G is clearly an automorphism, which acts as the identity element for $\operatorname{Aut}(G)$.

We now consider the map $f : G \to \operatorname{Aut}(G)$ given by $f(g) := \varphi_g : G \to G$ for all $g \in G$, where $\varphi_g(x) = gxg^{-1}$ for all $x \in G$. Let $x, g, h \in G$. Then $(\varphi_g \circ \varphi_h)(x) = \varphi_g(hxh^{-1}) = g(hxh^{-1})g^{-1} = (gh)x(gh)^{-1} = \varphi_{gh}(x)$. So for any $g, h \in G$ we have $f(g) \circ f(h) = f(gh)$. Thus f is a homomorphism, and by the First Isomorphism Theorem, we have ${}^{G}\!/_{\ker(f)} \cong \operatorname{im}(f) \leqslant \operatorname{Aut}(G)$. Note that φ_g is the inner automorphism $\operatorname{Ad}_{g^{-1}}$, hence

$\mathrm{im}(f)$ is precisely the set describing $\mathrm{Inn}(G)$, and φ_g is the identity map exactly when $gxg^{-1} = x$ for all $x \in G$, i.e. when $g \in Z(G)$.

Now suppose $\varphi_g \in \mathrm{Inn}(G)$ and $\psi \in \mathrm{Aut}(G)$. Then for $x \in G$ we have $(\psi^{-1}\varphi_g\psi)(x) = \psi^{-1}\left(\varphi_g\left(\psi(x)\right)\right) = \psi^{-1}\left(g\psi(x)g^{-1}\right) = \psi^{-1}(g)\psi^{-1}(\psi(x))\psi^{-1}(g^{-1}) = \psi^{-1}(g)x\psi^{-1}(g)^{-1} = \varphi_{\psi^{-1}(g)}(x) \in \mathrm{Inn}(G)$. Thus $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$.

Letting q be the quotient map $\psi \mapsto \psi\,\mathrm{Inn}(G)$ and i be the inclusion $Z(G) \hookrightarrow G$, we acquire a sequence $Z(G) \xrightarrow{i} G \xrightarrow{f} \mathrm{Aut}(G) \xrightarrow{q} {\mathrm{Aut}(G)}/{\mathrm{Inn}(G)}$ where $\ker(i) = 1$, $\mathrm{im}(i) = Z(G) = \ker(f)$, $\mathrm{im}(f) = \mathrm{Inn}(G) = \ker(q)$, and
$\mathrm{im}(q) = {\mathrm{Aut}(G)}/{\mathrm{Inn}(G)} =: \mathrm{Out}(G)$. ☐

We will usually write elements of $\mathrm{Out}(G)$ as upper-case Greek letters and their $\mathrm{Aut}(G)$ representatives as their lower-case counterparts, although if we are given $\psi \in \mathrm{Aut}(G)$ we may sometimes write $[\psi]$ for its class in $\mathrm{Out}(G)$.

Definition 1.3.5. We say a subgroup H of a group G is *characteristic* if for every automorphism $\varphi \in \mathrm{Aut}(G)$ we have $\varphi(H) \subseteq H$.

Since this definition requires that $\varphi(H) \subseteq H$ for **every** $\varphi \in \mathrm{Aut}(G)$, then it also holds for every $\varphi^{-1} \in \mathrm{Aut}(G)$. In particular, if H is characteristic in G, then for every $\varphi \in \mathrm{Aut}(G)$ we have $\varphi(H) \subseteq H$ and also $\varphi^{-1}(H) \subseteq H$. Then $H = \varphi\left(\varphi^{-1}(H)\right) \subseteq \varphi(H)$, hence $\varphi(H) = H$. So this condition is equivalent to requiring that $\varphi(H) = H$ for every $\varphi \in \mathrm{Aut}(G)$. However, for the purposes of easy verification, we use the definition with the weakest requirements.

1.4 Semidirect Products

We now give some equivalent definitions of a 'semidirect product'. The content below can be found in introductory textbooks such as D. Robinson's 'An Introduction to Abstract Algebra' [20, pp.75-76].

Definition 1.4.1. Let G be a group, $H \leqslant G$ a subgroup of G, and $N \trianglelefteq G$ a normal subgroup of G. If for all $g \in G$ there exist unique $n \in N$ and $h \in H$ such that $g = nh$, then we say that G is the *(internal) semidirect product* of N and H, denoted $G = N \rtimes H$.

Lemma 1.4.2. *Let G be a group, $H \leqslant G$ a subgroup of G, and $N \trianglelefteq G$ a normal subgroup of G. Then $G = N \rtimes H$ if and only if G is equal to the product $G = NH$, and $N \cap H = \{1\}$.*

Definition 1.4.3. Let H and N be two groups and let $\varphi : H \to \mathrm{Aut}(N)$, $h \mapsto \varphi_h$ be a group homomorphism. The *(external) semidirect product* of N and H with respect to φ, denoted $N \rtimes_\varphi H$, is the group whose underlying set is $N \times H$ with group operation $(n_1, h_1) \cdot (n_2, h_2) := (n_1\varphi_{h_1}(n_2), h_1h_2)$.

Lemma 1.4.4. *The concepts of external and internal semidirect products are equivalent in the following way:*

(i) If $G = N \rtimes_{\varphi} H$ is an external semidirect product of two groups N and H, then G is the internal semidirect product of the subgroups $(N, 1) = \{(n, 1) | n \in N\} \cong N$ and $(1, H) = \{(1, h) | h \in H\} \cong H$.

(ii) If $G = N \rtimes H$ is an internal semidirect product of two subgroups N and H, then G is isomorphic to the external semidirect product $G \cong N \rtimes_{\varphi} H$ where $\varphi : H \to \mathrm{Aut}(N)$ is given by $\varphi(h) = \varphi_h : n \mapsto hnh^{-1}$.

As promised earlier (Definition 1.3.3), we have the following connection with short exact sequences:

Lemma 1.4.5. *Let G, H, and N be groups. Then $G \cong N \rtimes H$ if and only if there exists a split short exact sequence:*

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, "\iota"] & G \arrow[r, "\pi"] & H \arrow[r] \arrow[l, bend left, "\psi"] & 1 \end{tikzcd}$$

That is, there exists a short exact sequence $1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 1$ *and a homomorphism $\psi : H \to G$ such that $\pi \circ \psi = \mathrm{Id}_H$ is the identity map on H. In this case $G \cong N \rtimes_{\varphi} H$ where $\varphi : H \to \mathrm{Aut}(N)$ is given by $\varphi(h) = \varphi_h : n \mapsto \iota^{-1}\left(\psi(h)\iota(n)\psi(h^{-1})\right)$.*

The following special case of a semidirect product, where $N = \mathbb{Z}$, provides a useful construction for studying group automorphisms:

Example 1.4.6 (Mapping Torus)**.** Let G be a group and $\varphi \in \mathrm{Aut}(G)$. The *mapping torus* of φ is the semidirect product $M_{\varphi} := G \rtimes_{\Phi} \mathbb{Z}$, where $\mathbb{Z} = \langle t \rangle$ and $\Phi : \mathbb{Z} \to \mathrm{Aut}(G)$ is given by $t^k \mapsto \Phi_{t^k} := \varphi^k$ for all $t^k \in \langle t \rangle$. It is common to write $G \rtimes_{\varphi} \mathbb{Z}$ to denote this construction.

According to Definition 1.4.3, M_{φ} is the group with underlying set $G \times \mathbb{Z}$ and group operation $(g_1, t^{k_1}) \cdot (g_2, t^{k_2}) := (g_1 \Phi_{t^{k_1}}(g_2), t^{k_1} t^{k_2}) = (g_1 \varphi^{k_1}(g_2), t^{k_1 + k_2})$.

Writing gt^k for the element $(g, t^k) \in G \times \mathbb{Z}$, it is easy to see from this that M_{φ} has the presentation $\langle G, t \mid tgt^{-1} = \varphi(g) \ \forall g \in G \rangle$.

By Lemma 1.4.4 and Defintion 1.4.1, elements of M_{φ} have a standard form gt^k where $g \in G$ and $k \in \mathbb{Z}$ are unique.

1.5 Free Products

Free products (or rather, their automorphisms) are the main focus of Papers 1 and 2, and are the first case considered in Paper 3. They may be thought of as a generalisation of free groups, and as with free groups, there are several ways of defining them. The following material is taken from A. Hatcher's 'Algebraic Topology' [17, Section 1.2], D. Cohen's 'Combinatorial Group Theory: A Topological Approach' [6, Section 1.3], and R. Lynodn and P. Schupp's 'Combinatorial Group Theory' [19, Chapter IV].

Hatcher [17] defines free products via reduced words, and concatenation and reduction, analagous to the free groups in Example 1.1.2:

Definition 1.5.1 (Hatcher [17])**.** Let $\{G_i\}_{i\in I}$ be a collection of groups. The *free product* of the G_i's, denoted $\mathop{\ast}\limits_{i\in I} G_i$, is the set of all words of the form $g_1 \dots g_n$ such that $n \in \mathbb{N}$ is finite and for each $j \in \{1, \dots, n\}$ there exists $i_j \in I$ such that $g_j \in G_{i_j}$ and $i_j \neq i_{j+1}$. The group operation here is concatenation and reduction, and the identity element is the empty word, denoted '1'.

More explicitly, suppose $g_1 \dots g_n$ and $h_1 \dots h_m$ are two words in $\ast_{i\in I} G_i$ (with respective lengths n and m), where each $g_j \in G_{i_j}$ and each $h_j \in G_{k_j}$ (for some $i_j, k_j \in I$). If $i_n \neq k_1$ then $g_1 \dots g_n \cdot h_1 \dots h_m$ is simply the concatenation $g_1 \dots g_n h_1 \dots h_m$ (a word of length $n + m$). On the other hand, if $i_n = k_1$ then $x := g_n h_1$ is an element of $G_{i_n} = G_{k_1}$. If $x \neq 1$ then $g_1 \dots g_n \cdot h_1 \dots h_m$ is the word $g_1 \dots g_{n-1} x h_2 \dots h_m$ (with length $n + m - 1$). If $x = 1$, we repeat this process looking at the indices i_{n-1} and k_2, and continue until we have a word in $\ast_{i\in I} G_i$.

Cohen [6] takes the view of a universal property, generalising Definition 1.1.1 for free groups. As before, Cohen does not initially require the G_i's to be subgroups of the group G, but later proves them to be so (or at least, shows his homomorphisms $G_i \to G$ are monomorphisms).

Definition 1.5.2 (Cohen [6])**.** Let $\{G_i\}_{i\in I}$ be a collection of subgroups of a group G. We say G is a *free product* of the groups G_i if for any group H and homomorphisms $\varphi_i : G_i \to H$, there exists a unique homomorphism $\varphi : G \to H$ such that the restriction $\varphi|_{G_i}$ of φ to G_i is equal to φ_i. That is, the following diagram commutes for all $i \in I$:

$$\begin{tikzcd}
G_i \arrow[r, hook, "\iota_i"] \arrow[dr, "\varphi_i"'] & G \arrow[d, dashed, "\exists!\varphi"] \\
& H
\end{tikzcd}$$

where $\iota_i : G_i \to G$ is the inclusion map.

Proposition 1.5.3 ([6, Proposition 23])**.** *Let $\{G_i\}_{i\in I}$ be a collection of subgroups of a group G. The following are equivalent:*

(i) G is the free product of the subgroups G_i.

(ii) Every non-identity element of G can be uniquely expressed as a product $g_1 \dots g_n$ with $n \geqslant 0$, $g_j \in G_{i_j}$, $g_j \neq 1$, and $i_j \neq i_{j+1}$.

(iii) *G is generated by the subgroups G_i, and $1 \in G$ cannot be written as a product $g_1 \dots g_n$ with $n > 0$, $g_j \in G_{i_j}$, $g_j \neq 1$, and $i_j \neq i_{j+1}$.*

Remark. Given a free product $G = G_1 * \dots * G_n$ and group elements $g_1 * \dots * g_n \in G$, we can form the free product $H = G_1^{g_1} * \dots * G_n^{g_n}$. To have $H \cong G$ here, it is enough that the subgroups $G_1^{g_1}, \dots, G_n^{g_n}$ of G generate the group G itself. [2]

A corollary of this Proposition is that Cohen's definition of a free product agrees with that of Hatcher.

Our final definition is that of Lyndon and Schupp [19], who define a free product of two groups via their group presentations. They do not define a free product of more than two groups, but the Lemma which follows shows how this may be extended inductively.

Definition 1.5.4 (Lyndon and Schupp [19])**.** Let A and B be groups with presentations $A = \langle a_1, \dots | r_1, \dots \rangle$ and $B = \langle b_1, \dots | s_1, \dots \rangle$ respectively. The *free product* of A and B, denoted $A * B$, is the group $A * B := \langle a_1, \dots, b_1, \dots | r_1, \dots, s_1, \dots \rangle$.

Lemma 1.5.5 ([6, p.25])**.** *Let G_1, G_2, and G_3 be groups. We then have:*

- $G_1 * G_2 = G_2 * G_1$
- $G_1 * (G_2 * G_3) = (G_1 * G_2) * G_3$

The following is a well-known theorem detailing how a group may be expressed more or less uniquely as a certain free product:

Theorem 1.5.6 (Grushko Decomposition Theorem)**.** *Let G be a non-trivial finitely generated group. Then there exist non-trivial, freely indecomposable, not infinite cyclic groups $G_1, \dots, G_n$ and a free group F_k of rank k such that $G = G_1 * \dots * G_n * F_k$. Moreover, if $G = H_1 * \dots * H_m * F_l$ for some non-trivial, freely indecomposable, not infinite cyclic groups $H_1, \dots, H_m$ and a free group F_l of rank l, then $l = k$, $n = m$, and there exists a permutation $\sigma \in S_n$ such that for each i, there exists $g_i \in G$ such that $H_{\sigma(i)} = G_i^{g_i}$. We call such a product a* Grushko decomposition *for G.*

2 Bass–Serre Theory

Bass–Serre Theory provides a framework for understanding the actions of groups on trees, via a correspondence with graphs of groups. It underpins all of the work in Papers 1, 2, and 3. We give an overview of the theory in this section, in a manner which is hopefully self-contained and notationally minimal. The material presented here is an

---

[2] This can be proved using normal forms of words, but is very long and tedious to do, so we omit the proof here in the interest of not making this thesis any longer than it already is.

amalgam of that found in Serre's book 'Trees' [21, Chapter I, Section 5] and Bass' article 'Covering theory for graphs of groups' [3].

Definition 2.0.1. A *graph* Γ is a set $V(\Gamma)$ (the vertex set of Γ), a set $E(\Gamma)$ (the edge set of Γ), a map $E(\Gamma) \to V(\Gamma) \times V(\Gamma)$ which assigns $e \mapsto (o(e), t(e))$, and a map $E(\Gamma) \to E(\Gamma)$ which sends $e \mapsto \bar{e}$, such that for all $e \in E(\Gamma)$ we have $\bar{\bar{e}} = e$ and $o(e) = t(\bar{e})$.

- For an edge $e \in E(\Gamma)$ we call $o(e)$ the *origin* of e and $t(e)$ the *terminus* of e.
- An *orientation* of Γ is a subset E^+ of $E(\Gamma)$ such that $E(\Gamma)$ is the disjoint union $E(\Gamma) = E^+ \sqcup \overline{E^+}$.
- For a vertex $v \in V(\Gamma)$, we define the *valency* of v to be the cardinality of the set $\{ e \in E(\Gamma) \mid o(e) = v\}$.

We will sometimes refer to this construction as a "graph in the sense of Serre", particularly when we are also considering similar concepts such as simplicial complexes or metric spaces with the structure of a graph. For this section however, since there is no ambiguity, we will simply call it a 'graph'.

Definition 2.0.2. Let Γ be a graph. A *path* (of length n) (from v_0 to v_n) is either a single vertex v_0 (in which case $n = 0$) or a sequence of edges $e_1 \dots e_n$ such that $v_i := t(e_i) = o(e_{i+1})$ for all $1 \leqslant i \leqslant n-1$ (and $v_0 := o(e_1)$, $v_n := t(e_n)$). Such a path is:

- *closed* if $v_n = v_0$.
- *reduced* if $e_{i+1} \neq \overline{e_i}$ for all $1 \leqslant i \leqslant n-1$.

Definition 2.0.3. Let Γ be a graph with vertex set $V(\Gamma)$.

- Γ is *locally finite* if for every vertex $v \in V(\Gamma)$ we have that the valency $\nu(v)$ of v is finite.
- Γ is *connected* if given any $v, w \in V(\Gamma)$ there exists a path in Γ from v to w.
- Γ is a *forest* if every closed, reduced path in Γ has length 0.
- Γ is a *tree* if it is a connected forest.

Remark. Paper 2 takes the approach that graphs are simplicial complexes and oriented graphs are ordered simplicial complexes (see Section 4.1). This notion is implicit in Paper 1 as well. This can be made equivalent to the approach of Bass and Serre (see Example 4.1.4), who seem to really be considering digraphs, rather than graphs in the traditional sense (they don't allow 'backwards traversal' of an edge, requiring an inverse edge to make such a journey).

2.1 Graphs of Groups

Definition 2.1.1. A *graph of groups* $\mathcal{G} = (G, \Gamma)$ consists of a connected non-empty graph Γ, a group G_v for each $v \in V(\Gamma)$, and a group G_e and monomorphism $\iota_e : G_e \to G_{o(e)}$ for each $e \in E(\Gamma)$, such that $G_e = G_{\bar{e}}$.

In Papers 1 and 2, all graphs of groups will be trees of groups (i.e. the underlying graph Γ is a tree), with trivial edge groups (i.e. $G_e = \{1\}$ for all $e \in E(\Gamma)$).

Definition 2.1.2. The *path group* $P(\mathcal{G})$ of a graph of groups $\mathcal{G} = (G, \Gamma)$ is the group generated by the groups G_v for $v \in V(\Gamma)$ and symbols $\hat{g}_e$ for $e \in E(\Gamma)$, subject to relations of the form $\hat{g}_{\bar{e}} = \hat{g}_e^{-1}$ and $\hat{g}_e \iota_{\bar{e}}(h) \hat{g}_e^{-1} = \iota_e(h)$ for $e \in E(\Gamma)$ and $h \in G_e$.

A *path* in $P(\mathcal{G})$ (from v_0 to v_n) is either an element $x_0 \in G_{v_0}$ for some $v_0 \in V(\Gamma)$, or an element of the form $x_0 \hat{g}_{e_1} x_1 \hat{g}_{e_2} \dots \hat{g}_{e_n} x_n$ where $e_1 e_2 \dots e_n$ is a path in $\mathcal{G} = (G, \Gamma)$ with $v_0 = o(e_1)$, $v_i = t(e_i) = o(e_{i+1})$ for all $1 \leqslant i \leqslant n-1$, and $v_n = t(e_n)$, such that for each $0 \leqslant i \leqslant n$ we have $x_i \in G_{v_i}$.

Definition 2.1.3. Let $\mathcal{G} = (G, \Gamma)$ be a graph of groups, let $v_0 \in V(\Gamma)$, and let T be a maximal tree of Γ (that is, $T \subseteq \Gamma$ is a tree with $V(T) = V(\Gamma)$). We define:

- $\pi_1(\mathcal{G}, v_0) := \{$ paths in $P(\mathcal{G})$ from v_0 to v_0 $\}$
- $\pi_1(\mathcal{G}, T) := {}^{P(\mathcal{G})}\big/_{\langle\langle\, \hat{g}_e \mid e \in E(T) \,\rangle\rangle_{P(\mathcal{G})}}$

Lemma 2.1.4. *$\pi_1(\mathcal{G}, v_0)$ is a subgroup of $P(\mathcal{G})$ isomorphic to $\pi_1(\mathcal{G}, T)$. We refer to either as the* fundamental group *of $\mathcal{G}$.*

In the case $\mathcal{G} = (G, \Gamma)$ where Γ is a tree and $G_e = \{1\}$ for all $e \in E(\Gamma)$, we have that

$$P(\mathcal{G}) = \big\langle\, \{G_v \mid v \in V(\Gamma)\} \cup \{\hat{g}_e \mid e \in E(\Gamma)\} \mid \hat{g}_{\bar{e}} = \hat{g}_e^{-1} \;\forall e \in E(\Gamma) \,\big\rangle$$

and the fundamental group of $\mathcal{G}$ is $\pi_1(\mathcal{G}, \Gamma) \cong \mathop{\ast}\limits_{v \in V(\Gamma)} G_v$, the free product of the vertex groups.

2.2 Universal Cover of a Graph of Groups

Let $\mathcal{G} = (G, \Gamma)$ be a graph of groups, T a maximal tree of Γ, and $E^+ \subset E(\Gamma)$ an orientation of Γ. We will build a graph $\widetilde{(\mathcal{G}, T)}$ with an associated action of $\pi_1(\mathcal{G}, T)$. We begin by defining sets $\mathcal{V}$ and $\mathcal{E}^+$ as follows:

$$\mathcal{V} := \bigsqcup_{v \in V(\Gamma)} \{g G_v | g \in \pi_1(\mathcal{G}, T)\}$$

$$\mathcal{E}^+ := \bigsqcup_{e \in E^+} \{g\iota_e(G_e) | g \in \pi_1(\mathcal{G}, T)\}$$

That is, $\mathcal{V}$ is the disjoint union of the $\pi_1(\mathcal{G}, T)$-cosets of the vertex groups G_v of $\mathcal{G}$, and $\mathcal{E}^+$ is the disjoint union of the $\pi_1(\mathcal{G}, T)$-cosets of the images of the edge groups G_e for edges $e \in E^+$ in $\mathcal{G}$. In particular, G_v and $\iota_e(G_e)$ are all subgroups of $\pi_1(\mathcal{G}, T)$, so these cosets are well-defined.

Recall that $\hat{g}_e$ is the generator of the path group $P(\mathcal{G})$ associated to the edge e. If Γ is a tree, then $\hat{g}_e = 1$ for all $e \in E(\Gamma)$.

Definition 2.2.1. We construct a graph $X = \widetilde{(\mathcal{G}, T)}$, with vertex set $V(X) := \mathcal{V}$ and edge set $E(X) := \mathcal{E}^+ \sqcup \overline{\mathcal{E}^+}$, where $o(g\iota_e(G_e)) := gG_{o(e)} = t(\overline{g\iota_e(G_e)})$ and $t(g\iota_e(G_e)) := g\hat{g}_e G_{t(e)} = o(\overline{g\iota_e(G_e)})$. We call X the *universal cover* of $\mathcal{G}$ (with respect to T).

Observation 2.2.2. The group $\pi_1(\mathcal{G}, T)$ acts naturally on $V(X)$ by $h \cdot gG_v := hgG_v$ for $gG_v \in V(X)$ and $h \in \pi_1(\mathcal{G}, T)$, and on $E(X)$ by $h \cdot g\iota_e(G_e) := hg\iota_e(G_e)$ and $h \cdot \overline{g\iota_e(G_e)} := \overline{hg\iota_e(G_e)}$ for $g\iota_e(G_e) \in \mathcal{E}^+$ and $h \in \pi_1(\mathcal{G}, T)$. This constitutes a well-defined action on X.

Under this action, we have that $\mathrm{Stab}_{\pi_1(\mathcal{G},T)}(gG_v) = gG_v g^{-1} \leqslant \pi_1(\mathcal{G}, T)$ for $gG_v \in V(X)$ and $\mathrm{Stab}_{\pi_1(\mathcal{G},T)}(g\iota_e(G_e)) = \mathrm{Stab}_{\pi_1(\mathcal{G},T)}(\overline{g\iota_e(G_e)}) = g\iota_e(G_e)g^{-1} \leqslant \pi_1(\mathcal{G}, T)$ for $g\iota_e(G_e) \in \mathcal{E}^+$.

Remark. Note that for any $e \in E^+$, $\iota_e(G_e) \subseteq G_{o(e)}$ and $\iota_{\bar{e}}(G_{\bar{e}}) = \iota_{\bar{e}}(G_e) = \hat{g}_e^{-1}\iota_e(G_e)\hat{g}_e \subseteq G_{t(e)}$, as subgroups of $\pi_1(\mathcal{G}, T)$. We then have that $g\iota_e(G_e)g^{-1} \subseteq gG_{o(e)}g^{-1}$ and $g\iota_e(G_e)g^{-1}$ $= g\left(\hat{g}_e\iota_{\bar{e}}(G_e)\hat{g}_e^{-1}\right)g^{-1} \subseteq g\hat{g}_e G_{t(e)}(g\hat{g}_e^{-1})$. That is, $\mathrm{Stab}_{\pi_1(\mathcal{G},T)}(g\iota_e(G_e))$ is a subset of both $\mathrm{Stab}_{\pi_1(\mathcal{G},T)}(o(g\iota_e(G_e)))$ and $\mathrm{Stab}_{\pi_1(\mathcal{G},T)}(t(g\iota_e(G_e)))$, as one would expect.

2.3 Quotient Graphs of Groups

Let X be a graph, and G be a group acting by isometries on X without inversion of the edges. We will build a graph of groups which encapsulates the action of X on G.

Definition 2.3.1. Given a graph X and an action (without inversion) of a group G on X, the *quotient graph* $G\backslash X$ is the graph with one vertex for every G-orbit of vertices in X, and one edge for every G-orbit of edges in X. If e is an edge in X, then the corresponding edge $[e]$ in $G\backslash X$ satisfies $\overline{[e]} = [\bar{e}]$, $o([e]) = t(\overline{[e]}) = [o(e)]$, and $t([e]) = o(\overline{[e]}) = [t(e)]$ (where $[v]$ is the vertex in $G\backslash X$ corresponding to the orbit of a vertex v in X).

Given such a quotient graph $\Gamma := G\backslash X$ and an orientation $E^+ \subset E(\Gamma)$ of Γ, we choose a maximal tree T of Γ, and a lift $\hat{T}$ of T in X. Note that $\hat{T}$ is a subtree of X (Serre [21, Chapter I, Proposition 14]).

For each $e \in E^+$-$E(T)$, we choose a lift $\hat{e}$ of e in X such that $o(\hat{e}) \in V(T)$, and we set $\hat{\bar{e}} := \bar{\hat{e}}$. We also choose an element $g_e \in G$ satisfying $t(\hat{e}) = g_e\widehat{t(e)}$, and set $g_{\bar{e}} := g_e^{-1}$. For $e \in E(T)$, we set $g_e := 1$.

Definition 2.3.2. Let G be a group acting on a graph X without inversion of the edges, and let $\Gamma = X\backslash G$ be the quotient graph, with maximal tree T, lifts $\hat{T}$ and $\hat{e}$ for each $e \in E(\Gamma)$-$E(T)$, and elements $g_e \in G$ for each $e \in E(\Gamma)$, as described above. The *quotient graph of groups* for the action of G on X is the graph of groups with underlying graph Γ, vertex groups $G_v := \mathrm{Stab}_G(\hat{v})$ for $v \in V(\Gamma)$, and for each $e \in E^+$, edge groups $G_e = G_{\bar{e}} = \mathrm{Stab}_G(\hat{e})$ and monomorphisms $\iota_e : G_e \to G_{o(e)}$ and $\iota_{\bar{e}} : G_e \to G_{t(e)}$ given by $\iota_e(h) = h$ and $\iota_{\bar{e}}(h) = g_e^{-1} h g_e$ for all $h \in G_e$.

Remark. Note that for $e \in E^+$ we have $G_{o(e)} = \mathrm{Stab}_G\left(\widehat{o(e)}\right) = \mathrm{Stab}_G\left(o(\hat{e})\right) \supseteq \mathrm{Stab}_G\left(\hat{e}\right) = G_e$ and $G_{t(e)} = \mathrm{Stab}_G\left(\widehat{t(e)}\right) = g_e^{-1}\,\mathrm{Stab}_G\left(t(\hat{e})\right) g_e \supseteq g_e^{-1}\,\mathrm{Stab}_G\left(\hat{e}\right) g_e = g_e^{-1} G_e g_e$. Thus ι_e and $\iota_{\bar{e}}$ as described above are indeed monomorphisms $G_e \to G_{o(e)}$ and $G_e \to G_{t(e)}$, respectively.

Note that we have $G_e = \{1\}\ \forall e \in E(\Gamma)$ if and only G acts freely on the set of edges of X. In this case, the monomorphisms ι_e and $\iota_{\bar{e}}$ are both trivial.

If the quotient graph Γ is a tree, then the maximal tree T is the whole of Γ, and we have $g_e = 1$ for every edge e of Γ.

2.4 Structure Theorem

The 'Structure Theorem' [21, Chapter I, Theorem 13], also known as the 'Fundamental Theorem of Bass–Serre Theory' shows that taking universal covers of graphs of groups and taking quotient graphs of groups for groups acting on trees are inverse operations. We summarise this result with the following two Theorems:

Theorem 2.4.1. *Let X be a (non-empty) tree, and G be a group acting on X without inversion of the edges. Choose a maximal tree T of the quotient graph $G\backslash X$, and let $\mathcal{G}$ be the associated quotient graph of groups. Then $\pi_1(\mathcal{G}, T) \cong G$ are isomorphic as groups, and the universal cover $\widetilde{(\mathcal{G}, T)}$ of $\mathcal{G}$ with respect to T is isomorphic to X (as graphs).*

Observation 2.4.2. Explicit isomorphisms here are given by $\varphi : \pi_1(\mathcal{G}, T) \to G$, $G_v \hookrightarrow G$, $\hat{g}_e \mapsto g_e$ and $\psi : \widetilde{(\mathcal{G}, T)} \to X$, $gG_v \mapsto \varphi(g)\hat{v}$, $g\iota_e(G_e) \mapsto \varphi(g)\hat{e}$, using the notation of the previous subsections.

Theorem 2.4.3. *Let $\mathcal{G} = (G, \Gamma)$ be a graph of groups, let T be a maximal tree of Γ, and let $\pi_1(\mathcal{G}, T)$ be the fundamental group of $\mathcal{G}$ with respect to T. Then the universal cover $X = \widetilde{(\mathcal{G}, T)}$ is a tree, there is a graph isomorphism $\rho : \Gamma \to \pi_1(\mathcal{G}, T)\backslash X$ (the quotient graph), and if $\mathcal{H}$ is the quotient graph of groups for the action of $\pi_1(\mathcal{G}, T)$ on X, then the fundamental groups $\pi_1(\mathcal{H}, \rho(T)) \cong \pi_1(\mathcal{G}, T)$ are isomorphic.*

Observation 2.4.4. The quotient graph $\pi_1(\mathcal{G}, T)\backslash X$ has one vertex for every orbit of vertices in $\widetilde{(\mathcal{G}, T)}$, which has one orbit of vertices for every vertex in Γ. Thus $\pi_1(\mathcal{G}, T)\backslash X$ has one vertex for every vertex of Γ. Similarly, it has one edge for every edge of Γ.

By making the 'obvious' choice of lifts (i.e. if v is a vertex of Γ with associated orbit $\{gG_v | g \in \pi_1(\mathcal{G}, T)\}$ in X, take the vertex $1G_v$, and similarly with edges), we recover that (after renaming) $\mathcal{H}$ is the graph of groups with underlying graph Γ, vertex groups G_v, and edge groups $\iota_e(G_e)$. Since ι_e is a monomorphism then $G_e \cong \iota_e(G_e)$ for each e, and the isomorphism of fundamental groups becomes clear.

3 Free Product Automorphisms

Let G be a group which splits as a free product $G_1 * \cdots * G_n$, where each G_i is non-trivial, not infinite cyclic, freely indecomposable, and $G_i \ncong G_j$ for $i \neq j$. We refer to each G_i as a *factor group*. According to the Grushko Decomposition Theorem (Theorem 1.5.6), any automorphism of G must send each factor group to a conjugate of itself. If factor groups are permitted to be pairwise isomorphic, automorphisms must send factor groups to conjugates of factor groups.

When $G_1 * \cdots * G_n$ is not a Grushko decomposition for G (i.e. factors may be infinite cyclic or freely decomposable), or when factor groups may be pairwise isomorphic, this need not hold. However, we can still study the automorphisms which do satisfy this, motivating the following definition:

Definition 3.0.1. We say $\psi \in \mathrm{Aut}(G)$ is a *pure symmetric automorphism* of the splitting $G_1 * \cdots * G_n$ if for each i there is some $g_i \in G$ such that $\psi(G_i) = G_i^{g_i} = g_i^{-1} G_i g_i$.

If one wished to allow permutation of the factor groups, one would be studying the 'symmetric' automorphisms instead.

When $G_1 * \cdots * G_n$ is a Grushko decomposition for a group G, we may categorise automorphisms according to their behaviour respecting the factor groups.

Definition 3.0.2. Let $\psi \in \mathrm{Aut}(G_1 * \cdots * G_n)$ where $G_1 * \cdots * G_n$ is a Grushko decomposition.

- We call ψ a *factor automorphism* if for each $i \in \{1, \ldots, n\}$ we have $\psi(G_i) = G_i$.
- We call ψ a *permutation automorphism* if there exists (non-trivial) $\sigma \in S_n$ such that for each $i \in \{1, \ldots, n\}$ we have $\psi(G_i) = G_{\sigma(i)}$.
- We call ψ a *Whitehead automorphism* if there exists $i \in \{1, \ldots, n\}$ and $x \in G_i$ such that for some (non-empty) subset $A \subseteq \{G_1, \ldots, G_n\} - \{G_i\}$ we have that for any $j \in \{1, \ldots, n\}$ and $g \in G_j$, $\psi(g) = \begin{cases} x^{-1}gx & \text{if } G_j \in A \\ g & \text{if } G_j \notin A \end{cases}$. We call G_i the *operating factor* and elements of the set A *dependent factors*.

- We call ψ a *multiple Whitehead automorphism* if ψ can be written as a product of Whitehead automorphisms with the same operating factor and pairwise disjoint sets of dependent factors.

We say an outer automorphism $\Psi \in \mathrm{Out}(G_1 * \cdots * G_n)$ is an *[adjective] outer automorphism* if it has a representative $\psi \in \Psi$ which is an [adjective] automorphism.

In Papers 1 and 2 we generalise these notions to categorise types of pure symmetric (outer) automorphism (although since we study pure symmetric automorphisms rather than just symmetric automorphisms, we have no permutation automorphisms).

4 Cell Complexes

We will not formally define a CW complex here, as the definition is cumbersome, and all of the complexes used in Papers 1 and 2 will be simplicial. We refer unfamiliar readers to [17, Appendix] for a full overview of the subject.

4.1 Simplicial Complexes

The material here can be found in textbooks such as A. Hatcher's 'Algebraic Topology' [17, Section 2.1] and J. Stillwell's 'Classical Topology and Combinatorial Group Theory' [22].

Definition 4.1.1. A *k-simplex* $\sigma = [v_0, \dots, v_k]$ is the convex hull of its $k+1$ *vertices* $v_0, \dots, v_k$. The *standard k-simplex* is the set $\{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} | \ x_0 + \cdots + x_k = 1, \ x_i > 0 \ \forall i = 0, \dots, k\}$. We may think of k-simplices as 'copies' of the standard k-simplex. Given two simplices $\sigma = [v_0, \dots, v_k]$ and $\mu = [u_0, \dots, u_m]$, we say that μ is a *face* of σ if $\{u_0, \dots, u_m\} \subseteq \{v_0, \dots, v_k\}$. Note that taking the convex hull of any non-empty subset of $\{v_0, \dots, v_k\}$ determines a face of σ.

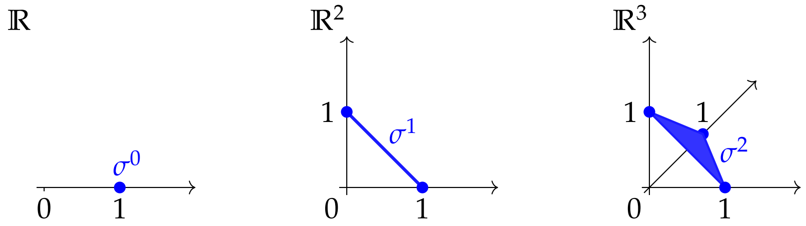


FIGURE 1: Standard k-simplices σ^k for $k \in \{0, 1, 2\}$

We will conflate the ideas of a geometric simplicial complex and an abstract simplicial complex, since these concepts are in one-to-one correspondence. Thus when we say 'simplicial complex', we mean the following:

Definition 4.1.2. A space K is a *simplicial complex* if K is a union of simplices satisfying:

(i) If σ is a simplex in K and μ is a face of σ, then μ is also a simplex in K

(ii) If σ_1 and σ_2 are two simplices in K with $\sigma_1 \cap \sigma_2 \neq \emptyset$, then $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2.

We will call 0-simplices 'vertices' and 1-simplices 'edges' of K.

Note that Condition (ii) implies that each simplex of K is uniquely determined by its vertices (i.e. its 0-simplex faces). We may thus denote a simplex in K by $[v_0, \ldots, v_k]$ where $[v_0], \ldots, [v_k]$ are its 0-simplex faces. Note that under this notation, if $\tau \in S_{k+1}$ is a permutation of the set $\{0, \ldots, k\}$, then $[v_{\tau(0)}, \ldots, v_{\tau(k)}] = [v_0, \ldots, v_k]$.

Definition 4.1.3. A *simplicial graph* X is a simplicial complex comprising only 0-cells and 1-cells. We refer to the 0-cells of X as 'vertices', and the 1-cells as 'edges'.

We call X a *simplicial tree* if X is a path connected simplicial graph such that for any cell σ, the space X-σ is not path connected (see Definition 4.2.1).

Example 4.1.4. Let Γ be a graph in the sense of Serre, with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. We can form a cellular graph $|\Gamma|$ (a 1-dimensional CW complex) whose 0-cells are the vertices of Γ, and with one 1-cell for every pair $\{e, \bar{e}\}$ of edges of Γ. Two cells are attached in $|\Gamma|$ if and only if they form a pair $(v, \{e, \bar{e}\})$ such that $v \in V(\Gamma)$, $e, \bar{e} \in E(\Gamma)$, and $v \in \{o(e), o(\bar{e})\}$. If Γ is a tree (in the sense of Serre), then $|\Gamma|$ is a simplicial tree.

Conversely, given a simplicial graph $|\Delta|$ with V its set of 0-cells and E its set of 1-cells, one can form a graph in the sense of Serre, Δ, with vertex set V and edge set $E \sqcup \bar{E}$ by choosing an orientation of the 1-cells of $|\Delta|$, i.e. given a 1-cell $\sigma \in E$ with endpoints u and v, assigning $\{o(\sigma), t(\sigma)\} := \{u, v\}$, and setting $o(\bar{\sigma}) := t(\sigma)$ and $t(\bar{\sigma}) := o(\sigma)$. If $|\Delta|$ is a simplicial tree, then Δ is a tree (in the sense of Serre).

Definition 4.1.5. We say a simplicial complex K is *flag* if for any collection $\{[v_0], \ldots, [v_k]\}$ of vertices in K, if $[v_0, \ldots, v_k]$ is not a simplex in K then there are some distinct $v_i, v_j \in \{v_0, \ldots, v_k\}$ such that $[v_i, v_j]$ is not an edge in K.

The following construction was introduced by Alexandroff in 'Über den allgemeinen Dimensionsbegriff und seine Beziehungen zur elementaren geometrischen Anschauung' [1] in order to better understand the properties of covers of topological spaces. We will use a similar construction in Paper 1 to show that a particular complex is simply connected.

Example 4.1.6 (Nerve Complex). Let X be a set, I be some indexing set, and for each $i \in I$, let $U_i \subseteq X$ be a subset of X. Denote $F := \{U_i | i \in I\}$. The nerve complex $N(F)$ associated to F is an abstract simplicial complex whose geometric realisation is determined

by the following: $[U_{i_0}, \ldots, U_{i_k}]$ is a k-simplex in $N(F)$ if and only if $i_0, \ldots, i_k \in I$ with $\bigcap_{j=0}^{k} U_{i_j} \neq \varnothing$.
We may then denote simplices of $N(F)$ by $[U_j | j \in J]$, where J is some finite (non-empty) subset of I satisfying $\bigcap_{j \in J} U_j \neq \varnothing$, and call $[U_j | j \in J]$ the simplex determined by J.

In particular, the vertices of $N(F)$ are in one-to-one correspondence with the sets U_i in F.

Note that if $J_1 \subset I$ is a finite set satisfying $\bigcap_{j \in J_1} U_j \neq \varnothing$ and $J_2 \subseteq J_1$, then we must have $\bigcap_{j \in J_1} U_j \subseteq \bigcap_{j \in J_2} U_j$, hence $\bigcap_{j \in J_2} U_j \neq \varnothing$. Thus if $J_1 \subset I$ is a finite non-empty set determining a simplex $[U_j | j \in J_1]$ of $N(F)$ and J_2 is any non-empty subset of J_1, then J_2 determines a face $[U_j | j \in J_2]$ of $[U_j | j \in J_1]$ and is also a simplex in $N(F)$.

Moreover, if $[U_j | j \in J_1]$ and $[U_j | j \in J_2]$ are two simplices in $N(F)$, then $J_1 \cap J_2$ is a subset of both J_1 and J_2, and by the above, $\bigcap_{j \in J_1 \cap J_2} U_j \neq \varnothing$.
Thus $[U_j | j \in J_1] \cap [U_j | j \in J_2] = [U_j | j \in J_1 \cap J_2]$ is a face of both $[U_j | j \in J_1]$ and $[U_j | j \in J_1]$.

Hence $N(F)$ is indeed a simplicial complex.

However, the complex $N(F)$ need not be flag. Indeed, let $U_1 := \{1, 2\}$, $U_2 := \{2, 3\}$ and $U_3 := \{1, 3\}$, and set $F = \{U_1, U_2, U_3\}$. Then the simplices of $N(F)$ are precisely $[U_1]$, $[U_2]$, $[U_3]$, $[U_1, U_2]$, $[U_2, U_3]$, and $[U_1, U_3]$. In particular, the vertices $[U_1]$, $[U_2]$, $[U_3]$ form a 3-clique in $N(F)$, but there is no 2-simplex $[U_1, U_2, U_3]$, since $U_1 \cap U_2 \cap U_3 = \varnothing$.

Definition 4.1.7. Let X be a set. A relation $\prec$ on X is called a *strict partial order* if:

(i) There is no $x \in X$ with $x \prec x$ (i.e. $\prec$ is irreflexive)

(ii) For all $x, y, z \in X$, if $x \prec y$ and $y \prec z$ then $x \prec z$ (i.e. $\prec$ is transitive)

A set X equipped with a strict partial order $\prec$ will be called a *poset*, sometimes denoted $(X, \prec)$.

Definition 4.1.8. Given a set X, a strict partial order $\prec$ on X is called a *total order* if for any distinct x and y in X we have that either $x \prec y$ or $y \prec x$. Given a poset $(X, \prec)$, a subset Y of X is called a *chain* if the restriction of $\prec$ to Y forms a total order.

Example 4.1.9 (Poset Complex)**.** Let X be a set equipped with a strict partial order $\prec$ (i.e. let $(X, \prec)$ be a poset).

The *poset complex* (or *order complex*) $P(X)$ associated to $(X, \prec)$ is the simplicial complex whose simplices are given by chains in X: that is, $[x_0, \ldots, x_k]$ is a k-simplex in $P(X)$ if and only if $x_0, \ldots, x_k \in X$ with $\tau(x_0) \prec \cdots \prec \tau(x_k)$ for some $\tau \in S_{k+1}$.

Equivalently, one may think of $P(X)$ as being the simplicial complex determined by the following properties:

1. There is a vertex (0-simplex) $[x]$ in $P(X)$ if and only if $x \in X$.
2. There is an edge (1-simplex) $[x, y]$ in $P(X)$ if and only if $x, y \in X$ with $x \prec y$ or $y \prec x$.
3. There is a k-simplex (for $k \geqslant 2$) $[x_0, \ldots, x_k]$ in $P(X)$ if and only if $[x_0], \ldots, [x_k]$ are vertices in $P(X)$ forming a $(k+1)$-clique in the 1-skeleton of $P(X)$ (that is, spanning a complete graph).

Condition 3 here implies that the complex $P(X)$ is flag.

Definition 4.1.10. Let K be a simplicial complex and $\prec$ a strict partial order on the vertices of K. We say that K is an *ordered* (or sometimes *oriented*) simplicial complex if for every simplex σ of K, $\prec$ restricts to a total order on the vertices of σ (that is, the vertices of σ are a chain in $(P(X), \prec)$).

Example 4.1.11. Let $(X, \ll)$ be a poset, and $P(X)$ be the associated poset complex. Then $P(X)$ is an ordered simplicial complex with respect to the relation $[x] \prec [y]$ in $P(X)^{(0)}$ if and only if $x \ll y$ in X.

Indeed, if $[x] \prec [x]$ in $P(X)$ then $x \ll x$ in X. But $\ll$ is irreflexive, so we cannot have that $[x] \prec [x]$ for any $[x] \in P(X)^{(0)}$. And if $[x] \prec [y]$ and $[y] \prec [z]$ in $P(X)$ then $x \ll y$ and $y \ll z$ in X, hence $x \ll z$ in X. So $[x] \prec [y]$ in $P(X)$. Thus $\prec$ defines a partial order on $P(X)$.

Moreover, if $[x_0, \ldots, x_k]$ is a simplex in $P(X)$ then by definition, $\{x_0, \ldots, x_k\}$ is a chain in $(X, \ll)$, and hence $\{[x_0], \ldots, [x_k]\}$ is a chain in $(P(X), \prec)$. So $\prec$ restricts to a total order on the vertices of any simplex in $P(X)$.

4.2 Connectivity Properties

The content here is taken from Chapter 0 and Chapter 1 Section 1.1 of Hatcher's 'Algebraic Topology' [17] (unless otherwise specified). Throughout this subsection, let $I := [0, 1]$ be the unit interval in $\mathbb{R}$.

Definition 4.2.1. Let X be a space and $x, y \in X$. A *path* from x to y is a continuous map $\gamma : I \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. If $x = y$ we say that γ is a *loop* based at x. We will often conflate the map with its image, also calling $\gamma(I) \subseteq X$ a path.

Given two paths $\gamma, \delta : I \to X$ with $\gamma(1) = \delta(0)$ we define the product $\gamma \cdot \delta : I \to X$ to be the path $(\gamma \cdot \delta)(s) := \begin{cases} \gamma(2s) & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ \delta(2s-1) & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$.

Definition 4.2.2. A non-empty space X is *path connected* if for any points x and y in X, there exists a path in X from x to y.

Definition 4.2.3. A *homotopy of paths* in a space X is a family of paths $\gamma_t : I \to X$, $0 \leqslant t \leqslant 1$ such that there exist $x_0, x_1 \in X$ with $\gamma_t(0) = x_0$ and $\gamma_t(1) = x_1$ for all $0 \leqslant t \leqslant 1$, and the associated map $F : I \times I \to X, (s,t) \mapsto \gamma_t(s)$ is continuous. We say that γ_0 and γ_1 are *homotopic*, denoted $\gamma_0 \simeq \gamma_1$.

Let X be a space and $x_0 \in X$. Then $\simeq$ is an equivalence relation on the set of paths in X ([17, Proposition 1.2]). We denote by $\pi_1(X, x_0)$ the set of all equivalence classes of loops in X based at x_0. This forms a group under the operation $[\gamma][\delta] = [\gamma \cdot \delta]$ [17, Proposition 1.3]. If $x_0, x_1 \in X$ and there exists some path in X from x_0 to x_1, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ [17, Proposition 1.5].

Definition 4.2.4. Let X be a path connected space and let $x_0 \in X$. We write $\pi_1(X)$ for the (isomorphism class of the) group $\pi_1(X, x_0)$, and call this the *fundamental group* of X.

Definition 4.2.5. A path connected space X is *simply connected* if $\pi_1(X) = \{1\}$ is trivial.

Lemma 4.2.6 ([17, Proposition 1.6])**.** *A non-empty path connected space X is simply connected if and only if given any two points x and y in X, there exists a unique homotopy class of paths from x to y.*

Definition 4.2.7. Let X and Y be spaces. A family of maps $f_t : X \to Y$ where $t \in I$ is called a *homotopy* if the associated map $F : X \times I \to Y, (x,t) \mapsto f_t(x)$ is continuous. If (f_t) is a homotopy, we say that f_0 is *homotopic* to f_1, and write $f_0 \simeq f_1$.

Definition 4.2.8. A non-empty space X is *contractible* if the identity map on X is homotopic to a constant map (i.e. it is *nullhomotopic*).

Definition 4.2.9 ([17, Section 4.1])**.** Let $k \in \mathbb{N}$ and let X be a non-empty space. We say that X is *k-connected* if for all integers $0 \leqslant d \leqslant k$, any continuous map $f : S^d \to X$ extends to a continuous map $\hat{f} : D^{d+1} \to X$ (equivalently, f is homotopic to a constant map on X).

Note that if X is k-connected for some $k \in \mathbb{N}$, then it is necessarily d-connected for every $0 \leqslant d \leqslant k$.

We have that a non-empty space X is 0-connected if and only if X is path connected, and 1-connected if and only if X is simply connected. If X is contractible, then it is k-connected for every $k \geqslant 0$, though the converse need not be true.

4.3 Brown's Theorem

In his 1984 paper 'Presentations for groups acting on simply-connected complexes' [4], Brown presents a method of finding generators and relations of a group G acting on a

CW-complex X. It is this method which Armstrong, Forrest, and Vogtmann exploit in [2] to give a presentation for $Aut(F_n)$, and which we use in Paper 1 to give presentations for automorphism groups of free products.

In short, this method involves constructing a group $\hat{G}$ which is the fundamental group of a quotient of the 1-skeleton of our space X, and then adding relations which can be read off from the 2-skeleton. To do this formally, we must first introduce some notation.

Setup and Notation

Let G be a group, and X a simply connected non-empty CW-complex on which G acts by permuting the cells. We assume that the action of G does not invert any of the 1-cells of X (if it does, we may simply subdivide the affected edges, producing two new edges neither of which is inverted under the G-action).

We now make a series of choices:

- We choose an orientation of the 1-cells of X equivariantly with respect to the G-action; we call an oriented 1-cell an *edge*, denoting it by a letter e, with its inverse (i.e. the same 1-cell with opposite orientation) denoted $\bar{e}$. Given two edges e_1 and e_2, we write $e_1 \simeq e_2$ if there exists $g \in G$ with $e_2 = g \cdot e_1$, and $e_1 \not\simeq e_2$ if no such g exists. Note then that, by assumption, $e \not\simeq \bar{e}$ for any edge e.
- We will now choose a tree of representatives T for the action of G on X, with vertex set $V(T) =: V \subset X^{(0)}$. Given two 0-cells/vertices v_1 and v_2 of X, we write $v_1 \simeq v_2$ if there exists $g \in G$ with $v_2 = g \cdot v_1$, and $v_1 \not\simeq v_2$ if no such g exists. When making this choice of T, we require that $u \not\simeq v$ for all vertices u, v of T. That is, $V = V(T)$ is a set of orbit representatives of the vertices of X such that all vertices of V belong to distinct orbits.
- Next we choose a set E of orbit representatives of the edges of X (such that all edges of E belong to distinct orbits, i.e. $e_1 \not\simeq e_2$ for all $e_1, e_2 \in E$). We require that every edge of T is in E, and that if $e \in E$ then $o(e) \in V$.
- Given $e \in E$, there is a unique $w_e \in V$ with $t(e) \simeq w_e$. For each $e \in E$, choose $g_e \in G$ such that $t(e) = g_e \cdot w_e$.
- Choose a set F of orbit representatives of the 2-cells in X in such a way that for each $\tau \in F$, there is some edge d in the boundary of τ with $o(d) \in V$.

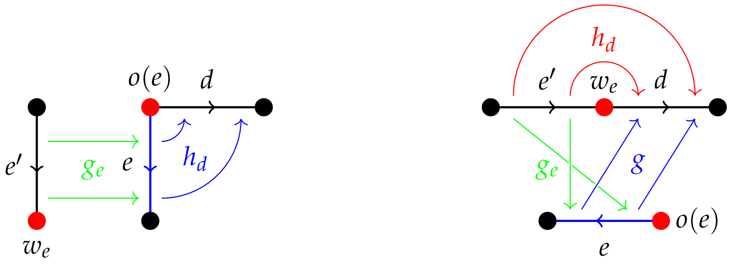
Note that for any edge $d \in X$ there is a unique edge $e \in E$ with either $d \simeq e$ or $d \simeq \bar{e}$. We define $\varepsilon_d := \begin{cases} +1 & \text{if } d \simeq e \\ -1 & \text{if } d \simeq \bar{e} \end{cases}$ where $e \in E$. Given a cell $\sigma \in X$ we will denote by G_σ the G-stabiliser $\mathrm{Stab}_G(\sigma)$.

Lemma 4.3.1. *Let d be an edge in X and suppose $o(d) \in V$. Then there exists $h_d \in G_{o(d)}$ such that $t(d) = h_d g_e^{\varepsilon_d} \cdot v$ for some $v \in V$, where $e \in E$ is such that $d \simeq e$ or $d \simeq \bar{e}$.*

Proof. Let d and e be as in the statement of the Lemma, and let w_e and g_e be as described above.

First suppose that $d \simeq e$. Take $h_d \in G$ such that $d = h_d \cdot e$. Note that since $o(d) \in V$ (by assumption) and $o(e) \in V$ (by design), then since $o(d) = h_d \cdot o(e)$ we must have $o(d) = o(e)$ and hence $h_d \cdot o(d) = o(d)$ (that is, $h_d \in G_{o(d)}$). Recall $t(e) = g_e \cdot w_e$, so $t(d) = h_d \cdot t(e) = h_d g_e \cdot w_e = h_d g_e^{\varepsilon_d} \cdot w_e$ with $w_e \in V$. We illustrate this process in Figure 2 (Case 1). Observe that in this case, d has the form $\begin{tikzcd} o(e) \arrow[r, "h_d \cdot e"] & h_d g_e \cdot w_e \end{tikzcd}$ where $e \in E$, $o(e), w_e \in V$, and $h_d \in G_{o(d)} = G_{o(e)}$.

Now suppose that $d \simeq \bar{e}$. Take $g \in G$ such that $d = g \cdot \bar{e}$, and set $e' := g_e^{-1} \cdot e$. Then $t(e') = w_e \in V$, and since $o(d) \in V$ with $o(d) = g \cdot t(e) = gg_e \cdot t(e') = gg_e \cdot w_e$, then we must have $o(d) = w_e$. Setting $h_d = gg_e$ we have $h_d \in G_{o(d)}$ and $t(d) = g \cdot o(e) = h_d g_e^{-1} \cdot o(e) = h_d g_e^{\varepsilon_d} \cdot o(e)$ where $o(e) \in V$. We illustrate this process in Figure 2 (Case 2). Observe that in this case, d has the form $\begin{tikzcd} w_e \arrow[r, "h_d g_e^{-1} \cdot \bar{e}"] & h_d g_e^{-1} \cdot o(e) \end{tikzcd}$ where $e \in E$, $o(e), w_e \in V$, and $h_d \in G_{o(d)} = G_{w_e}$.



Case 1: $d \simeq e$

Case 2: $d \simeq \bar{e}$

FIGURE 2: Finding Elements h_d and g_e for an edge d in X with $o(d) \in V$

In each case we have found $h_d \in G_{o(d)}$ with $t(d) = h_d g_e^{\varepsilon_d} \cdot v$ for some $v \in V$, as required. □

We now have that if d is an edge in X with $o(d) \in V$, and $e \in E$ is such that either $d \simeq e$ or $d \simeq \bar{e}$, then $t(d) \in h_d g_e^{\varepsilon_d} \cdot V$.

Next we consider a 2-cell $\tau \in F$. Suppose τ has boundary $x_1 \dots x_k$ where each x_i is an edge in X with $o(x_i) = t(x_{i-1})$, such that $t(x_k) = o(x_1) \in V$. We will recursively define edges d_i and e_i and group elements $g_i = h_{d_i} g_{e_i}^{\varepsilon_{d_i}}$.

Set $d_1 = x_1$ and let $e_1 \in E$ be the unique edge satisfying $d_1 \simeq e_1$ or $d_1 \simeq \bar{e}_1$. Take h_{d_1} and g_{e_1} as in Lemma 4.3.1, and set $g_1 = h_{d_1} g_{e_1}^{\varepsilon_{d_1}}$. Note that $t(x_1) = t(d_1) \in g_1 \cdot V$.

Now let $1 < i \leqslant k$ and suppose d_j and g_j are defined for all $j < i$, and that $t(x_{i-1}) \in g_1 \dots g_{i-1} \cdot V$. Set $d_i := g_{i-1}^{-1} \dots g_1^{-1} \cdot x_i$, and let $e_i \in E$ be the unique edge satisfying $d_i \simeq e_i$ or $d_i \simeq \bar{e}_i$. Note that $o(d_i) = g_{i-1}^{-1} \dots g_1^{-1} \cdot o(x_i) = g_{i-1}^{-1} \dots g_1^{-1} \cdot t(x_{i-1}) \in V$. Then Lemma 4.3.1 applies, and we can find corresponding group elements h_{d_i} and g_{e_i}. Setting $g_i = h_{d_i} g_{e_i}^{\varepsilon_{d_i}}$ we have that $t(d_i) \in g_i \cdot V$ and hence $t(x_i) \in g_1 \dots g_{i-1} g_i \cdot V$.

In particular, we note that $t(x_k) \in g_1 \dots g_k \cdot V$. But $t(x_k) = o(x_1) \in V$, and vertex representatives in V are unique. We must then have that $o(x_1) = t(x_k) = g_1 \dots g_k \cdot o(x_1)$, that is, $g_1 \dots g_k \in G_{o(x_1)}$.

Notation 4.3.2. Let $\tau \in F$ and suppose that τ has edge boundary $x_1 \dots x_k$, where $o(x_i) = t(x_{i-1})$ for each $1 < i \leqslant k$ and $t(x_k) = o(x_1) \in V$. For each i, let d_i and e_i be the edges described above, and let h_{d_i} and $g_i = h_{d_i} g_{e_i}^{\varepsilon_{d_i}}$ be the associated group elements given by Lemma 4.3.1. We now denote

$$g_\tau := g_1 \dots g_k = h_{d_1} g_{e_1}^{\varepsilon_{d_1}} \dots h_{d_k} g_{e_k}^{\varepsilon_{d_k}}$$

and

$$\hat{g}_\tau := h_{d_1} \hat{g}_{e_1}^{\varepsilon_{d_1}} \dots h_{d_k} \hat{g}_{e_k}^{\varepsilon_{d_k}}$$

where for each $e \in E$, $\hat{g}_e$ is a formal symbol. Note that g_τ is an element of $G_{o(x_1)}$ and $\hat{g}_\tau$ is an element of the free product $\left(\mathop{\ast}_{v \in V} G_v \right) * \left(\mathop{\ast}_{e \in E} \hat{g}_e \right)$.

The Theorem

We can now describe a presentation for G, using the above notation.

For $v \in V$, let G_v be the vertex stabiliser $\mathrm{Stab}_G(v)$ of v (a subgroup of G). For each $e \in E$, let $\hat{g}_e$ be a formal symbol, and let $G_e = \mathrm{Stab}_G(e) = \mathrm{Stab}_G(o(e)) \cap \mathrm{Stab}_G(t(e))$. Given a face $\tau \in F$ with boundary $x_1 \dots x_k$ starting at $v_0 := o(x_1) = t(x_k) \in V$, let $g_\tau \in G_{v_0}$ and $\hat{g}_\tau$ be the associated elements given in Notation 4.3.2.

Now the group G is generated by the set $\left\{ \bigcup_{v \in V} G_v \right\} \cup \{\hat{g}_e | e \in E\}$ subject to the relations:

(i) $\hat{g}_e = 1$ if e is an edge in T

(ii) $\hat{g}_e^{-1} i_e(g) \hat{g}_e = g_e^{-1} g g_e$ for any $e \in E$ and $g \in G_e$, where $i_e : G_e \to G_{o(e)}$ is the inclusion map.

(iii) $\hat{g}_\tau = g_\tau$ for all $\tau \in F$

as well as all relations from the groups G_v (for each $v \in V$).

That is to say:

Theorem 4.3.3 (Brown's Theorem)**.** *Let G be a group acting on a simply connected non-empty CW-complex X by permuting the cells, and without inversion of the 1-cells of X. Then with the above notation, we have:*

$$G = \left(\big(\mathop{\ast}_{v \in V} G_v \big) \ast \big(\mathop{\ast}_{e \in E} \hat{g}_e \big) \right) \Big/ \langle\langle \{ \textit{ relations (i)–(iii) } \} \rangle\rangle$$

Observation 4.3.4. If $F = \varnothing$ (that is, there are no 2-cells in X), then Relation (iii) is vacuous. Note that a CW complex with no 2-cells must be a 1-complex (i.e. comprise only 0-cells and 1-cells), making it a simplicial graph. The requirement that X be simply connected implies that in this case, X is a tree. Following the method in Example 4.1.4, we can make this into a tree in the sense of Serre, and applying Brown's Theorem, we recover the Structure Theorem from Bass–Serre Theory.

We will demonstrate this Theorem by applying it to some groups which we do understand, in order to see how it can help us in groups that we don't understand so well.

Example 4.3.5 (Action of S_3 on a Triangle)**.** Consider the group

$$S_3 = \{(1), (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\}$$

(where $(1) = 1$ is the identity) and recall that S_3 acts on the triangle X_0 with vertices v_1, v_2, v_3 (illustrated in Figure 3) by $\sigma \cdot v_i := v_{\sigma(i)}$ for $\sigma \in S_3$ and $i \in \{1, 2, 3\}$. We will use our understanding of this action to extract a presentation for S_3 (supposing for now that we did not already know one, or indeed, several).

The first obstacle we encounter is that the edges of X_0 are inverted under the action of S_3 — for example, the element $(1, 3)$ of S_3 setwise preserves the edge $v_1 — v_3$ of X_0, but swaps its endpoints. To overcome this, we simply subdivide each edge of X_0 to produce a new complex, X_1, illustrated in Figure 3, where the new vertices u_1, u_2, u_3 are labelled so that the inherited action of S_3 is $\sigma \cdot u_i = u_{\sigma(i)}$ for $\sigma \in S_3$ and $i \in \{1, 2, 3\}$.

Whilst X_1 is connected, Brown's Theorem requires that the complex is also simply connected. To remedy this, we insert a 2-cell with boundary

$$v_1 — u_3 — v_2 — u_1 — v_3 — u_2 — v_1$$

to create the complex X illustrated in Figure 3.

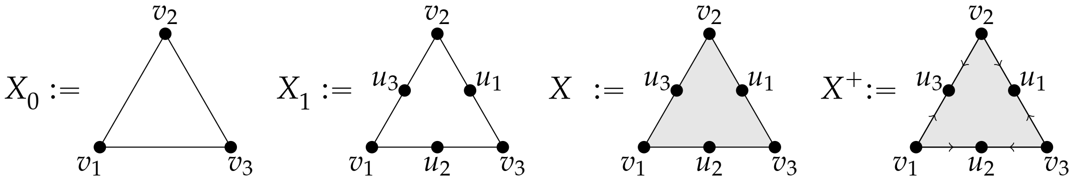


FIGURE 3: Building a CW-Complex on which S_3 Acts "Nicely"

We may now proceed. The first step is to choose an orientation of the edges of X. Since there is only one orbit of edges under the action of S_3, the orientation of one edge determines the orientation of all the edges of X. Thus we have two possible orientations; we choose the orientation X^+ illustrated in Figure 3 ("arrows point towards 'u' vertices").

The next step is to choose representative sets T, V, E, and F, for the orbits of cells in X. We begin by choosing a tree of representatives T with vertex set V such that the vertices of V belong to distinct orbits, and every orbit of vertices has a representative in V. We will let T be the tree with vertex set $V := \{v_1, u_2\}$ and edge set $\{v_1 — u_2 =: a\}$. Since there is only one orbit of edges in X, we are now forced to take our set E of edge representatives to be $E := \{a\}$. Finally, there is only one 2-cell in X, which we will label 'τ', and so we must take $F := \{\tau\}$. We illustrate these choices in Figure 4.

$V = \{v_1, u_2\}$
$E = \{a\}$
$F = \{\tau\}$

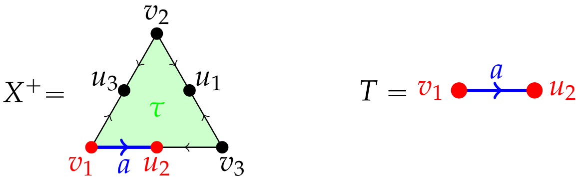


FIGURE 4: Representatives for the Action of S_3 on X

According to Brown's Theorem, S_3 is generated by the subgroups $G_{v_1} := \mathrm{Stab}_{S_3}(v_1)$ and $G_{u_2}\ \mathrm{Stab}_{S_3}(u_2)$ and an abstract element $\hat{g}_a$. It is easy to see that for $\{i, j, k\} = \{1, 2, 3\}$ we have $\mathrm{Stab}_{S_3}(v_i) = \mathrm{Stab}_{S_3}(u_i) = \{(1), (j\ k)\} \cong C_2$. By Relation (i) of Brown's Theorem, since the edge a belongs to the tree T, we have $\hat{g}_a = 1$. Thus S_3 is generated by $(2\ 3)$ and $(1\ 3)$, both of which are involutions (elements of order 2 in S_3).

Since S_3 acts freely on the edges of X (i.e. edge stabilisers are trivial), then Relation (ii) of Brown's Theorem is vacuous. Thus all that is left to do is consider Relation (iii) for the 2-cell τ. We proceed accordingly, labelling the boundary of τ in X by the edge path $x_1 x_2 \ldots x_6$ illustrated in Figure 5. Note that some of these labels are not positively oriented according to the orientation X^+ — this is necessary in this case to form a closed loop, and simply means that for some x_is we have $x_i = \bar{d}_i$ where d_is are positively oriented edges in X^+ (for example, $x_2 = \overline{(1\ 3)a} = (1\ 3)\bar{a}$).

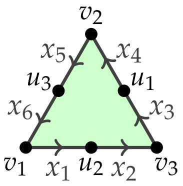


FIGURE 5: The Boundary of the 2-Cell τ in X

We now apply the inductive argument outlined in the Setup for Brown's Theorem to determine g_τ and $\hat{g}_\tau$, the results of which are summarised in Table 3. The illustration in Figure 6 may be a useful reference during this process.

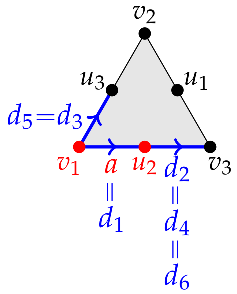


FIGURE 6: Illustration to Assist in Calculations for g_τ

We begin by noting that $g_a = (1)$ since $t(g_a) = u_2 \in V$, so for each i we will have $g_{e_i} = (1)$ (since $E = \{a\}$ meaning $e_i = a$), and thus $g_i = h_{d_i}$, where $h_{d_i} \in G_{o(d_i)}$ is such that $d_i = h_{d_i} \cdot a$ or $d_i = h_{d_i} \cdot \bar{a}$.

Firstly, according to Brown, we set $d_1 := x_1$. Noting that $d_1 = a$, we have $g_1 = h_{d_1} = (1)$. Secondly we set $d_2 := g_1^{-1} \cdot x_2 = (1) \cdot x_2 = x_2$, which we see is equal to $(1\ 3) \cdot \bar{a}$, hence $g_2 = h_{d_2} = (1\ 3)$. We now set $d_3 := g_2^{-1} g_1^{-1} \cdot x_3 = (1\ 3)(1) \cdot x_3$, which is the edge $v_1 — u_3 = (2\ 3) \cdot a$, meaning $g_3 = h_{d_3} = (2\ 3)$. Then $d_4 := g_3^{-1} g_2^{-1} g_1^{-1} \cdot x_4 = (2\ 3)(1\ 3)(1) \cdot x_4$ which we see is just the edge d_2, thus $g_4 = h_{d_4} = h_{d_2} = (1\ 3)$. Next $d_5 := g_4^{-1} g_3^{-1} g_2^{-1} g_1^{-1} \cdot x_5 = (1\ 3)(2\ 3)(1\ 3)(1) \cdot x_5 = d_3$, so $g_5 = h_{d_5} = h_{d_3} = (2\ 3)$. Finally we set $d_6 := g_5^{-1} g_4^{-1} g_3^{-1} g_2^{-1} g_1^{-1} \cdot x_6 = (2\,3)(1\,3)(2\,3)(1\,3)(1) \cdot x_6 = d_2$, and hence $g_6 = h_{d_6} = h_{d_2} = (1\,3)$.

x_i	d_i	e_i	g_{e_i}	h_{d_i}	ε_{d_i}	g_i
x_1	a	a	(1)	(1)	+1	(1)
x_2	$(1\,3) \cdot \bar{a}$	a	(1)	(1 3)	−1	(1 3)
x_3	$(2\,3) \cdot a$	a	(1)	(2 3)	+1	(2 3)
x_4	$(1\,3) \cdot \bar{a}$	a	(1)	(1 3)	−1	(1 3)
x_5	$(2\,3) \cdot a$	a	(1)	(2 3)	+1	(2 3)
x_6	$(1\,3) \cdot \bar{a}$	a	(1)	(1 3)	−1	(1 3)

TABLE 3: Summary of Data for Brown's Relation (iii) Applied to S_3

We note that $g_\tau = g_1 g_2 g_3 g_4 g_5 g_6 \cdot u_2 = (1)(1\ 3)(2\ 3)(1\ 3)(2\ 3)(1\ 3) \cdot u_2 = u_3 \neq u_2$, thus g_τ cannot be the identity in S_3. Thus since $g_\tau \in G_{v_0} = \{(1), (2\ 3)\}$ we must have that $g_\tau = (2\,3)$. Additionally, we have

$$\begin{aligned}
\hat{g}_\tau &= h_{d_1} \hat{g}_{e_1}^{\varepsilon_{d_1}} h_{d_2} \hat{g}_{e_2}^{\varepsilon_{d_2}} h_{d_3} \hat{g}_{e_3}^{\varepsilon_{d_3}} h_{d_4} \hat{g}_{e_4}^{\varepsilon_{d_4}} h_{d_5} \hat{g}_{e_5}^{\varepsilon_{d_5}} h_{d_6} \hat{g}_{e_6}^{\varepsilon_{6_1}} \\
&= (1)\hat{g}_a (1\,3) \hat{g}_a^{-1} (2\,3) \hat{g}_a (1\,3) \hat{g}_a^{-1} (2\,3) \hat{g}_a (1\,3) \hat{g}_a^{-1} \\
&= (1)(1)(1\,3)(1)(2\,3)(1)(1\,3)(1)(2\,3)(1)(1\,3)(1) \\
&= (1\,3)(2\,3)(1\,3)(2\,3)(1\,3).
\end{aligned}$$

Thus Brown's Relation (iii) (which puts $\hat{g}_\tau = g_\tau$) tells us that $(1\,3)(2\,3)(1\,3)(2\,3)(1\,3) = (2\,3)$, and from this we deduce that $\big((1\,3)(2\,3)\big)^3 = (1)$.

We may now conclude that:

$$S_3 = \left\langle\ (1\,3), (2\,3)\ \middle|\ (1\,3)^2, (2\,3)^2, \big((1\,3)(2\,3)\big)^3\ \right\rangle$$

This example was useful in demonstrating how vertex stabilisers are key to understanding Brown's Theorem, not just as generators, but also as part of the face relations. However, it is limited in that the fundamental domain was the whole space, and that the (single) edge in the set E belonged to the maximal tree T. We now give an example where this is not the case, in order to better understand the role of the generators $\hat{g}_e$ and the elements $g_e \in G$.

Example 4.3.6 (Action of $\mathbb{Z}^2$ on the Plane)**.** Consider the group $\mathbb{Z}^2 = \{\ [a,b]\ |\ a,b \in \mathbb{Z}\ \}$ (where the identity is $[0,0] = 1$ and the inverse of $[a,b]$ is $[-a,-b]$) which acts on the plane $\mathbb{R}^2 = \{\ (x,y)\ |\ x,y \in \mathbb{R}\ \}$ via $[a,b] \cdot (x,y) := (x+a, y+b)$.

Again, we will suppose that we do not know any presentation for the group $\mathbb{Z}^2$, but that we do somewhat understand its action on the plane $\mathbb{R}^2$. We will apply Brown's Theorem to this action and extract a presentation for $\mathbb{Z}^2$. We will use square brackets $[-,-]$ to denote elements of the group $\mathbb{Z}^2$ and round brackets $(-,-)$ to denote points in the plane $\mathbb{R}^2$.

We endow $\mathbb{R}^2$ with the CW structure where 0-cells are given by integer coordinates, 1-cells are horizontal and vertical lines joining points (a,b)—$(a+1,b)$ and (a,b)—$(a,b+1)$ respectively (with $a,b \in \mathbb{Z}$), and 2-cells have boundary (a,b)—$(a+1,b)$—$(a+1,b+1)$—$(a,b+1)$—(a,b). This makes $\mathbb{R}^2$ into a simply connected (infinite) CW-complex where the given $\mathbb{Z}^2$ action is both free and transitive. In particular, no edges (1-cells) are inverted under the action. Note that there are two orbits of edges, horizontal and vertical, and one orbit of vertices. We may now make the following choices (which we illustrate in Figure 7):

- We orient horizontal edges to "point right", and vertical edges to "point upwards".
- We set our tree of representatives (and set of vertex representatives) to be $T = V = \{(0,0)\}$.
- Letting a be the horizontal edge $(0,0)$—$(1,0)$ and b be the vertical edge $(0,0)$—$(0,1)$, we let our set of edge representatives be $E = \{a,b\}$.
- We take our set of 2-cell representatives to be $F = \{\tau\}$ where τ is the 2-cell whose boundary is $(0,0)$—$(0,1)$—$(1,1)$—$(1,0)$—$(0,0)$.

As noted, the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ is free, so all vertex and edge stabilisers are trivial. In particular, $G_{(0,0} = \mathrm{Stab}_{\mathbb{Z}^2}\big((0,0)\big) = \big\{[0,0]\big\} = \{1\}$. Thus $\mathbb{Z}^2$ is generated by two

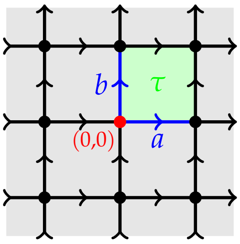


FIGURE 7: Representatives for the Action of $\mathbb{Z}^2$ on $\mathbb{R}^2$

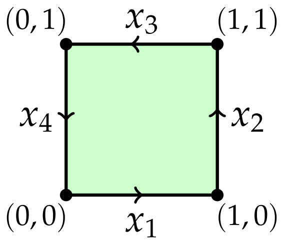


FIGURE 8: The Boundary of the 2-Cell τ in $\mathbb{R}^2$

abstract elements $\hat{g}_a$ and $\hat{g}_b$, with a single relation $g_\tau = \hat{g}_\tau$ coming from the 2-cell τ. Since $g_\tau \in G_{(0,0} = \{[0,0]\}$ we must have that $g_\tau = [0,0]$, the identity in $\mathbb{Z}^2$.

It now remains to calculate $\hat{g}_\tau$. We proceed by labelling the boundary of τ by $x_1x_2x_3x_4$ as illustrated in Figure 8, and following the procedure outlined in the Setup of Brown's Theorem to determine the elements g_{e_i} and h_{d_i} for each i. We summarise the results of this process in Table 4.

Note that $t(a) = (1,0) = [1,0] \cdot (0,0)$ and $t(b) = (0,1) = [0,1] \cdot (0,0)$, hence $g_a = [1,0]$ and $g_b = [0,1]$. Additionally, since $h_{d_i} \in G_{o(d_i)} = \{[0,0]\}$ we will have $h_{d_i} = [0,0]$ for each i. We will then have that $g_i = g_{e_i}^{\varepsilon_{d_i}}$, with $e_i \in \{a,b\}$.

We may now begin, setting $d_1 := x_1 = a$, thus $e_1 = a$ and $\varepsilon_{d_1} = 1$, so $g_1 = g_{e_1} = g_a = [1,0]$. Then setting $d_2 := g_1^{-1} \cdot x_2 = [-1,0] \cdot x_2 = b$, we have that $e_2 = b$ and $\varepsilon_{d_2} = 1$, hence $g_2 = g_{e_2} = g_b = [0,1]$. Next we set $d_3 := g_2^{-1}g_1^{-1} \cdot x_3 = [0,-1][-1,0] \cdot x_3 = [-1,0] \cdot \bar{a}$, hence $e_3 = a$, $\varepsilon_{d_3} = -1$, and $g_3 = g_{e_3}^{-1} = g_a^{-1} = [-1,0]$. Finally we set $d_4 := g_3^{-1}g_2^{-1}g_1^{-1} \cdot x_4 = [1,0][0,-1][-1,0] \cdot x_4 = [0,-1] \cdot \bar{b}$, thus $e_4 = b$ and $\varepsilon_{d_3} = -1$ (and so $g_4 = g_{e_4}^{-1} = g_b^{-1} = [0,-1]$).

x_i	d_i	e_i	g_{e_i}	h_{d_i}	ε_{d_i}	g_i
x_1	a	a	$[1,0]$	$[0,0]$	$+1$	$[1,0]$
x_2	b	b	$[0,1]$	$[0,0]$	$+1$	$[0,1]$
x_3	$[-1,0] \cdot \bar{a}$	a	$[1,0]$	$[0,0]$	-1	$[-1,0]$
x_4	$[0,-1] \cdot \bar{b}$	b	$[0,1]$	$[0,0]$	-1	$[0,-1]$

TABLE 4: Summary of Data for Brown's Relation (iii) Applied to $\mathbb{Z}^2$

We conclude that $\hat{g}_\tau = h_{d_1}\hat{g}_{e_1}^{\varepsilon_{d_1}} h_{d_2}\hat{g}_{e_2}^{\varepsilon_{d_2}} h_{d_3}\hat{g}_{e_3}^{\varepsilon_{d_3}} h_{d_4}\hat{g}_{e_4}^{\varepsilon_{d_4}} = [0,0]\hat{g}_a[0,0]\hat{g}_b[0,0]\hat{g}_a^{-1}[0,0]\hat{g}_b^{-1} = \hat{g}_a\hat{g}_b\hat{g}_a^{-1}\hat{g}_b^{-1}$.

Therefore our single relation $\hat{g}_\tau = g_\tau$ becomes $\hat{g}_a\hat{g}_b\hat{g}_a^{-1}\hat{g}_b^{-1} = [0,0] = 1$, and we conclude:

$$\mathbb{Z}^2 = \left\langle \hat{g}_a, \hat{g}_b \;\middle|\; \hat{g}_a\hat{g}_b\hat{g}_a^{-1}\hat{g}_b^{-1} \right\rangle$$

Or, more naturally, $\mathbb{Z}^2 = \langle a, b \mid ab = ba \rangle$.

In Paper 1, we apply this technique to a subcomplex of the spine of Guirardel and Levitt's 'Outer Space for a Free Product' (see Section 5) to obtain a presentation for the outer automorphism group of a free product.

In the case of three non-isomorphic free factors, we will use this to recover Collins and Gilbert's (algebraic) result [7, Proposition 4.1]. As we generalise the problem, Outer Space becomes too complicated to study efficiently. We then adapt the technique used in [18] (to reduce Culler-Vogtmann space to a subcomplex of its 2-skeleton) to obtain a somewhat simpler space, which preserves the information we require.

As it turns out, our complexes have 'strict' fundamental domains, allowing us to use a streamlined version of Brown's Theorem which ignores face relations. However, as is demonstrated in [2], once we start studying automorphisms of splittings with free rank (i.e. of the form $G_1 * \cdots * G_n * F_k$ with $k > 0$), we lose this 'strictness'. This adds a layer of complexity to the calculations required in this case, [3] but we are optimistic that it will still be manageable in future work.

5 Outer Space

In [15], Guirardel and Levitt give a deformation space for certain free products $G = G_1 * \cdots * G_n * F_k$ on which $\mathrm{Out}(G)$ acts, allowing us to study properties of the outer automorphism group of a free product. They call this space $\mathcal{O}$, the 'Outer Space' (for a free product). This is defined for groups $G = G_1 * \cdots * G_n * F_k$ with each G_i non-trivial, freely indecomposable, and not infinite cyclic (i.e. Grushko decompositions), where $n \geqslant 1$ and $n + k \geqslant 2$. We will be interested in subcomplexes of the 'barycentric spine' of $\mathcal{O}$.

Remark. This construction applies more generally to free factors which are not freely indecomposable. In this case it is not $\mathrm{Out}(G)$ which acts on the Outer Space, but the subgroup of $\mathrm{Out}(G)$ which preserves (up to conjugation) the free-factor splitting of G. Papers 1 and 2 consider this more general case, although with the restriction that $k = 0$ (the case $k > 0$ is ongoing work, very briefly introduced in Section 6).

---

[3] Current estimates are that there are 952 faces to consider for the fundamental domain of the complex $\mathcal{C}_{n,k,2}$ described in Section 6, but many of these are quickly ruled out as 'duplicates', and the rest are handled according to 8 cases, resulting in only 5 non-trivial generators from edges, each of the form $\hat{g}_e = \varphi$ for some automorphism φ in some vertex stabiliser.

5.1 Points in Outer Space

Let $G = G_1 * \cdots * G_n * F_k$ be a Grushko decomposition. A point in $\mathcal{O}$ is (up to equivariant isometries) a G-tree T satisfying:

(i) There is no proper subtree of T which is invariant under the action of G (i.e. the action of G on T is minimal);

(ii) G acts freely on the edge set of T (i.e. all edge stabilisers are trivial);

(iii) There is exactly one orbit of vertices with stabiliser conjugate to G_i for each $i \in \{1, \ldots, n\}$;

(iv) All other vertices have trivial stabiliser;

(v) If v is a valence-2 vertex of T then it is the fixed point of an element of G which exchanges the two edges incident to v (i.e. T has no redundant vertices).

We assign a positive length to each edge of T in such a way that all edges in the same orbit have the same length (possible since there are only finitely many edge orbits). This defines a metric on the tree, called the path metric.

Alternatively, points may be viewed as marked metric graphs of groups Γ, via Bass–Serre Theory. These will have fundamental group isomorphic to G, having one vertex group conjugate to G_i for each i, and all other vertex groups trivial (we will call such vertices 'trivial vertices' — note that some authors will refer to these as 'free vertices'). All edge groups of Γ are trivial, and given any maximal tree T_{max} of Γ, there will be precisely k edges not in T_{max} (k being the rank of the free component of our group). Instead of a group action, we equip our graph of groups with a marking (i.e. a map to the fundamental group). We assign to each edge a positive length (corresponding to the edge lengths above). Here, minimality means any leaves (vertices of valence 1) cannot be trivial vertices, and lack of redundancy further implies that trivial vertices must have valence at least 3.

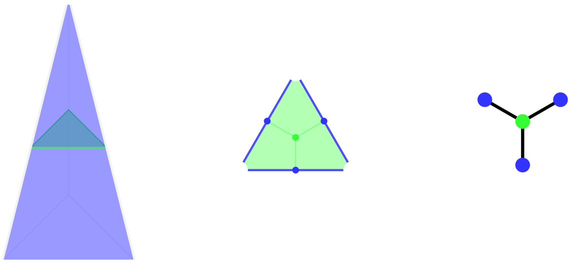
Note that two graphs of groups represent the same point in $\mathcal{O}$ if and only if they are isomorphic in the sense of Bass [3, Definition 2.1]. This is equivalent to requiring the existence of an equivariant isometry between their universal covers.

The quotient of $\mathcal{O}$ by the natural action of $(0, \infty)$ is the projectivised space, $P\mathcal{O}$. This equates to requiring that the sum of edge lengths in any given graph of groups is 1 (or any other positive number, but 1 is convenient), or that given a set of representatives of the edge orbits in a tree T the lengths of these edges sum to 1.

5.2 Structure of Outer Space

Given an underlying simplicial structure of one of our graphs of groups Γ, we can vary the assigned edge lengths. Assuming these always sum to 1, we can consider the resulting collection as an open m-simplex , where m is the number of edges of Γ. We will call this the '(open) simplex in $\mathcal{PO}$ associated to Γ'. If a graph Γ' can be obtained by collapsing one or more edges of Γ, then the simplex associated to Γ' can be viewed as a face of the simplex associated to Γ (equating to making the associated coordinate zero). In this way, we can put a simplicial structure on $\mathcal{PO}$. Note that this is (usually) not a CW-complex, as it may be missing faces. To obtain a CW-complex, one can simply take the barycentric subdivision of the structure, and linearly retract off the missing faces. This is called the *barycentric spine*, denoted $\mathcal{SO}$. We can put the Weak Topology on $\mathcal{SO}$ (i.e. open cells in $\mathcal{SO}$ are precisely the cells whose intersection with each simplex of $\mathcal{SO}$ is an open subset of said simplex).

A given graph of groups with underlying graph Γ will yield an (open) simplex in $\mathcal{PO}$ (edge lengths may vary but must always sum to 1), an (open) cone in $\mathcal{O}$ (edge lengths vary with no restriction), and a vertex in $\mathcal{SO}$ (edge lengths are all fixed). We illustrate this (for a group $G = G_1 * G_2 * G_3$) in Figure 9.



Cone in $\mathcal{O}$ Simplex in $\mathcal{PO}$ Vertex in $\mathcal{SO}$

FIGURE 9: Structures Associated with a Graph of Groups in $\mathcal{O}$, $\mathcal{PO}$, and $\mathcal{SO}$

One may wish to consider alternative topologies on $\mathcal{O}$ or $\mathcal{PO}$, such as the 'axes topology', which (viewing points as G-trees) is induced by length functions associated to the group actions. Guirardel and Levitt show that $\mathcal{PO}$ is contractible in both the Weak Topology and in the Axes Topology.

Structure of the Barycentric Spine $\mathcal{SO}$ of $\mathcal{O}$

Given two 0-cells Γ_1 and Γ_2 in our barycentric spine, we have a 1-cell $[\Gamma_1, \Gamma_2]$ whenever Γ_2 can be achieved by collapsing an edge or edges of Γ_1.

Whenever a collection of 0-cells $\Gamma_1, \ldots, \Gamma_m$ form an m-clique in the 1-skeleton (that is, whenever the restriction of the 1-skeleton to the vertices $\Gamma_1, \ldots, \Gamma_m$ forms a complete graph), we insert an $(m-1)$-cell $[\Gamma_1, \ldots, \Gamma_m]$.

Since the maximum number of edges such a graph of groups can have is $2n-3$ (when all non-trivial vertices have valence 1 and all trivial vertices have valence 3), and the minimum number is $n-1$ (when there are no trivial vertices), then the dimension of the barycentric spine of Outer Space is $(2n-3)-(n-1) = n-2$. Since $\mathcal{O}$ is contractible, and $\mathcal{O}$ deformation retracts onto $\mathcal{SO}$, then so too is $\mathcal{SO}$.

5.3 Action on Outer Space

The outer automorphism group $\mathrm{Out}(G)$ acts on $\mathcal{O}$, $\mathcal{PO}$ and $\mathcal{SO}$ by changing the marking of the associated graphs of groups.

Explicitly, if we consider points of $\mathcal{SO}$ to be actions $\psi : G \times T \to T, \psi(g,t) = g \cdot_\psi t$ on G-trees T, then for $\theta \in \mathrm{Out}(G)$, the action on $\mathcal{SO}$, $\theta \cdot (T, \psi)$, is defined by $\theta(\psi(g,t)) = \theta(g \cdot_\psi t) = \theta(g) \cdot_\psi t$.

We are interested in finding stabilisers of vertices in the barycentric spine $\mathcal{SO}$, which is equivalent to finding stabilisers of open simplices in $\mathcal{PO}$. Note that there are only finitely many orbits of simplices. Since an open simplex S contains only (and all possible) graphs of a given underlying simplicial structure, Γ, then the stabiliser of S in $\mathrm{Out}(G)$, $\mathrm{Stab}(S)$, is the group of automorphisms which 'preserves the decomposition of G as a graph of groups given by Γ'. That is, the automorphisms which stabilise a simplex (or vertex in the spine) represented by a graph Γ are precisely those which map an action on the universal cover T of Γ to another action on a tree T' whose quotient under the new action is Γ (upto equivariant isometries). If some of our vertex groups in Γ are isomorphic, we do allow these to be permuted.

If we restrict to the case where all our factors G_i are pairwise non-isomorphic and we have no free component, then considering points as actions of G on trees T, the stabiliser of a point T is precisely the group of automorphisms acting trivially on the quotient graph $\Gamma = {}^{T}\!\diagup_{G}$. This is the subgroup denoted by Guirdardel and Levitt as $\mathrm{Out}_0^S(G)$. If the vertex v_i of Γ represents the orbit of the vertex in T whose stabiliser is G_i, and p_i is the valence of v_i, then $\mathrm{Out}_0^S(G)$ is isomorphic to the direct product $\prod_{i=1}^{n}(G_i^{p_i-1} \rtimes \mathrm{Aut}(G_i))$ (where $\mathrm{Aut}(G_i)$ is identified with its projection in $\mathrm{Aut}(G)$ (or $\mathrm{Out}(G)$)). The precise details of this are found in [15].

6 The Complexes $\mathcal{C}_n$, $\mathcal{S}_n$, and $\mathcal{C}_{n,k,d}$

In this section, we will give a very brief overview of the complexes $\mathcal{C}_n$ and $\mathcal{S}_n$ used in Papers 1 and 2, respectively, and introduce a complex $\mathcal{C}_{n,k,d}$ for the purpose of one day generalising Paper 1.

The complexes $\mathcal{C}_n$ and $\mathcal{S}_n$ are defined for a free product $G = G_1 * \cdots * G_n$, assuming that the free factors are pairwise non-isomorphic (and non-trivial). They may be viewed as subcomplexes of $\mathcal{SO}$, the spine of Guirardel and Levitt's Outer Space (from [15]), by restricting to only those graphs of groups whose underlying graph has one of a few approved shapes. Equivalently, $\mathcal{C}_n$ and $\mathcal{S}_n$ can be viewed as order complexes associated to posets whose elements are certain graphs of groups, where the partial order is given by collapses (i.e. $\Gamma_1 < \Gamma_2$ iff Γ_2 can be achieved by collapsing edges of Γ_1). Again, these graphs of groups should have particular graph shapes as their underlying graph, and should have vertex groups $G_1^{g_1}, \ldots, G_n^{g_n}$ conjugate to the factor groups $G_1, \ldots, G_n$ such that G is the free product $G_1^{g_1} * \cdots * G_n^{g_n}$. All edge groups should be trivial, and they are permitted to also have additional vertices with trivial vertex group, so long as these have valency at least 3.

The graph shapes permitted for $\mathcal{C}_n$ are shown in Figure 10, where unlabelled (yellow) vertices have trivial vertex group, and a vertex with a label '$n - a$' for some a (and a blue ring) indicates that $n - a$ leaves have been supressed.

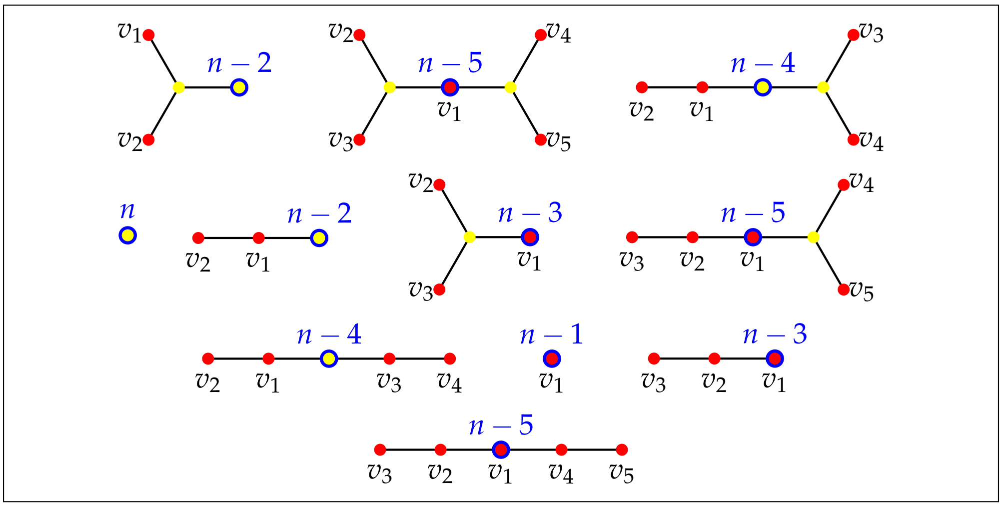
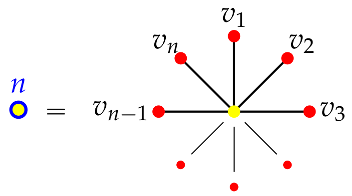
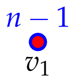
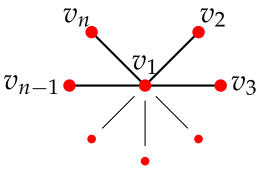


FIGURE 10: Graph Shapes Permitted for $\mathcal{C}_n$

The graph shapes permitted for $\mathcal{S}_n$ are $\alpha =$  and

$A =$ 

$=$ 

. Clearly then, for a fixed n, we have that $\mathcal{S}_n$ is a subcomplex of $\mathcal{C}_n$.

6.1 A 'Degree d' Complex for Outer Space

We now consider a group $G = G_1 * \cdots * G_n * F_k$ where $k > 0$ and again, the factor groups G_i are non-trivial and pairwise non-isomorphic [4].

In Hatcher and Vogtmann's 'Cerf theory for graphs' [18], the authors introduce a subcomplex of Auter/Outer Space (for a free group), called the 'degree d' complex, which is $(d-1)$-connected. In particular, the degree 2 complex is simply connected, which allows Armstrong, Forrest, and Vogtmann to apply Brown's Theorem in 'A presentation for Aut(F_n)' [2] and extract their result. We use a similar idea to produce a subcomplex of the outer space of a free product, which will hopefully satisfy nice connectivity properties.

Definition 6.1.1. Let Γ be a graph of groups (whose edge groups are all trivial). For a vertex v of Γ with vertex group G_v and valency $\nu(v)$, set $\mu(v) := \begin{cases} \nu(v) - 2 & \text{if } G_v = \{1\} \\ \nu(v) - 1 & \text{if } G_v \neq \{1\} \end{cases}$. Choose a vertex v_0 of Γ so that $\mu(v_0) \geqslant \mu(v)$ for all vertices v of Γ. Then the *degree* of Γ is $\widehat{\deg}(\Gamma) := \sum\limits_{v \neq v_0} \mu(v)$. We call v_0 the *basepoint* of Γ.

Remark. Note that the degree is clearly independent of the choice (if there is one) of v_0. For n and k large enough, and $\widehat{\deg}(\Gamma)$ small enough, we will have that v_0 is unique.

Observation 6.1.2. If we fix n and k and let Γ be a graph of groups with all edge groups trivial, n non-trivial vertex groups, and so that the underlying graph of Γ has fundamental group F_k (the free group of rank k), then since we have exactly n non-trivial vertex groups, we see that $\sum\limits_{v \in \Gamma} \mu(v) = \left(\sum\limits_{v \in \Gamma} \nu(v) - 2 \right) + n$. Moreover, since the underlying graph has fundamental group F_k, then $\sum\limits_{v \in \Gamma} (\nu(v) - 2) = 2k - 2$. Putting this together, we have that $\sum\limits_{v \in \Gamma} \mu(v) = 2k - 2 + n$. Hence $\widehat{\deg}(\Gamma) = 2k - 2 + n - \mu(v_0)$. In particular, for such a Γ we have that $\widehat{\deg}(\Gamma) \leqslant 2k - 2 + n$.

Definition 6.1.3. Let $G = G_1 * \cdots * G_n * F_k$ where the factor groups G_i are non-trivial and pairwise non-isomorphic, and let $\mathcal{SO}$ be the spine of the associated outer space.

---

[4] Extending to the case where some factors may be isomorphic to each other is easier done algebraically, rather than geometrically. It constitutes an action of a direct product of symmetric groups (or an action of a permutation group) dependent on the partition of $\{G_1, \ldots, G_n\}$ into sets of isomorphic factors.

We define $\mathcal{C}_{n,k,d}$ to be the subcomplex of $\mathcal{SO}$ obtained by restricting to graphs of groups Γ with $\widehat{\deg}(\Gamma) \leqslant d$. We call this the *degree d complex* of the Outer Space for a free product.

As with $\mathcal{C}_n$ and $\mathcal{S}_n$, one could alternatively view $\mathcal{C}_{n,k,d}$ as the order complex associated to a poset.

Remark. By Observation 6.1.2, every Γ in $\mathcal{SO}$ satisfies $\widehat{\deg}(\Gamma) \leqslant 2k - 2 + n$, hence for any $d \geqslant 2k - 2 + n$ we have that $\mathcal{C}_{n,k,d}$ is the whole spine $\mathcal{SO}$.

We now sketch the graph shapes for Γ with $\widehat{\deg}(\Gamma) \in \{0, 1, 2\}$. Figure 11 shows all graphs with $\widehat{\deg}(\Gamma) = 0$, assuming $n \geqslant 1$ and $k \geqslant 1$.

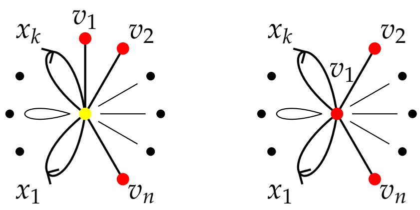


FIGURE 11: Graphs with $\widehat{\deg}(\Gamma) = 0$

We represent both of these graphs by a single blue vertex, ● . In the following figures (Figures 12 and 13), our diagrams will show vertices with non-trivial vertex group in red and vertices with trivial vertex group in yellow. We will show the 'basepoint' as a blue vertex, and assume that there are enough loops and leaves coming off this vertex so that the underlying graph has fundamental group F_k and so that there are n vertices with non-trivial vertex group. Note that the blue basepoint may represent a vertex with either trivial or non-trivial vertex group; in this way, each diagram represents two distinct graph structures. Figure 12 shows all graphs with $\widehat{\deg}(\Gamma) = 1$, assuming $n \geqslant 2$ and $k \geqslant 2$, and Figure 13 shows all graphs with $\widehat{\deg}(\Gamma) = 2$, assuming $n \geqslant 4$ and $k \geqslant 4$. If these conditions on n and k are not met, we simply remove the graph structures with 'too many' loops or non-trivial vertices.

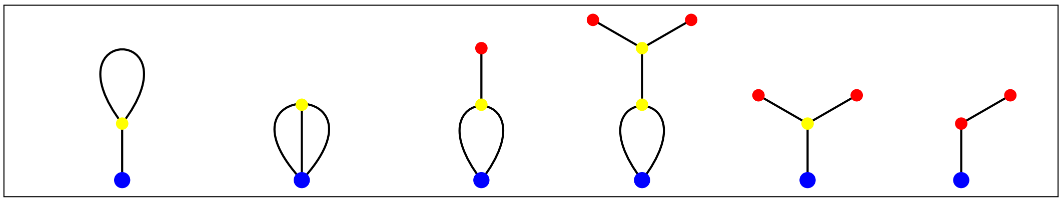


FIGURE 12: Graphs with $\widehat{\deg}(\Gamma) = 1$

Note then that $\mathcal{C}_{n,k,2}$ comprises all graphs in Figures 11, 12, and 13. For a fixed n, we have the following relationship between $\mathcal{C}_n$, $\mathcal{S}_n$, and $\mathcal{C}_{n,k,d}$:

$$\mathcal{C}_{n,0,0} = \mathcal{S}_n \subset \mathcal{C}_{n,0,1} \subset \mathcal{C}_n \subset \mathcal{C}_{n,0,2}$$

Observe that $\mathcal{C}_{0,k,d}$ is exactly the degree d complex $\mathbb{A}_{k,d}$ for Auter/Outer Space (i.e. Culler–Vogtmann Space), as described in Hatcher and Vogtmann's 'Cerf theory for graphs' [18]. Hence the presentation recovered $\mathcal{C}_{0,k,2} = \mathbb{A}_{k,2}$ will be that of Armstrong, Forrest, and Vogtmann in 'A presentation for Aut(F_n)' [2]. On the other hand, the presentation obtained from $\mathcal{C}_{n,0,2}$ should match that from $\mathcal{C}_n$ achieved in Paper 1. In this way, $\mathcal{C}_{n,k,2}$ should allow us to interpolate between these results [5] .

7 Twisted Conjugacy

Definition 7.0.1. For a group G and an automorphism $\varphi \in \mathrm{Aut}(G)$, we say that two elements are *φ-twisted conjugate* (denoted $x \sim_\varphi y$) in G if there exists $w \in G$ such that $\varphi(w)xw^{-1} = y$.

Lemma 7.0.2. *Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Then $\sim_\varphi$ defines an equivalence relation on G.*

Proof. We verify each of the three defining properties of an equivalence relation:

1. For any $x \in G$ we have that $\varphi(1)x1^{-1} = 1x1 = x$, hence $x \sim_\varphi x$.
2. Let $x, y \in G$ and suppose $x \sim_\varphi y$. Then there exists some $w \in G$ with $\varphi(w)xw^{-1} = y$. Set $v = w^{-1}$. Now $\varphi(v)yv^{-1} = \varphi(w^{-1})y(w^{-1})^{-1} = \varphi(w)^{-1}\left(\varphi(w)xw^{-1}\right)w = x$, hence $y \sim_\varphi x$.
3. Let $x, y, z \in G$ and suppose $x \sim_\varphi y$ and $y \sim_\varphi z$. Then there exist some $u, w \in G$ with $\varphi(w)xw^{-1} = y$ and $\varphi(u)yu^{-1} = z$. Set $v = uw$. Now $\varphi(v)xv^{-1} = \varphi(uw)x(uw)^{-1} = \varphi(u)\left(\varphi(w)xw^{-1}\right)u^{-1} = \varphi(u)yu^{-1} = z$, hence $x \sim_\varphi z$.

□

Observation 7.0.3. Other authors may define twisted conjugacy in different ways. The four 'obvious' ways to define when $x \sim_\varphi y$ are as follows:

(1) $\exists w \in G$ such that $y = \varphi(w)xw^{-1}$;

(2) $\exists w \in G$ such that $y = \varphi(w)^{-1}xw$;

(3) $\exists w \in G$ such that $y = w^{-1}x\varphi(w)$;

(4) $\exists w \in G$ such that $y = wx\varphi(w)^{-1}$.

---

[5] However, showing that $\mathcal{C}_{n,k,2}$ is simply connected, and that we can thus apply Brown's Theorem, is likely to be quite tricky.

The definition we will be using throughout is (1), but note that $y = \varphi(w)^{-1}xw = \varphi(w^{-1})x(w^{-1})^{-1}$ for some $w \in G$ if and only if $x \sim_\varphi y$ (in the sense of Definition (1)), so (1) and (2) are equivalent. Similarly, (3) and (4) are clearly equivalent. However,

$$\begin{aligned} &\exists w \in G \text{ such that } y = wx\varphi(w)^{-1} \\ \Leftrightarrow\ &\exists u = \varphi(w) \in G \text{ such that } y = \varphi^{-1}(u)xu^{-1} \\ \Leftrightarrow\ &x \sim_{\varphi^{-1}} y \text{ (in the sense of Definition (1)).} \end{aligned}$$

Thus (4) does not give the same equivalence classes as (1), but rather those of the inverse of the automorphism φ.

Definition 7.0.4 (Property R_∞)**.** Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Denote by $[x]_{\sim_\varphi}$ the $\sim_\varphi$ equivalence class of x in G, and let ${}^{G}\!/_{\sim_\varphi}$ be the set of all $\sim_\varphi$ equivalence classes in G. We say that G has *property* $R_\infty(\varphi)$ if the cardinality of ${}^{G}\!/_{\sim_\varphi}$ is infinite (i.e. $\sim_\varphi$ has infinitely many equivalence classes in G). We say that G has *property* R_∞ if G has $R_\infty(\varphi)$ for every $\varphi \in \mathrm{Aut}(G)$.

Lemma 7.0.5. *Let G be a group and let $\Phi \in \mathrm{Out}(G)$. Then for any $\varphi, \psi \in \Phi$, there exists a bijection $F : {}^{G}\!/_{\sim_\varphi} \to {}^{G}\!/_{\sim_\psi}$.*

Proof. Let G and Φ be as in the statement of the Lemma, and let $\varphi, \psi \in \Phi$. Then there is some inner automorphism $\iota \in \mathrm{Inn}(G)$ such that $\psi = \iota \circ \varphi$. Let $x \in G$ be such that $\iota(g) = xgx^{-1}$ for all $g \in G$. We now define a map $F : {}^{G}\!/_{\sim_\varphi} \to {}^{G}\!/_{\sim_\psi} = {}^{G}\!/_{\sim_{\iota\circ\varphi}}$ by $F([y]_{\sim_\varphi}) := [xy]_{\sim_{\iota\circ\varphi}} = [xy]_{\sim_\psi}$ for all $[y]_{\sim_\varphi} \in {}^{G}\!/_{\sim_\varphi}$.

Observe that for any $y, w \in G$, we have

$$x\varphi(w)yw^{-1} = x\varphi(w)(x^{-1}x)yw^{-1} = \iota(\varphi(w))xyw^{-1} = \psi(w)(xy)w^{-1} \sim_\psi xy$$

that is, $[x\varphi(w)yw^{-1}]_{\sim_\psi} = [xy]_{\sim_\psi}$. Thus F is well-defined.

Surjectivity of F is immediate upon noting that if $[z]_{\sim_\psi} \in {}^{G}\!/_{\sim_\psi}$ then $z \in G$ and hence $x^{-1}z \in G$, so $[x^{-1}z]_{\sim_\varphi} \in {}^{G}\!/_{\sim_\varphi}$ exists.

Now let $[y]_{\sim_\varphi}, [z]_{\sim_\varphi} \in {}^{G}\!/_{\sim_\varphi}$ and suppose $F([y]_{\sim_\varphi}) = F([z]_{\sim_\varphi})$, that is, $[xy]_{\sim_\psi} = [xz]_{\sim_\psi}$ (i.e. $xy \sim_\psi xz$). Then there exists $w \in G$ such that $xz = \psi(w)(xy)w^{-1} = \iota(\varphi(w))xyw^{-1} = x\varphi(w)x^{-1}xyw^{-1} = x\varphi(w)yw^{-1}$. In particular, $z = \varphi(w)yw^{-1}$, thus $y \sim_\varphi z$ and so $[y]_{\sim_\varphi} = [z]_{\sim_\varphi}$. Hence F is injective. □

Definition 7.0.6. Let G be a group and let $\Phi \in \mathrm{Out}(G)$. We say that G has property $R_\infty(\Phi)$ if G has property $R_\infty(\varphi)$ for some automorphism $\varphi \in \Phi$.

Remark. As a consequence of the above Lemma, we see that for a group G and an outer automorphism $\Phi \in \mathrm{Out}(G)$, G has property $R_\infty(\Phi)$ if and only if G has property $R_\infty(\varphi)$

for *every* automorphism $\varphi \in \Phi$. Thus G has property R_∞ if and only if G has property $R_\infty(\Phi)$ for all $\Phi \in \mathrm{Out}(G)$.

In Paper 3, we show that any accessible group with infinitely many ends (see Definition 8.2.3) has property R_∞. Some groups that **don't** have property R_∞ are:

- Finite groups (obviously).
- $(\mathbb{Z}, +)$ — Consider the automorphism $\varphi : a \mapsto -a$, then $a \sim_\varphi b$ if and only if $\frac{a-b}{2} \in \mathbb{Z}$, that is, a and b have the same parity. Thus $\sim_\varphi$ has two classes in $\mathbb{Z}$: the set of odd numbers and the set of even numbers.
- $(\mathbb{Z}^n, +)$ — Again, consider the automorphism $\varphi : g \mapsto -g$, in this case, $(a_1, \dots, a_n) \mapsto (-a_1, \dots, -a_n)$. Then $(a_1, \dots, a_n) \sim_\varphi (b_1, \dots, b_n)$ if and only if for each $i \in \{1, \dots, n\}$, a_i and b_i have the same parity. Thus $\sim_\varphi$ has 2^n classes in $\mathbb{Z}^n$.
- $F(X)$ where X is some infinite set (i.e. the free group of infinite rank); this is proved by Dekimpe–Gonçalves in [9, Proposition 5.4].

8 Ends

In this section we present several notions of an 'end' of a space, which are equivalent when the space in question is a locally finite graph. Given a group G with generating set S, the number of ends of G is the number of ends of the Cayley graph $\mathrm{Cay}(G, S)$ of G with respect to S. If S is finite, then $\mathrm{Cay}(G, S)$ is locally finite, and so we will direct our attention to finitely generated groups (although W. Dicks and M. Dunwoody extend the theory to all groups in their book 'Groups Acting on Graphs' [10]).

8.1 Ends of a Graph

Ends via Compact Sets

We begin with H. Freudenthal's Definition of the ends of a topological space X:

Definition 8.1.1 (Freudenthal [13])**.** Let X be a (Hausdorff) space and let $U_1 \supseteq U_2 \supseteq \cdots$ and $U'_1 \supseteq U'_2 \supseteq \cdots$ be sequences of non-empty connected open subsets of X such that each U_i (respectively U'_i) is a connected component of X-X_i (respectively X-X'_i) for some compact set X_i (respectively, X'_i).

We say that $U_1 \supseteq U_2 \supseteq \cdots$ and $U'_1 \supseteq U'_2 \supseteq \cdots$ are equivalent if for all i there exist j and k such that $U_i \supseteq U'_j$ and $U'_i \supseteq U_k$.

An *end* of X is an equivalence class of such sequences.

In the case where the space X is exhaustible by compact sets (i.e. there exists a sequence of compact sets $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$ such that $X = \bigcup_i K_i$), we obtain the below Definition for the number of ends of X, which is somewhat more intuitive:

Definition 8.1.2. Let X be a topological space which is covered by a nested sequence of compact sets $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$. We then have a sequence $X\text{-}K_0 \supseteq X\text{-}K_1 \supseteq X\text{-}K_2 \supseteq \cdots$, and letting c_i be the number of infinite connected components of $X\text{-}K_i$, we define the *number of ends* of X to be the supremum of the c_i's, $\sup\limits_{i \in \mathbb{N}} c_i$.

As R. Diestel and D. Kühn explain in their paper 'Graph-theoretical versus topological ends of graphs' [11], we can apply this concept to a graph Γ, by viewing Γ as 1-complex whose edges are homoemorphic to the real interval $[0, 1]$. Under this interpretation, an open neighbourhood of a vertex v is a union of half-open intervals $[v, x)$ where x is some point along an edge $[v, w]$, taken over all such edges. We then have that a subset of Γ is connected if and only if it is path connected, and a subset of Γ is compact if and only if it is closed (i.e. its compliment is open) and contains only finitely many vertices, and contains 'inner points' (points found within an edge and not a vertex) from only finitely many edges.

This roughly gives us the Definition of the number of ends of a graph used in Paper 3:

Definition 8.1.3 ([Paper 3, Definition 5.1.1])**.** The number of ends of a locally finite graph Γ is the supremum of the number of infinite components of the graphs $\Gamma\text{-}F$, where F ranges over all finite subgraphs of Γ.

Ends via Rays

A more direct approach concerning the ends of a graph is given by R. Halin [16], as equivalence classes of rays.

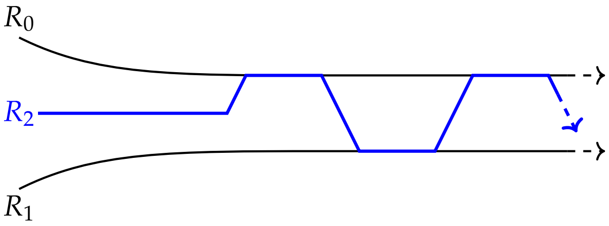
Definition 8.1.4. A *ray* in a simplicial graph Γ is an infinite sequence of vertices $(v_0, v_1, v_2, \ldots)$ such that no vertex is repeated, and consecutive vertices v_i and v_{i+1} are connected by an edge in Γ.

A *subray*, r, of a ray, $R = (v_0, v_1, v_2, \ldots)$, is a sequence of vertices $(u_0, u_1, u_2, \ldots)$ such that there exists some $n \in \mathbb{N}$ so that for all $i \in \mathbb{N}$, $u_i = v_{n+i}$.

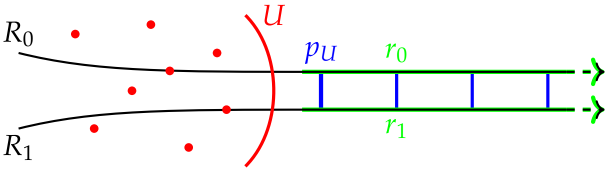
We now present two (equivalent) ways in which we may say rays are equivalent:

Definition 8.1.5. Let Γ be a simplicial graph.

1. Two rays R_0 and R_1 in Γ are equivalent if there exists a ray R_2 such that $|V(R_2) \cap V(R_0)|$ and $|V(R_2) \cap V(R_1)|$ are both infinite. That is, R_2 meets both R_0 and R_1 infinitely many times. We illustrate this below:



2. Two rays R_0 and R_1 in Γ are equivalent if for any finite subset U of $V(\Gamma)$, there exist subrays r_0 of R_0 and r_1 of R_1 which are rays in Γ-U and are connected by some path p_U in Γ-U. That is, no finite subset of Γ separates R_0 from R_1. We illustrate this below:



Definition 8.1.6 (Halin [16])**.** An *end* of a graph Γ is an equivalence class of rays in Γ.

Theorem 8.1.7 (Diestel–Kühn [11, Theorem 4.11])**.** *Let Γ be a locally finite graph. Then the set of topological ends (i.e. ends in the sense of Freudenthal [13]) is in one-to-one correspondence with the set of graph-theoretical ends (i.e. ends in the sense of Halin [16]).*

8.2 Ends of a Group

We are now ready to begin formally defining the number of ends of a group.

Definition 8.2.1. Let G be a finitely generated group with some finite generating set S. We define the *Cayley graph* $\mathrm{Cay}(G,S)$ of G with respect to S to be the graph with vertex set G and edge set $\{(g,gs)|g \in G, s \in S\}$.

Remark. Note that G acts on the vertex set of $\mathrm{Cay}(G,S)$ on the left via $h \cdot g = hg$ for $h \in G$ and $g \in V(\mathrm{Cay}(G,S)) = G$. This action preserves adjacency, and therefore extends to an action on $\mathrm{Cay}(G,S)$.

Lemma 8.2.2. *Let G be a finitely generated group and let S_1 and S_2 be two finite generating sets for G. Then the number of ends of $Cay(G,S_1)$ is equal to the number of ends of $Cay(G,S_2)$.*

Recall that if G is finitely generated (by some finite generating set S), then $\mathrm{Cay}(G,S)$ is locally finite, and Theorem 8.1.7 applies.

Definition 8.2.3. Let G be a finitely generated group and S be some finite generating set for G. The *number of ends* of G is defined to be the number of ends of $\mathrm{Cay}(G,S)$.

We now present J. Stallings' categorisation of finitely generated groups according to their ends, generalised to the case of all groups by W. Dicks and M. Dunwoody [10]:

Theorem 8.2.4 (Stallings' Theorem [10, Chapter IV: Proposition 6.5, Theorem 6.10, and Theorem 6.12])**.** *The number of ends of a group G is either* $0, 1, 2,$ *or* ∞*. Moreover:*

(i) G has 0 ends if and only if G is a finite group.

(ii) The following are equivalent:

- *G has 2 ends*
- *G is virtually infinite cyclic (i.e. G has a finite index subgroup isomorphic to* $\mathbb{Z}$*).*
- *G acts on a line with finite stabilisers and one orbit of edges.*
- *G has a finite normal subgroup N with* ${}^{G}\!\diagup_{N}$ *isomorphic to either* $\mathbb{Z}$ *or* $C_2 * C_2$*.*

(iii) G has infinitely many ends if and only if G isn't virtually cyclic and G acts on a tree with finite edge stabilisers and no global fixed point. Equivalently, G has infinitely many ends if and only if G isn't virtually cyclic and one of the following hold:

- *G is an amalgamated free product* $G = B *_C D$ *where* $B \neq C \neq D$ *and C is finite.*
- *G is an HNN extension* $G = B *_C x$ *where C is finite.*
- *G is countably infinite (has the cardinality of* $\mathbb{N}$*) and locally finite (every finitely generated subgroup of G is finite).*

(iv) G has 1 end if and only if G does not have 0, 2, or infinitely many ends (i.e. none of the above conditions are met).

Remark. Note that W. Dicks and M. Dunwoody work with an alternative definition of ends in [10], using Boolean algebra.

9 Groups Acting on $\mathbb{R}$-Trees

The main objects of study in Paper 3 are '$\mathbb{R}$-trees', which may be thought of as a generalisation of simplicial trees (see Definition 4.1.3). Culler and Morgan's 'Group actions on $\mathbb{R}$-trees' [8] provides a solid background on this topic, of which we give a brief overview in this section.

Definition 9.0.1. An $\mathbb{R}$-*tree* T is a non-empty metric space $T = (T, d)$ such that for any two points $x, y \in T$ there exists a unique arc $[x, y]$ from x to y which is isometric to the closed interval $[0, d(x, y)]$ in $\mathbb{R}$.

Equivalently, an $\mathbb{R}$-tree is a 0-hyperbolic geodesic metric space.

Observation 9.0.2. Let T be an $\mathbb{R}$-tree, and let $x \in T$. Denote by $\nu(x)$ the number of connected components of the space $T - \{x\}$.

If the set $\{x \in T \mid \nu(x) \geqslant 3\}$ is a discrete subset of T, we can form a 1-complex whose 0-cells form a discrete set V with $\{x \in T \mid \nu(x) \geqslant 3\} \subseteq V \subset T$, and whose 1-cells are the arcs $[x, y]$ in T such that $x, y \in V$ and $[x, y] \cap V = \{x, y\}$. Note that this is a simplicial tree with vertex set V and edge set the set of 1-cells.

Conversely, if X is a simplicial tree, we can form an $\mathbb{R}$-tree T_X by assigning a positive length to each edge of X, and letting T_X be the metric space with underlying structure that of X such that if x and y in T_X are the endpoints of an edge e in X then $d(x, y)$ is the length of the edge e.

Definition 9.0.3. Let G be a group acting by isometries on an $\mathbb{R}$-tree $T = (T, d)$, and let $g \in G$. The *translation length* of g, denoted $\|g\|$, is defined to be

$$\|g\| := \inf_{x \in T} d(x, g \cdot x)$$

Lemma 9.0.4 ([8, Lemma 1.3]). *Let G be a group acting on an $\mathbb{R}$-tree T with metric d, and let $g \in G$. Then the set $A_g := \{x \in T \mid d(x, g \cdot x) = \|g\|\}$ is a closed, non-empty subtree of T which is invariant under the action of g. Moreover,*

(i) if $\|g\| = 0$ then A_g is the fixed point set of g (we call g elliptic*);*

(ii) if $\|g\| > 0$ then A_g is isometric to the real line $\mathbb{R}$ and g acts on A_g by translation by $\|g\|$ (we call g hyperbolic *and call A_g the* axis *of g); and*

(iii) for any $x_0 \in T$, we have $d(x, g \cdot x) = \|g\| + 2d(x_0, A_g)$.

Note that some authors may call hyperbolic elements ‘loxodromic’, which avoids confusion with the concept of a hyperbolic group. Unless directly stated otherwise, when we say ‘hyperbolic’ in this thesis, we mean a group element which acts by translation on some line (i.e. the image of an isometric embedding of $\mathbb{R}$) in our space.

In Paper 3, we present the following Lemma, which allows one to easily determine when an element of a group acting on an $\mathbb{R}$-tree is hyperbolic:

Lemma 9.0.5 (Circle-Dot Lemma [Paper 3, Lemma 3.2.11]). *Let G be a group acting isometrically on an $\mathbb{R}$-tree, T. Suppose there exist distinct points $x, y \in T$ such that the arc $[x, g \cdot y]$ crosses the points y and $g \cdot x$ in that order (possibly with $y = g \cdot x$), that is, we have the following picture:*

Then g is hyperbolic and both x and y belong to the axis of g. In particular, $\|g\|_T = d(x, g \cdot x)$.

This generalises a result of Serre [21, Chapter I, Section 6.4, Proposition 25] applicable to simplicial trees.

9.1 Ends of an $\mathbb{R}$-Tree

As with graphs, one may define the concept of an end for an $\mathbb{R}$-tree T, via the idea of rays [8, Section 2]:

Definition 9.1.1. Let $\rho : [0, \infty) \to T$ be an isometric embedding of the half-line in $\mathbb{R}$ into T (note that this is necessarily injective). We call the image $R = \mathrm{im}(\rho)$ of ρ in T a *ray*, and say that it *emanates* from the point $\rho(0)$ in T. Given two such embeddings, ρ_1 and ρ_2, with images R_1 and R_2 respectively, we say that R_2 is a *subray* of R_1 if there is some $x \geqslant 0$ in $\mathbb{R}$ such that for all $a \in [0, \infty)$, $\rho_2(a) = \rho_1(a + x)$.

Since T is a tree, then the previous notions of equivalence of rays become the following:

Lemma 9.1.2. *Let T be an $\mathbb{R}$-tree, and let R_1 and R_2 be two rays in T. Then $R_1 \sim R_2$ if and only if $R_1 \cap R_2$ is a subray of both R_1 and R_2.*

Definition 9.1.3. Let T be an $\mathbb{R}$-tree. The *ends* of T are defined to be the equivalence classes of rays in T.

Remark. Note that for any point $p \in T$ there is precisely one ray emanating from p for each equivalence class of rays in T. So for any $p \in T$, the number of ends of T is equal to the number of rays emanating from p.

9.2 Reducibility

A key concept in the study of group actions on $\mathbb{R}$-trees is when such an action is 'irreducible'. We give a definition, together with some equivalent formulations, due to M. Culler and J. Morgan [8]. Note that being irreducible is not the same as being 'reduced' (see Section 10).

Definition 9.2.1 (Culler–Morgan [8])**.** Let G be a group acting by isometries on an $\mathbb{R}$-tree T. We say the action is *reducible* if one of the following hold:

(i) Every element of G fixes some point in T; or

(ii) T has a line which is invariant under the action of G; or

(iii) T has an end which is fixed by G.

We say the action is *irreducible* if it is not reducible.

Theorem 9.2.2 (Culler–Morgan [8, Theorem 2.7])**.** *Let G be a group acting on an $\mathbb{R}$-tree T and suppose there is some $g \in G$ with non-zero translation length (i.e. not every element of G fixes a point in T). Then the following are equivalent:*

(i) The action of G on T is irreducible.

(ii) There exist hyperbolic elements $g, h \in G$ such that the translation length $\|[g, h]\|$ of the commutator $[g, h] = g^{-1}h^{-1}gh$ is non-zero.

(iii) There exist hyperbolic elements $g, h \in G$ such that $A_g \cap A_h$ is an arc of finite positive length.

(iv) G contains a free group of rank 2 which acts freely and properly discontinuously on T.

10 Reduced Trees

In Paper 3, we give the following Definition:

Definition 10.0.1 ([Paper 3, Definition 3.3.4], [14, Definition 3.5])**.** A simplicial G-tree T is called *reduced* if whenever $e = (u, v)$ is an edge of T with $\mathrm{Stab}_G(e) = \mathrm{Stab}_G(v)$, then u and v are in the same G-orbit.

The purpose of this section is to motivate this definition. We adapt and expand upon Section 3 of V. Guirardel and G. Levitt's 'Deformation Spaces of Trees' [14] in order to provide this background context.

Let T be a (simplicial) tree and G be a group acting by isometries on T, without inversion of the edges. For an edge $e \in E(T)$ we let $G_e = \{g \in G | g \cdot e = e\} \leqslant G$ be the (setwise) stabiliser of e, called an edge stabiliser. For a vertex $v \in V(T)$ we let $G_v = \mathrm{Stab}_G(v)$ be the stabiliser of v, called a vertex stabiliser. Recall that for an edge e with endpoints $o(e)$ and $t(v)$, we have $G_e \leqslant G_{o(e)}$ and $G_e \leqslant G_{t(e)}$.

Given an edge e_0 of T, collapsing every edge in the G-orbit of e_0 yields a new tree T'. We say that T' is obtained by collapsing e_0.

The action of G on T induces an action of G on T'. If we label the vertex in T' resulting from the collapse of an edge $g \cdot e_0$ by $v_{g \cdot e_0}$, then $g \cdot v_{e_0} = v_{g \cdot e_0}$ and $G_{v_{g \cdot e_0}} = G_{o(e_0)} *_{G_{e_0}} G_{t(e_0)}$.

Note that if a subgroup H of G fixes a point in T, then H also fixes a point in T'. The converse need not hold.

Definition 10.0.2. Let T' be obtained by collapsing an edge (orbit) e_0 of T. We say e_0 is *collapsible* if for any subgroup H of G which fixes a point in T', H also fixes a point in T.

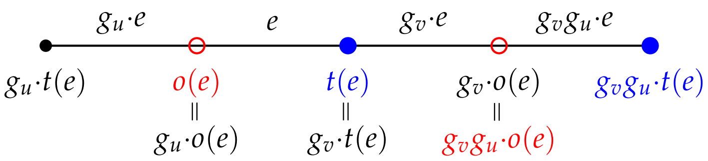
Remark. For those familiar with the subject, this is equivalent to saying that T and T' have the same elliptic subgroups in G, or that they belong to the same deformation space (in the sense of Guirardel and Levitt [14]).

Lemma 10.0.3. *An edge (orbit) e of T is collapsible if and only if $o(e)$ and $t(e)$ belong to distinct orbits and at least one of $G_{o(e)}$ or $G_{t(e)}$ is equal to G_e.*

Proof. Let T' be obtained by collapsing an edge (orbit) e of T, and let v_e be the resulting vertex in T'.

We first show that the conditions given are sufficient. Indeed, the only elliptic subgroups of T' which might not be elliptic in T are the vertex stabilisers $G_{v_{g\cdot e}} = G_{g\cdot v_e} = gG_{v_e}g^{-1}$. But if $o(e)$ and $t(e)$ belong to distinct orbits and (without loss of generality) $G_{t(e)}$ is equal to G_e, then $G_{v_e} = G_{o(e)} *_{G_e} G_{t(e)} = G_{o(e)} *_{G_{t(e)}} G_{t(e)} = G_{t(e)}$ is elliptic in T.

We now show that each of the conditions is necessary. If G_e is a proper subgroup of both $G_{o(e)}$ and $G_{t(e)}$ then we can find $g_u \in G_{o(e)}$ and $g_v \in G_{t(e)}$ such that $g_u \cdot o(e) = o(e)$ and $g_v \cdot t(e) = t(e)$ but $g_u \cdot e \neq e \neq g_v \cdot e$. Then g_vg_u is elliptic in T' (it fixes the vertex v_e), and we have the following setup in T:



Thus by the Circle-Dot Lemma (see Lemma 9.0.5, or equivalently, [21, Chapter I, Proposition 25]) we have that the element g_vg_u is hyperbolic in T.

On the other hand, if $o(e)$ and $t(e)$ belong to the same orbit in T, say $t(e) = g \cdot o(e)$ where $g \in G$, then $g \cdot v_e = v_e$ in T' (i.e. $g \in G_{v_e}$). But G acts without inversion on T, so we cannot have that g is elliptic in T, as if it were, it would have to stabilise the edge e (without stabilising each endpoint). ☐

Observation 10.0.4. In the context of the quotient graph of groups $\Gamma = T\backslash G$, collapsing an edge (orbit) in T corresponds to collapsing an edge in Γ. Let e be an edge (orbit) in T and let $[e]$ be the corresponding edge in Γ, with edge group $G_{[e]}$ (conjugate to the stabiliser G_e in G). The condition that $o(e)$ and $t(e)$ belong to distinct orbits in T corresponds to requiring that $[e]$ not be a loop, and the condition that one of $G_{o(e)}$ or $G_{t(e)}$ is equal to G_e corresponds to requiring that at least one of the monomorphisms $G_{o([e])} \hookleftarrow G_{[e]} \hookrightarrow G_{t([e])}$ is surjective.

Remark. In Papers 1 and 2, we say an edge of a graph of groups is collapsible if at least one of its endpoints has trivial vertex group. Since all of our graphs of groups in that context are trees (in particular, there are no edges which are loops) and all edge groups are trivial, then these notions of collapsibility agree.

By combining Lemma 10.0.3 with Definition 10.0.1 (and recalling that $P \implies Q$ is logically equivalent to $Q \vee \neg P$, which is the logical negation of $P \wedge \neg Q$), we obtain the following Corollary, giving a much more natural concept of a reduced tree:

Corollary 10.0.5 ([14, Defintion 3.5]). *A tree T is reduced if and only if it has no collapsible edges.*

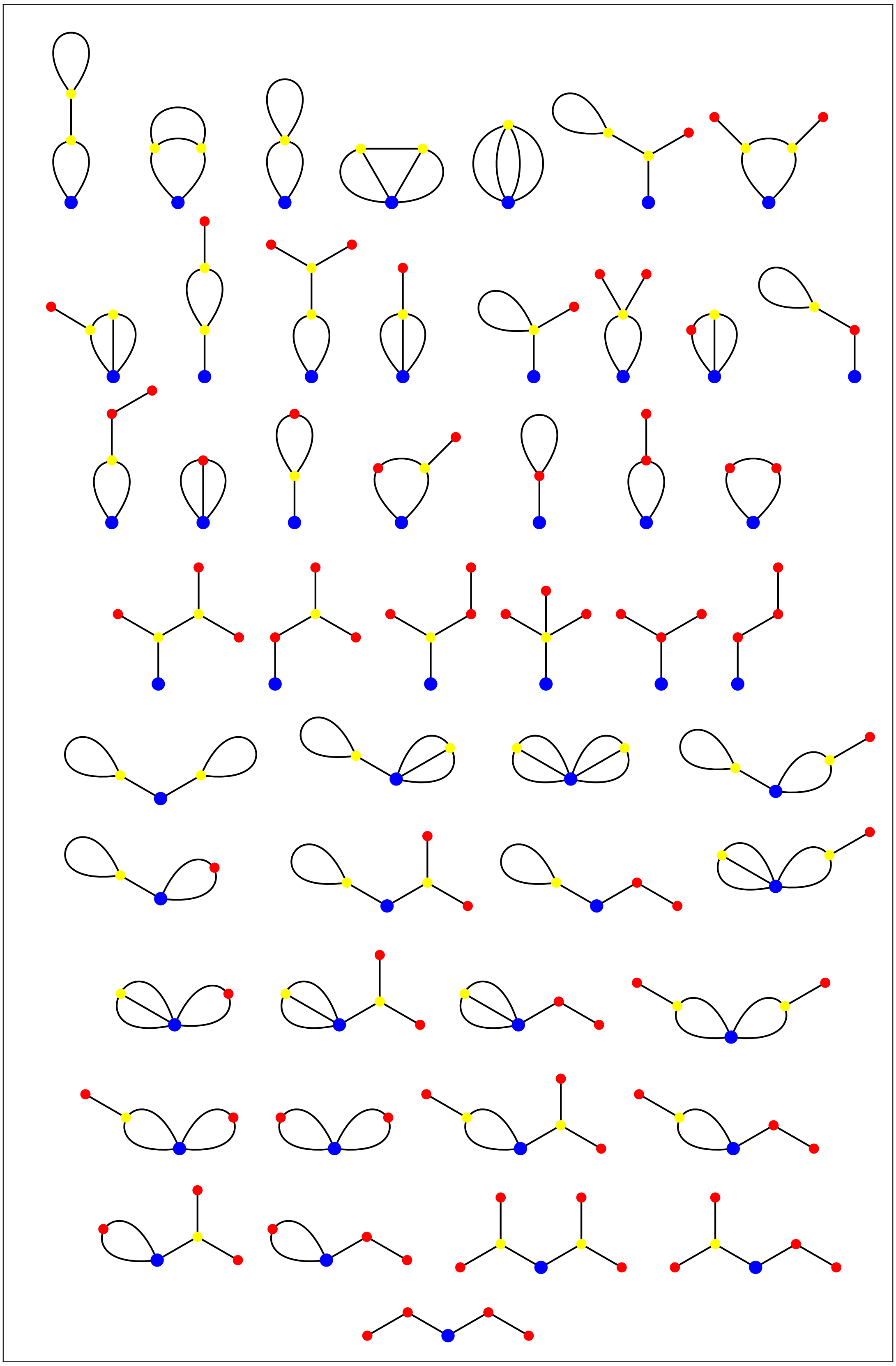


FIGURE 13: Graphs with $\widehat{\deg}(\Gamma) = 2$

References

[1] P. Alexandroff. Über den allgemeinen dimensionsbegriff und seine beziehungen zur elementaren geometrischen anschauung. *Mathematische Annalen*, 98:617–635, 1928.

[2] Heather Armstrong, Bradley Forrest, and Karen Vogtmann. A presentation for $\mathrm{Aut}(F_n)$. *Journal of Group Theory*, 11(2):267–276, 2008.

[3] Hyman Bass. Covering theory for graphs of groups. *Journal of Pure and Applied Algebra*, 89(1):3–47, 1993.

[4] Kenneth S. Brown. Presentations for groups acting on simply-connected complexes. *Journal of Pure and Applied Algebra*, 32:1–10, 1984.

[5] Daniel E. Cohen. *Combinatorial Group Theory: A Topological Approach*. Department of Pure Mathematics, Queen Mary College, University of London, 1978.

[6] Daniel E. Cohen. *Combinatorial Group Theory: A Topological Approach*. London Mathematical Society Student Texts. Cambridge University Press, 1989.

[7] Donald J. Collins and N. D. Gilbert. Structure and torsion in automorphism groups of free products. *The Quarterly Journal of Mathematics. Oxford Second Series*, 41(162):155–178, 1990.

[8] Marc Culler and John W. Morgan. Group actions on **r**-trees. *Proc. London Math. Soc. (3)*, 55(3):571?604, 1987.

[9] Karel Dekimpe and Daciberg Gonçalves. The R_∞ property for free groups, free nilpotent groups and free solvable groups. *Bulletin of the London Mathematical Society*, 46(4):737–746, 2014.

[10] Warren Dicks and M. J. Dunwoody. *Groups Acting on Graphs*, volume 17 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1989.

[11] Reinhard Diestel and Daniela Kühn. Graph-theoretical versus topological ends of graphs. *Journal of Combinatorial Theory, Series B*, 87(1):197–206, 2003.

[12] David Steven Dummit and Richard M. Foote. *Abstract Algebra.* John Wiley and Sons, Inc., third edition, 2004.

[13] Hans Freudenthal. Über die enden topologischer räume und gruppen. *Mathematische Zeitschrift*, 33:692–713, 1931.

[14] Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. *Groups Geom. Dyn.*, 1(2):135?181, 2007.

[15] Vincent Guirardel and Gilbert Levitt. The outer space of a free product. *Proceedings of the London Mathematical Society. Third Series*, 94(3):695–714, 2007.

[16] Rudolf Halin. Über unendliche wege in graphen. *Mathematische Annalen*, 157:125–137, 1964.

[17] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[18] Allen Hatcher and Karen Vogtmann. Cerf theory for graphs. *Journal of the London Mathematical Society*, 58(3):633–655, 1998.

[19] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete; bd. 89. Springer-Verlag, Berlin;, 1977.

[20] Derek John Scott Robinson. *An Introduction to Abstract Algebra*. Walter de Gruyter, 2003.

[21] J. P. Serre. *Trees*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1980.

[22] John Stillwell. *Classical Topology and Combinatorial Group Theory*. Graduate Texts in Mathematics. Springer New York, NY, 1993.

[23] Wikipedia contributors. Isomorphism theorems — Wikipedia, the free encyclopedia, 2025. [Online; accessed 15-August-2025].

Paper 1: **A Presentation for the Group of Pure Symmetric Outer Automorphisms of a Given Splitting of a Free Product**

Harry M. J. Iveson

ABSTRACT. We give a concise presentation for the group of pure symmetric outer automorphisms of a given splitting of a free product $G_1 * \cdots * G_n$. These are the (outer) automorphisms which preserve the conjugacy classes of the free factors G_i. This is achieved by considering the action of these automorphisms on a particular subcomplex of 'Outer Space', which we show to be simply connected. We then apply a theorem of K. S. Brown to extract our presentation.

Introduction

The study of group presentations, especially finite ones, is a core part of geometric group theory, dating back to the work of M. Dehn in the 1910's. Providing such group presentations is not only necessary for such study, but interesting in and of itself. Automorphism groups of free groups, and more generally, of free products, are natural objects to consider in this area. Different presentations may display various desirable properties, such as having few generators or relations, or highlighting some structure of the group.

In 2008, H. Armstrong, B. Forrest, and K. Vogtmann [2] gave a finite presentation for $\mathrm{Aut}(F_r)$, the automorphism group of a free group of rank r. They achieved this by applying a theorem of Brown [6, Theorem 1] to a subcomplex of a version of M. Culler and K. Vogtmann's 'Auter Space' [9] on which $\mathrm{Aut}(F_r)$ acts 'nicely'.

While finite presentations for $\mathrm{Aut}(F_r)$ were already known (for example, see the works of J. Nielsen [19] from 1924, whose presentation demonstrates the surjectivity of the map to $GL_n(\mathbb{Z})$, and B. Neumann [18] from 1933, whose presentation had only 2 generators, but many relations), Armstrong, Forrest, and Vogtmann [2] gave a presentation whose generators are all involutions, with a relatively small number of relations, making it straightforward to comprehend and apply.

In 1986, J. McCool [16] gave a concise presentation for the subgroup of $\mathrm{Aut}(F_r)$ comprising automorphisms which map each generator to a conjugate of itself. McCool's presentation comprised $r^2 - r$ generators, but only three (families of) relations.

There is a longstanding trend of generalising results from automorphisms of free groups to automorphisms of free products. In the 1940's, D. I. Fouxe-Rabinovitch [10], [11] gave a finite presentation for the automorphism group of a free product $G = G_1 * \cdots * G_n * F_k$, where F_k is the free group of rank k and where each G_i is non-trivial, freely indecomposable, and not infinite cyclic (i.e. $G_i \ncong \mathbb{Z}$). N. D. Gilbert [12, Theorem 2.20] gave an equivalent presentation for $\mathrm{Aut}(G)$ in 1987 with fewer relations, using 'peak reduction' methods of J. H. C. Whitehead [21] adapted to the free product case by D. J. Collins and H. Zieschang [8].

Main Result

We follow the methods of Armstrong, Forrest, and Vogtmann [2] to give a concise presentation for the group of pure symmetric outer automorphisms of a given splitting $G_1 * \cdots * G_n$ of free product G, denoted $\mathrm{Out}(G; G_1, \ldots, G_n)$. In our case, we have a 'strict fundamental domain' for the action of $\mathrm{Out}(G; G_1, \ldots, G_n)$, so can apply a more straightforward theorem of Brown [6, Theorem 3] to extract our presentation, given below:

Theorem 4.1.1. *Let $G_1 * \cdots * G_n$ be a free splitting of a group G where each G_i is non-trivial and $n \geqslant 5$. For $i \in [n] := \{1, \ldots, n\}$ and $j \in [n] - \{i\}$, let $f_{i_j} : G_i \to G_{i_j}$ be group isomorphisms, and for $g \in G_i$ let $\mathrm{Ad}_{G_i}(g)$ be the inner automorphism $x \mapsto gxg^{-1}$ of G_i. Then the group $\mathrm{Out}(G; G_1, \ldots, G_n)$ is generated by the $n(n-1)$ groups $G_{i_j} \cong G_i$ and $\Phi = \prod_{i=1}^{n} \mathrm{Aut}(G_i)$, subject to relations:*

1. $[f_{i_j}(g), f_{i_k}(h)] = 1 \; \forall g, h \in G_i$, *for all* $i \in [n]$, $j, k \in [n] - \{i\}$
2. $[f_{i_j}(g), f_{k_l}(h)] = 1 \; \forall g \in G_i, h \in G_k$, *for all distinct* $i, j, k, l \in [n]$
3. $[f_{j_k}(g), f_{i_j}(h) f_{i_k}(h)] = 1 \; \forall g \in G_j, h \in G_i$, *for all distinct* $i, j, k \in [n]$
4. $f_{i_{v_1}}(g) \ldots f_{i_{v_{n-1}}}(g) = \mathrm{Ad}_{G_i}(g^{-1}) \; \forall g \in G_i$, *for all* $i \in [n]$ *and* $\{v_1, \ldots, v_{n-1}\} = [n] - \{i\}$
5. $\varphi^{-1} f_{i_j}(g) \varphi = f_{i_j}(g\varphi) \; \forall g \in G_i$, *for all distinct* $i, j \in [n]$ *and all* $\varphi \in \Phi$

As well as all relations in G and Φ.

Note that here we assume $\mathrm{Aut}(G)$ acts on G on the right.

Corollary 4.1.2. *If a group G splits as a free product where the factor groups are non-trivial, freely indecomposable, not infinite cyclic, and pairwise non-isomorphic, then Theorem 4.1.1 gives a presentation for* $\mathrm{Out}(G)$.

Observation 0.0.1. In the case where some of the factor groups may be isomorphic, one may choose to study the symmetric automorphisms of the splitting. Then a finite direct product of symmetric groups, Π, acts on the splitting by permuting all possible isomorphic factors. The group of symmetric outer automorphisms of the splitting is then given by $\mathrm{Out}(G; G_1, \ldots, G_n) \rtimes \Pi$. While this is hard to see geometrically using the methods of this paper, it may be deduced algebraically.

The cases $n = 4$ and $n = 3$ are similar, and are given in Theorems 4.2.2 and 4.3.1 in Section 4.

If each of the groups G_i and $\mathrm{Aut}(G_i)$ is finitely presented, then one may extract a finite presentation for $\mathrm{Out}(G; G_1, \ldots, G_n)$ from this theorem by replacing each group G_{i_j} with a set of elements $\{f_{i_j}(g_1), \ldots, f_{i_j}(g_{m_i})\}$ such that the g_k's generate G_i, and replacing the group Φ with a generating set $\{\varphi_1, \ldots, \varphi_{m_\Phi}\}$. For conciseness, we do not make this more formal.

Our result may be considered to be a generalisation of McCool's presentation for the group of pure symmetric automorphisms of a free group.

Theorem 0.0.2 (McCool [16]). *Let $F_r = \langle x_1, \ldots, x_r \rangle$ be the free group on r generators. The group of pure symmetric automorphisms of F_r is generated by $r(r-1)$ elements $(x_i; x_j)$ (for $i, j \in \{1, \ldots, r\}$ and $i \neq j$), subject to commutation relations:*

1. $(x_i; x_j)(x_k; x_j) = (x_k; x_j)(x_i; x_j)$
2. $(x_i; x_j)(x_k; x_l) = (x_k; x_l)(x_i; x_j)$
3. $(x_i; x_j)(x_k; x_j)(x_i; x_k) = (x_i; x_k)(x_i; x_j)(x_k; x_j)$

(where i, j, k, l are assumed to be distinct).

Observe by comparing indices that these relations directly translate to our Relations 1–3. Our Relation 4 is an 'outer' relation so is not present in the automorphism group, and our Relation 5 describes automorphisms within a given factor, which are trivial in McCool's case.

In the case $n = 3$, we recover a special case of Gilbert's result [12, Theorem 2.20], given by D. J. Collins and N. D. Gilbert [7] in 1990, for three freely indecomposable, non-trivial, not infinite cyclic, pairwise non-isomorphic factors:

Theorem 0.0.3 (Collins–Gilbert [7, Proposition 4.1]). *For $G = X * Y * Z$ where each of X, Y, Z is freely indecomposable, non-trivial, not infinite cyclic, and where none of X, Y, Z are isomorphic to each other, we have that $\mathrm{Out}(G)$ is generated by*

$$\{(Y, x), (Z, y), (X, z), \varphi | x \in X, y \in Y, z \in Z, \varphi \in \Phi\}$$

where Φ is the set of factor automorphisms (see Definition 1.2.1), subject to relations:

- $(Y, x_1)(Y, x_2) = (Y, x_1 x_2)$
- $(Z, y_1)(Z, y_2) = (Z, y_1 y_2)$
- $(X, z_1)(X, z_2) = (X, z_1 z_2)$

- $\varphi^{-1}(Y,x)\varphi = (Y,x\varphi)$
- $\varphi^{-1}(Z,y)\varphi = (Z,y\varphi)$
- $\varphi^{-1}(X,z)\varphi = (X,z\varphi)$
- *All relations from* Φ

In particular, $\mathrm{Out}(G) \cong G \rtimes \Phi$.

Collins and Gilbert's result may be thought of as a presentation for the pure symmetric outer automorphisms preserving a free splitting structure $G_1 * G_2 * G_3$ where the only condition on the G_i's is that they are non-trivial. Thus our result may also be seen as both a special case of Gilbert's presentation [12, Theorem 2.20] and a generalisation of Collins and Gilbert's presentation [7, Proposition 4.1].

In future, we hope to generalise this further to free splittings of the form $G_1 * \cdots * G_n * F_k$ where automorphisms need not preserve the conjugacy classes of the generators for F_k. However this greatly increases the number of cells in our chosen subcomplex of Outer Space. Moreover the fundamental domain of the action ceases to be strict, meaning we can no longer apply the simplified version of Brown's theorem. These complications increase the complexity of the problem, though we hope that the end result will still be a pleasing presentation.

Methods and Techniques

To achieve our presentation, we choose a particular subcomplex of the 'Outer Space' for a free product introduced by Guirardel and Levitt in [13]. In order to study the symmetric automorphisms, we use the version where there is no free rank, so the Outer Space is similar to the poset complex introduced by D. McCullough and A. Miller in [17]. We work with the definition of the space provided by Guirardel and Levitt [13], since this interpolates between the Outer Spaces of Culler and Vogtmann [9] and of McCullough and Miller [17], which lends itself well to future work in the case of a splitting $G_1 * \cdots * G_n * F_k$.

We call our chosen complex $\mathcal{C}_n$, discussed in Section 2. In the cases $n = 3$ and $n = 4$, $\mathcal{C}_n$ is precisely the barycentric spine of Guirardel and Levitt's Outer Space for a free product whose Grushko decomposition has four non-isomorphic free factors and no free rank (see Section 2.4). Definition 2.2.4 details the construction of the complex $\mathcal{C}_n$ for $n \geqslant 5$.

In order to apply Brown's theorem [6, Theorem 3], we require that $\mathrm{Out}_{\mathfrak{S}}(G)$ acts cellularly on our complex $\mathcal{C}_n$ and with a strict fundamental domain, and that the complex

$\mathcal{C}_n$ is both connected and simply connected. We will also need suitable presentations for $\mathrm{Out}_{\mathfrak{S}}(G)$-stabilisers of vertices (graphs of groups) in $\mathcal{C}_n$.

The action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{C}_n$ and its fundamental domain are studied in Section 2.3, and the connectedness of the complex $\mathcal{C}_n$ is Corollary 3.1.4 of Section 3.1.

Vertex stabilisers are studied in Section 5.1.16, where we combine techniques of Guirardel and Levitt [13] with those of H. Bass and R. Jiang [4] to procure presentations which are both concise and precise (Propositions 2.4.3, 2.4.4, and 2.4.5).

Showing that the complex $\mathcal{C}_n$ is simply connected is highly non-trivial and is delayed until the second half of the paper, comprising Sections 5 and 6. We give a brief overview of the idea of the proof below.

In 1928, P. Alexandroff [1] introduced the notion of a 'nerve complex' associated to a cover of a space. In ideal conditions, this shares many of the same topological properties as the original space, while often being a much simpler object to understand.

We apply a similar concept in Section 5, introducing the 'Space of Domains' (see Definition 5.1.4) as a way of recording intersection patterns of $\mathrm{Out}_{\mathfrak{S}}(G)$-images of the fundamental domain in $\mathcal{C}_n$. Unlike Alexandroff's nerve complex, we are only interested in 2-way and 3-way intersections.

We show in Proposition 5.3.5 that in order to prove simple connectivity of the complex $\mathcal{C}_n$, it suffices to show that the Space of Domains is simply connected (having already shown that our fundamental domain of the $\mathrm{Out}_{\mathfrak{S}}(G)$-action on $\mathcal{C}_n$ is simply connected in Theorem 3.2.11 of Section 3.2).

Finally in Section 6 we apply 'peak reduction' techniques as used by Collins and Zieschang [8] and Gilbert [12] to deduce that the Space of Domains is simply connected (Theorem 6.3.2).

Acknowledgements

I am grateful to my supervisor, Armando Martino, for his continued help and support. I am also grateful to Naomi Andrew for her helpful and generous feedback on this manuscript.

1 Preliminaries

1.1 Some Useful Definitions and Notation

We adapt the notation for automorphisms used by Gilbert [12, Section 1]. Throughout, we consider a group G which splits as a free product $G_1 * \cdots * G_n$, where each G_i is non-trivial and $n \geqslant 3$. We refer to each G_i as a *factor group*. Note that throughout, we adopt the convention for conjugation that $g^x = xgx^{-1}$. Recall then that $g^{xy} = (xy)g(xy)^{-1} = x(ygy^{-1})x^{-1} = (g^y)^x$.

Notation 1.1.1. Let G be a group. We denote by $\mathrm{Aut}(G)$ the group of automorphisms of G, that is, isomorphisms from G to itself. We say $\psi \in \mathrm{Aut}(G)$ is an *inner automorphism* if there exists $x \in G$ so that for all $g \in G$, $\psi(g) = g^x = xgx^{-1}$. The collection of inner automorphisms forms a normal subgroup, $\mathrm{Inn}(G)$, of $\mathrm{Aut}(G)$. We then define $\mathrm{Out}(G) := {}^{\mathrm{Aut}(G)}\diagup_{\mathrm{Inn}(G)}$, and call this the *outer automorphism group* of G.

Definition 1.1.2 (Pure Symmetric Automorphism)**.** Let $G = G_1 * \cdots * G_n$ be a group which splits as a free product. We say $\psi \in \mathrm{Aut}(G)$ is a *pure symmetric automorphism* of the splitting $G_1 * \cdots * G_n$ if for each i there is some $g_i \in G$ such that $\psi(G_i) = G_i^{g_i} = g_i G_i g_i^{-1}$. We say $\hat{\psi} \in \mathrm{Out}(G)$ is a *pure symmetric outer automorphim* of the splitting if there is some $\psi \in \hat{\psi}$ which is a pure symmetric automorphism of the splitting.

Remark. It is easy to see that if ψ is a pure symmetric automorphism of some free splitting, and ι is an inner automorphism of the free product, then $\iota\psi$ is also a pure symmetric automorphism of the splitting. Thus the concept of 'pure symmetric outer automorphisms' is well-defined. It is not hard to verify that the collection of pure symmetric (outer) automorphisms forms a subgroup of $\mathrm{Aut}(G)$ (respectively, $\mathrm{Out}(G)$).

Notation 1.1.3. We denote by $\mathrm{Out}(G; G_1 * \cdots * G_n)$ the subgroup of $\mathrm{Out}(G)$ comprising pure symmetric outer automorphisms of the splitting $G_1 * \cdots * G_n$ of G. Given such a splitting, we may set $\mathfrak{S}$ to be the tuple $(G_1, \ldots, G_n)$ and let $\mathrm{Out}(G; G_1, \ldots, G_n) =: \mathrm{Out}_{\mathfrak{S}}(G)$, for brevity. We may similarly define $\mathrm{Aut}(G; G_1, \ldots, G_n)$ and $\mathrm{Aut}_{\mathfrak{S}}(G)$. We will sometimes refer to $\mathfrak{S}$ itself as the splitting, as opposed to the product $G_1 * \cdots * G_n$.

Observation 1.1.4. Given a splitting $G = G_1 * \cdots * G_n$ with $\mathfrak{S} = (G_1, \ldots, G_n)$, it is clear that $\mathrm{Inn}(G) \subseteq \mathrm{Aut}_{\mathfrak{S}}(G)$. Since $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ then $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}_{\mathfrak{S}}(G)$, and it follows that ${}^{\mathrm{Aut}_{\mathfrak{S}}(G)}\diagup_{\mathrm{Inn}(G)} \cong \mathrm{Out}_{\mathfrak{S}}(G)$, as one would expect.

Definition 1.1.5 (Factor Automorphism)**.** We say $\varphi \in \mathrm{Aut}(G; G_1, \ldots, G_n)$ is a *factor automorphism* if for each $i \in \{1, \ldots, n\}$, $\varphi|_{G_i}$ (that is, φ with domain restricted to the embedding of G_i in G) is an automorphism of G_i (i.e. $\varphi|_{G_i} \in \mathrm{Aut}(G_i)$). We will say $\hat{\varphi} \in \mathrm{Out}(G; G_1, \ldots, G_n)$ is a *factor automorphism* if $\hat{\varphi}$ has a representative $\varphi \in \mathrm{Aut}(G; G_1, \ldots, G_n)$ which is a factor automorphism. We will denote the set of factor automorphisms in $\mathrm{Out}(G; G_1, \ldots, G_n)$ by Φ.

The set of factor automorphisms Φ forms a subgroup of $\mathrm{Out}(G; G_1, \ldots, G_n)$, with $\Phi \cong \prod_{i=1}^{n} \mathrm{Aut}(G_i)$.

Notation 1.1.6. We write $\mathrm{Ad}_{G_i}(g)$ for the inner automorphism of G_i which conjugates each element of G_i by g (with $g \in G_i$), that is, $\mathrm{Ad}_{G_i}(g) : x \mapsto gxg^{-1}$. Since $\mathrm{Ad}_{G_i}(g) \in \mathrm{Inn}(G_i) \leqslant \mathrm{Aut}(G_i)$, then $\mathrm{Ad}_{G_i}(g) \in \Phi \leqslant \mathrm{Out}(G; G_1, \ldots, G_n)$. Note however that $\mathrm{Ad}_{G_i}(g)$ is **not** in $\mathrm{Inn}(G)$.

We will often abuse notation by writing ψ for both an automorphism in $\mathrm{Aut}(G)$ (or $\mathrm{Aut}(G; G_1, \ldots, G_n)$), and for the class it represents in $\mathrm{Out}(G)$ (or $\mathrm{Out}(G; G_1, \ldots, G_n)$).

Definition 1.1.7. Let $\mathfrak{S} = (G_1, \ldots, G_n)$ be the tuple associated to a group G which splits as a free product $G_1 * \cdots * G_n$, and let T be a finite tree on at least n vertices.
A free product $H_1 * \cdots * H_n$ is an $\mathfrak{S}$ *free factor splitting* for $G = G_1 * \cdots * G_n$ if for each i, there exists $g_i \in G$ so that $H_i = G_i^{g_i}$, and the subgroups $G_1^{g_1}, \ldots, G_n^{g_n}$ generate the group G. Note that by assumption $H_1 * \cdots * H_n \leqslant G$.
An $\mathfrak{S}$*-labelling* of T is an assignment of n vertex groups H_v to vertices $v \in V(T)$ so that $H_1 * \cdots * H_n$ is an $\mathfrak{S}$ free factor splitting for G.
Given an $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$ of T, we may consider the graph of groups $\mathbf{T} = (T, (H_1, \ldots, H_n))$ formed by associating the trivial group $\{1\}$ to any remaining vertices of T, and setting all edge groups to also be trivial.

Lemma 1.1.8. *Let G be a group with splitting $\mathfrak{S} = (G_1, \ldots, G_n)$ and let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for G. Then there exists $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ with $(G_i)\psi = H_i$ for each i.*

Proof. Since $H_1 * \cdots * H_n$ is an $\mathfrak{S}$ free factor splitting for G, for each i there exists $g_i \in G$ so that $H_i = G_i^{g_i}$. Let $\psi_i : G_i \to H_i$ be the map $(x)\psi_i = g_i x g_i^{-1}$ $\forall x \in G_i$. Clearly, ψ_i is an isomorphism of (sub)groups. By the universal property of free products, these isomorphisms ψ_i extend to an endomorphism $\psi : G \to G$. Since $H_1 * \cdots * H_n$ is an $\mathfrak{S}$ free factor splitting of $G_1 * \cdots * G_n$, then G is generated by the subgroups $H_1, \ldots, H_n$, and so ψ is surjective. Repeating this process on the maps $\psi_i^{-1} : H_i \to G_i$, we recover a surjective homomorphism $\varphi : G \to G$, which composes with ψ to give the identity map. Thus φ is an inverse for ψ, and so $\psi \in \mathrm{Aut}(G)$. Moreover, ψ restricts to ψ_i on each G_i, that is, $(G_i)\psi = G_i^{g_i} = H_i$, and so $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$, as required. □

Definition 1.1.9 (Whitehead Automorphism)**.** An automorphism in $\mathrm{Aut}(G; G_1, \ldots, G_n)$ which, for each j, either pointwise fixes G_j, or pointwise conjugates G_j by a given $x \in G$ is called a *Whitehead automorphism*. Given $x \in G_i$ and $A \subseteq \{G_1, \ldots, G_n\} - \{G_i\}$, we write (A, x) for the Whitehead automorphism which pointwise fixes any $G_j \notin A$, and pointwise conjugates by x any $G_j \in A$.

Given finite sequences $\mathbf{x} = (x_1, \ldots, x_k)$ with $x_1, \ldots, x_k \in G_i$ and $\mathbf{A} = (A_1, \ldots, A_k)$ with $A_1, \ldots, A_k \subseteq \{G_1, \ldots, G_n\} - \{G_i\}$ (with the A_j's pairwise disjoint), we write $(\mathbf{A}, \mathbf{x})$ for the composition $(A_1, x_1) \dots (A_k, x_k)$ (which should be read from left to right, since we consider the action of $\mathrm{Aut}(G)$ on G to be a right action). We call such a map a *multiple Whitehead automorphism*.

An element $\hat{\psi} \in \mathrm{Out}(G; G_1, \dots, G_n)$ will be called a (multiple) Whitehead automorphism if it has some representative $\psi \in \mathrm{Aut}(G; G_1, \dots, G_n)$ which is a (multiple) Whitehead automorphism.

Remark. Our notation differs from that of Gilbert [12] in that we decide not to include the operating factor G_i (see below) in the set A.

More detail on Whitehead automorphisms, including relative Whitehead automorphisms, can be found in Section 5.5.

Notation 1.1.10. Given factor groups G_i and G_j, we will write G_{i_j} (sometimes abbreviated as i_j) for the group generated by automorphisms (G_j, x) where $x \in G_i$. We call G_i the *operating factor* and G_j the *dependant factor*. Additionally, given factor groups G_i and $G_{v_1}, \dots, G_{v_k}$, we will write $i_{v_1 \dots v_k}$ (or $G_{i_{v_1 \dots v_k}}$) for the subgroup of $i_{v_1} \times \cdots \times i_{v_k}$ generated by the Whitehead automorphisms $(\{G_{v_1}, \dots, G_{v_k}\}, x)$ where $x \in G_i$. We think of this as the diagonal subgroup, and denote this by $i_{v_1 \dots v_k} \mathrel{\underline{\Diamond}} i_{v_1} \times \cdots \times i_{v_k}$.

Observation 1.1.11. We have a natural isomorphism $f_{i_j} : G_i \to G_{i_j}$ given by $f_{i_j}(x) = (G_j, x)$. Indeed, $G_j \cdot f_{i_j}(x) f_{i_j}(y) = G_j \cdot (G_j, x)(G_j, y) = x G_j x^{-1} \cdot (G_j, y) = xy G_j y^{-1} x^{-1} = G_j^{xy} = G_j \cdot f_{i_j}(xy)$.

1.2 Key Theorems

We will later make repeated use of the Seifert–van Kampen Theorem. As our simplicial complexes are all closed, and we usually only care about closed subcomplexes of these, we will use a 'closed version' of the theorem. Such a theorem can be found in some undergraduate Algebraic Topology notes, such as [22] delivered by H. Wilton at the University of Cambridge.

Theorem 1.2.1 (Seifert–Van Kampen (Closed Version))**.** *For closed sets A and B with A, B, and $A \cap B$ path-connected and such that there exist open sets $U \subset A$ and $V \subset B$ with $A \cap B$ a (strong) deformation retract of both U and V, we have that the diagram:*

$$\begin{tikzcd}
\pi_1(A \cap B) \arrow[r, "i_{A*}"] \arrow[d, "i_{B*}"] & \pi_1(A) \arrow[d, "j_{A*}"] \\
\pi_1(B) \arrow[r, "j_{B*}"] & \pi_1(A \cup B)
\end{tikzcd}$$

is a pushout, where $i_A : A \cap B \hookrightarrow A$, $i_B : A \cap B \hookrightarrow B$, $j_A : A \hookrightarrow A \cup B$, and $j_B : B \hookrightarrow A \cup B$ are inclusion maps. We will abuse notation and abbreviate this by writing:

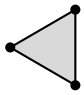
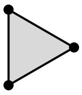
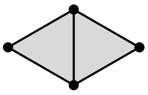
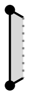
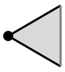
$$\pi_1(A \cup B) \cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

Remark. This closed version follows by noting that the diagram:

$$\begin{tikzcd}
\pi_1(U \cup V) \arrow[r] \arrow[d] & \pi_1(A \cup V) \arrow[d] \\
\pi_1(U \cup B) \arrow[r] & \pi_1(A \cup B)
\end{tikzcd}$$

is a pushout by the standard Seifert–van Kampen Theorem, where the corresponding components in the closed version are neighbourhood deformation retracts of those here, and hence have the same fundamental group.

Since our sets A, B, $A \cup B$, and $A \cap B$ will always be (finite) simplicial complexes, we will always have that $A \cap B$ is a neighbourhood deformation retract in both A and B. Indeed, we can take a union of open subsets of each simplex of A containing $A \cap B$, and similarly for B, and we will have open sets U and V satisfying this requirement. We illustrate this with an example:

Example 1.2.2. Let $A =$  and $B =$  be two (closed) simplices, and $X = A \cup B =$  a simplicial complex. Then in X we have that $A \cap B =$ . We can then take our sets $U \subseteq A$ and $V \subseteq B$ to $U =$  and $V =$ . Then $A - U =$  which is a closed set, hence U is open in A. Similarly, V is open in B, and it is clear that $A \cap B$ is a deformation retract of both U and V.

In [6], Brown presents a method for extracting a group presentation from its action on a CW complex. A streamlined version of this is given as Theorem 3 in [6] which holds when the action of the group on the complex has a strict fundamental domain:

Theorem 1.2.3 (Brown [6, Theorem 3])**.** *Let $\mathcal{G}$ act on a simply connected $\mathcal{G}$-CW complex X (without inversion on the 1-cells of X). Suppose there is a subcomplex W of X so that every cell of X is equivalent under the action of $\mathcal{G}$ to a unique cell of W. Then $\mathcal{G}$ is generated by the isotropy subgroups $\mathcal{G}_v$ ($v \in V(W)$) subject to edge relations $\iota_{o(e)}(g) = \iota_{t(e)}(g)$ for all $g \in \mathcal{G}_e$ ($e \in E(W)$) (where for any $e \in E(W)$, $\iota_{o(e)} : \mathcal{G}_e \to \mathcal{G}_{o(e)}$ and $\iota_{t(e)} : \mathcal{G}_e \to \mathcal{G}_{t(e)}$ are inclusions).*

It is this theorem that forms the basis of Section 4 in which we give a presentation for $\mathrm{Out}_{\mathfrak{G}}(G)$.

1.3 Outer Space for Free Products

In [13], Guirardel and Levitt give a description of a deformation space for certain free products $G = G_1 * \cdots * G_n * F_k$ on which $\mathrm{Out}(G)$ acts, allowing us to study properties of the outer automorphism group of a free product. They call this space $\mathcal{O}$, the 'Outer Space' (for a free product), and the projectivised space $\mathcal{PO}$.

The space $\mathcal{PO}$, while cellular, is not simplicial, due to 'missing' faces (faces 'at infinity'). To resolve this, we consider a construction called the *barycentric spine* of $\mathcal{PO}$ (denoted '$\mathcal{S}$'). This is obtained by taking the first barycentric subdivision of $\mathcal{PO}$, and then linearly retracting off the missing faces, to give a simplicial complex. This equates to taking the geometric realisation of the poset on the cells of $\mathcal{PO}$ given by $A < B$ if and only if A is a face of B.

Whilst their construction is defined for a Grushko decomposition (i.e. each factor group G_i is non-trivial, freely indecomposable, and not infinite cyclic), by considering instead the subgroup $\mathrm{Out}(G; G_1, \ldots, G_n, F_k)$ of $\mathrm{Out}(G)$ which preserves a given splitting of G, we can loosen these conditions. We may refer to this as a 'relative' Outer Space.

Since we are going to be interested in subcomplexes of the barycentric spine $\mathcal{S}$, we will now give an explicit description for it. We will restrict ourselves to the case where every factor group in the splitting of G acts elliptically (i.e. $k = 0$).

Points in the Barycentric Spine of Projectivised Outer Space

Let G be a group which splits as a free product $G_1 * \cdots * G_n$ where each G_i is non-trivial, and let $\mathfrak{G} = (G_1, \ldots, G_n)$ be the tuple associated to the splitting.

The barycentric spine $\mathcal{S}$ of $\mathcal{PO}$ is a simplicial complex whose 0-cells are graphs of groups Γ (with $\pi_1(\Gamma) \cong G$), as follows:

- The underlying graph structure of Γ is a tree
- Γ has one vertex with vertex group conjugate to G_i for each i
- All other vertex groups are trivial (vertices with trivial vertex group will be called 'trivial vertices')
- Any trivial vertex has valency at least 3
- All edge groups are trivial
- The vertex groups $G_1^{g_1}, \ldots, G_n^{g_n}$ generate the group G (that is, $G_1^{g_1} * \cdots * G_n^{g_n}$ is a free fractor splitting for G)

Note that two graphs of groups are equivalent if and only if they are isomorphic in the sense of Bass [3, Definition 2.1].

Via Bass–Serre Theory, we could equally consider points of $\mathcal{S}$ to be certain actions of G on trees T, up to equivariant isometry.

Structure of the Barycentric Spine $\mathcal{S}$ of $\mathcal{PO}$

Given two 0-cells Γ_1 and Γ_2 in our barycentric spine, we have a 1-cell $[\Gamma_1, \Gamma_2]$ whenever Γ_2 can be achieved by collapsing an edge or edges of Γ_1 (or vice versa).

Whenever a collection of 0-cells $\Gamma_1, \dots, \Gamma_m$ form an m-clique in the 1-skeleton (that is, whenever the restriction of the 1-skeleton to the vertices $\Gamma_1, \dots, \Gamma_m$ forms a complete graph), we have a unique $(m-1)$-cell $[\Gamma_1, \dots, \Gamma_m]$.

Since the maximum number of edges such a graph of groups can have is $2n-3$ (when all non-trivial vertices have valency 1 and all trivial vertices have valency 3), and the minimum number of edges is $n-1$ (when there are no trivial vertices), then the dimension of the barycentric spine of projectivised Outer Space is $(2n-3)-(n-1) = n-2$. Since $\mathcal{PO}$ is contractible [13, Theorem 4.2 and Corollary 4.4], and $\mathcal{PO}$ deformation retracts onto $\mathcal{S}$, then so too is $\mathcal{S}$.

Action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{S}$

If we consider points of $\mathcal{S}$ to be actions $\rho : G \to \mathrm{Isom}(T), \rho(g) = \rho_g : T \to T, \rho_g(x) = g \cdot_T x$ on G-trees T, then for $\varphi \in \mathrm{Aut}_{\mathfrak{G}}(G)$, the action on $\mathcal{S}$ is given by $(T, \rho) \cdot \varphi = (T, \varphi^{-1}\rho)$, that is, $g \cdot_{T\varphi} x = g\varphi^{-1} \cdot_T x$. This extends to a cellular action on $\mathcal{S}$. Since inner automorphisms act trivially on $\mathcal{S}$ then this defines an action of $\mathrm{Out}_{\mathfrak{G}}(G)$, where for any cell C of $\mathcal{S}$ and $\hat{\varphi} \in \mathrm{Out}_{\mathfrak{G}}(G)$, $C \cdot \hat{\varphi} = C \cdot \varphi$ for any automorphism $\varphi \in \hat{\varphi}$. In Section 2.3, we give a description of the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on our chosen subcomplex of $\mathcal{S}$ in terms of graphs of groups. Note that we have chosen notation so that G always acts on the left and $\mathrm{Aut}(G)$ always acts on the right.

We will be interested in finding $\mathrm{Out}_{\mathfrak{G}}(G)$-stabilisers of vertices in the barycentric spine. Considering points as actions of G on trees T, the stabiliser of a point T is precisely the group of automorphisms acting trivially on the quotient graph of groups $\Gamma = {}^{T}\!\diagup_{G}$. This is the subgroup denoted by Guirdardel and Levitt as $\mathrm{Out}_0^S(G)$.

If the vertex v_i of $\Gamma = {}^{T}\!\diagup_{G}$ represents the orbit of the vertex in T whose stabiliser is G_i, and μ_i is the valency of v_i in Γ, then $\mathrm{Out}_0^S(G)$ is isomorphic to the direct product $\prod_{i=1}^{n}(G_i^{\mu_i-1} \rtimes \mathrm{Aut}(G_i))$ (where $\mathrm{Aut}(G_i)$ is identified with its projection in $\mathrm{Aut}_{\mathfrak{G}}(G)$ (or $\mathrm{Out}_{\mathfrak{G}}(G)$)). The precise details of this are found in [13, Section 5]. We explore this more explicitly in Section 5.1.16.

2 The Complex $\mathcal{C}_n$

From now on, we fix a splitting $\mathfrak{G} = (G_1, \dots, G_n)$ of a group $G = G_1 * \cdots * G_n$, where each G_i is non-trivial. We will consider graphs of groups of G which respect the splitting

$\mathfrak{S}$ — note that these will all be trees, as each factor group acts elliptically in the relative Bass–Serre tree.

The barycentric spine of the projectivised relative Outer Space for G with respect to $\mathfrak{S}$ has a 'reasonably sized' quotient under the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ when $n = 3$ and $n = 4$ (4 vertices contributing to a total of 7 cells, and 32 vertices contributing to a total of 159 cells, respectively). As n grows, this quotient space quickly becomes unwieldy.

Definition 2.0.1. For $n = 3$ or $n = 4$, we define $\mathcal{C}_n$ to be the barycentric spine of Guirardel and Levitt's projectivised relative Outer Space associated to the splitting $\mathfrak{S}$ of G.

That is, $\mathcal{C}_3$ and $\mathcal{C}_4$ are the geometric realisations of the posets whose elements are simplices in the projectivised relative Outer Spaces for the splittings $G_1 * G_2 * G_3$ and $G_1 * G_2 * G_3 * G_4$, respectively, where $A \prec B$ if the simplex A is a face of the simplex B.

Lemma 2.0.2. *$\mathcal{C}_3$ and $\mathcal{C}_4$ are contractible. In particular, they are simply connected.*

Proof. This follows from contractibility of projectivised Outer Space, proven by Guirardel and Levitt [13, Theorem 4.2 and Corollary 4.4] , since projectivised Outer Space deformation retracts onto its spine. ☐

Our goal now is to construct a simplicial complex $\mathcal{C}_n$ for each $n \geqslant 5$ whose quotient under the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ remains 'reasonably sized'.

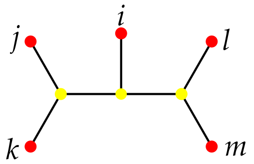
2.1 Restricting to a Subcomplex of Outer Space

For the rest of this section, we assume $n \geqslant 5$. In general, the barycentric spine of the Outer Space for n factors will be $(n-2)$-dimensional.

Since our graphs of groups are all trees, we will find that the stabilisers of higher-dimensional simplices in Outer Space are contained in the stabilisers of their faces. Hence restricting ourselves to lower dimensional simplices will not sacrifice information gathered from vertex stabilisers in the barycentric spine. We will thus restrict ourselves to the three lowest possible dimensions of simplex; then when we take the barycentyric spine of this restricted space, we will recover a 2-dimensional complex.

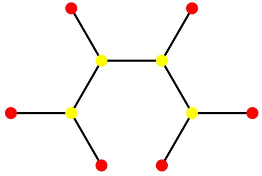
The lowest dimension of a simplex in Outer Space for n factors is $n-1$ (since our trees will have the minimal possible number of vertices, n, leading to $n-1$ edges). Thus we are interested in graphs of groups with $n-1$, n, or $n+1$ edges.

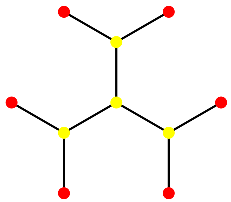
For $n = 5$, this means we just drop the top-dimensional simplices, which represent the graphs of groups of the form , where the labelled (red) vertices have vertex groups conjugate to the free factors of G, and the unlabelled (yellow) vertices have trivial vertex group (and all edge groups are trivial).



We will call such a graph (i.e. associated to a top-dimensional simplex) a *maximal graph*. Note that in our case, these are characterised by having precisely n leaves (all with non-trivial vertex group) and with all other vertices (each having trivial vertex group) having valency exactly 3.

As n increases, so too does the number of maximal graphs associated to 'top' simplices. For $n = 6$, there are two maximal graph structures, and . For $n = 7$ there are also two types of maximal graph, for $n = 8$ there are four, for $n = 9$ there are six, for $n = 10$ there are twelve, and for $n = 11$ there are eighteen. [1] Collapsing edges (passing to faces in the associated simplex in Outer Space) in each of these leads to a variety of structures.





Definition 2.1.1. In a graph of groups, we say that an edge is *collapsible* if it has at least one trivial endpoint (that is, at least one endpoint whose vertex group is the trivial group).

The process of replacing a collapsible edge (including its endpoints) by a single vertex whose vertex group is the free product of the vertex groups of the endpoints of said edge is called *collapsing*.

Given two graphs of groups T_1 and T_2, we will say T_2 is a *collapse* of T_1 if T_2 can be achieved as the result of successively collapsing edges of T_1.

Remark. Since a collapsible edge has at least one trivial endpoint, then one may think of the edge as collapsing to its other (potentially non-trivial) vertex.

That is, if $\overset{u}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{v}{\bullet}$ is a collapsible edge, with u being the trivial vertex and v having vertex group G_v (possibly also trivial), then in collapsing $\overset{u}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{v}{\bullet}$, we replace it with a vertex whose vertex group is equal to $\{1\} * G_v = G_v$. Thus we may think of collapsing

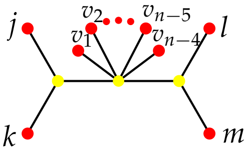
---

[1] This is somewhat analogous to alkane chains in organic chemistry, and the various isomers for these (if we were to pretend that carbon could make only three bonds, and not four).

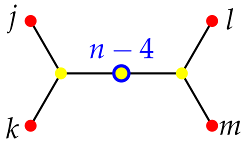
$\overset{u}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{v}{\bullet}$ as replacing it with the vertex v. Note that the new valency of v is equal to the old valency of v plus the valency of u minus 2.

Alternatively, the collapse of such an edge $\overset{u}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{v}{\bullet}$ in T_1 may be thought of as a map $f : T_1 \to T_2$ sending $\overset{u}{\bullet}\!\!-\!\!\!-\!\!\!-\!\!\overset{v}{\bullet}$ to $\overset{v}{\bullet}$ and acting as the identity on the rest of T_1. Collapses of multiple edges can be achieved by composing these maps.

Recall that we have already decided to limit ourselves to graphs with $n-1$, n, or $n+1$ edges. So we will restrict ourselves further to collapses of graphs of groups of the form



, which we will 'abbreviate' as



(where the '$n-4$' means we have suppressed $n-4$ leaves). We will use this method of abbreviation on a frequent basis. We will often refer to the blue-ringed vertex (with valency dependent on n) as the 'basepoint' of the graph.

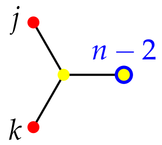
For $n = 5$ this is exactly as we have described, and results in taking the barycentric spine of the 3-skeleton of Outer Space. This graph shape provides a natural way to generalise to $n > 5$, without having to worry about the varying maximal graphs. Note that this means that for $n > 5$ there will be graphs of groups representing simplices of the 'correct' dimension (i.e. $n-1$, n, or $n+1$) in Outer Space which we do not include in our chosen complex.

Our complex $\mathcal{C}_n$ will be the geometric realisation of the poset whose elements are the graphs of groups we have selected above, where the order is given by collapsing. We formalise this in the following subsection.

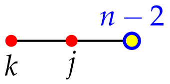
2.2 Points in the Complex

Table 1.1 summarises the graph shapes we will encounter, as well as a naming convention, the number we expect to see in a fundamental domain of the subcomplex we choose, and associated colours which are useful in drawing diagrams (though can largely be ignored).

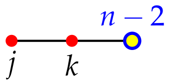
In general, subscripts separated by a comma are ordered, whereas subscripts not separated by a comma are not ordered. So ρ_{jk} and ρ_{kj} both refer to the tree



whereas $\beta_{j,k}$ and $\beta_{k,j}$ refer to distinct trees,



and



, respectively.

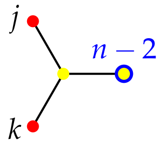
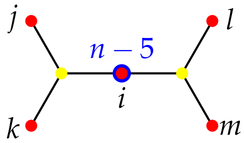
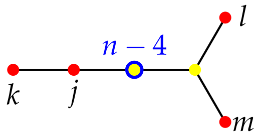
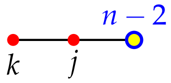
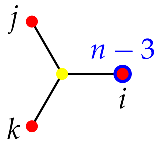
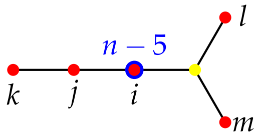
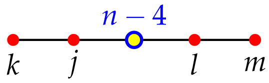
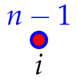
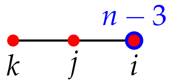
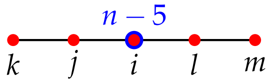
Tree	Name	No. per Domain	Colour
	ρ_{jk}	$\frac{n(n-1)}{2}$	●
	$\sigma_{i,jk,lm}$	$\frac{n!}{8\times(n-5)!}$	●
	$\tau_{j,k,lm}$	$\frac{n!}{2\times(n-4)!}$	●
	α	1	●
	$\beta_{j,k}$	$n(n-1)$	●
	$\gamma_{i,jk}$	$\frac{n!}{2\times(n-3)!}$	●
	$\delta_{i,j,k,lm}$	$\frac{n!}{2\times(n-5)!}$	●
	$\varepsilon_{j,k,l,m}$	$\frac{n!}{2\times(n-4)!}$	●
	A_i	n	●
	$B_{i,j,k}$	$\frac{n!}{(n-3)!}$	●
	$C_{i,j,k,l,m}$	$\frac{n!}{2\times(n-5)!}$	●

TABLE 1.1: Points in the Subcomplex for $n \geqslant 5$

There is some additional symmetry from our trees, so we also have that $\sigma_{i,jk,lm} = \sigma_{i,lm,jk}$, $\varepsilon_{j,k,l,m} = \varepsilon_{l,m,j,k}$, and $C_{i,j,k,l,m} = C_{i,l,m,j,k}$. It is always assumed that, for example, $\{i, j, k, l, v_1, \ldots, v_{n-4}\} = \{1, \ldots, n\}$ as sets.

Recall from Definition 2.1.1 that an $\mathfrak{S}$-labelling is an assignement of vertex groups $H_1, \ldots, H_n$ to a tree T so that $\pi_1(\mathbf{T}) \cong G$ which respects the splitting $\mathfrak{S}$ of G. For trees in Table 1.1, vertex groups are only assigned to named (red) vertices.

Definition 2.2.1. Let $H = (H_{v_1}, \dots, H_{v_n})$ and $H' = (H'_{v_1}, \dots, H'_{v_n})$ be two $\mathfrak{S}$-labellings of a tree T from Table 1.1 with $\{v_1, \dots, v_n\} \subseteq V(T)$. Then H and H' are *equivalent* as labellings (with respect to T) if for each $v \in V(T)$ (including trivial vertices) there exists $g_v \in G$ and (if v is not a trivial vertex) $\varphi_v \in \mathrm{Aut}(H_v)$ so that $H'_v = g_v(H_v)\varphi_v g_v^{-1}$, and moreover, for any edge $e \in E(T)$ we have $g_{o(e)}^{-1} g_{t(e)} \in H_{o(e)}$ (where $o(e)$ is the endpoint of e closest in T to the 'basepoint', and $t(e)$ is the further endpoint). If $o(e)$ does not have a vertex group assigned (i.e. $o(e)$ is a trivial vertex) then $g_{t(e)} = g_{o(e)}$.

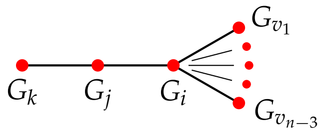
Two graphs of groups $\mathbf{T_1}$ and $\mathbf{T_2}$ are *equivalent* if they each have underlying graph isomorphic to some graph T, and their labellings are equivalent (with respect to T). We denote this equivalence by $\mathbf{T_1} \simeq \mathbf{T_2}$.

Considering the fundamental group of a labelled tree $\mathbf{T} = (T, H)$ to be $\mathop{\ast}\limits_{i=1}^{n} H_{v_i}$, this equivalence induces an isomorphism $H_{v_1} * \cdots * H_{v_n} \to H'_{v_1} * \cdots * H'_{v_n}$. Some basic manipulation of notation shows that this notion of equivalence corresponds to taking isomorphism classes of graphs of groups described by Bass [3, Section 2].

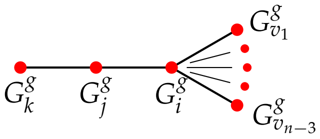
When considering $\mathcal{C}_n$, we assume a given splitting $\mathfrak{S}$ of our group G, and may simply refer to 'labellings' of trees.

Observation 2.2.2. If $\mathbf{T_1} \simeq \mathbf{T_2}$ are equivalent graphs of groups and f is a collapsing map of the underlying graph T, then $f(\mathbf{T_1}) \simeq f(\mathbf{T_1})$ are also equivalent.

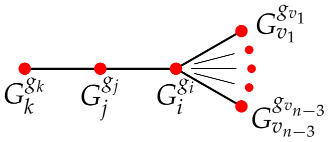
Example 2.2.3. Consider the labelled graph of groups $T :=$



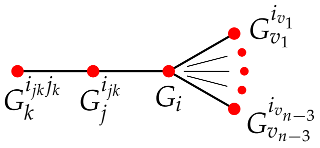
. The following labelled graphs of groups are all equivalent to T:

1. 

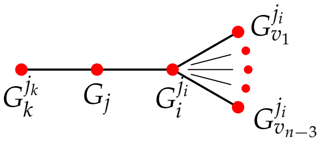
where $g \in G$ — since $gg^{-1} = 1 \in G_v$ for any v.

2. 

where each $g_v \in G_v$ — since for each v, $G_v \mapsto g_v G_v g_v^{-1}$ is an element of $\mathrm{Aut}(G_v)$.

3. 

where $i_{v_1}, \dots, i_{v_{n-3}}, i_{jk} \in G_i$ and $j_k \in G_j$ — since $i \in G_i \Rightarrow 1^{-1}i \in G_i$ and ${i_{jk}}^{-1}(i_{jk}j_k) \in G_j$.

4. 

where $j_k, j_i \in G_j$ — this is achieved by combining 1. (with $g = j_i$), 2. (conjugating $G_j^{j_i}$ by $j_i^{-1} \in G_j$), and 3. (conjugating $G_k^{j_i}$ by $j_k j_i^{-1} \in G_j$) above.

In general, elements in the equivalence class of T all have the form shown in Figure 1.1, where $g \in G$, $g_v \in G_v$ for each v, $i_{v_1}, \ldots, i_{v_{n-3}}, i_{jk} \in G_i$, and $j_k \in G_j$.

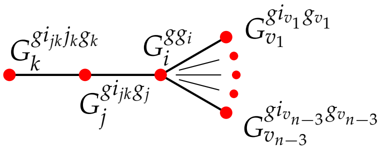
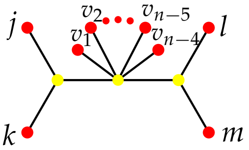


FIGURE 1.1: Equivalence Class of $\mathfrak{G}$-Labellings of T

We are now ready to define our complex.

Definition 2.2.4 (The Complex $\mathcal{C}_n$)**.** We build a (2-dimensional simplicial) complex called $\mathcal{C}_n$ as follows:

- Take one 0-simplex for each equivalence class of $\mathfrak{G}$-labellings of each tree in Table 1.1 (equivalently, take one 0-simplex for each equivalence class of $\mathfrak{G}$-labellings of each tree which is achieved by collapsing at least one edge of one of the trees 

for each of the $\frac{1}{2}\binom{n}{2}\binom{n-2}{2}$ unordered pairs of subsets $\{j, k\}$ and $\{l, m\}$ of $\{1, \ldots, n\}$).
- Given 0-simplices $[T_1]$ and $[T_2]$, insert a 1-simplex joining $[T_1]$ and $[T_2]$ if and only if some representative T_2 of $[T_2]$ is a collapse of some representative T_1 of $[T_1]$ or vice versa.
- Insert a 2-simplex wherever there is a 3-clique $[T_1]$—$[T_2]$—$[T_3]$—$[T_1]$ in the 1-skeleton.

We will often refer to simplices of $\mathcal{C}_n$ as cells. We will use these terms interchangeably. Additionally, we will sometimes refer to 0-cells as 'vertices', 1-cells as 'edges', and 2-cells as 'faces'.

Note that $\mathcal{C}_n$ is the barycentric spine of the subspace of Outer Space obtained by restricting to only simplices representing the above graph shapes. As such, we will sometimes refer to it as 'the/our complex', or 'the/our subcomplex'.

2.3 The Action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{C}_n$ and its Fundamental Domain $\mathcal{D}_n$

By considering the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on our complex $\mathcal{C}_n$, there is a natural idea of a quotient of $\mathcal{C}_n$ (two points are equivalent if they are in the same $\mathrm{Out}_{\mathfrak{S}}(G)$-orbit). We can then pick a 'fundamental domain' $\mathcal{D}_n$ for the action by choosing a lift of this quotient in $\mathcal{C}_n$.

Recall from Section 2.4 that $\mathrm{Out}_{\mathfrak{S}}(G)$ acts on the Spine of Outer Space. We restate this action in terms of graphs of groups (rather than G-trees), and verify that this is indeed an action on $\mathcal{C}_n$ when restricted to our chosen graphs. We begin by stating the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on the 0-skeleton $\mathcal{C}_n^{(0)}$ of $\mathcal{C}_n$, and then show this extends to an action on the full complex $\mathcal{C}_n$.

Definition 2.3.1 (Action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{C}_n^{(0)}$)**.** Let $[\psi] \in \mathrm{Out}_{\mathfrak{S}}(G)$ have representative $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ and let T be a point in $\mathcal{C}_n^{(0)}$ with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$. Then $T \cdot [\psi]$ is a graph of groups with the same underlying graph as T and labelling $((H_1)\psi, \ldots, (H_n)\psi)$ (where $(H_i)\psi$ is given by the usual action of $\mathrm{Aut}(G)$ on G, noting that $H_i \leqslant G$).

Remark. Given $[\psi_1] = [\psi_2] \in \mathrm{Out}_{\mathfrak{S}}(G)$, there is some $\iota_g : x \mapsto gxg^{-1} \in \mathrm{Inn}(G)$ so that $\psi_2 = \psi_1\iota$. Then for any point $T \in \mathcal{C}_n^{(0)}$ with labelling $(H_1, \ldots, H_n)$, we have $(H_i)\psi_2 = ((H_i)\psi_1)^g$. As noted in 1 of Example 2.2.3, $(((H_1)\psi_1)^g, \ldots, ((H_n)\psi_1)^g)$ and $((H_1)\psi_1, \ldots, (H_n\psi_1))$ are equivalent as labellings. So we really do have that $T \cdot [\psi_1] = T \cdot [\psi_2]$ — that is, the action here is well-defined. As such, we will often write $T \cdot \psi$ (or even $T\psi$) for $T \cdot [\psi]$.

Lemma 2.3.2. *Let $S, T \in \mathcal{C}_n^{(0)}$ such that S is a collapse of T, and let $f : T \to S$ be the collapsing map. Let $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$. Then $f(T \cdot \psi) = f(T) \cdot \psi$.*

Proof. Suppose T as a graph has labelling $(H_1, \ldots, H_n)$. Recall from Definition 2.1.1 that permitted collapses do not alter vertex groups in any way. Thus $(H_1, \ldots, H_n)$ must also be a labelling for S. Now $T \cdot \psi$ is a graph of groups with the same underlying graph as T, and labelling $((H_1)\psi, \ldots, (H_n)\psi)$. Similarly, $S \cdot \psi$ has the same underlying graph as S, with labelling $((H_1)\psi, \ldots, (H_n)\psi)$. Since $T \cdot \psi$ has the same underlying graph as T, applying f to $T \cdot \psi$ yields a graph of groups whose underlying graph is the same as that of S, and has $((H_1)\psi, \ldots, (H_n)\psi)$ as a labelling. But this exactly describes the graph of groups $S \cdot \psi$. That is, $f(T \cdot \psi) = S \cdot \psi = f(T) \cdot \psi$. □

Since $\mathcal{C}_n$ is a simplicial complex, then any cell is uniquely determined by its vertices (0-cells). We will thus denote a cell by $[T_0, \ldots, T_k]$ where $T_0, \ldots, T_k$ are its vertices. Note that for us we will only ever have $k = 1$ or $k = 2$ (or $k = 0$).

Proposition 2.3.3. *Let $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$. If $[T_0, \ldots, T_k]$ is a cell in $\mathcal{C}_n$, then so is $[T_0 \cdot \psi, \ldots, T_k \cdot \psi]$.*

Proof. This is true by definition of the action for $k = 0$.

Let $[T_0, T_1]$ be an edge in $\mathcal{C}_n$. Then T_0 and T_1 are graphs of groups with T_1 a collapse of T_0 — say $f : T_0 \to T_1$ is the collapsing map. We know that $T_0 \cdot \psi$ is a point in $\mathcal{C}_n^{(0)}$, and since it has the same underlying graph as T_0, then so is $f(T_0 \cdot \psi)$. So we have an edge $[T_0 \cdot \psi, f(T_0 \cdot \psi)] \in \mathcal{C}_n$. But by Lemma 2.3.2, $f(T_0 \cdot \psi) = f(T_0) \cdot \psi = T_1 \cdot \psi$. So if $[T_0, T_1]$ is a cell in $\mathcal{C}_n$, then so is $[T_0 \cdot \psi, T_1 \cdot \psi]$.

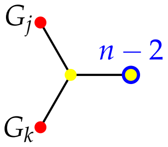
Now suppose $[T_0, T_1, T_2]$ is a 2-cell in $\mathcal{C}_n$. Then we must have a 3-clique $[T_0]$—$[T_1]$—$[T_2]$—$[T_0]$, so $[T_0, T_1]$, $[T_1, T_2]$, and $[T_0, T_2]$ are 1-cells in $\mathcal{C}_n$. Then $[T_0 \cdot \psi, T_1 \cdot \psi]$, $[T_1 \cdot \psi, T_2 \cdot \psi]$, and $[T_0 \cdot \psi, T_2 \cdot \psi]$ are 1-cells in $\mathcal{C}_n$ forming a 3-clique, hence by Definition 2.2.4 we must have a 2-cell $[T_0 \cdot \psi, T_1 \cdot \psi, T_2 \cdot \psi]$. □

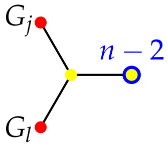
Definition 2.3.4 (Action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{C}_n$)**.** The action of an element $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ on a k-cell $[T_0, \ldots, T_k]$ of $\mathcal{C}_n$ is defined to be:

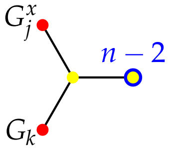
$$[T_0, \ldots, T_k] \cdot \psi := [T_0 \cdot \psi, \ldots, T_k \cdot \psi]$$

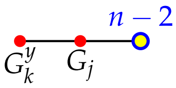
We now construct a fundamental domain for this action. The quotient space obtained from the action has one cell for each orbit of cells in $\mathcal{C}_n$. The obvious choice to make here is to take the lift to be the subcomplex supported by vertices which are all the graphs of groups (as listed in Table 1.1) whose vertex groups are precisely the factor groups $G_1, \ldots, G_n$. This is formalised below:

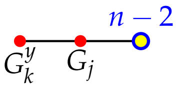
Definition 2.3.5 (Construction of $\mathcal{D}_n$)**.** We take the 0-skeleton $\mathcal{D}_n^{(0)}$ of $\mathcal{D}_n$ to be the set of graphs of groups T whose underlying graph is a tree from Table 1.1 so that, up to permuting the indices, T has a labelling $(G_1, \ldots, G_n)$. We now define $\mathcal{D}_n$ to be the subcomplex of $\mathcal{C}_n$ made up of all cells whose vertices are in $\mathcal{D}_n^{(0)}$.

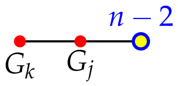
Example 2.3.6. Note that in our selection of graphs of groups, we still allow permutation of the vertex labels, just not conjugation. So 

and 

are both in $\mathcal{D}_n^{(0)}$, while 

is not (for $x \notin G_j$, i.e. $G_j \neq G_j^x$ as sets). Note however that (for $y \in G_j$) 

is in $\mathcal{D}_n^{(0)}$, since 

is equivalent to 

under Definition 2.1.2.

Notation 2.3.7. Given a vertex T in $\mathcal{C}_n$, we denote by $\mathrm{Stab}(T)$ the $\mathrm{Out}_{\mathfrak{S}}(G)$-stabiliser of T, that is, the set $\{\psi \in \mathrm{Out}_{\mathfrak{S}}(G) | T \simeq T \cdot \psi\}$, where $\simeq$ is the equivalence described in Definition 2.1.2. We will often abuse notation and write $T = S$ for $T \simeq S$.

Lemma 2.3.8. *Let T be a vertex in $\mathcal{C}_n$ (so T is a graph of groups) and let S be achieved by collapsing edges of T. Then* $\mathrm{Stab}(T) \subseteq \mathrm{Stab}(S)$.

Proof. Let $f : T \to S$ be the collapsing map, and let $\psi \in \mathrm{Stab}(T) \leqslant \mathrm{Out}_{\mathfrak{S}}(G)$. By Lemma 2.3.2, $S \cdot \psi = f(T) \cdot \psi = f(T \cdot \psi)$. Since $\psi \in \mathrm{Stab}(T)$ then $T \cdot \psi = T$, hence $S \cdot \psi = f(T) = S$. That is, $\psi \in \mathrm{Stab}(S)$. □

Proposition 2.3.9. *The subcomplex $\mathcal{D}_n$ of $\mathcal{C}_n$ described above is indeed a fundamental domain for the action of* $\mathrm{Out}_{\mathfrak{S}}(G)$.

Proof. We need to show that every orbit of cells in $\mathcal{C}_n$ is represented in $\mathcal{D}_n$. That is, if $C \in \mathcal{C}_n$ is a k-cell of $\mathcal{C}_n$ (for $k \in \{0, 1, 2\}$), then there is some $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ so that $C \cdot \psi^{-1} \in \mathcal{D}_n$.

Let $(T, (H_1, \ldots, H_n))$ be a point in $\mathcal{C}_n^{(0)}$. Since $H_1 * \cdots * H_n$ is an $\mathfrak{S}$ free factor splitting of $G_1 * \cdots * G_n$, then by Lemma 1.1.8, there exists $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ so that for each i, $(G_i)\psi = H_i$. Then $(T, (H_1, \ldots, H_n)) \cdot [\psi^{-1}] = (T, (G_1, \ldots, G_n)) \in \mathcal{D}_n$, with $[\psi^{-1}] \in \mathrm{Out}_{\mathfrak{S}}(G)$ as required.

Now let $[\mathbf{T}, \mathbf{S}]$ be an edge in $\mathcal{C}_n$ (so $\mathbf{S}$ is a collapse of $\mathbf{T}$), and choose $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ so that $\mathbf{T} \cdot \psi^{-1} \in \mathcal{D}_n$. Then $(G_1, \ldots, G_n)$ is an $\mathfrak{S}$-labelling for $\mathbf{T} \cdot \psi^{-1}$, and by Lemma 2.3.2, $\mathbf{S} \cdot \psi$ is a collapse of $\mathbf{T} \cdot \psi^{-1}$ and hence $(G_1, \ldots, G_n)$ is also an $\mathfrak{S}$-labelling for $\mathbf{S} \cdot \psi^{-1}$. That is, $[\mathbf{T} \cdot \psi^{-1}, \mathbf{S} \cdot \psi^{-1}]$ is an edge in $\mathcal{D}_n$.

Similarly, if $[\mathbf{T_0}, \mathbf{T_1}, \mathbf{T_2}]$ is a face in $\mathcal{C}_n$, then $\mathbf{T_2}$ is a collapse of $\mathbf{T_1}$, which in turn is a collapse of $\mathbf{T_0}$. Choosing $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ with $\mathbf{T_0} \cdot \psi^{-1} \in \mathcal{D}_n$, the above argument then yields that $[\mathbf{T_0} \cdot \psi^{-1}, \mathbf{T_1} \cdot \psi^{-1}, \mathbf{T_2} \cdot \psi^{-1}]$ is a face in $\mathcal{D}_n$. □

Proposition 2.3.10. *The fundamental domain $\mathcal{D}_n$ described above is* strict. *That is, it contains precisely one representative of each vertex, edge, and face (2-cell) orbit.*

Proof. First, note that for two vertices to share an $\mathrm{Out}_{\mathfrak{S}}(G)$-orbit, they must have the same underlying graph structure. Moreover, since our automorphisms are pure symmetric (i.e. do not permute factor groups), they must have the same indexing of vertices. Since our fundamental domain was chosen to allow only one labelling for each distinct graph structure, this precisely means that each vertex of $\mathcal{D}_n$ is in a distinct orbit.

Now suppose we have two faces in the fundamental domain, $[T_0, T_1, T_2]$ and $[S_0, S_1, S_2]$, which are in the same orbit. Then their vertices are also in the same respective orbits (i.e. T_i and S_i share an orbit for each i). Since our fundamental domain contains only one representative of each vertex orbit, we must have that $T_i = S_i$ for each $i = 1, 2, 3$. But when we constructed $\mathcal{C}_n$, we inserted only one 2-cell for each 3-clique. That is, a face is uniquely determined by its vertices, so $[T_0, T_1, T_2] = [S_0, S_1, S_2]$.

The same argument applies to edges (cells with the form $[T_0, T_1]$). Hence no two cells of our fundamental domain are in the same orbit, that is, we have a strict fundamental domain. $\square$

2.4 Stabilisers of Vertices in $\mathcal{D}_n$

To move through our complex $\mathcal{C}_n$, we consider 'collapse–expansion' paths, since two vertices (graphs of groups) are adjacent in $\mathcal{C}_n$ if and only if one is a collapse of the other. If $T_1 — T_2 — T_3$ is a path in $\mathcal{C}_n$ such that T_2 is a collapse of both T_1 and T_3, and T_1 and T_3 have the same underlying graph structure, then we will have that $T_3 = T_1 \cdot \psi$ for some $\psi \in \mathrm{Stab}(T_2)$. Thus understanding vertex stabilisers is key to understanding adjacency in $\mathcal{C}_n$. We will also need to understand vertex stabilisers in order to apply Brown's Theorem (Theorem 1.2.3).

Recall that given a point $T = (T, (H_1, \ldots, H_n)) \in \mathcal{C}_n^{(0)}$, we have that $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ is in the stabiliser $\mathrm{Stab}(T)$ of T if and only if $T \cdot \psi = T$, that is, $(H_1, \ldots, H_n)$ and $((H_1)\psi, \ldots, (H_n)\psi)$ are equivalent as labellings of T. Recall from Definition 2.1.2 that this means for each $i = 1, \ldots, n$ there exists $g_i \in G$ and $\varphi_i \in \mathrm{Aut}(H_i)$ so that $(H_i)\psi = \varphi_i(H_i)^{g_i}$, and moreover, for every edge $\overset{u}{\bullet}\!\longrightarrow\!\overset{v}{\bullet}$ of T we have $g_u^{-1} g_v \in H_u$.

We will only compute stabilisers of vertices in $\mathcal{D}_n$. However, if $T \cdot \chi \in \mathcal{C}_n$ (with $T \in \mathcal{D}_n$ and $\chi \in \mathrm{Out}_{\mathfrak{G}}(G)$), then $\mathrm{Stab}(T \cdot \chi) = \chi^{-1} \mathrm{Stab}(T) \chi = \mathrm{Stab}(T)^{\chi^{-1}}$. As such, we will assume for now that any graph of groups T has $(G_1, \ldots, G_n)$ as a labelling.

We will present several viewpoints on the stabiliser of a vertex, from Guirardel–Levitt [13] and Bass–Jiang [4], respectively. Note that Levitt [15] shows how these results are equivalent, and demonstrates how to translate between them.

The Guirardel–Levitt Approach

Recall from Definition 1.2.1 that $\Phi \leqslant \mathrm{Out}_{\mathfrak{G}}(G)$ is the group of factor automorphisms of $G_1 * \cdots * G_n$, with $\Phi = \prod_{i=1}^{n} \mathrm{Aut}(G_i)$.

Given a vertex v_i of a point (graph of groups) $T \in \mathcal{C}_n$, with vertex group G_{v_i} (assuming $G_{v_i} \neq \{1\}$, that is, v_i is not a trivial vertex), let μ_i be the valency of v_i in T.

In [13, Section 5], Guirardel and Levitt give the stabiliser of T in $\mathrm{Out}_{\mathfrak{G}}(G)$ as being isomorphic to:

$$\prod_{i=1}^{n}\left(G_i^{\mu_i-1} \rtimes \mathrm{Aut}(G_i)\right) = \left(\prod_{i=1}^{n} G_i^{\mu_i-1}\right) \rtimes \Phi$$

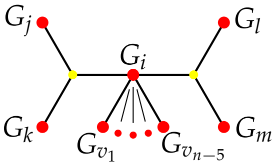
where the semidirect product relation is given by the natural action of Φ on each G_i.

Using this, we recover Table 1.2 showing the stabilisers (up to isomorphism) of points in $\mathcal{D}_n$. Recall that the graph structures of these points are shown in Table 1.1.

Vertex	Stabiliser
ρ_{jk}	Φ
$\sigma_{i,jk,lm}$	$G_i^{n-4} \rtimes \Phi$
$\tau_{j,k,lm}$	$G_j \rtimes \Phi$
α	Φ
$\beta_{j,k}$	$G_j \rtimes \Phi$
$\gamma_{i,jk}$	$G_i^{n-3} \rtimes \Phi$
$\delta_{i,j,k,lm}$	$(G_i^{n-4} \times G_j) \rtimes \Phi$
$\varepsilon_{j,k,l,m}$	$(G_j \times G_l) \rtimes \Phi$
A_i	$G_i^{n-2} \rtimes \Phi$
$B_{i,j,k}$	$(G_i^{n-3} \times G_j) \rtimes \Phi$
$C_{i,j,k,l,m}$	$(G_i^{n-4} \times G_j \times G_l) \rtimes \Phi$

TABLE 1.2: Vertex Stabilisers (up to isomorphism) using Guirardel–Levitt

This point of view corresponds to fixing a particular edge of a graph of groups T, and then twisting the remaining edges (outwards from the fixed edge). We demonstrate this with an example:



Example 2.4.1. Consider the graph of groups σ: (the graph $\sigma_{i,jk,lm}$ with labelling $\hat{G} := (G_1, \ldots, G_n)$). We will compute the stabiliser of σ by 'fixing' an edge in the graph of groups.

Let $\hat{H} = \left(G_1^{h_1}, \ldots, G_n^{h_n}\right)$ be an arbitrary labelling in the equivalence class of $\hat{G}$ of labellings of σ. Note that by Definition 2.1.2 we must also have elements $h_{jk}, h_{lm} \in G$ corresponding to the two trivial vertices of σ, and that $h_j = h_k = h_{jk}$ and $h_l = h_m = h_{lm}$. As noted in 1 of Example 2.2.3, inner automorphisms of G preserve equivalence classes of labellings. Thus the labelling $\hat{H}'$ achieved by replacing each $G_a^{h_a}$ by $(G_a^{h_a})h_{lm}^{-1} = G_a^{h_{lm}^{-1}h_a}$ is equivalent to $\hat{H}$. Note that in this labelling, the conjugator of G_i is $h_{lm}^{-1}h_i = (h_i^{-1}h_{lm})^{-1} \in G_i$ by Definition 2.1.2. Since this is an (inner) factor automorphism of G_i, then the labelling $\hat{H}''$ achieved by replacing $G_i^{h_i h_{lm}^{-1}}$ in $\hat{H}'$ with simply G_i is also equivalent to $\hat{H}$. We have now essentially 'fixed' the edge i—lm (i.e. the edge which separates G_l and G_m from G_i and all the other vertex groups) in σ.

Note that for any a we have $h_{lm}^{-1}h_a = (h_i^{-1}h_{lm})^{-1}(h_i^{-1}h_a) \in G_i$. So aside from factor automorphisms (i.e. replacing G_a with $(G_a)\varphi$ for $\varphi \in \Phi$), the only freedom we have left is to 'twist' along the remaining edges incident to i (the vertex in σ whose vertex group is G_i); that is, given each remaining edge e incident to i, to conjugate all vertex groups separated from i by e by an element of G_i. Note that for $g_i \in G_i$ and $\varphi \in \Phi$, we have $(G_a^{g_i})\varphi = (G_a\varphi)^{(g_i)\varphi} = G_a^{(g_i)\varphi}$. We will let G_{i_v} denote the group of Whitehead automorphisms which conjugate the vertex group G_v by elements of G_i (for $v = v_1, \ldots, v_{n-5}$), and similarly denote by $G_{i_{jk}}$ the group of Whitehead automorphisms which conjugate the vertex groups G_j and G_k simultaneously by elements of G_i. Note that $G_{i_{jk}} \cong G_{i_v} \cong G_i$ (for $v = v_1, \ldots, v_{n-5}$). Since twists along edges from i happen independently of each other, we then have that $\mathrm{Stab}(\sigma) = (G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi \cong G_i^{n-4} \rtimes \Phi$, as listed in Table 1.2.

Note that by choosing an edge to 'fix' in a graph of groups T and indexing the remaining edges as in the above example, we can similarly expand all the stabilisers listed in Table 1.2. While this works well for some graphs, it does lead to a lack of symmetry, and in graphs such as A_i, such a choice can feel entirely arbitrary.

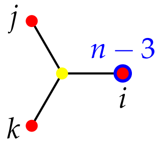
Even in the above example, we could have chosen to fix the edge from G_i leading to G_j and G_k (or even an edge G_i — G_v) instead of the edge from G_i leading to G_l and G_m . This means we must have that $\mathrm{Stab}(\sigma) = (G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi = (G_{i_{lm}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi \left(= (G_{i_{jk}} \times G_{i_{lm}} \times G_{i_{v_2}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi\right)$ as subgroups of $\mathrm{Out}_{\mathfrak{S}}(G)$.

To see why this holds, let $g_i \in G_i$, let ι_{g_i} be the element of $\mathrm{Inn}(G)$ which conjugates every element of G by g_i, and for groups $G_{w_1}, \ldots, G_{w_a}$ ($\{w_1, \ldots, w_a\} \subseteq \{1, \ldots, n\}$) let $(\{G_{w_1}, \ldots, G_{w_k}\}, g_i)$ be the Whitehead automorphism which conjugates each G_w by g_i (for $w = w_1, \ldots, w_a$). Then for $(\{G_l, G_m\}, g_i)$ an arbitrary element of $G_{i_{lm}}$, we have that $(\{G_l, G_m\}, g_i) = \left(\{G_i\}, g_i^{-1}\right)\left(\{G_j, G_k, G_{v_1}, \ldots, G_{v_{n-5}}\}, g_i^{-1}\right)\iota_{g_i}$, where $\left(\{G_i\}, g_i^{-1}\right) \in \mathrm{Inn}(G_i)$
$\leqslant \mathrm{Aut}(G_i) \leqslant \Phi$ and $\left(\{G_j, G_k, G_{v_1}, \ldots, G_{v_{n-5}}\}, g_i^{-1}\right) \in G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}$. That is, $G_{i_{lm}} \leqslant (G_{i_{jk}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi$. One can similarly show that $G_{i_{jk}} \leqslant (G_{i_{lm}} \times G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-5}}}) \rtimes \Phi$, as well as the inclusions required for the third claimed equality.

Thus while correct, and simple to write down, this method of determining stabilisers can obscure subgroups and other structure. To remedy this, we may consider initially fixing just a vertex, rather than an edge, in our graph of groups, or more generally, not 'fixing' anything at all.

The Bass–Jiang Approach

Given a vertex v_i of a point (graph of groups) $T \in \mathcal{C}_n^{(0)}$, with vertex group G_i (assuming $G_i \neq \{1\}$, that is, v_i is not a trivial vertex), let $E(v_i)$ be the set of edges of T with v_i

as an endpoint. We will index these edges by the vertices they separate from v_i (so for example in 

, the edges incident to the vertex i are indexed by $v_1, \dots, v_{n-3}$ and (jk)).

Bass and Jiang [4, Theorem 8.1] give a filtration explicitly describing the $\mathrm{Out}(G)$ stabiliser of a graph of groups. Since we are restricting to pure symmetric (outer) automorphisms, we have trivial edge stabilisers in our graphs of groups and no graph automorphisms (as our graphs of groups are trees, and we do not permit permutation of the vertex groups). So this filtration simplifies to a short exact sequence:

$$1 \longrightarrow \prod_{i=1}^{n} \left(\left(\prod_{e \in E(v_i)} G_{i_e} \right) \Big/ Z(G_i) \right) \longrightarrow \mathrm{Stab}(T) \longrightarrow \prod_{i=1}^{n} \mathrm{Out}(G_i) \longrightarrow 1$$

where $f_{i_e} : G_i \to G_{i_e}$ is an isomorphism and G_{i_e} is the group of Whitehead automorphisms which conjugate the vertex groups of vertices separated from v_i by e by elements of the vertex group G_i, and $Z(G_i)$ is the centre of G_i, with diagonal embedding. For brevity, given $T \in \mathcal{D}_n^{(0)}$ we will write M_T for the $\prod_{i=1}^{n} \left(\left(\prod_{e \in E(v_i)} G_{i_e} \right) \Big/ Z(G_i) \right)$ term of the above short exact sequence. This term corresponds to 'twisting' along each edge incident to each vertex in T.

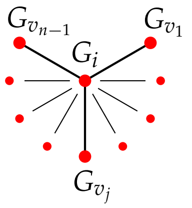
Writing v for the vertex group G_v, and v_e for the automorphism group G_{v_e} (with the indexing described above), we recover Table 1.3, showing the M_T term of the Bass–Jiang short exact sequence for each tree T of Table 1.1.

Observe that for any $a = 2, \dots, n-1$ we can embed the group $i_{v_1 \dots v_a}$ diagonally into the direct product $i_{v_1} \times \cdots \times i_{v_a}$. We write $i_{v_1 \dots v_a} \mathrel{\underline{\Diamond}} i_{v_1} \times \cdots \times i_{v_a}$ to indicate that we consider $i_{v_1 \dots v_a}$ to be the diagonal subgroup of $i_{v_1} \times \cdots \times i_{v_a}$. If $g_i \in G_i$ then $f_{i_{v_1 \dots v_{n-1}}}(g_i) \in i_{v_1 \dots v_{n-1}}$ and $\iota_{g_i^{-1}}$ is the inner automorphism which conjugates all elements of G by g_i^{-1}. Now $f_{i_{v_1 \dots v_{n-1}}}(g_i)\iota_{g_i^{-1}}$ conjugates all elements of G_i by g_i^{-1} and fixes all other elements of G. That is, $f_{i_{v_1 \dots v_{n-1}}}(g_i)\iota_{g_i} \in \mathrm{Inn}(G_i)$. So we have that $i_{v_1 \dots v_{n-1}} \mathrm{Inn}(G) = \mathrm{Inn}(G_i)\,\mathrm{Inn}(G)$ as cosets in $\mathrm{Out}_{\mathfrak{S}}(G)$.

More generally, if $A \sqcup B$ partitions $\{1, \dots, n\} - \{i\}$ then $G_{i_A} = G_{i_B}$ in $\mathrm{Out}_{\mathfrak{S}}(G)$.

T	M_T
ρ_{jk}	$\prod_{v} \mathrm{Inn}(v)$
$\sigma_{i,jk,lm}$	$\prod_{v\neq i} \mathrm{Inn}(v) \times (i_{v_1} \times \cdots \times i_{v_{n-5}} \times i_{jk} \times i_{lm}) \big/ Z(i)$
$\tau_{j,k,lm}$	$\prod_{v\neq j} \mathrm{Inn}(v) \times (j_{v_1 \dots v_{n-4} lm} \times j_k) \big/ Z(j)$
α	$\prod_{v} \mathrm{Inn}(v)$
$\beta_{j,k}$	$\prod_{v\neq j} \mathrm{Inn}(v) \times (j_{v_1 \dots v_{n-2}} \times j_k) \big/ Z(j)$
$\gamma_{i,jk}$	$\prod_{v\neq i} \mathrm{Inn}(v) \times (i_{v_1} \times \cdots \times i_{v_{n-3}} \times i_{jk}) \big/ Z(i)$
$\delta_{i,j,k,lm}$	$\prod_{v\neq i,j} \mathrm{Inn}(v) \times (i_{v_1} \times \cdots \times i_{v_{n-5}} \times i_{jk} \times i_{lm}) \big/ Z(i) \times (j_{v_1 \dots v_{n-5} ilm} \times j_k) \big/ Z(j)$
$\varepsilon_{j,k,l,m}$	$\prod_{v\neq j,l} \mathrm{Inn}(v) \times (j_{v_1 \dots v_{n-4} lm} \times j_k) \big/ Z(j) \times (l_{v_1 \dots v_{n-4} jk} \times l_m) \big/ Z(l)$
A_i	$\prod_{v\neq i} \mathrm{Inn}(v) \times (i_{v_1} \times \dots i_{v_{n-1}}) \big/ Z(i)$
$B_{i,j,k}$	$\prod_{v\neq i,j} \mathrm{Inn}(v) \times (i_{v_1} \times \cdots \times i_{v_{n-3}} \times i_{jk}) \big/ Z(i) \times (j_{iv_1 \dots v_{n-3}} \times j_k) \big/ Z(j)$
$C_{i,j,k,l,m}$	$\prod_{v\neq i,j,l} \mathrm{Inn}(v) \times (i_{v_1} \times \cdots \times i_{v_{n-5}} \times i_{jk} \times i_{lm}) \big/ Z(i) \times (j_{iv_1 \dots v_{n-5} lm} \times j_k) \big/ Z(j)$ $\times (l_{iv_1 \dots v_{n-5} jk} \times l_m) \big/ Z(l)$

TABLE 1.3: M_T Terms of Vertices T from Bass–Jiang Short Exact Sequence

Example 2.4.2. Consider the graph of groups A: 

(the graph A_i with labelling $\hat{G} := (G_1, \dots, G_n)$). We explore two ways of determining $\mathrm{Stab}(A)$.

1. We will first deduce $\mathrm{Stab}(A)$ from the Bass–Jiang filtration. Note that this filtration allows us to compute the M_A term of the Bass–Jiang short exact sequence, rather than the stabiliser of A itself.

   We begin by considering 'twisting' from a vertex v_j (with vertex group G_{v_j}) for some $j = 1, \dots, n-1$. Since this is a vertex of valency 1, we conjugate all other vertex groups by the same element g_j of G_{v_j}, to achieve a labelling (up to appropriate reordering) $\left(G_{v_j}, G_i^{g_j}, G_{v_1}^{g_j}, \dots, G_{v_{n-1}}^{g_j}\right)$. However, by applying the inner automorphism $\iota_{g_j^{-1}} \in \mathrm{Inn}(G)$ which conjugates all elements of G by g_j^{-1}, we see that this is equivalent to the labelling $\left(G_{v_j}^{g_j^{-1}}, G_i, G_{v_1}, \dots, G_{v_{n-1}}\right)$, which equates to having applied the inner factor automorphism which conjugates G_{v_j} by an element of itself. Thus twisting from a valency 1 vertex v_j simply yields $\mathrm{Inn}(G_{v_j})$ at this stage of the Bass–Jiang filtration.

We now consider twists from the vertex i with vertex group G_i. This has valency $n-1$, and so we get $n-1$ groups $i_{v_j} \cong G_i$, which equate to conjugating the vertex group G_{v_j} by elements of G_i. Since twisting along edges incident to i is independent of the order in which we twist them, this forms a direct product of groups. Note that if we were to twist all edges by the same element g_i of G_i, this is equivalent (up to an inner automorphism of G) to just conjugating G_i by g_i^{-1}. That is, $\mathrm{Inn}(G_i) = i_{v_1 \dots v_{n-1}} \mathrel{\underline{\Diamond}} i_{v_1} \times \cdots \times i_{v_{n-1}}$, the diagonal subgroup. Moreover, if $g_i \in Z(G_i)$ is central in G_i, then conjugation of G_i by g_i^{-1} is the identity map on G_i, and so twisting along all edges incident to i by g_i is equivalent to the identity automorphism. Thus we must quotient out by the centre of G_i, embedded diagonally into $i_{v_1 \dots v_{n-1}} \mathrel{\underline{\Diamond}} i_{v_1} \times \cdots \times i_{v_{n-1}}$.

Hence we have the short exact sequence:

$$1 \to \prod_{v \neq i} \mathrm{Inn}(v) \times {}^{\left(i_{v_1} \times \dots i_{v_{n-1}}\right)} \big/ {}_{Z(i)} \to \mathrm{Stab}(A) \to \prod_{v \neq i} \mathrm{Out}(v) \times \mathrm{Out}(i) \to 1$$

We deduce from this that $\mathrm{Stab}(A)$ is generated by $\Phi = \prod_{a=1}^{n} \mathrm{Aut}(G_a)$ and groups $G_{i_a} \cong G_i$ for $a \in \{1, \dots, n\} - \{i\}$, so that $[G_{i_a}, G_{i_b}] = 1$ (each element of G_{i_a} commutes with each element of G_{i_b}) for every $a, b \in \{1, \dots, n\} - \{i\}$, and where $G_{i_{v_1 \dots v_{n-1}}} \mathrel{\underline{\Diamond}} G_{i_{v_1}} \times \cdots \times G_{i_{v_{n-1}}}$ is the diagonal subgroup, we have ${}^{G_{i_{v_1 \dots v_{n-1}}}} \big/ {}_{Z(G_i)} = \mathrm{Inn}(G_i)$. We observe that for $a, b \neq i$, $[\mathrm{Aut}(G_a), G_{i_b}] = 1$. However, for $a \neq i$, $(\{G_a\}, g_i) \in G_{i_a}$, and $\varphi_i \in \mathrm{Aut}(G_i)$, we have $(\{G_a\}, g_i)\, \varphi_i = \varphi_i\, (\{G_a\}, \varphi_i(g_i))$. These relations on the given generators are enough to fully determine $\mathrm{Stab}(A)$ as a subgroup of $\mathrm{Out}(G)$.

2. Alternatively, we can calculate $\mathrm{Stab}(A)$ from Definition 2.1.2 by considering equivalent labellings on A and the automorphisms which lead to these. This corresponds to 'twisting outwards from i'. Note that in this method, we implicitly 'fix' the 'basepoint' i. Recall that A has labelling $\hat{G} := (G_1, \dots, G_n)$, and let $\hat{H} := \left(G_1^{h_1}, \dots, G_n^{h_n}\right)$ be an arbitrary labelling in the equivalence class of $\hat{G}$ of labellings of A.

   Observe that as in 1 of Example 2.2.3, we can apply the inner automorphism $\iota_{h_i^{-1}} \in \mathrm{Inn}(G)$ which conjugates each element of G by h_i^{-1}. Thus we obtain the labelling $\hat{H}' := \left(G_i, G_{v_1}^{h_i^{-1} h_{v_1}}, \dots, G_{v_{n-1}}^{h_i^{-1} h_{v_{n-1}}}\right)$ which (up to appropriate reordering) is equivalent to $\hat{H}$.

   By Definition 2.1.2, we have that for each $a \in \{1, \dots, n-1\}$, $h_i^{-1} h_{v_a} \in G_i$. Hence (aside from factor automorphisms) our only freedom in labellings is to conjugate each non-G_i vertex group by an element of G_i, i.e. to 'twist' along each of the edges incident to the vertex i (with vertex group G_i) in A.

Thus $\mathrm{Stab}(A)$ is generated by Φ and by $n-1$ groups $G_{i_a} \cong G_i$. Note that this is exactly as determined above, and the same arguments can be made to determine relations, resulting in the same presentation for $\mathrm{Stab}(A)$.

While the Bass–Jiang approach deals with the removal of symmetry which occurs by making specific choices in the Guirardel–Levitt approach, we lose the ability to concisely write down stabilisers. As such, we do not wish to replace the Guirardel–Levitt approach with this one, but rather enhance it.

Vertex Stabilisers for Common Use

We will now detail presentations for the stabilisers of vertices in $\mathcal{D}_n$ (graphs of groups with structure listed in Table 1.1 and labelling $(G_1, \ldots, G_n)$) which will be useful throughout the paper, but especially in Sections 4.1 and 5.2. For brevity, we write i_j for the group $G_{i_j} \cong G_i$ of Whitehead automorphisms which conjugate the factor group G_j by elements of the factor group G_i. Recall that there is an isomorphism $f_{i_j} : G_i \to i_j$ given by $f_{i_j}(g) = (\{G_j\}, g)$. When the isomorphism is relevant, we may write $f_{i_j}(G_i)$ for the group i_j. We will consider the centre $Z(G_i)$ of G_i to be embedded in $i_{v_1 \ldots v_k} \Diamond i_{v_1} \times \cdots \times i_{v_k}$ via the isomorphisms f_{i_j}.

We divide the vertices of $\mathcal{D}_n$ into three categories, according to which method(s) we will use to compute their stabilisers.

First, are vertices which have graph structure well-suited to the Guirardel–Levitt approach:

Proposition 2.4.3. *As subgroups of* $\mathrm{Out}_{\mathfrak{G}}(G)$ *we have:*

- $\mathrm{Stab}(\rho_{jk}) = \mathrm{Stab}(\alpha) = \Phi$
- $\mathrm{Stab}(\tau_{j,k,lm}) = \mathrm{Stab}(\beta_{j,k}) = j_k \rtimes \Phi$
- $\mathrm{Stab}(\varepsilon_{j,k,l,m}) = (j_k \times l_m) \rtimes \Phi$

where the semidirect relation is given by $\big(\{G_k\}, g_j\big) \circ \varphi = \varphi \circ (\{G_k\}, (g)\varphi)$ *for any* j, k *with* $(\{G_k\}, g) \in j_k$ *and* $\varphi \in \Phi$. *In other words,* $\varphi^{-1} f_{j_k}(g) \varphi = f_{j_k}((g)\varphi)$ *for* $g \in G_j$.

Proof. These are lifted directly from Table 1.2, utilising Example 2.4.1 which follows it to index the groups of automorphisms by the vertices they act on. ☐

Our second category is that of vertices whose graph structures are well-suited to the Bass–Jiang approach. We write $i_{v_1 \dots v_k}$ for the group of automorphims $\{(\{G_{v_1}, \dots, G_{v_k}\}, g) \mid g \in G_i\}$.

Proposition 2.4.4. *As subgroups of* $\mathrm{Out}_{\mathfrak{G}}(G)$ *we have:*

- $\mathrm{Stab}(\sigma_{i,jk,lm})$ *is generated by* ${}^{\left(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}}\right)}\!\big/_{Z(G_i)}$ *and* Φ
- $\mathrm{Stab}(\gamma_{i,jk})$ *is generated by* ${}^{\left(i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-3}}\right)}\!\big/_{Z(G_i)}$ *and* Φ
- $\mathrm{Stab}(A_i)$ *is generated by* ${}^{\left(i_{v_1} \times \cdots \times i_{v_{n-1}}\right)}\!\big/_{Z(G_i)}$ *and* Φ

each subject to the relations $f_{i_{w_1}}(g) \dots f_{i_{w_{n-1}}}(g) = \mathrm{Ad}_{G_i}(g^{-1})$ *(with* $\mathrm{Ad}_{G_i}(g^{-1})$ *as in Notation 1.1.6) and* $\varphi^{-1} f_{i_v}(g) \varphi = f_{i_v}((g)\varphi)$*, where* $\{w_1, \dots, w_{n-1}\} = \{1, \dots, n\} - \{i\}$*,* $v \in \{w_1, \dots, w_{n-1}\}$*, and* $\varphi \in \Phi$*. That is,* ${}^{i_{w_1 \dots w_{n-1}}}\!\big/_{Z(G_i)} = \mathrm{Inn}(G_i)$ *and* $f_{i_v}((G_i)\varphi)^{\varphi} = i_v$*.*

Proof. These are deduced from the M_T terms of the Bass–Jiang short exact sequences listed in Table 1.3. Example 2.4.2 explicitly details how to recover relations for $\mathrm{Stab}(A_i)$, and the others follow similarly. □

Finally, our third category is that of vertices whose graph structures are not well-suited to either approach, and we thus work directly from Definition 2.1.2:

Proposition 2.4.5. *As subgroups of* $\mathrm{Out}_{\mathfrak{G}}(G)$ *we have:*

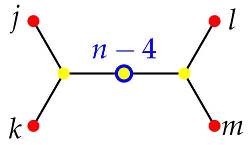
- $\mathrm{Stab}(\delta_{i,j,k,lm})$ *is generated by* j_k*,* ${}^{\left(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}}\right)}\!\big/_{Z(G_i)}$*, and* Φ*, subject to the relations:*
  1. ${}^{i_{jklmv_1 \dots v_{n-5}}}\!\big/_{Z(G_i)} = \mathrm{Inn}(G_i)$
  2. $\left[j_k, {}^{\left(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}}\right)}\!\big/_{Z(G_i)}\right] = 1$
  3. $f_{j_k}((G_j)\varphi)^{\varphi} = j_k$ *and* $f_{i_x}((G_i)\varphi)^{\varphi} = i_x$ *for each* $x \in \{jk, lm, v_1, \dots, v_{n-5}\}$
- $\mathrm{Stab}(B_{i,j,k})$ *is generated by* j_k*,* ${}^{\left(i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-3}}\right)}\!\big/_{Z(G_i)}$*, and* Φ*, subject to the relations:*
  1. ${}^{i_{jkv_1 \dots v_{n-3}}}\!\big/_{Z(G_i)} = \mathrm{Inn}(G_i)$
  2. $\left[j_k, {}^{\left(i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-3}}\right)}\!\big/_{Z(G_i)}\right] = 1$
  3. $f_{j_k}((G_j)\varphi)^{\varphi} = j_k$ *and* $f_{i_x}((G_i)\varphi)^{\varphi} = i_x$ *for each* $x \in \{jk, v_1, \dots, v_{n-3}\}$
- $\mathrm{Stab}(C_{i,j,k,l,m})$ *is generated by* j_k*,* l_m*,* ${}^{\left(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}}\right)}\!\big/_{Z(G_i)}$*, and* Φ*, subject to the relations:*
  1. ${}^{i_{jklmv_1 \dots v_{n-5}}}\!\big/_{Z(G_i)} = \mathrm{Inn}(G_i)$

2. $\left[a_b, {}^{(i_{jk} \times i_{lm} \times i_{v_1} \times \cdots \times i_{v_{n-5}})}\big/_{Z(G_i)}\right] = 1$ *for each of* $a_b = j_k$ *and* $a_b = l_m$
3. $f_{j_k}((G_j)\varphi)^{\varphi} = j_k$, $f_{l_m}((G_l)\varphi)^{\varphi} = l_m$, *and* $f_{i_x}((G_i)\varphi)^{\varphi} = i_x$ *for each* $x \in \{jk, lm, v_1, \ldots, v_{n-5}\}$
4. $[j_k, l_m] = 1$

Proof. We calculate these using Definition 2.1.2. The equivalence class of labellings for $B_{i,j,k}$ is explicitly described in Example 2.2.3. The classes for $\delta_{i,j,k,lm}$ and $C_{i,j,k,l,m}$ follow similarly. From here, deduction of the $\mathrm{Out}_{\mathfrak{S}}(G)$ stabilisers follows in much the same way as in the second part of Example 2.4.2. □

3 Properties of the Fundamental Domain $\mathcal{D}_n$

Before proving any structural statements about the fundamental domain $\mathcal{D}_n$, we provide some illustrations to aid in understanding this subcomplex. We describe some of the substructures found within the 1-skeleton $\mathcal{D}_n^{(1)}$ of the fundamental domain of $\mathcal{C}_n$. These will be particularly useful in determining edge inclusion relations in Theorem 4.1.1, as well as in showing that $\mathcal{D}_n$ is simply connected in Section 3.2.

The subcomplex of the fundamental domain obtained by restricting to collapses of a given graph 

will be referred to as a 'spike' of the fundamental domain.

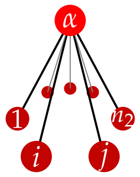


FIGURE 1.2: The α-A–Star

Figure 1.2 shows the α-A–Star. The circle with label 'i' represents the vertex A_i. The circle with label 'n_2' represents the vertex $A_{v_{n-2}}$. Where the 3 small dots at the back are, one should imagine "many", that is, that there are in fact $n-4$ A-vertices there. This structure appears precisely once in the fundamental domain, and is present in every 'spike' (of which there are $\frac{n!}{8(n-4)!}$).

Figure 1.3 shows a ρ–Book (**the** ρ–Book associated to the graph ρ_{jk}). The circle with label '1' represents the vertex $\gamma_{v_1,jk}$. The circle with label 'jk' represents the vertex ρ_{jk}, and the circle with label 'j,k' represents the vertex $\beta_{j,k}$. The orange circle adjacent to both $\beta_{j,k}$ and $\gamma_{v_1,j,k}$ represents $B_{v_1,j,k}$. This structure appears $\frac{n(n-1)}{2}$ times in the fundamental

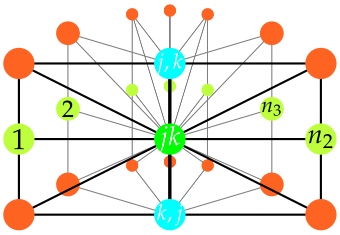


FIGURE 1.3: The ρ–Book

domain. There are two ρ–Books per 'spike', and each ρ–Book appears in $\frac{(n-2)(n-3)}{2}$ spikes. Two distinct ρ–Books appear in only one spike together, and only if they are associated to ρ_{jk} and ρ_{lm} where j, k, l, m are all distinct.

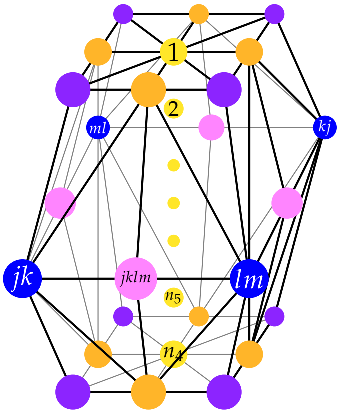


FIGURE 1.4: The τ-ε–Box

Figure 1.4 shows a τ-ε–Box. It appears precisely once in each spike, and is unique to its spike. It has $n-4$ layers, each associated to a σ-vertex. A given layer will be called a σ–Slice. The cycle left when removing all σ–Slices is called a τ-ε–Square. The yellow circle with label 'n_5' represents $\sigma_{v_{n-5},jk,lm}$. The blue circle with label 'jk' represents $\tau_{j,k,lm}$, and the pink circle with label '$jklm$' represents $\varepsilon_{j,k,l,m}$.

The remainder of this section will be spent proving connectivity results (namely path connectivity and simple connectivity) for the fundamental domain $\mathcal{D}_n$, which will be key to proving respective theorems for the complex $\mathcal{C}_n$.

3.1 Connectedness of the Fundamental Domain

We first show that the fundamental domain $\mathcal{D}_n$ is (path) connected. We will do this by finding a path from an arbitrary vertex $T \in \mathcal{D}_n$ to the vertex $\alpha \in \mathcal{D}_n$ (see Table 1.1). Then any two arbitrary vertices of $\mathcal{D}_n$ will be connected via α.

Lemma 3.1.1. *The fundamental domain $\mathcal{D}_n$ of $\mathcal{C}_n$ is path connected.*

Proof. We refer to Table 1.1 for the naming convention of vertices in $\mathcal{D}_n$ (where each vertex group is precisely one of the factor groups $G_1, \ldots, G_n$). Note that α is adjacent to every ρ_{jk} and every A_i in $\mathcal{D}_n$. Every β and γ graph is a collapse of some ρ graph, hence any β or γ vertex in $\mathcal{D}_n$ is path connected to α. Additionally, every B graph is the collapse of some β graph, so B vertices are also connected. Note that $\sigma_{i,jk,lm}$ and $\tau_{j,k,lm}$ collapse to A_i and A_j, respectively (which are both adjacent to α). Any C graph is the collapse of some σ graph, and every δ and ε graph is the collapse of some τ graph. Hence any vertex in $\mathcal{D}_n$ has a path in the fundamental domain to α. Thus the fundamental domain is (path) connected. Explicit paths are listed in Table 1.4. □

α
$\rho_{jk} — \alpha$
$A_i — \alpha$
$\beta_{j,k} — \rho_{jk} — \alpha$
$\gamma_{i,jk} — \rho_{jk} — \alpha$
$B_{i,j,k} — \beta_{j,k} — \rho_{jk} — \alpha$
$\sigma_{i,jk,lm} — A_i — \alpha$
$\tau_{j,k,lm} — A_j — \alpha$
$C_{i,j,k,l,m} — \sigma_{i,jk,lm} — A_i — \alpha$
$\delta_{i,j,k,lm} — \tau_{j,k,lm} — A_j — \alpha$
$\varepsilon_{j,k,l,m} — \tau_{j,k,lm} — A_j — \alpha$

TABLE 1.4: Paths Between Vertices in the Fundamental Domain

Corollary 3.1.2. *Any vertex T in the complex $\mathcal{C}_n$ is connected via an edge path to some α graph.*

Proof. This follows by noting that T sits in at least one copy of the fundamental domain, and that the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ preserves adjacency, so the above argument applies. □

In [12, Section 4], Gilbert gives a summary of Fouxe-Rabinovitch's presentation for $\mathrm{Aut}(G)$ described in [10] and [11]. Restricting to the pure symmetric automorphisms $\mathrm{Aut}_{\mathfrak{S}}(G)$ of a splitting $\mathfrak{S}$ of G, this states that $\mathrm{Aut}_{\mathfrak{S}}(G)$ is generated by factor automorphisms and Whitehead automorphisms only (see Definitions 1.2.1 and 1.3.1). We can use this to give a quick proof that our full complex $\mathcal{C}_n$ is path connected.

Proposition 3.1.3. *Any two α-graphs in the complex $\mathcal{C}_n$ are connected via a path which travels only via α and A graphs.*

Proof. Let α_0 be the α-graph in the fundamental domain $\mathcal{D}_n$ and let $\alpha_0 \cdot \hat{\psi}$ be an arbitrary α-graph in $\mathcal{C}_n$ (with $\hat{\psi} \in \mathrm{Out}_{\mathfrak{S}}(G)$ and $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ a representative for $\hat{\psi}$). By [10] and [11], transcribed in [12, Section 4], we can write ψ as $\psi_0\psi_1 \ldots \psi_m$ for some $m \in \mathbb{N}$ where $\psi_0 \in \Phi$ is a factor automorphism, and for each $1 \leqslant i \leqslant m$, ψ_i is a Whitehead automorphism of the form (S_i, x_i) with $x_i \in G_{j(i)}$ for some $j(i) \in \{1, \ldots, n\}$ and $S_i \subseteq \{G_1, \ldots, G_n\} - \{G_{j(i)}\}$.

By Proposition 2.4.3, $\psi_0 \in \mathrm{Stab}(\alpha_0)$, that is, $\alpha_0 \cdot \psi_0 = \alpha_0$. We now write $\psi_1 \ldots \psi_m = \psi_m(\psi_{m-1}^{\psi_m})(\psi_{m-2}^{\psi_{m-1}\psi_m}) \ldots (\psi_2^{\psi_3 \ldots \psi_m})(\psi_1^{\psi_2 \ldots \psi_m})$. Observe that for each $1 \leqslant i < m$, $\psi_i^{\psi_{i+1} \ldots \psi_m}$ acts on $\alpha_0 \cdot \psi_{i+1} \ldots \psi_m$ to produce the graph $\alpha_0 \cdot \psi_i \ldots \psi_m$ (and ψ_m acts on $\alpha_0 = \alpha_0 \cdot \psi_0$ producing $\alpha_0 \cdot \psi_m$). Moreover, if ψ_i has operating factor $G_{j(i)}$ and $A_{j(i)}$ is the A-graph in $\mathcal{D}_n$ whose central vertex has stabiliser $G_{j(i)}$, then by Proposition 2.4.4, $\psi_i \in \mathrm{Stab}(A_{j(i)})$ and thus $\psi_i^{\psi_{i+1} \ldots \psi_m} \in \mathrm{Stab}(A_{j(i)} \cdot \psi_{i+1} \ldots \psi_m)$ (and $\psi_m \in \mathrm{Stab}(A_{j(m)})$). Thus both the graphs $\alpha_0 \cdot \psi_i \ldots \psi_m$ and $\alpha_0 \cdot \psi_{i+1} \ldots \psi_m$ collapse to the graph $A_{j(i)} \cdot \psi_{i+1} \ldots \psi_m$.

We therefore have a path $\alpha_0 — A_{j(m)} — \alpha_0 \cdot \psi_m — A_{j(m-1)} \cdot \psi_m — \alpha_0 \cdot \psi_{m-1}\psi_m — \ldots — \alpha_0 \cdot \psi_2 \ldots \psi_m — A_{j(1)} \cdot \psi_2 \ldots \psi_m — \alpha_0 \cdot \psi_1 \ldots \psi_m = \alpha_0 \cdot \psi_0\psi_1 \ldots \psi_m = \alpha_0 \cdot \psi$, as required. ☐

We give an alternative proof of this in [14], which does not rely on already having a presentation for $\mathrm{Aut}_{\mathfrak{G}}(G)$ or $\mathrm{Out}_{\mathfrak{G}}(G)$, and rather uses the geometry of $\mathcal{C}_n$.

Corollary 3.1.4. *The complex $\mathcal{C}_n$ is (path) connected.*

Proof. This follows immediately by combining Corollary 3.1.2 with Proposition 3.1.3. ☐

3.2 Simple Connectivity of the Fundamental Domain

We will now show that the fundamental domain $\mathcal{D}_n$ of the space $\mathcal{C}_n$ (and hence each $\mathrm{Out}_{\mathfrak{G}}(G)$-image of $\mathcal{D}_n$) is simply connected. This will be the main result of this section, and is given as Theorem 3.2.11.

We will consider nested subcomplexes of $\mathcal{D}_n$, adding 'types' of 0-cell at each stage. We will show that the first of these subcomplexes is simply connected, and then apply a corollary of the Seifert–van Kampen Theorem to see that each successive subcomplex is also simply connected.

Corollary 3.2.1. *Let X and Y both be simply connected (simplicial) complexes. If we (suitably[2]) glue X and Y together along a path connected collection of edges, then $X \cup Y$ is simply connected.*

Proof. Note that in the complex $X \cup Y$, the subset $X \cap Y$ is precisely the collection of edges we have glued along. Since this is stipulated to be path connected, then by the Seifert–van Kampen Theorem (Theorem 1.2.1), we have $\pi_1(X \cup Y) \cong \pi_1(X) *_{\pi_1(X \cap Y)} \pi_1(Y) = \{1\} *_{\pi_1(X \cap Y)} \{1\} = \{1\}$. ☐

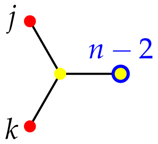
---

[2] i.e. so that $X \cup Y$ is still a simplicial complex

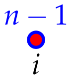
Recall that we describe a vertex $[T]$ of $\mathcal{C}_n$ as 'collapsing' to another vertex $[S]$ if a graph represented by $[T]$ has an edge (or edges) which can be collapsed to form a graph associated to $[S]$. That is, the vertices $[T]$ and $[S]$ are adjacent in $\mathcal{C}_n$.

Definition 3.2.2. Let $\mathcal{T}$ be a subset of the graph structures shown in Table 1.1. Denote by $\mathcal{D}_n[\mathcal{T}]$ the subcomplex of the fundamental domain of $\mathcal{C}_n$ obtained by restricting to simplices whose 0-cells are those associated to graph structures in $\mathcal{T}$.

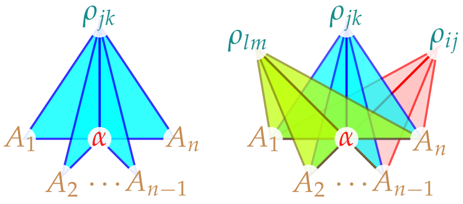
Lemma 3.2.3. *$\mathcal{D}_n[\{\alpha, \rho, A\}]$ is simply connected.*

Proof. Note that any ρ-vertex 

collapses to α, 

(by collapsing the edge whose endpoints are both trivial), and that α in turn collapses to any A-vertex 

(including $i = j$ or $i = k$). According to how we constructed the space $\mathcal{C}_n$, this means that for each pair (ρ, A) we have a 2-cell $[\rho, \alpha, A]$. So the subcomplex $\mathcal{D}_n[\{\alpha, \rho, A\}]$ comprises these 2-cells, glued along 'matching' edges. Given a particular ρ_{ij}-graph, we have a cone on a star at α (where the leaves of the star are the various A graphs). As we vary ρ, we get copies of this cone, all glued along the star formed by the α and A vertices.



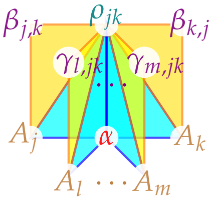
Clearly each cone is simply connected. Since the intersection of these cones is (the star based at α with leaves the A vertices, which is) path connected, we can iteratively apply Corollary 3.2.1 to 'add in' each cone (of which there are finitely many). Hence the structure $\mathcal{D}_n[\{\alpha, \rho, A\}]$ is simply connected. □

Lemma 3.2.4. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ is simply connected.*

Proof. By the previous lemma, we have that $\pi_1(\mathcal{D}_n[\{\alpha, \rho, A\}]) = \{1\}$. We will (iteratively) apply Corollary 3.2.1 to $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ taking X to be $\mathcal{D}_n[\{\alpha, \rho, A\}]$ (or the union of this with successive Y's) and Y to be the neighbourhood in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ of β or γ.

Each ρ graph collapses to 2 β graphs, and $n - 2$ γ graphs. These βs and γs are unique to the given ρ (that is, two distinct ρ graphs cannot both collapse to the same β or γ. Specifically, ρ_{jk} collapses to $\beta_{j,k}$, $\beta_{k,j}$ and $\gamma_{i,jk}$ for $i \neq j, k$. In turn, $\beta_{j,k}$ collapses to A_j, $\beta_{k,j}$

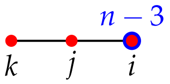
to A_k, and $\gamma_{i,jk}$ to A_i. Thus the neighbourhood of $\beta_{j,k}$ (or $\gamma_{i,jk}$) is a 2-cell $[\rho_{jk}, \beta_{j,k}, A_j]$ (or $[\rho_{jk}, \gamma_{i,jk}, A_i]$, respectively). Note that any 2-cell is simply connected. The intersection of each of these neighbourhoods with any of the spaces X is an edge $\rho — A$ (which is, in particular, path connected). So by Corollary 3.2.1, $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ is simply connected.

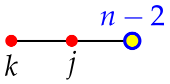


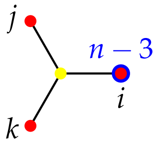
□

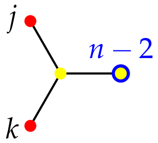
We now consider how to attach the B-vertices.

Lemma 3.2.5. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is simply connected.*

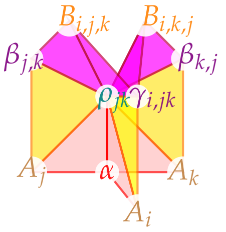
Proof. Each $B_{i,j,k}$ 

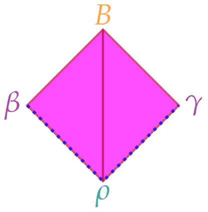
grows to a unique $\beta_{j,k}$ 

, a unique $\gamma_{i,jk}$ 

, and a unique ρ_{jk} 

. So $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is the subcomplex $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma\}]$ with the addition of 2-cells $[\rho_{jk}, \beta_{j,k}, B_{i,j,k}]$ and $[\rho_{jk}, \gamma_{i,jk}, B_{i,j,k}]$ for all possible values of i, j and k (by gluing along the ρ-β and ρ-γ edges that already exist, and additionally gluing our two 2-cells along the ρ-β edge they both share):

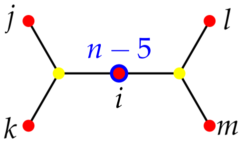


That is, given a specific ρ-vertex in our structure, for every β and γ we see adjacent to said ρ, we glue in a 'fin' 

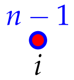
along the dotted line. The edge path $\beta — \rho — \gamma$ is path connected, so by repeated applications of Corollary 3.2.1, we see that $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is simply connected. □

Lemma 3.2.6. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$ is simply connected.*

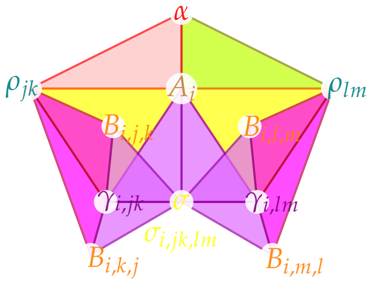
Proof. Given a vertex $\sigma_{i,jk,lm}$ in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$, with associated graph



, we see there is precisely one A-vertex it is adjacent to,



. We also see that $\sigma_{i,jk,lm}$ collapses to two γ-vertices, $\gamma_{i,jk}$ and $\gamma_{i,lm}$. Further, $\gamma_{i,jk}$ collapses to both $B_{i,j,k}$ and $B_{i,k,j}$ (similarly for $\gamma_{i,lm}$). To create $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma\}]$, in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ at any A_i-vertex we attach two 2-cells $[\sigma_{i,jk,lm}, \gamma_{i,jk}, A_i]$ and $[\sigma_{i,jk,lm}, \gamma_{i,lm}, A_i]$ (glued to each other along the shared edge $A_i — \sigma_{i,jk,lm}$) wherever we see two vertices $\gamma_{i,jk}$ and $\gamma_{i,lm}$ adjacent to A_i with j, k, l and m (and i) distinct. We then glue in additional 2-cells of the form $[\sigma, \gamma, B]$ wherever we see a path $\sigma — \gamma — B$.

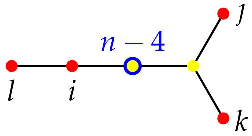


That is, each $\sigma_{i,jk,lm}$ has a simply connected neighbourhood, and the intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B\}]$ is the collection of edges $\gamma_{i,jk} — B_{i,j,k}$, $\gamma_{i,jk} — B_{i,k,j}$, $\gamma_{i,lm} — B_{i,l,m}$, $\gamma_{i,lm} — B_{i,m,l}$, $\gamma_{i,jk} — A_i$, and $\gamma_{i,lm} — A_i$, which is path conected. The result follows from Corollary 3.2.1. □

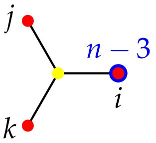
The process of adding in τ-vertices to our structure will be very similar to that for σ-vertices.

Lemma 3.2.7. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$ is simply connected.*

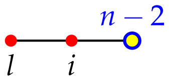
Proof. The vertex $\tau_{i,l,jk}$ in $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$ with graph



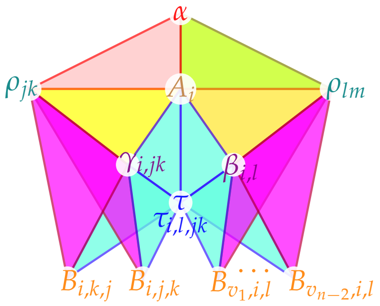
collapses to $\gamma_{i,jk}$



and $\beta_{i,l}$



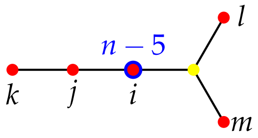
. In turn, $\beta_{i,l}$ and $\gamma_{i,jk}$ both collapse to A_i. Additionally $\gamma_{i,jk}$ collapses to $B_{i,j,k}$ and $B_{i,k,j}$, and $\beta_{i,l}$ collapses to $n - 2$ vertices of the form $B_{v,i,l}$ (for $v \neq i, l$). and in our structure so far, wherever we see a path $\gamma_{i,jk} — A_i — \beta_{i,l}$ with i, j, k, l distinct, we glue in a pair of 2-cells of the form $[\tau, \gamma, A]$ and $[\tau, \beta, A]$ (glued together along their common edge $A — \tau$). As before with σ, we must also glue in all possible 2-cells of the form $[\tau, \gamma, B]$ and $[\tau, \beta, B]$ as determined by the relative collapses of γ and β.

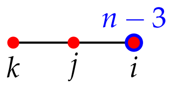


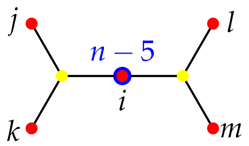
We have then identified the neighbourhood of $\tau_{i,l,jk}$ inside $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau\}]$, and found the intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma\}]$. By the previous lemma, $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma\}]$ is simply connected, and clearly the neighbourhood of τ (shown in pale blue) is simply connected. Moreover, the intersection of these subsets is the collection of edges $\gamma_{i,jk} — A_i$, $\gamma_{i,jk} — B_{i,j,k}$, $\gamma_{i,jk} — B_{i,k,j}$, $\beta_{i,l} — A_i$, $\beta_{i,l} — B_{v,i,l}$ (for $v \neq i,j,k,l$). Since this is path connected, then by Corollary 3.2.1, $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau\}]$ is simply connected. □

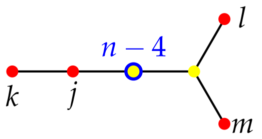
Lemma 3.2.8. *$\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta\}]$ is simply connected.*

Proof. Each δ in $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta\}]$ collapses to a unique B and 'grows' to a unique σ and a unique τ.

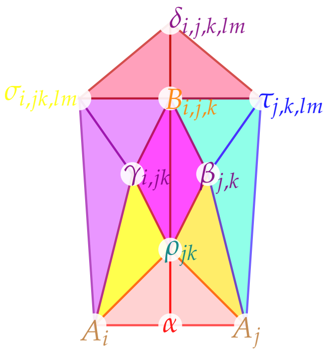
Precisely, $\delta_{i,j,k,lm}$ 

collapses to $B_{i,j,k}$ 

and grows to $\sigma_{i,jk,lm}$ 

and $\tau_{j,k,lm}$ 

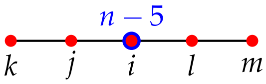
. As such, we see that a given $\sigma_{i,jk,lm}$ and $\tau_{j,k,lm}$ are both adjacent to $B_{i,j,k}$ (and share no other common B adjacency). So whenever we see a path $\sigma_{i,jk,lm} — B_{i,j,k} — \tau_{j,k,lm}$ in our structure, we will glue in 2-cells $[\sigma,\delta,B]$ and $[\tau,\delta,B]$ (gluing them along their shared δ-B edge).



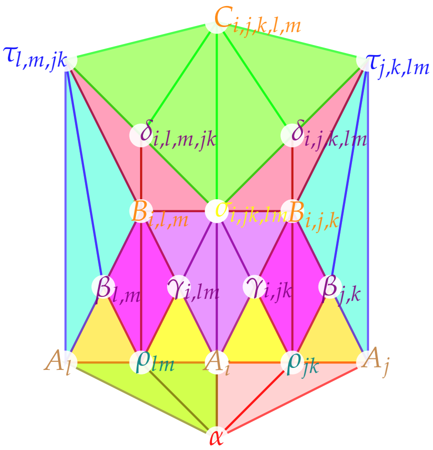
That is, each $\delta_{ij,k,lm}$ has a neighbourhood in $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta\}]$ comprising two 2-cells glued along a single edge. The intersection of this neighbourhood with

$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau\}]$ is the edge path $\sigma_{i,jk,lm} — B_{i,j,k} — \tau_{j,k,lm}$. So successive applications of Corollary 3.2.1 tells us that $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta\}]$ is simply connected. □

Lemma 3.2.9. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C\}]$ is simply connected.*

Proof. Each $C_{i,j,k,l,m}$  grows to $\delta_{i,j,k,lm}$, $\delta_{i,l,m,jk}$, $\tau_{l,m,jk}$, $\tau_{j,k,lm}$, and $\sigma_{i,jk,lm}$. So the neighbourhood of $C_{i,j,k,l,m}$ inside $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C\}]$ is four 2-cells $[\tau_{l,m,jk}, \delta_{i,l,m,jk}, C_{i,j,k,l,m}]$, $[\sigma_{i,jk,lm}, \delta_{i,l,m,jk}, C_{i,j,k,l,m}]$, $[\tau_{j,k,lm}, \delta_{i,j,k,lm}, C_{i,j,k,l,m}]$, and $[\sigma_{i,jk,lm}, \delta_{i,j,k,lm}, C_{i,j,k,l,m}]$, glued along their common edges.

The intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta\}]$ is the edge path $\tau_{l,m,jk} — \delta_{i,l,m,jk} — \sigma_{i,jk,lm} — \delta_{i,j,k,lm} — \tau_{j,k,lm}$.
By repeated applications of Corollary 3.2.1 (and by the previous lemma), $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C\}]$ is simply connected.

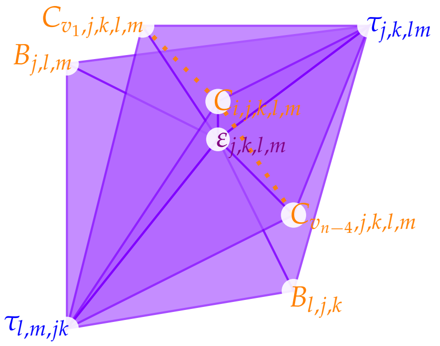


□

Finally, we add ε-vertices to our complex.

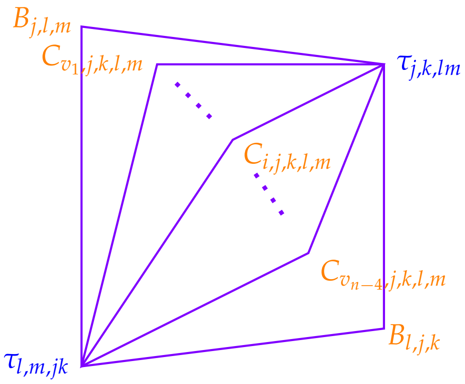
Lemma 3.2.10. *$\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}]$ is simply connected.*

Proof. Note that the subcomplex neighbourhood around $\varepsilon_{j,k,l,m}$ in our fundamental domain (equal to $\mathcal{D}_n[\{\alpha, \rho, A, \beta, \gamma, B, \sigma, \tau, \delta, C, \varepsilon\}]$) is:



That is, $\varepsilon_{j,k,l,m}$ grows to $\tau_{j,k,lm}$ and $\tau_{l,m,jk}$ and collapses to $B_{j,l,m}$, $B_{l,j,k}$, and $n-4$ vertices of the form $C_{v,j,k,l,m}$ (for $v \neq j,k,l,m$).

The intersection of this neighbourhood with $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta,C\}]$ is the boundary of the neighbourhood.



Since this is path connected, Corollary 3.2.1 applies, and $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta,C,\varepsilon\}]$ is simply connected. $\square$

Finally, we have proved:

Theorem 3.2.11. *The fundamental domain $\mathcal{D}_n$ of the complex $\mathcal{C}_n$ is simply connected.*

Proof. By Lemma 3.2.10, $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta,C,\varepsilon\}]$ is simply connected. But $\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta,C,\varepsilon\}$ covers all of the graph structures in Table 1.1. So by Definitions 3.2.2 and 2.3.5 $\mathcal{D}_n[\{\alpha,\rho,A,\beta,\gamma,B,\sigma,\tau,\delta,C,\varepsilon\}]$ is precisely the fundamental domain $\mathcal{D}_n$ of $\mathcal{C}_n$. $\square$

4 A Presentation for $\mathrm{Out}_{\mathfrak{S}}(G)$

This section is the main result of the paper.

We recall from Section 1 the theorem of Brown [6] which we will use to determine a presentation for $\mathrm{Out}_{\mathfrak{S}}(G)$ (where $G = G_1 * \cdots * G_n$, $\mathfrak{S} = (G_1, \ldots, G_n)$, and $\mathcal{G} = \mathrm{Out}_{\mathfrak{S}}(G)$):

Theorem 1.2.3 (Brown [6, Theorem 3])**.** *Let $\mathcal{G}$ act on a simply connected $\mathcal{G}$-CW complex X (without inversion on the 1-cells of X). Suppose there is a subcomplex W of X so that every cell of X is equivalent under the action of $\mathcal{G}$ to a unique cell of W. Then $\mathcal{G}$ is generated by the isotropy subgroups $\mathcal{G}_v$ ($v \in V(W)$) subject to edge relations $\iota_{o(e)}(g) = \iota_{t(e)}(g)$ for all $g \in \mathcal{G}_e$ ($e \in E(W)$) (where for any $e \in E(W)$, $\iota_{o(e)} : \mathcal{G}_e \to \mathcal{G}_{o(e)}$ and $\iota_{t(e)} : \mathcal{G}_e \to \mathcal{G}_{t(e)}$ are inclusions).*

The complex X we will use is $\mathcal{C}_n$, and the subcomplex W is $\mathcal{D}_n$. Since $\mathcal{C}_3$ and $\mathcal{C}_4$ are just the barycentric spine of Guirardel and Levitt's Outer Space (for $n = 3$ and $n = 4$ respectively), results satisfying the restrictions on X and W are assumed from [13]. It is highly non-trivial to show that $\mathcal{C}_n$ is simply connected for $n \geqslant 5$, so we delay the proof of this to Sections 5 and 6. The required result here is:

Corollary 6.3.3. *The space $\mathcal{C}_n$ (for $n \geqslant 5$) is simply connected.*

That $\mathcal{D}_n$ for $n \geqslant 5$ satisfies the strictness condition on W is the result of Propositions 2.3.9 and 2.3.10.

The isotropy subgroups $\mathcal{G}_v$ here are the vertex stabilisers $\mathrm{Stab}(T)$ for $T \in \mathcal{D}_n^{(0)}$, which are detailed in Propositions 2.4.3, 2.4.4, and 2.4.5.

Note that Lemma 2.3.8 implies that the edge relations in Brown's Theorem become: '$g = \iota_{o(e)}^{-1}\left(\iota_{t(e)}(g)\right)$ for all $g \in \mathcal{G}_{o(e)}$' (that is, that vertex stabilisers $\mathrm{Stab}(T)$ are identified with their natural images under inclusion in the stabilisers $\mathrm{Stab}(S)$ of any vertices S to which the original vertex T collapses).

Notation 4.0.1. We summarise the notational shorthand we have adopted thus far:

$[A, B] = 1$**:** For subgroups A and B, 'A commutes with B', in the sense that for all $a \in A$ and for all $b \in B$ we have $ab = ba$.

G_{i_j}**:** The group of (outer) automorphisms which act by conjugating all elements of the factor group G_j by an element of the factor group G_i.

f_{i_j}**:** The isomorphism $f_{i_j} : G_i \to G_{i_j}$ which maps an element $g \in G_i$ to the element in G_{i_j} which conjugates each element of G_j by g.

$G_{i_{v_1 \ldots v_k}}$**:** The group $\left\{ \left(f_{i_{v_1}}(g_i), \ldots, f_{i_{v_k}}(g_i)\right) \middle| g_i \in G_i \right\}$ which is the diagonal subgroup of $f_{i_{v_1}}(G_i) \times \cdots \times f_{i_{v_k}}(G_i) = G_{i_{v_1}} \times \cdots \times G_{i_{v_k}}$.
We often denote this by $G_{i_{v_1 \ldots v_k}} \Diamond G_{i_{v_1}} \times \cdots \times G_{i_{v_k}}$.

$Z(G_i)$**:** The centre of the group G_i, i.e. the subgroup $\{g \in G_i \mid gh = hg \ \forall h \in G_i\}$. We will often identify $Z(G_i)$ with its images $f_{i_j}(Z(G_i))$.

$\mathrm{Aut}(G_i)$**:** Often considered to be the subgroup $\{(1,\dots,1,\mathrm{Aut}(G_i),1,\dots,1)\}$ of $\Phi = \prod_{j=1}^{n} \mathrm{Aut}(G_j)$.

$\mathrm{Ad}_{G_i}(g)$**:** The element of $\mathrm{Inn}(G_i)$ which conjugates each element of G_i by g (where $g \in G_i$).

$G_{i_j}^{\varphi}$**:** The group of automorphisms $\left\{ \varphi \circ f_{i_j}(g) \circ \varphi^{-1} \,\middle|\, g \in G_i \right\}$ (where $\varphi \in \Phi$).

$\varphi(G_{i_j})$**:** The group of automorphisms $\left\{ f_{i_j}(\varphi(g)) \,\middle|\, g \in G_i \right\}$ (where $\varphi \in \Phi$).

We now split into cases dependent on the number n of factors in our splitting $G = G_1 * \cdots * G_n$.

4.1 The Case $n \geqslant 5$

We have all the pieces required to build our presentation for $\mathrm{Out}_{\mathfrak{G}}(G)$.

Theorem 4.1.1. *Let $G_1 * \cdots * G_n$ be a free splitting of a group G where each G_i is non-trivial and $n \geqslant 5$. For $i \in [n] := \{1,\dots,n\}$ and $j \in [n] - \{i\}$, let $f_{i_j} : G_i \to G_{i_j}$ be group isomorphisms, and for $g \in G_i$ let $\mathrm{Ad}_{G_i}(g)$ be the inner automorphism $x \mapsto gxg^{-1}$ of G_i. Then the group $\mathrm{Out}(G; G_1,\dots,G_n)$ is generated by the $n(n-1)$ groups $G_{i_j} \cong G_i$ and $\Phi = \prod_{i=1}^{n} \mathrm{Aut}(G_i)$, subject to relations:*

1. $[f_{i_j}(g), f_{i_k}(h)] = 1$ $\forall g,h \in G_i$, *for all* $i \in [n]$, $j,k \in [n] - \{i\}$
2. $[f_{i_j}(g), f_{k_l}(h)] = 1$ $\forall g \in G_i, h \in G_k$, *for all distinct* $i,j,k,l \in [n]$
3. $[f_{j_k}(g), f_{i_j}(h) f_{i_k}(h)] = 1$ $\forall g \in G_j, h \in G_i$, *for all distinct* $i,j,k \in [n]$
4. $f_{i_{v_1}}(g) \dots f_{i_{v_{n-1}}}(g) = \mathrm{Ad}_{G_i}(g^{-1})$ $\forall g \in G_i$, *for all* $i \in [n]$ *and* $\{v_1,\dots,v_{n-1}\} = [n] - \{i\}$
5. $\varphi^{-1} f_{i_j}(g) \varphi = f_{i_j}(g\varphi)$ $\forall g \in G_i$, *for all distinct* $i,j \in [n]$ *and all* $\varphi \in \Phi$

As well as all relations in G and Φ.

Proof. We apply Brown's Theorem (Theorem 1.2.3) to the fundamental domain $\mathcal{D}_n$ of the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{C}_n$. As previously noted, Proposition 2.3.10 states that the strictness requirement on $\mathcal{D}_n$ to apply Brown's Theorem is satisfied. We also require $\mathcal{C}_n$ to be simply connected. We delay the proof of this until after this section. The desired result here is Corollary 6.3.3.

We now have that $\mathrm{Out}_{\mathfrak{G}}(G)$ is generated by $\left\{ \mathrm{Stab}(T) \,\middle|\, T \in \mathcal{D}_n^{(0)} \right\}$, such that if $[S,T]$ is an edge in $\mathcal{D}_n^{(1)}$ (i.e. $S, T \in \mathcal{D}_n^{(0)}$ with T a collapse of S) then we have inclusions $\mathrm{Stab}(S) \hookrightarrow \mathrm{Stab}(T)$. We use the descriptions of $\mathrm{Stab}(T)$ from Propositions 2.4.3, 2.4.4, and 2.4.5. We proceed by examining the structure of $\mathcal{D}_n^{(1)}$. Recall that graph shapes for $T \in \mathcal{D}_n^{(0)}$ are listed in Table 1.1.

We first observe that every ρ_{ij} collapses to α. Since abstractly, $\mathrm{Stab}(\rho_{ij}) = \mathrm{Stab}(\alpha)$ (Proposition 2.4.3), then each $\mathrm{Stab}(\rho_{ij})$ is identified with $\mathrm{Stab}(\alpha) = \Phi$ in $\mathrm{Out}_{\mathfrak{S}}(G)$. Similarly, for each i, j we have that every $\mathrm{Stab}(\tau_{i,j,kl})$ ($k, l \in [n] - \{i, j\}$) is identified with $\mathrm{Stab}(\beta_{i,j})$ in $\mathrm{Out}_{\mathfrak{S}}(G)$.

Since ρ_{jk} collapses to $\beta_{j,k}$, $\gamma_{i,jk}$, A_v, and $B_{i,j,k}$ (for any $v \in \{1, \dots, n\}$ and any $i \notin \{j, k\}$), we immediately deduce that the Φ contribution from any $\mathrm{Stab}(\beta)$, $\mathrm{Stab}(\gamma)$, $\mathrm{Stab}(A)$, or $\mathrm{Stab}(B)$ is identified with $\mathrm{Stab}(\alpha)$.

We now consider the τ-ε–Square (Figure 1.4). Note that for each $\varepsilon_{i,j,k,l}$ there are precisely two τ graphs, $\tau_{i,j,kl}$ and $\tau_{k,l,ij}$, which collapse to $\varepsilon_{i,j,k,l}$. We may 'replace' $\tau_{i,j,kl}$ with $\beta_{i,j}$ and $\tau_{k,l,ij}$ with $\beta_{k,l}$, and recalling that ρ_{ij} collapses to $\beta_{i,j}$ and ρ_{kl} to $\beta_{k,l}$, also 'replace' ρ_{ij} and ρ_{kl} with α. We thus have a diagram:

$$\begin{tikzcd}
\mathrm{Stab}(\tau_{i,j,kl}) \arrow[rr, hook] \arrow[d, equal] & & \mathrm{Stab}(\varepsilon_{i,j,k,l}) & & \mathrm{Stab}(\tau_{k,l,ij}) \arrow[ll, hook'] \arrow[d, equal] \\
\mathrm{Stab}(\beta_{i,j}) \arrow[urr, dashed, hook] & & & & \mathrm{Stab}(\beta_{k,l}) \arrow[ull, dashed, hook'] \\
\mathrm{Stab}(\rho_{ij}) \arrow[u, hook] \arrow[rr, equal] & & \mathrm{Stab}(\alpha) \arrow[rr, equal] \arrow[ull, dashed, hook'] \arrow[urr, dashed, hook] & & \mathrm{Stab}(\rho_{kl}) \arrow[u, hook]
\end{tikzcd}$$

where the dashed inclusions are naturally induced by the 'replacements' we made. Note that $\mathrm{Stab}(\beta_{i,j})$ and $\mathrm{Stab}(\beta_{k,l})$ 'cover' $\mathrm{Stab}(\varepsilon_{i,j,k,l})$ in the sense that $\mathrm{Stab}(\varepsilon_{i,j,k,l}) \subseteq \mathrm{Stab}(\beta_{i,j}) \times \mathrm{Stab}(\beta_{k,l})$. Since $\mathrm{Stab}(\beta_{i,j}) \cap \mathrm{Stab}(\beta_{k,l}) = \Phi = \mathrm{Stab}(\alpha)$, the dashed inclusions form something akin to a pushout diagram, and we may conclude that $\mathrm{Stab}(\varepsilon) = \mathrm{Stab}(\beta_{i,j}) \times_{\Phi} \mathrm{Stab}(\beta_{k,l})$.

Next, we consider $\mathrm{Stab}(A_i)$. Observe that given $i \in [n]$, every $\beta_{i,j}$ for $j \in [n] - \{i\}$ collapses to A_i, thus we have $n - 1$ inclusions $\mathrm{Stab}(\beta_{i,j}) \hookrightarrow \mathrm{Stab}(A_i)$. Then the G_{i_j} contribution from $\mathrm{Stab}(A_i)$ is identified in $\mathrm{Out}_{\mathfrak{S}}(G)$ with the G_{i_j} contribution from $\mathrm{Stab}(\beta_{i,j})$, and we can consider $\mathrm{Stab}(A_i)$ to be generated by $\mathrm{Stab}(\beta_{i_{v_1}}) \times_{\Phi} \cdots \times_{\Phi} \mathrm{Stab}(\beta_{i_{v_{n-1}}})$ subject to the relations in Proposition 2.4.4, as well as the relation $Z(G_{i_{v_1 \dots v_{n-1}}}) = \{1\}$ (since the $\beta_{i,j}$'s 'cover' A_i).

A similar principle applies to $\mathrm{Stab}(\gamma_{i,jk})$. Given $i, j, k \in [n]$, we have that for any $l \in [n] - \{i, j, k\}$, the graph $\tau_{i,l,jk}$ collapses to $\gamma_{i,jk}$. Noting that $\mathrm{Stab}(\tau_{i,l,jk})$ is identified in $\mathrm{Out}_{\mathfrak{S}}(G)$ with $\mathrm{Stab}(\beta_{i,j})$, we have $n - 3$ inclusions of the form $\mathrm{Stab}(\beta_{i,l}) \hookrightarrow \mathrm{Stab}(\gamma_{i,jk})$. Recall from Proposition 2.4.4 that abstractly, $\mathrm{Stab}(\gamma_{i,jk})$ is generated by $G_{i_{jk}} \times G_{i_{l_1}} \times \cdots \times G_{i_{l_{n-3}}} \big/ Z(G_i)$ and Φ. However, by manipulation of relations in $\mathrm{Stab}(\gamma_{i,jk})$ (or by considering the Guirardel–Levitt approach to computing stabilisers), we have that the $G_{i_{jk}}$ component is redundant as a generator. Specifically, for $f_{i_{jk}}(g_i) \in G_{i_{jk}}$ (with

$g_i \in G_i$), we have that $f_{i_{jk}}(g_i) = \iota_{g_i} f_{i_{l_1 \dots l_{n-3}}}(g_i^{-1}) f_i(g_i^{-1})$, where $\iota_{g_i} \in \mathrm{Inn}(G)$ conjugates every element of G by g_i, and $f_i : G_i \to \mathrm{Inn}(G_i)$ is the canonical homomorphism. Thus we can consider $\mathrm{Stab}(\gamma_{i,jk})$ to be generated by $\mathrm{Stab}(\beta_{i_{l_1}}) \times_{\Phi} \cdots \times_{\Phi} \mathrm{Stab}(\beta_{i_{l_{n-3}}})$ (with relations similar to $\mathrm{Stab}(A_i)$).

Given a 'top' vertex in $\mathcal{D}_n^{(0)}$ (i.e. a ρ, σ, or τ graph), we can reach a 'bottom' vertex (A, B, or C graph) by successively collapsing two edges. By changing the order in which we collapse these edges, we produce square (or 'diamond') diagrams of inclusions

$$\begin{tikzcd} & W \arrow[dl, hook] \arrow[dr, hook] & \\ X \arrow[dr, hook] & & Y \arrow[dl, hook] \\ & Z & \end{tikzcd}$$

where so long as X and Y 'cover' Z (in the sense that $\mathrm{Stab}(Z) \subseteq \mathrm{Stab}(X) \times \mathrm{Stab}(Y)$), we will have $\mathrm{Stab}(Z) = \mathrm{Stab}(X) \times_{\mathrm{Stab}(W)} \mathrm{Stab}(Y)$, where $\mathrm{Stab}(W) = \mathrm{Stab}(X) \cap \mathrm{Stab}(Y)$. Figure 1.5 illustrates some such diagrams which are of particular use.

$$\begin{tikzcd} & \sigma_{i,jk,lm} \arrow[dl, hook] \arrow[dr, hook] & \\ \gamma_{i,jk} \arrow[dr, hook] & \boxed{1} & \gamma_{i,lm} \arrow[dl, hook] \\ & A_i & \end{tikzcd} \qquad \begin{tikzcd} & \rho_{jk} \arrow[dl, hook] \arrow[dr, hook] & \\ \beta_{j,k} \arrow[dr, hook] & \boxed{2} & \gamma_{i,jk} \arrow[dl, hook] \\ & B_{i,j,k} & \end{tikzcd}$$

$$\begin{tikzcd} & \sigma_{i,jk,lm} \arrow[dl, hook] \arrow[dr, hook] & \\ \gamma_{i,jk} \arrow[dr, hook] & \boxed{3} & \delta_{i,j,k,lm} \arrow[dl, hook] \\ & B_{i,j,k} & \end{tikzcd} \qquad \begin{tikzcd} & \sigma_{i,jk,lm} \arrow[dl, hook] \arrow[dr, hook] & \\ \delta_{i,j,k,lm} \arrow[dr, hook] & \boxed{4} & \delta_{i,l,m,jk} \arrow[dl, hook] \\ & C_{i,j,k,l,m} & \end{tikzcd}$$

FIGURE 1.5: Inclusion Diagrams in $\mathcal{D}_n^{(1)}$

Diagram $\boxed{1}$ may be thought of as akin to a pullback diagram, in that $\sigma_{i,jk,lm}$ is uniquely determined by $\gamma_{i,jk}$ and $\gamma_{i,lm}$. Since $\mathrm{Stab}(A_i) \subseteq \mathrm{Stab}(\gamma_{i,jk}) \times \mathrm{Stab}(\gamma_{i,lm})$, we deduce that $\mathrm{Stab}(\sigma_{i,jk,lm}) = \mathrm{Stab}(\gamma_{i,jk}) \cap \mathrm{Stab}(\gamma_{i,lm})$ in $\mathrm{Out}_{\mathfrak{S}}(G)$. The remaining diagrams are more akin to pushouts than pullbacks.

Diagram $\boxed{2}$ is from the ρ–Book of Figure 1.3, and is enough to uniquely determine a given $B_{i,j,k}$. Recalling that $\mathrm{Stab}(\rho_{jk}) = \mathrm{Stab}(\alpha) = \Phi$ in $\mathrm{Out}_{\mathfrak{S}}(G)$, we conclude that $\mathrm{Stab}(B_{i,j,k}) = \mathrm{Stab}(\beta_{j,k}) \times_{\Phi} \mathrm{Stab}(\gamma_{i,jk})$.

Diagram $\boxed{3}$ implies that $\mathrm{Stab}(B_{i,j,k}) = \mathrm{Stab}(\delta_{i,j,k,lm}) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} \mathrm{Stab}(\gamma_{i,jk})$, thus $\mathrm{Stab}(\beta_{j,k}) \times_{\Phi} \mathrm{Stab}(\gamma_{i,jk}) = \mathrm{Stab}(\delta_{i,j,k,lm}) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} \mathrm{Stab}(\gamma_{i,jk})$. From this, we deduce that $\mathrm{Stab}(\delta_{i,j,k,lm}) = \mathrm{Stab}(\beta_{j,k}) \times_{\Phi} \mathrm{Stab}(\sigma_{i,jk,lm})$.

Diagram $\boxed{4}$ is from the σ–Slice of Figure 1.4, and is enough to uniquely determine a given $C_{i,j,k,l,m}$. We then have that:

$$\mathrm{Stab}(C_{i,j,k,l,m}) = \mathrm{Stab}(\delta_{i,j,k,lm}) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} \mathrm{Stab}(\delta_{i,l,m,jk})$$

$$= \big(\mathrm{Stab}(\beta_{jk}) \times_\Phi \mathrm{Stab}(\sigma_{i,jk,lm})\big) \times_{\mathrm{Stab}(\sigma_{i,jk,lm})} \big(\mathrm{Stab}(\beta_{l,m}) \times_\Phi \mathrm{Stab}(\sigma_{i,jk,lm})\big)$$
$$= \mathrm{Stab}(\beta_{j,k}) \times_\Phi \mathrm{Stab}(\beta_{l,m}) \times_\Phi \mathrm{Stab}(\sigma_{i,jk,lm})$$

We have now shown that any $\mathrm{Stab}(T)$ for $T \in \mathcal{D}_n^{(0)}$ can be written in terms of $\Phi = \mathrm{Stab}(\alpha)$ and $\mathrm{Stab}(\beta_{i,j})$ (allowing i and j to vary over $\{1,\dots,n\}$). Thus $\mathrm{Out}_{\mathfrak{G}}(G)$ is generated by $\{\mathrm{Stab}(\beta_{i,j}) \,|\, i \in [n], j \in [n] - \{i\}\}$, that is, $\mathrm{Out}_{\mathfrak{G}}(G)$ is generated by
$\left\{ G_{i_j} \,\middle|\, i \in \{1,\dots,n\}, j \in \{1,\dots,n\} - \{i\} \right\} \cup \Phi$.

From $\mathrm{Stab}(A_i)$ (Proposition 2.4.4), we see that $[G_{i_j}, G_{i_k}] = 1$ and ${}^{G_{i_{v_1\dots v_{n-1}}}}\!\!\diagup_{Z(G_i)} = \mathrm{Inn}(G_i) \cong {}^{G_i}\!\diagup_{Z(G_i)}$. Using the isomorphisms f_{i_j} for preciseness, we recover Relations 1 and 4. Note that if $g \in Z(G_i)$ then Relation 4 gives $f_{i_{v_1}}(g) \dots f_{i_{v_{n-1}}}(g) = 1$. We deduce from $\mathrm{Stab}(\varepsilon_{i,j,k,l})$ (Proposition 2.4.3) that $[G_{i_j}, G_{k_l}] = 1$, from $\mathrm{Stab}(B_{i,j,k})$ (Proposition 2.4.5) that $[G_{j_k}, G_{i_{jk}}] = 1$, and from $\mathrm{Stab}(\beta_{i,j})$ (Proposition 2.4.3) that $G_{i_j}^{\varphi^{-1}} = \varphi(G_{i_j})$. We now recover Relations 2,3, and 5 by substituting the appropriate isomorphisms f_{i_j} into the above formulae. All other relations found in vertex stabilisers are subsumed by these five. □

Remark. Note that n of these generators are 'redundant', in that for each $i \in [n]$ and $j \in [n] - \{i\}$, $G_{i_j} \leqslant (G_{i_{v_1}} \times \dots \times G_{i_{v_{n-5}}}) \rtimes \Phi$. However, consistently choosing generators to remove without overcomplicating the relations is tricky, so we elect not to do this.

Corollary 4.1.2. *If a group G splits as a free product where the factor groups are non-trivial, freely indecomposable, not infinite cyclic, and pairwise non-isomorphic, then Theorem 4.1.1 gives a presentation for* $\mathrm{Out}(G)$.

Proof. Note that the splitting $G_1 * \dots * G_n$ described is a Grushko decomposition for G, and so every automorphism must preserve the conjugacy classes of the factor groups. That is, $\mathrm{Out}(G; G_1, \dots, G_n) = \mathrm{Out}(G)$. □

4.2 The Case $n = 4$

Let $G = G_1 * G_2 * G_3 * G_4$ be a free splitting of a group G, and let $\mathfrak{G} = (G_1, G_2, G_3, G_4)$.

By Definition 2.0.1, our complex $\mathcal{C}_4$ is the barycentric spine of Guirardel and Levitt's Outer Space relative to $\mathfrak{G}$, which we build by taking only the graph shapes from Table 1.1 which have at most four non-trivial (red) vertices. This leaves us with graph shapes ρ, α, β, γ, A, and B. Note however that for $\{i,j,k,l\} = \{1,2,3,4\}$, we have $\gamma_{i,kl} = \beta_{i,j}$. Also, $\rho_{ij} = \rho_{kl}$ and $B_{i,j,k} = B_{j,i,l}$. Note additionally that we did not take τ or ε graphs (despite these only displaying four non-trivial vertices) since the trivial 'basepoint' must have valency at least three here, implying at least one suppressed non-trivial vertex in each case.

Thus the vertex set of the fundamental domain $\mathcal{D}_4^{(0)}$ of $\mathcal{C}_4$ consists of $\frac{1}{2}\binom{4}{2} = 3$ ρ vertices, 1 α vertex, $4 \times 3 = 12$ β vertices, 4 A vertices, and $\frac{4!}{2} = 12$ B vertices. These form three ρ_{ij} 'spikes' (for $\{i, j\} \subseteq \{1, 2, 3, 4\}$) in $\mathcal{D}_4^{(1)}$, shown in Figure 1.6, which are identified

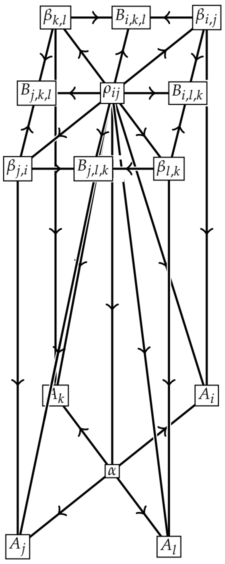
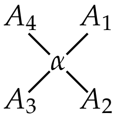


FIGURE 1.6: A 'Spike' in $\mathcal{D}_4^{(1)}$

along the α-A–Star  .

Proposition 4.2.1. *We have the following for* $\{i, j, k, l\} = \{1, 2, 3, 4\}$*:*

1. $\mathrm{Stab}(\rho_{ij}) = \mathrm{Stab}(\alpha) = \Phi$
2. $\mathrm{Stab}(\beta_{i,j}) = G_{i_j} \rtimes \Phi$
3. $\mathrm{Stab}(A_i)$ *is generated by* ${}^{(G_{i_j} \times G_{i_k} \times G_{i_l})}\!\big/_{Z(G_i)}$ *and* Φ *with relations* ${}^{G_{i_{jkl}}}\!\big/_{Z(G_i)} = \mathrm{Inn}(G_i)$ *and* $(\{G_a\}, g_i) \circ \varphi = \varphi \circ (\{G_a\}, (g_i)\varphi)$ *for* $\varphi \in \Phi$ *and* $(\{G_a\}, g_i) \in i_a$ *for each* $a \in \{j, k, l\}$
4. $\mathrm{Stab}(B_{i,k,l}) = (G_{i_j} \times G_{k_l}) \rtimes \Phi$

Proof. The first three items are the same as the presentations listed for the $n \geqslant 5$ case. The arguments there also hold for $n = 4$. We use the Guirardel–Levitt approach (see Example 2.4.1) to compute $\mathrm{Stab}(B_{i,k,l})$, noting that $B_{i,k,l} =$  and fixing the edge between vertex groups G_i and G_k. □

Theorem 4.2.2. *Let* $G_1 * G_2 * G_3 * G_4$ *be a free splitting of a group* G *where each* G_i *is non-trivial, and let* $\mathfrak{G} = (G_1, G_2, G_3, G_4)$*. Writing* f_{i_j} *for the isomorphism* $G_i \to G_{i_j}$*,* $\mathrm{Out}_{\mathfrak{G}}(G)$ *is generated by the twelve groups* G_{i_j} *for* $i, j \in \{1, 2, 3, 4\}$ *distinct, and* $\Phi = \prod_{k=1}^{4} \mathrm{Aut}(G_k)$*, subject to relations:*

1. $\left[f_{i_j}(g), f_{i_k}(h)\right] = 1\ \forall g, h \in G_i$
2. $\left[f_{i_j}(g), f_{k_l}(h)\right] = 1\ \forall g \in G_i, h \in G_k$
3. $f_{i_j}(g) f_{i_k}(g) f_{i_l}(g) = \mathrm{Ad}_{G_i}(g^{-1})\ \forall g \in G_i$
4. $\varphi^{-1} f_{i_j}(g)\varphi = f_{i_j}((g)\varphi)\ \forall g \in G_i$ *for all* $\varphi \in \Phi$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Proof. We apply Theorem 1.2.3 to the complex $\mathcal{C}_4$. Note that Lemma 2.0.2 tells us $\mathcal{C}_4$ is simply connected, and it is not hard to deduce that the fundamental domain $\mathcal{D}_4$ meets the strictness requirements here. Then $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by the stabiliser groups described in Proposition 4.2.1 for each combination of $\{i, j, k, l\} = \{1, 2, 3, 4\}$. We now consider the edge relations from $\mathcal{D}_4$. Since each ρ_{ij} collapses to α, we have that $\mathrm{Stab}(\rho_{12})$, $\mathrm{Stab}(\rho_{13})$, $\mathrm{Stab}(\rho_{14})$, and $\mathrm{Stab}(\alpha)$ all generate the same subgroup Φ of $\mathrm{Out}(G)$. Moreover, since every vertex is a collapse of some ρ_{ij} then the Φ contribution from each stabiliser in Proposition 4.2.1 is identified with $\mathrm{Stab}(\alpha)$. Each 'spike' of $\mathcal{D}_4$ gives a diagram of inclusions:

$$\begin{tikzcd}
(G_{k_l}) \rtimes \Phi \arrow[r] \arrow[d] & (G_{i_j} \times G_{k_l}) \rtimes \Phi & (G_{i_j}) \rtimes \Phi \arrow[l] \arrow[d] \\
(G_{j_i} \times G_{k_l}) \rtimes \Phi & \Phi \arrow[l] \arrow[r] \arrow[u] \arrow[d] \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & (G_{i_j} \times G_{l_k}) \rtimes \Phi \\
(G_{j_i}) \rtimes \Phi \arrow[r] \arrow[u] & (G_{j_i} \times G_{l_k}) \rtimes \Phi & (G_{l_k}) \rtimes \Phi \arrow[l] \arrow[u]
\end{tikzcd}$$

and each A_i neighbourhood gives a further diagram of inclusions:

$$\begin{tikzcd}
& G_{i_l} \rtimes \Phi \arrow[d] & \\
& \mathrm{Stab}(A_i) & \\
G_{i_j} \rtimes \Phi \arrow[ur] & & G_{i_k} \rtimes \Phi \arrow[ul]
\end{tikzcd}$$

Hence any G_{i_j} contribution from $\mathrm{Stab}(A_i)$ or $\mathrm{Stab}(B_{i,k,l})$ is identified with the G_{i_j} contribution from $\beta_{i,j}$. Thus $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated $\mathrm{Stab}(\beta_{i,j})$ for each pair (i, j) and α, with the Φ contribution from each $\mathrm{Stab}(\beta_{i,j})$ identified with $\mathrm{Stab}(\alpha)$. That is, $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by G_{i_j} for each pair (i, j) and Φ. The Relations 1–4 come from the groups in

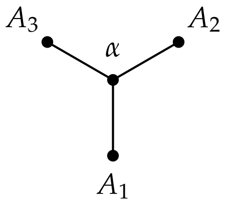
Proposition 4.2.1 (specifically, 1 and 3 are from $\mathrm{Stab}(A_i)$, 2 is from $\mathrm{Stab}(B_{i,k,l})$, and 4 is from $\mathrm{Stab}(\beta_{i,j})$). □

Observe that the Relation '$\left[f_{j_k}(g), f_{i_j}(h)f_{i_k}(h)\right] = 1$' from the case $n \geqslant 5$ may be deduced from Relations 2 and 3 here (since writing k_k for $\mathrm{Inn}(k)$ yields $k_{ij} = k_{kl}$). Thus Theorem 4.1.1 in fact holds for $n \geqslant 4$.

One should note that one of each of the groups $G_{i_j} \cong G_i$ is 'redundant' in the sense that only any 2 of the groups $G_{i_j}, G_{i_k}, G_{i_l}$ are independent of each other. It is possible to consistently choose 4 such groups to eliminate from the list of generators, but doing so would somewhat complicate the relations.

4.3 The Case $n = 3$

For $G = G_1 * G_2 * G_3$, the only graphs which respect the splitting $\mathfrak{S} = (G_1, G_2, G_3)$ are α and A graphs. Thus the fundamental domain $\mathcal{D}_3$ of our complex $\mathcal{C}_3$ is the following tripod:



where A_i is the graph  for $i = 1,2,3$ and $\{j,k\} = \{1,2,3\} - \{i\}$.

As before, we have that $\mathrm{Stab}(\alpha) = \Phi = \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2) \times \mathrm{Aut}(G_3)$, and that for each $i \in \{1,2,3\}$ with $\{j,k\} = \{1,2,3\} - \{i\}$, $\mathrm{Stab}(A_i)$ is generated by ${}^{(G_{i_j} \times G_{i_k})}\!\big/_{Z(G_i)}$ and Φ such that ${}^{G_{i_{jk}}}\!\big/_{Z(G_i)} = \mathrm{Inn}(G_i)$ and $(\{G_a\}, g_i) \circ \varphi = \varphi \circ (\{G_a\}, (g_i)\varphi)$ (for $a = j,k$) for all $(\{G_a\}, g_i) \in G_{i_a}$ and $\varphi \in \Phi$.

Theorem 4.3.1. *Let $G = G_1 * G_2 * G_3$ be a free splitting of a group G where each G_i is non-trivial, and let $\mathfrak{S} = (G_1, G_2, G_3)$. Then writing f_{i_j} for the isomorphism $G_i \to G_{i_j}$, $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by the six groups G_{i_j} for $i, j \in \{1,2,3\}$ distinct, and Φ, subject to relations:*

1. $\left[f_{i_j}(g), f_{i_k}(h)\right] = 1 \ \forall g, h \in G_i$
2. $f_{i_j}(g)f_{i_k}(g) = \mathrm{Ad}_{G_i}(g) \ \forall g \in G_i$
3. $\varphi^{-1}f_{i_j}(g)\varphi = f_{i_j}((g)\varphi) \ \forall g \in G_i$ *for all* $\varphi \in \Phi$

for all $i = 1,2,3$ and $\{j,k\} = \{1,2,3\} - \{i\}$.

Proof. By Theorem 1.2.3, $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by $\mathrm{Stab}(\alpha)$, $\mathrm{Stab}(A_1)$, $\mathrm{Stab}(A_2)$, and $\mathrm{Stab}(A_3)$. The structure of $\mathcal{D}_3$ means that the Φ contribution from each $\mathrm{Stab}(A_i)$ is identified with $\mathrm{Stab}(\alpha)$. That is, $\mathrm{Out}_{\mathfrak{S}}(G) = \mathrm{Stab}(A_1) *_{\Phi} \mathrm{Stab}(A_2) *_{\Phi} \mathrm{Stab}(A_3)$. The result then follows by examining each $\mathrm{Stab}(A_i)$. □

Note that ${G_{i_{jk}}}\big/{Z(i)} = \mathrm{Inn}(G_i)$ means $G_{i_j}\,\mathrm{Inn}(G_i) = G_{i_k}\,\mathrm{Inn}(G_i)$ as cosets in $\mathrm{Stab}(A_i)$. Thus we can write $\mathrm{Stab}(A_i) = G_{i_j} \rtimes \Phi = G_{i_k} \rtimes \Phi$ (using the Guirardel–Levitt approach to computing stabilisers demonstrated in Example 2.4.1). Since the only relation between vertex groups here is the amalgamation over Φ, we can in this case obtain a much simpler presentation:

Corollary 4.3.2. *For $G = G_1 * G_2 * G_3$ as above, we have:*

$$\mathrm{Out}_{\mathfrak{S}}(G) = (G_{1_2} * G_{2_3} * G_{3_1}) \rtimes \Phi \cong G \rtimes \Phi$$

Proof. As noted above, in this case we have that $\mathrm{Out}_{\mathfrak{S}}(G) = \mathrm{Stab}(A_1) *_{\Phi} \mathrm{Stab}(A_2) *_{\Phi} \mathrm{Stab}(A_3)$. We now observe that $(G_{1_2} \rtimes \Phi) *_{\Phi} (G_{2_3} \rtimes \Phi) *_{\Phi} (G_{3_1} \rtimes \Phi) = (G_{1_2} * G_{2_3} * G_{3_1}) \rtimes \Phi$. □

If each of G_1, G_2, and G_3 is additionally freely indecomposable, not infinite cyclic, and pairwise non-isomorphic, then we have that $\mathrm{Out}_{\mathfrak{S}}(G) = \mathrm{Out}(G)$. In this case, this is exactly the presentation for $\mathrm{Out}(G)$ given by Collins and Gilbert [7, Propositions 4.1,4.2].

5 The Space of Domains

The rest of the paper will be spent proving that for $n \geqslant 5$, $\mathcal{C}_n$ is simply connected.

In order to study global properties of our complex $\mathcal{C}_n$, we will define a new space, akin to a nerve complex (first introduced by Alexandroff in [1]).

A nerve complex is an abstract simplicial complex built using information on the intersections within a family of sets. The sets we will choose are copies of the fundamental domain $\mathcal{D}_n$, which form a (closed) cover of $\mathcal{C}_n$. However, in order to keep the dimension low, we will only consider k-wise intersections of sets for $k \leqslant 3$.

It is not necessary to have background knowledge of nerve complexes in order to understand our space or arguments.

5.1 Defining the Space of Domains

We will call a subset of our complex a *domain* if it is of the form $\mathcal{D}_n \cdot \psi$ for some $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$, where we think of $\mathcal{D}_n$ as a set and $\mathcal{D}_n \cdot \psi = \{x \cdot \psi | x \in \mathcal{D}_n\}$. Since the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{C}_n$ preserves adjacency, then $\mathcal{D}_n \cdot \psi$ has the same topological structure as $\mathcal{D}_n$.

Definition 5.1.1. The *Graph of Domains* is a graph whose vertex set contains one vertex for every domain in our complex, and whose edge set contains an edge joining distinct vertices u and v if and only if the intersection of the domains associated to u and v is non-empty.

Since the graph $\overset{n}{\circ}$ (denoted α) occurs precisely once per domain, taking one vertex per domain equates to taking one vertex for every point of the form $\overset{n}{\circ}$ in our complex $\mathcal{C}_n$. We will thus often denote vertices in the Graph of Domains by α. Now any two distinct vertices α_1 and α_2 in the Graph of Domains are joined by an edge precisely when the intersection of the domain containing the graph α_1 and the domain containing the graph α_2 is non-empty.

Note that the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ on $\mathcal{C}_n$ induces a natural $\mathrm{Out}_{\mathfrak{G}}(G)$-action on the Graph of Domains.

Definition 5.1.2. The *splitting associated to a domain* $\mathcal{D}_n \cdot \psi$ is the labelling $((G_1)\psi, \ldots, (G_n)\psi)$, which is equivalent to any labelling $(H_1, \ldots, H_n)$ of the α-graph contained within the domain $\mathcal{D}_n \cdot \psi$.

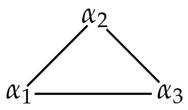
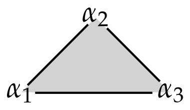
Note that $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ is only unique up to factor automorphisms — that is, if $\varphi \in \Phi$ then $\mathcal{D}_n \cdot \psi = \mathcal{D}_n \cdot \varphi\psi$.

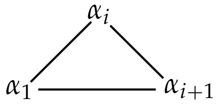
Corollary 5.1.3. *The Graph of Domains is (path) connected.*

Proof. This follows immediately from Proposition 3.1.3. ☐

Remark. Note that the Graph of Domains is **not** locally finite.

To show our complex $\mathcal{C}_n$ is simply connected, we will need the following construction:

Definition 5.1.4. Wherever we have a 3-cycle  in our Graph of Domains, we will insert a 2-simplex  if and only if $\alpha_1 \cap \alpha_2 \cap \alpha_3 \neq \varnothing$ (in the complex $\mathcal{C}_n$). We call the resulting CW-complex the *Space of Domains*.

Note that given a cycle $\alpha_1 — \ldots — \alpha_{n-1} — \alpha_n = \alpha_1$ with $\alpha_1 \cap \cdots \cap \alpha_{n-1} \neq \emptyset$, we can split this up into 3-cycles 

for $i = 2, \ldots, n-2$, with each $\alpha_1 \cap \alpha_i \cap \alpha_{i+1} \neq \emptyset$, so any such loop is contractible in our Space of Domains.

The idea behind this definition is that, since we have shown that the fundamental domain $\mathcal{D}_n$ is simply connected, then if we had that any pairwise intersection of domains is either empty or path-connected, then any non-trivial loop in $\mathcal{C}_n$ would be projected to a non-trivial loop in the Space of Domains.

In particular, if $\alpha_1 \cup \alpha_2$ were simply connected (assuming $\alpha_1 \cap \alpha_2 \neq \emptyset$) then there would be no non-trivial loops in $\mathcal{C}_n$ which would appear as a forwards-and-backwards traversal of an edge when projected to the Space of Domains, and similarly for $\alpha_1 \cup \alpha_2 \cup \alpha_3$ (or the 2-cell $\alpha_1 — \alpha_2 — \alpha_3 — \alpha_1$ in the Space of Domains).

Then to show that our space $\mathcal{C}_n$ is simply connected, it would suffice to show that our Space of Domains is simply connected.

Unfortunately, it will not be quite this simple, but the general idea will remain the same. We will formalise (and resolve) this in Sections 5.2 and 5.3.

5.2 Pairwise Intersections

Here we would hope to show that the intersection of two adjacent domains is path-connected.

Then we could deduce using the Seifert–van Kampen Theorem that the union of two adjacent domains is simply connected. This would ensure, for example, that there are no non-trivial loops in $\mathcal{C}_n$ of the form $\alpha_1 — A — \alpha_2 — A — \alpha_1$, and would justify the use of a single edge between adjacent vertices in our Graph of Domains.

As it turns out, not quite all such intersections are path-connected, but we show that the case where this does not hold can be circumvented. This is deduced in Propositions 5.2.10 and 5.2.11, the main results of this subsection.

To avoid confusion regarding domains and graphs within domains, we will temporarily break from the convention of naming domains α. So let $\aleph_1$ and $\aleph_2$ be two arbitrary domains. Assume $\aleph_1 \cap \aleph_2 \neq \emptyset$. Without loss of generality, we may assume $\aleph_1$ is the fundamental domain $\mathcal{D}_n$.

Observation 5.2.1. Note that if $\aleph_2 = \aleph_1 \cdot \psi$ then for $T \in \aleph_1$, we have $T \in \aleph_2$ if and only if $T = T' \cdot \psi$ for some $T' \in \aleph_1$. But each domain contains precisely one element of each orbit, so we must have $T = T'$. Then $\psi \in \mathrm{Stab}(T)$. Moreover, for $T \in \aleph_1$ and $\psi \in \mathrm{Stab}(T)$, we have $T = T \cdot \psi \in \aleph_1 \cdot \psi$. That is, for $T \in \aleph_1$, we have $T \in \aleph_2$ if and only if $\aleph_2 = \aleph_1 \cdot \psi$ for some $\psi \in \mathrm{Stab}(T)$.

Lemma 5.2.2. *Let T_1, T_2, and T_3 be vertices in the complex $\mathcal{C}_n$. If $T_1, T_2 \in \aleph_1 \cap \aleph_2$ and $T_3 \in \aleph_1$ with* $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) \subseteq \mathrm{Stab}(T_3)$ *then $T_3 \in \aleph_1 \cap \aleph_2$.*

Proof. By Observation 5.2.1, $T \in \aleph_1 \cap \aleph_2$ if and only if $\aleph_2 = \aleph_1 \cdot \psi$ for some $\psi \in \mathrm{Stab}\, T$. Thus if $T_1, T_2 \in \aleph_1 \cap \aleph_2$ then $\aleph_2 = \aleph_1 \cdot \psi_1 = \aleph_1 \cdot \psi_2$ for some $\psi_1 \in \mathrm{Stab}\, T_1$ and $\psi_2 \in \mathrm{Stab}\, T_2$. Note then that $\aleph_1 = \aleph_1 \cdot \psi_1\psi_2^{-1}$, so $\psi_1\psi_2^{-1} \in \mathrm{Stab}(\aleph_1) = \mathrm{Stab}(\alpha) = \Phi \subseteq \mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2)$. In particular, $\psi_1 = (\psi_1\psi_2^{-1})\psi_2 \in \mathrm{Stab}(T_2)$, so $\aleph_2 = \aleph_1 \cdot \psi_1$ with $\psi_1 \in \mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2)$. Additionally, if for some $T_3 \in \aleph_1$ we have $\psi_1 \in \mathrm{Stab}(T_3)$, then $T_3 \in \aleph_1 \cap \aleph_2$. This last condition holds if (but not only if) $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) \subseteq \mathrm{Stab}(T_3)$. □

Corollary 5.2.3. *Let T_1 and T_2 be vertices in the intersection $\aleph_1 \cap \aleph_2 \subseteq \mathcal{C}_n$. If* $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) = \Phi$, *then $\aleph_1 \cap \aleph_2 = \aleph_1 = \aleph_2$.*

Proof. Recall from Proposition 2.4.3 that $\mathrm{Stab}(\alpha) = \Phi$. Thus if $\mathrm{Stab}(T_1) \cap \mathrm{Stab}(T_2) = \Phi$ for some $T_1, T_2 \in \aleph_1 \cap \aleph_2$, then $\alpha \in \aleph_1 \cap \aleph_2$. But each α vertex appears in exactly one domain, hence $\aleph_1 = \aleph_2$. □

Suppose T_1 and T_2 are distinct vertices in the complex $\mathcal{C}_n$. By Lemma 2.3.8, if T_1 is a collapse of T_2, then $\mathrm{Stab}(T_2) \subseteq \mathrm{Stab}(T_1)$, so $T_2 \in \aleph_1 \cap \aleph_2 \implies T_1 \in \aleph_1 \cap \aleph_2$. Since every graph collapses to at least one of A_i, $B_{i,j,k}$, or $C_{i,j,k,l,m}$ (for some i, j, k, l, m), then to show path connectivity of intersections, it suffices to find paths in the intersection $\aleph_1 \cap \aleph_2$ with endpoints as the following six cases:

1. $A_i — A_p$
2. $B_{i,j,k} — A_p$
3. $B_{i,j,k} — B_{p,q,r}$
4. $C_{i,j,k,l,m} — A_p$
5. $C_{i,j,k,l,m} — B_{p,q,r}$
6. $C_{i,j,k,l,m} — C_{p,q,r,s,t}$

(where $i, j, k, l, m, p, q, r, s, t$ need not be distinct, unless appearing together as indices of a single vertex.) In our proofs, we will assume the 'left' vertex has fixed indices, and allow the indices of the 'right' vertex to vary.

Writing i_j for the group G_{i_j}, we recall that the stabiliser of A_i is a quotient of $(i_{v_1} \times \cdots \times i_{v_{n-1}}) \rtimes \Phi$, the stabiliser of $B_{i,j,k}$ is a quotient of $(i_{jk} \times j_k \times i_{v_1} \times \cdots \times i_{v_{n-3}}) \rtimes \Phi$, and the stabiliser of $C_{i,j,k,l,m}$ is a quotient of $(i_{lm} \times l_m \times i_{jk} \times j_k \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$. In the group of automorphisms G_{i_j}, we call G_i the *operating factor* and G_j the *dependent factor*. We say a graph has operating and dependent factors if the same is true of its stabiliser. So A_i has one operating factor, $B_{i,j,k}$ has two, and $C_{i,j,k,l,m}$ has three distinct operating factors. In each case, only one operating factor has more than one dependent factor. We now proceed through the Cases 1–6:

Lemma 5.2.4 (Case 1). *If A_i and A_p are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from A_i to A_p.*

Proof. We have $\mathrm{Stab}(A_i) \cap \mathrm{Stab}(A_p) \neq \Phi$ if and only if $i = p$. But each domain contains only one A_i-graph for each $i \in \{1, \ldots, n\}$. So either $\alpha \in \aleph_1 \cap \aleph_2$ (in which case $\alpha_1 = \alpha_2$), or $A_i = A_p$.

So if A_i and A_p are points in $\aleph_1 \cap \aleph_2$ for any $i, p \in \{1, \ldots, n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.5 (Case 2). *If $B_{i,j,k}$ and A_p are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $B_{i,j,k}$ to A_p.*

Proof. If $p \notin \{i, j\}$ then $B_{i,j,k}$ and A_p share no common operating factors, hence $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_p) = \Phi$ and $\alpha \in \aleph_1 \cap \aleph_2$.

If $p = i$ then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_p)$ contains only one operating factor, with $n - 3$ dependent factors. That is, $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_i) = (i_{v_1} \times \cdots \times i_{v_{n-3}}) \rtimes \Phi$ for $\{v_1, \ldots, v_{n-3}\} = \{1, \ldots, n\} - \{i, j, k\}$. This is precisely the stabiliser of $\gamma_{i,jk}$, hence $\gamma_{i,jk} \in \aleph_1 \cap \aleph_2$. Moreover, $\gamma_{i,jk}$ collapses to both A_i and $B_{i,j,k}$, so we have a path $B_{i,j,k} — \gamma_{i,jk} — A_i$.

For $p = j$, we have $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(A_j) = j_k \rtimes \Phi = \mathrm{Stab}(\beta_{j,k})$, and so $B_{i,j,k} — \beta_{j,k} — A_j$ is a path in $\aleph_1 \cap \aleph_2$.

So if $B_{i,j,k}$ and A_p are points in $\aleph_1 \cap \aleph_2$ for any $i, j, k, l \in \{1, \ldots, n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.6 (Case 3). *If $B_{i,j,k}$ and $B_{p,q,r}$ are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $B_{i,j,k}$ to $B_{p,q,r}$.*

Proof. In order to have $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) \neq \Phi$, we must have that $\{i, j\} \cap \{p, q\} \neq \emptyset$. We will thus assume this holds.

If $\{i, j\} = \{p, q\}$ and additionally $r = k$, then either $B_{p,q,r} = B_{i,j,k}$ and we are done, or we have $B_{p,q,r} = B_{j,i,k}$, in which case $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$. So we may assume $r \neq k$ in this case.

If $p = j$ and $q = i$ (with $r \neq k$) then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = (i_r \times j_k) \rtimes \Phi = \mathrm{Stab}(\varepsilon_{i,r,j,k})$, and $B_{i,j,k} — \varepsilon_{i,r,j,k} — B_{j,i,r}$ is a path in $\aleph_1 \cap \aleph_2$.

If $p = i$ and $q = j$ (with $r \neq k$) then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = (i_{v_1} \times \cdots \times i_{v_{n-4}}) \rtimes \Phi$ where $\{i_{v_1}, \ldots, i_{v_{n-4}}\} = \{1, \ldots, n\} - \{i, j, k, r\}$. This is contained within $\mathrm{Stab}(A_i)$, hence $A_i \in \aleph_1 \cap \aleph_2$. Then by Case 2 (Lemma 5.2.5), there is some path from $B_{i,j,k}$ to A_i and some path from A_i to $B_{i,j,r}$ in $\aleph_1 \cap \aleph_2$.

We will now consider $\{i, j\} \neq \{p, q\}$. Then $|\{i, j\} \cap \{p, q\}| = 1$ and so $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r})$ has at most one operating factor (with at most $n - 4$ dependent factors). If there is no common operating factor, then $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$. Otherwise, $\mathrm{Stab}(B_{i,j,k}) \cap \mathrm{Stab}(B_{p,q,r}) \subset A_v$ for some $v \in \{i, j, p, q\}$. Then we are reduced to Case 2.

So if $B_{i,j,k}$ and $B_{p,q,r}$ are points in $\aleph_1 \cap \aleph_2$ for any $i, j, k, p, q, r \in \{1, \ldots, n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.7 (Case 4). *If $C_{i,j,k,l,m}$ and A_p are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to A_p.*

Proof. In order to satisfy $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(A_p) \neq \Phi$, we require that $p \in \{i, j, l\}$. Note that by symmetry, $C_{i,j,k,l,m} = C_{i,l,m,j,k}$, so we only need to consider one of $p = j$ and $p = l$.

If $p = j$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(A_p) = j_k \rtimes \Phi = \mathrm{Stab}(\tau_{j,k,lm})$. We have that $\tau_{j,k,lm}$ collapses to both A_j and $C_{i,j,k,l,m}$, so $C_{i,j,k,l,m} — \tau_{j,k,lm} — A_j$ is a path in $\aleph_1 \cap \aleph_2$.

If $p = i$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(A_p) = (i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$ (where $\{v_1, \ldots, v_{n-5}\} = \{1, \ldots, n\} - \{i, j, k, l, m\}$). This is contained in the stabiliser of $\sigma_{i,jk,lm}$, which is a graph that collapses to both A_i and $C_{i,j,k,l,m}$, hence $C_{i,j,k,l,m} — \sigma_{i,jk,lm} — A_i$ is a path in $\aleph_1 \cap \aleph_2$.

So if $C_{i,j,k,l,m}$ and A_p are points in $\aleph_1 \cap \aleph_2$ for any $i, j, k, l, m, p \in \{1, \ldots, n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.8 (Case 5). *If $C_{i,j,k,l,m}$ and $B_{p,q,r}$ are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to $B_{p,q,r}$.*

Proof. To satisfy $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) \neq \Phi$, we require $\{p, q\} \cap \{i, j, l\} \neq \varnothing$.

Suppose $p = j$ and $q = l$. If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$. If $r = m$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = (j_k \times l_m) \rtimes \Phi = \mathrm{Stab}(\varepsilon_{j,k,l,m})$, and $\varepsilon_{j,k,l,m}$ collapses to both $B_{j,l,m}$ and $C_{i,j,k,l,m}$, so these points are connected by a path in $\aleph_1 \cap \aleph_2$. If $r \notin \{k, m\}$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = j_k \rtimes \Phi \subset \mathrm{Stab}(A_j)$. Thus $A_j \in \aleph_1 \cap \aleph_2$. By Case 2 (Lemma 5.2.5) there is a path in $\aleph_1 \cap \aleph_2$ from $B_{j,l,r}$ to A_j, and by Case 4 (Lemma 5.2.7) there is a path in $\aleph_1 \cap \aleph_2$ from A_j to $C_{i,j,k,l,m}$.

By symmetry of $C_{i,j,k,l,m} = C_{i,l,m,j,k}$, we do not need to consider the case $p = l$ and $q = j$.

We may now assume $\{p, q\} \neq \{j, l\}$. Again by the symmetry of C-vertices, we need only consider $\{p, q\} \cap \{i, j\} \neq \varnothing$.

Suppose $p = i$ and $q = j$ (or $q = l$ by symmetry). If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r})$ is either $(j_k \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$ or $(j_k \times i_{jk} \times i_{v_1} \times \cdots \times i_{v_{n-5}}) \rtimes \Phi$, in either case this is contained in $\mathrm{Stab}(\delta_{i,j,k,lm})$. Since $\delta_{i,j,k,lm}$ collapses to both $B_{i,j,k}$ and $C_{i,j,k,l,m}$ then this provides a path in $\aleph_1 \cap \aleph_2$. If $r \neq k$ then the only operating factor in $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r})$ is i, hence $A_i \in \aleph_1 \cap \aleph_2$ and thus by Cases 2 and 4 there is a path from $C_{i,j,k,l,m}$ to $B_{i,j,r}$ in $\aleph_1 \cap \aleph_2$.

Suppose $q = i$ and $p = j$ (or $p = l$ by symmetry). If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = \Phi$ and $\alpha \in \aleph_1 \cap \aleph_2$. If $r \in \{l, m\}$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = j_k \rtimes \Phi \subset \mathrm{Stab}(A_j)$, hence we are reduced to Cases 2 and 4. If $r \notin \{k, l, m\}$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r}) = (i_r \times j_k) \rtimes \Phi \subset \mathrm{Stab}(B_{i,j,k})$. In the previous paragraph we showed there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to $B_{i,j,k}$ via a δ-graph, and by Case 3 (Lemma 5.2.6), there is a path in $\aleph_1 \cap \aleph_2$ from $B_{i,j,k}$ to $B_{j,i,r} = B_{p,q,r}$.

If $|\{p, q\} \cap \{i, j, l\}| = 1$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(B_{p,q,r})$ has at most one operating factor (with at most $n - 6$ dependent factors). So either $\alpha \in \aleph_1 \cap \aleph_2$, or there is some A-vertex in $\aleph_1 \cap \aleph_2$, and we are reduced to Cases 2 and 4.

So if $C_{i,j,k,l,m}$ and $B_{p,q,r}$ are points in $\aleph_1 \cap \aleph_2$ for any $i, j, k, l, m, p, q, r \in \{1, \ldots, n\}$ then there is a path in $\aleph_1 \cap \aleph_2$ connecting them. □

Lemma 5.2.9 (Case 6). *If $C_{i,j,k,l,m}$ and $C_{p,q,r,s,t}$ are vertices in $\aleph_1 \cap \aleph_2$, then there is a path in $\aleph_1 \cap \aleph_2$ from $C_{i,j,k,l,m}$ to $C_{p,q,r,s,t}$ if and only if $r \neq k \implies t \neq m$.*

Proof. Suppose we have $C_{i,j,k,l,m} \in \aleph_1 \cap \aleph_2$ and $C_{p,q,r,s,t} \in \aleph_1 \cap \aleph_2$.

If $\{p, q, s\} \cap \{i, j, l\} = \varnothing$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) = \Phi$ and $\alpha \in \aleph_1 \cap \aleph_2$.

If $|\{p, q, s\} \cap \{i, j, l\}| = 1$ then we have at most one operating factor in $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t})$ (with at most $n - 7$ dependent factors), so either $\alpha \in \aleph_1 \cap \aleph_2$ (and we are done) or there is some A-vertex in $\aleph_1 \cap \aleph_2$, and we are reduced to Case 4 (Lemma 5.2.7).

Suppose $|\{p, q, s\} \cap \{i, j, l\}| = 2$. Then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t})$ has at most two operating factors. If $p \neq i$ then all of these operating factors have at most one dependent factor. Since any B-vertex has two operating factors, each with at least one dependent, then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) \subset \mathrm{Stab}(B)$ for some B-vertex. Then we are reduced to Case 5 (Lemma 5.2.8). If $p = i$ then we have at most one operating factor with at most $n - 5$ dependent factors, and the possible other operating factor has at most one dependent factor. So again there is some B-vertex in $\aleph_1 \cap \aleph_2$, and by Case 5, there must be some path between our two C-vertices.

Now suppose $\{p, q, s\} = \{i, j, l\}$. If $p = i$ then by symmetry of C-vertices we may assume $q = j$ and $s = l$. If in addition we have $r = k$ and $t = m$ then $C_{p,q,r,s,t} = C_{i,j,k,l,m}$. So suppose $\{r, t\} \neq \{k, m\}$. If $|\{r, t\} \cap \{k, m\}| = 1$ then either $B_{i,j,k}$ or $B_{i,l,m}$ is in $\aleph_1 \cap \aleph_2$, which reduces us to Case 5. If $|\{r,t\} \cap \{k,m\}| = 0$ then $A_i \in \aleph_1 \cap \aleph_2$, and we are reduced to Case 4.

Finally, suppose $p = j$, $q = i$, and $s = l$. By permuting indices in accordance with the symmetries of $C_{i,j,k,l,m}$ and $C_{p,q,r,s,t}$, this covers all cases where $\{p,q,s\} = \{i,j,l\}$ and $p \neq i$. If $r = k$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) \subseteq l_t \rtimes \Phi \subset \mathrm{Stab}(A_l)$, so by Case 4 we are done. If $r \neq k$ and $t \neq m$ then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) \subseteq (j_k \times i_r) \rtimes \Phi \subset \mathrm{Stab}(B_{i,j,k})$, so by Case 5 we are done.

A problem arises when $r \neq k$ but $t = m$. Then $\mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{p,q,r,s,t}) = \mathrm{Stab}(C_{i,j,k,l,m}) \cap \mathrm{Stab}(C_{j,i,r,l,m}) = (j_k \times l_m \times i_r) \rtimes \Phi$. The only graph T in our complex (besides $C_{i,j,k,l,m}$ and $C_{j,i,r,l,m}$) with $(j_k \times l_m \times i_r) \rtimes \Phi \subseteq \mathrm{Stab}(T)$ is $C_{l,i,r,j,k}$. So in this case $\aleph_1 \cap \aleph_2$ consists of three distinct non-adjacent points. Note that this is the only case where $\aleph_1 \cap \aleph_2$ is not path-connected.

So if $C_{i,j,k,l,m}$ and $C_{p,q,r,s,t}$ are points in $\aleph_1 \cap \aleph_2$ for any $i,j,k,l,m,p,q,r,s,t \in \{1,\ldots,n\}$, and we don't have that $r \neq k$ and $t = m$, then there is a path in $\aleph_1 \cap \aleph_2$ connecting $C_{p,q,r,s,t}$ to $C_{i,j,k,l,m}$. □

Having dealt with all six cases, we may now conclude:

Proposition 5.2.10. *If $\aleph_1$ and $\aleph_2$ are two domains with $\aleph_1 \cap \aleph_2 \neq \varnothing$ such that $\aleph_1 \cap \aleph_2$ contains some vertex which is **not** of the form $C_{i,j,k,l,m}$, then $\aleph_1 \cap \aleph_2$ is path-connected.*

Proof. Note that in the proof of Lemma 5.2.9 we showed that if $C_{i,j,k,l,m} \in \aleph_1 \cap \aleph_2$ and $C_{p,q,r,s,t} \in \aleph_1 \cap \aleph_2$ then there exists some non-C vertex in $\aleph_1 \cap \aleph_2$ if and only if $r \neq k \implies t \neq m$. The result then follows from Lemmas 5.2.4, 5.2.5, 5.2.6, 5.2.7, 5.2.8, and 5.2.9. □

Remark. Note that this statement is not saying that $\aleph_1 \cap \aleph_2$ cannot contain a vertex $C_{i,j,k,l,m}$ if it is to be path connected, just that it must also contain some other vertex as well which is not a C-vertex.

While it is not ideal that such an intersection containing only C-vertices is not path-connected, this can be handled via the following:

Proposition 5.2.11. *If $\aleph_1$ and $\aleph_2$ are two domains with $\aleph_1 \cap \aleph_2 \neq \varnothing$ such that $\aleph_1 \cap \aleph_2$ contains **only** vertices of the form $C_{i,j,k,l,m}$, then $\aleph_1 \cap \aleph_2 = \{C_{i,j,k,l,m}, C_{j,l,m,i,p}, C_{l,i,p,j,k}\}$ for some (distinct) $i,j,k,l,m,p \in \{1,\ldots,n\}$. Moreover, there exists a domain $\aleph_3$ with $\aleph_1 \cap \aleph_2 \subset \aleph_3$, such that $\aleph_1 \cup \aleph_2 \cup \aleph_3$ is simply connected.*

Proof. As noted in Case 6 (Lemma 5.2.9) above, if $\aleph_1 \cap \aleph_2$ does not contain any vertices except C-vertices, then there exist some distinct $i,j,k,l,m,p \in \{1,\ldots,n\}$ such that $\aleph_1 \cap \aleph_2 = \{C_{i,j,k,l,m}, C_{j,l,m,i,p}, C_{l,i,p,j,k}\}$. Further, we have that $\aleph_2 = \aleph_1\psi$ for some $\psi \in (i_p \times j_k \times l_m) \rtimes \Phi$ (and $\psi \notin (i_p \times j_k) \rtimes \Phi \cup (j_k \times l_m) \rtimes \Phi \cup (l_m \times i_p) \rtimes \Phi$). That is,

$\psi = (G_p, i_0)(G_k, j_0)(G_m, l_0)\phi$ for some $i_0 \in G_i$, $j_0 \in G_j$, $l_0 \in G_l$, and $\phi \in \Phi$. Define $\aleph_3$ to be $\aleph_1(G_p, i_0)$. Note that $(G_p, i_0) \in i_p \rtimes \Phi = \mathrm{Stab}(A_i)$, so we have $A_i \in \aleph_1 \cap \aleph_3$. Then $\aleph_2 = \aleph_1(G_p, i_0)(G_k, j_0)(G_m, l_0)\phi = \aleph_3(G_k, j_0)(G_m, l_0)\phi$. Note that $(G_k, j_0)(G_m, l_0)\phi \in (j_k \times l_m) \rtimes \Phi \subseteq \mathrm{Stab}(B_{j,l,m})$, so we have $B_{j,l,m} \in \aleph_2 \cap \aleph_3$. Moreover, $(G_p, i_0), (G_k, j_0)(G_m, l_0)\phi \in (i_p \times j_k \times l_m) \rtimes \Phi$. So $C_{i,j,k,l,m}, C_{j,l,m,i,p}, C_{l,i,p,j,k} \in \aleph_3$.

By Proposition 5.2.10, we have that both $\aleph_1 \cap \aleph_3$ and $\aleph_2 \cap \aleph_3$ are path-connected, and by Theorem 3.2.11, $\aleph_1$, $\aleph_2$, and $\aleph_3$ are each simply connected. Then by the Seifert–Van Kampen Theorem (Theorem 1.2.1), $\aleph_1 \cup \aleph_3$ and $\aleph_2 \cup \aleph_3$ are both simply connected. Then we can again apply the Seifert–Van Kampen Theorem to the sets $A = \aleph_1 \cup \aleph_3$ and $B = \aleph_2 \cup \aleph_3$. Since we have that $\pi_1(A) = \pi_1(B) = \{1\}$, and $A \cap B = (\aleph_1 \cap \aleph_2) \cup \aleph_3 = \aleph_3$ is path-connected by Corollary 3.1.2, then we get that $\pi_1(\aleph_1 \cup \aleph_2 \cup \aleph_3) = \pi_1(A \cup B) = \{1\}$. That is, $\aleph_1 \cup \aleph_2 \cup \aleph_3$ is simply connected. ☐

In plain language, this means if there is an edge in the Space of Domains which does not represent a simply connected subset of the complex $\mathcal{C}_n$, then it must be in the boundary of a 2-cell which **does** represent a simply connected subset of $\mathcal{C}_n$.

Observation 5.2.12. In fact, the proof of Proposition 5.2.11 shows that if $\aleph_1$ and $\aleph_2$ are two domains with $\aleph_1 \cap \aleph_2 \neq \varnothing$ such that $\aleph_1 \cap \aleph_2$ contains only vertices of the form $C_{i,j,k,l,m}$, then there exists a domain $\aleph_3$ with $\aleph_1 \cap \aleph_3$ containing a vertex of the form A_i and $\aleph_2 \cap \aleph_3$ containing a vertex of the form $B_{j,l,m}$, such that $\aleph_1 \cup \aleph_2 \cup \aleph_3$ is simply connected.

Corollary 5.2.13. *Let $\aleph_1$ and $\aleph_2$ be two domains so that $\aleph_1 \cap \aleph_2 \neq \varnothing$. Then there exists $U \subset \mathcal{C}_n$ with $\aleph_1 \cup \aleph_2 \subseteq U$ such that U is simply connected.*

Proof. Note that any pair of domains with non-empty intersection fall under precisely one of Proposition 5.2.10 or Proposition 5.2.11. In the second case, we take $U = \aleph_1 \cup \aleph_2 \cup \aleph_3$ with $\aleph_3$ as in Proposition 5.2.11. In the first case, we take $U = \aleph_1 \cup \aleph_2$. By Theorem 3.2.11 we have that each of $\aleph_1$ and $\aleph_2$ is simply connected, and since in this case $\aleph_1 \cap \aleph_2$ is path connected, then by the Seifert–van Kampen Theorem (Theorem 1.2.1), we must have that $\aleph_1 \cup \aleph_2$ is simply connected. ☐

5.3 A Map from the Space of Domains to the Complex $\mathcal{C}_n$

In this subsection, we will define a (continuous) map from the Space of Domains to the complex $\mathcal{C}_n$. This will allow us to conclude that simple connectivity of $\mathcal{C}_n$ can be deduced from simple connectivity of the Space of Domains, as desired.

Remark. If we had that pairwise intersections of domains were always path connected, then since each domain is simply connected, a theorem of Bjorner [5, Theorem 6] would tell us that if the Space of Domains is simply connected then so too is $\mathcal{C}_n$. However, since we do not have this, we must do some extra work to deduce this.

Definition 5.3.1. We define a map F from the 1-skeleton of the Space of Domains (equivalently, from the Graph of Domains) to $\mathcal{C}_n$ as follows:

- A vertex $\aleph$ in the 0-skeleton of the Space of Domains is mapped under F to the α-graph contained in the domain $\aleph \subseteq \mathcal{C}_n$.
- Let $\aleph_1 — \aleph_2$ be some edge in the Space of Domains, and set $\alpha_1 = F(\aleph_1)$ and $\alpha_2 = F(\aleph_2)$. Choose an edge path $\lambda_{12} : [0,1] \to \aleph_1 \cup \aleph_2 \subset \mathcal{C}_n$ with $\lambda_{12}(0) = \alpha_1$ and $\lambda_{12}(1) = \alpha_2$ and let $|\lambda_{12}|$ be its image. We define $F(\aleph_1 — \aleph_2) := |\lambda_{12}|$.
- Given λ_{12}, we require that λ_{21} (the chosen path in $\aleph_1 \cup \aleph_2$ from α_2 to α_1) be defined by $\lambda_{21}(t) := \lambda_{12}(1-t)$ for all $t \in [0,1]$; that is, that $F(\aleph_1 — \aleph_2) = F(\aleph_2 — \aleph_1)$.

Observation 5.3.2. Note that by Definition 5.1.4, if $\aleph_1 — \aleph_2$ is an edge in the Space of Domains, then we have that $\aleph_1 \cap \aleph_2 \neq \varnothing$ as a subset of $\mathcal{C}_n$. By Lemma 3.1.1, each of $\aleph_1$ and $\aleph_2$ is a path-connected subset of $\mathcal{C}_n$, hence so too is $\aleph_1 \cup \aleph_2 \subset \mathcal{C}_n$. Thus there must exist some path in $\aleph_1 \cup \aleph_2$ connecting $F(\aleph_1)$ and $F(\aleph_2)$.

Remark. If one wishes to be more explicit, given an edge $\aleph_1 — \aleph_2$, one may choose $T \in \aleph_1 \cap \aleph_2 \subset \mathcal{C}_n$ and $\psi, \varphi \in \mathrm{Out}_{\mathfrak{S}}(G)$ with $\aleph_1 = \mathcal{D}_n \cdot \psi$ and $\aleph_2 = \mathcal{D}_n \cdot \varphi$ so that $\psi^{-1}\varphi \in \mathrm{Stab}(T)$, where $\mathcal{D}_n$ is the fundamental domain. Denote the path in Table 1.4 from $T \cdot \psi^{-1} = T \cdot \varphi^{-1}$ to α_0 by p, where α_0 is the unique α-graph contained in $\mathcal{D}_n$, and set $\lambda_{12} := (p^{-1} \cdot \psi)(p \cdot \varphi)$. Then $F(\aleph_1 — \aleph_2) = |p| \cdot \psi \cup |p| \cdot \varphi$. It may well be that this construction leads to an $\mathrm{Out}_{\mathfrak{S}}(G)$-equivariant map, but one would need to carefully handle the choice of $T \in \aleph_1 \cap \aleph_2$ to make it so.

Lemma 5.3.3. *Any loop in $\mathcal{C}_n$ is homotopic to a loop which is the image under F of a loop in the Graph of Domains.*

Proof. Let $\lambda' = T'_0 — T'_1 — \cdots — T'_{m'} — T'_0$ be a loop in the complex $\mathcal{C}_n$. By Corollary 3.1.4, we can write down a path p in $\mathcal{C}_n$ from T'_0 to the α graph in our fundamental domain (which we call α_0). By setting $\lambda = p^{-1}\lambda' p$ we now have a based loop in $\mathcal{C}_n$ (i.e. a loop containing the 'basepoint' α_0) which is homotopic to λ'. Say $\lambda = T_0 — T_1 — \cdots — T_m$, where $T_m = T_0 = \alpha_0$.

We will now describe how to associate a domain $\aleph_i$ to each vertex T_i in λ with $T_i \in \aleph_i$. This will allow us to construct a path in the Graph of Domains whose image under F we will show to be homotopic to λ.

First set $\aleph_0$ to be $\mathcal{D}_n$, the fundamental domain. If T_i is associated to $\aleph_i$ and $T_{i+1} \in \aleph_i$, we set $\aleph_{i+1} := \aleph_i$. In particular, this is the case whenever T_{i+1} is a collapse of T_i. Note that for any edge $S — T$ in $\mathcal{C}_n$ we have that either S is a collapse of T, or T is a collapse of S. Now suppose we have a domain $\aleph_i \ni T_i$ and $T_{i+1} \notin \aleph_i$. We must then have that T_i is a collapse of T_{i+1}. Choose any domain $\aleph_{i+1}$ containing T_{i+1}. Then $T_i \in \aleph_{i+1}$, hence $T_i — T_{i+1} \subseteq \aleph_{i+1}$, and $T_i \in \aleph_i \cap \aleph_{i+1}$.

For a given domain $\aleph_i$, let $\alpha_i \in \mathcal{C}_n$ be the unique α-graph contained in $\aleph_i$. Since $\aleph_i \cap \aleph_{i+1} \neq \emptyset$, then either $\aleph_i = \aleph_{i+1}$ or $\aleph_i — \aleph_{i+1}$ is an edge in the Graph of Domains. We define paths $\mu_{i,i+1}$ as follows: if $\aleph_i = \aleph_{i+1}$, set $\mu_{i,i+1}$ to be the constant path at $\alpha_i = \alpha_{i+1}$; otherwise, set $\mu_{i,i+1}$ to be the path $\lambda_{i,i+1}$ from α_i to α_{i+1} such that $F(\aleph_i — \aleph_{i+1}) = |\lambda_{i,i+1}|$.

Then the concatenation $\mu := \mu_{0,1}\mu_{1,2} \ldots \mu_{m-1,m}$ is equal to the concatenation $F(\aleph_{\sigma(0)} — \aleph_{\sigma(1)})F(\aleph_{\sigma(1)} — \aleph_{\sigma(2)}) \ldots F(\aleph_{\sigma(k)} — \aleph_{\sigma(0)})$ which is $F(\aleph_{\sigma(0)} — \aleph_{\sigma(1)} — \aleph_{\sigma(2)} — \cdots — \aleph_{\sigma(k)} — \aleph_{\sigma(0)})$, where $\sigma(0) = 0$, and given $\sigma(i)$, $\sigma(i+1)$ is the next index such that $\alpha_{\sigma(i+1)} \neq \alpha_{\sigma(i)}$.

We now prove that λ (and hence also λ') is homotopic in $\mathcal{C}_n$ to μ.

Given T_i and its associated domain $\aleph_i$ with graph α_i, let ν_i be a path contained in $\aleph_i$ from T_i to α_i (to be explicit, one may take the correct Out(G)-image of the relevant path listed in Table 1.4). Additionally, let e_i denote the (oriented) edge (path) $T_i — T_{i+1}$. By Corollary 5.2.13, there exists a simply connected neighbourhood $U_i \subset \mathcal{C}_n$ containing $\aleph_i \cup \aleph_{i+1}$ (if $\aleph_{i+1} = \aleph_i$, instead set $U_i = \aleph_i$, and note that by Theorem 3.2.11 this is simply connected).

Then the loop $\nu_i\mu_i\nu_{i+1}^{-1}\bar{e_i}$ is contained in U_i, hence said loop is contractible in $\mathcal{C}_n$. In other words, the edge $T_i — T_{i+1}$ is homotopic in $\mathcal{C}_n$ to the path $\nu_i\mu_i\nu_{i+1}^{-1}$. Since this holds for all i, it follows that λ is homotopic in $\mathcal{C}_n$ to μ, the image under F of a loop in the Graph of Domains. □

Lemma 5.3.4. *The map F described in Definition 5.3.1 extends to a continuous map $\bar{F}$ from the Space of Domains to $\mathcal{C}_n$.*

Proof. Let $[\aleph_1, \aleph_2, \aleph_3]$ be a face (2-cell) in the Space of Domains. Then as subsets of $\mathcal{C}_n$, we have that $\aleph_1 \cap \aleph_2 \cap \aleph_3 \neq \emptyset$. Let λ_{12}, λ_{23}, and λ_{31} be paths such that each λ_{ij} is an edge path in $\aleph_i \cup \aleph_j$ from α_i to α_j with $F(\aleph_i — \aleph_j) = |\lambda_{ij}|$ (for $i, j \in \{1, 2, 3\}$ distinct). To show that F extends to a map $\bar{F}$, it will suffice to show that the concatenated path $\lambda_{12}\lambda_{23}\lambda_{31}$ is the boundary of some simply connected subset of $\mathcal{C}_n$, that is, that the loop $\lambda_{12}\lambda_{23}\lambda_{31}$ is contractible in $\mathcal{C}_n$. We illustrate the following process in Figure 1.7.

Step 1: Since $[\aleph_1, \aleph_2, \aleph_3]$ is a 2-cell in the Space of Domains, then by Definition 5.1.4, there must exist some point $x \in \aleph_1 \cap \aleph_2 \cap \aleph_3 \subseteq \mathcal{C}_n$. Let λ_{03} be a path in $\aleph_3 \subseteq \mathcal{C}_n$ from x to α_3 and let $\lambda_{30} = \overline{\lambda_{03}}$ be its reverse path (i.e. the same set of edges, but read from α_3 to x). By Theorem 3.2.11, each domain is simply connected, so $\lambda_{12}\lambda_{23}\lambda_{30}\lambda_{03}\lambda_{31}$ is a path in $\mathcal{C}_n$ which is homotopic to the image $\lambda_{12}\lambda_{23}\lambda_{31}$ of the loop $\aleph_1 — \aleph_2 — \aleph_3 — \aleph_1$ in the Space of Domains.

Step 2: By Corollary 5.2.13, there exists a simply connected neighbourhood in $\mathcal{C}_n$ containing $\aleph_2 \cup \aleph_3$. Thus the subpath $\lambda_{23}\lambda_{30}$ from α_2 to x is contained within a simply connected subset of $\mathcal{C}_n$ containing $\aleph_2$, hence is homotopic to some path λ_2 from α_2 to x fully contained in $\aleph_2$.

Step 3: By the same reasoning as in Step 2, the subpath $\lambda_{03}\lambda_{31}$ from x to α_1 is homotopic in $\mathcal{C}_n$ to some path λ_1 from x to α_1 fully contained in $\aleph_1$. We have so far shown that $\lambda_{12}\lambda_{23}\lambda_{31}$ is homotopic in $\mathcal{C}_n$ to the loop $\lambda_{12}\lambda_2\lambda_1$.

Step 4: We have that $\lambda_{12}\lambda_2\lambda_1$ is a loop contained in $\aleph_1 \cup \aleph_2$. Again, by Corollary 5.2.13, the loop $\lambda_{12}\lambda_2\lambda_1$ is contained within a simply connected subset of $\mathcal{C}_n$, hence it must be contractible. □

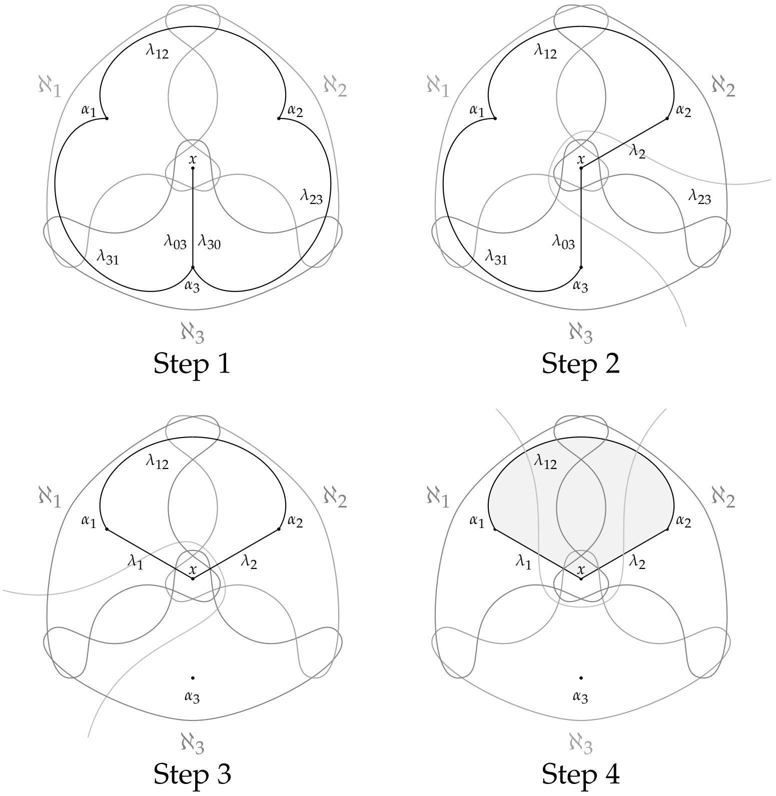


FIGURE 1.7: Illustration Contracting the Image Under F of a 3-Cycle

Proposition 5.3.5. *Suppose that the Space of Domains is simply connected. Then so too is the complex $\mathcal{C}_n$.*

Proof. Let $\lambda : \mathbb{S}^1 \to \mathcal{C}_n$ be an arbitrary loop in $\mathcal{C}_n$. We temporarily denote the Space of Domains by $\mathcal{S}$, and its 1-skeleton (the Graph of Domains) by $\mathcal{S}^{(1)}$. By Lemma 5.3.3, λ is homotopic in $\mathcal{C}_n$ to some loop $\mu : \mathbb{S}^1 \to F(\mathcal{S}^{(1)})$ which lifts to a loop $M : \mathbb{S}^1 \to \mathcal{S}$ in the Space of Domains with $\mu = F \circ M$, where $F : \mathcal{S}^{(1)} \to \mathcal{C}_n$ is the map described in Definition 5.3.1. If $\mathcal{S}$ is simply connected, then the loop M is contractible. That is, there exists a continuous map $f : \mathbb{D}^2 \to \mathcal{S}$ so that $f|_{\mathbb{S}^1} = M$. By Lemma 5.3.4, F extends to a continuos map $\bar{F} : \mathcal{S} \to \mathcal{C}_n$. Now $\bar{F} \circ f : \mathbb{D}^2 \to \mathcal{C}_n$ is a continuous map, and $\bar{F} \circ f|_{\mathbb{S}^1} = F \circ M = \mu : \mathbb{S}^1 \to \mathcal{C}_n$. Thus μ is contractible, and hence λ is as well. □

5.4 Edges in the Space of Domains

Here we will consider adjacency in the Graph/Space of Domains. We will use α to represent both a vertex in the Space of Domains (i.e. a domain, as a subspace of $\mathcal{C}_n$), and a graph in a domain (i.e. a vertex in $\mathcal{C}_n^{(0)}$).

Let $\alpha_1 — \alpha_2$ be an edge in the Space of Domains, that is, let $\alpha_1 \subset \mathcal{C}_n$ and $\alpha_2 \subset \mathcal{C}_n$ be two domains such that $\alpha_1 \cap \alpha_2 \neq \emptyset$. Then there is some vertex $T \in \mathcal{C}_n$ in the intersection $\alpha_1 \cap \alpha_2 \subset \mathcal{C}_n$ and, as noted in Section 5.2, some $\varphi \in \mathrm{Stab}(T)$ such that $\alpha_2 = (\alpha_1)\varphi$.

Thus the edge $\alpha_1 — \alpha_2$ in the Space of Domains can be described in two ways: according to some vertex $T \in \alpha_1 \cap \alpha_2$, or according to some (pure symmetric outer) automorphism $\varphi \in \mathrm{Out}_{\mathfrak{S}}(G)$ satisfying $\alpha_2 = (\alpha_1)\varphi$ (and hence also $\alpha_1 = (\alpha_2)\varphi^{-1}$). We may then label the edge $\alpha_1 — \alpha_2$ by either $\alpha_1 \overset{T}{—} \alpha_2$ or $\alpha_1 \overset{\varphi}{\rightarrow} \alpha_2$, depending on our viewpoint.

This subsection considers the former viewpoint, i.e. points in the intersection $\alpha_1 \cap \alpha_2$. As such, this subsection can be considered the Space of Domains analogue to Section 5.2. The latter viewpoint (automorphisms satisfying $\alpha_2 = (\alpha_1)\varphi$) will be discussed in Section 5.5

Definition 5.4.1. We will say an edge $\alpha_1 — \alpha_2$ in the Graph/Space of Domains is of *Type T* if there is some tree T in the intersection $\alpha_1 \cap \alpha_2$ in the complex $\mathcal{C}_n$.

Note that edges can be of more than one Type. In particular, if an edge is of Type T_1, and T_2 is a collapse of T_1, then the edge is also of Type T_2. However an edge can be of Type T_1 and Type T_2 even if neither is a collapse of the other. Recall from Section 5.2 that if $T \in \alpha_1 \cap \alpha_2$, then at least one of A_i, $B_{i,j,k}$, or $C_{i,j,k,l,m}$ is in $\alpha_1 \cap \alpha_2$ for some $i, j, k, l, m \in \{1, \ldots, n\}$, hence every edge in the Space of Domains is at least one of Type A, Type B, or Type C.

Some earlier results may be summarised using this new terminology:

Proposition 3.1.3: Any two vertices in the Graph of Domains are connected via a path whose edges are all of Type A.

Proposition 5.2.10: If an edge $\alpha_1 — \alpha_2$ in the Graph of Domains is of Type A or Type B, then $\alpha_1 \cap \alpha_2$ is path connected in the complex $\mathcal{C}_n$.

Proposition 5.2.11: If an edge $\alpha_1 — \alpha_2$ in the Graph of Domains is of Type C but not of Type A or Type B, then $\alpha_1 \cap \alpha_2$ is not path connected in the complex $\mathcal{C}_n$. However, there exists a domain $\alpha_3 \supseteq \alpha_1 \cap \alpha_2$ so that $[\alpha_1, \alpha_2, \alpha_3]$ is a 2-cell in the Space of Domains and $\alpha_1 \cup \alpha_2 \cup \alpha_3$ is simply connected in the complex $\mathcal{C}_n$.

We will now deduce that simple connectivity of the Space of Domains can be proved considering only edges of Type A.

Proposition 5.4.2. *Suppose* $\alpha_1 — \alpha_2$ *is an edge in the Space of Domains of Type B but not of Type A. Then there exists some domain* α_3 *and edges* $\alpha_1 — \alpha_3$ *and* $\alpha_3 — \alpha_2$ *of Type A such that* $\alpha_1 — \alpha_2$ *is homotopic to* $\alpha_1 — \alpha_3 — \alpha_2$ *in the Space of Domains.*

Proof. Suppose $B_{i,j,k}$ is a B-graph in the intersection $\alpha_1 \cap \alpha_2 \subset \mathcal{C}_n$.
Let $\{i, j, k, v_1, \ldots, v_{n-3}\} = \{1, \ldots, n\}$ and let $\left(H_i, H_j, H_k, H_{v_1}, \ldots, H_{v_{n-3}}\right)$ be an $\mathfrak{S}$-labelling for the α-graph in the domain α_1 of $\mathcal{C}_n$. Then $\left(H_i, H_j, H_k, H_{v_1}, \ldots, H_{v_{n-3}}\right)$ is an $\mathfrak{S}$-labelling for $B_{i,j,k}$, and by Example 2.2.3, any equivalent labelling for $B_{i,j,k}$ must be of the form $\left(H_i^{gg_i}, H_j^{gi_{jk}g_j}, H_k^{gi_{jk}j_kg_k}, H_{v_1}^{gi_{v_1}g_{v_1}}, \ldots, H_{v_{n-3}}^{gi_{v_{n-3}}g_{v_{n-3}}}\right)$ for some $g \in G = H_1 * \ldots H_n$, $g_i \in H_i$, $g_j \in H_j$, $g_k \in H_k$, $j_k \in H_j$, $i_{jk} \in H_i$, with $g_v \in H_v$ and $i_v \in H_i$ for each $v = v_1, \ldots, v_{n-3}$.

We may now assume that the α-graph in α_2 has an $\mathfrak{S}$-labelling of the form $\left(H_i, H_j^{i_{jk}}, H_k^{i_{jk}} j_k, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$ (since both inner automorphisms and relative factor automorphisms stabilise α). Since the edge $\alpha_1 — \alpha_2$ is stipulated to not be of Type A, then we must have that $j_k \neq 1$ and $i_a \neq 1$ for some $a \in \{jk, v_1, \ldots, v_{n-3}\}$.

Let α_3 be the domain whose α-graph has $\mathfrak{S}$-labelling $\left(H_i, H_j^{i_{jk}}, H_k^{i_{jk}}, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$, and let A_i and A_j be the A-graphs in α_3 with central vertex H_i and $H_j^{i_{jk}}$, respectively. Observe that $A_i \in \alpha_3 \cap \alpha_1$, thus $\alpha_1 — \alpha_3$ is an edge of Type A in the Space of Domains. Further, note that since $\left(H_i, H_j^{i_{jk}}, H_k^{i_{jk}}, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$ is an $\mathfrak{S}$-labelling for A_j, then by Definition 2.1.2, so too is $\left(H_i, H_j^{i_{jk}}, \left(H_k^{i_{jk}}\right)^{\left(j_k^{i_{jk}}\right)}, H_{v_1}^{i_{v_1}}, \ldots, H_{v_{n-3}}^{i_{v_{n-3}}}\right)$. But $\left(H_k^{i_{jk}}\right)^{\left(j_k^{i_{jk}}\right)} = H_k^{\left(i_{jk}^{-1} j_k i_{jk}\right) i_{jk}} = H_k^{i_{jk}j_k}$. Thus $A_j \in \alpha_3 \cap \alpha_2$, and so $\alpha_2 — \alpha_3$ is an edge of Type A in the Space of Domains.

Finally, since $B_{i,j,k} \in \alpha_1 \cap \alpha_2 \cap \alpha_3$ then $[\alpha_1, \alpha_2, \alpha_3]$ is a 2-cell in the Space of Domains, and so $\alpha_1 — \alpha_2$ is homotopic to the path $\alpha_1 — \alpha_3 — \alpha_2$ whose edges are both of Type A. □

Proposition 5.4.3. *Suppose* $\alpha_1 — \alpha_2$ *is an edge in the Space of Domains of Type C, but which is not of Type A or Type B. Then there exist domains* α_3 *and* α_4 *and edges* $\alpha_1 — \alpha_3$*,* $\alpha_3 — \alpha_4$ *and* $\alpha_4 — \alpha_2$ *of Type A such that* $\alpha_1 — \alpha_2$ *is homotopic in the Space of Domains to* $\alpha_1 — \alpha_3 — \alpha_4 — \alpha_2$*.*

Proof. By Proposition 5.2.11 (or rather, Observation 5.2.12), there exists a domain α_3 so that $\alpha_1 — \alpha_2$ is homotopic to $\alpha_1 — \alpha_3 — \alpha_2$, where $\alpha_1 — \alpha_3$ is an edge of Type A and $\alpha_3 — \alpha_2$ is an edge of Type B. Then by Proposition 5.4.2, there exists a domain α_4 with $\alpha_3 — \alpha_2$ homotopic to $\alpha_3 — \alpha_4 — \alpha_2$, where $\alpha_3 — \alpha_4$ and $\alpha_4 — \alpha_2$ are both edges of Type A. Now $\alpha_1 — \alpha_2$ is homotopic to $\alpha_1 — \alpha_3 — \alpha_4 — \alpha_2$. □

Corollary 5.4.4. *Any path in our Graph of Domains is homotopic (in the Space of Domains) to a path whose edges are all of Type A.*

Proof. This follows from Propositions 5.4.2 and 5.4.3, recalling that any edge in the Graph of Domains is at least one of Type A, Type B, or Type C. ☐

5.5 Relative Whitehead Automorphisms

In this subsection, we consider how to move through the Graph of Domains (i.e. how to move along edges in the Space of Domains). By Corollary 5.4.4, we need only consider edges of Type A.

In Definition 1.3.1, we define Whitehead automorphisms and multiple Whitehead automorphisms, which provide a convenient way to discuss elements of the stabilisers of vertices in the fundamental domain $\mathcal{D}_n$. These are the same (differing only in notation) as those used by Gilbert [12] and Collins and Zieschang [8]. However, to examine domains other than the fundamental domain (i.e. vertices in the Space of Domains), it will prove useful to discuss 'relative' Whitehead automorphisms, which may act on groups beyond just the factor groups G_i of the splitting $\mathfrak{G}$.

We will find that moving along edges of Type A in the Space of Domains is achieved by applying 'relative multiple Whitehead automorphisms', or equivalently, by collapsing and expanding edges of graphs of groups in $\mathcal{C}_n$ (i.e. travelling along $\alpha — A — \alpha$ paths in $\mathcal{C}_n$).

Definition 5.5.1 (Relative Whitehead Automorphism)**.** Let $H_1 * \cdots * H_n$ be an $\mathfrak{G}$ free factor splitting for $G = G_1 * \cdots * G_n$. A *relative Whitehead automorphism* (with respect to the splitting $H_1 * \cdots * H_n$) is a map ψ for which there exists $x \in H_i$ for some i and $A \subseteq \{H_1, \ldots, H_n\} - \{H_i\}$ so that ψ pointwise conjugates H_j by x for each $H_j \in A$, and pointwise fixes H_k for each $H_k \notin A$. We denote such a map ψ by (A, x). If $|A| = 1$, i.e. $A = \{H_j\}$ for some j, we may abuse notation and write (H_j, x) for $(\{H_j\}, x)$.

If $\mathbf{x} = (x_1, \ldots, x_k) \subset H_i$ for some i and $\mathbf{A} = (A_1, \ldots, A_k)$ where each $A_j \subseteq \{H_1, \ldots, H_n\} - \{H_i\}$ and $A_{j_1} \cap A_{j_2} = \varnothing$ for $j_1 \neq j_2$, then we denote by $(\mathbf{A}, \mathbf{x})$ the composition $(A_1, x_1) \ldots (A_k, x_k)$. Such a map is called a *relative multiple Whitehead automorphism*. We denote the union $A_1 \cup \cdots \cup A_k$ by $\hat{A}$, or, for longer expressions, by $\bigcup \mathbf{A}$.

The Whitehead automorphisms of Definition 1.3.1 may be thought of as relative Whitehead automorphisms with respect to the initial splitting $G_1 * \cdots * G_n$ of G. However it should be noted that they behave quite differently under composition.

Lemma 5.5.2. *Let $H_1 * \cdots * H_n$ be an $\mathfrak{G}$ free factor splitting for $G_1 * \cdots * G_n$. If $\psi \in \mathrm{Out}(G)$ is a relative Whitehead automorphism (with respect to $H_1 * \cdots * H_n$), then $\psi \in \mathrm{Out}_{\mathfrak{G}}(G)$.*

Proof. By Lemma 1.1.8, there exists $\varphi \in \mathrm{Out}_{\mathfrak{G}}(G)$ with $(G_i)\varphi = H_i = G_i^{g_i}$ for each i (for some $g_i \in G$). Since ψ is a relative Whitehead automorphism for $H_1 * \cdots * H_n$, for each

i there exists $h_i \in G$ such that $(H_i)\psi = H_i^{h_i}$. Now let $\chi \in \mathrm{Out}_{\mathfrak{S}}(G)$ be such that for each i, $\chi : g \mapsto g^{(h_i)\varphi^{-1}}$ for all $g \in G_i$. Then for each i, $((G_i)\chi)\varphi = \left((h_i)\varphi^{-1}G_i(h_i^{-1})\varphi^{-1}\right)\varphi = h_i(G_i)\varphi h_i^{-1} = h_i g_i G_i g_i^{-1} h_i^{-1} = G_i^{h_i g_i}$, and the following diagram commutes:

$$\begin{tikzcd}
G_1 * \cdots * G_n \arrow[r, "\varphi"] \arrow[d, "\chi"] & G_1^{g_1} * \cdots * G_n^{g_n} \arrow[r, equal] \arrow[d, "\psi"] & H_1 * \cdots * H_n \arrow[d, "\psi"] \\
G_1^{(h_1)\varphi^{-1}} * \cdots * G_n^{(h_n)\varphi^{-1}} \arrow[r, "\varphi"] & G_1^{h_1 g_1} * \cdots * G_n^{h_n g_n} \arrow[r, equal] & H_1^{h_1} * \cdots * H_n^{h_n}
\end{tikzcd}$$

Thus in $\mathrm{Out}(G)$ we have $\psi = \varphi^{-1}\chi\varphi$ and since $\varphi, \chi \in \mathrm{Out}_{\mathfrak{S}}(G) \leqslant \mathrm{Out}(G)$ then $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$. ☐

Lemma 5.5.3. *If $\alpha_1 — \alpha_2$ is an edge in the Space of Domains of Type A, then there exists some relative multiple Whitehead automorphism $(\mathbf{A}, \mathbf{x})$ such that $\alpha_2 = \alpha_1 \cdot (\mathbf{A}, \mathbf{x})$.*

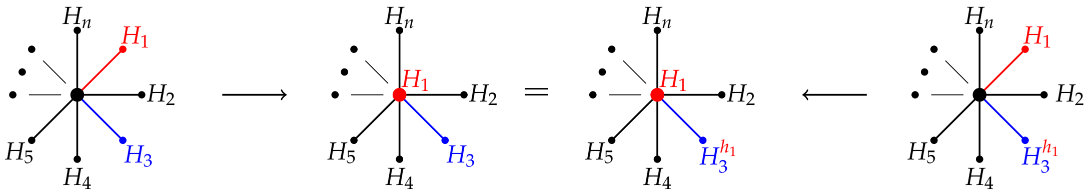
Proof. If $\alpha_1 — \alpha_2$ is an edge of Type A, then there is some A-graph $A_i \in \alpha_1 \cap \alpha_2 \subset \mathcal{C}_n$. Suppose the α-graph in the domain α_1 has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n) = (G_1^{g_1}, \ldots, G_n^{g_n})$ for some $g_1, \ldots, g_n \in G$. Then so too does A_i, and the α-graph in α_2 must have labelling $((H_1)\chi, \ldots, (H_n)\chi)$ for some $[\chi] \in \mathrm{Stab}(A_i)$.

Let $[\psi] \in \mathrm{Out}_{\mathfrak{S}}(G)$ be such that $H_k = (G_k)\psi$ for all k (that is, $\alpha_1 = \mathcal{D}_n \cdot \psi$). If $\underline{A_i}$ is the A_i-graph in $\mathcal{D}_n$, the fundamental domain, then $\mathrm{Stab}(A_i) = \psi^{-1}\,\mathrm{Stab}(\underline{A_i})\psi$. Recall from Proposition 2.4.4 and Example 2.4.2 that $\mathrm{Stab}(\underline{A_i})$ comprises elements of the form $(G_{v_1}, y_1) \ldots (G_{v_{n-1}}, y_{n-1})\varphi$ where $y_1, \ldots, y_{n-1} \in G_i$ and $\varphi \in \Phi = \prod_{k=1}^n \mathrm{Aut}(G_k)$.

Since $\varphi \in \mathrm{Stab}(\underline{\alpha})$ (where $\underline{\alpha}$ is the α-graph in $\mathcal{D}_n$), then we may assume that $\alpha_2 \cdot \psi^{-1} = \mathcal{D}_n \cdot (G_{v_1}, y_1) \ldots (G_{v_{n-1}}, y_{n-1})$ for some $y_1, \ldots, y_{n-1} \in G_i$. Now for a given factor group G_j (with $j \neq i$), we have $(G_j)(G_j, y_j)\psi = (G_j^{y_j})\psi = ((G_j)\psi)^{((y_j)\psi)} = \left(G_j^{g_j}\right)^{\left(y_j^{g_i}\right)} = (H_j)\left(H_j, y_j^{g_i}\right)$. Observe that since $y_j \in G_i$ then $y_j^{g_i} \in G_i^{g_i} = H_i$.

Thus setting $x_j = y_j^{g_i}$ for all $j \neq i$, we have $\alpha_2 = \alpha_1 \cdot \psi^{-1}(G_{v_1}, y_1) \ldots (G_{v_{n-1}}, y_{n-1})\psi = \alpha_1 \cdot (H_{v_1}, x_1) \ldots (H_{v_{n-1}}, x_{v_{n-1}})$. Grouping together terms for which $x_j = x_k$, we may then write this in the form $\alpha_2 = \alpha_1 \cdot (\mathbf{A}, \mathbf{x})$ with $\hat{A} = \bigcup \mathbf{A} \subseteq \{H_1, \ldots, H_n\} - \{H_i\}$ and $\mathbf{x} \subseteq H_i$, as required. Moreover, $(\mathbf{A}, \mathbf{x}) \in \mathrm{Stab}(A_i)$, as one would expect. ☐

Example 5.5.4. Let α_1 be the α-graph with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, and let $h_1 \in H_1$. Then $\alpha_2 := \alpha_1 \cdot (H_3, h_1)$ is the α-graph with $\mathfrak{S}$-labelling $(H_1, H_2, H_3^{h_1}, H_4, \ldots, H_n)$. Moreover, $\alpha_1 — \alpha_2$ is an edge in the Space of Domains of Type A (writing α_i for the domain containing the α-graph α_i). We demonstrate this via the following collapse–expansion path in $\mathcal{C}_n$:



It is not hard to see using this example how to extend Lemma 5.5.3 to an "if and only if" statement. Additionally, we see that relative (multiple) Whitehead automorphisms must obey the relations of the stabiliser of the relevant A-graph in $\mathcal{C}_n$.

Observation 5.5.5. In order to apply this kind of geometric argument, we must ensure that all automorphisms are written relative to the domain on which they act. This includes automorphisms written within a composition. Thus, continuing Example 5.5.4, if $h_2 \in H_2$ say, we have $(H_3, h_2h_1) = (H_3, h_1)(H_3^{h_1}, h_2)$.

Lemma 5.5.6. *Let $(H_1, \ldots, H_n)$ be an $\mathfrak{S}$-labelling for some α-graph $\alpha \in \mathcal{C}_n$. Suppose $x \in H_i$ for some i, and $A, B \subseteq \hat{H} = \{H_1, \ldots, H_n\}$ with $H_i \notin A \cup B$. If $A = \{H_{a_1}, \ldots, H_{a_m}\}$, set $A^x := \{H_{a_1}^x, \ldots, H_{a_m}^x\}$.*

1. *For $x_1, x_2 \in H_i$, we have $\alpha \cdot (A, x_1)(A^{x_1}, x_2) = \alpha \cdot (A, x_2x_1)$.*
2. *We have $\alpha \cdot (A, x)(A^x, x^{-1}) = \alpha$. We will thus write $(A^x, x^{-1}) = (A, x)^{-1}$.*
3. *If $A \cap B = \varnothing$, then $\alpha \cdot (A, x)(B, x) = \alpha \cdot (B, x)(A, x)$, which we may write as $\alpha \cdot (A \cup B, x)$.*

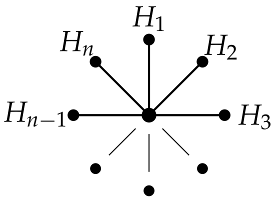
Proof. Let α be the α-graph in $\mathcal{C}_n$ with $(H_1, \ldots, H_n)$ as a labelling. We partition $\{H_1, \ldots, H_n\}$ as $\{H_{a_1}, \ldots, H_{a_m}\} \cup \{H_{i_1}, \ldots, H_{i_s}\}$ where $m + s = n$ and $H_a \in A$ for each $a \in \{a_1, \ldots, a_m\}$.

1. By Definition 2.4.3, we have that $\alpha \cdot (A, x_1)$ is the α-graph, say α_1, with labelling $(H_{a_1}^{x_1}, \ldots, H_{a_m}^{x_1}, H_{i_1}, \ldots, H_{i_s})$. Then $\alpha_1 \cdot (A^{x_1}, x_2)$ is the α-graph, say α_2, with labelling $((H_{a_1}^{x_1})^{x_2}, \ldots, (H_{a_m}^{x_1})^{x_2}, H_{i_1}, \ldots, H_{i_s}) = (H_{a_1}^{x_1x_2}, \ldots, H_{a_m}^{x_1x_2}, H_{i_1}, \ldots, H_{i_s})$. On the other hand, we clearly have that $\alpha \cdot (A, x_1x_2) = \alpha_2$.
2. This follows immediately from 1 by setting $x_1 = x$ and $x_2 = x^{-1}$ and noting that $xx^{-1} = 1$ and $(A, 1)$ is the identity for any A.
3. Since $A \cap B = \varnothing$, we will partition $\{H_1, \ldots, H_n\}$ as $\{H_{a_1}, \ldots, H_{a_p}\} \cup \{H_{b_1}, \ldots, H_{b_q}\} \cup \{H_{i_1}, \ldots, H_{i_r}\}$ where $p + q + r = n$, $H_a \in A$ for each $a \in \{a_1, \ldots, a_p\}$, and $H_b \in B$ for each $b \in \{b_1, \ldots, b_q\}$. Set $\alpha_1 := \alpha \cdot (A, x)$ and $\alpha_2 := \alpha \cdot (B, x)$. Then α_1 is the α-graph in $\mathcal{C}_n$ with labelling $(H_{a_1}^x, \ldots, H_{a_p}^x, H_{b_1}, \ldots, H_{b_q}, H_{i_1}, \ldots, H_{i_r})$, and α_2 is the α-graph in $\mathcal{C}_n$ with labelling $(H_{a_1}, \ldots, H_{a_p}, H_{b_1}^x, \ldots, H_{b_q}^x, H_{i_1}, \ldots, H_{i_r})$. Now $\alpha_3 := \alpha_1 \cdot (B, x)$ is the α-graph in $\mathcal{C}_n$ with labelling $(H_{a_1}^x, \ldots, H_{a_p}^x, H_{b_1}^x, \ldots, H_{b_q}^x, H_{i_1}, \ldots, H_{i_r})$. But

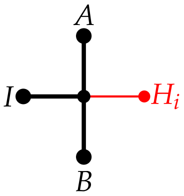
$\alpha_4 := \alpha_2 \cdot (A, x)$ is also an α-graph in $\mathcal{C}_n$, with the same labelling as α_3. Thus α_3 and α_4 belong to the same $\mathrm{Out}(G)$-orbit of the fundamental domain, $\mathcal{D}_n$. Since $\mathcal{D}_n$ contains a unique α-graph, then we must have that $\alpha_3 = \alpha_4$. That is, $\alpha \cdot (A, x)(B, x) = \alpha \cdot (B, x)(A, x)$.

□

Remark. Let $\hat{H} = \{H_1, \ldots, H_n\}$. For $I = \hat{H} - (A \cup B \cup \{H_i\})$ with $A \cap B = \varnothing$ and $H_i \notin A \cup B$, we can 'partition' the α-graph with labelling $(H_1, \ldots, H_n)$



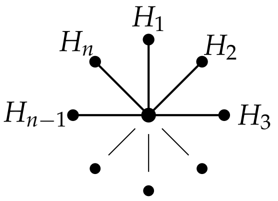
as



. One should think of this still as an α-graph, but 'abbreviated' — instead of drawing individual edges for each leaf H_j of A, we draw one wider edge (similarly for B and I).

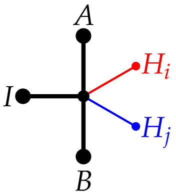
Lemma 5.5.7. *Let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for G, and let α be the α-graph (and the domain containing it) with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$. Suppose there are elements $x \in H_i$ and $y \in H_j$ for some $i \neq j$, and subsets $A, B \subseteq \{H_1, \ldots, H_n\} - \{H_i, H_j\}$. If $A \cap B = \varnothing$, then $\alpha \cdot (A, x)(B, y) = \alpha \cdot (B, y)(A, x)$.*

Proof. We have that α is the graph of groups



.

For $I = \{H_1, \ldots, H_n\} - (A \cup B \cup \{H_i, H_j\})$, we 'partition' α as



, noting that by construction $A \sqcup B \sqcup \{H_i, H_j\} \sqcup I$ forms a disjoint partition of the labelling $(H_1, \ldots, H_n)$ for α. Then the diagram in Figure 1.8 commutes.

That is, in the Space of Domains there is a loop

$$\begin{tikzcd}
\alpha \arrow[r, "{(A,x)}"] \arrow[d, "{(B,y)}"] & \alpha \cdot (A, x) \arrow[d, "{(B,y)}"] \\
\alpha \cdot (B, y) \arrow[r, "{(A,x)}"] & \alpha \cdot (A, x)(B, y)
\end{tikzcd}$$

which can be seen algebraically by considering the labellings of each α-graph. Thus we have $\alpha \cdot (A, x)(B, y) = \alpha \cdot (B, y)(A, x)$. □

Remark. With notation as in Lemma 5.5.7, if instead we have $H_j \in A$, then $\alpha \cdot (A, x)(B, y)$ is **not** well-defined geometrically — we would instead need to write $\alpha \cdot (A, x)(B, y^x)$. On the other hand, if $A \cap B \neq \varnothing$, say $B \subseteq A$, then $\alpha \cdot (A, x)(B, y)$ would also not be

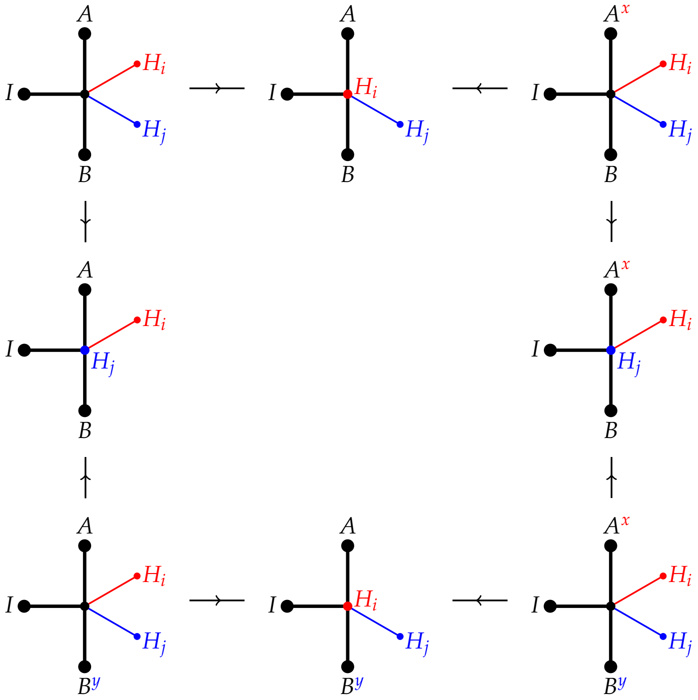


FIGURE 1.8: Commuting Diagram of α and A Graphs

well-defined geometrically. Instead, we would write $\alpha \cdot (A, x)(B^x, y)$. Note that this differs from the notation used for non-relative Whitehead automorphisms, where if α_0 is the α-graph in $\mathcal{D}_n$ (i.e. with labelling $(G_1, \ldots, G_n)$) and $x, y \in G_n$, we would have that $\alpha_0 \cdot (G_1, x)(G_1, y)$ is the α-graph with labelling $(G_1^{yx}, G_2, \ldots, G_n)$. For the remainder of the paper, all automorphisms will be assumed to be relative (multiple) Whitehead automorphisms, unless otherwise specified.

Despite our relative (multiple) Whitehead automorphisms being different objects than the Whitehead automorphisms used by Gilbert [12], we borrow some notation introduced in [12, Section 2]:

Notation 5.5.8. Let $H_1 * \cdots * H_n$ be an $\mathfrak{G}$ free factor splitting for G and set $\hat{H} := \{H_1, \ldots, H_n\}$. Given subsets $A_1, \ldots, A_k \subseteq \hat{H}$ where $A_i \cap A_j = \varnothing$ for $i \neq j$, set $\mathbf{A} = (A_1, \ldots, A_k)$ and let $\hat{A} := A_1 \cup \cdots \cup A_k$. Suppose B is an arbitrary subset of $\hat{H} - \{H_i\}$, let $x_1 \ldots, x_k, y \in H_i$ (where $H_i \notin \hat{A}$) and set $\mathbf{x} = (x_1, \ldots, x_k)$. Also set $\bar{A} := (\hat{H} - \hat{A}) - \{H_i\}$. We will adopt the following notation:

- $(\mathbf{A} \cap B, \mathbf{x}) := (A_1 \cap B, x_1) \ldots (A_k \cap B, x_k)$
- $(\mathbf{A} - B, \mathbf{x}) := (A_1 - B, x_1) \ldots (A_k - B, x_k)$

- $$\begin{aligned}(\mathbf{A} +_j B, \mathbf{x}) :=& (A_1 - B, x_1)\dots(A_{j-1} - B, x_{j-1})(A_j \cup B, x_j)\\ &(A_{j+1} - B, x_{j+1})\dots(A_k - B, x_k)\\ =& (A_1 - B, x_1)\dots(A_j \cup B, x_j)\dots(A_k - B, x_k)\end{aligned}$$
- $(\mathbf{A}, y\mathbf{x}) := (A_1, yx_1)\dots(A_k, yx_k)$
- $(\mathbf{A}, \mathbf{x}y) := (A_1, x_1y)\dots(A_k, x_ky)$
- $$\begin{aligned}(\bar{\mathbf{A}}_j, \mathbf{x}) :=& (A_1, x_1)\dots(A_{j-1}, x_{j-1})(\bar{A}, x_j)(A_{j+1}, x_{j+1})\dots(A_k, x_k)\\ =& (A_1, x_1)\dots(\bar{A}, x_j)\dots(A_k, x_k)\end{aligned}$$
- $$\begin{aligned}(\mathbf{A}, \tilde{\mathbf{x}}_j) :=& (A_1, x_1)\dots(A_{j-1}, x_{j-1})(A_j, 1)(A_{j+1}, x_{j+1})\dots(A_k, x_k)\\ =& (A_1, x_1)\dots(A_j, 1)\dots(A_k, x_k)\end{aligned}$$
- $[\mathbf{A}]_j := A_j$, $[\mathbf{x}]_j := x_j$, and $[(\mathbf{A}, \mathbf{x})]_j := (A_j, x_j)$
- If $A = \{H_{a_1}, \dots, H_{a_m}\}$ then $A^x := \{H_{a_1}^x, \dots, H_{a_m}^x\}$ and we define $\mathbf{A}^{\mathbf{x}} := (A_1^{x_1}, \dots, A_k^{x_k})$

Note that $(\mathbf{A} \cap B, \mathbf{x})$ and $(\mathbf{A} - B, \mathbf{x})$ may still be defined when $H_i \in B$.

We find that similar (though not identical) properties hold for us as are used by Gilbert [12]. In particular, part 4 of the following Proposition is adapted from [12, Lemma 2.10].

Proposition 5.5.9. *With the above notation, we have that:*

1. $(\mathbf{A}, \mathbf{x}) = (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x}) = (\mathbf{A} \cap B, \mathbf{x})(\mathbf{A} - B, \mathbf{x})$
2. $(\mathbf{A} +_j B, \mathbf{x}) = (\mathbf{A} - B, \mathbf{x})(B, x_j)$
3. $(\bar{\mathbf{A}}_j, x_j^{-1}\tilde{\mathbf{x}}_j) = (\mathbf{A}, x_j^{-1}\mathbf{x})(\bar{A}, x_j^{-1})$ *and* $(\bar{\mathbf{A}}_j, \tilde{\mathbf{x}}_j x_j^{-1}) = (\mathbf{A}, \mathbf{x}x_j^{-1})(\bar{A}, x_j^{-1})$
4. $(\mathbf{A}, \mathbf{x}) = (\mathbf{A} +_j B, \mathbf{x})((\bar{\mathbf{A}}_j \cap B)^{x_j}, \tilde{\mathbf{x}}_j x_j^{-1}) = (\bar{\mathbf{A}}_j \cap B, x_j^{-1}\tilde{\mathbf{x}}_j)\left((\mathbf{A} +_j B)', \mathbf{x}\right)$
   where $[(\mathbf{A} +_j B)']_a := [(\mathbf{A} +_j B)]_a = A_a - B$ *for* $a \in \{1, \dots, k\} - \{j\}$
   and $$\begin{aligned}[(\mathbf{A} +_j B)']_j :=& \left([(\mathbf{A} +_j B)]_j - \bigcup(\bar{\mathbf{A}}_j \cap B)\right) \cup \bigcup\left([(\mathbf{A} +_j B)]_j \cap (\bar{\mathbf{A}}_j \cap B)\right)^{x_j^{-1}\tilde{\mathbf{x}}_j}\\ =& (A_j - B) \cup \bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}} \cup (B - \hat{A})^{x_j^{-1}}\end{aligned}$$

Proof. 1. Given arbitrary sets A and B, we have $A = (A - B) \cup (A \cap B)$, which is a disjoint partition of the set A. Now

$$\begin{aligned}(\mathbf{A}, \mathbf{x}) &= (A_1, x_1)\dots(A_k, x_k)\\ &= ((A_1 - B) \cup (A_1 \cap B), x_1)\dots((A_k - B) \cup (A_k \cap B), x_k)\\ &= (A_1 - B, x_1)(A_1 \cap B, x_1)\dots(A_k - B, x_k)(A_k \cap B, x_k)\\ &= (A_1 - B, x_1)\dots(A_k - B, x_k)(A_1 \cap B, x_1)\dots(A_k \cap B, x_k)\\ &= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x}).\end{aligned}$$

Moreover, we also have $A = (A \cap B) \cup (A - B)$, and a similar argument yields $(\mathbf{A}, \mathbf{x}) = (\mathbf{A} \cap B, \mathbf{x})(\mathbf{A} - B, \mathbf{x})$.

2. Given arbitrary sets A and B, we have $A \cup B = (A - B) \cup B$, which is a disjoint partition of the set $A \cup B$. Now
$$\begin{aligned}(\mathbf{A} +_j B, \mathbf{x}) &= (A_1 - B, x_1) \dots (A_j \cup B, x_j) \dots (A_k - B, x_k)\\ &= (A_1 - B, x_1) \dots ((A_j - B) \cup B, x_j) \dots (A_k - B, x_k)\\ &= (A_1 - B, x_1) \dots (A_j - B, x_j)(B, x_j) \dots (A_k - B, x_k)\\ &= (A_1 - B, x_1) \dots (A_j - B, x_j) \dots (A_k - B, x_k)(B, x_j)\\ &= (\mathbf{A} - B, \mathbf{x})(B, x_j).\end{aligned}$$

3. We have
$$\begin{aligned}(\bar{\mathbf{A}}_j, x_j^{-1}\tilde{\mathbf{x}}_j) &= (A_1, x_j^{-1}x_1) \dots (\bar{A}, x_j^{-1}1) \dots (A_k, x_j^{-1}x_k)\\ &= (A_1, x_j^{-1}x_1) \dots (A_j, 1)(\bar{A}, x_j^{-1}) \dots (A_k, x_j^{-1}x_k)\\ &= (A_1, x_j^{-1}x_1) \dots (A_j, 1) \dots (A_k, x_j^{-1}x_k)(\bar{A}, x_j^{-1})\\ &= (A_1, x_j^{-1}x_1) \dots (A_j, x_j^{-1}x_j) \dots (A_k, x_j^{-1}x_k)(\bar{A}, x_j^{-1})\\ &= (\mathbf{A}, x_j^{-1}\mathbf{x})(\bar{A}, x_j^{-1}).\end{aligned}$$
Similarly,
$$\begin{aligned}(\bar{\mathbf{A}}_j, \tilde{\mathbf{x}}_j x_j^{-1}) &= (A_1, x_1 x_j^{-1}) \dots (\bar{A}, x_j^{-1}) \dots (A_k, x_k x_j^{-1})\\ &= (A_1, x_1 x_j^{-1}) \dots (A_j, x_j x_j^{-1})(\bar{A}, x_j^{-1}) \dots (A_k, x_k x_j^{-1})\\ &= (A_1, x_1 x_j^{-1}) \dots (A_j, x_j x_j^{-1}) \dots (A_k, x_k x_j^{-1})(\bar{A}, x_j^{-1})\\ &= (\mathbf{A}, \mathbf{x}x_j^{-1})(\bar{A}, x_j^{-1}).\end{aligned}$$

4. By 2. and 3. above, we have
$$\begin{aligned}&(\mathbf{A} +_j B, \mathbf{x})((\bar{\mathbf{A}}_j \cap B)^{x_j}, \tilde{\mathbf{x}}_j x_j^{-1})\\ &= (\mathbf{A} - B, \mathbf{x})(B, x_j)((\mathbf{A} \cap B)^{x_j}, \mathbf{x}x_j^{-1})((\bar{A} \cap B)^{x_j}, x_j^{-1})\\ &= (\mathbf{A} - B, \mathbf{x})(B \cap \hat{A}, x_j)(B - \hat{A}, x_j)((\mathbf{A} \cap B)^{x_j}, \mathbf{x}x_j^{-1})((B - \hat{A})^{x_j}, x_j^{-1})\\ &= (\mathbf{A} - B, \mathbf{x})(\hat{A} \cap B, x_j)((\mathbf{A} \cap B)^{x_j}, \mathbf{x}x_j^{-1})(B - \hat{A}, x_j)((B - \hat{A})^{x_j}, x_j^{-1})\\ &= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x}x_j^{-1}x_j)(B - \hat{A}, x_j^{-1}x_j)\\ &= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x})\\ &= (\mathbf{A}, \mathbf{x}).\end{aligned}$$
Similarly,
$$\begin{aligned}&(\bar{\mathbf{A}}_j \cap B, x_j^{-1}\tilde{\mathbf{x}}_j)((\mathbf{A} +_j B)', \mathbf{x})\\ &= (\mathbf{A} \cap B, x_j^{-1}\mathbf{x})(B - \hat{A}, x_j^{-1})(\mathbf{A} - B, \mathbf{x})((\bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}}) \cup (B - \hat{A})^{x_j^{-1}}), x_j)\\ &= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, x_j^{-1}\mathbf{x})(B - \hat{A}, x_j^{-1})(\bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}}, x_j)(B - \hat{A})^{x_j^{-1}}, x_j)\\ &= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, x_j^{-1}\mathbf{x})(\bigcup(\mathbf{A} \cap B)^{x_j^{-1}\mathbf{x}}, x_j)(B - \hat{A}, x_j^{-1})(B - \hat{A})^{x_j^{-1}}, x_j)\\ &= (\mathbf{A} - B, \mathbf{x})(\mathbf{A} \cap B, \mathbf{x})\\ &= (\mathbf{A}, \mathbf{x}).\end{aligned}$$

□

6 Peak Reduction in the Space of Domains

In this section we will prove that the Space of Domains is simply connected. We will do this via 'peak reduction'. The idea of this is that given a (based) loop in the Space of Domains, any 'peaks' in the loop can be reduced, until the loop is just the basepoint. This is roughly illustrated in Figure 1.9.

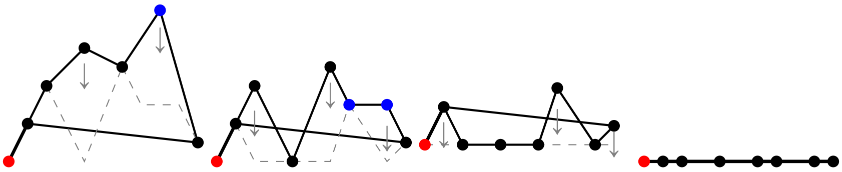


FIGURE 1.9: The Idea of 'Squashing Loops' by Reducing Peaks

We will follow the outline below, which is largely based on the method used by Gilbert [12, Section 2] (which in turned is based on the work of Collins and Zieschang [8, Section 2])[3]:

- Define a concept of 'height' of a vertex/domain α, and define a 'peak' of a loop in the Graph/Space of Domains
- By Corollary 5.4.4, we may solely consider loops comprising edges of Type A, so a 'peak' looks like $\alpha \overset{A}{—} \alpha \overset{A}{—} \alpha$
- Split into four cases of $\alpha \overset{A_i}{—} \alpha \overset{A_j}{—} \alpha$ for various conditions on i and j
- For a given path $\alpha \overset{A_i}{—} \alpha \overset{A_j}{—} \alpha$ show there is either a 4-cycle or 5-cycle in the Graph of Domains (whose edges are all of Type A) with $\alpha \overset{A_i}{—} \alpha \overset{A_j}{—} \alpha$ as a subpath
- Given such a loop in the Graph of Domains, show that it is contractible in the Space of Domains (that is, that $\alpha \overset{A_i}{—} \alpha \overset{A_j}{—} \alpha$ is homotopic to a path of length 2 or 3 with the same endpoints)
- Show that if $\alpha \overset{A_i}{—} \alpha \overset{A_j}{—} \alpha$ was a peak in some loop in the Space of Domains, then it is homotopic to a path whose 'middle' is 'smaller' than that of $\alpha \overset{A_i}{—} \alpha \overset{A_j}{—} \alpha$

Note that we are only interested in loops in the Space of Domains whose endpoints are vertices and who strictly follow edge paths (with no backtracking etc.). This is permissible, as any loop can be deformed into such a loop quite easily. It also means we can easily consider the loop in the Graph of Domains (which is just the one-skeleton of the Space of Domains).

---

[3] Note however that the objects Gilbert as well as Collins and Zieschang study are words in a group, as opposed to geometric objects, thus while the overall structure of the idea is similar, the details vary greatly.

6.1 Defining Height

Notation 6.1.1. Let α be an α-graph $\overset{n}{\circ}$ in $\mathcal{C}_n$ with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$. We denote by $\hat{\alpha}$ the G-tree which is the universal cover relative to α (according to Serre [20]), that is, the G-tree satisfying (up to equivariant isometry) $\hat{\alpha}\big/G = \alpha$ viewing α here as a quotient graph of groups (via Bass–Serre theory). We will label vertices of $\hat{\alpha}$ by their stabiliser in G, so, for example, $g \cdot G_i = G_i^g$ for $g \in G$ and $G_i \in \hat{\alpha}$. When edges in $\hat{\alpha}$ are given labels, we will write the action of G multiplicatively, so, for example, $g \cdot e = ge$.

Remark. Note that each labelling $(H_1', \ldots, H_n')$ in the equivalence class of $(H_1, \ldots, H_n)$ of labellings of α determines a lift of α in $\hat{\alpha}$ (by taking the convex hull of the vertices in $\hat{\alpha}$ with stabilisers $H_1', \ldots, H_n'$). Since this lift has the same 'shape' as α (deduced from equivalence of the labellings), we may consider α with its labelling $(H_1, \ldots, H_n)$ to be a subgraph of $\hat{\alpha}$, acting as a fundamental domain for the action of G.

Let α_0 be the fundamental domain $\mathcal{D}_n$ (the domain with the graph in Figure 1.10 at its centre). Let α be an arbitrary domain (with the graph in Figure 1.11 at its centre).

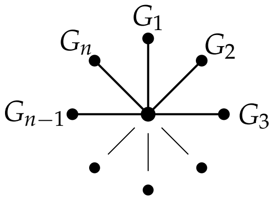


FIGURE 1.10: The α Graph at the Centre of the Fundamental Domain

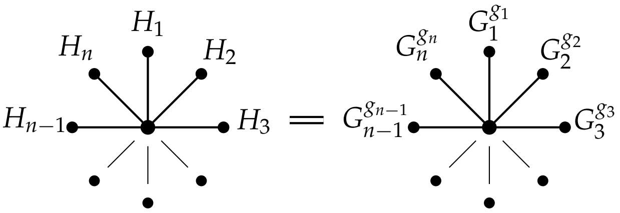


FIGURE 1.11: An Arbitrary α Graph

Let $\mathcal{W}$ be the set of pairs $\{G_i, G_j\}$ of (distinct) elements of $\{G_1, \ldots, G_n\}$. Note that $|\mathcal{W}| = \binom{n}{2} = \frac{1}{2}n(n-1)$.

Set $\left|\{G_i, G_j\}\right|_\alpha$ to be the length of the edge path from the vertex labelled G_i to the vertex labelled G_j in $\hat{\alpha}$ (note that this is symmetric, so we don't need to worry about the order of our pair).

Definition 6.1.2. We define the *height* of the domain α to be

$$||\alpha|| := \sum_{w \in \mathcal{W}} (|w|_\alpha - 2)$$

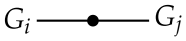
Note that this can only take (non-negative) integer values.

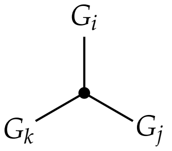
Lemma 6.1.3. *We have* $||\alpha|| = 0$ *if and only if* $\alpha = \alpha_0$.

Proof. On the one hand, in $\hat{\alpha_0}$ we have that $|w|_{\alpha_0} = 2 \; \forall w \in \mathcal{W}$, so clearly $||\alpha_0|| = 0$.

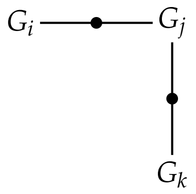
On the other hand, observe that for any $w \in \mathcal{W}$, $|w|_\alpha \geqslant 2 \; \forall \alpha$. So

$$\begin{aligned} ||\alpha|| = ||\alpha_0|| &\implies \sum_{w \in \mathcal{W}} (|w|_\alpha - 2) = 0 \\ &\implies \sum_{w \in \mathcal{W}} |w|_\alpha = 2|\mathcal{W}| \\ &\implies |w|_\alpha = 2 \quad \forall w \in \mathcal{W} \end{aligned}$$

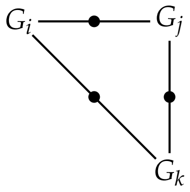
We claim that this implies $\alpha = \alpha_0$. Indeed, if there is some α such that for all $i, j \in \{1, \ldots, n\}$ we have $|\{G_i, G_j\}|_\alpha = 2$, then the path in $\hat{\alpha}$ between the vertex labelled G_i and the vertex labelled G_j must be  (for each pair $\{G_i, G_j\}$). Suppose for some $i, j, k \in \{1 \ldots, n\}$ that $\hat{\alpha}$ does not contain the tripod



. Then $\hat{\alpha}$ must contain the path

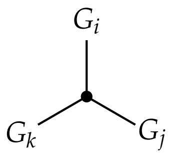


. But the only way for the length of (G_i, G_k) to be 2 now is to have a cycle



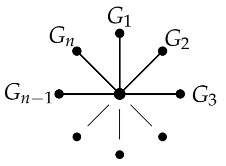
. But $\hat{\alpha}$ is a tree, so this cannot happen.

Hence $\hat{\alpha}$ must contain every tripod of the form



for all $i, j, k \in \{1, \ldots, n\}$.

But this precisely means that $\hat{\alpha}$ contains as a subgraph the star



.

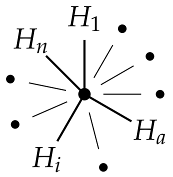
Thus if $|w|_\alpha = 2 \; \forall w \in \mathcal{W}$, then we must have $\alpha = \alpha_0$.

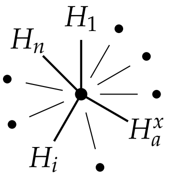
□

Note that we think of $\{G_i, G_j\}$ as both a pair of groups, and a pair of vertices in the universal cover of the graph of groups. Since the universal cover is a tree, there is a unique path from the vertex whose stabiliser is G_i to the vertex whose stabiliser is G_j, so we may also use $\{G_i, G_j\}$ to refer to the edge path connecting them (in a given $\hat{\alpha}$).

Definition 6.1.4. Let w be a sequence of edges forming a path in a G-tree $\hat{\alpha}$, and let u be any subpath of w. Denote by $\Lambda_w(u)$ the number of times the subword u (or some G-translation $z \cdot u$, or inverse $\overline{z \cdot u}$) appears in w. Define $|w|_\alpha$ to be the reduced path length of w in $\hat{\alpha}$.

Convention 6.1.5. Let α_1 be the α-graph with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, let $\psi \in \mathrm{Out}_{\mathfrak{S}}(G)$, and set $\alpha_2 := \alpha_1 \cdot \psi$. Recall that $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are the G-trees associated to α_1 and α_2, respectively, with vertices labelled by their G-stabiliser. Call the vertex with trivial stabiliser in the convex hull of the vertices $H_1, \ldots, H_n$ in $\hat{\alpha}_1$ 'v', and the vertex with trivial stabiliser in the convex hull of the vertices $(H_1)\psi, \ldots, (H_n)\psi$ in $\hat{\alpha}_2$ 'v''. We equivariantly label the edges of $\hat{\alpha}_1$ by assigning an edge $v \rightarrowtail H_j$ the label 'e_j', and its G-images 'xe_j' where $x \in G$. Similarly, we label the edges of $\hat{\alpha}_2$ of the form $v' \rightarrowtail (H_j)\psi$ 'f_j', and equivariantly extend this to a labelling of all the edges of $\hat{\alpha}_2$. We now define an equivariant map $\varphi_\psi : \hat{\alpha}_1 \to \hat{\alpha}_2$ with $(v)\varphi_\psi = v'$ so that the vertex in $\hat{\alpha}_1$ whose stabiliser is H_j is mapped to the vertex in $\hat{\alpha}_2$ whose stabiliser is H_j.

Example 6.1.6. Let α_1 be 

and $\alpha_2 = \alpha_1(H_a, x)$ (with $x \in H_i$) be 

.

Consider (the subgraphs of) $\hat{\alpha}_1$ and $\hat{\alpha}_2$ as illustrated in Figure 1.12, labelled according to Convention 6.1.5. Observe that $(e_a)\varphi_{(H_a,x)} = f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$, where $x^{-1}f_i$ is the image in $\hat{\alpha}_2$ of f_i under the action of x^{-1}, and $\overline{x^{-1}f_i}$ is its inverse edge (the same 1-cell, with opposite orientation), and for $j \neq a$ we have $(e_j)\varphi_{(H_a,x)} = f_j$. Moreover, we see that $(e_i(\overline{xe_i})(xe_a))\varphi_{(H_a,x)} = f_a$. Thus $\varphi_{(H_a,x)}$ expands some edge paths, contracts some edge paths, and does not change the length of other edge paths.

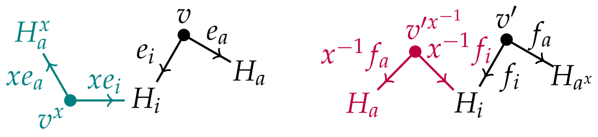


FIGURE 1.12: Subgraphs of $\hat{\alpha}_1$ (left) and $\hat{\alpha}_2$ (right)

We now prove some technical lemmas. Unless otherwise stated, Λ_w will always concern the word w relating to α_1. The following lemmas (and proofs) are similar in structure to a lemma (and proof) of Collins and Zieschang [8, Lemma 1.5], in that we count 'subwords' to determine how an automorphism changes the height of a domain. However, since our arguments are applied to different objects, we recover quite different formulae.

Lemma 6.1.7. *Let* α_1 *and* $\alpha_2 = \alpha_1(H_a, x)$ *be as in Example 6.1.6, and let* w *be an edge path in* $\hat{\alpha}_1$*. Then*

$$|(w)\varphi_{(H_a,x)}|_{\alpha_2} = |w|_{\alpha_1} + 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e}_i e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a)$$

Proof. Given any word u, let $l(u)$ be its unreduced length. We may assume that w is a reduced word, that is if $w = d_1 d_2 \ldots d_m$, then for any d_j, we have that $d_{j+1} \neq \overline{d_j}$. Then $l(w) = \sum_{k=1}^{n} \Lambda_w(e_k) = |w|_{\alpha_1} = m$. Since w is an edge path in $\hat{\alpha}_1$, then given a letter (edge) d_j, there is some $k \in \{1, \ldots, n\}$ and some $y \in G$ so that either $d_j = ye_k$ or $d_j = \overline{ye_k}$. Equivariance of $\varphi_{(H_a,x)}$ means that $(ye_k)\varphi_{(H_a,x)} = y(e_k)\varphi_{(H_a,x)}$, and $(\overline{e_k})\varphi_{(H_a,x)} = \overline{(e_k)\varphi_{(H_a,x)}}$.

Let w' be the unreduced word $(w)\varphi_{(H_a,x)}$ in α_2. That is, $w' = (d_1)\varphi_{(H_a,x)} \ldots (d_m)\varphi_{(H_a,x)}$ and $l(w') = l((d_1)\varphi_{(H_a,x)}) + \cdots + l((d_m)\varphi_{(H_a,x)})$. We will say $d_j \simeq e_k$ if $d_j = ye_k$ or $d_j = \overline{ye_k}$ for some $k \in \{1, \ldots, n\}$ and some $y \in G$. If $d_j \simeq e_k$ for some $k \neq a$ then $(d_j)\varphi_{(H_a,x)} \simeq f_k$ and $l((d_j)\varphi_{(H_a,x)}) = l(d_j) = 1$. If on the other hand $d_j \simeq e_a$, then $(d_j)\varphi_{(H_a,x)} \simeq f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$, and $l((d_j)\varphi_{(H_a,x)}) = 3l(d_j)$. Thus:

$$l(w') = \sum_{k \neq a} \Lambda_w(e_k) + 3\Lambda(e_a) = l(w) + 2\Lambda(e_a) = |w|_{\alpha_1} + 2\Lambda_w(e_a)$$

We now consider reductions to w'. Note that $(\bar{e}_i)\varphi_{(H_a,x)}(e_a)\varphi_{(H_a,x)} = \bar{f}_i f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$, which reduces to $(\overline{x^{-1}f_i})(x^{-1}f_a)$. Let w'' be the result of applying all such reductions to w' (including inversions and G-translations of the subword $\bar{f}_i f_i$ resulting from images $(\bar{e}_i e_a)\varphi_{(H_a,x)}$). Then the length of $(\bar{e}_i e_a)\varphi_{(H_a,x)}$ (and its inversions and G-translations) is 2 less in w'' than it is in w'. Hence $l(w'') = l(w') - 2\Lambda_w(\bar{e}_i e_a)$.

We also have that $(x^{-1}e_i)\varphi_{(H_a,x)}(\bar{e}_i e_a)\varphi_{(H_a,x)} = (x^{-1}f_i)(\overline{x^{-1}f_i})(x^{-1}f_a)$, which reduces to $x^{-1}f_a$. Let w''' be the result of applying all such reductions to w'' (including inversions and G-translations of the subword $((x^{-1}e_i)\bar{e}_i e_a)\varphi_{(H_a,x)})$. Then the length of $((x^{-1}e_i)\bar{e}_i e_a)\varphi_{(H_a,x)}$ (and its inversions and G-translations) is 2 less in w''' than it is in w''. So $l(w''') = l(w'') - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a)$. Since w was assumed to be reduced, there are no further reductions we can apply to w''', hence $l(w''') = |(w)\varphi_{(H_a,x)}|_{\alpha_2}$. We therefore have:

$$\begin{aligned}
|(w)\varphi_{(H_a,x)}|_{\alpha_2} = l(w''') &= l(w'') - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a) \\
&= l(w') - 2\Lambda_w(\bar{e}_i e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a) \\
&= |w|_{\alpha_1} + 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e}_i e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a).
\end{aligned}$$

□

Lemma 6.1.8. *Let* α_1 *be the* α*-graph with* $\mathfrak{S}$*-labelling* $(H_1, \ldots, H_n)$*, let* $x \in H_i$ *for some* i*, and let* $A \subseteq \{H_1, \ldots, H_n\} - \{H_i\}$*. Set* $\alpha_2 := \alpha_1(A, x)$*, label* $\hat{\alpha}_1$ *and* $\hat{\alpha}_2$ *according to Convention 6.1.5, and let* $\varphi_{(A,x)} : \hat{\alpha}_1 \to \hat{\alpha}_2$ *be the equivariant map described in Convention 6.1.5. Given an* *edge path w in $\hat{\alpha}_1$, we have:*

$$|(w)\varphi_{(A,x)}|_{\alpha_2} = |w|_{\alpha_1} + 2\sum_{a:H_a\in A}\left(\Lambda_w(e_a) - \Lambda_w(\bar{e}_i e_a) - \Lambda_w((x^{-1}e_i)\bar{e}_i e_a) - \sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\bar{e}_a e_b)\right)$$

Proof. As in Lemma 6.1.7, we let $l(u)$ be the unreduced length of a given word u, and we assume that w is a reduced word $w = d_1d_2\ldots d_m$ with $l(w) = m = |w|_{\alpha_1}$.

Let w' be the unreduced word $(w)\varphi_{(A,x)}$ in $\hat{\alpha}_2$. That is, $w' = (d_1)\varphi_{(A,x)}\ldots(d_m)\varphi_{(A,x)}$ and $l(w') = l((d_1)\varphi_{(A,x)}) + \cdots + l((d_m)\varphi_{(A,x)})$. Extended from Lemma 6.1.7, we have that $(e_a)\varphi_{(A,x)} = f_i(\overline{x^{-1}f_i})(x^{-1}f_a)$ for any a such that $H_a \in A$, and $(e_k)\varphi_{(A,x)} = f_k$ for any k such that $H_k \notin A$. Thus if $d_j \simeq e_a$ where $H_a \in A$ then $l((d_j)\varphi_{(A,x)}) = 3 = l(d_j) + 2$ and if $d_j \simeq e_k$ where $H_k \notin A$ then $l((d_j)\varphi_{(A,x)}) = 1 = l(d_j)$. Now:

$$l(w') = \sum_{j=1}^{m} l((d_j)\varphi_{(A,x)}) = 3\sum_{a:H_a\in A}\Lambda_w(e_a) + \sum_{k:H_k\notin A}\Lambda_w(e_k) = l(w) + 2\sum_{a:H_a\in A}\Lambda(e_a)$$

We now consider reductions to the word w'. As in Lemma 6.1.7, we have that for any a where $H_a \in A$, $(\bar{e}_i)\varphi_{(A,x)}(e_a)\varphi_{(A,x)} = \bar{f}_i f_i(\overline{x^{-1}f_i})(x^{-1}f_a) = (\overline{x^{-1}f_i})(x^{-1}f_a)$ and $(x^{-1}e_i)\varphi_{(A,x)}(\bar{e}_i e_a)\varphi_{(A,x)} = (x^{-1}f_i)(\overline{x^{-1}f_i})(x^{-1}f_a) = x^{-1}f_a$. Let w'' be the result of applying all such reductions to w'. Then:

$$l(w'') = l(w') - 2\sum_{a:H_a\in A}\Big(\Lambda_w(\bar{e}_i e_a) + \Lambda_w((x^{-1}e_i)\bar{e}_i e_a)\Big)$$

Contrary to Lemma 6.1.7, w'' is not yet fully reduced. Indeed, observe that for any distinct a and b with $H_a, H_b \in A$, we have that $(\bar{e}_a e_b)\varphi_{(A,x)} = (\overline{x^{-1}f_a})(x^{-1}f_i)\bar{f}_i f_i(\overline{x^{-1}f_i})(x^{-1}f_b) = (\overline{x^{-1}f_a})(x^{-1}f_b)$. Let w''' be the result of applying all such reductions to w'' (including inversions and G-translations). Then for each distinct a and b with $H_a, H_b \in A$, the length of $(\bar{e}_a e_b)\varphi_{(A,x)}$ is 4 less in w''' than it is in w''. Note that for any a and b we have $\Lambda_w(\bar{e}_a e_b) = \Lambda_w(\overline{\bar{e}_a e_b}) = \Lambda_w(\bar{e}_b e_a)$. Thus:

$$l(w''') = l(w'') - \frac{1}{2}\sum_{a:H_a\in A}\sum_{b:H_b\in A-\{H_a\}}4\Lambda_w(\bar{e}_a e_b) = l(w'') - 2\sum_{a:H_a\in A}\sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\bar{e}_a e_b)$$

Since w was assumed to be reduced, there are now no further reductions we can apply to w''', hence $l(w''') = |(w)\varphi_{(A,x)}|_{\alpha_2}$. We therefore have:

$$\begin{aligned}
&|(w)\varphi_{(A,x)}|_{\alpha_2}\\
=&l(w''')\\
=&l(w'') - 2\sum_{a:H_a\in A}\sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\bar{e}_a e_b)
\end{aligned}$$

$$
\begin{aligned}
&=l(w') - 2\sum_{a:H_a\in A}\Big(\Lambda_w(\overline{e_i}e_a)+\Lambda_w((x^{-1}e_i)\overline{e_i}e_a)\Big) - 2\sum_{a:H_a\in A}\ \sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\\
&=l(w) + 2\sum_{a:H_a\in A}\Lambda(e_a) - 2\sum_{a:H_a\in A}\left(\Lambda_w(\overline{e_i}e_a)+\Lambda_w((x^{-1}e_i)\overline{e_i}e_a)+\sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right)\\
&=|w|_{\alpha_1} + 2\sum_{a:H_a\in A}\left(\Lambda(e_a)-\Lambda_w(\overline{e_i}e_a)-\Lambda_w((x^{-1}e_i)\overline{e_i}e_a)-\sum_{b:H_b\in A-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right).
\end{aligned}
$$

□

Lemma 6.1.9. *Let* α_1 *be the* α*-graph with* $\mathfrak{S}$*-labelling* $(H_1,\ldots,H_n)$*, and let* $(\mathbf{A},\mathbf{x})$ *be a relative multiple Whitehead automorphism with respect to* α_1*, where* $\mathbf{x}\subset H_i$ *for some* i*. For brevity, we will write* $\sum_{H_a\in A_j}$ *for* $\sum_{a:H_a\in A_j}$ *(etc.). If* $\alpha_2=\alpha_1(\mathbf{A},\mathbf{x})$*, then* $||\alpha_2||-||\alpha_1||$ *is equal to:*

$$
2\sum_{w\in\mathcal{W}}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(e_a)-\Lambda_w(\overline{e_i}e_a)-\Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a)-\sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\right)
$$

Proof. We have that $(\mathbf{A},\mathbf{x})=(A_1,x_1)\ldots(A_K,x_K)$ for some disjoint subsets $A_1,\ldots,A_K\subset\{H_1,\ldots,H_n\}-\{H_i\}$ and some distinct $x_1,\ldots,x_K\in H_i$. We consider a word $w=d_1\ldots d_m$ in $\hat{\alpha}_1$ and its unreduced image $w'=(d_1)\varphi_{(\mathbf{A},\mathbf{x})}\ldots(d_m)\varphi_{(\mathbf{A},\mathbf{x})}$ in $\hat{\alpha}_2$. For any word u, let $l(u)$ be its unreduced length. As in Lemmas 6.1.7 and 6.1.8, we have that for any a with $H_a\in A_j\in\mathbf{A}$, $(e_a)\varphi_{(\mathbf{A},\mathbf{x})}=f_i(\overline{x_j^{-1}f_i})(x_j^{-1}f_a)$, and for any k with $H_k\notin\hat{A}$, $(e_k)\varphi_{(\mathbf{A},\mathbf{x})}=f_k$. Thus $l(w')=l(w)+2\sum_{H_a\in\hat{A}}\Lambda_w(e_a)=|w|_{\alpha_1}+2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Lambda_w(e_a)$.

As in Lemma 6.1.8, each (A_j,x_j) leads to reductions of the forms:

(i) $(\overline{e_i}e_a)\varphi_{(\mathbf{A},\mathbf{x})}=\overline{f_i}f_i(\overline{x_j^{-1}f_i})(x_j^{-1}f_a)=(\overline{x_j^{-1}f_i})(x_j^{-1}f_a)$,

(ii) $((x_j^{-1}e_i)\overline{e_i}e_a)\varphi_{(\mathbf{A},\mathbf{x})}=(x_j^{-1}f_i)(\overline{x_j^{-1}f_i})(x_j^{-1}f_a)=x_j^{-1}f_a$, and

(iii) $(\overline{e_a}e_b)\varphi_{(\mathbf{A},\mathbf{x})}=(\overline{x_j^{-1}f_a})(x_j^{-1}f_i)\overline{f_i}f_i(\overline{x_j^{-1}f_i})(x_j^{-1}f_b)=(\overline{x_j^{-1}f_a})(x_j^{-1}f_b)$,

where a and b are such that H_a and H_b are distinct elements of A_j. Let w'' be the result of applying all such reductions to w', and observe then that:

$$
\begin{aligned}
l(w'')&=l(w')-2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(\overline{e_i}e_a)+\Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a)+\sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right)\\
&=|w|_{\alpha_1}+2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(e_a)-\Lambda_w(\overline{e_i}e_a)-\Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a)-\sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right)
\end{aligned}
$$

We now consider further reductions to w'' which come from interactions between distinct (A_j, x_j) and (A_k, x_k). Suppose that $H_a \in A_j$ and $H_c \in A_k$, and observe that

$$\begin{aligned}
(\overline{e_a}e_c)\varphi_{(\mathbf{A},\mathbf{x})} &= (\overline{e_a})\varphi_{(\mathbf{A},\mathbf{x})}(e_c)\varphi_{(\mathbf{A},\mathbf{x})} \\
&= (\overline{x_j^{-1}f_a})(x_j^{-1}f_i)\overline{f_i}f_i(\overline{x_k^{-1}f_i})(x_k^{-1}f_c) \\
&= (\overline{x_j^{-1}f_a})(x_j^{-1f_i})(\overline{x_k^{-1}f_i})(x_k^{-1}f_c).
\end{aligned}$$

Let w''' be the result of applying all such reductions to w'', and note that the length of $(\overline{e_a}e_c)\varphi_{(\mathbf{A},\mathbf{x})}$ is 2 less in w''' than it is in w''. Recall that for any a and c, $\Lambda_w(\overline{e_a}e_c) = \Lambda_w(\overline{e_c}e_a)$. Thus $l(w''') = l(w'') - \frac{1}{2}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\sum_{H_c\in\hat{A}-A_j} 2\Lambda_w(\overline{e_a}e_c)$.

Since w was assumed to be reduced, we now have that there are no further reductions to w'''. Thus:

$$\begin{aligned}
&|(w)\varphi_{(\mathbf{A},\mathbf{x})}|_{\alpha_2} \\
=&l(w''') \\
=&l(w'') - 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\sum_{H_c\in\hat{A}-A_j}\frac{1}{2}\Lambda_w(\overline{e_a}e_c) \\
=&|w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\left(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)\right) \\
&-2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\sum_{H_c\in\hat{A}-A_j}\frac{1}{2}\Lambda_w(\overline{e_a}e_c) \\
=&|w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b) \\
&-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg)
\end{aligned}$$

Now:

$$\begin{aligned}
||\alpha_2|| =&\sum_{w\in\mathcal{W}}(|w|_{\alpha_2}) - 2|\mathcal{W}| \\
=&\sum_{w\in\mathcal{W}}\Bigg(|w|_{\alpha_1} + 2\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b) \\
&-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg)\Bigg) - 2|\mathcal{W}| \\
=&2\sum_{w\in\mathcal{W}}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a) - \Lambda_w(\overline{e_i}e_a) - \Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a) - \sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)
\end{aligned}$$

$$-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg)+\sum_{w\in\mathcal{W}}(|w|_{\alpha_1})-2|\mathcal{W}|$$

$$=2\sum_{w\in\mathcal{W}}\sum_{A_j\in\mathbf{A}}\sum_{H_a\in A_j}\Bigg(\Lambda_w(e_a)-\Lambda_w(\overline{e_i}e_a)-\Lambda_w((x_j^{-1}e_i)\overline{e_i}e_a)-\sum_{H_b\in A_j-\{H_a\}}\Lambda_w(\overline{e_a}e_b)$$

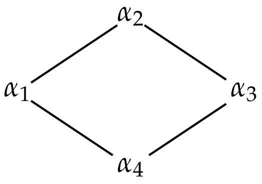
$$-\frac{1}{2}\sum_{H_c\in\hat{A}-A_j}\Lambda_w(\overline{e_a}e_c)\Bigg)+||\alpha_1||.$$

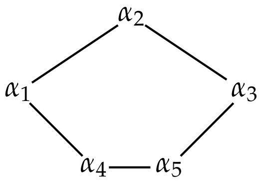
□

Remark. Since an edge path w in $\hat{\alpha}_1$ is uniquely defined by its endpoints, which are preseved by the map $\varphi_{(\mathbf{A},\mathbf{x})}$, we will often write $|w|_{\alpha_2}$ for $|(w)\varphi_{(\mathbf{A},\mathbf{x})}|_{\alpha_2}$.

6.2 Reducible Peaks

Definition 6.2.1. We will say a path $\alpha_1 — \alpha_2 — \alpha_3$ in our Graph/Space of Domains is a *peak* if $||\alpha_2|| \geqslant ||\alpha_1||$ and $||\alpha_2|| \geqslant ||\alpha_3||$, and either $||\alpha_2|| > ||\alpha_1||$ or $||\alpha_2|| > ||\alpha_3||$ (or both). Equivalently, $\alpha_1 — \alpha_2 — \alpha_3$ is a peak if $||\alpha_2|| \geqslant \max(||\alpha_1||, ||\alpha_3||)$ and $||\alpha_2|| > \min(||\alpha_1||, ||\alpha_3||)$.

In this section, we claim that given a path $\alpha_1 — \alpha_2 — \alpha_3$, there exist domains α_4 and α_5 such that 

or 

forms a loop in the Graph of Domains.

Moreover, we claim that this loop is contractible in our Space of Domains. Further, we claim that if $\alpha_1 — \alpha_2 — \alpha_3$ was a peak, then $\alpha_1 — \alpha_4 — \alpha_3$ or $\alpha_1 — \alpha_4 — \alpha_5 — \alpha_3$ is a reduction; that is, $||\alpha_4|| < ||\alpha_2||$ and (if we are in the second of these cases) $||\alpha_5|| < ||\alpha_2||$. In other words, we are claiming that the peak is *reducible*.

Later in this section we will encounter many lemmas, divided into multiple cases. The series of lemmas in each case will roughly follow the structure outlined above. These will often correspond to lemmas used by Gilbert [12, Section 2], but as Gilbert is reducing (cyclic) words of G and we are reducing domains in $\mathcal{C}_n$, the proofs are quite different. Recall as well that the notation we use differs subtly to that used by Gilbert.

Definition 6.2.2. We say a peak $\alpha_1 — \alpha_2 — \alpha_3$ is *reducible* (or 'can be reduced') if the path $\alpha_1 — \alpha_2 — \alpha_3$ is homotopic (in the Space of Domains) to some path $\alpha_1 = \chi_0 — \chi_1 — \cdots — \chi_{k-1} — \chi_k = \alpha_3$ where $||\chi_i|| < ||\alpha_2||$ for every $1 \leqslant i \leqslant k-1$.

Proposition 6.2.3. *Suppose* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak in the Space of Domains (whose edges are both of Type A), where the* α*-graph in the domain* α_2 *has* $\mathfrak{S}$*-labelling* $(H_1, \dots, H_n)$*. If there is some* $i \in \{1, \dots, n\}$ *so that* $\mathbf{x}, \mathbf{y} \subset H_i$*, then this peak is reducible.*

This proposition corresponds to [12, Lemma 2.4].

Proof. If $(\mathbf{A}, \mathbf{x}) = (\mathbf{B}, \mathbf{y})$, then $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x}) = \alpha_2(\mathbf{B}, \mathbf{y}) = \alpha_3$. Then our peak is really the loop $\alpha_1 — \alpha_2 — \alpha_1$. Since this is a forwards and backwards traversal of a single edge in the Space of Domains, then this is clearly contractible to the point α_1. By Definition 6.2.1, we must have that $||\alpha_1|| < ||\alpha_2||$. Thus the constant 'path' at α_1 is a reduction of the peak $\alpha_1 — \alpha_2 — \alpha_1$.

Now suppose $(\mathbf{A}, \mathbf{x}) \neq (\mathbf{B}, \mathbf{y})$. Since $\mathbf{x}, \mathbf{y} \subset H_i$, then the A-graph in the domain α_2 with central vertex group H_i (call it A_i) belongs to both $\alpha_2 \cap \alpha_1$ and $\alpha_2 \cap \alpha_3$. In particular, $A_i \in \alpha_1 \cap \alpha_3$, so there is an edge $[\alpha_1, \alpha_3]$ between α_1 and α_3 in the space of domains. Moreover, $A_i \in \alpha_1 \cap \alpha_2 \cap \alpha_3$, so there is a 2-cell $[\alpha_1, \alpha_2, \alpha_3]$ in the Space of Domains. Hence the path $\alpha_1 — \alpha_2 — \alpha_3$ is homotopic in the Space of Domains to the single edge path $\alpha_1 — \alpha_3$. The condition in Definition 6.2.2 is vacuously satisfied here, thus $\alpha_1 — \alpha_3$ is a reduction of the peak $\alpha_1 — \alpha_2 — \alpha_3$. ☐

From now on, we will be considering peaks of the form $\alpha_1 \overset{A_i}{—} \alpha_2 \overset{A_j}{—} \alpha_3$ where $i \neq j$, that is, paths $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ where $\mathbf{x}$ and $\mathbf{y}$ belong to different factor groups of the splitting associated to α_2. If the α-graph contained in the domain α_2 has $\mathfrak{S}$-labelling $(H_1, \dots, H_n)$, let i and j be the (distinct) elements of $\{1, \dots, n\}$ such that $\mathbf{x} \subset H_i$ and $\mathbf{y} \subset H_j$. Recall from Definition 5.5.1 that $\mathbf{A} = (A_1, \dots, A_k)$, a disjoint partition of a subset of $\{H_1, \dots, H_n\}$, and $\hat{A} := A_1 \cup \dots \cup A_k$. Similarly, $\mathbf{B} = (B_1, \dots, B_l)$ is another disjoint partition of some subset of $\{H_1, \dots H_n\}$ and $\hat{B} := B_1 \cup \dots \cup B_l$.

Observation 6.2.4. Note that we necessarily have $H_i \notin \hat{A}$ and $H_j \notin \hat{B}$. We adopt the four cases used by Gilbert [12, Lemma 2.12]:

Case 1: $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$

Case 2: $H_i \in \hat{B}$ and $H_j \notin \hat{A}$

Case 3: $H_i \notin \hat{B}$ and $H_j \in \hat{A}$

Case 4: $H_i \in \hat{B}$ and $H_j \in \hat{A}$

As Cases 2 and 3 are symmetric, we will not consider Case 3 (since after renaming, this will be identical to Case 2). Additionally, if $H_j \in \hat{A}$ then there exists $A_p \in \mathbf{A}$ with $H_j \in A_p$, and if $H_i \in \hat{B}$, then there exists $B_q \in \mathbf{B}$ with $H_i \in B_q$. As Gilbert does in [12, Lemma 2.12], we further split the remaining cases as follows:

Case 1(a): $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$, with $\hat{A} \cap \hat{B} = \varnothing$

Case 1(b): $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$, with $\hat{A} \cap \hat{B} \neq \varnothing$

Case 2(a): $H_i \in \hat{B}$ (say $H_i \in B_q$) and $H_j \notin \hat{A}$, with $\hat{A} \subseteq B_q$

Case 2(b): $H_i \in \hat{B}$ (say $H_i \in B_q$) and $H_j \notin \hat{A}$, with $\hat{A} \nsubseteq B_q$

Case 4: $H_i \in \hat{B}$ and $H_j \in \hat{A}$ (say $H_i \in B_q$ and $H_j \in A_p$)

We now present a series of lemmas in order to prove that in each of the above cases, the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is reducible.

Case 1(a): $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$, with $\hat{A} \cap \hat{B} = \varnothing$

The lemmas for this case are adapted from [12, Lemma 2.6].

Lemma 6.2.5. *If* $H_i \notin \hat{B}$ *and* $H_j \notin \hat{A}$ *with* $\hat{A} \cap \hat{B} = \varnothing$*, then* $(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = (\mathbf{B},\mathbf{y})(\mathbf{A},\mathbf{x})^{-1}$.

That is, there exists a vertex α_4 in our Graph of Domains such that

$$\begin{tikzcd}
& \alpha_2 \arrow[dl, "(\mathbf{A},\mathbf{x})"'] \arrow[dr, "(\mathbf{B},\mathbf{y})"] & \\
\alpha_1 \arrow[dr, "(\mathbf{B},\mathbf{y})"'] & & \alpha_3 \arrow[dl, "(\mathbf{A},\mathbf{x})"] \\
& \alpha_4 &
\end{tikzcd}$$

is a loop.

Proof. By assumption, we have $\alpha_1 = \alpha_2(\mathbf{A},\mathbf{x})$. Since $H_j \notin \hat{A}$, then the $\mathfrak{G}$-labelling for the α-graph α_1 contained in the domain α_1 contains the group H_j. Thus as vertices of $\mathcal{C}_n$, we can collapse an edge of α_1 to achieve an A-graph with the group H_j at its centre. Then $(\mathbf{B},\mathbf{y})$ is in the stabiliser of this A-graph, meaning there is an edge in the Graph of Domains from α_1 to $\alpha_1(\mathbf{B},\mathbf{y}) = \alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})$, which we will call α_4.

Now by Definition 5.5.1 we can write $(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})$ as $(A_1,x_1)\dots(A_k,x_k)(B_1,y_1)\dots(B_l,y_l)$ for some k and l in $\mathbb{N}$. So by repeated applications of Lemma 5.5.7, we have $(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y}) = (\mathbf{B},\mathbf{y})(\mathbf{A},\mathbf{x})$, noting that each A_a and B_b are pairwise disjoint.

Hence $\alpha_3 \cdot (\mathbf{A},\mathbf{x}) = \alpha_2 \cdot (\mathbf{B},\mathbf{y})(\mathbf{A},\mathbf{x}) = \alpha_2 \cdot (\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y}) = \alpha_4$. □

Lemma 6.2.6. *The loop*

$$\begin{tikzcd}
& \alpha_2 \arrow[dl, "(\mathbf{A},\mathbf{x})"'] \arrow[dr, "(\mathbf{B},\mathbf{y})"] & \\
\alpha_1 \arrow[dr, "(\mathbf{B},\mathbf{y})"'] & & \alpha_3 \arrow[dl, "(\mathbf{A},\mathbf{x})"] \\
& \alpha_4 &
\end{tikzcd}$$

described in Lemma 6.2.5 (with $\mathbf{x} \subset H_i$, $\mathbf{y} \subset H_j$, $H_i \notin \hat{B}$, $H_j \notin \hat{A}$*, and* $\hat{A} \cap \hat{B} = \varnothing$*) is contractible in our Space of Domains.*

$$\begin{tikzcd}
\hat{\alpha}_{0,0} \arrow[r] \arrow[d] \arrow[rrrrr, bend left, "(\mathbf{B},\mathbf{y})"] \arrow[ddddd, bend right, "(\mathbf{A},\mathbf{x})"'] & \cdots \arrow[r] & \hat{\alpha}_{0,b} \arrow[r, "(\mathbf{B},\mathbf{y})|_{b+1}"'] \arrow[d] & \hat{\alpha}_{0,b+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{\alpha}_{0,L} \arrow[d] \arrow[ddddd, bend left, "(\mathbf{A},\mathbf{x})"] \\
\vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\
\hat{\alpha}_{a,o} \arrow[r] \arrow[d, "(\mathbf{A},\mathbf{x})|_{a+1}"] & \cdots \arrow[r] & \hat{\alpha}_{a,b} \arrow[r] \arrow[d] & \hat{\alpha}_{a,b+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{\alpha}_{a,L} \arrow[d] \\
\hat{\alpha}_{a+1,0} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{\alpha}_{a+1,b} \arrow[r] \arrow[d] & \hat{\alpha}_{a+1,b+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \hat{\alpha}_{a+1,L} \arrow[d] \\
\vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\
\hat{\alpha}_{K,0} \arrow[r] \arrow[rrrrr, bend right, "(\mathbf{B},\mathbf{y})"'] & \cdots \arrow[r] & \hat{\alpha}_{K,b} \arrow[r] & \hat{\alpha}_{K,b+1} \arrow[r] & \cdots \arrow[r] & \hat{\alpha}_{K,L}
\end{tikzcd}$$

FIGURE 1.13: Lattice Describing $(A, x)(B, y) = (B, y)(A, x)$ in Case 1a

Proof. Let $\hat{A} = \{H_{A_1}, \ldots, H_{A_K}\}$ and $\hat{B} = \{H_{B_1}, \ldots, H_{B_L}\}$. For $a \in \{1, \ldots, K\}$, write $(\mathbf{A}, \mathbf{x})|_a := (\mathbf{A} \cap \{H_{A_a}\}, \mathbf{x})$. Set $\hat{\alpha}_{0,0} := \alpha_2$, and recursively define $\hat{\alpha}_{a+1,b} := \hat{\alpha}_{a,b}(\mathbf{A}, \mathbf{x})|_{a+1}$ and $\hat{\alpha}_{a,b+1} := \hat{\alpha}_{a,b}(\mathbf{B}, \mathbf{y})|_{b+1}$. Note then that $\alpha_1 = \hat{\alpha}_{K,0}$, $\alpha_3 = \hat{\alpha}_{0,L}$, and $\alpha_4 = \hat{\alpha}_{K,L}$. We can now build the lattice depicted in Figure 1.13.

Note that for each $a \in \{0, \ldots, K-1\}$ and $b \in \{0, \ldots, L-1\}$, the square

$$\begin{tikzcd}
\hat{\alpha}_{a,b} \arrow[r, "(\mathbf{B},\mathbf{y})|_{b+1}"] \arrow[d, "(\mathbf{A},\mathbf{x})|_{a+1}"'] & \hat{\alpha}_{a,b+1} \arrow[d, "(\mathbf{A},\mathbf{x})|_{a+1}"] \\
\hat{\alpha}_{a+1,b} \arrow[r, "(\mathbf{B},\mathbf{y})|_{b+1}"'] & \hat{\alpha}_{a+1,b+1}
\end{tikzcd}$$

is such that the B-graph $B_{i,j,B_{b+1}}$ in the domain $\hat{\alpha}_{a,b}$ (and similarly the graph $B_{j,i,A_{a+1}}$) lives in the intersection $\hat{\alpha}_{a,b} \cap \hat{\alpha}_{a+1,b} \cap \hat{\alpha}_{a,b+1} \cap \hat{\alpha}_{a+1,b+1}$, hence this intersection is non-empty. By Definition 5.1.4, this means that the square is contractible in the Space of Domains. In the same way, loops $\hat{\alpha}_{0,b} - \cdots - \hat{\alpha}_{K,b} - \hat{\alpha}_{0,b}$ are contractible via the A-graph A_j in $\hat{\alpha}_{0,b}$, and similarly $\hat{\alpha}_{a,0} - \cdots - \hat{\alpha}_{a,L} - \hat{\alpha}_{a,0}$ via A_i. Since every 'cell' in our lattice is contractible in our Space of Domains, then so too is our initial loop (for which the lattice is akin to a tiling). $\square$

Note that we only actually needed the first and last columns (or rows) of this lattice, since the graph B_{j,i,A_a} (or B_{i,j,B_b}) lies in the intersection of all the domains in row a (or column b).

Lemma 6.2.7. *Suppose we have a loop*

$$\begin{tikzcd}
& \alpha_2 \arrow[dl, "{(H_a,x)}"'] \arrow[dr, "{(H_b,y)}"] & \\
\alpha_1 \arrow[dr, "{(H_b,y)}"'] & & \alpha_3 \arrow[dl, "{(H_a,x)}"] \\
& \alpha_4 &
\end{tikzcd}$$

where $x \in H_i$, $y \in H_j$, $H_a \neq H_b$, *and* $\{H_a, H_b\} \cap \{H_i, H_j\} = \varnothing$. *Then for any edge path* w *in* α_2, *we have* $|w|_{\alpha_4} - |w|_{\alpha_1} = |w|_{\alpha_3} - |w|_{\alpha_2}$.

Proof. Let w be a reduced edge path in $\hat{\alpha}_2$, the G-tree associated to the α-graph α_2 contained in the domain α_2. By assumption, α_2 is the (domain containing the) α-graph with $\mathfrak{S}$-labelling $(H_1, \ldots, H_n) = (H_a, H_b, H_i, H_j, H_{v_1}, \ldots, H_{v_{n-4}})$. It follows that α_4 is the (domain containing the) α-graph with $\mathfrak{S}$-labelling $(H_a^x, H_b^y, H_i, H_j, H_{v_1}, \ldots, H_{v_{n-4}})$, where $x \in H_i$ and $y \in H_j$. Suppose the edges in $\hat{\alpha}_2$ are labelled with e's, and the edges in $\hat{\alpha}_4$ are labelled with h's. Let $\varphi_{24} : \hat{\alpha}_2 \to \hat{\alpha}_4$ be the equivariant map described in Convention 6.1.5. Then $(e_a)\varphi_{24} = h_i(\overline{x^{-1}h_i})(x^{-1}h_a)$, $(e_b)\varphi_{24} = h_j(\overline{y^{-1}h_j})(y^{-1}h_b)$, and $(e_k)\varphi_{24} = h_k$ for any $k \neq a, b$.

Utilising the ideas from the proof of Lemma 6.1.7, we will let $l(u)$ be the unreduced length of a word u, and let w' be the unreduced word $(w)\varphi_{24}$ in $\hat{\alpha}_4$. Then $l(w') = l(w) + 2\Lambda_w(e_a) + 2\Lambda_w(e_b) = |w|_{\alpha_2} + 2\left(\Lambda_w(e_a) + \Lambda_w(e_b)\right)$.

Let w'' be the result of applying all reductions of the forms

$$\begin{aligned}
(\bar{e}_i e_a)\varphi &= \bar{h}_i h_i(\overline{x^{-1}h_i})(x^{-1}h_a) = (\overline{x^{-1}h_i})(x^{-1}h_a) \\
((e_i x^{-1})\bar{e}_i e_a)\varphi &= (x^{-1}h_i)(\overline{x^{-1}h_i})(x^{-1}h_a) = x^{-1}h_a \\
(\bar{e}_j e_b)\varphi &= \bar{h}_j h_j(\overline{y^{-1}h_j})(y^{-1}h_b) = (\overline{y^{-1}h_j})(y^{-1}h_b) \\
((e_j y^{-1})\bar{e}_j e_b)\varphi &= (y^{-1}h_j)(\overline{y^{-1}h_j})(y^{-1}h_b) = y^{-1}h_b,
\end{aligned}$$

as in the proof of Lemma 6.1.7. Then:

$$\begin{aligned}
l(w'') =& l(w') - 2\Lambda_w(\bar{e}_i e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a) - 2\Lambda_w(\bar{e}_j e_b) - 2\Lambda_w((y^{-1}e_j)\bar{e}_j e_b) \\
=& |w|_{\alpha_2} + 2\Big(\Lambda_w(e_a) + \Lambda_w(e_b) - \Lambda_w(\bar{e}_i e_a) - \Lambda_w(\bar{e}_j e_b) - \Lambda_w((x^{-1}e_i)\bar{e}_i e_a) \\
& - \Lambda_w((y^{-1}e_j)\bar{e}_j e_b)\Big).
\end{aligned}$$

Since H_i, H_j, H_a, and H_b are all distinct, then there are no possible 'cross-reductions' to w'' as found in the proofs of Lemmas 6.1.8 and 6.1.9. Since w was assumed to be reduced in $\hat{\alpha}_2$, this now implies that w'' is reduced in $\hat{\alpha}_4$, hence $|w|_{\alpha_4} = l(w'')$.

Thus by Lemma 6.1.7, we have:

$$\begin{aligned}
& |w|_{\alpha_1} + |w|_{\alpha_3} - |w|_{\alpha_2} \\
=& |w|_{\alpha_2} + 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e}_i e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a)
\end{aligned}$$

$$
\begin{aligned}
&+|w|_{\alpha_2} + 2\Lambda_w(e_b) - 2\Lambda_w(\bar{e}_j e_b) - 2\Lambda_w((y^{-1}e_j)\bar{e}_j e_b) - |w|_{\alpha_2} \\
=&|w|_{\alpha_2} + 2\left(\Lambda_w(e_a) - \Lambda_w(\bar{e}_i e_a) - \Lambda_w((x^{-1}e_i)\bar{e}_i e_a) + \Lambda_w(e_b) - \Lambda_w(\bar{e}_j e_b) - \Lambda_w((y^{-1}e_j)\bar{e}_j e_b)\right) \\
=&|w|_{\alpha_4}.
\end{aligned}
$$

□

Observation 6.2.8. Recall from the proof of Lemma 6.2.6 that we can write the loop

$$\begin{tikzcd}
 & \alpha_2 \arrow[dl, "{(\mathbf{A},\mathbf{x})}"'] \arrow[dr, "{(\mathbf{B},\mathbf{y})}"] & \\
\alpha_1 \arrow[dr, "{(\mathbf{B},\mathbf{y})}"'] & & \alpha_3 \arrow[dl, "{(\mathbf{A},\mathbf{x})}"] \\
 & \alpha_4 &
\end{tikzcd}$$

as a lattice, each 'cell' of which has the form of the loop in Lemma 6.2.7. Using the notation from Lemma 6.2.6, for any edge path w in $\alpha_2 = \hat{\alpha}_{0,0}$, we have that:

$$
\begin{aligned}
&|w|_{\alpha_4} - |w|_{\alpha_1} \\
=&|w|_{\hat{\alpha}_{K,L}} - |w|_{\hat{\alpha}_{K,0}} \\
=&|w|_{\hat{\alpha}_{K,L}} - |w|_{\hat{\alpha}_{K,L-1}} + |w|_{\hat{\alpha}_{K,L-1}} - \cdots - |w|_{\hat{\alpha}_{K,1}} + |w|_{\hat{\alpha}_{K,1}} - |w|_{\hat{\alpha}_{K,0}} \\
=&|w|_{\hat{\alpha}_{K-1,L}} - |w|_{\hat{\alpha}_{K-1,L-1}} + |w|_{\hat{\alpha}_{K-1,L-1}} - \cdots - |w|_{\hat{\alpha}_{K-1,1}} + |w|_{\hat{\alpha}_{K-1,1}} - |w|_{\hat{\alpha}_{K-1,0}} \\
&\vdots \\
=&|w|_{\hat{\alpha}_{0,L}} - |w|_{\hat{\alpha}_{0,L-1}} + |w|_{\hat{\alpha}_{0,L-1}} - \cdots - |w|_{\hat{\alpha}_{0,1}} + |w|_{\hat{\alpha}_{0,1}} - |w|_{\hat{\alpha}_{0,0}} \\
=&|w|_{\hat{\alpha}_{0,L}} - |w|_{\hat{\alpha}_{0,0}} \\
=&|w|_{\alpha_3} - |w|_{\alpha_2}.
\end{aligned}
$$

Lemma 6.2.9. *If $H_i \notin \hat{B}$ and $H_j \notin \hat{A}$ and $\hat{A} \cap \hat{B} = \varnothing$ then $||\alpha_4|| - ||\alpha_1|| = ||\alpha_3|| - ||\alpha_2||$, where $\alpha_4 = \alpha_1(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})$.*

Proof. By Lemma 6.2.7 and the Observation 6.2.8,

$$
\begin{aligned}
||\alpha_4|| - ||\alpha_1|| &= \sum_{w \in \mathcal{W}} \left(|w|_{\alpha_4} - 2\right) - \sum_{w \in \mathcal{W}} \left(|w|_{\alpha_1} - 2\right) \\
&= \sum_{w \in \mathcal{W}} \left(|w|_{\alpha_4} - 2 - |w|_{\alpha_1} + 2\right) \\
&= \sum_{w \in \mathcal{W}} \left(|w|_{\alpha_3} - |w|_{\alpha_2}\right) \\
&= \sum_{w \in \mathcal{W}} \left(|w|_{\alpha_3} - 2\right) - \sum_{w \in \mathcal{W}} \left(|w|_{\alpha_2} - 2\right) \\
&= ||\alpha_3|| - ||\alpha_2||.
\end{aligned}
$$

□

Lemma 6.2.10. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak in the Graph of Domains, where* $H_i \notin \hat{B}$ *and* $H_j \notin \hat{A}$ *with* $\hat{A} \cap \hat{B} = \emptyset$*, then* $||\alpha_2 \cdot (\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})|| < ||\alpha_2||$.

Proof. By Lemma 6.2.9, $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})|| = ||\alpha_4|| = ||\alpha_3|| + ||\alpha_1|| - ||\alpha_2||$. Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then $||\alpha_2|| \geqslant \max(||\alpha_1||, ||\alpha_3||)$ and $||\alpha_2|| > \min(||\alpha_1||, ||\alpha_3||)$. Now:

$$\begin{aligned} ||\alpha_4|| &= \max(||\alpha_1||, ||\alpha_3||) + \min(||\alpha_1||, ||\alpha_3||) - ||\alpha_2|| \\ &< ||\alpha_2|| + ||\alpha_2|| - ||\alpha_2|| \\ &= ||\alpha_2||. \end{aligned}$$

□

Proposition 6.2.11. *Let* $H_1 * \cdots * H_n$ *be an* $\mathfrak{S}$ *free factor splitting for* G*. Let* $(\mathbf{A},\mathbf{x})$ *and* $(\mathbf{B},\mathbf{y})$ *be relative multiple Whitehead automorphisms with* $\mathbf{x} \subset H_i$ *and* $\mathbf{y} \subset H_j$ *and* $\hat{A}, \hat{B} \subset \{H_1, \dots, H_n\} - \{H_i, H_j\}$ *such that* $\hat{A} \cap \hat{B} = \emptyset$*. Let* α_2 *be the domain whose* α*-graph has* $\mathfrak{S}$*-labelling* $(H_1, \dots, H_n)$*, and let* $\alpha_1 = \alpha_2(\mathbf{A},\mathbf{x})$ *and* $\alpha_3 = \alpha_2(\mathbf{B},\mathbf{y})$*. If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak, then it is reducible.*

Proof. By Lemmas 6.2.5 and 6.2.6, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to the path $\alpha_1 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y}) \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_3$, and by Lemma 6.2.10, $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B},\mathbf{y})|| < ||\alpha_2||$. Thus by Definition 6.2.2, the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is reducible. □

Case 1(b): $H_i \notin \hat{B}$ **and** $H_j \notin \hat{A}$**, with** $\hat{A} \cap \hat{B} \neq \emptyset$

Lemma 6.2.12. *If* $H_i \notin \hat{B}$ *and* $H_j \notin \hat{A}$ *with* $\hat{A} \cap \hat{B} \neq \emptyset$*, then* $(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = (\mathbf{A} \cap \hat{B},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y})(\mathbf{A} - \hat{B},\mathbf{x})^{-1}$ *and* $(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = (\mathbf{B} - \hat{A},\mathbf{y})(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B} \cap \hat{A},\mathbf{y})$.

That is, there exist vertices α_4, α_5, α_4', α_5' in our Graph of Domains such that

$$\begin{tikzcd}
& & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[ddl, dashed, "{(\mathbf{A}-\hat{B},\mathbf{x})}"] & & \\
\alpha_1 & & & & \alpha_3 \arrow[dl, "{(\mathbf{A}-\hat{B},\mathbf{x})}"] \\
& \alpha_4 \arrow[ul, "{(\mathbf{A}\cap\hat{B},\mathbf{x})}"] \arrow[rr, "{(\mathbf{B},\mathbf{y})}"'] & & \alpha_5 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[ddr, dashed, "{(\mathbf{B}-\hat{A},\mathbf{y})}"'] & & \\
\alpha_1 \arrow[dr, "{(\mathbf{B}-\hat{A},\mathbf{y})}"'] & & & & \alpha_3 \\
& \alpha_4' & & \alpha_5' \arrow[ll, "{(\mathbf{A},\mathbf{x})}"] \arrow[ur, "{(\mathbf{B}\cap\hat{A},\mathbf{y})}"'] &
\end{tikzcd}$$

are both loops.

Proof. By Proposition 5.5.9 (1), $(\mathbf{A},\mathbf{x})(\mathbf{A} \cap \hat{B},\mathbf{x})^{-1} = (\mathbf{A} - \hat{B},\mathbf{x})$. Note that $\widehat{\mathbf{A} - \hat{B}}$ and $\hat{B}$ are disjoint, and we have $H_i \notin \hat{B}$ and $H_j \notin \hat{A} - \hat{B}$. Thus by Lemma 6.2.5, $(\mathbf{A} - \hat{B},\mathbf{x})(\mathbf{B},\mathbf{y}) = (\mathbf{B},\mathbf{y})(\mathbf{A} - \hat{B},\mathbf{x})$. The second statement follows similarly, by appropriately switching A and B and x and y. □

Lemma 6.2.13. *The loops*

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[ddl, dashed, "{(\mathbf{A}-\hat{B},\mathbf{x})}"] & & \\
\alpha_1 & & & & \alpha_3 \arrow[dl, "{(\mathbf{A}-\hat{B},\mathbf{x})}"] \\
 & \alpha_4 \arrow[ul, "{(\mathbf{A}\cap\hat{B},\mathbf{x})}"] \arrow[rr, "{(\mathbf{B},\mathbf{y})}"'] & & \alpha_5 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[ddr, dashed, "{(\mathbf{B}-\hat{A},\mathbf{y})}"'] & & \\
\alpha_1 \arrow[dr, "{(\mathbf{B}-\hat{A},\mathbf{y})}"'] & & & & \alpha_3 \\
 & \alpha_4' & & \alpha_5' \arrow[ll, "{(\mathbf{A},\mathbf{x})}"] \arrow[ur, "{(\mathbf{B}\cap\hat{A},\mathbf{y})}"'] &
\end{tikzcd}$$

are contractible in our Space of Domains.

Proof. Note that the α_1–α_2–α_4 triangle can be 'filled' with an A_i graph (that is to say, since $(\mathbf{A},\mathbf{x})$, $(\mathbf{A}\cap\hat{B},\mathbf{x})$ and $(\mathbf{A}-\hat{B},\mathbf{x})$ all live in the stabiliser of the A_i-graph with H_i at its centre in the domain α_2, then by Definition 5.1.4, we have a 2-cell $[\alpha_1,\alpha_2,\alpha_4]$). Similarly, the α_2–α_3–α_5' triangle can be 'filled' with an A_j graph. Now $(\widehat{\mathbf{A}-\hat{B}})\cap\hat{B} = \varnothing = \hat{A}\cap(\widehat{B-\hat{A}})$, so by Lemma 6.2.6 (Case 1a) the squares $\alpha_4 — \alpha_2 — \alpha_3 — \alpha_5 — \alpha_4$ and $\alpha_1 — \alpha_2 — \alpha_5' — \alpha_4' — \alpha_1$ are both contractible. $\square$

Lemma 6.2.14. *Let $H' = (H_1',\ldots,H_n')$, and suppose $\mathbf{u}\subset H_i'$, $\mathbf{v}\subset H_j'$, $\mathbf{C},\mathbf{D}\subset\hat{H}'$ with $\hat{C}=\hat{D}$ and $H_i',H_j'\notin\hat{C}$. If $\|\alpha(\mathbf{C},\mathbf{u})\|-\|\alpha\|\leqslant 0$ and $\|\alpha(\mathbf{D},\mathbf{v})\|-\|\alpha\|\leqslant 0$ then $\|\alpha(\mathbf{C},\mathbf{u})\|-\|\alpha\| = \|\alpha(\mathbf{D},\mathbf{v})\|-\|\alpha\| = 0$.*

Proof. Suppose $\|\alpha(\mathbf{C},\mathbf{u})\|-\|\alpha\|\leqslant 0$ and $\|\alpha(\mathbf{D},\mathbf{v})\|-\|\alpha\|\leqslant 0$. By Lemma 6.1.9:

$$\begin{aligned}\|\alpha(\mathbf{C},\mathbf{u})\|-\|\alpha\| =& 2\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Bigg(\Lambda_w(e_c)-\Lambda_w(\overline{e_i}e_c)-\Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c)\\ &-\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)-\frac{1}{2}\sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c)\Bigg).\end{aligned}$$

Note that we can write $\hat{C}-\{H_c\} = (\hat{C}-C_k)\sqcup(C_k-\{H_c\})$. Thus $\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c) = \sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c)+\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)$ for all c such that $H_c\in\hat{C}$.

Since Λ_w counts occurrences of subwords of w, we must have that $\Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c)\leqslant\Lambda_w(\overline{e_i}e_c)\leqslant\Lambda_w(e_c)$ for every k such that $C_k\in\mathbf{C}$ and every c such that $H_c\in C_k$. Since e_i, e_j, and e_a (where $H_a\in\hat{C}$) are distinct, we must also have that

$$\Lambda_w(\overline{e_i}e_c)+\Lambda_w(\overline{e_j}e_c)+\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\leqslant\Lambda_w(e_c) \tag{1}$$

holds for every c such that $H_c\in\hat{C}$. By assumption, $\|\alpha(\mathbf{C},\mathbf{u})\|-\|\alpha\|\leqslant 0$, that is:

$$\begin{aligned}&\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Bigg(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c)-\frac{1}{2}\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\Bigg)\\ \leqslant&\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\Big(\Lambda_w(\overline{e_i}e_c)+\Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c)\Big).\end{aligned}$$

We now deduce the following system of inequalities:

$$\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right) \tag{2}$$

$$=\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right) \tag{3}$$

$$=\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c)-\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right) \tag{4}$$

$$\leqslant\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c)-\frac{1}{2}\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right) \tag{5}$$

$$\leqslant\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\left(\Lambda_w(\overline{e_i}e_c)+\Lambda_w((e_iu_k^{-1})\overline{e_i}e_c)\right) \tag{6}$$

$$\leqslant\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}(2\Lambda_w(\overline{e_i}e_c)) \tag{7}$$

$$=2\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c). \tag{8}$$

The same argument yields that $\sum_{w\in\mathcal{W}}\sum_{H_d\in\hat{D}}\left(\Lambda_w(e_d)-\sum_{H_b\in\hat{D}-\{H_d\}}\Lambda_w(\overline{e_b}e_d)\right)$ $\leqslant 2\sum_{w\in\mathcal{W}}\sum_{H_d\in\hat{D}}\Lambda_w(\overline{e_j}e_d)$. Since it is assumed that $\hat{D}=\hat{C}$, we can rewrite this to give $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right)\leqslant 2\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c)$.

Now $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c)\geqslant\frac{1}{2}\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right)$ and $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c)\geqslant\frac{1}{2}\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right)$. Summing these terms, and comparing with the inequality (1), gives that $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c)$ $+\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c)=\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right)$.
We must then in fact have that $\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_i}e_c)=\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\Lambda_w(\overline{e_j}e_c)$ $=\frac{1}{2}\sum_{w\in\mathcal{W}}\sum_{H_c\in\hat{C}}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right)$. This in turn forces each line of (2)–(8) to be an equality. In particular, $\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\left(\Lambda_w(e_c)-\sum_{H_a\in\hat{C}-C_k}\Lambda_w(\overline{e_a}e_c)\right.$ $\left.-\frac{1}{2}\sum_{H_a\in C_k-\{H_c\}}\Lambda_w(\overline{e_a}e_c)\right)=\sum_{w\in\mathcal{W}}\sum_{C_k\in\mathbf{C}}\sum_{H_c\in C_k}\left(\Lambda_w(\overline{e_i}e_c)+\Lambda_w((u_k^{-1}e_i)\overline{e_i}e_c)\right)$. That is, $\|\alpha(\mathbf{C},\mathbf{u})\|-\|\alpha\|=0$. The same argument applies to see that we must also have $||\alpha(\mathbf{D},\mathbf{v})||-||\alpha||=0$. □

Lemma 6.2.15. *Suppose* $\alpha_1\xleftarrow{(\mathbf{A},\mathbf{x})}\alpha_2\xrightarrow{(\mathbf{B},\mathbf{y})}\alpha_3$ *is a peak. If* $H_i\notin\hat{B}$ *and* $H_j\notin\hat{A}$ *with* $\hat{A}\cap\hat{B}\neq\emptyset$, *then either* $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{A}\cap\hat{B},\mathbf{x})^{-1}||<||\alpha_2||$ *and* $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}-\hat{B},\mathbf{x})||<||\alpha_2||$ *, or* $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{B}-\hat{A},\mathbf{y})||<||\alpha_2||$ *and* $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{B}\cap\hat{A},\mathbf{y})^{-1}||<||\alpha_2||$ *.*

That is, $||\alpha_2||>\max(||\alpha_4||,||\alpha_5||)$ or $||\alpha_2||>\max(||\alpha_4'||,||\alpha_5'||)$.

$$\begin{tikzcd}
& & \alpha_2 \arrow[dll, "(\mathbf{A},\mathbf{x})"'] \arrow[dl, "(\mathbf{U},\mathbf{x})"] \arrow[drr, "(\mathbf{B},\mathbf{y})"] \arrow[dddr, bend left=60, "(\mathbf{V},\mathbf{y})"] & & \\
\alpha_1 \arrow[ddr, "(\mathbf{V},\mathbf{y})"'] & \alpha_4 \arrow[l, "(\mathbf{W},\mathbf{x})"] \arrow[rr, "(\mathbf{B},\mathbf{y})"] \arrow[dr, "(\mathbf{V},\mathbf{y})"'] & & \alpha_5 & \alpha_3 \arrow[l, "(\mathbf{U},\mathbf{x})"'] \\
& & \alpha_6 \arrow[ur, "(\mathbf{W}^{\backprime},\mathbf{y})"] \arrow[dl, "(\mathbf{W},\mathbf{x})"'] & & \\
& \alpha_4' & & \alpha_5' \arrow[ll, "(\mathbf{A},\mathbf{x})"] \arrow[ul, "(\mathbf{U},\mathbf{x})"'] \arrow[uur, "(\mathbf{W}^{\backprime},\mathbf{y})"'] &
\end{tikzcd}$$

FIGURE 1.14: Commuting Diagram for Case 1b

Proof. Let $\alpha_6 = \alpha_2(\mathbf{A} - \hat{B}, \mathbf{x})(\mathbf{B} - \hat{A}, \mathbf{y})$. For brevity, set $\mathbf{U} = \mathbf{A} - \hat{B}$, $\mathbf{V} = \mathbf{B} - \hat{A}$, $\mathbf{W} = \mathbf{A} \cap \hat{B}$, and $\mathbf{W}^{\backprime} = \mathbf{B} \cap \hat{A}$. Note that $\hat{W} = \hat{W}^{\backprime} = \hat{A} \cap \hat{B}$, $\hat{U} = \hat{A} - \hat{B} = \hat{A} \cap \bar{\hat{B}}$, and $\hat{V} = \hat{B} - \hat{A} = \hat{B} \cap \bar{\hat{A}}$. Observe that $\hat{U} \cap \hat{V} = (\hat{A} - \hat{B}) \cap (\hat{B} - \hat{A}) = (\hat{A} \cap \bar{\hat{B}}) \cap (\hat{B} \cap \bar{\hat{A}}) = \hat{A} \cap \bar{\hat{A}} \cap \bar{\hat{B}} \cap \hat{B} = \varnothing \cap \varnothing = \varnothing$. We leave it to the reader to verify that we also have $\hat{U} \cap \hat{B} = \hat{U} \cap \hat{W} = \hat{V} \cap \hat{W} = \hat{V} \cap \hat{A} = \varnothing$; these follow in much the same way. Then by Proposition 5.5.9 (1), the diagram in Figure 1.14 commutes.

By applying Lemma 6.2.9 to each of the squares $\alpha_1 — \alpha_4 — \alpha_6 — \alpha_4' — \alpha_1$ and $\alpha_3 — \alpha_5' — \alpha_6 — \alpha_5 — \alpha_1$ (which both fall under Case 1a), we recover that $\|\alpha_1\| - \|\alpha_4\| = \|\alpha_4'\| - \|\alpha_6\|$ and $\|\alpha_3\| - \|\alpha_5'\| = \|\alpha_5\| - \|\alpha_6\|$. By considering the triangles $\alpha_1 — \alpha_2 — \alpha_4 — \alpha_1$ and $\alpha_3 — \alpha_2 — \alpha_5' — \alpha_3$, we see that $\|\alpha_1\| - \|\alpha_2\| = (\|\alpha_1\| - \|\alpha_4\|) + (\|\alpha_4\| - \|\alpha_2\|)$ and $\|\alpha_3\| - \|\alpha_2\| = (\|\alpha_3\| - \|\alpha_5'\|) + (\|\alpha_5'\| - \|\alpha_2\|)$. By Lemma 6.2.14, either $\max(\|\alpha_4'\| - \|\alpha_6\|, \|\alpha_5\| - \|\alpha_6\|) > 0$ or $\|\alpha_4'\| - \|\alpha_6\| = \|\alpha_5\| - \|\alpha_6\| = 0$. Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then $\max(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) \leqslant 0$ and $\min(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) < 0$.

We claim that $\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|) < 0$. If $\|\alpha_4'\| - \|\alpha_6\| = \|\alpha_5\| - \|\alpha_6\| = 0$, then:

$$\begin{aligned}
&\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|) \\
=&\min\big((\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_1\| - \|\alpha_4\|), (\|\alpha_3\| - \|\alpha_2\|) - (\|\alpha_3\| - \|\alpha_5'\|)\big) \\
=&\min\big((\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_4'\| - \|\alpha_6\|), (\|\alpha_3\| - \|\alpha_2\|) - (\|\alpha_5\| - \|\alpha_6\|)\big) \\
=&\min(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) \\
<&0.
\end{aligned}$$

On the other hand, if $\max(\|\alpha_4'\| - \|\alpha_6\|, \|\alpha_5\| - \|\alpha_6\|) > 0$ (without loss of generality, say $\|\alpha_4'\| - \|\alpha_6\| > 0$ — a symmetrically identical argument holds if $\|\alpha_5\| - \|\alpha_6\| > 0$), then:

$$\begin{aligned}
\|\alpha_4\| - \|\alpha_2\| &= (\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_1\| - \|\alpha_4\|) \\
&= (\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_4'\| - \|\alpha_6\|)
\end{aligned}$$

$$\begin{aligned} &< \|\alpha_1\| - \|\alpha_2\| \\ &\leqslant 0. \end{aligned}$$

In either case, we have that $\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|) < 0$.

Without loss of generality, assume $\|\alpha_4\| - \|\alpha_2\| < 0$. Then $\alpha_4 — \alpha_2 — \alpha_3$ is a peak falling under Case 1a, and by Lemma 6.2.10, $\|\alpha_5\| < \|\alpha_2\|$. An identical (symmetric) argument holds if instead $\|\alpha_5'\| - \|\alpha_2\| < 0$. Thus $\min\big(\max(\|\alpha_4\|, \|\alpha_5\|), \max(\|\alpha_4'\|, \|\alpha_5'\|)\big) < \|\alpha_2\|$, as required. □

Proposition 6.2.16. *Let* $H_1 * \ldots H_n$ *be an* $\mathfrak{G}$ *free factor splitting for* G. *Let* $(\mathbf{A}, \mathbf{x})$ *and* $(\mathbf{B}, \mathbf{y})$ *be relative multiple Whitehead automorphisms with* $\mathbf{x} \subset H_i$ *and* $\mathbf{y} \subset H_j$ *and* $\hat{A}, \hat{B} \subset \{H_1, \ldots, H_n\} - \{H_i, H_j\}$ *such that* $\hat{A} \cap \hat{B} \neq \varnothing$. *Let* α_2 *be the domain whose* α*-graph has* $\mathfrak{G}$*-labelling* $(H_1, \ldots, H_n)$, *and let* $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ *and* $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak, then it is reducible.*

Proof. By Lemmas 6.2.12 and 6.2.13, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to each of the paths

$$\alpha_1 \xleftarrow{(\mathbf{A}\cap\hat{B},\mathbf{x})} \alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{A} \cap \hat{B}, \mathbf{x})^{-1} \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{A} - \hat{B}, \mathbf{x}) \xleftarrow{(\mathbf{A}-\hat{B},\mathbf{x})} \alpha_3 \quad \text{and}$$

$$\alpha_1 \xrightarrow{(\mathbf{B}-\hat{A},\mathbf{y})} \alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{B} - \hat{A}, \mathbf{y}) \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{B} \cap \hat{A}, \mathbf{y})^{-1} \xrightarrow{(\mathbf{B}\cap\hat{A},\mathbf{y})} \alpha_3 \quad .$$

By Lemma 6.2.15, either $||\alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{A} \cap \hat{B}, \mathbf{x})^{-1}|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{A} - \hat{B}, \mathbf{x})|| < ||\alpha_2||$, or $||\alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{B} - \hat{A}, \mathbf{y})|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{B} \cap \hat{A}, \mathbf{y})^{-1}|| < ||\alpha_2||$ Thus by Definition 6.2.2, one of the above paths is a reduction for the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$. □

Case 2(a): $H_i \in \hat{B}$ **(say** $H_i \in B_q$**) and** $H_j \notin \hat{A}$**, with** $\hat{A} \subseteq B_q$

The lemmas for this case are adapted from [12, Lemma 2.7].

Lemma 6.2.17. *If* $H_i \in B_q$ *and* $H_j \notin \hat{A}$ *with* $\hat{A} \subseteq B_q$, *then* $(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y}) = (\mathbf{B}', \mathbf{y})(\mathbf{A}^{y_q}, \mathbf{x}^{y_q})^{-1}$ *where* $[\mathbf{B}']_b := B_b$ *for* $b \in \{1, \ldots, l\} - \{q\}$ *and* $[\mathbf{B}']_q := (B_q - \hat{A}) \cup (\widehat{B_q \cap \mathbf{A}})^{\mathbf{x}} = (B_q - \hat{A}) \cup \bigcup_{a=1}^{k} A_a^{x_a}$.

That is, there exists a vertex α_4 in our Graph of Domains such that

$$\begin{tikzcd} & \alpha_2 \arrow[dl, "(\mathbf{A}, \mathbf{x})"'] \arrow[dr, "(\mathbf{B}, \mathbf{y})"] & \\ \alpha_1 \arrow[dr, "(\mathbf{B}', \mathbf{y})"'] & & \alpha_3 \arrow[dl, "(\mathbf{A}^{y_q}, \mathbf{x}^{y_q})"] \\ & \alpha_4 & \end{tikzcd}$$

is a loop.

Proof. We have that $B_q = (B_q - \hat{A}) \sqcup \hat{A}$. In particular, $\hat{A} \cap B_b = \varnothing$ for any $b \neq q$. By Lemma 5.5.6 (3) we have that $(B_q, y_q) = (B_q - \hat{A}, y_q)(\hat{A}, y_q)$. By (1) and (2) of Lemma 5.5.6, we have that $(\mathbf{A}^{y_q}, \mathbf{x}^{y_q}) = (\hat{A}^{y_q}, y_q^{-1})(\mathbf{A}^{y_q^{-1} y_q}, y_q \mathbf{x}) = (\hat{A}, y_q)^{-1}(\mathbf{A}, y_q \mathbf{x})$. Now

$$
\begin{aligned}
(\mathbf{A}, \mathbf{x})(\mathbf{B}', \mathbf{y}) &= (\mathbf{A}, \mathbf{x})(B_1, y_1) \ldots ((B_q - \hat{A}) \cup \widehat{\mathbf{A}^{\mathbf{x}}}, y_q) \ldots (B_l, y_l) \\
&= (\mathbf{A}, \mathbf{x})(B_1, y_1) \ldots (B_q - \hat{A}, y_q)(\widehat{\mathbf{A}^{\mathbf{x}}}, y_q) \ldots (B_l, y_l) \\
&= (B_1, y_1) \ldots (B_q - \hat{A}, y_q) \ldots (B_l, y_l)(\mathbf{A}, \mathbf{x})(\widehat{\mathbf{A}^{\mathbf{x}}}, y_q) \\
&= (\mathbf{B} - \hat{A}, \mathbf{y})(\mathbf{A}, y_q \mathbf{x}) \\
&= (\mathbf{B} - \hat{A}, \mathbf{y})(\hat{A}, y_q)(\mathbf{A}^{y_q}, \mathbf{x}^{y_q}) \\
&= (\mathbf{B}, \mathbf{y})(\mathbf{A}^{y_q}, \mathbf{x}^{y_q}).
\end{aligned}
$$

□

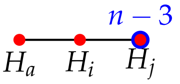
Lemma 6.2.18. *The loop*

$$\begin{tikzcd}
& \alpha_2 \arrow[dl, "(\mathbf{A},\mathbf{x})"'] \arrow[dr, "(\mathbf{B},\mathbf{y})"] & \\
\alpha_1 \arrow[dr, "(\mathbf{B}',\mathbf{y})"'] & & \alpha_3 \arrow[dl, "(\mathbf{A}^{y_q},\mathbf{x}^{y_q})"] \\
& \alpha_4 &
\end{tikzcd}$$

(where $H_i \in B_q$ and $H_j \notin \hat{A}$ with $\hat{A} \subseteq B_q$, and $\mathbf{B}'$ is given by $[\mathbf{B}']_b = B_b$ for $b \neq q$ and $[\mathbf{B}']_q = (B_q - \hat{A}) \cup \widehat{\mathbf{A}^{\mathbf{x}}}$) is contractible in our Space of Domains.

Proof. Let $\hat{A} = \{H_{A_1}, \ldots, H_{A_K}\}$. Recall from Lemma 6.2.6 that we denote $(\mathbf{A}, \mathbf{x})|_a := (\mathbf{A} \cap \{H_a\}, \mathbf{x})$. Then there is some $x_{A_a} \in \mathbf{x}$ so that $(\mathbf{A}, \mathbf{x})|_a = (H_a, x_{A_a})$. Let $\tilde{\alpha}_{0,0} := \alpha_2$ and for $a = 1, \ldots, K$, recursively define $\tilde{\alpha}_{a,0} := \tilde{\alpha}_{a-1,0}(\mathbf{A}, \mathbf{x})|_a$. Then for each a, $\tilde{\alpha}_{a,0}$ is the (domain whose) α-graph has $\mathfrak{S}$-labelling comprising the groups $H_{A_1}^{x_{A_1}}, \ldots, H_{A_a}^{x_{A_a}}, H_{A_{a+1}}, \ldots, H_{A_K}, H_{v_1}, \ldots, H_{v_{n-K}}$. For each a, define $\mathbf{B^{a}}'$ by $[\mathbf{B^{a}}']_b := B_b$ and $[\mathbf{B^{a}}']_q := (B_q - \{H_{A_1}, \ldots, H_{A_a}\}) \cup \{H_{A_1}^{x_{A_1}}, \ldots, H_{A_a}^{x_{A_a}}\}$. Note that $\mathbf{B^{K}}' = \mathbf{B}'$. Again for each $a = 0, \ldots, K$, we define $\tilde{\alpha}_{a,L} := \tilde{\alpha}_{a,0}(\mathbf{B^{a}}', \mathbf{y})$. Then $\tilde{\alpha}_{a+1,L} = \tilde{\alpha}_{a,L}(H_{A_a}^{y_q}, x_{A_a}^{y_q})$. Observe that $\tilde{\alpha}_{K,0} = \alpha_1$, $\tilde{\alpha}_{0,L} = \alpha_3$, and $\tilde{\alpha}_{K,L} = \alpha_4$. Thus we have constructed a lattice as depicted in Figure 1.15. As in the proof of Lemma 6.2.6, the loops $\tilde{\alpha}_{0,0} — \ldots — \tilde{\alpha}_{K,0} — \tilde{\alpha}_{0,0}$ and $\tilde{\alpha}_{0,L} — \ldots — \tilde{\alpha}_{K,L} — \tilde{\alpha}_{0,L}$ are contractible via A_i-graphs living in domains $\tilde{\alpha}_{0,0}$ and $\tilde{\alpha}_{0,L}$, respectively. Also, for each $a \in \{0, \ldots, K-1\}$ the square

$$\begin{tikzcd}
\tilde{\alpha}_{a,0} \arrow[r, "(\mathbf{B^{a}}',\mathbf{y})"] \arrow[d, "(\mathbf{A},\mathbf{x})|_{a+1}"'] & \tilde{\alpha}_{a,L} \arrow[d, "(H_{A_a}^{y_q}, x_{A_a}^{y_q})"] \\
\tilde{\alpha}_{a+1,0} \arrow[r, "(\mathbf{B^{a+1}}',\mathbf{y})"'] & \tilde{\alpha}_{a+1,L}
\end{tikzcd}$$

is contractible via the graph $B_{j,i,a}$ in the domain $\tilde{\alpha}_{a,0}$ (that is, the graph 

lives in the intersection $\tilde{\alpha}_{a,0} \cap \tilde{\alpha}_{a,L} \cap \tilde{\alpha}_{a+1,0} \cap \tilde{\alpha}_{a+1,L}$). □

$$\begin{tikzcd}
\tilde{\alpha}_{0,0} \arrow[r, "(\mathbf{B},\mathbf{y})"] \arrow[d, "(\mathbf{A},\mathbf{x})|_1"] \arrow[ddddd, bend right, "(\mathbf{A},\mathbf{x})"'] & \tilde{\alpha}_{0,L} \arrow[d] \arrow[ddddd, bend left, "(\mathbf{A}^{y_q},\mathbf{x}^{y_q})"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
\tilde{\alpha}_{a,0} \arrow[r, "(\mathbf{B}^{\mathbf{a}'},\mathbf{y})"] \arrow[d, "(\mathbf{A},\mathbf{x})|_{a+1}"] & \tilde{\alpha}_{a,L} \arrow[d] \\
\tilde{\alpha}_{a+1,0} \arrow[r, "(\mathbf{B}^{\mathbf{a+1}'},\mathbf{y})"'] \arrow[d] & \tilde{\alpha}_{a+1,L} \arrow[d] \\
\vdots \arrow[d, "(\mathbf{A},\mathbf{x})|_K"] & \vdots \arrow[d] \\
\tilde{\alpha}_{K,0} \arrow[r, "(\mathbf{B}',\mathbf{y})"'] & \tilde{\alpha}_{K,L}
\end{tikzcd}$$

FIGURE 1.15: Lattice Describing $(A,x)(B',y) = (B,y)(A^{y_q}, x^{y_q})$ in Case 2a

Lemma 6.2.19. *Suppose we have a loop*

$$\begin{tikzcd}
 & \alpha_2 \arrow[dl, "(H_a, x)"'] \arrow[dr, "(B_q, y_q)"] & \\
\alpha_1 \arrow[dr, "(B'_q, y_q)"'] & & \alpha_3 \arrow[dl, "(H_a^{y_q}, x^{y_q})"] \\
 & \alpha_4 &
\end{tikzcd}$$

where $B'_q = \{H_a^x\} \cup (B_q - \{H_a\})$*, and* $y_q \in H_j \neq H_a$*,* $x \in H_i \in B_q$*, and* $H_a \in B_q$*. Then for any edge path* w *in* $\hat{\alpha}_2$*, we have* $|w|_{\alpha_4} - |w|_{\alpha_1} = |w|_{\alpha_3} - |w|_{\alpha_2}$.

Proof. Set $\tilde{B}_q := B_q - \{H_i, H_a\}$. By Lemma 6.1.7, we have that

$$|w|_{\alpha_1} = |w|_{\alpha_2} + 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e}_i e_a) - 2\Lambda_w((x^{-1}e_i)\bar{e}_i e_a).$$

By Lemma 6.1.8, we have that

$$\begin{aligned}
|w|_{\alpha_3} =& |w|_{\alpha_2} + 2\sum_{H_b \in B_q}\left(\Lambda_w(e_b) - \Lambda_w(\bar{e}_j e_b) - \Lambda_w((y_q^{-1}e_j)\bar{e}_j e_b) - \sum_{H_c \in B_q - \{H_b\}}\Lambda_w(\bar{e}_c e_b)\right) \\
=& |w|_{\alpha_2} + 2\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(e_b) - \Lambda_w(\bar{e}_j e_b) - \Lambda_w((y_q^{-1}e_j)\bar{e}_j e_b) - \sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\bar{e}_c e_b)\right) \\
&+ 2\Lambda_w(e_i) - 2\Lambda_w(\bar{e}_j e_i) - 2\Lambda_w((y_q^{-1}e_j)\bar{e}_j e_i) - 4\sum_{H_c \in \tilde{B}_q}\Lambda_w(\bar{e}_c e_i) \\
&+ 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e}_j e_a) - 2\Lambda_w((y_q^{-1}e_j)\bar{e}_j e_a) - 4\sum_{H_c \in \tilde{B}_q}\Lambda_w(\bar{e}_c e_a) \\
&- 4\Lambda_w(\bar{e}_i e_a).
\end{aligned}$$

We will follow the methods used in Section 6.1 to compute $|w|_{\alpha_4}$.

Let $\varphi_{24} := \varphi_{(H_a,x)(B'_q,y_q)} : \hat{\alpha}_2 \to \hat{\alpha}_4$ be the equivariant map described in Convention 6.1.5. If the edges of $\hat{\alpha}_2$ are labelled by e's, and the edges of $\hat{\alpha}_4$ are labeled by h's, then we have:

$$\begin{aligned}(e_a)\varphi_{24} &= h_j(\overline{y_q^{-1h_j}})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)\\ (e_i)\varphi_{24} &= h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)\\ (e_b)\varphi_{24} &= h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b)\\ (e_k)\varphi_{24} &= h_k\end{aligned}$$

for all b with $H_b \in \tilde{B}_q$, and for all k with $H_k \notin B_q$. In particular, since $H_j \notin B_q$, we have $(e_j)\varphi_{24} = h_j$.

Given a word (edge path) w in $\hat{\alpha}_2$, set w' to be the unreduced word $(w)\varphi_{24}$ in $\hat{\alpha}_4$. If $l(u)$ is the unreduced length of a given word u, then $l(w') = l(w) + 4\Lambda_w(e_a) + 2\Lambda_w(e_i) + 2\sum_{H_b\in\tilde{B}_q}\Lambda_w(e_b)$.

Observe that for any b with $H_b \in \tilde{B}_q$ we have $(\overline{e_j}e_b)\varphi_{24} = \overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) = (\overline{y_q^{-1}h_j})(y_q^{-1}h_b)$. We also have that $(\overline{e_j}e_a)\varphi_{24} = \overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$ $= (\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$ and $(\overline{e_j}e_i)\varphi_{24} = \overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i) =$ $(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)$. Let w'' be the result of performing all such reductions (i.e. of the form $(\overline{e_j}e_b)\varphi_{24}$ where $H_b \in B_q$) to w'. Then for each b with $H_b \in B_q$, we have that the length of $(\overline{e_j}e_b)\varphi_{24}$ is 2 less in w'' than it is in w'. Thus $l(w'') = l(w') - 2\sum_{H_b\in B_q}\Lambda_w(\overline{e_j}e_b) = l(w') - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w(\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w(\overline{e_j}e_b)$.

We now observe that for any b with $H_b \in \tilde{B}_q$, we have $((y_q^{-1}e_j)\overline{e_j}e_b)\varphi_{24} = (y_q^{-1}h_j)(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) = (y_q^{-1}h_b)$, and similarly, $((y_q^{-1}e_j)\overline{e_j}e_i)\varphi_{24} = (y_q^{-1}h_j)(\overline{y_q^{-1}h_j})(y_q^{-1}h_i) = (y_q^{-1}h_i)$. Also note that $((y_q^{-1}e_j)\overline{e_j}e_a)\varphi_{24} = (y_q^{-1}h_j)(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a) = (y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$. Let w''' be the result of applying all such reductions to w'', and note that for each b with $H_b \in B_q$, the length of $((y_q^{-1}e_j)\overline{e_j}e_b)\varphi_{24}$ is 2 less in w''' than it is in w''. Now

$$\begin{aligned}l(w''') &= l(w'') - 2\sum_{H_b\in B_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b)\\ &= l(w'') - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 2\sum_{H_b\in\tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b).\end{aligned}$$

We now consider 'cross-reductions' between elements of B_q. Let b and c be such that $H_b \in \tilde{B}_q$ and $H_c \in \tilde{B}_q - \{H_b\}$. Then $(\overline{e_c}e_b)\varphi_{24} = (\overline{y_q^{-1}h_c})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) = (\overline{y_q^{-1}h_c})(y_q^{-1}h_b)$. Additionally, $(\overline{e_i}e_b)\varphi_{24} = (\overline{y_q^{-1}h_i})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b) =$ $(\overline{y_q^{-1}h_i})(y_q^{-1}h_b)$, and $(\overline{e_a}e_b)\varphi_{24} = (\overline{x^{-1}y_q^{-1}h_a})(x^{-1}y_q^{-1}h_i)(\overline{y_q^{-1}h_i})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_b)$ $= (\overline{x^{-1}y_q^{-1}h_a})(x^{-1}y_q^{-1}h_i)(\overline{y_q^{-1}h_i})(y_q^{-1}h_b)$. Finally, $(\overline{e_i}e_a)\varphi_{24} =$ $(\overline{y_q^{-1}h_i})(y_q^{-1}h_j)\overline{h_j}h_j(\overline{y_q^{-1}h_j})(y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a) = (\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a)$. Letting

w'''' be the result of applying all such reductions to w''', we see that for b and c with $H_b, H_c \in \tilde{B}_q$ distinct, the length of $(\overline{e_c}e_b)\varphi_{24}$ is 4 less in w'''' than it is in w''', the lengths of $(\overline{e_i}e_b)\varphi_{24}$ and $(\overline{e_a}e_b)\varphi_{24}$ are each 4 less in w'''' than in w''', and the length of $(\overline{e_i}e_a)\varphi_{24}$ is 6 less in w'''' than it is in w'''. Observe that $\Lambda_w(\overline{e_c}e_b) = \Lambda_w(\overline{\overline{e_c}e_b}) = \Lambda_w(\overline{e_b}e_c)$. Thus $l(w'''') = l(w''') - 6\Lambda_w(\overline{e_i}e_a) - 4\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right)$.

Assuming w was reduced to begin with, then there is only one final type of reduction we can apply to w''''. We have that $((x^{-1}e_i)\overline{e_i}e_a)\varphi =$ $(x^{-1}h_j)(\overline{x^{-1}y_q^{-1}h_j})(x^{-1}y_q^{-1}h_i)(\overline{x^{-1}y_q^{-1}h_i})(x^{-1}y_q^{-1}h_a) = (x^{-1}h_j)(\overline{x^{-1}y_q^{-1}h_j})(x^{-1}y_q^{-1}h_a)$. Letting w''''' be the result of applying all such reductions to w'''', we see that the length of $((x^{-1}e_i)\overline{e_i}e_a)\varphi$ is 2 less in w''''' than it is in w''''. Then $l(w''''') = l(w'''') - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a)$.

Since w''''' and w are both fully reduced, we have:

$$
\begin{aligned}
&|w|_{\alpha_4} = |(w)\varphi_{24}|_{\alpha_4} = l(w''''') \\
=&l(w'''') - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w''') - 6\Lambda_w(\overline{e_i}e_a) - 4\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w'') - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 2\sum_{H_b \in \tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - 6\Lambda_w(\overline{e_i}e_a) \\
&- 4\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w') - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w(\overline{e_j}e_a) - 2\sum_{H_b \in \tilde{B}_q}\Lambda_w(\overline{e_j}e_b) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) \\
&- 2\sum_{H_b \in \tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - 6\Lambda_w(\overline{e_i}e_a) - 4\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) \\
&- 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&l(w) + 4\Lambda_w(e_a) + 2\Lambda_w(e_i) + 2\sum_{H_b \in \tilde{B}_q}\Lambda_w(e_b) - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w(\overline{e_j}e_a) - 2\sum_{H_b \in \tilde{B}_q}\Lambda_w(\overline{e_j}e_b) \\
&- 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_a) - 2\sum_{H_b \in \tilde{B}_q}\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - 6\Lambda_w(\overline{e_i}e_a) \\
&- 4\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(\overline{e_a}e_b) + \Lambda_w(\overline{e_i}e_b) + \frac{1}{2}\sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
=&|w|_{\alpha_2} + 2\Lambda_w(e_a) - 2\Lambda_w(\overline{e_i}e_a) - 2\Lambda_w((x^{-1}e_i)\overline{e_i}e_a) \\
&+ 2\sum_{H_b \in \tilde{B}_q}\left(\Lambda_w(e_b) - \Lambda_w(\overline{e_j}e_b) - \Lambda_w((y_q^{-1}e_j)\overline{e_j}e_b) - \sum_{H_c \in \tilde{B}_q - \{H_b\}}\Lambda_w(\overline{e_c}e_b)\right) \\
&+ 2\Lambda_w(e_i) - 2\Lambda_w(\overline{e_j}e_i) - 2\Lambda_w((y_q^{-1}e_j)\overline{e_j}e_i) - 4\sum_{H_c \in \tilde{B}_q}\Lambda_w(\overline{e_c}e_i)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\Lambda_w(e_a) - 2\Lambda_w(\bar{e}_j e_a) - 2\Lambda_w((y_q^{-1} e_j)\bar{e}_j e_a) - 4\sum_{H_c \in \tilde{B}_q} \Lambda_w(\bar{e}_c e_a) \\
&- 4\Lambda_w(\bar{e}_i e_a) \\
=&|w|_{\alpha_1} + |w|_{\alpha_3} - |w|_{\alpha_2}.
\end{aligned}
$$

□

Lemma 6.2.20. *If* $\mathbf{x} \subset H_i \in B_q$ *and* $\mathbf{y} \subset H_j \notin \hat{A}$ *and* $\hat{A} \subseteq B_q$ *then* $||\alpha_4|| - ||\alpha_1|| = ||\alpha_3|| - ||\alpha_2||$*, where* $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$*,* $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$*, and* $\alpha_4 = \alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{A}^{y_q}, \mathbf{x}^{y_q})$*.*

Proof. By Lemma 6.2.19,

$$
\begin{aligned}
||\alpha_4|| - ||\alpha_1|| &= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_4} - 2\Big) - \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_1} - 2\Big) \\
&= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_4} - 2 - |w|_{\alpha_1} + 2\Big) \\
&= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_3} - |w|_{\alpha_2}\Big) \\
&= \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_3} - 2\Big) - \sum_{w \in \mathcal{W}} \Big(|w|_{\alpha_2} - 2\Big) \\
&= ||\alpha_3|| - ||\alpha_2||.
\end{aligned}
$$

□

Lemma 6.2.21. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak with* $H_i \in \hat{B}$*,* $H_j \notin \hat{A}$*, and* $\hat{A} \subseteq B_q$*, then* $||\alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{A}^{y_q}, \mathbf{x}^{y_q})|| < ||\alpha_2||$*.*

Proof. By Lemma 6.2.20, $||\alpha_2(\mathbf{B}, \mathbf{y})(\mathbf{A}^{y_q}, \mathbf{x}^{y_q})|| = ||\alpha_4|| = ||\alpha_3|| + ||\alpha_1|| - ||\alpha_2||$. Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then $||\alpha_2|| \geqslant \max(||\alpha_1||, ||\alpha_3||)$ and $||\alpha_2|| > \min(||\alpha_1||, ||\alpha_3||)$. Now:

$$
\begin{aligned}
||\alpha_4|| &= \max(||\alpha_1||, ||\alpha_3||) + \min(||\alpha_1||, ||\alpha_3||) - ||\alpha_2|| \\
&< ||\alpha_2|| + ||\alpha_2|| - ||\alpha_2|| \\
&= ||\alpha_2||.
\end{aligned}
$$

□

Proposition 6.2.22. *Let* $H_1 * \cdots * H_n$ *be an* $\mathfrak{S}$ *free factor splitting for* G*. Let* $(\mathbf{A}, \mathbf{x})$ *and* $(\mathbf{B}, \mathbf{y})$ *be relative multiple Whitehead automorphisms with* $\mathbf{x} \subset H_i$ *and* $\mathbf{y} \subset H_j$*,* $\hat{A} \subset \{H_1, \ldots, H_n\} - \{H_i, H_j\}$*,* $\hat{B} \subset \{H_1, \ldots, H_n\} - \{H_j\}$ *such that* $\{H_i\} \cup \hat{A} \subset B_q$ *for some* q*. Let* α_2 *be the domain whose* α*-graph has* $\mathfrak{S}$*-labelling* $(H_1, \ldots, H_n)$*, and let* $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ *and* $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$*. If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak, then it is reducible.*

Proof. By Lemmas 6.2.17 and 6.2.18, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to the path $\alpha_1 \xrightarrow{(\mathbf{B}',\mathbf{y})} \alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q}) \xleftarrow{(\mathbf{A}^{y_q},\mathbf{x}^{y_q})} \alpha_3$ where $\mathbf{B}' = (B_1,\ldots,(B_q - \hat{A}) \cup \widehat{\mathbf{A}^{\mathbf{x}}},\ldots,B_k)$. By Lemma 6.2.21, $||\alpha_2(\mathbf{B},\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q})|| < ||\alpha_2||$. Thus by Definition 6.2.2, the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is reducible. □

Case 2(b): $H_i \in \hat{B}$ (say $H_i \in B_q$) and $H_j \notin \hat{A}$, with $\hat{A} \nsubseteq B_q$

Lemma 6.2.23. *If $H_i \in \hat{B}$ and $H_j \notin \hat{A}$ with $\hat{A} \nsubseteq B_q$, then*

$$(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = (\mathbf{A} - B_q,\mathbf{x})^{-1}(\mathbf{B}',\mathbf{y})((\mathbf{A} \cap B_q)^{y_q},\mathbf{x}^{y_q})^{-1}$$

and

$$(\mathbf{A},\mathbf{x})^{-1}(\mathbf{B},\mathbf{y}) = ((\mathbf{B} +_q \hat{A})',\mathbf{y})(\mathbf{A}^{y_q},\mathbf{x}^{y_q})^{-1}((\bar{\mathbf{B}}_q \cap \hat{A})^{y_q},\tilde{\mathbf{y}}_q y_q^{-1})$$

, where $[\mathbf{B}']_q := (B_q - \hat{A}) \cup \bigcup\left((\mathbf{A} \cap B_q)^{\mathbf{x}}\right)$ and $[\mathbf{B}']_k := B_k$ for $k \neq q$, and $[(\mathbf{B} +_q \hat{A})']_q := ([(\mathbf{B} +_q \hat{A})]_q - \hat{A}) \cup \bigcup([(\mathbf{B} +_q \hat{A})]_q \cap \mathbf{A})^{\mathbf{x}} = (B_q - \hat{A}) \cup \bigcup \mathbf{A}^{\mathbf{x}}$ and $[(\mathbf{B} +_q \hat{A})']_k := [(\mathbf{B} +_q \hat{A})]_k = B_k - \hat{A}$ for $k \neq q$.

That is, there exist vertices α_4, α_5, α_4', α_5' in our Graph of Domains such that

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[dl, dashed, "{(\mathbf{A} \cap B_q,\mathbf{x})}"] & & \\
\alpha_1 & & & & \alpha_3 \arrow[dl, "{((\mathbf{A} \cap B_q)^{y_q},\mathbf{x}^{y_q})}"] \\
 & \alpha_4 \arrow[ul, "{(\mathbf{A} - B_q,\mathbf{x})}"] \arrow[rr, "{(\mathbf{B}',\mathbf{y})}"'] & & \alpha_5 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[dr, dashed, "{(\mathbf{B} +_q \hat{A},\mathbf{y})}"'] & & \\
\alpha_1 \arrow[dr, "{((\mathbf{B} +_q \hat{A})',\mathbf{y})}"'] & & & & \alpha_3 \\
 & \alpha_4' & & \alpha_5' \arrow[ll, "{(\mathbf{A}^{y_q},\mathbf{x}^{y_q})}"] \arrow[ur, "{((\bar{\mathbf{B}}_q \cap \hat{A})^{y_q},\tilde{\mathbf{y}}_q)y_q^{-1}}"'] &
\end{tikzcd}$$

are both loops.

Proof. By Proposition 5.5.9 (1), $(\mathbf{A},\mathbf{x})(\mathbf{A} - B_q,\mathbf{x})^{-1} = (\mathbf{A} \cap B_q,\mathbf{x})$. Writing $\bigcup \mathbf{A} = \hat{A}$, we have that $\bigcup(\mathbf{A} \cap B_q) = \hat{A} \cap B_q \subseteq B_q$. Similarly, by Proposition 5.5.9 (4), $(\mathbf{B},\mathbf{y})((\bar{\mathbf{B}}_q \cap \hat{A})^{y_q},\tilde{\mathbf{y}}_q y_q^{-1})^{-1} = (\mathbf{B} +_q \hat{A},\mathbf{y})$. Note that $[\mathbf{B} +_q \hat{A}]_q = B_q \cup \hat{A} \supseteq \hat{A}$. We have now reduced both problems to the form required by Case 2a, so the result follows from Lemma 6.2.17. □

Lemma 6.2.24. *The loops*

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[dl, dashed, "{(\mathbf{A} \cap B_q,\mathbf{x})}"] & & \\
\alpha_1 & & & & \alpha_3 \arrow[dl, "{((\mathbf{A} \cap B_q)^{y_q},\mathbf{x}^{y_q})}"] \\
 & \alpha_4 \arrow[ul, "{(\mathbf{A} - B_q,\mathbf{x})}"] \arrow[rr, "{(\mathbf{B}',\mathbf{y})}"'] & & \alpha_5 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "{(\mathbf{A},\mathbf{x})}"'] \arrow[drr, "{(\mathbf{B},\mathbf{y})}"] \arrow[dr, dashed, "{(\mathbf{B} +_q \hat{A},\mathbf{y})}"'] & & \\
\alpha_1 \arrow[dr, "{((\mathbf{B} +_q \hat{A})',\mathbf{y})}"'] & & & & \alpha_3 \\
 & \alpha_4' & & \alpha_5' \arrow[ll, "{(\mathbf{A}^{y_q},\mathbf{x}^{y_q})}"] \arrow[ur, "{((\bar{\mathbf{B}}_q \cap \hat{A})^{y_q},y_q^{-1}\tilde{\mathbf{y}}_q)}"'] &
\end{tikzcd}$$

are both contractible in our Space of Domains.

Proof. As in Lemma 6.2.13, the α_1–α_2–α_4 triangle can be 'filled' with an A_i graph, and the α_2–α_3–α_5' triangle can be 'filled' with an A_j graph. Now $\hat{A} \cap B_q \subseteq B_q$ and $\hat{A} \subseteq$

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "(\mathbf{A},\mathbf{x})"'] \arrow[dl, "(\mathbf{Q},\mathbf{x})"] \arrow[drr, "(\mathbf{B},\mathbf{y})"] \arrow[dddr, bend left, "(\mathbf{R},\mathbf{y})"] & & \\
\alpha_1 \arrow[ddr, "(\mathbf{R}',\mathbf{y})"'] & \alpha_4 \arrow[l, "(\mathbf{P},\mathbf{x})"] \arrow[rr, "(\mathbf{B}',\mathbf{y})"] \arrow[dr, "(\mathbf{R}'',\mathbf{y})"'] & & \alpha_5 & \alpha_3 \arrow[l, "(\mathbf{Q}^{y_q},\mathbf{x}^{y_q})"] \\
 & & \alpha_6 \arrow[ur, "(\mathbf{S}^{y_q},\tilde{\mathbf{y}}_q y_q^{-1})"] \arrow[dl, "(\mathbf{P}^{y_q},\mathbf{x}^{y_q})"'] & & \\
 & \alpha_4' & & \alpha_5' \arrow[ll, "(\mathbf{A}^{y_q},\mathbf{x}^{y_q})"] \arrow[ul, "(\mathbf{Q}^{y_q},\mathbf{x}^{y_q})"] \arrow[ur, "(\mathbf{S}^{y_q},\tilde{\mathbf{y}}_q y_q^{-1})"'] &
\end{tikzcd}$$

FIGURE 1.16: Commuting Diagram for Case 2b

$[\mathbf{B} +_q \hat{A}]_q = B_q \cup \hat{A}$, so by Lemma 6.2.18 (Case 2a), the squares $\alpha_4 — \alpha_2 — \alpha_3 — \alpha_5 — \alpha_4$ and $\alpha_1 — \alpha_2 — \alpha_5' — \alpha_4' — \alpha_1$ are both contractible. □

Lemma 6.2.25. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak with* $H_i \in \hat{B}$, $H_j \notin \hat{A}$, *and* $\hat{A} \nsubseteq B_q$, *then either* $||\alpha_2(\mathbf{A},\mathbf{x})(\mathbf{A} - B_q,\mathbf{x})^{-1}|| < ||\alpha_2||$ *and* $||\alpha_2(\mathbf{A} \cap B_q,\mathbf{x})(\mathbf{B}',\mathbf{y})|| < ||\alpha_2||$*, or* $||\alpha_2(\mathbf{A},\mathbf{x})((\mathbf{B} +_q \hat{A})',\mathbf{y})|| < ||\alpha_2||$ *and* $||\alpha_2(\mathbf{B} +_q \hat{A},\mathbf{y})|| < ||\alpha_2||$ *.*

That is, $||\alpha_2|| > \max(||\alpha_4||, ||\alpha_5||)$ or $||\alpha_2|| > \max(||\alpha_4'||, ||\alpha_5'||)$.

Proof. It will suffice to show that either $||\alpha_4|| < ||\alpha_2||$ or $||\alpha_5'|| < ||\alpha_2||$. The problem then reduces to Case 2a (Lemma 6.2.21).

Let $\alpha_6 = \alpha_2(\mathbf{A} \cap B_q, \mathbf{x})(\mathbf{B} +_q \hat{A}, \mathbf{y})$. For brevity, let $\mathbf{P} = \mathbf{A} - B_q$, $\mathbf{Q} = \mathbf{A} \cap B_q$, $\mathbf{R} = \mathbf{B} +_q \hat{A}$, and $\mathbf{S} = \bar{\mathbf{B}}_q \cap \hat{A}$. For $b \neq q$ set $[\mathbf{R}'']_b = [\mathbf{R}']_b := B_b - \hat{A} = [\mathbf{R}]_b$, $[\mathbf{R}'']_q := (B_q - \hat{A}) \sqcup (\widehat{B_q \cap \mathbf{A}})^{\mathbf{x}} \sqcup (\hat{A} - B_q)$, and $[\mathbf{R}']_q := (B_q - \hat{A}) \sqcup (\widehat{\mathbf{A}})^{\mathbf{x}}$. Then the diagram in Figure 1.16 commutes.

Since $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, we have that $\min(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) < 0$ and $\max(\|\alpha_1\| - \|\alpha_2\|, \|\alpha_3\| - \|\alpha_2\|) \leqslant 0$.

Observe that $\widehat{\mathbf{Q}^{y_q}} \cap \widehat{\mathbf{S}^{y_q}} = (\hat{A} \cap B_q)^{y_q} \cap (\hat{A} - \hat{B})^{y_q} = \varnothing$. Additionally, $\tilde{\mathbf{y}}_q y_q^{-1} \in H_j \notin \hat{A}$ (so $\tilde{\mathbf{y}}_q y_q^{-1} \notin \widehat{\mathbf{Q}^{y_q}}$), and $\mathbf{x}^{y_q} \in H_i^{y_q} \in B_q^{y_q}$ (so $\mathbf{x}^{y_q} \notin \widehat{\mathbf{S}^{y_q}}$). Thus $\alpha_5' — \alpha_6 — \alpha_5 — \alpha_3 — \alpha_5'$ is a square falling under Case 1a, so by Lemma 6.2.9, we have that $\|\alpha_3\| - \|\alpha_5'\| = \|\alpha_5\| - \|\alpha_6\|$.

Also observe that $\hat{P} = \hat{A} - B_q \subseteq [\mathbf{R}'']_q$, $\mathbf{x} \in H_i \in B_q - \hat{A} \subset [\mathbf{R}'']_q$, and $\mathbf{y} \in H_j \notin \hat{A} - B_q = \hat{P}$. So $\alpha_4 — \alpha_6 — \alpha_4' — \alpha_1 — \alpha_4$ is a square falling under Case 2a, and by Lemma 6.2.20, we have that $\|\alpha_1\| - \|\alpha_4\| = \|\alpha_4'\| - \|\alpha_6\|$.

Since $\widehat{\mathbf{P}^{y_q}} = (\hat{A} - B_q)^{y_q} = \widehat{\mathbf{S}^{y_q}}$ (and $H_i^{y_q}, H_j^{y_q} \notin \widehat{\mathbf{P}^{y_q}}$) then by Lemma 6.2.14, we have that either $\max(\|\alpha_4'\| - \|\alpha_6\|, \|\alpha_5\| - \|\alpha_6\|) > 0$ or $\|\alpha_4'\| - \|\alpha_6\| = \|\alpha_5\| - \|\alpha_6\| = 0$.

Finally, we note that $\|\alpha_4\| - \|\alpha_2\| = (\|\alpha_1\| - \|\alpha_2\|) - (\|\alpha_1\| - \|\alpha_4\|)$, and similarly, $\|\alpha_5'\| - \|\alpha_2\| = (\|\alpha_3\| - \|\alpha_2\|) - (\|\alpha_3\| - \|\alpha_5'\|)$.

As in the proof of Lemma 6.2.15, we now deduce from this information that $\min(\|\alpha_4\| - \|\alpha_2\|, \|\alpha_5'\| - \|\alpha_2\|) < 0$, as required. $\square$

Proposition 6.2.26. *Let $H_1 * \cdots * H_n$ be an $\mathfrak{S}$ free factor splitting for G. Let $(\mathbf{A}, \mathbf{x})$ and $(\mathbf{B}, \mathbf{y})$ be relative multiple Whitehead automorphisms with $\mathbf{x} \subset H_i$, $\mathbf{y} \subset H_j$, $\hat{A} \subset \{H_1, \ldots, H_n\} - \{H_i, H_j\}$, $\hat{B} \subset \{H_1, \ldots, H_n\} - \{H_j\}$, $H_i \in B_q$ for some Q, and $\hat{A} \not\subset B_q$. Let α_2 be the domain whose α-graph has $\mathfrak{S}$-labelling $(H_1, \ldots, H_n)$, and let $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ and $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$. If $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is a peak, then it is reducible.*

Proof. By Lemmas 6.2.23 and 6.2.24, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to each of the paths

$$\alpha_1 \xleftarrow{(\mathbf{A}-B_q,\mathbf{x})} \alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{A} - B_q, \mathbf{x})^{-1} \xrightarrow{(\mathbf{B}',\mathbf{y})} \alpha_2(\mathbf{A} \cap B_q, \mathbf{x})(\mathbf{B}', \mathbf{y}) \xleftarrow{((\mathbf{A} \cap B_q)^{y_q},\mathbf{x}^{y_q})} \alpha_3$$

and $\alpha_1 \xrightarrow{((\mathbf{B}+_q\hat{A})',\mathbf{y})} \alpha_2(\mathbf{A}, \mathbf{x})((\mathbf{B} +_q \hat{A})', \mathbf{y}) \xleftarrow{(\mathbf{A}^{y_q},\mathbf{x}^{y_q})} \alpha_2(\mathbf{B} +_q \hat{A}, \mathbf{y}) \xrightarrow{((\bar{\mathbf{B}}_q \cap \hat{A})^{y_q},\tilde{\mathbf{y}}_q y_q^{-1})} \alpha_3$.

By Lemma 6.2.25, either $||\alpha_2(\mathbf{A}, \mathbf{x})(\mathbf{A} - B_q, \mathbf{x})^{-1}|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{A} \cap B_q, \mathbf{x})(\mathbf{B}', \mathbf{y})|| < ||\alpha_2||$, or $||\alpha_2(\mathbf{A}, \mathbf{x})((\mathbf{B} +_q \hat{A})', \mathbf{y})|| < ||\alpha_2||$ and $||\alpha_2(\mathbf{B} +_q \hat{A}, \mathbf{y})|| < ||\alpha_2||$. Thus by Definition 6.2.2, one of the above paths is a reduction for the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$. $\square$

Case 4: $H_i \in \hat{B}$ **and** $H_j \in \hat{A}$ **(say** $H_i \in B_q$ **and** $H_j \in A_p$**)**

Lemma 6.2.27. *If $H_i \in B_q$ and $H_j \in A_p$, then*
$(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y}) = ((\mathbf{A} +_p B_q)', \mathbf{x})^{-1}(\mathbf{B}', \mathbf{y})((\bar{\mathbf{A}}_p \cap B_q)^{y_q}, (x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})^{-1}$ and
$(\mathbf{A}, \mathbf{x})^{-1}(\mathbf{B}, \mathbf{y}) = ((\bar{\mathbf{B}}_q \cap A_p)^{x_p}, (y_q^{-1}\tilde{\mathbf{y}}_q)^{x_p})(\mathbf{A}', \mathbf{x})^{-1}((\mathbf{B} +_q A_p)', \mathbf{y})$, where $(\mathbf{A} +_p B_q)'$ (and $(\mathbf{B} +_q A_p)'$) are as defined in Proposition 5.5.9 (4), and $\mathbf{A}'$ and $\mathbf{B}'$ are defined similarly.

That is, there exist vertices α_4, α_5, α_4', and α_5' in our Graph of Domains such that

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "(\mathbf{A},\mathbf{x})"'] \arrow[drr, "(\mathbf{B},\mathbf{y})"] \arrow[ddl, dashed, "(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)"] & & \\
\alpha_1 & & & & \alpha_3 \arrow[dl, "((\bar{\mathbf{A}}_p \cap B_q)^{y_q}, (x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})"] \\
 & \alpha_4 \arrow[ul, "((\mathbf{A} +_p B_q)', \mathbf{x})"] \arrow[rr, "(\mathbf{B}', \mathbf{y})"'] & & \alpha_5 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \alpha_2 \arrow[dll, "(\mathbf{A},\mathbf{x})"'] \arrow[drr, "(\mathbf{B},\mathbf{y})"] \arrow[ddr, dashed, "(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)"] & & \\
\alpha_1 \arrow[dr, "((\bar{\mathbf{B}}_q \cap A_p)^{x_p}, (y_q^{-1}\tilde{\mathbf{y}}_q)^{x_p})"'] & & & & \alpha_3 \\
 & \alpha_4' & & \alpha_5' \arrow[ll, "(\mathbf{A}', \mathbf{x})"] \arrow[ur, "((\mathbf{B} +_q A_p)', \mathbf{y})"'] &
\end{tikzcd}$$

are both loops.

Proof. By Proposition 5.5.9 (4), $(\mathbf{A},\mathbf{x}) = (\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)((\mathbf{A} +_p B_q)', \mathbf{x})$. Now $x_p^{-1}\tilde{\mathbf{x}}_p \in H_i \in B_q$ still, and as $H_j \notin \mathbf{B}$ then $H_j \notin \bar{\mathbf{A}}_p \cap B_q$. Also $\bar{\mathbf{A}}_p \cap B_q \subseteq B_q$, so by Case 2a (Lemma 6.2.17), $(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}',\mathbf{y}) = (\mathbf{B},\mathbf{y})((\bar{\mathbf{A}}_p \cap B_q)^{y_q}, (x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})$. The second loop is achieved by renaming A to B and x to y, and vice versa. □

Lemma 6.2.28. *The loops*

$$\begin{tikzcd}
& & \alpha_2 \arrow[dll, "(\mathbf{A},\mathbf{x})"'] \arrow[drr, "(\mathbf{B},\mathbf{y})"] \arrow[ddl, dashed, "(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)"] & & \\
\alpha_1 & & & & \alpha_3 \arrow[dl, "((\bar{\mathbf{A}}_p \cap B_q)^{y_q}, (x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})"] \\
& \alpha_4 \arrow[ul, "((\mathbf{A} +_p B_q)', \mathbf{x})"] \arrow[rr, "(\mathbf{B}',\mathbf{y})"'] & & \alpha_5 &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& & \alpha_2 \arrow[dll, "(\mathbf{A},\mathbf{x})"'] \arrow[drr, "(\mathbf{B},\mathbf{y})"] \arrow[ddr, dashed, "(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)"'] & & \\
\alpha_1 \arrow[dr, "((\bar{\mathbf{B}}_q \cap A_p)^{x_p}, (y_q^{-1}\tilde{\mathbf{y}}_q)^{x_p})"'] & & & & \alpha_3 \\
& \alpha_4' & & \alpha_5' \arrow[ll, "(\mathbf{A}',\mathbf{x})"] \arrow[ur, "((\mathbf{B} +_q A_p)', \mathbf{y})"'] &
\end{tikzcd}$$

are both contractible in our Space of Domains.

Proof. As in Lemma 6.2.13, the α_1–α_2–α_4 triangle can be 'filled' with an A_i graph, and the α_2–α_3–α_5' triangle can be 'filled' with an A_j graph. Now $\bigcup(\bar{\mathbf{A}}_p \cap B_q) \subseteq B_q$ and $\bigcup(\bar{\mathbf{B}} \cap A_p) \subseteq A_p$, so (after relabelling) by Lemma 6.2.18 (Case 2a), the squares $\alpha_4 — \alpha_2 — \alpha_3 — \alpha_5 — \alpha_4$ and $\alpha_1 — \alpha_2 — \alpha_5' — \alpha_4' — \alpha_1$ are both contractible. □

Lemma 6.2.29. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak with* $H_i \in \hat{B}$ *and* $H_j \in \hat{A}$, *then (up to relabelling)* $||\alpha_2(\mathbf{A},\mathbf{x})((\mathbf{A} +_p B_q)', \mathbf{x})^{-1}|| < ||\alpha_2||$ *and* $||\alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}',\mathbf{y})|| < ||\alpha_2||$.

Proof. First, note that inner automorphisms stabilise each point of $\mathcal{C}_n$, and hence each domain in the Space of Domains. Writing $\gamma(z)$ for the inner automorphism which conjugates everything by z, we then see that $\alpha_1 = \alpha_2(\mathbf{A},\mathbf{x}) = \alpha_2(\mathbf{A},\mathbf{x})\gamma(x_p^{-1}) = \alpha_2(\bar{\mathbf{A}}_p, x_p^{-1}\tilde{\mathbf{x}}_p)$. Similarly, $\alpha_3 = \alpha_2(\bar{\mathbf{B}}_q, y_q^{-1}\tilde{\mathbf{y}}_q)$. Note that $\hat{\bar{\mathbf{A}}}_p = \bar{A}_p$ and $\hat{\bar{\mathbf{B}}}_q = \bar{B}_q$. Since $\mathbf{x} \subset H_i \in B_q$ and $\mathbf{y} \subset H_j \in A_p$, then $x_p^{-1}\tilde{\mathbf{x}}_p \subset H_i \notin \hat{\bar{\mathbf{A}}}_p$ and $y_q^{-1}\tilde{\mathbf{y}}_q \subset H_j \notin \hat{\bar{\mathbf{B}}}_q$.

If $A_p \cup B_q \neq \hat{H}$ then by Lemma 6.2.15, either $\|\alpha_2(\bar{\mathbf{A}}_p - \bar{B}_q, x_p^{-1}\tilde{\mathbf{x}}_p)\| < \|\alpha_2\|$ or $\|\alpha_2(\bar{\mathbf{B}}_q - \bar{A}_p, y_q^{-1}\tilde{\mathbf{y}}_q)\| < \|\alpha_2\|$. But given arbitrary sets C and D, $C - \bar{D} = C \cap D$. Hence either $\|\alpha_4\| < \|\alpha_2\|$ or $\|\alpha_5'\| < \|\alpha_2\|$.

If $A_p \cup B_q = \hat{H}$ then $\alpha_4 = \alpha_1(\mathbf{A} +_p B_q, \mathbf{x})^{-1} = \alpha_1\gamma(x_p^{-1}) = \alpha_1$. Similarly, $\alpha_5' = \alpha_3$, and since $\alpha_1 — \alpha_2 — \alpha_3$ is a peak, then $\|\alpha_2\| > \min(\|\alpha_4\|, \|\alpha_5'\|)$.

Now one of $\alpha_4 — \alpha_2 — \alpha_3$ or $\alpha_1 — \alpha_2 — \alpha_5'$ is a peak satisfying Case 2a, and the result follows from Lemma 6.2.21. □

Proposition 6.2.30. *Let* $H_1 * \cdots * H_n$ *be an* $\mathfrak{S}$ *free factor splitting for* G. *Let* $(\mathbf{A},\mathbf{x})$ *and* $(\mathbf{B},\mathbf{y})$ *be relative multiple Whitehead automorphisms with* $\mathbf{x} \subset H_i$ *and* $\mathbf{y} \subset H_j$ *and* $\hat{A} \subset \{H_1, \ldots, H_n\} - \{H_i\}$, $\hat{B} \subset \{H_1, \ldots, H_n\} - \{H_j\}$ *such that for some* p *and* q *we have* $H_i \in B_q$

and $H_j \in A_p$. *Let* α_2 *be the domain whose* α*-graph has* $\mathfrak{S}$*-labelling* $(H_1, \dots, H_n)$, *and let* $\alpha_1 = \alpha_2(\mathbf{A}, \mathbf{x})$ *and* $\alpha_3 = \alpha_2(\mathbf{B}, \mathbf{y})$. *If* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *is a peak, then it is reducible.*

Proof. By Lemmas 6.2.27 and 6.2.28, the path $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ is homotopic in the Space of Domains to each of the paths

$$\alpha_1 \xleftarrow{((\mathbf{A}+_p B_q)',\mathbf{x})} \alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p) \xrightarrow{(\mathbf{B}',\mathbf{y})} \alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}', \mathbf{y}) \xleftarrow{((\bar{\mathbf{A}}_p \cap B_q)^{y_q},(x_p^{-1}\tilde{\mathbf{x}}_p)^{y_q})} \alpha_3$$

and

$$\alpha_1 \xrightarrow{((\bar{\mathbf{B}}_q \cap A_p)^{x_p},(y_q^{-1}\tilde{\mathbf{y}}_q)^{x_p})} \alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)(\mathbf{A}', \mathbf{x}) \xleftarrow{(\mathbf{A}',\mathbf{x})} \alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q) \xrightarrow{((\mathbf{B}+_q A_p)',\mathbf{y})} \alpha_3 \,.$$

By Lemma 6.2.29, either $||\alpha_2(\mathbf{A}, \mathbf{x})((\mathbf{A} +_p B_q)', \mathbf{x})^{-1}|| < ||\alpha_2||$ and $||\alpha_2(\bar{\mathbf{A}}_p \cap B_q, x_p^{-1}\tilde{\mathbf{x}}_p)(\mathbf{B}', \mathbf{y})|| < ||\alpha_2||$, or $||\alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)(\mathbf{A}', \mathbf{x})|| < ||\alpha_2||$ and $||\alpha_2(\bar{\mathbf{B}}_q \cap A_p, y_q^{-1}\tilde{\mathbf{y}}_q)|| < ||\alpha_2||$. Thus by Definition 6.2.2, one of the above paths is a reduction for the peak $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$. ☐

6.3 Simple Connectivity

We have now done all the required work to conclude:

Proposition 6.3.1. *Every peak* $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ *(whose edges are both of Type A) in the Space of Domains is reducible (to a path of length* 2 *or* 3 *whose edges are all of Type A).*

Proof. Let $\alpha_1 \xleftarrow{(\mathbf{A},\mathbf{x})} \alpha_2 \xrightarrow{(\mathbf{B},\mathbf{y})} \alpha_3$ be a peak in the Space of Domains whose edges are both of Type A. Suppose α_2 has $\mathfrak{S}$-labelling $(H_1, \dots, H_n)$. Then for some i and j we have $\mathbf{x} \subset H_i$ and $\mathbf{y} \subset H_j$. By assumption, $H_i \notin \hat{A}$ and $H_j \notin \hat{B}$. If $i = j$ then by Proposition 6.2.3, the peak is reducible. Otherwise, our peak falls into one of the Cases 1–4 as described in Observation 6.2.4, and by Propositions 6.2.11, 6.2.16, 6.2.22, 6.2.26, and 6.2.30, we are done (after renaming, if we fell under Case 3). ☐

We can now use this Peak Reduction Proposition to argue that the Space of Domains, and hence the complex $\mathcal{C}_n$, is simply connected.

Theorem 6.3.2. *Our Space of Domains is simply connected.*

Proof. Let λ be a loop in the Space of Domains. Note that any loop is homotopic to a based loop, so without loss of generality, we assume λ contains the basepoint α_0. By Corollary 5.4.4 and Proposition 6.3.1, λ is homotopic (in the Space of Domains) to a peak reduced loop λ'. But any peak reduced loop must have constant height (else it would contain some 'heighest' point, i.e. a peak). Since the basepoint has height 0 (Lemma 6.1.3) then every point in λ' must have height 0. But again by Lemma 6.1.3, the only point with height 0 is the basepoint. Hence λ' is actually the constant 'loop' at the basepoint, α_0. Thus any loop λ is homotopic to a constant loop, hence the Space of Domains is simply connected. □

Corollary 6.3.3. *The space $\mathcal{C}_n$ (for $n \geqslant 5$) is simply connected.*

Proof. This follows directly from Theorem 6.3.2 and Proposition 5.3.5. □

References

[1] P. Alexandroff. Über den allgemeinen dimensionsbegriff und seine beziehungen zur elementaren geometrischen anschauung. *Mathematische Annalen*, 98:617–635, 1928.

[2] H. Armstrong, B. Forrest, and K. Vogtmann. A presentation for Aut(F_n). *Journal of Group Theory*, 11(2):267–276, 2008.

[3] H. Bass. Covering theory for graphs of groups. *Journal of Pure and Applied Algebra*, 89(1):3–47, 1993.

[4] H. Bass and R. Jiang. Automorphism groups of tree actions and of graphs of groups. *Journal of Pure and Applied Algebra*, 112(2):109–155, 1996.

[5] A. Bjorner. Nerves, fibers, and homotopy groups. *Journal of Combinatorial Theory, Series A*, 102:88–93, 2003.

[6] K. S. Brown. Presentations for groups acting on simply-connected complexes. *Journal of Pure and Applied Algebra*, 32:1–10, 1984.

[7] D. J. Collins and N. D. Gilbert. Structure and torsion in automorphism groups of free products. *The Quarterly Journal of Mathematics. Oxford Second Series*, 41(162):155–178, 1990.

[8] D. J. Collins and H. Zieschang. Rescuing the Whitehead method for free products, I: Peak reduction. *Mathematische Zeitschrift*, 185:487–504, 1984.

[9] M. Culler and K. Vogtmann. Moduli of graphs and automorphisms of free groups. *Inventiones mathematicae*, 84:91–120, 1986.

[10] D. I. Fouxe-Rabinovitch. Über die Automorphismengruppen der freien Produkte. I. *Rec. Math. [Mat. Sbornik] N.S.*, 8 (50):265–276, 1940.

[11] D. I. Fouxe-Rabinovitch. Über die Automorphismengruppen der freien Produkte. II. *Rec. Math. [Mat. Sbornik] N. S.*, 9 (51):183–220, 1941.

[12] N. D. Gilbert. Presentations of the automorphism group of a free product. *Proceedings of the London Mathematical Society*, s3-54(1):115–140, 01 1987.

[13] V. Guirardel and G. Levitt. The outer space of a free product. *Proceedings of the London Mathematical Society. Third Series*, 94(3):695–714, 2007.

[14] H. Iveson. Generators for the pure symmetric outer automorphisms of a free product. `arXiv:2507.16662` Preprint, 2025.

[15] G. Levitt. Automorphisms of Hyperbolic Groups and Graphs of Groups. *Geometriae Dedicata*, 114:49–70, 2005.

[16] J. McCool. On basis-conjugating automorphisms of free groups. *Canadian Journal of Mathematics*, 38(6):1525–1529, 1986.

[17] D. McCullough and A. Miller. *Symmetric Automorphisms of Free Products*. American Mathematical Society: Memoirs of the American Mathematical Society. American Mathematical Society, 1996.

[18] B. Neumann. Die Automorphismengruppe der freien Gruppen. *Math. Ann.*, 107(1):367–386, 1933.

[19] J. Nielsen. Die Isomorphismengruppe der freien Gruppen. *Math. Ann.*, 91:169–209, 1924.

[20] J. P. Serre. *Trees*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1980.

[21] J. H. C. Whitehead. On equivalent sets of elements in a free group. *Ann. of Math. (2)*, 37(4):782–800, 1936.

[22] H. Wilton. Part II Algebraic Topology lecture notes. `https://www.dpmms.cam.ac.uk/~hjrw2/teaching.html`, November 2019. Accessed September 2024.

Paper 2: **Generators for the Pure Symmetric Outer Automorphisms of a Free Product**

Harry M J Iveson

ABSTRACT. We construct a 'nice' subcomplex of the Outer Space for a free product in order to give a geometric proof that the pure symmetric outer automorphisms of a given splitting of a free product are generated by factor outer automorphisms and Whitehead outer automorphisms relative to the splittting.

Introduction

The study of presentations of automorphism groups of free products dates back to the 1940's with Fouxe-Rabinovitch [3]. This may be viewed as a generalisation of the study of automorphism groups of free groups, investigated by Nielsen in the 1920's, Neumann and Whitehead (independently) in the 1930's, and many others. The focus on symmetric automorphisms came later, studied by McCool in the free group case in the 1980's and McCullough–Miller [8] in the free product case in the 1990's, who used the geometry of a particular CW complex to achieve their results. Such complexes encode various data on these automorphism groups, for example, connectivity may be used to extract generators, and simple connectivity provides group presentations. They may also be used for explicit homology/cohomology calculations, amongst other group invariants and properties.

This paper complements the paper 'A Presentation for the Group of Pure Symmetric Outer Automorphisms of a Given Splitting of a Free Product' [6], which uses simple connectivity of a particular subcomplex "$\mathcal{C}_n$" of Guirardel–Levitt's 'Outer Space' [5] to extract a group presentation using a method of Brown [2]. In that paper, we take a shortcut in proving that $\mathcal{C}_n$ is connected, using a result of Gilbert [4] that the group in question is generated by Whitehead (and factor) automorphisms. Proposition 4.2.12 of this paper gives a direct proof of [6, Proposition 3.1.3], a key step in proving connectivity of $\mathcal{C}_n$, without relying on any prior assumptions.

As a result of this, we achieve an alternative method for determining generators of the pure symmetric outer automorphism group (described in Subsection 1.1) of a given splitting of a free product. We show that this group is generated by the subgroup comprising factor outer automorphisms (described in Subsection 1.2) relative to the splitting and Whitehead outer automorphisms (described in Subsection 1.3) relative to the splitting.

Theorem 5.0.5 (Main Theorem). *Let* $G = G_1 * \cdots * G_n$ *be a group which splits as a free product of* $n \geqslant 3$ *non-trivial factors, and let* $\mathfrak{G} = (G_1, \ldots, G_n)$. *Then any pure symmetric* *outer automorphism* $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ *of the splitting* $G_1 * \cdots * G_n$ *can be written as a product of factor outer automorphisms relative to* $\mathfrak{S}$ *and Whitehead outer automorphisms relative to* $\mathfrak{S}$*.*

We do this by showing that a particular subcomplex $\mathcal{S}_n$ (see Definition 3.2.1) of Guirardel–Levitt's [5] Outer Space is path connected.

Corollary 4.2.14. *The subcomplex* $\mathcal{S}_n$ *is path-connected.*

Our key ingredients are Proposition 4.2.12 and Proposition 5.0.3. The former is a statement about our subcomplex $\mathcal{S}_n$ of Outer Space, detailing how one may find certain paths within the subcomplex. The latter turns this into a statement about (outer) automorphisms, and how Whitehead and factor (outer) automorphisms are sufficient to traverse these paths, and hence the subcomplex.

The paper begins with some background definitions and preliminary results. The varying types of automorphism we will encounter are introduced in Section 1, although for the most part these will not be used until Section 5, where we prove our main theorem. In order to discuss the 'Outer Space' for our free product splitting (Subsection 2.4), we present some Bass–Serre theory regarding graphs of groups in Section 2. In Section 3 we introduce our subcomplex $\mathcal{S}_n$ of Outer Space relative to our chosen splitting of our free product, and discuss some of its basic properties. The key ideas in this paper are found in Section 4, where we use a reduction argument to show that our subcomplex $\mathcal{S}_n$ is path connnected. Finally in Section 5 we use this result to determine generators for the pure symmetric outer automorphism group of our free product splitting.

1 Automorphisms of $G_1 * \cdots * G_n$

Below is a summary of definitions and notation regarding automorphisms adapted from the notation of Gilbert [4] and McCool [7].

Throughout, we consider a group G which splits as a free product $G_1 * \cdots * G_n$, where each G_i is non-trivial and $n \geqslant 3$. We refer to each G_i as a *factor group*.

Notation 1.0.1. Let G be a group. We denote by $\mathrm{Out}(G)$ the outer automorphism group of G. Note that $\mathrm{Out}(G) = {}^{\mathrm{Aut}(G)}\!\big/_{\mathrm{Inn}(G)}$, where $\mathrm{Aut}(G)$ is the group of automorphisms of G and $\mathrm{Inn}(G)$ is the normal subgroup of $\mathrm{Aut}(G)$ comprising all inner automorphisms of G, that is, automorphisms of the form $g \mapsto h^{-1}gh =: g^h$ for some fixed $h \in G$.

We will usually write elements of $\mathrm{Out}(G)$ as upper-case Greek letters and their $\mathrm{Aut}(G)$ representatives as their lower-case counterparts, although if we are given $\psi \in \mathrm{Aut}(G)$ we may sometimes write $[\psi]$ for its class in $\mathrm{Out}(G)$.

1.1 Pure Symmetric Automorphisms

Definition 1.1.1. Let $G = G_1 * \cdots * G_n$ be a group which splits as a non-trivial free product, and denote $\mathfrak{S} := (G_1, \ldots, G_n)$.

- We say $\psi \in \mathrm{Aut}(G)$ is a *pure symmetric automorphism* of the splitting $G_1 * \cdots * G_n$ if for each i there is some $g_i \in G$ such that $\psi(G_i) = G_i^{g_i} = g_i^{-1}G_ig_i$. We denote the subgroup of $\mathrm{Aut}(G)$ comprising these pure symmetric automorphisms by $\mathrm{Aut}_{\mathfrak{S}}(G)$.
- We say $\Psi \in \mathrm{Out}(G)$ is a *pure symmetric outer automorphism* of the splitting if there is some $\psi \in \Psi$ which is a pure symmetric automorphism of the splitting. We denote the subgroup of $\mathrm{Out}(G)$ comprising these pure symmetric outer automorphisms by $\mathrm{Out}_{\mathfrak{S}}(G)$.
- Let $H_1, \ldots, H_n$ be subgroups of $G = G_1 * \cdots * G_n$. If for each i there exists $g_i \in G$ such that $H_i = G_i^{g_i}$, and moreover, G is generated by $H_1 \cup \cdots \cup H_n$, then the free product $H_1 * \cdots * H_n$ is equal to the group G. We summarise this by saying that $H_1 * \cdots * H_n$ is an *$\mathfrak{S}$-free splitting* for $G = G_1 * \cdots * G_n$.

Observation 1.1.2. If $G = G_1 * \cdots * G_n$ where the factor groups G_i are non-trivial, not infinite cylic, and freely indecomposable, then $G_1 * \cdots * G_n$ is a Grushko decomposition for G, and by the Grushko Decomposition Theorem, the G_i's are unique up to permutation of their conjugacy classes in G. In particular, if the G_i's are additionally pairwise non-isomorphic, we deduce that every automorphism of G is a pure symmetric automorphism of the splitting. That is, $\mathrm{Aut}_{\mathfrak{S}}(G) = \mathrm{Aut}(G)$ and $\mathrm{Out}_{\mathfrak{S}}(G) = \mathrm{Out}(G)$ (where $\mathfrak{S} = (G_1, \ldots, G_n)$).

Remark. Note that $\mathrm{Inn}(G) \leqslant \mathrm{Aut}_{\mathfrak{S}}(G) \leqslant \mathrm{Aut}(G)$. We then have that ${}^{\mathrm{Aut}_{\mathfrak{S}}(G)}\!\diagup_{\mathrm{Inn}(G)} = \mathrm{Out}_{\mathfrak{S}}(G)$ and moreover, if $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$, then *every* representative of Ψ in $\mathrm{Aut}(G)$ is in fact in $\mathrm{Aut}_{\mathfrak{S}}(G)$.

Lemma 1.1.3. *Let $G = G_1 * \cdots * G_n$ and $\mathfrak{S} = (G_1, \ldots, G_n)$. Then $H_1 * \cdots * H_n$ is an $\mathfrak{S}$-free splitting for G if and only if there exists $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ such that $H_i = \psi(G_i)$ for each $i \in \{1, \ldots, n\}$.*

Proof. First, suppose $H_1 * \cdots * H_n$ is an $\mathfrak{S}$-free splitting for G. Then there exist $g_1, \ldots, g_n \in G$ such that for each i, $H_i = G_i^{g_i}$. We may now define a map ψ by setting $\psi(g) := g^{g_i}$ for each $g \in G_i$ and each $i \in \{1, \ldots, n\}$. By the universal property for free products, ψ extends to an endomorphism of $G_1 * \cdots * G_n = G$. Similarly, the map φ given by $\varphi(g) := g^{g_i^{-1}}$ for each $g \in G_i^{g_i}$ and each $i \in \{1, \ldots, n\}$ extends to an endomorphism of $G_1^{g_1} * \cdots * G_n^{g_n} = G$. Noting that φ is an inverse for ψ, we see that ψ is bijective. Hence $\psi \in \mathrm{Aut}(G)$ and thus $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$.

Now suppose $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$. Then there exist $g_1, \ldots, g_n \in G$ such that for each i, $\psi(G_i) = G_i^{g_i}$. Observe that since ψ is an automorphism of G $\psi(G_1) * \cdots * \psi(G_n) = \psi(G_1 * \cdots * G_n) = G_1 * \cdots * G_n$. Hence $\psi(G_1) * \cdots * \psi(G_n)$ is an $\mathfrak{S}$-free splitting for G. □

1.2 Factor Automorphisms

Definition 1.2.1. Let $G = G_1 * \cdots * G_n$ and $\mathfrak{S} = (G_1, \ldots, G_n)$. We say $\varphi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ is a *factor automorphism* relative to $\mathfrak{S}$ if for each $i \in \{1, \ldots, n\}$, $\varphi|_{G_i}$ (i.e. φ with domain restricted to the embedding of G_i in G) is an automorphism of G_i, that is, $\varphi|_{G_i} \in \mathrm{Aut}(G_i)$.

We say $\Phi \in \mathrm{Out}_{\mathfrak{S}}(G)$ is a *factor outer automorphism* relative to $\mathfrak{S}$ if Φ has a representative $\varphi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ which is a factor automorphism relative to $\mathfrak{S}$.

When $\mathfrak{S}$ is understood we will simply say φ (or Φ) is a factor (outer) automorphism.

Remark. The set of factor automorphisms relative to $\mathfrak{S}$ forms a subgroup of $\mathrm{Aut}_{\mathfrak{S}}(G)$ which is isomorphic to $\prod_{i=1}^{n} \mathrm{Aut}(G_i)$, via the map $\varphi \mapsto (\varphi|_{G_1}, \ldots, \varphi|_{G_n})$. Note that the only factor automorphism relative to $\mathfrak{S}$ which is also an inner automorphism is the identity, so by the Second Isomorphism Theorem, we also have that the set of factor outer automorphisms relative to $\mathfrak{S}$ in $\mathrm{Out}_{\mathfrak{S}}(G)$ is isomorphic to $\prod_{i=1}^{n} \mathrm{Aut}(G_i)$.

1.3 Whitehead Automorphisms

Definition 1.3.1. Let $G = G_1 * \cdots * G_n$ and $\mathfrak{S} = (G_1, \ldots, G_n)$, take $i \in \{1, \ldots, n\}$, fix some $x \in G_i$, and let $Y \subseteq \{G_1, \ldots, G_n\} - \{G_i\}$.

We denote by (Y, x) the automorphism in $\mathrm{Aut}_{\mathfrak{S}}(G)$ which for each $j \in \{1, \ldots, n\}$ maps $g \in G_j$ by $g \mapsto \begin{cases} x^{-1}gx & \text{if } G_j \in Y \\ g & \text{if } G_j \notin Y \end{cases}$. We call such an automorphism a *Whitehead automorphism* relative to $\mathfrak{S}$, and say that G_i is its *operating factor*.

An element $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ will be called a *Whitehead outer automorphism* relative to $\mathfrak{S}$ if it has some representative $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ which is a Whitehead automorphism relative to $\mathfrak{S}$.

When $\mathfrak{S}$ is understood we will simply say ψ (or Ψ) is a Whitehead (outer) automorphism.

Remark. Our notation differs from that of Gilbert [4] in that we do not include the operating factor G_i in the set Y.

We write g^x for the conjugation $x^{-1}gx$. If $Y = \{G_j\}$ we will often write (G_j, x) for $(\{G_j\}, x)$.

Observation 1.3.2. Note that Whitehead automorphisms relative to $\mathfrak{G}$ leave their operating factor pointwise fixed. If $\iota_h \in \mathrm{Aut}(G)$ is the inner automorphism $g \mapsto h^{-1}gh = g^h$ and $(\{G_1, \ldots, G_m\}, g_0)$ is a Whitehead automorphism relative to $\mathfrak{G}$ with $g_0 \in G_i$ (and $i > m$ for ease of notation), then $(\{G_1, \ldots, G_m\}, g_0)\iota_g$ cannot pointwise fix any factor G_j unless either $g = 1$ or $g = g_0^{-1}$, in which case $g \in G_i$ so $g \notin G_j$. Hence there is no candidate for operating factor unless $g = 1$, and so the only Whitehead automorphism relative to $\mathfrak{G}$ in the outer automorphism class $[(\{G_1, \ldots, G_m\}, g_0)]$ is $(\{G_1, \ldots, G_m\}, g_0)$ itself. Thus if $\Psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ is a Whitehead outer automorphism relative to $\mathfrak{G}$, then Ψ has a unique representative $\psi \in \mathrm{Aut}_{\mathfrak{G}}(G)$ which is a Whitehead automorphism relative to $\mathfrak{G}$.

We will therefore say that if $\psi \in \mathrm{Aut}_{\mathfrak{G}}(G)$ is a Whitehead automorphism relative to $\mathfrak{G}$ with operating factor G_i, then $[\psi] \in \mathrm{Out}_{\mathfrak{G}}(G)$ has operating factor G_i.

2 Graphs of Groups for $G_1 * \cdots * G_n$

We now give a brief background in Bass–Serre theory with the aim of constructing an $\mathrm{Out}_{\mathfrak{G}}(G)$-invariant complex for the splitting $G = G_1 * \cdots * G_n$ where $n \geqslant 3$. Vertices (0-cells) in this complex will be equivalence classes of certain graphs of groups. We will require that all of our graphs of groups are trees, and have trivial edge groups. To aid in our construction, we define (n, k)-trees below, which will satisfy these conditions. Note that our idea of a graph differs from that of Serre [9], in that we have a single unoriented edge between adjacent vertices, which may later be assigned an orientation.

Definition 2.0.1.

- An (n, k)-*tree* T is a finite simplicial tree with vertex set $V(T) = \{v_1, \ldots v_n, u_1, \ldots, u_k\}$ (with the possibility that $k = 0$, i.e. $V(T) = \{v_1, \ldots, v_n\}$) and edge set $E(T) \subseteq V(T) \times V(T)$ such that each vertex u_j has valency at least 3. We consider (w_1, w_2) and (w_2, w_1) to be the same edge. We will refer to the u_j's as 'trivial vertices'.
- An *orientation* of T is a pair of maps $o : E(T) \to V(T)$ and $t : E(T) \to V(T)$ such that for any edge e of T with endpoints w_1 and w_2, we have $\{o(e), t(e)\} = \{w_1, w_2\}$ as sets.
- An (n, k)-*automorphism* of T is a graph automorphism γ of T (i.e. a bijection $T \to T$ which sends vertices to vertices and edges to edges, and preserves adjacency) which satisfies $\gamma(v_i) \in \{v_1, \ldots, v_n\}$ and $\gamma(u_j) \in \{u_1, \ldots, u_k\}$ for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, k\}$.

2.1 $\mathfrak{G}$-Labellings

Now that we have defined our underlying graphs, we can construct our graphs of groups by equipping our graphs with '$\mathfrak{G}$-labellings':

Definition 2.1.1. Let $G = G_1 * \cdots * G_n$ be a free product with $\mathfrak{G} = (G_1, \dots, G_n)$, let $H_1 * \cdots * H_n$ be an $\mathfrak{G}$-free splitting for G, and let $\sigma \in S_n$ be some permutation of $\{1, \dots, n\}$ (so $\{1, \dots, n\} = \{\sigma(1), \dots, \sigma(n)\}$ as sets). Let T be an (n, k)-tree and γ an (n, k)-automorphism of T.

We define $(\gamma(T) : H_{\sigma(1)}, \dots, H_{\sigma(n)})$ to be the graph of groups whose edge groups are all trivial, each vertex $\gamma(u_j)$ for $j \in \{1, \dots, k\}$ has trivial vertex group, and each vertex $\gamma(v_i)$ for $i \in \{1, \dots, n\}$ has vertex group $H_{\sigma(i)}$. We call $(\gamma(T) : H_{\sigma(1)}, \dots, H_{\sigma(n)})$ an *$\mathfrak{G}$-labelling* of $\gamma(T)$.

Remark. Since γ is an automorphism of T, then as abstract graphs we have $\gamma(T) = T$. Moreover, if $\eta \in S_n$ is such that for each $i \in \{1, \dots, n\}$, $\gamma(v_i) = v_{\eta(i)}$, then $(T : H_{\sigma(\eta^{-1}(1))}, \dots, H_{\sigma(\eta^{-1}(n))})$ and $(\gamma(T) : H_{\sigma(1)}, \dots, H_{\sigma(n)})$ describe identical constructions. We will thus often suppress γ and subsume η into the permutation σ.

Definition 2.1.2. Let $\sigma, \tau \in S_n$ and let $T_H := (T : H_{\sigma(1)}, \dots, H_{\sigma(n)})$ and $T_K := (T : K_{\tau(1)}, \dots, K_{\tau(n)})$ be two $\mathfrak{G}$-labellings of an (n, k)-tree T. For $i \in \{1, \dots, n\}$, set $H_{v_i} := H_{\sigma(i)}$ and $K_{v_i} := K_{\tau(i)}$, and for $j \in \{1, \dots, k\}$, set $H_{u_j} := \{1\}$ and $K_{u_j} := \{1\}$.

We say that T_H and T_K are *equal* as $\mathfrak{G}$-labellings of T, written $T_H = T_K$, if (and only if) the map $v_j \mapsto v_{(\sigma^{-1}(\tau(j))}$ extends to an (n, k)-automorphism of T, and for each $i \in \{1, \dots, n\}$ there exists $h_i \in H_i$ so that $K_i = H_i^{h_i}$. In particular, $(T : G_1^{h_1}, \dots, G_n^{h_n}) = (T : G_{\sigma(1)}^{k_{\sigma(1)}}, \dots, G_{\sigma(n)}^{k_{\sigma(n)}})$ if and only if the map $v_j \mapsto v_{\sigma^{-1}(j)}$ extends to an (n, k)-automorphism of T, and for each $i \in \{1, \dots, n\}$ there exists $g_i \in G_i$ such that $k_i = g_i h_i$.

We say that T_H and T_K are *equivalent* as $\mathfrak{G}$-labellings of T, written $T_H \simeq T_K$, if (and only if) the map $v_j \mapsto v_{(\sigma^{-1}(\tau(j))}$ extends to an (n, k)-automorphism of T, and there exists an orientation of T such that for every $w \in V(T)$ there exists $h_w \in G = G_1 * \cdots * G_n$ with $K_w = H_w^{h_w}$ and moreover $h_{t(e)} {h_{o(e)}}^{-1} \in H_{o(e)}$.

Remark. This idea of equivalence is a simplified version of taking isomorphism classes of graphs of groups, as described by Bass in [1, Section 2]. Our simplification relies on having trivial edge groups, and the fact that our graphs of groups are trees. Note that $\simeq$ is indeed an equivalence relation on the set of $\mathfrak{G}$-labellings of T. We will denote the equivalence class of an $\mathfrak{G}$-labelling T_H by $[T_H]$.

Observation 2.1.3. Let T be an (n, k)-tree and let $H_1 * \cdots * H_n$ be an $\mathfrak{G}$-free splitting for $G = G_1 * \cdots * G_n$ (where $\mathfrak{G} = (G_1, \dots, G_n)$).

Let $\varphi \in \mathrm{Aut}_{\mathfrak{G}}(G)$ be a factor automorphism. Then for each $i \in \{1, \dots, n\}$, $\varphi(G_i) = G_i$, and we have $(T : \varphi(G_{\sigma(1)}), \dots, \varphi(G_{\sigma(n)})) = (T : G_{\sigma(1)}, \dots, G_{\sigma(n)})$.

Now let $x \in G$ and let $\iota_x \in \mathrm{Aut}(G)$ be the inner automorphism $g \mapsto x^{-1}gx = g^x$. Then $(T : \iota_x(H_{\sigma(1)}), \ldots, \iota_x(H_{\sigma(n)})) = (T : H^x_{\sigma(1)}, \ldots, H^x_{\sigma(n)}) \simeq (T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ since $x \in G$, and regardlesss of the orientation chosen for T, we have $h_{t(e)}h^{-1}_{o(e)} = xx^{-1} = 1 \in H_{o(e)}$ for each edge e of T.

That is to say, inner automorphisms preserve equivalence classes of $\mathfrak{G}$-labellings, and factor automorphisms preserve $\mathfrak{G}$-labellings whose vertex groups are precisely the groups $G_1, \ldots, G_n$.

2.2 Collapses

In order to build our complex, we will need some kind of relation between our graphs of groups; we may then have edges (1-cells) in our complex whenever their endpoints are related. The relation we use is 'collapsing':

Definition 2.2.1. Let $G = G_1 * \cdots * G_n$ be a free product with $\mathfrak{G} = (G_1, \ldots, G_n)$ and let $(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ be an $\mathfrak{G}$-labelling of some (n, k)-tree T.

- We say an edge of $(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ is *collapsible* if it has at least one trivial endpoint (i.e. at least one endpoint whose vertex group is the trivial group).
- The process of replacing a collapsible edge (including its endpoints) of $(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ by a single vertex whose vertex group is the free product of the vertex groups of the endpoints of said edge is called *collapsing*. This new vertex group will always be of the form $H * \{1\}$, which we will write simply as H (for $H \in \{ \{1\}, H_{\sigma(1)}, \ldots, H_{\sigma(n)}\}$).
- We say $(T' : H'_{\tau(1)}, \ldots, H'_{\tau(n)})$ is a *collapse* of $(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ if $(T' : H'_{\tau(1)}, \ldots, H'_{\tau(n)})$ can be achieved as the result of successively collapsing edges of $(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$. Note that we do not consider a graph of groups to be a collapse of itself.

Observation 2.2.2. Suppose we have $(T_1 : H_{\sigma(1)}, \ldots, H_{\sigma(n)}) \simeq (T_1 : H'_{\tau(1)}, \ldots, H'_{\tau(n)})$ for some (n, k)-tree T_1. Fix some collapsible edge e of T_1, and let T_2 be the tree resulting from the collapse of e in T_1. Then T_2 is an $(n, k-1)$-tree and $(T_2 : H_{\sigma(1)}, \ldots, H_{\sigma(n)}) \simeq (T_2 : H'_{\tau(1)}, \ldots, H'_{\tau(n)})$.

We may thus consider collapses on equivalence classes of $\mathfrak{G}$-labellings in the natural way, i.e. $[T']$ is a collapse of $[T]$ if there is some representative $(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ of $[T]$ and some representative $(T' : H'_{\tau(1)}, \ldots, H'_{\tau(n)})$ of $[T']$ such that $(T' : H'_{\tau(1)}, \ldots, H'_{\tau(n)})$ is a collapse of $(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$. Note that if $[(T' : H'_{\tau(1)}, \ldots, H'_{\tau(n)})]$ is a collapse of $[(T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})]$ then $(T' : H'_{\tau(1)}, \ldots, H'_{\tau(n)}) \simeq (T' : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$.

2.3 Universal Cover of an $\mathfrak{S}$-Labelling

The Fundamental Theorem of Bass–Serre Theory tells us that we have a correspondence between graphs of groups and certain G-trees (namely, their universal covers), which we will later exploit. We adapt the following construction from Serre [9, Chapter I Section 5.3], restricted to our case of (n,k)-trees.

Definition 2.3.1 (Serre [9]). Let $G = G_1 * \cdots * G_n$ be a free product with $\mathfrak{S} = (G_1, \ldots, G_n)$ and let $T_H := (T : H_1, \ldots, H_n)$ be an $\mathfrak{S}$-labelling of some (n,k)-tree T with vertex set $V(T) = \{v_1, \ldots v_n, u_1, \ldots, u_k\}$ and edge set $E(T)$. For $i \in \{1, \ldots, n\}$, set $H_{v_i} := H_i$ and for $j \in \{1, \ldots, k\}$, set $H_{u_j} := \{1\}$. The *universal cover* of T_H, denoted $\widetilde{T_H}$, is the infinite tree with vertex set $V(\widetilde{T_H}) = \bigsqcup_{\substack{g \in G \\ w \in V(T)}} H_w \cdot g$ and edge set $E(\widetilde{T_H}) = \bigsqcup_{\substack{g \in G \\ e \in E(T)}} e \cdot g$ (with $o(e \cdot g) = H_{o(e)} \cdot g$), where $H_w \cdot g$ is the coset $\{hg \mid h \in H_w\} \subset G$.

Observation 2.3.2. We have a natural isometric action of G on $\widetilde{T_H}$ given by $(x \cdot g) \cdot h = x \cdot gh$ for $x \cdot g \in V(\widetilde{T_H}) \cup E(\widetilde{T_H})$ and $h \in G$. Under this action we have that the G-stabiliser of each edge in $\widetilde{T_H}$ is trivial (since all edge groups in T_H are trivial), and the G-stabiliser of a vertex $H_w \cdot g$ in $\widetilde{T_H}$ is precisely the subgroup $g^{-1} H_w g = {H_w}^g$ of G.

Remark. If a vertex $w \in V(T)$ has valency N in T, then the vertex $H_w \cdot g \in V(\widetilde{T_H})$ will have valency $N|H_w|$ in $\widetilde{T_H}$. In particular, if $H_w = \{1\}$ then these valencies must be equal. However, if H_w is infinite, then $H_w \cdot g$ will have infinite valency in $\widetilde{T_H}$.

2.4 Outer Space for a Free Product

There are several equivalent formulations of an 'Outer Space' on which $\mathrm{Out}_{\mathfrak{S}}(G)$ acts. Originally, the study of the (pure) symmetric automorphisms of a free product by their action on a cellular complex was due to McCullough–Miller [8]. Later, Guirardel–Levitt [5] introduced their deformation space for studying the outer automorphisms of a Grushko decomposition of a free product. This space retracts onto a cellular complex which, in the case where there is no free rank, is equivalent to that of McCullough–Miller. Since our decomposition need not be a Grushko decomposition, the Outer Space we use is due to McCullough–Miller. However, the description we give is more in the spirit of Guirardel–Levitt.

Let $G = G_1 * \cdots * G_n$ be a free product with $n \geqslant 3$ where each factor group G_i is non-trivial, and set $\mathfrak{S} = (G_1, \ldots, G_n)$.

Remark. The relation '$(T_1 : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ is a collapse of $(T_2 : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$' defines a strict partial order on the set of collapses of equivalence classes of $\mathfrak{S}$-labellings of a given (n,k)-tree. This extends to encompass *all* (n,k)-trees, even when k is allowed to vary.

Definition 2.4.1. Let $G = G_1 * \cdots * G_n$ and $\mathfrak{S} = (G_1, \ldots, G_n)$ be as above. The complex $\mathcal{O}_n(\mathfrak{S})$ is the geometric realisation of the poset of equivalence classes of $\mathfrak{S}$-labellings of (n, k)-trees (where $n \geqslant 3$ is fixed and $k \geqslant 0$ varies), with $T_1 < T_2$ if (and only if) T_2 is a collapse of T_1.

In this paper, we will always have $\mathfrak{S} = (G_1, \ldots, G_n)$, and so we will write $\mathcal{O}_n$ for $\mathcal{O}_n(\mathfrak{S})$.

Remark. Note that $\mathcal{O}_n$ is a simplicial complex. Moreover for $m \geqslant 2$, for each $(m+1)$-clique in $\mathcal{O}_n^{(1)}$ there is a unique m-cell in $\mathcal{O}_n$ which contains precisely the vertices of that clique. That is to say, $\mathcal{O}_n$ is the flag complex on $\mathcal{O}_n^{(1)}$.

Notation 2.4.2. We will call cells in $\mathcal{O}_n^{(0)}$ 'vertices'; these are equivalence classes of $\mathfrak{S}$-labellings of (n, k)-trees. We will call cells in $\mathcal{O}_n^{(1)}$ 'edges'; there is an edge $[T_1]$—$[T_2]$ precisely when $[T_2]$ is a collapse of $[T_1]$. We may abuse notation and also write $[T_1]$—$[T_2]$ when $[T_1]$ is a collapse of $[T_2]$.

By Observation 1.1.2, if the factor groups G_i are additionally freely indecomposable and not infinite cyclic, then $G_1 * \cdots * G_n$ is a Grushko decomposition for G, and our description of $\mathcal{O}_n$ is precisely the barycentric spine of Guirardel and Levitt's 'Outer Space for a Free Product' [5] (where the barycentric spine is a CW-complex resulting from taking the first barycentric subdivision of Outer Space and linearly retracting off the 'missing' faces).

We thus have that $\mathcal{O}_n$ is isometric to the barycentric spine of Guirdardel and Levitt's Outer Space for a free product of n non-trivial, freely indecomposable, not infinite cyclic, pairwise non-isomorphic factors (even when our factor groups do not satisfy these conditions — since we want an action of $\mathrm{Out}_{\mathfrak{S}}(G)$, not $\mathrm{Out}(G)$).

Definition 2.4.3. Let $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ have representative $\psi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ and let $T_0 = (T : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ be a representative of some point $[T_0]$ in $\mathcal{O}_n^{(0)}$. We define:

$$\Psi \cdot [T_0] := [(T : \psi(H_{\sigma(1)}), \ldots, \psi(H_{\sigma(n)}))]$$

Remark. Note that by Lemma 1.1.3, $\Psi \cdot [T_0]$ is indeed a point in $\mathcal{O}_n$. One ought to additionally check that this action is independent of both the choice of representative for $\Psi \in \mathrm{Out}_{\mathfrak{S}}\, G$ and the choice of representative for $[T_0] \in \mathcal{O}_n^{(0)}$. The former point is addressed by Observation 2.1.3 and the latter point is a similarly straightforward exercise.

Observation 2.4.4. From Observation 2.2.2 we deduce that if $[T_0]$—$[T_1]$ is an edge in $\mathcal{O}_n$, then so too is $(\Psi \cdot [T_0])$—$(\Psi \cdot [T_1])$ for any $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$. Thus the action of $\mathrm{Out}_{\mathfrak{S}}(G)$ on $\mathcal{O}_n^{(0)}$ preserves adjacency, and we may extend this to an action on $\mathcal{O}_n$ by sending a cell corresponding to an $(m+1)$-clique $\{V_0, \ldots, V_m\}$ to the cell corresponding to the $\mathrm{Out}_{\mathfrak{S}}(G)$-image of the $(m+1)$-clique, $\{\Psi \cdot V_0, \ldots, \Psi \cdot V_m\}$.

3 A Subcomplex $\mathcal{S}_n$ of Outer Space

In this paper, we care about two specific types of vertices in $\mathcal{O}_n^{(0)}$: α-graph classes and A-graph classes. We will then consider the subcomplex $\mathcal{S}_n$ of $\mathcal{O}_n$ spanned by these vertices. This subcomplex will encode Whitehead outer automorphisms of the splitting $G_1 * \cdots * G_n$. Throughout, we assume that $n \geqslant 3$ (as neither α nor A graphs make sense for $n < 3$).

3.1 α-Graphs and A-Graphs

We begin by describing 'α-graphs' and 'A-graphs', and their basic properties (equivalence and adjacency).

Definition 3.1.1. We define α to be the $(n, 1)$-tree in Figure 2.1. We say $[\alpha_0] \in \mathcal{O}_n^{(0)}$ is an α-graph class if it has a representative $\alpha_0 = (\alpha : H_{\sigma(1)}, \dots, H_{\sigma(n)})$ which is an $\mathfrak{S}$-labelling of α (for some $\sigma \in S_n$). We call α_0 itself an α-graph.

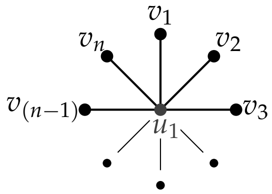


FIGURE 2.1: The Graph 'α' with vertices $v_1, \dots, v_n, u_1$

Remark. Note that if $n < 3$ then α is not well-defined as an (n, k)-tree, since we require u_1 to have valency at least 3.

Observation 3.1.2. Given $\sigma \in S_n$, we can define a map γ_σ on α by $v_j \mapsto v_{\sigma(j)}$, $u_1 \mapsto u_1$, and $u_1 — v_j \mapsto u_1 — v_{\sigma(j)}$ for each $j \in \{1, \dots, n\}$. Clearly γ_σ is a graph automorphism of α and satisfies $\gamma_\sigma(v_j) \in \{v_1, \dots, v_n\}$, that is, γ_σ is an $(n, 1)$-automorphism of α. Now for any $\mathfrak{S}$-labelling $(\alpha : H_{\sigma(1)}, \dots, H_{\sigma(n)})$ of α, we have:

$$(\alpha : H_{\sigma(1)}, \dots, H_{\sigma(n)}) = (\alpha : H_1, \dots, H_n)$$

Lemma 3.1.3. *Two $\mathfrak{S}$-labellings, $(\alpha : H_1, \dots, H_n)$ and $(\alpha : K_1, \dots, K_n)$, of α are equivalent if and only if there exists some $g \in G$ so that for each $j \in \{1, \dots, n\}$, $K_j = H_j^g$.*

Proof. Suppose $(\alpha : H_1, \dots, H_n) \simeq (\alpha : K_1, \dots, K_n)$. Then there exist $h_u, h_1, \dots, h_n \in G$ so that for all $j \in \{1, \dots, n\}$, $K_j = H_j^{h_j}$ (and $\{1\}^{h_u} = \{1\}$). Moreover, for some fixed orientation of α, we have $h_{t(e)} h_{o(e)}^{-1} \in H_{o(e)}$ for every edge e of α (where $H_{u_1} = \{1\}$). Every edge of α is of the form $u_1 — v_j =: e_j$. If $o(e_j) = u_1$ then $h_j h_u^{-1} \in \{1\}$, that is, $h_j = h_u$ and so $K_j = H_j^{h_j} = H_j^{h_u}$. If $o(e_j) = v_j$ then $h_u h_j^{-1} \in H_j$ and $K_j = H_j^{h_j} = H_j^{h_u}$. Hence taking $g = h_u$, we have that for every $j \in \{1, \dots, n\}$, $K_j = H_j^g$.

Now suppose there exists $g \in G$ such that for every $j \in \{1, \ldots, n\}$, $K_j = H_j^g$. Note that $K_{u_1} = \{1\} = \{1\}^g = H_{u_1}^g$. Set $h_w = g$ for every vertex w of α. Since $gg^{-1} = 1$, then regardless of orientation, we have $h_{t(e)}h_{o(e)}^{-1} = 1 \in H_{o(e)}$ for every edge e of α. Hence $(\alpha : H_1, \ldots, H_n) \simeq (\alpha : K_1, \ldots, K_n)$. □

Definition 3.1.4. We define A to be the $(n, 0)$-tree in Figure 2.2. We say $[A_0] \in \mathcal{O}_n^{(0)}$ is an A-graph class if it has a representative $A_0 = (A : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ which is an $\mathfrak{S}$-labelling of A (for some $\sigma \in S_n$). We call A_0 itself an A-graph.

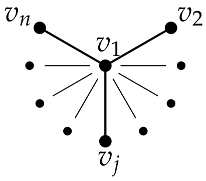


FIGURE 2.2: The Graph 'A' with Vertices $v_1, v_2, \ldots, v_n$

Remark. While the graph A may be defined when $n = 2$, the idea is to have a single vertex with valency strictly greater than 1, so to keep within the spirit of this, we assume $n \geqslant 3$.

Observation 3.1.5. Note that if $\eta \in S_n$ fixes 1 (i.e. $\eta(1) = 1$), then η extends to a graph automorphism of A via the same method as Observation 3.1.2. Now fix $\sigma \in S_n$ with $\sigma(1) = i$ for some $i \in \{1, \ldots, n\}$ and let $(A : H_{\sigma(1)}, \ldots, H_{\sigma(n)})$ be an $\mathfrak{S}$-labelling of A. Taking $\tau := (i\; i-1\; \ldots\; 2\; 1) \in S_n$, we have that $\sigma^{-1}\tau \in S_n$ and $\sigma^{-1}(\tau(1)) = \sigma^{-1}(i) = 1$. Thus $(A : H_{\sigma(1)}, H_{\sigma(2)}, \ldots, H_{\sigma(n)}) = (A : H_{\tau(1)}, H_{\tau(2)}, \ldots, H_{\tau(i)}, H_{\tau(i+1)}, \ldots, H_{\tau(n)}) = (A : H_i, H_1, \ldots, H_{i-1}, H_{i+1}, \ldots, H_n)$.

Notation 3.1.6. We will write $(A : H_i; H_1, \ldots, H_n)$ for the $\mathfrak{S}$-labelling $(A : H_i, H_1, \ldots, H_{i-1}, H_{i+1}, \ldots, H_n)$ of A. Note that for any $\sigma \in S_n$ with $\sigma(i) = i$, we have:

$$(A : H_i; H_1, \ldots, H_n) = (A : H_i; H_{\sigma(1)}, \ldots, H_{\sigma(n)})$$

Lemma 3.1.7. *Two $\mathfrak{S}$-labellings, $(A : H_i; H_1, \ldots, H_n)$ and $(A : K_k; K_1, \ldots, K_n)$, of A are equivalent if and only if $k = i$ and there exist $g \in G$ and $g_1, \ldots, g_n \in H_i$ so that for each $j \in \{1, \ldots, n\}$, $K_j = H_j^{g_j g}$.*

Proof. Suppose $(A : H_i; H_1, \ldots, H_n) \simeq (A : K_k; K_1, \ldots, K_n)$. Then there exist $h_1, \ldots, h_n \in G$ so that for all $j \in \{1, \ldots, n\}$, $K_j = H_j^{h_j}$. Moreover, since $n \geqslant 3$, for some fixed orientation of A, we have $h_{t(e)}h_{o(e)}^{-1} \in H_{o(e)}$ for every edge e of A. Since any $(n, 0)$-automorphism of A must fix the vertex v_1, we deduce that $k = i$. To ease notation, we will relabel the vertices of A by $w_j := \begin{cases} v_{j+1} & \text{if } j < i \\ v_1 & \text{if } j = i \\ v_j & \text{if } j > i \end{cases}$. Now each edge of A is of the form $w_i - w_j =: e_j$ for $j \in \{1, \ldots, n\} - \{i\}$. If $o(e_j) = w_j$ then $h_i h_j^{-1} \in H_j$ and we have

$K_j = H_j^{h_j} = H_j^{h_i}$. If $o(e_j) = w_i$ then $h_j h_i^{-1} \in H_i$. Note that $K_j = H_j^{h_j} = H_j^{h_j h_i^{-1} h_i}$. Now taking $g := h_i$, $g_i := 1$, and for $j \in \{1, \dots, n\} - \{i\}$, $g_j := \begin{cases} 1 & \text{if } o(e_j) = w_j \\ h_j h_i^{-1} & \text{if } o(e_j) = w_i \end{cases}$, we have that for each $j \in \{1, \dots, n\}$, $K_j = H_j^{g_j g}$, with $g_1, \dots, g_n \in H_i$.

Now suppose there exist $g \in G$ and $g_1, \dots, g_n \in H_i$ such that for every $j \in \{1, \dots, n\}$, $K_j = H_j^{g_j g}$. We choose an orientation of A by setting $o(e_j) := w_i$ (where $e_j := w_i — w_j$) for each $j \in \{1, \dots, n\} - \{i\}$, and we set $h_j := g_j g$ for each $j \in \{1, \dots, n\}$. Now for each $j \in \{1, \dots, n\} - \{i\}$, we have $h_{t(e)} h_{o(e)}^{-1} = (g_j g)(g_i g)^{-1} = g_j g g^{-1} g_i^{-1} = g_j g_i^{-1} \in H_i$ since $g_i, g_j \in H_i$. Thus $(A : H_i; H_1, \dots, H_n) \simeq (A : K_i; K_1, \dots, K_n)$. □

Lemma 3.1.8. *Let $[\alpha_0] \in \mathcal{O}_n^{(0)}$ be an α-graph class with representative $(\alpha : H_1, \dots, H_n)$. Then $[T] \in \mathcal{O}_n^{(0)}$ is a collapse of $[\alpha_0]$ if and only $[T]$ is an A-graph class with representative $(A : H_i; H_1, \dots, H_n)$ for some $i \in \{1, \dots, n\}$.*

Proof. First, note that α_0 has only one trivial vertex, so any collapse of α_0 is the result of collapsing precisely one edge in α_0. For $i \in \{1, \dots, n\}$, denote the edge in α_0 between $\{1\}$ and H_i by e_i. Observe that every edge of α_0 is of this form, and so every edge of α_0 is collapsible. Collapsing an edge e_i of α_0 results in the graph of groups $(A : H_i; H_1, \dots, H_n\}$. Hence T is a collapse of α_0 if and only if T results from collapsing some edge e_i of α_0, so T is a collapse of α_0 if and only if $T = (A : H_i; H_1, \dots, H_n)$ for some $i \in \{1, \dots, n\}$. □

3.2 Construction of the Subcomplex $\mathcal{S}_n$

We are now ready to build our subcomplex $\mathcal{S}_n$ of $\mathcal{O}_n$. We will then see how we can move from α-graph class to α-graph class in $\mathcal{S}_n$ by applying elements of $\mathrm{Out}_{\mathfrak{S}}(G)$ found in the $\mathrm{Out}_{\mathfrak{S}}(G)$-stabilisers of A-graph classes.

Definition 3.2.1. For $n \geqslant 3$, we define $\mathcal{S}_n$ to be the subcomplex of $\mathcal{O}_n$ comprising α-graph classes and A-graph classes, as well as all edges whose endpoints are α-graph classes or A-graph classes. That is, $\mathcal{S}_n$ is the subcomplex of $\mathcal{O}_n$ spanned by α-graph classes and A-graph classes.

Observation 3.2.2. By Lemma 3.1.8, we see that $\mathcal{S}_n$ is a 1-dimensional simplicial complex (i.e. a graph). Moreover, since α is an $(n, 1)$-tree and A is an $(n, 0)$-tree and (n, k)-trees collapse to $(n, k-1)$-trees, then no α-graph can collapse to another α-graph, and no A-graph can collapse to another A-graph. Thus $\mathcal{S}_n$ is bipartite between the set of α-graph classes and the set of A-graph classes.

Notation 3.2.3. We denote $\underline{\alpha} := (\alpha : G_1, \dots, G_n)$ and for $i \in \{1, \dots, n\}$,
$\underline{A^i} := (A : G_i; G_1, \dots, G_n) = (A : G_i, G_1, \dots, G_{i-1}, G_{i+1}, \dots, G_n)$.

Observation 3.2.4. From Lemma 3.1.8 we see that the collapses of $[\underline{\alpha}]$ are precisely the vertices $[\underline{A^1}], \ldots, [\underline{A^n}]$ in $\mathcal{S}_n^{(0)}$.

Observation 3.2.5. The subcomplex $\mathcal{S}_n$ inherits the $\mathrm{Out}_{\mathfrak{G}}(G)$ action from $\mathcal{O}_n$. Note that by Lemma 1.1.3, the subgraph of $\mathcal{S}_n$ comprising the α-graph class $[\underline{\alpha}]$, the A-graph classes $\underline{A^1}, \ldots, \underline{A^n}$, and the edges joining them forms a strict fundamental domain for the action of $\mathrm{Out}_{\mathfrak{G}}(G)$ restricted to $\mathcal{S}_n$. In particular, there are precisely $n+1$ $\mathrm{Out}_{\mathfrak{G}}(G)$-orbits of vertices and n $\mathrm{Out}_{\mathfrak{G}}(G)$-orbits of edges in $\mathcal{S}_n$.

Lemma 3.2.6. *Suppose we have an edge path* $[\alpha_0]$—$[A_1]$—$[\alpha_1]$ *in* $\mathcal{S}_n$*, where* $[\alpha_0]$ *and* $[\alpha_1]$ *are* α*-graph classes and* $[A_1]$ *is an* A*-graph class. Then there exists* $\Psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ *with* $\Psi \cdot [A_1] = [A_1]$ *and* $[\alpha_1] = \Psi \cdot [\alpha_0]$.

Proof. Suppose $[\alpha_0]$ has representative $\alpha_0 = (\alpha : H_1, \ldots, H_n)$ and $[\alpha_1]$ has representative $\alpha_1 = (\alpha : K_1, \ldots, K_n)$. Then $H_1 * \cdots * H_n$ and $K_1 * \cdots * K_n$ are both $\mathfrak{G}$-free splittings for G, so by Lemma 1.1.3, there exist $\varphi, \psi \in \mathrm{Aut}_{\mathfrak{G}}(G)$ so that for each $j \in \{1, \ldots, n\}$, $H_j = \varphi(G_j)$ and $K_j = \psi(G_j)$ (hence $K_j = \psi(\varphi^{-1}(H_j))$). In particular, $[\alpha_1] = [\psi\varphi^{-1}] \cdot [\alpha_0]$ where $[\psi\varphi^{-1}] \in \mathrm{Out}_{\mathfrak{G}}(G)$ is the outer automorphism class containing $\psi\varphi^{-1}$. Now by Observation 2.2.2, since $[A_1]$ is a collapse of both $[\alpha_0]$ and $[\alpha_1]$, we have $A_1 \simeq (A : H_i; H_1, \ldots, H_n) \simeq (A : K_i; K_1, \ldots, K_n)$ for some $i \in \{1, \ldots, n\}$. Hence

$$\begin{aligned}
[\psi\varphi^{-1}] \cdot [A_1] &= [\psi\varphi^{-1}] \cdot [(A : H_i; H_1, \ldots, H_n)] \\
&= [(A : \psi(\varphi^{-1}(H_i)); \psi(\varphi^{-1}(H_1)), \ldots, \psi(\varphi^{-1}(H_n)))] \\
&= [(A : K_i; K_1, \ldots, K_n)] \\
&= [A_1]
\end{aligned}$$

□

4 Connectedness of the Subcomplex $\mathcal{S}_n$

Recall that $n \geqslant 3$, $G = G_1 * \cdots * G_n$, $\mathfrak{G} = (G_1, \ldots, G_n)$, and $\underline{\alpha} = (\alpha : G_1, \ldots, G_n)$. The main goal of this Section is to prove Proposition 4.2.12 — that is, given an arbitrary α-graph class $[\alpha_0]$ in $\mathcal{S}_n$, we can find a path in $\mathcal{S}_n$ from $[\alpha_0]$ to $[\underline{\alpha}]$. An immediate corollary of this is that $\mathcal{S}_n$ is path connected (Corollary 4.2.14). Proposition 4.2.12 will be key to proving our main theorem, Theorem 5.0.5.

4.1 The Universal Cover of $\underline{\alpha}$

The majority of the arguments in this Section will take place in the universal cover of $\underline{\alpha}$. We will thus begin by establishing some notation regarding this, as well as some useful (but technical) lemmas.

Notation 4.1.1. Let $U := \{1\}$ be the vertex group in $\underline{\alpha}$ for the vertex u_1 of the graph α (see Figure 2.1). From Definition 2.3.1, we see that the universal cover of $\underline{\alpha}$, denoted $\widetilde{\underline{\alpha}}$, has vertex set $V(\widetilde{\underline{\alpha}}) = \bigsqcup_{\substack{g\in G\\ i\in\{1,\dots,n\}}} G_i{\cdot}g \sqcup \bigsqcup_{g\in G} U{\cdot}g$ and edge set $E(\widetilde{\underline{\alpha}}) = \bigsqcup_{\substack{g\in G\\ j\in\{1,\dots,n\}}} e_j{\cdot}g$ where for each $j \in \{1,\dots,n\}$, e_j is the edge joining $\{1\} = U$ and G_j in $\underline{\alpha}$. Note then that $e_j{\cdot}g$ has endpoints $U{\cdot}g$ and $G_j{\cdot}g$. We will call vertices of the form $G_i{\cdot}g$ 'C-vertices' ('C' for coset) and vertices of the form $U{\cdot}g$ 'U-vertices'. Given $H \in \{U, G_1, \dots, G_n\}$, we will often write $H{\cdot}1 = H$.

Observation 4.1.2. Note that $\widetilde{\underline{\alpha}}$ is bipartite between the set of C-vertices and the set of U-vertices, that is, every neighbour of a U-vertex is a C-vertex, and vice versa. Every edge of $\widetilde{\underline{\alpha}}$ has trivial G-stabiliser, and so does every U-vertex. A C-vertex $G_i{\cdot}g$ of $\widetilde{\underline{\alpha}}$ has stabiliser G_i^g in G. Note that every U-vertex has valency exactly n; in particular the neighbours of a vertex $U{\cdot}g$ of $\widetilde{\underline{\alpha}}$ are precisely the vertices $G_1{\cdot}g, \dots, G_n{\cdot}g$. A C-vertex $G_i{\cdot}g$ has valency equal to $|G_i|$ — this will often be infinite.

We sketch part of the universal cover $\widetilde{\underline{\alpha}}$ of $\underline{\alpha}$ in Figure 2.3.

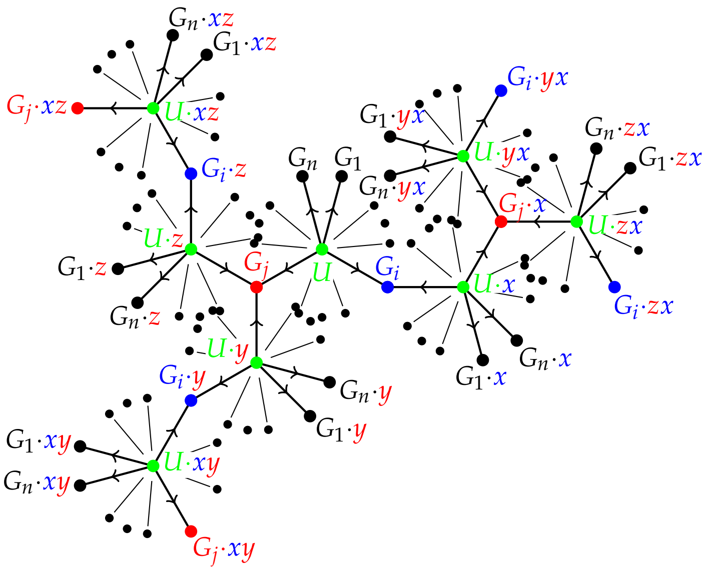


FIGURE 2.3: A Small Part of $\widetilde{\underline{\alpha}}$, with $x \in G_i$ and $y, z \in G_j$

Observation 4.1.3. Fix $i \in \{1,\dots,n\}$ and let $g_i, h \in G = G_1 * \cdots * G_n$. As cosets we have that $G_i{\cdot}g_ih = G_i{\cdot}g_i \Leftrightarrow G_i{\cdot}g_ihg_i^{-1} = G_i \Leftrightarrow g_ihg_i^{-1} \in G_i \Leftrightarrow h \in g_i^{-1}G_ig_i = G_i^{g_i}$. Moreover, since G_i is malnormal in G (as it is a free factor), we have that $G_i^{g_ih} = G_i^{g_i} \Leftrightarrow G_i^{g_ihg_i^{-1}} = G_i \Leftrightarrow g_ihg_i^{-1} \in G_i \Leftrightarrow h \in g_i^{-1}G_ig_i = G_i^{g_i}$. Thus $G_i{\cdot}g_ih = G_i{\cdot}g_i$ as cosets if and only if $G_i^{g_ih} = G_i^{g_i}$ as subgroups of G.

Notation 4.1.4. We will write $[P, Q]$ for the geodesic in $\widetilde{\underline{\alpha}}$ from the vertex P to the vertex Q. We write $[P_1, P_2, \dots, P_q]$ to mean that the vertices $P_2, \dots, P_{q-1}$ lie on the geodesic $[P_1, P_q]$, in the order given. We allow the possibility that $P_i = P_{i+1}$ for $i \in \{1, \dots, q-1\}$. The length of a geodesic $[P, Q]$ is simply the number of edges it contains (equivalently, one less than the number of vertices it contains), written $|[P, Q]|$.

Lemma 4.1.5. *Let $G_j{\cdot}g_j$ be a C-vertex in $\underline{\widetilde{\alpha}}$ (so $j \in \{1, \dots, n\}$ and $g_j \in G$) and suppose $h \in G_k^{g_k}$ is non-trivial, for some $k \in \{1, \dots, n\}$ and some $g_k \in G$ with $G_k^{g_k} \neq G_j^{g_j}$. Then the C-vertex $G_k{\cdot}g_k$ in $\underline{\widetilde{\alpha}}$ lies halfway along the $\underline{\widetilde{\alpha}}$-geodesic $[G_j{\cdot}g_j, G_j{\cdot}g_jh]$.*

Remark. This Lemma holds more generally whenever we have a tree equipped with an edge-free action and an elliptic element. Nevertheless, it will be useful to familiarise ourselves with the details of the tree $\underline{\widetilde{\alpha}}$ by proving it in this specific case.

Proof. We consider the $\underline{\widetilde{\alpha}}$-geodesic $[G_j{\cdot}g_j, G_k{\cdot}g_k]$. Since $G_k^{g_k} \neq G_j^{g_j}$ then by Observation 4.1.3, $G_j{\cdot}g_j \neq G_k{\cdot}g_k$, and are thus distinct C-vertices in $\underline{\widetilde{\alpha}}$. Due to the bipartite structure of $\underline{\widetilde{\alpha}}$, there is some U-vertex $U{\cdot}x$ (where $x \in G$) on the geodesic $[G_j{\cdot}g_j, G_k{\cdot}g_k]$. Let $U{\cdot}y$ be the closest such vertex to $G_k{\cdot}g_k$ (i.e. its neighbour).

Note that since $h \in G_k^{g_k}$ then h stabilises $G_k{\cdot}g_k$, that is, $G_k{\cdot}g_k \cdot h = G_k{\cdot}g_kh = G_k{\cdot}g_k$. By the action of G on $\underline{\widetilde{\alpha}}$, we have $[G_j{\cdot}g_j, G_k{\cdot}g_k] \cdot h = \left[G_j{\cdot}g_j \cdot h, G_k{\cdot}g_k \cdot h\right] = [G_j{\cdot}g_jh, G_k{\cdot}g_k]$. Moreover, the vertex $U{\cdot}yh$ in $\underline{\widetilde{\alpha}}$ lies on $[G_j{\cdot}g_jh, G_k{\cdot}g_k]$ and is adjacent to $G_k{\cdot}g_k$. Since each U-vertex has trivial stabiliser (and h is assumed to be non-trivial) then $U{\cdot}y$ and $U{\cdot}yh$ are distinct vertices in $\underline{\widetilde{\alpha}}$. Thus $[G_j{\cdot}g_j, G_k{\cdot}g_k] \cap [G_j{\cdot}g_jh, G_k{\cdot}g_k] = \{G_k{\cdot}g_k\}$.

Since $\underline{\widetilde{\alpha}}$ is a tree, we conclude that the $\underline{\widetilde{\alpha}}$-geodesic $[G_j{\cdot}g_j, G_j{\cdot}g_jh]$ is the concatenation of the geodesics $[G_j{\cdot}g_j, G_k{\cdot}g_k]$ and $[G_j{\cdot}g_j, G_k{\cdot}g_k] \cdot h = [G_k{\cdot}g_k, G_j{\cdot}g_jh]$. Hence $G_k{\cdot}g_k$ lies halfway along $[G_j{\cdot}g_j, G_j{\cdot}g_jh]$, as required. □

The following Lemma is a technical result required for the proof of Lemma 4.2.9 (which in turn is required for our main ingredient of Proposition 4.2.12, Lemma 4.2.10).

Lemma 4.1.6. *Suppose $G_1^{g_1} * \cdots * G_n^{g_n}$ is an $\mathfrak{S}$-free splitting of G. Let $k_0 \in \{1, \dots, n\}$ and let $G_{k_0}{\cdot}g_{k_0}h_1 \dots h_m$ be a vertex in $\underline{\widetilde{\alpha}}$ such that $h_1 \dots h_m$ is a reduced word with respect to the splitting $G_1^{g_1} * \cdots * G_n^{g_n}$, and $h_1 \notin G_{k_0}^{g_{k_0}}$ (that is, for each $i \in \{1, \dots, m\}$, h_i belongs to a factor group $G_{k_i}^{g_{k_i}}$ with $k_i \neq k_{i-1}$). Then for some $j, l \in \{0, \dots, m\}$ with $k_j \neq k_l$ we have that the vertex $G_{k_j}{\cdot}g_{k_j}$ in $\underline{\widetilde{\alpha}}$ lies on the $\underline{\widetilde{\alpha}}$-geodesic $[G_{k_0}{\cdot}g_{k_0}h_1 \dots h_m, G_{k_l}{\cdot}g_{k_l}]$, that is, we have $[G_{k_0}{\cdot}g_{k_0}h_1 \dots h_m, G_{k_j}{\cdot}g_{k_j}, G_{k_l}{\cdot}g_{k_l}]$.*

Proof. For bookkeeping purposes, we set $h_0 = 1$, and for brevity, we will write C_{k_j} for the coset $G_{k_j}{\cdot}g_{k_j}$ and q_j for the element $h_0 \dots h_j$. Note then that $q_{j+1} = q_jh_{j+1}$, $g_{k_0}h_1 \dots h_m = g_{k_0}q_m$, and $g_{k_0}h_0 = g_{k_0}$.

Suppose for contradiction that the Lemma never holds, that is, we never have
$[C_{k_0}q_m, C_{k_j}, C_{k_l}]$ for $j, l \in \{0, \dots, m\}$ with $k_j \neq k_l$.

We claim that we must then have that for each $i \in \{1, \dots, m\}$, the vertex $C_{k_{m-(i-1)}}$ in $\underline{\widetilde{\alpha}}$ lies on the $\underline{\widetilde{\alpha}}$-geodesic $[C_{k_0}q_m, C_{k_0}q_{m-i}]$, that is, we have $[C_{k_0}q_m, C_{k_{m-(i-1)}}, C_{k_0}q_{m-i}]$. We proceed by induction on i.

Since $q_m = q_{m-1}h_m$, then by Lemma 4.1.5, we have $[C_{k_0}q_m, C_{k_m}, C_{k_0}q_{m-1}]$. Thus the claim holds for $i = 1$.

Now take $1 \leqslant i \leqslant m-1$ and suppose we have $[C_{k_0}q_m, C_{k_{m-(i-1)}}, C_{k_0}q_{m-i}]$. By Lemma 4.1.5, we have $[C_{k_0}q_{m-(i+1)}h_{m-i}, C_{k_{m-i}}, C_{k_0}q_{m-(i+1)}]$, with $q_{m-(i+1)}h_{m-i} = q_{m-i}$. Take V_i to be the vertex in $\underline{\widetilde{\alpha}}$ satisfying $[C_{k_0}q_{m-i}, C_{k_0}q_m] \cap [C_{k_0}q_{m-i}, C_{k_0}q_{m-(i+1)}] = [C_{k_0}q_{m-i}, V_i]$. Note that $[C_{k_0}q_m, C_{k_{m-(i-1)}}, V_i]$ would imply $[C_{k_0}q_m, C_{k_{m-(i-1)}}, C_{k_{m-i}}]$. Similarly, $[V_i, C_{k_{m-i}}, C_{k_0}q_{m-i}]$ would imply either $[C_{k_0}q_m, C_{k_{m-(i-1)}}, C_{k_{m-i}}]$ or $[C_{k_0}q_m, C_{k_{m-i}}, C_{k_{m-(i-1)}}]$. Since we assumed that $[C_{k_0}q_m, C_{k_j}, C_{k_l}]$ never holds for $k_j \neq k_l$, then we must have $[C_{k_0}q_m, V_i, C_{k_{m-i}}, C_{k_0}q_{m-(i+1)}]$. We illustrate this in Figure 2.4. In particular, we note that $C_{k_{m-i}}$ lies on the geodesic in $\underline{\widetilde{\alpha}}$ from $C_{k_0}q_m$ to $C_{k_0}q_{m-(i+1)}$, that is, we have $[C_{k_0}q_m, C_{k_{m-i}}, C_{k_0}q_{m-(i+1)}]$. Hence if the claim holds for $i \in \{1, \ldots, m-1\}$, then it holds for $i+1$.

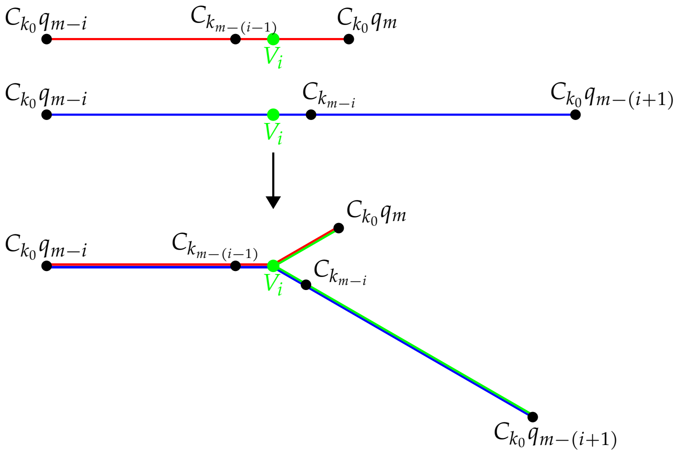


FIGURE 2.4: Inductive Step in Proof of Lemma 4.1.6

Since the claim holds for $i = 1$, we conclude that the claim holds for all $i \in \{1, \ldots, m\}$. In particular, we have $[C_{k_0}q_m, C_{k_1}, C_{k_0}h_0]$. However, $C_{k_0}h_0 = C_{k_0}$, contradicting that we never have $[C_{k_0}q_m, C_{k_j}, C_{k_l}]$. □

Lemma 4.1.7. *Let $G_i{\cdot}g_i$ be a C-vertex in $\underline{\widetilde{\alpha}}$ (for some $i \in \{1, \ldots, n\}$ and $g_i \in G$). Let $U{\cdot}x$ and $U{\cdot}y$ be two neighbours of $G_i{\cdot}g_i$ in $\underline{\widetilde{\alpha}}$ (that is, $x, y \in G$). Then $x^{-1}y$ stabilises $G_i{\cdot}g_i$ (i.e. $G_i{\cdot}g_i x^{-1}y = G_i{\cdot}g_i$ and $x^{-1}y \in G_i^{g_i}$).*

Proof. Let $G_i{\cdot}g_i$, $U{\cdot}x$, and $U{\cdot}y$ be as in the statement of the Lemma. Since $G_i{\cdot}g_i$ is adjacent in $\underline{\widetilde{\alpha}}$ to $U{\cdot}x$, then $(G_i{\cdot}g_i) \cdot x^{-1}y$ must be adjacent in $\underline{\widetilde{\alpha}}$ to $(U{\cdot}x) \cdot x^{-1}y = U{\cdot}xx^{-1}y = U{\cdot}y$. Note that the neighbours of $U{\cdot}y$ in $\underline{\widetilde{\alpha}}$ are the vertices $G_1{\cdot}h_1, \ldots, G_n{\cdot}h_n$ for some $h_1, \ldots, h_n \in G$. Then we must have that $(G_i{\cdot}g_i) \cdot x^{-1}y = G_i{\cdot}h_i$. But $G_i{\cdot}g_i$ is also adjacent to $U{\cdot}y$ in $\underline{\widetilde{\alpha}}$, and so we must also have that $G_i{\cdot}g_i = G_i{\cdot}h_i$. That is, $(G_i{\cdot}g_i) \cdot x^{-1}y = G_i{\cdot}g_i$ and so $x^{-1}y \in \mathrm{Stab}_G(G_i{\cdot}g_i) = G_i^{g_i}$. □

4.2 'Volumes' of α-Graphs

In order to show that we can find paths in $\mathcal{S}_n$ from arbitrary α-graph classes to $[\underline{\alpha}]$, we define quantities 'volume' associated to α-graphs, and show that this quantity is somehow 'reducible' to that of $\underline{\alpha} = (\alpha : G_1, \dots, G_n)$.

Definition 4.2.1. Let $\alpha_0 = (\alpha : G_1^{g_1}, \dots, G_n^{g_n})$ be a representative of an α-graph class $[\alpha_0]$ in $\mathcal{S}_n^{(0)}$ and let $x \in G$. We call the geodesic $[U{\cdot}x, G_i{\cdot}g_i]$ in $\widetilde{\underline{\alpha}}$ from the vertex $U{\cdot}x$ to the vertex $G_i{\cdot}g_i$ the *i^{th} $U{\cdot}x$-spoke of* α_0 . We now construct the *$U{\cdot}x$-spoke graph of* α_0 by taking the disjoint union of all n $U{\cdot}x$-spokes of α_0 and identifying them at the vertex $U{\cdot}x$. We denote this graph $\mathrm{Sp}_{U\cdot x}(\alpha_0)$.

Example 4.2.2. Let $G = W * X * Y * Z$ where W, X, Y, and Z are any groups satisfying $|W| \geqslant 3$, $|X| \geqslant 3$, $|Y| \geqslant 2$, and $|Z| \geqslant 2$, and let $\mathfrak{G} := (W, X, Y, Z)$. Let $1 \in G$ be the identity element, let $w_1, w_2 \in W - \{1\}$, $x_1, x_2 \in X - \{1\}$, and $z_1 \in Z - \{1\}$, and let $\varphi \in \mathrm{Aut}_{\mathfrak{G}}(G)$ be the automorphism generated by: $g \mapsto \begin{cases} g & \text{if } g \in W \\ w_1^{-1}z_1^{-1}gz_1w_1 & \text{if } g \in X \\ w_2^{-1}x_1^{-1}gx_1w_2 & \text{if } g \in Y \\ w_1^{-1}z_1^{-1}x_2^{-1}gx_2z_1w_1 & \text{if } g \in Z \end{cases}$
We will take for granted that this is a pure symmetric automorphism of $W * X * Y * Z$ and hence that $W * X^{z_1w_1} * Y^{x_1w_2} * Z^{x_2z_1w_1}$ is an $\mathfrak{G}$-free splitting for G. Let $\underline{\alpha} = (\alpha : W, X, Y, Z)$ and $\alpha_0 = (\alpha : W, X^{z_1w_1}, Y^{x_1w_2}, Z^{x_2z_1w_1}) = \varphi \cdot \underline{\alpha}$. We sketch part of the tree $\widetilde{\underline{\alpha}}$ in Figure 2.6. Note that we have highlighted the $U{\cdot}z_1w_1$-spokes of α_0 in pink; we will return to the orange highlighting at a later point (Example 4.2.6).

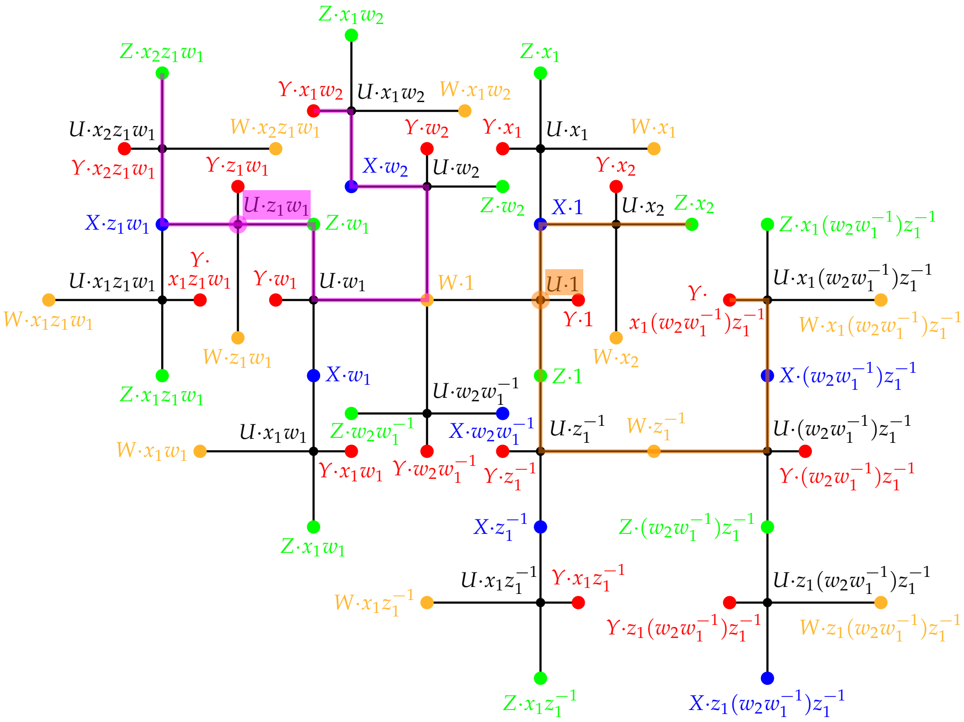


FIGURE 2.5: Various $U{\cdot}g$-spokes in the Tree $\widetilde{\underline{\alpha}}$ for $G = W * X * Y * Z$

Using Figure 2.5, we construct the spoke graphs $\mathrm{Sp}_{U\cdot 1}(\alpha_0)$ and $\mathrm{Sp}_{U\cdot z_1w_1}(\alpha_0)$, illustrated in Figure 2.6. Observe how even a small change in 'basepoint' (i.e. the vertex $U\cdot g$ chosen for $\mathrm{Sp}_{U\cdot g}(\alpha_0)$) can drastically alter the resulting spoke graph.

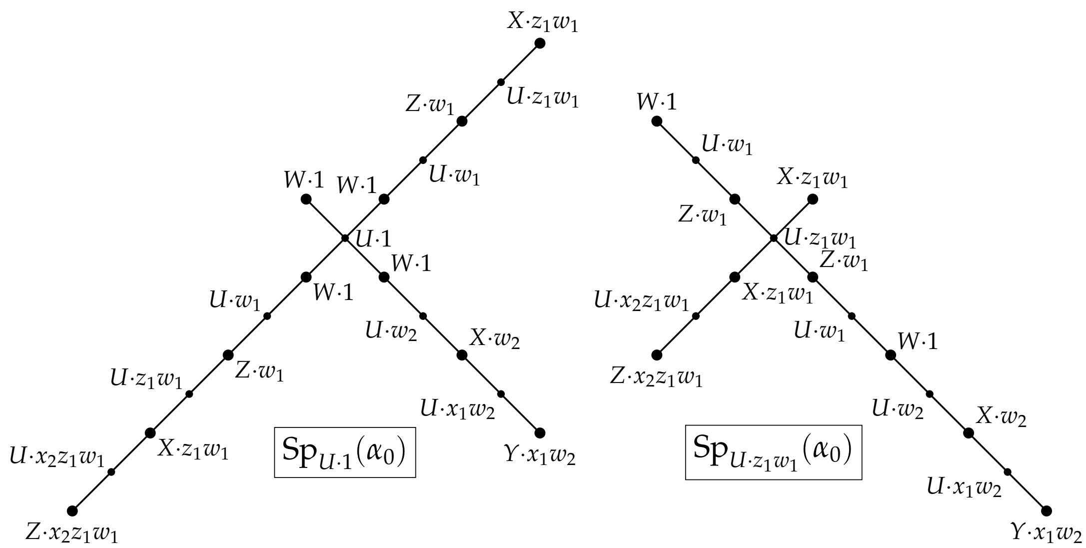


FIGURE 2.6: The Spoke Graphs $\mathrm{Sp}_{U\cdot 1}(\alpha_0)$ and $\mathrm{Sp}_{U\cdot z_1w_1}(\alpha_0)$

Definition 4.2.3. Let $\alpha_0 = (\alpha : G_1^{g_1}, \ldots, G_n^{g_n})$ and let $x \in G$. We define the *$U\cdot x$-volume of* α_0 , denoted $|\alpha_0|_{U\cdot x}$, to be the total number of edges in the $U\cdot x$-spoke graph $\mathrm{Sp}_{U\cdot x}(\alpha_0)$.

Example 4.2.4. Note that this definition is dependent on the choice of U-vertex $U\cdot x$ in $\underset{\sim}{\widetilde{\alpha}}$ (equivalently, the choice of $x \in G$). Indeed, let $G = W * X * Y * Z$ and $\alpha_0 = (\alpha : W, X^{z_1w_1}, Y^{x_1w_2}, Z^{x_2z_1w_1})$, as in Example 4.2.2. Using Figure 2.6, we easily compute $|\alpha_0|_{U\cdot 1} = 1 + 5 + 5 + 7 = 18$ and $|\alpha_0|_{U\cdot z_1w_1} = 3 + 1 + 7 + 3 = 14$.

Observation 4.2.5. Let $\alpha_0 = (\alpha : G_1^{g_1}, \ldots, G_n^{g_n})$ and $\alpha_1 = (\alpha : G_1^{g_1h}, \ldots, G_n^{g_nh})$ be two equivalent $\mathfrak{G}$-labellings of an α-graph class in $\mathcal{S}_n$, and let $x \in G$. Since G acts on $\underset{\sim}{\widetilde{\alpha}}$ by isometries, we have that the i^{th} $U\cdot xh$-spoke of α_1 in $\underset{\sim}{\widetilde{\alpha}}$ is the h-image of the i^{th} $U\cdot x$-spoke of α_0 (i.e. $[U\cdot xh, G_i\cdot g_ih] = [U\cdot x, G_i\cdot g_i]\cdot h$). That is, $\mathrm{Sp}_{U\cdot xh}(\alpha_1)$ 'looks like' $\mathrm{Sp}_{U\cdot x}(\alpha_0)$. In particular, $|\alpha_1|_{U\cdot xh} = |\alpha_0|_{U\cdot x}$.

Thus $U\cdot x$-volume is a property of α-graphs, rather than α-graph classes, although volumes of equivalent α-graphs are comparable by changing the choice of $x \in G$. One may adapt this to define a definitive volume of an α-graph class (for example by taking a minimum or a sum over all $U\cdot x$-volumes), but it will not be necessary for our arguments.

Example 4.2.6. Let $G = W * X * Y * Z$ and $\alpha_0 = (\alpha : W, X^{z_1w_1}, Y^{x_1w_2}, Z^{x_2z_1w_1})$, as in Example 4.2.2. Further, let $\alpha_1 = (\alpha : W^{(w_1^{-1}z_1^{-1})}, X^{z_1w_1(w_1^{-1}z_1^{-1})}, Y^{x_1w_2(w_1^{-1}z_1^{-1})}, Z^{x_2z_1w_1(w_1^{-1}z_1^{-1})}) = (\alpha : W^{z_1^{-1}}, X, Y^{x_1(w_2w_1^{-1})z_1^{-1}}, Z^{x_2})$. Note that the $U\cdot 1$-spokes of α_1 are highlighted in orange in Figure 2.5. Using this, we construct the spoke graph $\mathrm{Sp}_{U\cdot 1}(\alpha_1)$ in Figure 2.7, and compare it with $\mathrm{Sp}_{U\cdot z_1w_1}(\alpha_0)$.

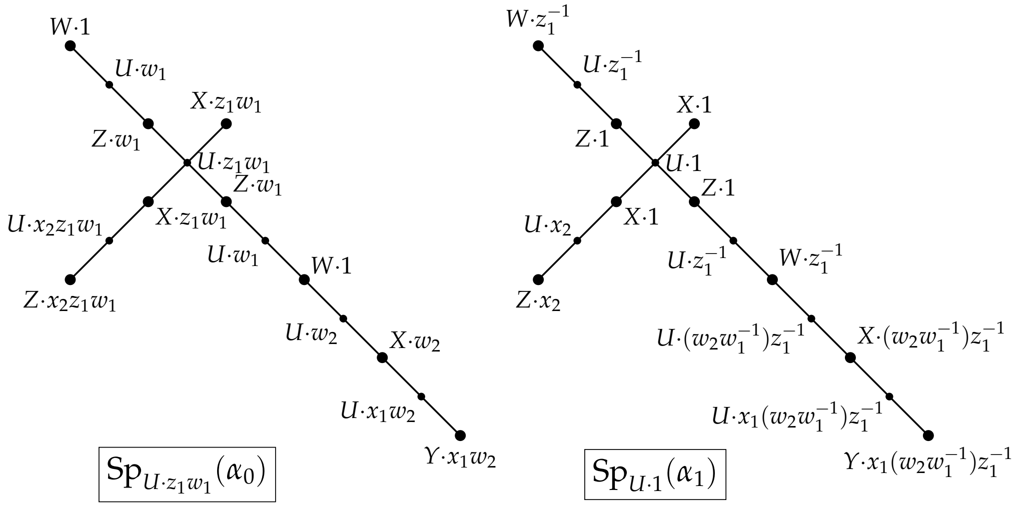


FIGURE 2.7: The Spoke Graphs $\mathrm{Sp}_{U\cdot z_1w_1}(\alpha_0)$ and $\mathrm{Sp}_{U\cdot 1}(\alpha_1)$

Letting $\iota_{w_1^{-1}z_1^{-1}} \in \mathrm{Inn}(G)$ be the inner automorphism $g \mapsto z_1w_1gw_1^{-1}z_1^{-1}$, we see that $\alpha_1 = \iota_{w_1^{-1}z_1^{-1}} \cdot \alpha_0$. Thus in $\mathcal{S}_4$, we have $[\alpha_1] = [\alpha_0]$, and $\mathrm{Sp}_{U\cdot(z_1w_1)}(\alpha_0) \cong \mathrm{Sp}_{U\cdot 1}(\iota_{(z_1w_1)^{-1}} \cdot \alpha_0)$.

Lemma 4.2.7. *Let α_0 be a representative of an α-graph class in $\mathcal{S}_n^{(0)}$. We have that $\alpha_0 \simeq \underline{\alpha}$ if and only if there is some $x \in G$ so that $|\alpha_0|_{U\cdot x} = n$.*

Proof. Recall that $\underline{\alpha} := (\alpha : G_1, \ldots, G_n)$. Thus if $\alpha_0 \simeq \underline{\alpha}$ then by Lemma 3.1.3, there exists $g \in G$ such that $\alpha_0 = (\alpha : G_1^g, \ldots, G_n^g)$. For each i we have that $U — G_i$ is an edge in $\underline{\widetilde{\alpha}}$, thus so too is $U{\cdot}g — G_i{\cdot}g$. We then have that $\mathrm{Sp}_{U\cdot g}(\alpha_0)$ is the star with central vertex $U{\cdot}g$ and n leaves $G_1{\cdot}g, \ldots, G_n{\cdot}g$. Hence $|\alpha_0|_{U\cdot g} = n$.

Now let $\alpha_0 = (\alpha : G_1^{g_1}, \ldots, G_n^{g_n})$ and suppose $|\alpha_0|_{U\cdot x} = n$ for some $x \in G$. Since $\mathrm{Sp}_{U\cdot x}(\alpha_0)$ comprises n distinct $U{\cdot}x$-spokes, we must have that each $U{\cdot}x$-spoke has length 1 in $\underline{\widetilde{\alpha}}$. That is, for each i, $U{\cdot}x — G_i{\cdot}g_i$ is an edge in $\underline{\widetilde{\alpha}}$. Then for each i, $U — G_i{\cdot}g_ix^{-1}$ must also be an edge of $\underline{\widetilde{\alpha}}$. However, the only neighbours of U in $\underline{\widetilde{\alpha}}$ are the vertices $G_1, \ldots, G_n$. So we must have that $G_i{\cdot}g_ix^{-1} = G_i$ as cosets, and thus by Observation 4.1.3, $G_i^{g_ix^{-1}} = G_i$. Now $\alpha_0 = (\alpha : G_1^{g_1}, \ldots, G_n^{g_n}) \simeq (\alpha : G_1^{g_1x^{-1}}, \ldots, G_n^{g_nx^{-1}}) = (\alpha : G_1, \ldots, G_n) = \underline{\alpha}$. □

Observation 4.2.8. By choosing $U{\cdot}x$ to be suitably 'far away' from the vertices $G_1{\cdot}g_1, \ldots, G_n{\cdot}g_n$ in $\underline{\widetilde{\alpha}}$, one can take $|\alpha_0|_{U\cdot x}$ to be as large as desired. However, since $\mathrm{Sp}_{U\cdot x}(\alpha_0)$ always comprises n $U{\cdot}x$-spokes, each of which have integer length at least 1, we deduce that $|\alpha_0|_{U\cdot x} \geqslant n$ for any α_0 and any $x \in G$ (and that $|\alpha_0|_{U\cdot x}$ must always take integer values).

Lemma 4.2.9. *Suppose $G_1^{g_1} * \cdots * G_n^{g_n}$ is an $\mathfrak{S}$-free splitting of G. Let $\alpha_0 = (\alpha : G_1^{g_1}, \ldots, G_n^{g_n})$ and let $x \in G$ be such that $|\alpha_0|_{U\cdot x} > n$. Then there exist distinct j and l in $\{1, \ldots, n\}$ so that the l^{th} $U{\cdot}x$-spoke of α_0 contains the C-vertex $G_j{\cdot}g_j$. That is, in $\underline{\widetilde{\alpha}}$ the vertex $G_j{\cdot}g_j$ lies on the geodesic $[U{\cdot}x, G_l{\cdot}g_l]$, i.e. we have $[U{\cdot}x, G_j{\cdot}g_j, G_l{\cdot}g_l]$.*

Proof. Since $|\alpha_0|_{U\cdot x} > n$, then the $U\cdot x$-spoke graph of α_0 contains more than n edges. In particular, there is some i so that the i^{th} $U\cdot x$-spoke of α_0 in $\underline{\tilde{\alpha}}$ contains more than one edge. Then there exist $k_0 \in \{1, \ldots, n\}$ and $h \in G$ so that $G_{k_0}\cdot g_{k_0}h$ lies on the i^{th} $U\cdot x$-spoke, with $G_{k_0}\cdot g_{k_0}h \neq G_i\cdot g_i$. Recall that g_{k_0} is the exponent in $G_{k_0}^{g_{k_0}}$ in the $\mathfrak{S}$-labelling α_0. By Observation 4.1.2, we may assume without loss of generality that $G_{k_0}\cdot g_{k_0}h$ is adjacent to $U\cdot x$ in $\underline{\tilde{\alpha}}$.

If $h \in G_{k_0}^{g_{k_0}}$ then $G_{k_0}\cdot g_{k_0}h = G_{k_0}\cdot g_{k_0}$, hence $G_{k_0}\cdot g_{k_0}$ lies on the i^{th} $U\cdot x$-spoke and the Lemma is satisfied (in this case, the assumption that the i^{th} $U\cdot x$-spoke has length strictly greater than 1 means we have that $G_{k_0}\cdot g_{k_0} \neq G_i\cdot g_i$). We will thus assume $h \notin G_{k_0}^{g_{k_0}}$.

Since $G_1^{g_1} * \cdots * G_n^{g_n}$ is a free splitting for G, we may write $h = h_1 \ldots h_m$ where for each $a \in \{1, \ldots, m\}$, h_a belongs to a factor group $G_{k_a}^{g_{k_a}}$. We will assume $g_{k_0}h_1 \ldots h_m$ is a reduced word — that is, $k_a \neq k_{a-1}$ for each $a \in \{1, \ldots, m\}$.

By Lemma 4.1.6, there exist j and l in $\{0, \ldots, m\}$ with $k_j \neq k_l$ such that $[G_{k_0}\cdot g_{k_0}h, G_{k_j}\cdot g_{k_j}, G_{k_l}\cdot g_{k_l}]$ holds. Note that $k_j \neq k_l$ means that $G_{k_j}\cdot g_{k_j} \neq G_{k_l}\cdot g_{k_l}$, and $h \notin G_{k_0}^{g_{k_0}}$ means $G_{k_0}\cdot g_{k_0}h \neq G_{k_0}\cdot g_{k_0}$ and hence $G_{k_0}\cdot g_{k_0}h \neq G_{k_a}\cdot g_{k_a}$ for any $k_a \in \{1, \ldots, n\}$. In particular, $G_{k_0}\cdot g_{k_0}h \neq G_{k_j}\cdot g_{k_j}$.

Observe that since $G_{k_0}\cdot g_{k_0}h$ is adjacent to $U\cdot x$ in $\underline{\tilde{\alpha}}$ and $[G_{k_0}\cdot g_{k_0}h, G_{k_j}\cdot g_{k_j}]$ has length at least 2 (since $\underline{\tilde{\alpha}}$ is bipartite), then $[G_{k_0}\cdot g_{k_0}h, G_{k_j}\cdot g_{k_j}, G_{k_l}\cdot g_{k_l}]$ implies either $[U\cdot x, G_{k_0}\cdot g_{k_0}h, G_{k_j}\cdot g_{k_j}, G_{k_l}\cdot g_{k_l}]$ or $[G_{k_0}\cdot g_{k_0}h, U\cdot x, G_{k_j}\cdot g_{k_j}, G_{k_l}\cdot g_{k_l}]$. In either case, we have that the vertex $G_{k_j}\cdot g_{k_j}$ in $\underline{\tilde{\alpha}}$ lies on the $\underline{\tilde{\alpha}}$-geodesic $[U\cdot x, G_{k_l}\cdot g_{k_l}]$, as required. □

We are now able to provide the key ingredient required for this Section: that $U\cdot x$-volumes are 'reducible':

Lemma 4.2.10. *Let $[\alpha_0]$ be an α-graph class in $\mathcal{S}_n^{(0)}$ with representative $\alpha_0 = (\alpha : G_1^{g_1}, \ldots, G_n^{g_n})$, and fix $x \in G$ so that $|\alpha_0|_{U\cdot x} > n$. Then there exist an α-graph class $[\alpha_1]$ and an A-graph class $[A_1]$ in $\mathcal{S}_n^{(0)}$ so that $[\alpha_0]$—$[A_1]$—$[\alpha_1]$ is a path in $\mathcal{S}_n$, and $[\alpha_1]$ has a representative α_1 with $|\alpha_1|_{U\cdot x} < |\alpha_0|_{U\cdot x}$.*

Proof. Since $|\alpha_0|_{U\cdot x} > n$ then by Lemma 4.2.9, we have $[U\cdot x, G_i\cdot g_i, G_j\cdot g_j]$ in $\underline{\tilde{\alpha}}$ for some distinct $i, j \in \{1, \ldots, n\}$. Take y and z in G such that $U\cdot y$ and $U\cdot z$ are adjacent to $G_i\cdot g_i$ in $\underline{\tilde{\alpha}}$ and satisfy $[U\cdot x, U\cdot y, G_i\cdot g_i, U\cdot z, G_j\cdot g_j]$ (possibly with $U\cdot y = U\cdot x$, but never $U\cdot y = U\cdot z$). Set $\alpha_1 := (\alpha : G_1^{g_1}, \ldots, G_{j-1}^{g_{j-1}}, G_j^{g_j z^{-1} y}, G_{j+1}^{g_{j+1}}, \ldots, G_n^{g_n})$ and $A_1 := (A : G_i^{g_i}; G_1^{g_1}, \ldots, G_j^{g_j}, \ldots, G_n^{g_n})$. By Lemma 4.1.7, we have that $z^{-1}y \in G_i^{g_i}$, and so by Lemma 3.1.7 we have $(A : G_i^{g_i}; G_1^{g_1}, \ldots, G_j^{g_j}, \ldots, G_n^{g_n}) \simeq (A : G_i^{g_i}; G_1^{g_1}, \ldots, G_j^{g_j z^{-1} y}, \ldots, G_n^{g_n})$. We now see that by Lemma 3.1.8, $[\alpha_0]$—$[A_1]$—$[\alpha_1]$ is a path in $\mathcal{S}_n$.

Observe that $\mathrm{Sp}_{U\cdot x}(\alpha_0)$ and $\mathrm{Sp}_{U\cdot x}(\alpha_1)$ differ only in the j^{th} $U\cdot x$-spoke. Let V be the vertex in $\underline{\tilde{\alpha}}$ satisfying $[U\cdot y, U\cdot x] \cap [U\cdot y, G_j\cdot g_j z^{-1} y] = [U\cdot y, V]$. Note that we may have

$V \in \{U{\cdot}y, U{\cdot}x, G_j{\cdot}g_jz^{-1}y\}$ (or we may have $V \notin \{U{\cdot}y, U{\cdot}x, G_j{\cdot}g_jz^{-1}y\}$). For brevity, let $k := |[U{\cdot}z, G_j{\cdot}g_j]|$, $l := |[U{\cdot}y, V]|$, and $m := |[U{\cdot}x, U{\cdot}y]|$. We illustrate this in Figure 2.8.

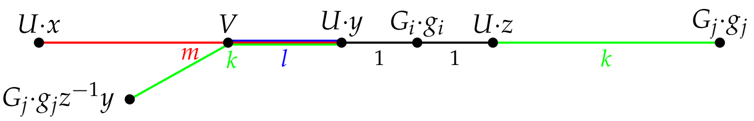


FIGURE 2.8: j^{th} $U{\cdot}x$-Spokes of α_0 and α_1 in $\underline{\tilde{\alpha}}$

Note that $0 \leqslant l \leqslant \min\{k, m\}$. Since G acts on $\underline{\tilde{\alpha}}$ by isometries, then $|[U{\cdot}y, G_j{\cdot}g_jz^{-1}y]| = |[U{\cdot}z, G_j{\cdot}g_j]| = k$. We see that $|[U{\cdot}x, G_j{\cdot}g_j]| = m + k + 2$, while $|[U{\cdot}x, G_j{\cdot}g_jz^{-1}y]| = m + k - 2l$. Thus $|\alpha_1|_{U\cdot x} = |\alpha_0|_{U\cdot x} - 2 - 2l \leqslant |\alpha_0|_{U\cdot x} - 2 < |\alpha_0|_{U\cdot x}$, as required. ☐

Example 4.2.11. Let $G = W * X * Y * Z$ and $\alpha_0 = (\alpha : W, X^{z_1w_1}, Y^{x_1w_2}, Z^{x_2z_1w_1})$ as in Example 4.2.2, and let $\alpha_1 = \iota_{w_1^{-1}z_1^{-1}} \cdot \alpha_0 = (\alpha : W^{z_1^{-1}}, X, Y^{x_1(w_2w_1^{-1})z_1^{-1}}, Z^{x_2})$ as in Example 4.2.6. We will now iteratively 'reduce' the spoke graph $\mathrm{Sp}_{U\cdot 1}(\alpha_1)$ using the technique outlined in the proof of Lemma 4.2.10, until we reach the $U{\cdot}1$-spoke graph for $\underline{\alpha} = (\alpha : W, X, Y, Z)$; we illustrate this process in Figure 2.9.

We begin with $\mathrm{Sp}_{U\cdot 1}(\alpha_1)$ (noting that $|\alpha_1|_{U\cdot 1} = 14$), and observe that in $\underline{\tilde{\alpha}}$ we have $[U{\cdot}1, U{\cdot}z_1^{-1}, W{\cdot}z_1^{-1}, U{\cdot}(w_2w_1^{-1})z_1^{-1}, Y{\cdot}x_1(w_2w_1^{-1})z_1^{-1}]$ with $W{\cdot}z_1^{-1}$ adjacent to both $U{\cdot}z_1^{-1}$ and $U{\cdot}(w_2w_1^{-1})z_1^{-1}$. We compute:

$$\begin{aligned}&(x_1(w_2w_1^{-1})z_1^{-1}))((w_2w_1^{-1})z_1^{-1})^{-1}(z_1^{-1})\\=&x_1w_2w_1^{-1}z_1^{-1}z_1w_1w_2^{-1}z_1^{-1}\\=&x_1z_1^{-1}\end{aligned}$$

and thus set $\alpha_2 := (\alpha : W^{z_1^{-1}}, X, Y^{x_1z_1^{-1}}, Z^{x_2})$. Constructing $\mathrm{Sp}_{U\cdot 1}(\alpha_2)$, we see $|\alpha_2|_{U\cdot 1} = 12$. We now observe that in $\underline{\tilde{\alpha}}$ we have $[U{\cdot}1, X, U{\cdot}x_2, Z{\cdot}x_2]$, so we may set $\alpha_3 := (\alpha : W^{z_1^{-1}}, X, Y^{x_1z_1^{-1}}, Z)$. We have $|\alpha_3|_{U\cdot 1} = 10$ and $[U{\cdot}1, Z, U{\cdot}z_1^{-1}, W{\cdot}z_1^{-1}]$ in $\underline{\tilde{\alpha}}$, thus we set $\alpha_4 := (\alpha : W, X, Y^{x_1z_1^{-1}}, Z)$ (which has $|\alpha_4|_{U\cdot 1} = 8$). Continuing in this fashion, we set $\alpha_5 := (\alpha : W, X, Y^{x_1}, Z)$ (with $|\alpha_5|_{U\cdot 1} = 6$), and finally, $\alpha_6 = (\alpha : W, X, Y, Z) = \underline{\alpha}$, with $|\alpha_6|_{U\cdot 1} = 4$.

If we were to perform this same sequence of steps on $\mathrm{Sp}_{U\cdot z_1w_1}(\alpha_0)$, we would instead terminate at $(\alpha : W^{z_1w_1}, X^{z_1w_1}, Y^{z_1w_1}, Z^{z_1w_1}) =: \alpha'$. Note that $\alpha' \simeq \underline{\alpha}$, and $|\alpha'|_{U\cdot z_1w_1} = 4 = |\underline{\alpha}|_{U\cdot 1}$.

Finally, we can prove our main result of this Section: that paths to $[\underline{\alpha}]$ always exist in $\mathcal{S}_n$.

Proposition 4.2.12. *If $[\alpha_0] \in \mathcal{S}_n^{(0)}$ is an α-graph class then there exists a path in $\mathcal{S}_n$ of the form:*

$$[\alpha_0] - [A_1] - [\alpha_1] - \ldots - [\alpha_{m-1}] - [A_m] - [\alpha_m] = [\underline{\alpha}]$$

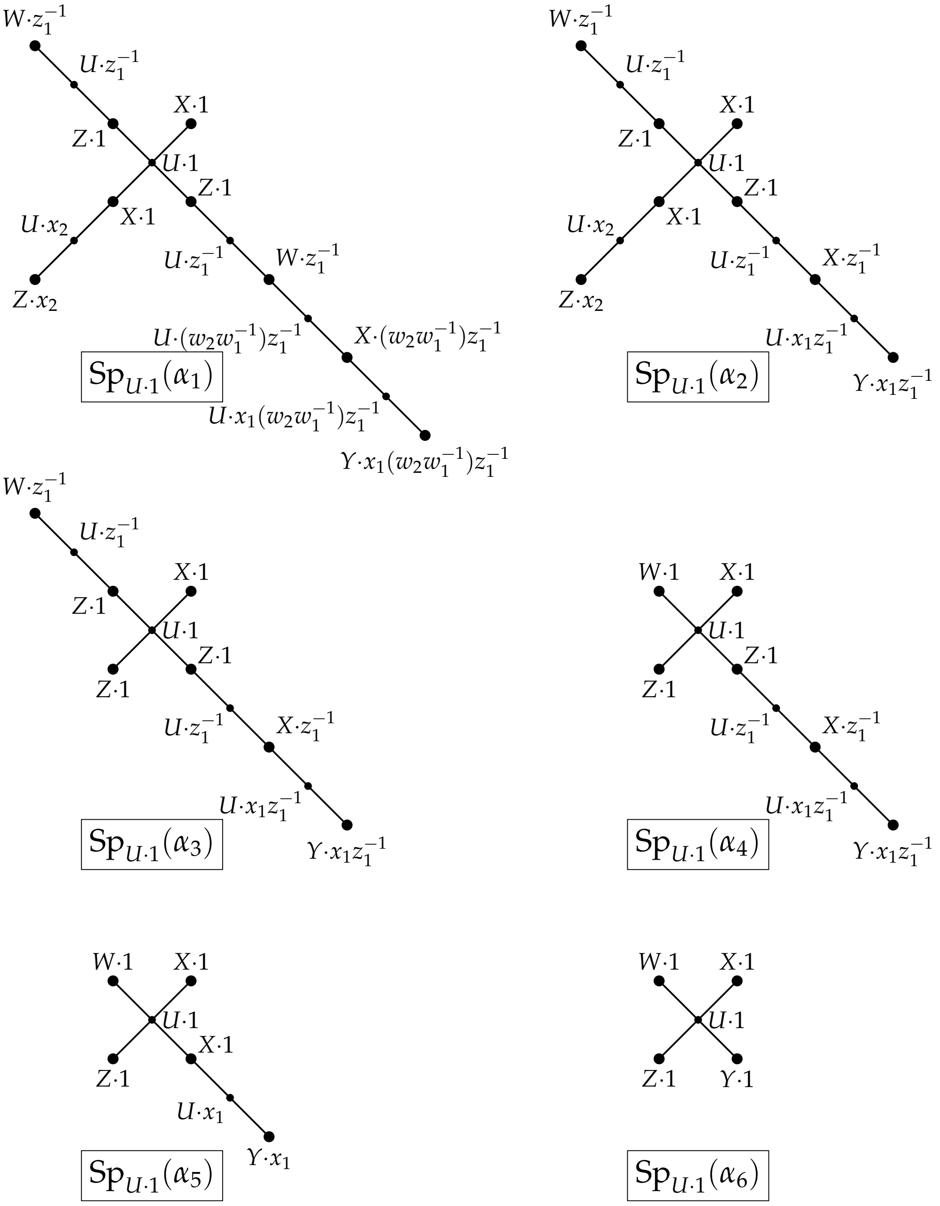


FIGURE 2.9: The Spoke Graphs $\mathrm{Sp}_{U\cdot 1}(\alpha_i)$ for $i = 1, \ldots, 6$

Proof. Let $[\alpha_0]$ be an arbitrary α-graph class in $\mathcal{S}_n$ with representative α_0. We will prove by (strong) induction on $|\alpha_0|_U$ that there is some (finite) path in $\mathcal{S}_n$ from $[\alpha_0]$ to $[\underline{\alpha}]$. Recall from Observation 4.2.8 that $|\alpha_0|_U$ is an integer with $|\alpha_0|_U \geqslant n$.

Suppose $|\alpha_0|_U = n$. By Lemma 4.2.7, we then have $\alpha_0 \simeq \underline{\alpha}$. Thus $[\alpha_0] = [\underline{\alpha}]$ is a path (of length 0) and we are done.

Fix $N \geqslant n$ and suppose that for all α-graphs $\hat{\alpha}$ with $|\hat{\alpha}|_U \leqslant N$ there is some path p in $\mathcal{S}_n$ from $[\hat{\alpha}]$ to $[\underline{\alpha}]$.

Now suppose $|\alpha_0|_U = N + 1$. Then $|\alpha_0|_U > N \geqslant n$, and by Lemma 4.2.10, there exist $[A_1], [\alpha_1] \in \mathcal{S}_n^{(0)}$ so that $[\alpha_0]$—$[A_1]$—$[\alpha_1]$ is a path in $\mathcal{S}_n$ and $|\alpha_1|_U < |\alpha_0|_U = N + 1$. In particular, $|\alpha_1|_U \leqslant N$, and so by hypothesis there is some path p in $\mathcal{S}_n$ from $[\alpha_1]$ to $[\underline{\alpha}]$.

Thus by concatenating the paths $[\alpha_0]$—$[A_1]$—$[\alpha_1]$ and p, we have found a path in $\mathcal{S}_n$ from $[\alpha_0]$ to $[\underline{\alpha}]$.

Since the statement holds for $|\alpha_0|_U = n$, and the statement holding for all $|\alpha_0|_U \leqslant N$ (for some $N \geqslant n$) implies it holds for $|\alpha_0|_U = N + 1$, we conclude that the statement holds for all possible values of $|\alpha_0|_U$. Thus given an arbitrary α-graph class $[\alpha_0]$, we can find a path in $\mathcal{S}_n$ from $[\alpha_0]$ to $[\underline{\alpha}]$.

That our path is of the form $[\alpha_0]$—$[A_1]$—$[\alpha_1]$—...—$[\alpha_{m-1}]$—$[A_m]$—$[\alpha_m] = [\underline{\alpha}]$ follows immediately by recalling from Observation 3.2.2 that $\mathcal{S}_n$ is a bipartite graph between the set of α-graph classes and the set of A-graph classes. ☐

Example 4.2.13. Let $G = W * X * Y * Z$ and $\alpha_1 = (\alpha : W^{z_1^{-1}}, X, Y^{x_1(w_2w_1^{-1})z_1^{-1}}, Z^{x_2})$ as in Example 4.2.11. From Figure 2.9 we can immediately read off a path in $\mathcal{S}_4$ from $[\alpha_1] = [\alpha_0]$ to $[\alpha_6] = [\underline{\alpha}]$, which we illustrate in Figure 2.10. Additionally, by considering the steps we took in Example 4.2.11, we can pinpoint the automorphisms in $\mathrm{Aut}(W * X * Y * Z)$ which translate between each α-graph, denoted in Figure 2.10 by dotted arrows. Note however that at this point it is not obvious that these should represent elements of $\mathrm{Out}_{\mathfrak{S}}(W * X * Y * Z)$. We address this in Section 5.

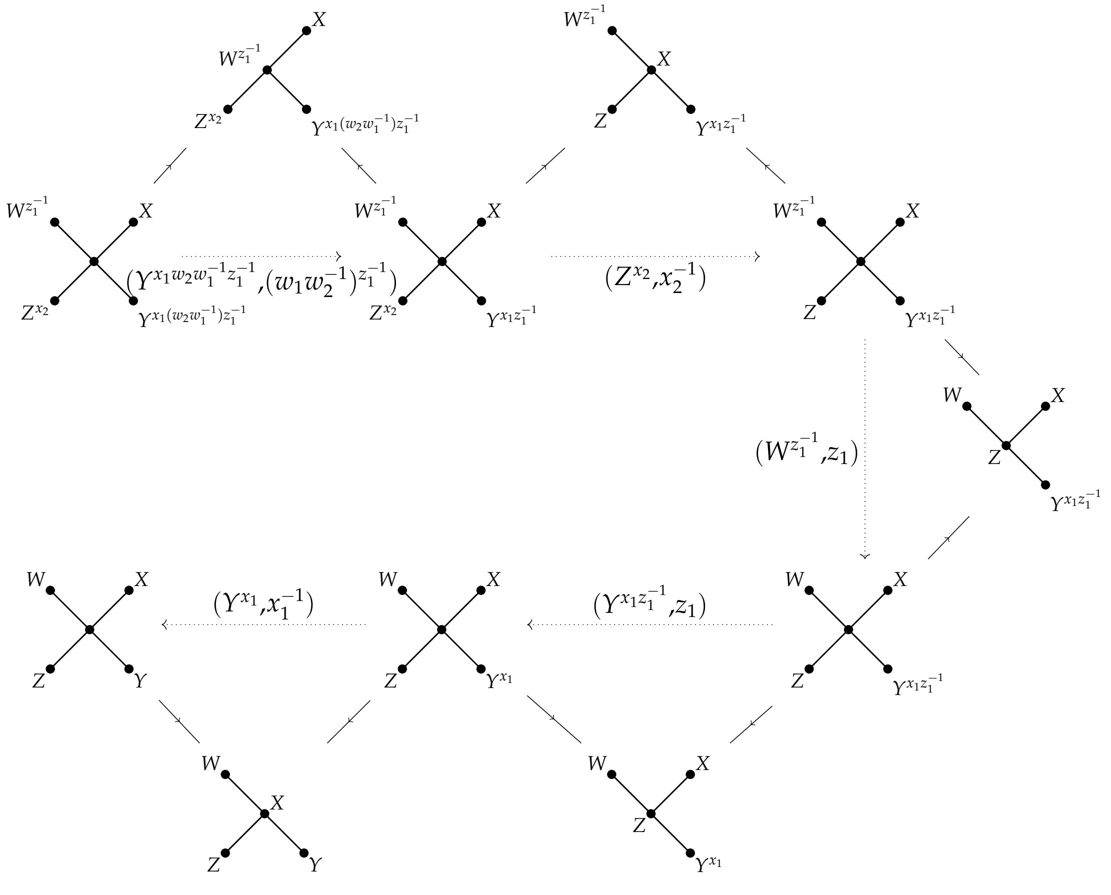


FIGURE 2.10: Path in $\mathcal{S}_4$ from $[\alpha_1]$ to $[\underline{\alpha}]$

Corollary 4.2.14. *The subcomplex $\mathcal{S}_n$ is path-connected.*

Proof. Recall from Observation 3.2.2 that every edge of $\mathcal{S}_n$ is of the form $[\alpha_0]$—$[A_0]$. Moreover, from Observation 3.2.5 we have that every A-graph class $[A_1]$ is adjacent in $\mathcal{S}_n$ to some α-graph class $[\alpha_1]$. Thus by Proposition 4.2.12, given any point P in $\mathcal{S}_n$, there is a path in $\mathcal{S}_n$ from P to the vertex $[\underline{\alpha}]$. Hence $\mathcal{S}_n$ is path connected. □

5 Generators for $\mathrm{Out}_{\mathfrak{S}}(G_1 * \cdots * G_n)$

Let $G = G_1 * \cdots * G_n$ and $\mathfrak{S} = (G_1, \ldots, G_n)$. Recall that $\underline{\alpha}$ is the $\mathfrak{S}$-labelling $(\alpha : G_1, \ldots, G_n)$ of the graph α in Figure 2.1 and $\underline{A^i}$ is the $\mathfrak{S}$-labelling $(A : G_i; G_1, \ldots, G_n) = (A : G_i, G_1, \ldots, G_{i-1}, G_{i+1}, \ldots, G_n)$ of the graph A in Figure 2.2.

The purpose of this Section (and the paper as a whole) is to prove Theorem 5.0.5, that is, to show that any pure symmetric outer automorphism (Definition 1.1.1) of the splitting $G_1 * \cdots * G_n$ is a product of factor outer automorphisms (Definiton 1.2.1) and Whitehead outer automorphisms (Definition 1.3.1) relative to $\mathfrak{S}$. Equivalently, we will show that $\mathrm{Out}_{\mathfrak{S}}(G)$ is generated by factor outer automorphisms and Whitehead outer automorphisms relative to $\mathfrak{S}$.

We begin by determining the $\mathrm{Out}_{\mathfrak{S}}(G)$ stabilisers of the vertices $[\underline{\alpha}]$ and $[\underline{A^i}]$ in $\mathcal{S}_n$.

Proposition 5.0.1. *Let $\Phi \in \mathrm{Out}_{\mathfrak{S}}(G)$. Then $\Phi \cdot [\underline{\alpha}] = [\underline{\alpha}]$ if and only if Φ is a factor outer automorphism.*

Proof. If $\Phi \in \mathrm{Out}_{\mathfrak{S}}(G)$ is a factor outer automorphism, then there exists $\varphi \in \Phi$ which is a factor automorphism and by Observation 2.1.3, we have:

$$\begin{aligned}\Phi \cdot [\underline{\alpha}] &= \Phi \cdot [(\alpha : G_1, \ldots, G_n)] \\ &= [(\alpha : \varphi(G_1), \ldots, \varphi(G_n))] \\ &= [(\alpha : G_1, \ldots, G_n)] \\ &= [\underline{\alpha}]\end{aligned}$$

Now suppose $\Phi \in \mathrm{Out}_{\mathfrak{S}}(G)$ is such that $\Phi \cdot [\underline{\alpha}] = [\underline{\alpha}]$. Then for any $\varphi \in \Phi$, we have $(\alpha : \varphi(G_1), \ldots, \varphi(G_n)) \simeq (\alpha : G_1, \ldots, G_n)$. Fix $\varphi \in \Phi$. By Lemma 3.1.3 we deduce that there exists some $g \in G$ so that for each $i \in \{1, \ldots, n\}$, $\varphi(G_i) = G_i^g$. Let $\iota_{g^{-1}} \in \mathrm{Aut}(G)$ be the inner automorphism $x \mapsto gxg^{-1}$ for all $x \in G$. Then $\iota_{g^{-1}}\varphi \in \Phi$ and for each $i \in \{1, \ldots, n\}$, we have $\iota_{g^{-1}}(\varphi(G_i)) = \iota_{g^{-1}}(G_i^g) = G_i$. Letting φ_i be the map $\iota_{g^{-1}}\varphi$ with domain restricted to the subgroup G_i, we have that $\varphi_i \in \mathrm{Aut}(G_i)$. Hence $(\varphi_1, \ldots, \varphi_n) \in \prod_{i=1}^{n} \mathrm{Aut}(G_i)$ so $\iota_g\varphi$ is a factor automorphism, and thus Φ is a factor outer automorphism. □

Proposition 5.0.2. *Let* $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$*. Then* $\Psi \cdot [\underline{A^i}] = [\underline{A^i}]$ *if and only if* Ψ *can be written as a product of factor outer automorphisms and Whitehead outer automorphisms with operating factor* G_i*.*

Proof. If $\Phi \in \mathrm{Out}_{\mathfrak{S}}(G)$ is a factor outer automorphism, then there exists $\varphi \in \Phi$ which is a factor automorphism and by Observation 2.1.3, we have:

$$\begin{aligned}\Phi \cdot [\underline{A^i}] &= \Phi \cdot [(A : G_i; G_1, \ldots, G_n)]\\ &= [(A : \varphi(G_i); \varphi(G_1), \ldots, \varphi(G_n))]\\ &= [(A : G_i; G_1, \ldots, G_n)]\\ &= [\underline{A^i}]\end{aligned}$$

If $\Omega \in \mathrm{Out}_{\mathfrak{S}}(G)$ is a Whitehead outer automorphism with operating factor G_i, then there is a (unique) $\omega \in \Omega$ which is a Whitehead automorphism with operating factor G_i. Now for some $j \in \{1, \ldots, n\} - \{i\}$ and some $x \in G_i$, we have $\omega = (G_j, x)$. Setting $g_j = x$ and $g_k = 1$ for $k \in \{1, \ldots, n\} - \{j\}$, we have that for each $k \in \{1, \ldots, n\}$, $\omega(G_k) = G_k^{g_k}$. Note that $g_1, \ldots, g_n \in G_i$, so by Lemma 3.1.7, we have:

$$\begin{aligned}\Omega \cdot [\underline{A^i}] &= [(A : \omega(G_i); \omega(G_1), \ldots, \omega(G_n))]\\ &= [(A : G_i^{g_i}; G_1^{g_1}, \ldots, G_n^{g_n})]\\ &= [(A : G_i; G_1, \ldots, G_n)]\\ &= [\underline{A^i}]\end{aligned}$$

Therefore any factor outer automorphism or Whitehead outer automorphism with operating factor G_i stabilises $[\underline{A^i}]$, and hence so too must any product of these.

Now suppose $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ is such that $\Psi \cdot [\underline{A^i}] = [\underline{A^i}]$. Then for all $\psi \in \Psi$ we have $(A : \psi(G_i); \psi(G_1), \ldots, \psi(G_n)) \simeq (A : G_i; G_1, \ldots, G_n) = \underline{A^i}$. By Lemma 3.1.7 and Observation 2.1.3 there is some $\psi \in \Psi$ such that there exist $g_1, \ldots, g_n \in G_i$ with $\psi(G_j) = G_j^{g_j}$ for all $j \in \{1, \ldots, n\}$.

Fix $j \in \{1, \ldots, n\}$. Since $\psi(G_j) = G_j^{g_j}$, then for each $x \in G_j$ there exists $y_x \in G_j^{g_j}$ such that $\psi(x) = y_x$ (and for each $z \in G_j^{g_j}$ there exists $x \in G_j$ with $z = \psi(x)$). Define a map φ_j with domain G_j by $x \mapsto g_j y_x g_j^{-1}$ for all $x \in G_j$. Note that since $y_x \in G_j^{g_j}$ then $g_j y_x g_j^{-1} \in G_j$. Thus $\varphi_j \in \mathrm{Aut}(G_j)$. Now $\psi(x) = g_j^{-1}\varphi_j(x)g_j = (G_j, g_j)(\varphi_j(x))$ where (G_j, g_j) is a Whitehead automorphism with operating factor G_i for all $j \neq i$ and $(G_i, g_i) \in \mathrm{Aut}(G_i)$ is the inner automorphism of G_i which conjugates each element of G_i by g_i. Identifying φ_j (and (G_i, g_i) when $j = i$) with its image in $\displaystyle\prod_{i=1}^{n} \mathrm{Aut}(G_i)$ under the natural inclusion $\mathrm{Aut}(G_j) \hookrightarrow \displaystyle\prod_{i=1}^{n} \mathrm{Aut}(G_i)$, we note that if $z \in G_k$ for any $k \neq j$ then $(G_j, g_j)(z) = z = \varphi_j(z)$.

Since this applies for each $j \in \{1, \ldots, n\}$, we conclude that $\psi = (G_1, g_1)\varphi_1 \ldots (G_n, g_n)\varphi_n$ is a product of factor automorphisms and Whitehead automorphisms. Thus $\Psi = [\psi] = [(G_1, g_1)\varphi_1 \ldots (G_n, g_n)\varphi_n] = [(G_1, g_1)][\varphi_1] \ldots [(G_n, g_n)][\varphi_n]$ is a product of factor outer automorphisms and Whitehead outer automorphisms. □

In Proposition 4.2.12, we established the existence of paths in $\mathcal{S}_n$ from arbitrary α-graph classes to $[\underline{\alpha}]$. We will now show how these paths can be used to generate (outer) automorphisms.

Proposition 5.0.3. *Let* $[\underline{\alpha}] = [\alpha_0] — [A_1] — [\alpha_1] — [A_2] — \ldots — [A_m] — [\alpha_m]$ *be a path in* $\mathcal{S}_n$. *Then there exists* $\Psi \in \mathrm{Out}_{\mathfrak{S}}(G)$ *with* $[\alpha_m] = \Psi \cdot [\underline{\alpha}]$ *such that* Ψ *is a product of factor outer automorphisms and Whitehead outer automorphisms.*

Proof. We prove the statement of the Lemma by induction on m.

First, suppose $m = 0$. Then our path consists of a single point $[\underline{\alpha}] = [\alpha_0]$, and by Proposition 5.0.1, any factor outer automorphism Ψ will satisfy $\Psi \cdot [\underline{\alpha}] = [\underline{\alpha}] = [\alpha_0]$. In particular, the outer automorphism class of the identity automorphism is such a factor outer automorphism.

Now let $N > 0$ and suppose the statement holds for all such paths with $m = N$. We will show that the statement must also hold for all such paths with $m = N + 1$.

Consider an arbitrary path

$$[\underline{\alpha}] = [\alpha_0] — [A_1] — [\alpha_1] — [A_2] — \ldots — [A_N] — [\alpha_N] — [A_{N+1}] — [\alpha_{N+1}]$$

in $\mathcal{S}_n$. By hypothesis, there is some $\Phi \in \mathrm{Out}_{\mathfrak{S}}(G)$ with $[\alpha_N] = \Phi \cdot [\underline{\alpha}]$ such that Φ is a product of factor outer automorphisms and Whitehead outer automorphisms. Since $[\alpha_N] — [A_{N+1}] — [\alpha_{N+1}]$ is a path in $\mathcal{S}_n$, then by Observation 2.4.4 so too is $[\underline{\alpha}] = \Phi^{-1} \cdot [\alpha_N] — \Phi^{-1} \cdot [A_{N+1}] — \Phi^{-1} \cdot [\alpha_{N+1}]$. By Observation 3.2.4 we deduce that $\Phi^{-1} \cdot [A_{N+1}] = [\underline{A^i}]$ for some $i \in \{1, \ldots, n\}$. Now by Lemma 3.2.6 there exists $\Omega \in \mathrm{Out}_{\mathfrak{S}}(G)$ with $\Phi^{-1} \cdot [\alpha_{N+1}] = \Omega \cdot [\underline{\alpha}]$ and $\Omega \cdot [\underline{A^i}] = [\underline{A^i}]$, moreover from Proposition 5.0.2 we see that Ω is a product of factor outer automorphisms and Whitehead outer automorphisms. Hence $[\alpha_{N+1}] = \Phi\Omega \cdot [\underline{\alpha}]$ where $\Phi\Omega \in \mathrm{Out}_{\mathfrak{S}}(G)$ is a product of factor outer automorphisms and Whitehead outer automorphisms, as required. □

Example 5.0.4. As in Example 4.2.2, we will let $G = W * X * Y * Z$ where W, X, Y, and Z are any groups satisfying $|W| \geqslant 3$, $|X| \geqslant 3$, $|Y| \geqslant 2$, and $|Z| \geqslant 2$; let $\mathfrak{S} := (W, X, Y, Z)$ and $\underline{\alpha} = (\alpha : WX, Y, Z)$; let $1 \in G$ be the identity element, let $w_1, w_2 \in W$-$\{1\}$, $x_1, x_2 \in X$-$\{1\}$, and $z_1 \in Z$-$\{1\}$, and let $\varphi \in \mathrm{Aut}_{\mathfrak{S}}(G)$ be such that $\varphi \cdot \underline{\alpha} = (\alpha : W, X^{z_1 w_1}, Y^{x_1 w_2}, Z^{x_2 z_1 w_1}) =: \alpha_0$ (φ is more precisely defined in Example 4.2.2). As in Examples 4.2.6 and 4.2.11, let $\alpha_1 = \iota_{w_1^{-1} z_1^{-1}} \cdot \alpha_0$ where $\iota_{w_1^{-1} z_1^{-1}}$ is an inner automorphism.

From Example 4.2.13, we have a path in $\mathcal{S}_4$

$$\alpha_1 — A_1 — \alpha_2 — A_2 — \alpha_3 — A_3 — \alpha_4 — A_4 — \alpha_5 — A_5 — \alpha_6 = \underline{\alpha}$$

which is explicitly drawn in Figure 2.10. From this, we see that $\alpha_5 = (Y, x) \cdot \alpha_6 = (Y, x) \cdot \underline{\alpha}$, with $(Y, x) \in \mathrm{Stab}(A_5)$ a Whitehead automorphism relative to $\mathfrak{S}$. We set $\psi_1 := (Y, x)$, and proceed to 'pull back' our path using ψ_1, creating a path $\psi_1^{-1} \cdot \alpha_1 — \psi_1^{-1} \cdot A_1 — \psi_1^{-1} \cdot \alpha_2 — \psi_1^{-1} \cdot A_2 — \psi_1^{-1} \cdot \alpha_3 — \psi_1^{-1} \cdot A_3 — \psi_1^{-1} \cdot \alpha_4 — \psi_1^{-1} \cdot A_4 — \psi_1^{-1} \cdot \alpha_5 = \underline{\alpha}$, with $\psi_1^{-1} \cdot \alpha_4 = (\alpha : W, X, Y^{z_1^{-1}}, Z)$. We see that $\psi_1^{-1} \cdot \alpha_4 = (Y, z_1^{-1}) \cdot \underline{\alpha}$ with (Y, z_1^{-1}) a Whitehead automorphism relative to $\mathfrak{S}$. Repeating this process, we obtain the commutative diagram in Figure 2.11, where $\psi_2 = (Y, z_1^{-1})$, $\psi_3 = (W, z_1^{-1})$, $\psi_4 = (Z, x_2)$, and $\psi_5 = (Y, w_2 w_1^{-1})$.

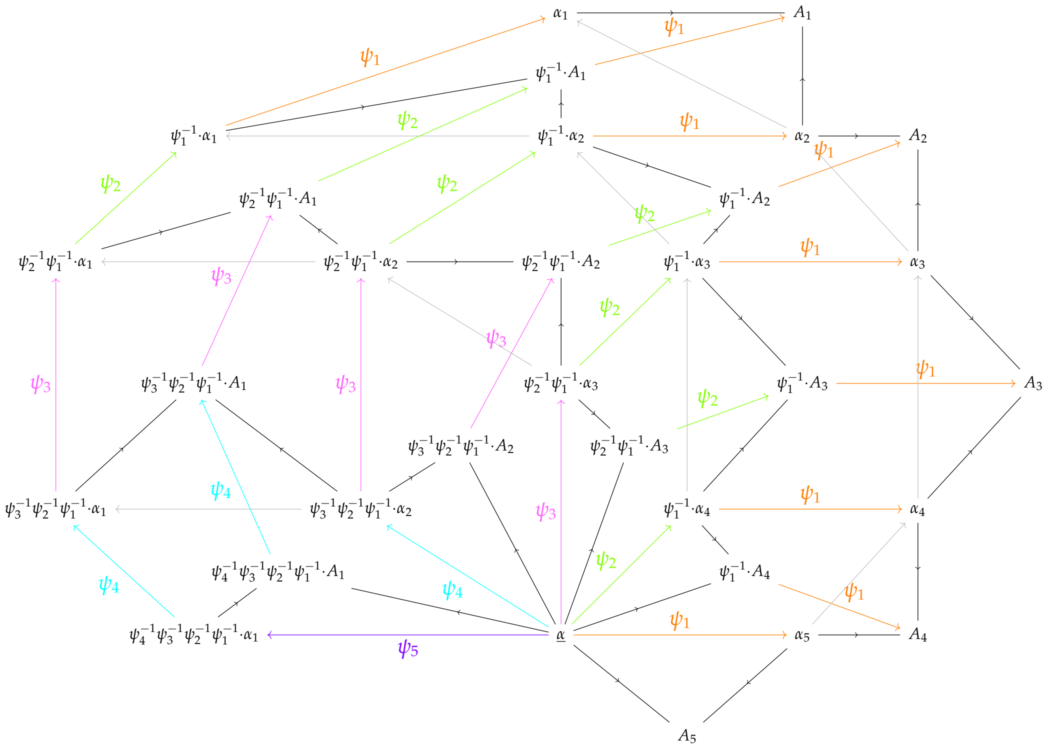


FIGURE 2.11: Commuting Diagram of Paths in $\mathcal{S}_4$

Now

$$\begin{aligned}\underline{\alpha} &= \psi_5^{-1}\psi_4^{-1}\psi_3^{-1}\psi_2^{-1}\psi_1^{-1} \cdot \alpha_1 \\ &= \psi_5^{-1}\psi_4^{-1}\psi_3^{-1}\psi_2^{-1}\psi_1^{-1}\iota_{w_1^{-1}z_1^{-1}} \cdot \alpha_0 \\ &= \psi_5^{-1}\psi_4^{-1}\psi_3^{-1}\psi_2^{-1}\psi_1^{-1}\iota_{w_1^{-1}z_1^{-1}}\varphi \cdot \underline{\alpha}\end{aligned}$$

thus for some $\theta \in \mathrm{Stab}(\underline{\alpha})$, we have $\psi_5^{-1}\psi_4^{-1}\psi_3^{-1}\psi_2^{-1}\psi_1^{-1}\iota_{w_1^{-1}z_1^{-1}}\varphi = \theta$. We conclude that $\varphi = \iota_{z_1 w_1}\psi_1\psi_2\psi_3\psi_4\psi_5\theta$ where θ is a factor automorphism relative to $\mathfrak{S}$ and each ψ_i is a Whitehead automorphism relative to $\mathfrak{S}$. Hence $[\varphi]$ is a product of Whitehead outer automorphisms and factor outer automorphisms.

Theorem 5.0.5 (Main Theorem)**.** *Let* $G = G_1 * \cdots * G_n$ *be a group which splits as a free product of* $n \geqslant 3$ *non-trivial factors, and let* $\mathfrak{G} = (G_1, \ldots, G_n)$*. Then any pure symmetric outer automorphism* $\Psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ *of the splitting* $G_1 * \cdots * G_n$ *can be written as a product of factor outer automorphisms relative to* $\mathfrak{G}$ *and Whitehead outer automorphisms relative to* $\mathfrak{G}$*.*

Proof. Let $\Psi \in \mathrm{Out}_{\mathfrak{G}}(G)$ and set $[\alpha_0] := \Psi \cdot [\underline{\alpha}] \in \mathcal{S}_n$. By Proposition 4.2.12, there exists some edge path

$$[\alpha_0] \text{—} [A_1] \text{—} [\alpha_1] \text{—} [A_2] \text{—} \ldots \text{—} [A_m] \text{—} [\alpha_m] = [\underline{\alpha}]$$

in $\mathcal{S}_n$ (where each $[\alpha_i]$ is an α-graph class and each $[A_i]$ is an A-graph class). Then by Proposition 5.0.3 there exists $\Theta \in \mathrm{Out}_{\mathfrak{G}}(G)$ with $[\alpha_0] = \Theta \cdot [\underline{\alpha}]$ such that Θ is a product of factor outer automorphisms and Whitehead outer automorphisms. Now $\Theta^{-1}\Psi \cdot [\underline{\alpha}] = [\underline{\alpha}]$ so by Proposition 5.0.1 there is some factor outer automorphism $\Phi \in \mathrm{Out}_{\mathfrak{G}}(G)$ with $\Theta^{-1}\Psi = \Phi$. Hence $\Psi = \Theta\Phi$ is a product of factor outer automorphisms and Whitehead outer automorphisms. ☐

Corollary 5.0.6. *If* $G_1 * \cdots * G_n$ *is a Grushko decomposition for* G *and moreover the factor groups* G_i *are pairwise non-isomorphic, then* $\mathrm{Out}(G)$ *is generated by factor outer automorphisms and Whitehead outer automorphisms.*

Proof. This follows immediately by considering Theorem 5.0.5 in the context of Observation 1.1.2. ☐

References

[1] H. Bass. Covering theory for graphs of groups. *Journal of Pure and Applied Algebra*, 89(1):3–47, 1993.

[2] K. S. Brown. Presentations for groups acting on simply-connected complexes. *Journal of Pure and Applied Algebra*, 32:1–10, 1984.

[3] D. I. Fouxe-Rabinovitch. Über die Automorphismengruppen der freien Produkte. I. *Rec. Math. [Mat. Sbornik] N.S.*, 8 (50):265–276, 1940.

[4] N. D. Gilbert. Presentations of the automorphism group of a free product. *Proceedings of the London Mathematical Society*, s3-54(1):115–140, 01 1987.

[5] V. Guirardel and G. Levitt. The outer space of a free product. *Proceedings of the London Mathematical Society. Third Series*, 94(3):695–714, 2007.

[6] H. Iveson. A presentation for the group of pure symmetric outer automorphisms of a given splitting of a free product. `arXiv:2412.04250` Preprint, 2024.

[7] J. McCool. On basis-conjugating automorphisms of free groups. *Canadian Journal of Mathematics*, 38(6):1525–1529, 1986.

[8] D. McCullough and A. Miller. *Symmetric Automorphisms of Free Products*. American Mathematical Society: Memoirs of the American Mathematical Society. American Mathematical Society, 1996.

[9] J. P. Serre. *Trees*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1980.

Paper 3: **Property R_∞ for groups with infinitely many ends**

Harry Iveson, Armando Martino, Wagner Sgobbi, and Peter Wong, with an Appendix by Francesco Fournier-Facio

ABSTRACT. We show that an accessible group with infinitely many ends has property R_∞. That is, it has infinitely many twisted conjugacy classes for any twisting automorphism. We deduce that having property R_∞ is undecidable amongst finitely presented groups.

We also show that the same is true for a wide class of relatively hyperbolic groups, filling in some of the gaps in the literature. Specifically, we show that a non-elementary, finitely presented relatively hyperbolic group with finitely generated peripheral subgroups which are not themselves relatively hyperbolic, has property R_∞.

In an appendix, Francesco Fournier-Facio shows that a group with a non-zero Aut-invariant homogeneous quasimorphism has property R_∞, which applies to many groups with hyperbolic features.

1 Introduction

Property R_∞ is a group theoretic property with connections to fixed point theory and which has been studied extensively by many authors. A non-exhaustive list might include [14], [16], [34], [35], [54] and [55].

It is a generalisation of the property of having infinitely many conjugacy classes. Instead, one asks that there are infinitely many twisted conjugacy classes, where one twists one side of the conjugacy by an automorphism. Property R_∞ then asks that a group have infinitely many twisted conjugacy classes for any automorphism — see Definition 2.1.5.

Our approach is to extend the techniques of [42], who proved that any non-elementary hyperbolic group has property R_∞. Our main theorem is that any accessible group with infinitely many ends has R_∞.

Theorem 5.1.18. *Any accessible group G with infinitely many ends has property R_∞. In particular, any finitely presented group with infinitely many ends has property R_∞.*

We note that, intuitively, the ends of a group is the number of components of the group at infinity. More concretely, a group with infinitely many ends acts non-trivially on a simplicial tree with finite edge stabilisers. See Theorem 5.1.2 for more detail, or [17] for a more extensive reference source. In particular free products of groups have infinitely many ends (except for the case of the infinite Dihedral groups) and we start by first proving the result for free products.

In fact, from the result on free products we quickly deduce that the property of having R_∞ is undecidable amongst finitely presented groups.

Corollary 4.1.11. *The property of being R_∞ is undecidable amongst finitely presented groups.*

At this point we should mention the result [20, Theorem 3.3] where it is claimed that any non-elementary relatively hyperbolic group has property R_∞. The proof strategy is modelled on the proof in [42], but the technical details are all but absent and to that extent it is very hard to verify the proof given. Specifically, the technique used there is due to Paulin [49] and [50] where one takes a limit of hyperbolic spaces to produce a limiting $\mathbb{R}$-tree, extended by [1] to the relatively hyperbolic case.

Already there is an issue that the limiting tree is not projectively fixed by the automorphism, when using the results in [1] (which is required in the proof). The result [50, Theoreme A] does provide a projectively fixed $\mathbb{R}$-tree but that result is not proved in the relatively hyperbolic case and it seems that the result there would require that the peripheral subgroups be left invariant by the automorphism. In fact, this is a hypothesis that we require in our result about relatively hyperbolic groups in Theorem 6.1.1 and we suspect that it is an essential one for these proof techniques.

There are then various technical details missing in the treatment from [20]. For instance, finite generation of the parabolic subgroups is not mentioned in [20], although it does appear as a hypothesis in [1] and also appears in our Theorem 6.1.2. Leaving aside the invariance of the limiting tree, the next step in is then to invoke [42, Proposition 3.2] (and also [42, Proposition 3.1] in the case '$\lambda = 1$').

These Propositions have various hypotheses but the main ones for [42, Proposition 3.2] are that the limiting tree should be (i) Irreducible (ii) Have finite arc stabilisers and (iii) Admit finitely many orbits of directions at each branch point.

The first of these is very plausible but merely asserted in [20]. For the second, smallness is invoked but without due care. Certainly, the argument given in [49] is not sufficient to produce virtually cyclic stabilisers as one needs to worry about parabolic subgroups. In fact, [1, Theorem 1.2] specifically proves that arc stabilisers are elementary, by which they include the parabolic subgroups. While it is possible that this technical obstacle might be overcome with a specific construction - our Lemma 4.1.8 does prove that arc stabilisers are not parabolic when one takes a limit of edge-free trees - it really merits a careful argument. In short, this hypothesis on arc stabilisers is claimed in [20] but is not supported by the literature.

Finally, the only time (iii) is addressed is by reference to the paper [3] which does not seem to be relevant to that issue (that there are finitely many orbits of directions under the stabiliser of a branch point). It is possible that the paper meant was [2] as referenced by [42], but this is also not a correct reference for that fact; however, in [42] there are other ways of obtaining this result which do not seem available in the relatively hyperbolic case.

That said, the broad strokes of the argument are correct and we address these technical issues carefully to produce our main result about groups with infinitely many ends as well as a fairly general result about relatively hyperbolic groups.

Theorem 6.1.2. *Let G be a non-elementary finitely presented relatively hyperbolic group G whose peripheral subgroups are finitely generated but not relatively hyperbolic. Then G has property R_∞.*

Remark. We note that this Theorem recovers the result of [42] that any non-elementary hyperbolic group has property R_∞, although we really use their argument except for the fact that we make use of JSJ decompositions at the final stage.

Our proof uses train track methods for free products so that we can refer to the literature to address the technical issues mentioned above. We then show that groups with infinitely many ends have R_∞ by extending the results for free products using the tree of cyclinders construction of Guirardel and Levitt, [37] and [38]. The idea here is that a group with infinitely many ends either admits a finite normal subgroup so that the quotient is a free product or admits a tree which is invariant under all automorphisms. See Theorem 5.1.17.

Finally, we prove the Theorem for relatively hyperbolic groups using the canonical JSJ splittings of [40].

After placing this article on the arXiv we received communications from Francesco Fournier-Facio and Anthony Genevois independently observing that it is possible to broaden the results in our paper. We are hugely grateful for these observations and insights.

The most general way to do this seems to be via quasi-morphisms and Francesco Fournier-Facio has written us an Appendix to this paper outlining how to do so. The most general statement of these results is as follows.

Corollary 8.1.6. *Let G be a finitely generated group satisfying one of the following properties.*

1. *G is a non-elementary hyperbolic group.*
2. *G is non-elementary hyperbolic relative to a collection of finitely generated subgroups, none of which is relatively hyperbolic.*
3. *G has infinitely many ends.*

4. *G is a graph product of groups over a finite graph that does not decompose non-trivially as a join and is not reduced to a single vertex.*

5. *G is a graph product of abelian groups and it is not virtually abelian.*

Then G has property R_∞.

We note that the main body of this paper constructs translation length functions arising from trees which are constant on twisted conjugacy classes, whereas the method via quasi-morphisms construct such functions which are bounded on twisted conjugacy classes (albeit for all automorphisms at the same time). Nevertheless, the methods should be seen as analogous both in their overall strategy and in the details of their technical construction.

2 Twisted Conjugacy and the R_∞ Property

2.1 Twisted Conjugacy

Definition 2.1.1. Let G be a group and $\varphi \in \mathrm{Aut}(G)$ an automorphism of G. We define a relation $\sim_\varphi$ on G by $x \sim_\varphi y$ if and only if $\exists w \in G$ with $(w\varphi)xw^{-1} = y$. If $x \sim_\varphi y$, we say that x and y are twisted conjugates in G.

Remark. We take the action of $\mathrm{Aut}(G)$ on G to be a right action, writing $g \cdot \varphi$ or $g\varphi$ for the image of $g \in G$ under the automorphism $\varphi \in \mathrm{Aut}(G)$.

Remark. Note that there is an asymmetry in the definition of twisted conjugacy. The side on which we place the inverse makes no difference, but the side on which we put the automorphism does. However,

$$wx(w\varphi)^{-1} = (u\varphi^{-1})xu^{-1}, \text{ where } u = w\varphi.$$

This shows the 'left' and 'right' versions of twisted conjugacy are related at the cost of changing the automorphism to its inverse. Since the property of having R_∞ - Definition 2.1.3 - is about all possible automorphisms, it makes no difference which we choose.

The following is a standard fact and an easy exercise.

Lemma 2.1.2. *Twisted conjugacy ($\sim_\varphi$) is an equivalence relation on G.*

Definition 2.1.3 (R_∞)**.** Let G be a group and $\varphi \in \mathrm{Aut}(G)$. We say that G has property $R_\infty(\varphi)$ if $\sim_\varphi$ has infinitely many equivalence classes in G. We say that G has property R_∞ if G has $R_\infty(\varphi)$ for all $\varphi \in \mathrm{Aut}(G)$.

Lemma 2.1.4. *If $\iota \in \mathrm{Inn}(G)$ then G has $R_\infty(\varphi)$ if and only if G has $R_\infty(\varphi\iota)$.*

Proof. Let $x \in G$ and let $\iota_x \in \mathrm{Inn}(G)$ be the automorphism $g \mapsto x^{-1}gx = g^x$. Let $y, z \in G$. Then:

$$\begin{aligned} & y \sim_\varphi z \\ \Leftrightarrow\ & \exists w \in G \text{ such that } z = (w\varphi)yw^{-1} \\ \Leftrightarrow\ & \exists w \in G \text{ such that } xz = x(w\varphi)(x^{-1}x)yw^{-1} = (w\varphi)^x(xy)w^{-1} = (w\varphi\iota_x)(xy)w^{-1} \\ \Leftrightarrow\ & xy \sim_{\varphi\iota_x} xz. \end{aligned}$$

Thus we have a bijection between the equivalence classes of $\sim_\varphi$ and those of $\sim_{\varphi\iota_x}$. Hence G has $R_\infty(\varphi)$ if and only if G has $R_\infty(\varphi\iota_x)$. □

Definition 2.1.5. Let G be a group and let $\Phi \in \mathrm{Out}(G)$. We say that G has property $R_\infty(\Phi)$ if G has $R_\infty(\varphi)$ for some (and hence every) automorphism $\varphi \in \Phi$.

Remark. By Lemma 2.1.4, this concept of $R_\infty(\Phi)$ for $\Phi \in \mathrm{Out}(G)$ is well-defined, and it follows that G has R_∞ (i.e. G has $R_\infty(\varphi)$ for all $\varphi \in \mathrm{Aut}(G)$) if and only if G has $R_\infty(\Phi)$ for all $\Phi \in \mathrm{Out}(G)$.

2.2 Mapping Torus

Our argument about twisted conjugacy classes uses the standard technique of converting questions about twisted conjugacy in a group G to ones concerning genuine conjugacy in a mapping torus of G. We therefore recall the definition of a mapping torus.

Definition 2.2.1 (Mapping torus)**.** Let G be a group and $\varphi \in \mathrm{Aut}(G)$. The mapping torus of φ is the semi-direct product $M_\varphi := G \rtimes_\varphi \mathbb{Z} = \langle G, t | g^t = g\varphi\ \forall g \in G \rangle$.

Remark. Since M_φ is a semi-direct product, elements of M_φ have a standard form gt^k where $g \in G$ and $k \in \mathbb{Z}$ are unique. Note that for any $h \in G$ we have $t^{-1}h = h^t t^{-1} = (h\varphi)t^{-1}$, thus an element $t^k h$ can be written in the alternate standard form $(h \cdot \varphi^{-k})t^k$.

A key observation is that twisted conjugacy is realised as standard conjugacy in the mapping torus and that having $R_\infty(\varphi)$ amounts to there being infinitely many conjugacy classes of a certain type in the mapping torus.

Lemma 2.2.2. *Let G be a group and $\varphi \in \mathrm{Aut}(G)$. Then G has $R_\infty(\varphi)$ if and only if the set $\{tx \mid x \in G\}$ has infinitely many M_φ conjugacy classes.*

Proof. Let $x, y \in G$. We have that tx and ty are conjugate in M_φ if and only if there exists $u = gt^k \in M_\varphi$ so that $ty = (tx)^u = u^{-1}(tx)u$. Observe that given $x \in G$ we can always find $h \in G$ so that $u = gt^k = (tx)^k h$. Then $(tx)^u = (tx)^{(tx)^k h} = (tx)^h$. Now for any $x, y \in G$, we have:

$$\begin{aligned} & tx \text{ and } ty \text{ are conjugate in } M_\varphi \\ \Longleftrightarrow & \exists h \in G \text{ with } ty = (tx)^h = h^{-1}txh = t(h^{-1}\varphi)xh \\ \Longleftrightarrow & \exists \hat{h} \in G \text{ with } y = (\hat{h}\varphi)x\hat{h}^{-1} \\ \Longleftrightarrow & y \sim_\varphi x. \end{aligned}$$

□

3 Trees and Group Actions

3.1 Simplicial trees and ℝ-trees

We refer the reader to [15] for a treatment on ℝ-trees and to [12] for a more general textbook on Λ-trees. The definitions and results here are mainly based on [15] and [12]. We note that in the case $\Lambda = \mathbb{R}$ these concepts coincide, whereas the case $\Lambda = \mathbb{Z}$ corresponds to the case of a simplicial tree below.

Definition 3.1.1. A simplicial graph, Γ, is a 4-tuple, (V, E, σ, τ) where:

- V is a set called the vertices of T,
- $E \subseteq V \times V$ is the set of oriented edges of the graph,
- $\sigma : E \to V$ and $\tau : E \to V$ are incidence maps, defined by $\sigma(u, v) = u$ and $\tau(u, v) = v$.

Moreover, for every $e = (u, v) \in E$ we always have that $(v, u) \in E$; we call this the inverse edge, denoted, $\bar{e}$.

Definition 3.1.2. An edge path in a graph, $\Gamma = (V, E, \sigma, \tau)$ is a sequence of vertices, $v_0, v_1, \ldots, v_k$ where for each $0 \leqslant i \leqslant k-1$, $(v_i, v_{i+1}) \in E$. We allow the edge path to consist of a single vertex, v_0.

- The edge path is called trivial when it consists of a single vertex, and non-trivial otherwise.
- An edge path $v_0, v_1, \ldots, v_k$ is said to start at v_0 and end at v_k.
- A non-trivial edge path may also be described as a sequence of edges, $e_0 \ldots e_k$, where $\tau(e_i) = \sigma(e_{i+1})$ for $0 \leqslant i \leqslant k-1$.

- An edge path, $e_0 \dots e_k$ is called reduced if, for all $0 \leqslant i \leqslant k-1$, $e_i \neq \overline{e_{i+1}}$. A trivial path is always considered reduced.

Definition 3.1.3. A simplicial tree is a simplicial graph where between any two vertices there is a unique reduced edge path starting at one and ending at the other.

For a tree, we let $[u, v]$ denote the unique reduced edge path from u to v; this is called the segment from u to v.

Definition 3.1.4 (Culler–Morgan [15])**.** An $\mathbb{R}$-tree T is a path-connected non-empty metric space so that for any points $x, y \in T$, there is a unique arc $[x, y] \subseteq T$ joining x and y, which is isometric to the interval $[0, d(x, y)] \subseteq \mathbb{R}$ (where d is the metric on T).

Equivalently, an $\mathbb{R}$-tree is a 0-hyperbolic geodesic metric space, in the sense of Gromov.

Definition 3.1.5. Given two points, u, v in an $\mathbb{R}$-tree T, we denote by $[u, v]$ the unique geodesic from u to v in T. This is called the segment from u to v.

Definition 3.1.6. Given an $\mathbb{R}$-tree T and a point $x \in T$, a direction at x is a connected component of $T - \{x\}$. We say a point $x \in T$ is a branch point if there are at least three directions at x.

Definition 3.1.7.

- We say that an $\mathbb{R}$-tree is a metric simplicial tree (or simply a simplicial $\mathbb{R}$-tree) if the set of branch points is a discrete subset of the tree.
- If T is a simplicial $\mathbb{R}$-tree and V a discrete subset of T which includes all the branch points (but may include more points), then the edges are all the segments $[u, v]$ between elements of V where $[u, v] \cap V = \{u, v\}$. Any non-trivial segment between vertices may then be given as an edge-path, $e_1 \dots e_k$ where each e_i is an edge.
- Conversely, a simplicial tree T given as a set of vertices and edges may be made into a simplicial $\mathbb{R}$-tree by assigning a positive length to each edge and making T into a metric space via the corresponding path metric.

Definition 3.1.8. Let T be an $\mathbb{R}$-tree. Then for any three points, $u, v, w \in T$, the Y-point, $Y(u, v, w) = y \in T$, is given by:

$$[u, v] \cap [u, w] = [u, y].$$

Lemma 3.1.9 ([12, Chapter 2, Lemma 1.2])**.** *Let T be an $\mathbb{R}$-tree. Then for any three points, $u, v, w \in T$, the Y-point is unique and does not depend on the order of the points. Moreover, $Y(u, v, w)$ is the point on $[u, v]$ whose distance from u is given by the Gromov product, $(v.w)_u = \frac{1}{2}(d_T(u, v) + d_T(u, w) - d_T(v, w))$.*

3.2 Group actions on $\mathbb{R}$-trees

Throughout this subsection, we will consider a group, G, acting isometrically on an $\mathbb{R}$-tree, T. We note that if one starts with a simplicial tree and a group action sending vertices to vertices and edges to edges then the process of making this a simplicial $\mathbb{R}$-tree described above allows one to extend the group action to an isometric action; that is, a group of automorphisms of a tree preserves the induced path metric.

Definition 3.2.1 ([15, p. 576 and Definition 1.4]). Suppose that G is a group acting isometrically on an $\mathbb{R}$-tree, T.

(i) For any $g \in G$ we define the translation length of g (with respect to T) to be,

$$\|g\|_T := \inf_{x \in T}\{d_T(x, xg)\}.$$

We write $\|g\|$ for $\|g\|_T$ if T is understood.

(ii) Define the characteristic set of g to be,

$$A_g = \{x \in T \ : \ d_T(x, xg) = \|g\|\}.$$

(iii) $g \in G$ is called ellipic if $\|g\|_T = 0$ and hyperbolic if $\|g\|_T > 0$.

Lemma 3.2.2 ([15, Lemma 1.3 (p.576)]). *Let G act isometrically on an $\mathbb{R}$-tree, T and let $g \in G$.*

(i) There exists an $x \in T$ such that $d_T(x, xg) = \|g\|$. That is, the infimum in Definition 3.2.1(i) is a minimum .

(ii) The set $A_g = \{x \in T \ : d_T(x, xg) = \|g\| \ \}$ is non-empty (by the previous part). It is also a closed subtree of T, invariant under the action of g.

(iii) If g is elliptic, then A_g is the fixed point set of g, $\mathrm{Fix}(g)$.

(iv) If g is hyperbolic, then A_g is called the axis of g and is isometric to the real line. It is the smallest g-invariant subtree of T. The element g acts on A_g as a translation by the real number $\|g\|$.

(v) If g is hyperbolic and $0 \neq n \in \mathbb{Z}$, then $A_{g^n} = A_g$ and $\|g^n\| = |n|\|g\|$.

Lemma 3.2.3. *Let G act isometrically on an $\mathbb{R}$-tree T. Let $g \in G$ and $x \in T$ any point.*

(i) The midpoint of the segment $[x, xg]$ lies in A_g.

(ii) For any hyperbolic $g \in G$, we have $Y(xg^{-1}, x, xg) \in A_g$.

Proof. (i) This is just [12, Chapter 3 (Lemma 1.1 for the elliptic case and Theorem 1.4 for the hyperbolic case)] .

(ii) This is [12, Chapter 3, Theorem 1.4].

□

Observation 3.2.4. It follows easily that $A_{g^h} = A_g \cdot h$. Hence if g and h commute then h preserves A_g. Further, if g and h commute and are both hyperbolic then $A_g = A_h$, since h preserves the line A_g but A_h is the smallest h-invariant subtree of T.

Definition 3.2.5. Let G act isometrically on an $\mathbb{R}$-tree, T. Then the length function of this action is the function, $l : G \to \mathbb{R}$ given by $l(g) = \|g\|_T$. This is also called the translation length function.

Definition 3.2.6 (Culler–Morgan [15])**.** Let T be an $\mathbb{R}$-tree equipped with an action of a group G by isometries. We say that T is:

- irreducible, if there is no point, line or end of T which is invariant under the action of G. Equivalently, there exist a pair of groups elements, $g, h \in G$ which are hyperbolic and whose axes meet in an arc of finite positive length (See [15, Theorem 2.7] for this equivalence).
- minimal, if there is no proper G-invariant subtree of T.
- non-trivial, if there is no global fixed point (i.e. no $x \in T$ such that $x \cdot G = x$).
- small, if for any non-trivial arc $[x, y]$, the pointwise stabiliser $\mathrm{Stab}([x, y]) = \{g \in G \ : \ zg = z, \ \forall z \in [x, y]\} \leqslant G$ does not contain a free subgroup of rank 2.

Remark. It is also true that the action of G on T is irreducible if and only if there are a pair of isometries whose axes are disjoint [15, Lemmas 2.1 and 1.5 and 1.6]. Concretely, if g, h are hyperbolic and $A_g \cap A_h$ meets in a segment of finite poisitive length then, for some large $n \in \mathbb{Z}$, the axes of g and g^{h^n} will be disjoint.

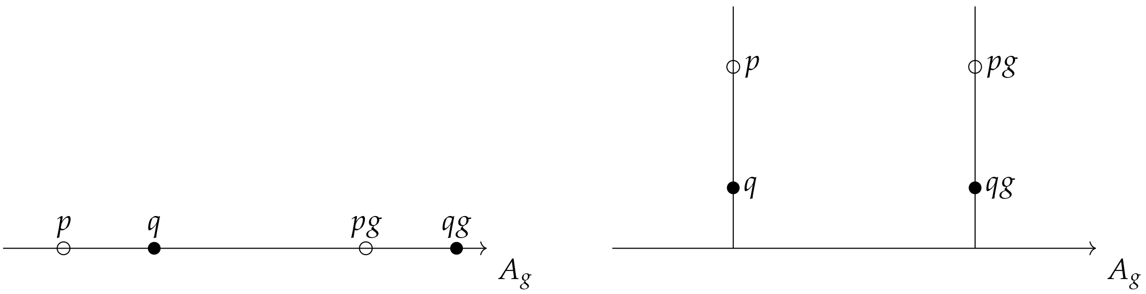
We shall also need the following.

Lemma 3.2.7 (Culler-Morgan [15], 1.5 and 1.6)**.** *Let g, h be hyperbolic isometries of an $\mathbb{R}$-tree T whose axes A_g, A_h are disjoint. Then $\|gh\| = \|hg\| = \|g\| + \|h\| + 2d(A_g, A_h)$. In particular gh and hg are hyperbolic.*

Proposition 3.2.8 ([15, Proposition 3.1])**.** *If G acts on an $\mathbb{R}$-tree T and some element $g \in G$ acts hyperbolically, then T admits a unique, minimal G-invariant subtree. This subtree is exactly the union of all the hyperbolic axes of elements of G.*

Lemma 3.2.9 (Serre [53, Proposition 25 (p.63)])**.** *If G is finitely generated and acts isometrically on an $\mathbb{R}$-tree, T, then the action is non-trivial if and only if there exists a hyperbolic element in G. That is, if there exists some $g \in G$ with $\|g\|_T > 0$.*

Remark. The result in [53] concerns simplicial trees, but the same proof works for $\mathbb{R}$-trees without change.



On the axis:
○ — ● — ○ — ●

Not on the axis:
○ — ● — ● — ○

The Circle-Dot Lemma

Definition 3.2.10. Let T be a tree. For points $x_1, \dots, x_n$ we write $[x_1, \dots, x_n]$ to mean that the unique segment from x_1 to x_n crosses the points $x_2, \dots, x_{n-1}$ in the order given.

Lemma 3.2.11 (Circle-Dot Lemma)**.** *Let G act isometrically on an $\mathbb{R}$-tree, T. Suppose we have distinct points, $p, q \in T$ such that $[p, q, pg, qg]$. That is, the segment from p to qg crosses the points q and pg in that order.*

Then g is hyperbolic and both p and q belong to the axis of g. In particular, $\|g\|_T = d(p, pg)$.

Remark. We allow the possibility that $q = pg$ in this Lemma.

Proof. We first show that g is not elliptic. We argue by contradiction; if g is elliptic then the midpoint of $[p, pg]$ is fixed by g by Lemmas 3.2.2(iii) and 3.2.3(i). Call this point w. Then $d_T(w, q) = d_T(w, qg)$. Since $w \in [p, pg]$, this forces $w \in [q, qg]$. But now w is both the midpoint of $[p, pg]$ and $[q, qg]$ which is impossible if $p \neq q$. Hence g is hyperbolic. Therefore, by Lemma 3.2.3(ii) it is enough to show that $p = Y(pg^{-1}, p, pg)$. Indeed, since $[p, q, pg, qg]$ then $[pg^{-1}, qg^{-1}, p, q]$, and hence $[pg^{-1}, qg^{-1}, p, q, pg, qg]$. That is, $p \in [pg^{-1}, pg]$ and therefore $p = Y(pg^{-1}, p, pg)$, as required. ☐

We will also need the following,

Lemma 3.2.12 (Paulin's Lemma [48, Lemma 4.3])**.** *Let G act minimally, non-trivially, isometrically and irreducibly on an $\mathbb{R}$-tree, T. Then for any $a, b \in T$ there exists a hyperbolic element $g \in G$ whose axis contains the segment $[a, b]$.*

This allows us to deduce the R_∞ property when we have an action of the mapping torus on a tree. Thus the following Lemmas will be a key tool for deducing property R_∞. We note that the existence of the action required by this Lemma will be discussed in Corollary 3.5.5.

Lemma 3.2.13. *Let G be a group and let $\varphi \in \mathrm{Aut}(G)$. If M_φ acts on a tree T minimally, isometrically, irreducibly then G also acts minimally and irreducibly on T.*

Proof. We can restrict the action to G and get an isometric action of G on T. We first claim that G admits a hyperbolic element with respect to this action.

Since M_φ acts irreducibly on T we get that (by the remark after Definition 3.2.6) we have two hyperbolic isometries, $m_1, m_2 \in M_\varphi$ whose axes are disjoint. If either of these are in G, then we have a hyperbolic isometry in G. Otherwise, we may write $m_1 = t^a u, m_2 = t^b v$, where $0 \neq a, b \in \mathbb{Z}$ and $u, v \in G$. Now consider, $m_1^b = t^{ab} g, m_2^{-a} = t^{-ab} h$ for some $g, h \in G$. By Lemma 3.2.2(iv) the axes of m_1 and m_1^b are equal, as are the axes of m_2 and m_2^{-a}. Hence the axes of $t^{ab} g$ and $t^{-ab} h$ are disjoint. Therefore, by Lemma 3.2.7, $(t^{ab} g)(t^{-ab} h) \in G$ is hyperbolic. This proves the claim.

Now, by Lemma 3.2.8, G admits a minimal G-invariant subtree. But since G is a normal subgroup of M_φ, this is also a M_φ-invariant subtree and so is the whole of T, as M_φ acts minimally. This proves that G also acts minimally on T.

Finally, to prove that G acts irreducibly on T notice that if there were an invariant line, then T would have to be a line by minimality, contradicting the fact that M_φ acts minimally and irreducibly. If G were to admit a fixed end, then any hyperbolic axes A_g, A_h of hyperbolic elements of G would intersect in this end. However, if $g \in G$ is hyperbolic and $m, \in M_\varphi$, then $g^m \in G$ is also hyperbolic and $A_{g^m} = (A_g)m$, showing that the end would need to be invariant by all elements of M_φ, again contradicting the fact that M_φ acts irreducibly. □

Lemma 3.2.14. *Let G be a group and let $\varphi \in \mathrm{Aut}(G)$. If M_φ acts on a tree T minimally, isometrically, irreducibly, then G has $R_\infty(\varphi)$.*

Proof. Notice that by Lemma 3.2.13, G also acts on T minimally and irreducibly. In particular, Paulin's Lemma 3.2.12 applies to the action of G on T.

We will prove the Lemma by showing that we have infinitely many conjugacy classes of the form tx, where $x \in G$, and deduce the result by Lemma 2.2.2.

Case (i): First suppose that t acts hyperbolically on T. Choose a in the axis of t and let $b = at^2$. Then, by Paulin's Lemma, there exists a hyperbolic $g \in G$ whose axis contains the segment, $[a, b]$.

As the axes of t and g intersect non-trivially, we can assume (by replacing g with g^{-1} if needed) that t and g translate in the same direction along their intersection. In that case, using the Circle-Dot Lemma (Lemma 3.2.11) with $p = a, q = at$, we have that $\|tg^n\| = \|t\| + n\|g\|$ for all positive integers n. Hence these elements are all in distinct conjugacy classes, since translation length is a conjugacy invariant.

Case(ii)(a): Next suppose that t is elliptic and that $|\operatorname{Fix}(t)| > 1$. Here choose $a \neq b \in \operatorname{Fix}(t)$ and let $g \in G$ be a hyperbolic element whose axis contains $[a, b]$. By replacing g with a sufficiently large (but possibly negative) power, we may assume that $b \in [a, ag]$.

It is straightforward to verify using the Circle-Dot Lemma (Lemma 3.2.11) with $p = a, q = b$ that $\|tg^n\| = n\|g\|$ for all positive integers n and hence, again by Lemma 2.2.2, we are done.

Case(ii)(b): Lastly suppose that t is elliptic and fixes a unique point in the tree. Since the action is G-irreducible, G admits two hyperbolic elements whose axes are disjoint. In particular, we may find a hyperbolic $g \in G$ whose axis does not contain the unique fixed point of T.

Let p be the fixed point of t and let q be the point on the axis of g closest to p. The the Circle-Dot Lemma (3.2.11) gives us that $\|tg^n\| = 2d(p,q) + |n|\|g\|$. Hence these elements fall into infinitely many conjugacy classes. □

Remark. Note that when using Lemma 3.2.14 we refer to Lemma 3.2.13 to deduce that the action of G on the same tree is also irreducible and minimal. In practice, however, we will start with a minimal irreducible action of G and extend to one on the mapping torus via Corollary 3.5.5. In that sense, Lemma 3.2.13 is redundant but we record it here for completeness.

3.3 Deformation spaces and reduced trees

In this section we wish to explore the concepts of a 'reduced' simplicial tree, which is different to that of an irreducible tree.

We start with the definition of a deformation space. This discussion is taken from [37], to which we refer the reader for a fuller account.

Definition 3.3.1 ([37, Definition 3.2]). Let T, S be two simplicial G-trees.

- We say that T and S have the same elliptic subgroups if for every subgroup H of G which fixes a point of T, H also fixes a point of S, and vice versa.
- T and S are then said to be in the same deformation space if they have the same elliptic subgroups.
- The deformation space $\mathcal{D}$ containing T is then the set of all G-trees (up to equivariant isometry) which have the same elliptic subgroups as T.

Remark. Note that the notion of equivariant isometry makes sense since we can regard our simplicial trees T as simplicial $\mathbb{R}$-trees by assiging length 1 to each edge.

Definition 3.3.2 ([37, Definition 3.12]). Given a deformation space $\mathcal{D}$ and a collection of subgroups $\mathcal{A}$ of G (closed under conjugacy and passing to subgroups) we define $\mathcal{D}_\mathcal{A} \subseteq \mathcal{D}$ as the set of G-trees $T \in \mathcal{D}$ whose edge stabilizers belong to $\mathcal{A}$. We call $\mathcal{D}_\mathcal{A}$ a restricted deformation space, or simply a deformation space.

Proposition 3.3.3 ([37, Proposition 3.10 (1)]). *Let $\mathcal{D}$ be a deformation space. If one tree $T \in \mathcal{D}$ is irreducible, then all trees in $\mathcal{D}$ are irreducible.*

Definition 3.3.4 ([37, Definition 3.5]). A simplicial G-tree T is called reduced if whenever $e = (u, v)$ is an edge of T with $\mathrm{Stab}_G(e) = \mathrm{Stab}_G(v)$, then u and v are in the same G-orbit.

Observation 3.3.5. Note that any deformation space contains a reduced tree. See [37], Definitions 3.3, 3.4 and 3.5.

Lemma 3.3.6. *Let T be a simplicial reduced G-tree. Then T is also minimal.*

Proof. We prove the contrapositive. So suppose that T is not minimal; we will deduce that it is not reduced.

We consider some proper, G-invariant subtree, S. Let $e = (u, v)$ be an edge of T such that $u \in S$ but $v \notin S$. Then $v \notin S \supseteq u.G$, as S is G-invariant and so u and v are not in the same G orbit. It then suffices to show that $\mathrm{Stab}_G(v) \subseteq \mathrm{Stab}_G(e)$ (as we always have that $\mathrm{Stab}_G(e) \subseteq \mathrm{Stab}_G(v)$).

Consider $g \in \mathrm{Stab}_G(v)$. Then the edge path, $e, \bar{e}g$ starts at u and ends at ug. Since neither e nor eg is an edge of S and $u, ug \in S$ are connected by a path in S, we must have that this path fails to be reduced (and $u = ug$). Thus $e = eg$, as required. ☐

Observation 3.3.7 ([37]). Let G be a finitely generated group acting on a simplicial tree, T.

(i) An edge e of T is called collapsible if collapsing e to a point produces a G-tree in the same deformation space as T, [37, Definition 3.3].

(ii) We can collapse all collapsible edges of T to produced a reduced G-tree, $r(T)$, the reduction of T. Any edge stabiliser of $r(T)$ stabilises an edge of T, [37, Definition 3.5].

(iii) T and $r(T)$ lie in the same deformation space. In particular, the action on T is non-trivial if and only if $r(T)$ is not a point, [37, Theorem 3.8] and Lemma 3.3.6.

(iv) If T is reduced and the edge stabilisers of T are finite, then all reduced trees in the same deformation space as T have the same edge stabilisers as T, [37, Proposition 7.1 (3) and Corollary 7.3]

3.4 Irreducible actions on trees

We write down a fairly simple criterion for an action of a group on a tree to be irreducible. This will be useful in what follows.

First we will need the following Lemma.

Lemma 3.4.1. *Suppose G is a finitely generated group and the derived subgroup $[G, G]$ is finite. Then G is virtually abelian.*

Proof. Any element g of G has finitely many conjugates since $x^{-1}gx = gg^{-1}x^{-1}gx \in g[G, G]$. Therefore, the centraliser of any element has finite index. Hence the intersection of all the centralisers of the generators has finite index, as G is finitely generated. But this intersection is clearly central in G and hence abelian. Therefore G is virtually abelian. □

Proposition 3.4.2. *Let G be a finitely generated group acting minimally and non-trivially on a tree, T. Suppose further that the action is reducible. Then,*

(i) If T is simplicial and edge stabilisers are finite, then G is virtually cyclic and T is a line.

(ii) If T is an $\mathbb{R}$-tree and (pointwise) arc stabilisers are trivial, then G is virtually abelian and T is a line.

Proof. If the action is reducible and non-trivial then we can, up to taking an index 2 subgroup of G, suppose that there is an invariant end. (See [15, Corollary 2.3 and Theorem 2.5 (ii)]; for a dihedral action take the index two subgroup corresponding to the orientation preserving isometries of $\mathbb{R}$.)

(i) Consider first the case where T is simplicial and edge stabilisers are finite.

Let R be a ray representing this invariant end. Invariance of the end means that $R \cap Rg$ is an infinite set (in fact, a subray) for every $g \in G$.

Next observe that for any vertex $p \in R$ there exist only finitely many pairs of vertices $x, y \in R$ such that p is the midpoint of $[x, y]$. This is because both x and y must be equidistant from p and be distinct. But now $d(x, p) = d(y, p) \leqslant d(\omega, p)$ where ω is the intial point of the ray, R. There are clearly only finitely many pairs satisfying this on the ray.

Armed with this observation, we proceed as follows. Note that for any $g \in G$, $R \cap Rg^{-1}$ is infinite. If we take any point $x \in R \cap Rg^{-1}$ then $x, xg \in R$. But the midpoint of $[x, xg]$ lies in $R \cap A_g$ by Lemma 3.2.3(i). Since we have infinitely many possible values for x this implies that $R \cap A_g$ is infinite for any $g \in G$.

Therefore if g is elliptic it fixes some subray of R (but potentially a different subray for each g) and if g is hyperbolic then the axis of g intersects R in a subray.

In any case it is then easy to see that the commutator of any two elements is elliptic with fixed point set containing an infinite subray of R. But now any finite set of commutators must fix some common infinite subray of R (one can take the intersection of the fixed subrays for each commutator). Hence any finite set of commutators are contained in some edge stabiliser; for instance the first edge of the common fixed subray. In particular, any finite set of commutators are contained in some edge stabiliser and so generate a finite group as edge stabilisers are finite.

While it is possible that each finite set of commutators is contained in a different edge stabiliser we will argue that the derived subgroup is finite (and contained in some edge stabiliser) as follows. The action of G on T is co-compact, since G is finitely generated. Hence there exists a constant, M, such that any edge stabiliser has order at most M. Therefore any finitely generated subgroup of $[G, G]$ has order bounded by M which implies that $[G, G]$ has order bounded by M. (As $[G, G]$ is countable and generated by commutators, we can realise it as a countable union of a chain of finitely generated subgroups. Since the order of these subgroups is bounded, the chain of subgroups stabilises and hence $[G, G]$ is finite.)

Therefore $[G, G]$ is finite and hence G is virtually abelian by Lemma 3.4.1. This implies that any two hyperbolic elements of G have positive powers which commute, since sufficiently large positive powers will lie in an abelian subgroup. Therefore, by Lemma 3.2.2 (v) and Observation 3.2.4, all hyperbolic elements have the same axis. This axis is now a G-invariant subtree of T by Propostion 3.2.8 and so is equal to T. Therefore T is a line.

To finish we note that some finite index subgroup of G is torsion-free abelian and acts on T with finite (and hence trivial) edge stabilisers. Thus the torsion-free subgroup of G is both free and free abelian (and non-trivial since if G were finite then the action on T would be trivial). Therefore G is virtually infinite cyclic.

(ii) The case where T is an $\mathbb{R}$-tree with trivial arc stabilisers is similar but easier. In this case, once descending to an index two subgroup of G admitting an invariant end, we deduce that any two hyperbolic axes intersect in an infinite ray. Since arc stabilisers are trivial, this means that any two hyperbolic elements commute which implies that their axes are equal. Hence T is isometric to the real line and G is virtually abelian.

□

3.5 Actions and automorphisms

Definition 3.5.1. Let G be a group, T a G-tree and $\varphi \in \mathrm{Aut}(G)$. Then φT is the same underlying tree but with the G-action twisted by φ. Specifically,

$$x \cdot_{(\varphi T)} g := x \cdot_T (g\varphi).$$

Equivalently, if the action on T is given by a homomorphism, $G \xrightarrow{\pi} Isom(T)$, then the action on φT is given by the pre-composition by φ; $G \xrightarrow{\varphi} G \xrightarrow{\pi} Isom(T)$.

Remark. Since we are writing our action of $\mathrm{Aut}(G)$ on G as a right action, this action of $\mathrm{Aut}(G)$ on the space of G-trees is a left action. Were we to write automorphisms of G on the left, this would be a right action.

Definition 3.5.2. Let G be a group and $\varphi \in \mathrm{Aut}(G)$. A topological representative of φ is an equivariant map $f : T \to \varphi T$, from a G-tree, T, which sends vertices to vertices and edges to edge paths.

Observation 3.5.3. Note that $f : T \to \varphi T$ being G-equivariant means that $f(xg) = f(x)(g\varphi)$ for all $x \in T, g \in G$.

However, note that if f is a topological representative for φ, then f_w is a topological representative for $\varphi Ad(w)$, where $f_w(x) = f(x)w$, and $g(\varphi Ad(w)) = w^{-1}(g\varphi)w$. Thus having a topological representaive is really a property of outer automorphisms.

Theorem 3.5.4 (Culler–Morgan [15, Theorem 3.7 (p.586)]). *If T_1 and T_2 are two minimal, irreducible G-trees with $\|g\|_{T_1} = \|g\|_{T_2}$ for all $g \in G$, then there is a unique equivariant isometry, $f : T_1 \to T_2$.*

Remark. Irreducible is a stronger hypothesis than needed but suffices for our purposes and makes the statement a little cleaner.

Corollary 3.5.5. *Let T be a minimal irreducible G-tree and $\Phi \in \mathrm{Out}(G)$. Then M_Φ has an isometric action on T which restricts to the G-action on T if and only if $\|g\Phi\|_T = \|g\|_T$ for all $g \in G$.*

Remark. Note that the isomorphism class of M_Φ and the value of $\|g\Phi\|_T$ only depend on the outer automorphism class, so this makes sense. One can think of $g\Phi$ as a conjugacy class rather than a single element.

Proof. Suppose first that M_Φ acts isometrically on T in a way which extends the G-action. Then, since translation length is a conjugacy invariant,

$$\|g\|_T = \|g^t\|_T = \|g\Phi\|_T \text{ for all } g \in G.$$

Conversely, suppose that $\|g\Phi\|_T = \|g\|_T$ for all $g \in G$. Then, by definition, $\|g\|_{\Phi T} = \|g\Phi\|_T$, which equals $\|g\|_T$ by hypothesis.

Choose some $\varphi \in \Phi$. Then, by Theorem 3.5.4, there is an isometry $f : T \to T$ which is equivariant when viewed as an isometry $f : T \to \varphi T$. But now we can extend the G action to an $M_\Phi = M_\varphi$ action by simply having t act as f. The only thing to check is that $t^{-1}gt$ and $g\varphi$ act in the same way on T. But this is readily checked to be equivalent to saying that $f : T \to \varphi T$ is equivariant. □

We can now state the main tools we will use to prove R_∞.

The first we have already stated in Lemma 3.2.14.

Proposition 3.5.6 (Levitt–Lustig [42, Proposition 3.1])**.** *Let G be a finitely generated group, fix $\Phi \in \mathrm{Out}(G)$. Let l be the length function of an irreducible action of G on an $\mathbb{R}$-tree. If $l \circ \Phi = l$, then G has $R_\infty(\Phi)$.*

Proposition 3.5.7 (Levitt–Lustig [42, Proposition 3.2])**.** *Let G be a finitely generated group, fix $\Phi \in \mathrm{Out}(G)$. Let l be the length function of an irreducible action of G on an $\mathbb{R}$-tree T and suppose $l \circ \Phi = \lambda l$ where $\lambda > 1$. If arc stabilisers of T are finite and the number of* $\mathrm{Stab}(x)$*-orbits of directions at branch points x are uniformly bounded, then G has $R_\infty(\Phi)$.*

4 Train Track Maps and free products

In this section we show that free products of groups have the R_∞ property using the machinery of (Relative) Train Track maps. These were first used in [6] in the context of free group automorphisms and then in [4] to define and analyse the stable (or forward) limit tree. The methods in the former paper were extended to free products in [13]. However, we are using the treatment in [24], [25] and [26] because of the more detailed relation to the stable limit tree. We also use some results from [41] to deal with the technical conditions mentioned in the introduction; specifically, this allows us to verify the condition that stabilisers of branch point act on the set of directions with finitely many orbits.

As always, the goal is to achieve condition R_∞ by looking at the action of the mapping torus on some tree and applying the argument of [42, Propositions 3.1 and 3.2].

4.1 Free factor systems

Definition 4.1.1 (Free Factor System)**.** Let G be a group which splits as a free product. A free factor system of G is a pair $\mathcal{F} = (\{G_1, \ldots, G_k\}, r)$ such that $G = G_1 * \cdots * G_k * F_r$ where each G_i is non-trivial and F_r is the free group of rank r.

Following [37, Definitions 3.2 and 3.12] we make the following definition.

Definition 4.1.2. Let G be a group which splits as a free product and $\mathcal{F} = (\{G_1, \ldots, G_k\}, r)$ a free factor system for G. We denote by $D(\mathcal{F})$ the set of all simplicial G-trees (up to equivariant isometry) whose elliptic subgroups are subgroups of conjugates of the G_i. That is $T \in D(\mathcal{F})$ if and only if for every $H \leqslant G$, H fixes a point in T exactly when H is a subgroup of some conjugate of some G_i.

Further, we let $D_1(\mathcal{F})$ denote the subset of $D(\mathcal{F})$ consisting of those trees whose edge stabilisers are all trivial.

Definition 4.1.3. Let $\mathcal{F} = (\{G_1, \ldots, G_k\}, r)$ be a free factor system of some group G, and let $\varphi \in \mathrm{Aut}(G)$. We say that $\mathcal{F}$ is:

- a Grushko decomposition of G if each G_i is freely indecomposable and not infinite cyclic.
- proper if $k + r \geqslant 2$.
- maximal if for any proper free factor system $\mathcal{F}' = (\{G'_1, \ldots, G'_l\}, s)$ of G, there exists some i so that G_i is not contained in any conjugate of any G'_j.
- φ-invariant if for each i there is some j and some $g_j \in G$ so that $\varphi(G_i) = G_j^{g_j}$.

We say that φ is irreducible with respect to $\mathcal{F}$ if $\mathcal{F}$ is a maximal, proper, φ-invariant free factor system of G.

Remark. Observe that a Grushko decomposition of a group G is φ-invariant for any $\varphi \in \mathrm{Aut}(G)$. See [45, III, Proposition 3.7] for a statement of Grushko's Theorem.

The goal is to produce a very well behaved topological representative (see Definition 3.5.2) for an automorphism of a free product.

Definition 4.1.4. Let T be a simplicial G-tree with finitely many (oriented) edge orbits, rerpresented by edges $e_1, \ldots, e_n$. Suppose that $f : T \to \varphi T$ is a topological representative. Then the transition matrix of f is the $n \times n$ matrix, M_f, whose (i, j)-entry is the number of times the edge path $f(e_i)$ crosses the orbit of e_j (in either direction).

Definition 4.1.5. Let T be a simplicial G-tree with finitely many orbits of edges and $f : T \to \varphi T$ a topological representative for some $\varphi \in \mathrm{Aut}(G)$. Then f is called a train track map if for every $k \geqslant 1$, f^k is locally injective on the interior of edges.

It is called a metric train track map if, in addition, T admits a G-equivariant path metric such that the length of $f(e)$ is λ times the length of e, where e is any edge of T for some fixed $\lambda \in \mathbb{R}_{\geqslant 0}$.

We refer the reader to [6] for the background on how irreducible matrices and their Perron–Frobenius eigenvalues are used in the theory of train tracks.

From a train track we can construct the (forward) stable limit tree as in [4], although we will base the following more closely on [28]. Namely, given a metric train track map, $f : T \to \varphi T$ with associated constant $\lambda > 1$, set $d_\infty(x, y) = \lim_{n\to\infty} \frac{d_T(f^n(x), f^n(y))}{\lambda^n}$, for $x, y \in T$. This converges due to monotonicity and is a pseudo-metric on T. Therefore on taking a suitable quotient, one obtains a metric space T_∞ with metric (which we will still name), d_∞.

It is straightforward to see that the quotient map, $T \to T_\infty$ is G-equivariant and 1-Lipschitz and that T_∞ is a 0-hyperbolic metric space on which G acts isometrically. In other words, T_∞ is an $\mathbb{R}$-tree on which G acts. Moreover, the train track property immediately implies that this quotient map is an isometry restricted to the edges of T.

It also follows quickly that for any $\gamma \in G$, $\|\gamma\|_{T_\infty} = \lim_{n\to\infty} \frac{\|\gamma\varphi^n\|_T}{\lambda^n}$. It is well known that given a (metric) train track there is a hyperbolic $g \in G$ such that f^k is injective on A_g for all $0 < k \in \mathbb{Z}$, [24, Corollary 8.12] (this is also found in [6]). Such a g is called a 'legal' group element. It now follows that if g is legal, $0 < \|g\|_{T_\infty} = \|g\|_T$.

One can now proceed as in [28, Lemma II.7], to see that T_∞ is a minimal and non-trivial G-tree. In fact we record more.

Theorem 4.1.6. *Let G be a non-trivial free product which is not the infinite dihedral group. Let $\Phi \in \mathrm{Out}(G)$, and let $\varphi \in \Phi$.*

Then there exists a maximal, proper, φ-invariant free factor system $\mathcal{F}$ and a simplicial G-tree $T \in D_1(\mathcal{F})$ admitting a train track representative f for φ, with associated constant $\lambda \geqslant 1$. Furthermore:

- *(i) λ is the Perron–Frobenius eigenvalue of the transition matrix for f.*
- *(ii) $\lambda = 1$ if and only if f is an isometry. In this case the mapping torus M_φ acts isometrically on T.*
- *(iii) If $\lambda > 1$ then $\lim_{n\to\infty} \frac{\varphi^n T}{\lambda^n}$ is a G-tree, T_∞, called the stable limit tree for φ.*
- *(iv) For all $g \in G$, $\|g\varphi\|_{T_\infty} = \lambda\|g\|_{T_\infty}$. In particular, there exists a G-equivariant homothety, $H : T_\infty \to \varphi T_\infty$ of stretching factor λ, by Theorem 3.5.4.*
- *(v) The stable limit tree is a non-trivial, irreducible $\mathbb{R}$-tree.*
- *(vi) If $1 \neq g \in G$ acts elliptically on T, then g fixes a unique point of T_∞. In particular, non-trivial elements of arc stabilisers of T_∞ act hyperbolically on T.*
- *(vii) There exist finitely many segments, $f_1, \ldots, f_r$ in T_∞ whose union of G-orbits covers the whole of T_∞.*

Proof. As in [24, Corollary 8.25], there exists a maximal φ-invariant free factor system, $\mathcal{F}$ so that φ is $\mathcal{F}$-irreducible. Again using [24, Theorem 8.24], φ will admit a train track representative. Just as in [6], the transition matrix of this map has a Perron–Frobenius eigenvalue, $\lambda \geqslant 1$, which is equal to the Lipschitz constant of the train track map (in [6] one must choose lengths for the edges, whereas in [24] this is implicit in the definition. This is the distinction between topological and metric train track maps).

Moreover, the theory of irreducible matrices implies that $\lambda = 1$ if and only if the transition matrix is a permutation matrix, implying that f is an isometry.

This proves (i) and (ii). For (iii) and (iv) we refer the reader to [26, Lemma 2.14.1]. Point (v) is well known to experts but we can also deduce it from Lemma 4.1.8 and Theorem 3.4.2, since the only non-trivial virtually abelian free product is the infinite dihedral group. Point (vi) is [26, Lemma 2.13.6].

Finally, for part (vii), we argue as above (following [28], but see also [26, Section 2.13]). That is, since the quotient map $T \to T_\infty$ is an isometry restricted to edges of T we may take the f_i to be the isometric images of the edges of T.

□

The following are also well-known to experts and the proofs are the same as in [28], adapted to the free product case.

Lemma 4.1.7. *For any $p \in \mathbb{N}$ there exists a constant L_p such that any arc σ in T_∞ whose length is greater than L_p, will contain at least p non-trivial, disjoint subsegments which are all in the same G-orbit.*

Proof. By Theorem 4.1.6 (vii), there are finitely many arcs, $f_1, \dots, f_r$ whose G-orbits cover T_∞. Moreover, any arc segment σ can be subdivided into subsegments, $\sigma = z_1 \cup \dots \cup z_n$ such that for each i there is a group element g_i such that $z_i g_i$ is a subsegment of one of $f_1, \dots, f_r$. As this is a subdivision, the length of σ is equal to the sum of the lengths of the z_i and two distinct z_i can intersect in at most a single point. (If numbered sequentially, z_i intersects precisely z_{i-1} and z_{i+1} in a single point and is disjoint from the rest).

Now, after possibly further subdividing, we may assume that for $i \neq j$ either $z_i g_i = z_j g_j$ or their intersection is at most a single point.

Next let us suppose that there is an integer p such that,

$$\max_i |\{j \ : z_i g_i = z_j g_j \}| \leqslant p.$$

It is then clear that the sum of the lengths of the $z_i g_i$ is at most $p(l_1 + \dots + l_r)$, where l_j is the length of f_j. But since each g_i is an isometry, this means that the length of σ is also at most $p(l_1 + \dots + l_r)$.

Therefore, if σ is any segment of length greater than $3p(l_1 + \dots + l_r)$ then we can ensure that σ contains at least $3p$ subsegments, $c_1, \dots, c_{3p}$ which are all in the same G orbit. Since each z_i meets at most z_{i-1} and z_{i+1}, we deduce that at least p of these must be disjoint. □

Lemma 4.1.8. *If G is finitely generated, then arc stabilisers in T_∞ are trivial.*

Remark. The hypothesis that G be finitely generated is not needed here. It is used to deduce that the limit of small actions is small. However, this will always be true of a limit of edge-free actions and we include the hypothesis for convenience of referencing.

Remark. Note that we are implicitly assuming that the λ from Theorem 4.1.6 is greater than 1.

Proof. Consider some non-trivial arc c of T_∞ and let S be the pointwise stabiliser of c. By Theorem 4.1.6(vi), every non-trivial element of S acts on T as a hyperbolic isometry. In particular, S acts freely on the simplicial tree, T, and so is a free group. Thus, in order to show that S is trivial it is enough to show that S is finite.

We therefore argue by contradiction and suppose that S is infinite.

Since $T \in \mathcal{D}_1(\mathcal{F})$ then edge stabilisers in T are trivial, hence the action of G on T (and also on each $\varphi^n T$) is small. Since a limit of small actions is itself small, by Culler–Morgan [15, Theorem 5.3], S cannot contain a free subgroup of rank 2.

However, we have already noted that S is a free group and so for S to be infinite it would have to be an infinite cyclic group. Therefore the minimal S-invariant subtree of T is a line L on which S acts by translation. (Note that all the non-trivial elements of S have axis equal to L, by Lemma 3.2.2 (v)).

Let $\bar{S}$ be the (setwise) stabiliser of L. Then $\bar{S}$ is virtually cyclic as its minimal invariant tree is L, by Lemma 3.2.2 (iv). Moreover, it is also a maximal virtually cyclic group as any virtually cyclic group containing S must preserve L, since any hyperbolic element in such a subgroup has a power which lies in S and hence has axis equal to L. (Recall that an invariant subtree for a subgroup is the union of all the hyperbolic axes, by Proposition 3.2.8.)

Let $H : T_\infty \to \varphi T_\infty$ be the homothety described in Theorem 4.1.6(iv) and consider $H^k(c)$ for some k. Let p be the index of S in $\bar{S}$.

By Lemma 4.1.7, we may choose k sufficiently large so that $H^k(c)$ contains $p+1$ disjoint non-trivial arcs which are in the same G-orbit. Since H is equivariant, c will also contain $p+1$ disjoint arcs, $c_0, \ldots, c_p$ in the same G orbit. Hence there exist group elements, $g_1, \ldots, g_p$ such that $c_0 g_i = c_i$. (Set $g_0 = 1$.)

Let C_i denote the pointwise stabiliser of c_i. As argued above, each C_i is cyclic (since it is small and acts freely on T). Note that S is a subgroup of C_i and hence C_i is a subgroup of $\bar{S}$, since the latter is a maximal virtually cyclic subgroup of G.

Moreover, since $c_0 g_i = c_i$ we also have that $C_i = C_0^{g_i}$. But now $S \leqslant C_0^{g_i} \leqslant \bar{S}^{g_i}$. By maximality, $\bar{S} = \bar{S}^{g_i}$ for all i. However, any hyperbolic element of $\bar{S}$ has axis equal to L, whereas any hyperbolic element of $\bar{S}^{g_i}$ has axis equal to $L g_i$. Hence $L = L g_i$ and $g_i \in \bar{S}$ for all i. But since the c_i are disjoint, $g_0, \ldots, g_p$ are in distinct cosets of C_0 in $\bar{S}$, contradicting the fact that $S \leqslant C_0$ has index p in $\bar{S}$. This contradiction shows that S must in fact be trivial.

□

Using this, we can bound the number of orbits of directions using the following result.

Proposition 4.1.9 (Horbez [41, Proposition 4.4])**.** *Let G be a countable group, $\mathcal{F}$ a free factor system for G, and T an $\mathbb{R}$-tree equipped with an isometric G action such that any subgroup in $\mathcal{F}$ fixes a point in T. (The subgroups in $\mathcal{F}$ are elliptic in T).*

Then if pointwise arc stabilisers in T are trivial, there is a uniform bound on the number of $\mathrm{Stab}(x)$-orbits of directions at branch points x in T.

Remark. Horbez [41, Proposition 4.4] gives a bound on the 'index' of a tree T, a component of which is the sum over G-orbits of points x in T of the number of $\mathrm{Stab}(x)$-orbits of directions from x in T with trivial stabiliser. The proposition then follows immediately.

Combining this with Proposition 3.5.7 we get,

Theorem 4.1.10. *Let G be a finitely generated group which splits as a non-trivial free product of finitely many indecomposable groups. Then G has property R_∞.*

Remark. The only times we need the hypothesis of being finitely generated are when we invoke the [15, Theorem 5.3] in Lemma 4.1.8 and [41, Proposition 4.4] in Proposition 4.1.9. However, both results are true in general without this hypothesis.

Proof. The case where $G = D_\infty$ is known and can be dealt with separately (see Theorem 7.1.2).

In all other cases, every tree in $D_1(\mathcal{F})$, for any proper free factor system $\mathcal{F}$ is irreducible.

Consider some $\varphi \in \mathrm{Aut}(G)$. We now use Theorem 4.1.6. If $\lambda = 1$, we get an isometric action of M_φ on some $T \in D_1(\mathcal{F})$ and hence we are done by Proposition 3.5.6 (or Lemma 3.2.14). If $\lambda > 1$ we form the stable limit tree, T_∞.

All that is left is to verify the hypotheses of Proposition 3.5.7 in the case where $\lambda > 1$. This is done in Theorem 4.1.6, Lemma 4.1.8 and Proposition 4.1.9. ☐

Corollary 4.1.11. *The property of being R_∞ is undecidable amongst finitely presented groups.*

Proof. It is well known [45, Chapter IV, Theorem 4.1] that the property of being trivial is undecidable amongst finitely presented groups. (It is clear that triviality is a Markov property).

Suppose we are given a finite presentation of a group, $G = \langle X : R \rangle$.

Then we can form the finitely presented group, $\Gamma = \mathbb{Z} * G = \langle X, t : R \rangle$, where t is a letter which does not appear in X.

The group Γ is isomorphic to $\mathbb{Z}$ precisely when G is trivial. In which case Γ would not have R_∞. ($\mathbb{Z}$ has exactly two twisted conjugacy classes with respect to the automorphism sending $n \mapsto -n$).

On the other hand, if G were non-trivial, then Γ would be a finitely generated group which splits as a non-trivial free product of finitely many indecomposable groups and so has R_∞ by Theorem 4.1.10.

Therefore, if we could decide R_∞ for Γ we could also decide triviality for G. $\square$

5 R_∞ for Groups with Infinitely Many Ends

5.1 Accessible groups

In this section we extend the result to groups with infinitely many ends. We recall the definition of an end. (See also [17, IV, Definition 6.4], for an alternate but equivalent formulation.)

Definition 5.1.1. Let G be a finitely generated group with finite generating set S. Let Γ be the Cayley graph of G with respect to S. Then the number of ends of G is the supremum of the number of infinite components of the graphs, $\Gamma \backslash F$, where F ranges over all finite subgraphs of Γ.

It turns out that the number of ends does not depend on S and is an invariant of the group. In fact, more can be said via the classical Stallings Theorem on ends, which can be found in [17, IV, Theorem 6.10 and Theorem 6.12]

Theorem 5.1.2. *Let G be a finitely generated group.*

- *The number of ends of G is $0, 1, 2$ or ∞.*
- *If G has more than one end, then G acts non-trivially on a simplicial tree with finite edge stabilisers.*
- *The number of ends of G is 2 if and only if G has a infinite cyclic subgroup of finite index.*

Thus a group with infinitely many ends can be 'split' along finite subgroups. One could then ask whether the vertex stabilisers of such an action split further and whether this process terminates. The groups for which it does terminate are called *accessible*.

Definition 5.1.3 (See [52, p.189] and [17, IV, Definition 7.1])**.** Given a group G a simplicial G-tree T is called terminal if the vertex stabilisers of T have at most one end and all edge stabilisers are finite.

A finitely generated group G is called accessible if it admits a terminal G-tree.

Observation 5.1.4. We insist that an accessible group is finitely generated as in [52] but contrary to [17].

Note that the vertex groups of an accessible group G acting minimally on a terminal G-tree T are finitely generated, by [17, III, Lemma 8.1].

Definition 5.1.5 ([17, VI, Definition 4.1]). A group G is called almost finitely presented if it is of type FP_2 over $\mathbb{Z}_2$. In particular, every finitely presented group is almost finitely presented.

Remark. Recall that a reduced G-tree (Definition 3.3.4) is also minimal by Lemma 3.3.6.

Theorem 5.1.6 ([17], VI, Theorem 6.3). *Let G be an almost finitely presented group. Then G has a reduced terminal G-tree T. Moreover, the action on T is co-compact.*

Proof. The existence of a terminal G-tree follows from [17, VI, Theorem 6.3]. The fact that we can take the tree to be reduced follows from [37]. The action can then be taken to be minimal by Lemma 3.2.9 and Proposition 3.2.8 (as G is finitely generated). Finally, the action is co-compact, as is any action of a finitely generated group acting minimally on a simplicial tree. For instance, see [17, I, Proposition 4.13]. □

We will also need the following, which is just a restatement of the 'blowing up' construction given in [17, IV, Section 7].

Proposition 5.1.7. *Let H be an accessible group acting on a simplicial tree S with finite edge stabilisers. Then there exists a terminal H-tree, X so that every edge stabiliser of S is equal to some edge stabiliser of X.*

Moreover, if the action on S is non-trivial then so too is the action on X.

Proof. The idea here is to 'blow up' S to produce X. Just as in Observation 5.1.4, all the vertex stabilisers in S are finitely generated and hence accessible, since H is. If we write H as a graph of groups using S, we can then replace each vertex group, H_v, with a graph of groups whose edge stabilisers are finite and whose vertex stabilisers have at most one end. In particular, since all edge stabilisers in S are finite, each edge group must fix a vertex in any incident H_v terminal tree and so we get a well defined graph of groups giving us the result. □

Proposition 5.1.8. *Suppose that a finitely generated group H acts minimally on a simplicial tree T with finite edge stabilisers and consider the reduction, $r(T)$ obtained from T by collapsing all collapsible edges. Then,*

(i) If T is not a point, then $r(T)$ is also not a point.

(ii) For any edge e of $r(T)$ there exists an edge e' of T such that $|\operatorname{Stab}_H(e')| = |\operatorname{Stab}_H(e)|$.

Proof. Since T and $r(T)$ are in the same deformation space by Observation 3.3.7, if $r(T)$ is a point then the whole of H is elliptic in $r(T)$ and hence also in T, proving (i).

For (ii), we can simply regard the edge set of $r(T)$ as a subset of the edge set for T, from which the result immediately follows. □

Definition 5.1.9. A subgroup H of a group G is called characteristic if for every $\varphi \in \mathrm{Aut}(G)$ we have that $(H)\varphi \leqslant H$.

Observation 5.1.10. It is an easy exercise to see that if H is a characteristic subgroup of a group G, then the following all hold:

1. For any $\varphi \in \mathrm{Aut}(G)$ we have $(H)\varphi = H$.
2. H is a normal subgroup of G.
3. Every automorphism of G induces an automorphism of G/H.
4. If G has a unique subgroup K of a given order, then K is characteristic.

Lemma 5.1.11. *Suppose a group G has a characteristic subgroup N. If G/N has R_∞, then so too does G.*

Proof. Let $\varphi \in \mathrm{Aut}(G)$. Since N is characteristic, φ induces an automorphism φ_* on G/N given by $(g \cdot N)\varphi_* = (g)\varphi \cdot N$. Given $xN, yN \in G/N$, we have that

$$\begin{aligned}
& xN \sim_{\varphi_*} yN \\
\iff & \exists wN \in G/N \text{ such that } (wN)\varphi_* xN(wN)^{-1} = yN \\
\iff & \exists w \in G \text{ such that } (w\varphi)xw^{-1}N = yN \\
\iff & \exists w \in G \text{ and } \exists n \in N \text{ such that } (w\varphi)xw^{-1} = ny \\
\iff & \exists n \in N \text{ such that } x \sim_\varphi ny.
\end{aligned}$$

In particular, if xN and yN belong to different $\sim_{\varphi_*}$ classes in G/N, then x and y belong to different $\sim_\varphi$ classes in G (since $1 \in N$). So if G/N has $R_\infty(\varphi_*)$, then we must have that G has $R_\infty(\varphi)$. This holds for all $\varphi \in \mathrm{Aut}(G)$, hence if G/N has R_∞ then so too must G. □

Lemma 5.1.12. *Let G be a group acting on a terminal minimal tree T, and let $N = \{g \in G | xg = x \ \forall x \in T\}$ be the kernel of this action. Then N is the maximal (normal and finite) subgroup of G. In particular, N is a finite characteristic subgroup of G.*

Proof. First, observe that N must be a subgroup of every edge stabiliser of T. Since T is terminal, it has finite edge stabilisers, hence N must be a finite group. Since N is a kernel, then it must be normal.

We will now show that N is the maximal normal finite subgroup of G. Indeed, let K be another finite normal subgroup in G. Since K is finite, then it must fix some point p in the tree T. Let $g \in G$ and $k \in K$. Since K is normal then there exists $k' \in K$ with $g^{-1}k'g = k$. Then $(p \cdot g) \cdot k = p \cdot gk = p \cdot k'g = p \cdot G$. Thus K pointwise fixes $p \cdot G$, and so must also fix the convex hull of the points $p \cdot G$. Since T is assumed to be minimal, we must have that this convex hull is equal to T, and hence $K \leqslant N$. Hence N is the maximal normal finite subgroup of G.

Now by Observation 5.1.10 (4), we must have that N is characteristic in G. □

Corollary 5.1.13. *Let G be a group acting on a terminal minimal G-tree T, and let $N = \{g \in G | xg = x \ \forall x \in T\}$ be the kernel of this action. If G/N has R_∞, then so too does G.*

Proof. This follows immediately from Lemmas 5.1.11 and 5.1.12. □

Remark. Since N is finite and G is assumed to be finitely generated, then G is quasi-isometric to G/N. Thus G has infinitely many ends if and only if G/N has infinitely many ends.

Definition 5.1.14. Let G be an accessible group and T a reduced terminal G-tree.

(i) Let $q := q(G)$ be the order of the smallest edge stabiliser of T. By [37, Corollary 7.3] this does not depend on T.

(ii) Let $T(q)$ be the tree obtained from T by collapsing every edge whose stabiliser has order greater than q.

(iii) Let V_0 be the set of vertices $x \in T(q)$ for which there exist incident edges e_1 and e_2 with $\mathrm{Stab}(e_1) \neq \mathrm{Stab}(e_2)$ as subgroups of G. Let V_1 be the set of subgroups Y of G which occur as an edge stabiliser in $T(q)$. We construct a bipartite tree $T(q)_c$ whose vertex set is $V(T(q)_c) = V_0 \sqcup V_1$, and where there is an edge (x, Y) between some $x \in V_0$ and some $Y \in V_1$ if and only if there is some edge $e \in T(q)$ with endpoint x and stabiliser Y.

Remark. Note that $T(q)_c$ is the tree of cylinders constructed by Guirardel and Levitt [38, Section 4.1], where the admissible relation is equality of edge stabilisers (as in [38, Example 3.6]).

We refer the reader to [38] for the construction of the tree of cylinders. However we note the following straightforward exercise.

Proposition 5.1.15. *Let G be a group acting on a simplicial tree T, subject to some admissable relation $\sim$ with corresponding tree of cylinders, T_c.*

(i) If the action on T is minimal, then so is the action on T_c

(ii) *If the action on T is minimal and non-trivial, then the action on T_c is trivial exactly when all edges belong to a single cylinder. Equivalently, this is exactly when all edge groups are equivalent under $\sim$.*

(iii) *If the action on T_c is non-trivial and reducible, then so is the action on T.*

Proof. (i) Each vertex of T_c is either a vertex of T belonging to more than one cylinder (the V_0 vertices) or a cylinder in T (the V_1 vertices). It is straightforward to show that the action of G on T then induces an action of G on T_c. For minimality see [38, Section 4.1, p.13] or [36, Lemma 4.9].

(ii) If the action of G on T_c is minimal, then the action on T_c is trivial if and only if T_c is a single point, i.e. the vertex set of T_c consists of a single cylinder $Y \in V_1$.

(iii) We briefly argue that if T_c admits a fixed end or an invariant line, then so too must T. Suppose that T_c admits an invariant line L. Since T_c is bi-partite, we can look at the vertices of T_c which are vertices of T belonging to more than one cylinder. (That is, the vertices in V_0). Thus we can think of $L \cap V_0$ as a set of vertices of T. It is clear that the convex hull of these is a line, $\tilde{L}$ in T. The invariance of L implies the invariance of $\tilde{L}$.

The case of an end is similar.

□

Proposition 5.1.16. *Let G be an accessible group, T a reduced terminal G-tree and $T(q)$ the G-tree obtained from T as in Definition 5.1.14. Then,*

(i) *Every edge stabiliser of $T(q)$ is finite of order q.*

(ii) *$T(q)$ is a reduced G-tree.*

(iii) *Every vertex stabiliser of $T(q)$ is accessible.*

(iv) *Suppose that H is a vertex stabiliser of $T(q)$ and $\varphi \in \mathrm{Aut}(G)$. Then $H\varphi$ also fixes a vertex of $T(q)$*

Proof. Let F denote the subforest of T consisting of all edges whose full stabiliser has order greater than q and their incident vertices. We can think of the edges of $T(q)$ as simply being the edges of T which are not in F, proving (i).

Further, note that F is G-invariant. Hence if C is a component of F and $g \in G$, then $C \cap Cg \neq \varnothing$ implies that $C = Cg$, since Cg must be another component of F. The vertex stabilisers of $T(q)$ are then either vertex stabilisers of T or stabilisers of components of F. It is then easy to see that $T(q)$ is reduced and so (ii) holds.

Next we want to argue that vertex stabilisers of $T(q)$ are finitely generated. This is clear if the vertex of $T(q)$ is equal to a vertex in T.

Otherwise, consider a component C of F and let H be its setwise stabiliser - this is a vertex stabiliser in $T(q)$. As above, $C \cap Cg \neq \emptyset$ implies that $C = Cg$. In particular, G-orbits on C must equal H-orbits and G-stabilisers in C must equal H stabilisers. Therefore the action of H on C is reduced and hence minimal, so C is the minimal H-invariant subtree T_H. Moreover, since G acts with finitely many orbits, so does H on T_H, and since G-stabilisers are finitely generated, so are H stabilisers in C. Hence H is finitely generated by the fundamental theorem of Bass–Serre Theory (see e.g. [17, I, Theorem 4.1]).

We also argue that each vertex stabiliser of $T(q)$ is accessible. If the vertex stabiliser of $T(q)$ equals the vertex stabiliser of some vertex in T, then it has at most one end and the result is clear. Otherwise, a vertex stabiliser H will be the setwise stabiliser of some component C of F. But the action of H on C is co-compact, has finite edge stabilisers and the vertex stabilisers have at most one end. This proves (iii).

Finally we prove (iv). First consider the action of H on T. If H is a vertex stabiliser of T, then it has at most one end and is finitely generated. Hence $H\varphi$ is also finitely generated and has at most one end. Therefore, since $T(q)$ has finite edge stabilisers, $H\varphi$ must fix a vertex of $T(q)$.

If H does not fix a vertex of T, then it is the setwise stabiliser of some component, C of F and, as argued above, is finitely generated and $C = T_H$, the minimal invariant subtree of H which is reduced and whose edge stabilisers have order greater than q.

Now, consider the action of $H\varphi$ on $T(q)$. Suppose that $H\varphi$ does not fix a vertex of $T(q)$, then as $H\varphi$ is finitely generated it admits an invariant subtree, S. By Proposition 5.1.7, $H\varphi$ admits a non-trivial terminal tree X whose edge stabilisers have order at most q (since G-edge stabilisers in $T(q)$ have order q).

Next we reduce X to obtain $r(X)$. By Proposition 5.1.8, the action of $H\varphi$ on $r(X)$ is reduced and non-trivial and edge stabilisers have order at most q. But since H is isomorphic to $H\varphi$, this gives us two reduced terminal H-trees with different edge stabilisers, contradicting Observation 3.3.7. This proves (iv). □

Theorem 5.1.17. *Let G be an accessible group with infinitely many ends and let N be the maximal normal finite subgroup of G. Then:*

- *(i) For any reduced terminal G-tree T the deformation space of $T(q)$ is $\mathrm{Aut}(G)$ invariant.*
- *(ii) If G/N does not split as a non-trivial free product then the tree of cylinders $T(q)_c$ is a non-trivial, irreducible, minimal G-tree which is invariant under the action of $\mathrm{Aut}(G)$. (Here the admissable relation is equality.)*

Proof. Let $\mathcal{D}$ be a deformation space of G-trees and $\varphi \in \mathrm{Aut}(G)$. Then $\mathcal{D}\varphi$ consists of all simiplicial G-trees for which $H\varphi$ is elliptic for any $\mathcal{D}$-elliptic subgroup H.

Therefore, to prove (i), it is enough to show that for vertex stabiliser H of T_q and any automorphism φ of G that $H\varphi$ also stabilises a vertex of T_q. This is Proposition 5.1.16(iv). This proves (i).

To prove (ii), we first consider the general situation.

Note that by [38], equality is an admissable relation since all edge stabilisers in $T(q)$ have the same order. By Proposition 5.1.16, we know that $T(q)$ is reduced and hence minimal by Lemma 3.3.6. Therefore, $T(q)_c$ is minimal. If $T(q)_c$ were trivial, this would imply that all edge stabilisers are equivalent, which in our context means equal (by Proposition 5.1.15). In particular, any edge stabiliser would be normal and so G/N would act on the tree T_q with trivial edge stabilisers.

Hence, if G/N does not split as a free product, then the action of G on $T(q)_c$ is non-trivial. Furthermore, we know that the action of G on $T(q)$ is irreducible by Proposition 3.4.2 and hence the action on $T(q)_c$ is also irreducible by Proposition 5.1.15. The fact that $T(q)_c$ is invariant under the action of $\mathrm{Aut}(G)$ follows from the fact that the deformation space of $T(q)$ is $\mathrm{Aut}(G)$ invariant and [38, Proposition 4.11] . □

Theorem 5.1.18. *Any accessible group G with infinitely many ends has property R_∞. In particular, any finitely presented group with infinitely many ends has property R_∞.*

Proof. Let N be the maximal normal finite subgroup of G. If G/N is a non-trivial free product then G has R_∞ by Corollary 5.1.13 and Theorem 4.1.10.

Otherwise, $T(q)_c$ is a non-trivial, irreducible, minimal G-tree which is invariant under the action of $\mathrm{Aut}(G)$ by Theorem 5.1.17 and hence G has R_∞ by Proposition 3.5.6. □

6 Relatively hyperbolic groups

6.1 JSJ decompositions and invariant trees

Theorem 6.1.1. *Let G be a non-elementary finitely presented hyperbolic group relative to a family $\mathcal{P} = \{P_1, \ldots, P_n\}$ of finitely generated groups. Let $\mathrm{Out}(G;\mathcal{P})$ denote the subgroup of $\mathrm{Out}(G)$ preserving the conjugacy classes of the P_i, but allowing permutation of the P_i.*

Then G has infinitely many twisted conjugacy classes with respect to any $\Phi \in \mathrm{Out}(G;\mathcal{P})$.

Remark. Note that our definition of $\mathrm{Out}(G;\mathcal{P})$ differs from that of [39] in that we are allowing permutations. However, this does not effect the invariance of the JSJ tree and so this is a benign change to make. We also note that the hypothesis that the P_i are finitely generated arises since we use the JSJ decomposition of [40], where that is required.

Proof. In our context G being non-elementary means that it is not virtually cyclic and it is not equal to a P_i.

If G has infinitely many ends, we can invoke Theorem 5.1.18 to deduce that G has R_∞. If Φ has finite order, we can use the argument of Delzant as in [42, Proposition 3.3 and Lemma 3.4]. We note that the argument there is for hyperbolic, rather than relatively hyperbolic groups, but the same arguments work since relative hyperbolicity is a commensurability (in fact a quasi-isometry) invariant by [18, Theorem 1.2], and one can replace the arguments about infinite order elements with ones about loxodromic elements in the relative hyperbolic setting.

Thus the remaining case is where $\mathrm{Out}(G;\mathcal{P})$ has an element of infinite order and G is one-ended. By [40, Corollary 9.20], there is a canonical relative JSJ tree, T. The fact that it is relatively canonical means that it is $\mathrm{Out}(G;\mathcal{P})$ invariant. In particular, we are done by Lemma 3.2.14 and Corollary 3.5.5 as long as T is G-irreducible.

However, if T were reducible then either G fixes a point of T or G admits an invariant line or end. If G were to fix a point of T, then G would equal a vertex stabiliser. Since G is not elementary, the vertex stabiliser cannot be parabolic or virtually cyclic. And since $\mathrm{Out}(G;\mathcal{P})$ is infinite, it cannot be rigid. Hence it must be a flexible quadratically hanging group with finite fibre. Therefore G would be hyperbolic and we would be done by [42, Theorem 3.5]. (This case could also be dealt with more concretely.)

So we can assume that G does not fix a point of T and in particular T is not a point and has an edge. We proceed to argue that T does not admit an invariant end or line. Consider N the kernel of the action. If the action were reducible we would get that G/N is either infinite cyclic or infinite dihedral by [15, Corollary 2.3 and Theorem 2.5]. (We note that in [15], those results construct homomorphisms to $\mathbb{R}$ and $Isom(\mathbb{R})$, but in the context of a simplicial tree, these land in $\mathbb{Z}$ and $Isom(\mathbb{Z})$ instead.).

On the other hand, as in [40, Corollary 9.20], edge stabilisers of T are elementary meaning that they are either virtually cyclic or contained in (a conjugate of) a parabolic subgroup P_i. However, elementary subgroups are almost malnormal (for instance, see [39, Corollary 3.2]), which implies that N is finite and if T were reducible, then G would have 0 or 2 ends and hence be elementary. Therefore T is G-irreducible and G has infinitely many twisted conjugacy classes with respect to any $\Phi \in \mathrm{Out}(G;\mathcal{P})$. □

Theorem 6.1.2. *Let G be a non-elementary finitely presented relatively hyperbolic group G whose peripheral subgroups are finitely generated but not relatively hyperbolic. Then G has property R_∞.*

Proof. Since the peripheral subgroups are not relatively hyperbolic, they are $\mathrm{Aut}(G)$ invariant by [47, Lemma 3.2] and so $\mathrm{Out}(G;\mathcal{P}) = \mathrm{Out}(G)$. Therefore G has property R_∞ by Theorem 6.1.1. □

7 Appendix A: R_∞ for the infinite Dihedral Group D_∞

The following result is well-known but we include it for completeness. See [35, Proposition 2.3].

Note that the infinite Dihedral group is the only group which can be written as a non-trivial free product but having finitely many ends (it has two ends).

7.1 The infinite dihedral group

The group D_∞ is the free product of two cyclic groups of order 2 and hence has presentation,

$$\langle x, y \mid x^2 = y^2 = 1 \rangle.$$

However, we will use the alternate generating set $\{x, t = xy\}$, which gives the corresponding presentation,

$$\langle x, t \mid x^2 = 1, t^x = t^{-1} \rangle.$$

This second presentation expresses the fact that D_∞ is a semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}_2$, where the monodromy is given by the (unique) non-identity automorphism of $\mathbb{Z}$.

With this second description is is straightforward to see that every element of D_∞ can be written as either t^n or $t^n x = xt^{-n}$ for some $n \in \mathbb{Z}$, and that this representation is unique.

Lemma 7.1.1. *The outer automorphism group of the infinite dihedral group, $D_\infty = \langle x, y \mid x^2 = y^2 = 1 \rangle$ has order 2. The non-identity outer automorphism has a representative given by the map:*

$$x \mapsto y, y \mapsto x.$$

Remark. It is easy to see from the universal property of free products that the map above defines an endomorphism of D_∞ whose square is the identity. Hence it defines an automorphism.

Proof. It is convenient to work with the alternate generating set, $x, t = xy$. First note that every nontrivial element of $\langle t \rangle$ has infinite order whereas every element of the coset $x\langle t \rangle$ has order 2. Therefore any automorphism of D_∞ must restrict to an automorphism of $\langle t \rangle$ and hence send t to $t^{\pm 1}$.

Notice that since $t^{-1} = t^x$ we may assume, up to inner automorphisms, that any automorphism fixes t. Also, since it has order 2, the image of x must therefore be xt^n for some $n \in \mathbb{Z}$. However, $(xt^n)^{t^k} = xt^{n+2k}$ and since $t^{t^k} = t$ we deduce that, up to inner automorphisms, any automorphism is represented by the maps $t \mapsto t, x \mapsto x$ and $t \mapsto t, x \mapsto xt$. It is a simple calculation to see that these maps are in the same outer automorphism classes as the identity map and the map given in the statement of this Lemma, respectively. ☐

Theorem 7.1.2. *The group D_∞ has the R_∞ property.*

Proof. The outer automorphism group of D_∞ has order 2 by Lemma 7.1.1, so by Lemma 2.1.4, it is sufficient to check that two automorphisms which represent these classes have infinitely many twisted conjugacy classes.

Case 1: φ is the identity

When φ is the identity, twisted conjugacy is simply standard conjugacy. We then note that,

$$(t^n)^t = t^n \quad \text{and} \quad (t^n)^x = t^{-n}.$$

It follows that t^n and t^m are conjugate if and only if $|m| = |n|$. Hence, in this case, the infinitely many elements t^n, where n is a positive integer, are all in different (twisted) conjugacy classes.

Case 2: $t\varphi = t^{-1}, x\varphi = xt.$

This automorphism can be seen from the other presentation as the one which interchanges x and y. In particular it is not inner and so represents the other outer automorphism class. Here we note that

$$(t\varphi)(t^n x)t^{-1} = t^{-1}t^n x t^{-1} = t^n x \quad \text{and} \quad (x\varphi)(t^n x)x = xt^{n+1} = t^{-n-1}x.$$

It follows that $t^n x$ is twisted φ-conjugate to $t^m x$ if and only if $m = n$ or $m = -n - 1$. Hence in this case, the infinitely many elements $t^n x$ where n is a positive integer are all in different twisted conjugacy classes. ☐

8 Appendix B: Property R_∞ via quasimorphisms (by Francesco Fournier-Facio)

In this appendix [1] we present a more indirect approach to prove property R_∞, which applies to the groups considered in this paper, and more.

8.1 Quasimorphisms

Definition 8.1.1. Let G be a group, and $f\colon G \to \mathbb{R}$ a function. The *defect* of f is

$$D(f) := \sup_{x,y \in G} |f(xy) - f(x) - f(y)|.$$

If $D(f) < \infty$, we say that f is a *quasimorphism*. If moreover $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$, we say that f is *homogeneous*.

Quasimorphisms are useful tools for the interactions of group theory with various subjects, such as bounded cohomology [27], stable commutator length [11], knot theory [46], symplectic geometry [51] and dynamics [33]. Quasimorphisms are abundant among groups with hyperbolic features: this originates from the Brooks construction for free groups [9], and culminated in the Bestvina–Fujiwara construction for acylindrically hyperbolic groups [5]. Here we are only concerned with quasimorphisms with an additional special property.

Definition 8.1.2. Let $f\colon G \to \mathbb{R}$ be a quasimorphism and $\varphi \in \mathrm{Aut}(G)$. We say that f is *φ-invariant* if $f(x\varphi) = f(x)$ for all $x \in G$. We say that f is Aut-*invariant* if it is φ-invariant for every $\varphi \in \mathrm{Aut}(G)$.

Every quasimorphism is at a bounded distance from a homogeneous one [11, Lemma 2.21], so it is common to restrict to those. Homogeneous quasimorphisms are easily seen to be conjugacy invariant [11, Section 2.2.3], therefore φ-invariance of a homogeneous quasimorphism is really a property of the corresponding outer automorphism.

Theorem 8.1.3. *Let G be a group, let $\varphi \in \mathrm{Aut}(G)$, and suppose that there exists a non-zero φ-invariant homogeneous quasimorphism on G. Then G has property $R_\infty(\varphi)$.*

Proof. Let $f\colon G \to \mathbb{R}$ be as in the statement. Let $\approx_D$ denote an equality up to $D(f)$, which is finite by assumption. Then we estimate:

$$f(wx(w\varphi)^{-1}) \approx_D f(wx) + f((w\varphi)^{-1}) \approx_D f(w) + f(x) + f((w\varphi)^{-1}) = f(x),$$

---

[1] FFF is supported by the Herchel Smith Postdoctoral Fellowship Fund. He thanks Yuri Santos Rego for useful comments. Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK. `ff373@cam.ac.uk`

where we used homogeneity and φ-invariance in the last equality. In other words:

$$x \sim_\varphi y \qquad \Longrightarrow \qquad |f(x) - f(y)| \leqslant 2D(f).$$

Now let $x \in G$ be such that $f(x) \neq 0$: this exists by assumption. Up to replacing x by a suitable power, using homogeneity, we may assume that $|f(x)| > 2D(f) \geqslant 0$. Then for all $i \neq j \in \mathbb{Z}$:

$$|f(x^i) - f(x^j)| = |i - j||f(x)| \geqslant |f(x)| > 2D(f),$$

and so x^i and x^j cannot be in the same $\sim_\varphi$-class. □

We immediately obtain:

Corollary 8.1.4. *If there exists a non-zero* Aut*-invariant homogeneous quasimorphism on G, then G has property R_∞.*

Let us note that the proof is showing more: if there exists a non-zero φ-invariant homogeneous quasimorphism, then there exists $x \in G$ whose powers are in pairwise distinct $\sim_\varphi$ classes. Similarly, if there exists a non-zero Aut-invariant homogeneous quasimorphism, then there exists $x \in G$ whose powers are in pairwise distinct $\sim_\varphi$ classes, for all φ simultaneously.

Remark. Of course homomorphisms are homogeneous quasimorphisms, therefore our result recovers [43, Theorem 5.4], which states that groups with non-zero Aut-invariant homomorphisms have property R_∞. There, it was applied to prove property R_∞ for a large family of Thompson-like groups, including the piecewise linear F-like groups of Bieri–Strebel [7], the piecewise projective groups of Lodha–Moore [44], and the braided Thompson group F_{br} [8]. In the first two cases, our more general criterion does not help: if G is a piecewise linear or piecewise projective group of the line, then every homogeneous quasimorphism is a homomorphism [**?**, 10]. In the third case, there exist many homogeneous quasimorphisms that are not homomorphisms [22], but the construction does not give information about Aut-invariance. The interest of our more general criterion is for applications to groups with hyperbolic features: already in free groups, it is easy to see that there exist no Aut-invariant homomorphisms, but Aut-invariant homogeneous quasimorphisms do exist [23].

Such quasimorphisms have been constructed in several cases. The most widely applicable criterion is the following:

Theorem 8.1.5 ([23, Theorem E])**.** *Let G be a group such that* $\mathrm{Inn}(G)$ *is infinite and* $\mathrm{Aut}(G)$ *is acylindrically hyperbolic. Then there exists an infinite-dimensional space of* Aut*-invariant homogeneous quasimorphisms.*

Acylindrical hyperbolicity of automorphism groups has been proved in several cases (in fact, it is still an open question whether it holds for *all* finitely generated acylindrically hyperbolic groups [29, Question 1.1]), and so we obtain the following list of examples of groups with property R_∞:

Corollary 8.1.6. *Let G be a finitely generated group satisfying one of the following properties.*

1. *G is a non-elementary hyperbolic group.*
2. *G is non-elementary hyperbolic relative to a collection of finitely generated subgroups, none of which is relatively hyperbolic.*
3. *G has infinitely many ends.*
4. *G is a graph product of groups over a finite graph that does not decompose non-trivially as a join and is not reduced to a single vertex.*
5. *G is a graph product of abelian groups and it is not virtually abelian.*

Then G has property R_∞.

Proof. Groups in the second item have acylindrically hyperbolic automorphism group [31, Theorem 1.3]. Therefore, they admit non-zero Aut-invariant quasimorphisms by Theorem 8.1.5, and hence have property R_∞ by Corollary 8.1.4. The groups in the first and third item are special cases (see also [29] and [31, Theorem 1.1]). Similarly, the groups in the fourth item have acylindrically hyperbolic automorphism group [19, Theorem A.27] (see also [32] and [30]), with the exception of the infinite dihedral group, which is treated separatedly in Theorem 7.1.2. The groups in the fifth item need not have acylindrically hyperbolic automorphism group (because we are allowing joins), but they still admit Aut-invariant homogeneous quasimorphisms [23, Theorem B(4)], so they have property R_∞ by Corollary 8.1.4. □

The first item recovers [42]. The second item recovers Theorem 6.1.2 (and removes the hypothesis of finite presentability). The third item recovers Theorem 5.1.18 (and removes the hypothesis of accessibility). The last item recovers the R_∞ property for non-abelian right angled Artin groups [55].

Remark. The same argument as Theorem 8.1.5 (see [23, Corollary 4.4] and the proof of [23, Theorem 5.1]) shows that, if $\mathrm{Inn}(G)$ is infinite, and $\langle \mathrm{Inn}(G), \varphi \rangle < \mathrm{Aut}(G)$ is acylindrically hyperbolic, then G admits an infinite-dimensional space of φ-invariant homogeneous quasimorphism. By Theorem 8.1.3, this statement is enough to deduce property $R_\infty(\varphi)$. It is possible that, for some groups, proving that $\langle \mathrm{Inn}(G), \varphi \rangle$ is acylindrically hyperbolic for all φ is easier than proving that $\mathrm{Aut}(G)$ itself is acylindrically hyperbolic.

References

[1] I. Belegradek and A. Szczepański. Endomorphisms of relatively hyperbolic groups. With appendix by Oleg V. Belegradek. *International Journal of Algebra and Computation*, 18(1):97–110, 2008.

[2] M. Bestvina and M. Feighn. Bounding the complexity of simplicial group actions on trees. *Inventiones Mathematicae*, 103(3):449–469, 1991.

[3] M. Bestvina and M. Feighn. A combination theorem for negatively curved groups. *Journal of Differential Geometry*, 35(1):85–102, 1992.

[4] M. Bestvina, M. Feighn, and M. Handel. The Tits alternative for $\mathrm{Out}(F_n)$. I: Dynamics of exponentially-growing automorphisms. *Annals of Mathematics. Second Series*, 151(2):517–623, 2000.

[5] M. Bestvina and K. Fujiwara. Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.*, 6:69–89, 2002.

[6] M. Bestvina and M. Handel. Train tracks and automorphisms of free groups. *Annals of Mathematics. Second Series*, 135(1):1–51, 1992.

[7] R. Bieri and R. Strebel. *On groups of PL-homeomorphisms of the real line*, volume 215 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2016.

[8] T. Brady, J. Burillo, S. Cleary, and M. Stein. Pure braid subgroups of braided Thompson's groups. *Publ. Mat.*, 52(1):57–89, 2008.

[9] R. Brooks. Some remarks on bounded cohomology. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, Ann. of Math. Stud., No. 97, pages 53–63. Princeton Univ. Press, Princeton, NJ, 1981.

[10] D. Calegari. Stable commutator length in subgroups of $\mathrm{PL}^+(I)$. *Pacific J. Math.*, 232(2):257–262, 2007.

[11] D. Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.

[12] I. Chiswell. *Introduction to* Λ*-trees*. Singapore: World Scientific, 2001.

[13] D. J. Collins and E. C. Turner. Efficient representatives for automorphisms of free products. *Michigan Mathematical Journal*, 41(3):443–464, 1994.

[14] C. G. Cox. A note on the R_∞ property for groups $\mathrm{FAlt}(X) \leqslant G \leqslant \mathrm{Sym}(X)$. *Communications in Algebra*, 47(3):978–989, 2019.

[15] M. Culler and J. W. Morgan. Group actions on $\mathbb{R}$-trees. *Proc. London Math. Soc. (3)*, 55(3):571?604, 1987.

[16] K. Dekimpe and D. Gonçalves. The R_∞ property for free groups, free nilpotent groups and free solvable groups. *Bulletin of the London Mathematical Society*, 46(4):737–746, 2014.

[17] W. Dicks and M. J. Dunwoody. *Groups acting on graphs.*, volume 17 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, reprint of the 1989 hardback ed. edition, 2010.

[18] C. Druţu. Relatively hyperbolic groups: geometry and quasi-isometric invariance. *Commentarii Mathematici Helvetici*, 84(3):503–546, 2009.

[19] A. Escalier and C. Horbez. Graph products and measure equivalence: classification, rigidity, and quantitative aspects. *arXiv preprint arXiv:2401.04635*, 2024.

[20] A. Fel'shtyn. New directions in Nielsen-Reidemeister theory. *Topology and its Applications*, 157(10-11):1724–1735, 2010.

[21] F. Fournier-Facio and Y. Lodha. Second bounded cohomology of groups acting on 1-manifolds and applications to spectrum problems. *Adv. Math.*, 428:Paper No. 109162, 42, 2023.

[22] F. Fournier-Facio, Y. Lodha, and M. C. B. Zaremsky. Braided Thompson groups with and without quasimorphisms. *Algebr. Geom. Topol.*, 24(3):1601–1622, 2024.

[23] F. Fournier-Facio and R. D. Wade. Aut-invariant quasimorphisms on groups. *Trans. Amer. Math. Soc.*, 376(10):7307–7327, 2023.

[24] S. Francaviglia and A. Martino. Metric properties of Outer Space. *Publicacions Matemàtiques*, 55(2):433 – 473, 2011.

[25] S. Francaviglia and A. Martino. Stretching factors, metrics and train tracks for free products. *Illinois Journal of Mathematics*, 59(4):859–899, February 2017.

[26] S. Francaviglia, A. Martino, and D. Syrigos. On the action of relatively irreducible automorphisms on their train tracks, 2024.

[27] R. Frigerio. *Bounded cohomology of discrete groups*, volume 227 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017.

[28] D. Gaboriau, A. Jaeger, G. Levitt, and M. Lustig. An index for counting fixed points of automorphisms of free groups. *Duke Mathematical Journal*, 93(3):425–452, 1998.

[29] A. Genevois. Negative curvature in automorphism groups of one-ended hyperbolic groups. *J. Comb. Algebra*, 3(3):305–329, 2019.

[30] A. Genevois. Automorphisms of graph products of groups and acylindrical hyperbolicity. *Mem. Amer. Math. Soc.*, 301(1509):vi+127, 2024.

[31] A. Genevois and C. Horbez. Acylindrical hyperbolicity of automorphism groups of infinitely ended groups. *J. Topol.*, 14(3):963–991, 2021.

[32] A. Genevois and A. Martin. Automorphisms of graph products of groups from a geometric perspective. *Proc. Lond. Math. Soc. (3)*, 119(6):1745–1779, 2019.

[33] E. Ghys. Groupes d'homéomorphismes du cercle et cohomologie bornée. In *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 81–106. Amer. Math. Soc., Providence, RI, 1987.

[34] D. Gonçalves, P. Sankaran, and P. Wong. Twisted conjugacy in fundamental groups of geometric 3-manifolds. *Topology and its Applications*, 293:11, 2021. Id/No 107568.

[35] D. Gonçalves and P. Wong. Twisted conjugacy classes in nilpotent groups. *Journal für die Reine und Angewandte Mathematik*, 633:11–27, 2009.

[36] V. Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. *Geometry & Topology*, 8:1427–1470, 2004.

[37] V. Guirardel and G. Levitt. Deformation spaces of trees. *Groups Geom. Dyn.*, 1(2):135?181, 2007.

[38] V. Guirardel and G. Levitt. Trees of cylinders and canonical splittings. *Geometry & Topology*, 15(2):977 – 1012, 2011.

[39] V. Guirardel and G. Levitt. Splittings and automorphisms of relatively hyperbolic groups. *Groups, Geometry, and Dynamics*, 9(2):599–663, 2015.

[40] V. Guirardel and G. Levitt. Jsj decompositions of groups. *Astérisque No. 395 (2017), vii+165 pp*, Feb. 2016.

[41] C. Horbez. The boundary of the outer space of a free product. *Israel Journal of Mathematics*, 221(1):179–234, 2017. 43 pages, 9 figures.

[42] G. Levitt and M. Lustig. Most automorphisms of a hyperbolic group have very simple dynamics. *Annales scientifiques de l'École Normale Supérieure*, 33(4):507–517, 2000.

[43] P. M. Lins de Araujo, A. S. de Oliveira-Tosti, and Y. Santos Rego. Thompson-like groups, Reidemeister numbers, and fixed points. *Geom. Dedicata*, 217(3):Paper No. 54, 22, 2023.

[44] Y. Lodha and J. T. Moore. A nonamenable finitely presented group of piecewise projective homeomorphisms. *Groups Geom. Dyn.*, 10(1):177–200, 2016.

[45] R. C. Lyndon and P. E. Schupp. *Combinatorial group theory*, volume 89 of *Ergeb. Math. Grenzgeb.* Springer-Verlag, Berlin, 1977.

[46] A. V. Malyutin. Writhe of (closed) braids. *Algebra i Analiz*, 16(5):59–91, 2004.

[47] A. Minasyan and D. Osin. Fixed subgroups of automorphisms of relatively hyperbolic groups. *The Quarterly Journal of Mathematics*, 63(3):695–712, 2012.

[48] F. Paulin. The gromov topology on $\mathbb{R}$-trees. *Topology Appl.*, 32(3):197?221, 1989.

[49] F. Paulin. Outer automorphisms of hyperbolic groups and small actions on $\mathbb{R}$- trees. In *Arboreal group theory. Proceedings of a workshop, held September 13-16, 1988, in Berkeley, CA (USA)*, pages 331–343. New York etc.: Springer-Verlag, 1991.

[50] F. Paulin. Sur les automorphismes extérieurs des groupes hyperboliques. *Annales Scientifiques de l'École Normale Supérieure*, 30(2):147–167, 1997.

[51] L. Polterovich and D. Rosen. *Function theory on symplectic manifolds*, volume 34 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2014.

[52] P. Scott and T. Wall. Topological methods in group theory. Homological group theory, Proc. Symp., Durham 1977, Lond. Math. Soc. Lect. Note Ser. 36, 137-203 (1979)., 1979.

[53] J. P. Serre. *Trees*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1980.

[54] I. Soroko and N. Vaskou. Property r_∞ for new classes of artin groups. *arXiv preprint arXiv:2409.18123*, 2024.

[55] T. Witdouck. The R_∞-property for right-angled Artin groups and their nilpotent quotients. Preprint, arXiv:2304.01077 [math.GR] (2023), 2023.