Article

A Gravity Tensor and Gauge Equations for Newtonian Dynamics

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Abstract

It is revealed that the material derivative of a variable in gravity field is its directional derivative, from which and energy/complementary-energy conservations with exterior derivatives, two sets of gauge equations of Newton's dynamic gravity field are derived, which has same mathematical structure with the gauge ones for the Maxwell equations in electromagnetic fields, revealing that gravity force and curl momentum in Newton's gravity field, respectively, play the roles like the electric E and the magnetic B of the Maxwell equations in the electromagnetic field. The gravity tensor of Newton's gravitational field is constructed, and an example is given to validate it. This finding allows Newton's gravity to be governed by a gauge theory, addressing the historic issue that "Newton's gravitation is an exception to the Yang–Mills gauge theory".

Keywords: gravity tensor; gauge equations of the gravity field; Newton's gravity field; exterior derivatives; energy conservation; energy-flow equation

MSC: 83-10

1. Introduction

Yang–Mills theory [1,2] has been successful in understanding *electromagnetism, weak, and strong forces*. Unfortunately, "Newton's gravitation (NG) revealing planet motions [3] is an *exception*". Einstein created an idea considering gravitation caused by geometrical deformation of the field [4], following which there are publications for its gauge field equations (GFE) [2–7]. Book [7] presented commentaries on the articles reported over about 50 years until 2013. It is considered that NG can be recast in a form of Newton–Cartan geometry theory, and General Relativity can be reformulated in a manner resembling a gravity theory, as *teleparallel gravity*. Concerning this, Knox [6], based on examinations, concluded that "the spacetime geometry involved in a gravitational theory is not a straightforward consequence of anything internal to that theory as a theory of gravity". It has been found that there are many publications to tackle/extent the above gravitational theories. Duel et al. [8] focus on reformulating Newtonian gravity as a fully covariant geometric theory, of which the key extensions involve incorporating torsion, lifting the theory to higher-dimensional Bargmann structures, and linking it to relativistic theories, condensed matter systems, and Hořava–Lifshitz gravity. A called Gravitomagnetism/Gravitoelectromagnetism (GEM) theory often revolves around foundational General Relativity texts [9] and specific applications to plasma cosmology or anomaly analysis [10]. Rabounski [11] developed a theory of vortical gravitational fields referring to the rotational component of gravitational-inertial fields described as a "vortical gravitational field". This concept extends General Relativity to include Maxwell-like equations for gravity, where the rotor of the gravitational-inertial force plays a key role. The above-mentioned key references with their cited publications

may provide very useful concepts and theoretical foundations for readers to compare them with the theory in this paper to explore the physical and geometrical meanings of the related gravitational variables defined in the paper.

Based on the energy flow theory developed by refs. [12,13], the author has investigated the functionals for NG theory and the two geometric ones compared in Appendix A and has considered that spacetime geometry gravity is not exactly equal to NG. The key reason is that the geometric theory restricts the force perpendicular to particle velocity, so that it does not work to change the kinetic energy of the mass moving in the gravity field, which implies the mass moves along a geometric line, i.e., a zero-energy flow line. To explain this, we consider the functional with the constraint for the Newton–Cartan theory given in Appendix A, which is as follows:

$$H[\boldsymbol{r}] = \int_{t_1}^{t_2} T(\dot{\boldsymbol{r}})\,dt, \quad T = \frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}, \quad \delta\boldsymbol{r}(t_1) = 0 = \delta\boldsymbol{r}(t_2), \quad \delta\boldsymbol{r} = \dot{\boldsymbol{r}}\delta t, \tag{1}$$

from which, when taking a variation, it follows

$$\begin{aligned}
\delta H[\boldsymbol{r}] &= \int_{t_1}^{t_2} \dot{\boldsymbol{r}}\cdot\delta\dot{\boldsymbol{r}}\,dt = \int_{t_1}^{t_2}\left[\frac{d}{dt}(\dot{\boldsymbol{r}}\cdot\delta\boldsymbol{r}) - \ddot{\boldsymbol{r}}\cdot\delta\boldsymbol{r}\right]dt \\
&= (\dot{\boldsymbol{r}}\cdot\delta\boldsymbol{r})\big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left[\ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}\delta t\right]dt = -\int_{t_1}^{t_2}\left[\frac{dT}{dt}\delta t\right]dt = 0,
\end{aligned} \tag{2}$$

implying $dT/dt = 0$, so that the gravity force, along the acceleration line $\ddot{\boldsymbol{r}}$ perpendicular to the velocity $\dot{\boldsymbol{r}}$, does not do work to change the kinetic energy of the system. The $dT/dt = 0$ is also shown for Einstein's geometric gravity by a simple example in Appendix A.

Therefore, publications on GFE only concern the geometry theory, and the one for the NG field is still unknown. In the abstract presented in ICTAM2024 [14], the author developed a gravity tensor and gauge equations for Newton's nonlinear dynamic gravity field, which is further minor-modified and described in more detail in this paper.

2. Dynamic Interactions of Newton's Gravity Fields of Two Masses

In a Cartesian system $O - x^1x^2x^3$ of base-vectors $\boldsymbol{g}_\alpha$, $(\alpha = 1, 2, 3)$ in space with time t, $x^0 = ct$, c light speed, using summation convention [15], we have a position $\boldsymbol{x} = x^\alpha\boldsymbol{g}_\alpha$. For the Cartesian system, the base vector is constant; it is not needed to distinguish between upper and lower indices, such as $x^\alpha = x_\alpha$, so that for our convenience in formulations, the upper or lower index of the same variable is freely chosen.

In this system, two masses m^I and m^J respectively located at the point $\boldsymbol{x}^I$ and $\boldsymbol{x}^J$ at time t. Following NG theory, there exists a gravitational force

$$\boldsymbol{f}^{IJ}(\boldsymbol{r}^{IJ}) = -\tilde{\mathrm{G}}\frac{\boldsymbol{r}^{IJ}}{r_{IJ}^3} = \tilde{\mathrm{G}}\frac{\boldsymbol{r}^{JI}}{r_{IJ}^3} = -\boldsymbol{f}^{JI}(\boldsymbol{r}^{JI}), \quad \boldsymbol{r}^{IJ} = \boldsymbol{x}^I - \boldsymbol{x}^J, \quad r_{IJ} = (\boldsymbol{r}^{IJ}\cdot\boldsymbol{r}^{IJ})^{1/2}, \mathrm{G}m^Im^J = \tilde{\mathrm{G}}, \tag{3}$$

between them. Here, G is a gravitational constant, and $\boldsymbol{f}^{IJ}$ denotes the force acting on mass m^I from mass m^J. From Newton's second law, the time change rate of momentum $\boldsymbol{p}^I(\boldsymbol{x}^I)$ of mass m^I is governed by the following:

$$\boldsymbol{f}^{IJ} = \frac{d\boldsymbol{p}^I(\boldsymbol{x}^I)}{dt} = \frac{\partial\boldsymbol{p}^I}{\partial t}\bigg|_X, \quad \boldsymbol{x}^I = \boldsymbol{x}^I(\boldsymbol{X}^I, t), \quad \boldsymbol{p}^I(\boldsymbol{x}^I) = \boldsymbol{p}^I(\boldsymbol{X}^I, t) = m^I\boldsymbol{v}^I(\boldsymbol{x}^I), \quad \boldsymbol{v}^I = \frac{d\boldsymbol{x}^I}{dt} = \frac{\partial\boldsymbol{x}^I}{\partial t}\bigg|_X, \tag{4}$$

where $\boldsymbol{x}^I$, $\boldsymbol{v}^I$ and $\boldsymbol{p}^I$, respectively, denote the position, velocity, and momentum of mass m^I at time t, $\boldsymbol{X}^I$ is a material coordinate to identify the mass m^I, such as its initial position $\boldsymbol{x}_0^I$ for the *total material coordinate*, and $\boldsymbol{X}^I = \boldsymbol{x}^I$ at instant time for the *updated Lagrange coordinate*, the time derivative d/dt represents the material derivative taken for the mass m^I, i.e., fixed $\boldsymbol{X}^I$.

Newton's law was originally developed concerning the two masses, so that the related variables are considered as material variables. However, it can also be understood as a field equation. Appendix C gives the physical and geometrical meanings of the variables in the two descriptions, which will be helpful to read this paper.

In the following discussion of this paper, we follow the original material description of Newton's gravity and adopt the updated Lagrange coordinate $X^I = x^I$ to identify the momentum vector of mass m^I and the gravitational force between the two masses. However, for mathematical convenience, the field function $p^I(x^I, t)$ may also be used to derive its material derivative. This can be understood as follows. The gravity intensity at the point x^I in the gravity field of mass m^J is the gravity force acting on an *imaginary unit mass* at x^I, which is a defined physical field quantity without caring if there is or is not a unit mass at that point. Therefore, from Newton's gravity, the image unit mass at x^I has its local momentum $\hat{p}^I(x^I, t)$ changing with time, a defined local momentum. If the mass m^I is physically located at x^I, its local momentum will be the product of the defined local momentum multiplied by the mass, i.e., $p^I(x^I, t) = m^I \hat{p}^I(x^I, t)$. Since different positions of field points have different gravity intensities, the momentum $p^I(x^I, t)$ is a field function.

2.1. Directional/Material Derivatives of Variables in NG Field

If both masses move in space, each mass has its dynamic gravity field and momentum, and therefore it must be dynamic interactions. As shown in Equations (3) and (4) and in Appendix C, the gravitational force and the time change rates of momenta of two masses are fully determined by the positions of the two masses in space. Generally, the gravitational force is expressed as a two-point vector $f^{IJ}(x^I, x^J)$, and the momentum as a one-mass point vector $p^I(x^I) = p^I(X^I = x^I, t) = p^I(x^I, t)$, where x^I is not only a field point but also the updated material coordinate. The directional/material derivatives of these field variables are calculated in the following forms:

$$\frac{df^{IJ}}{dt} = \frac{\partial f^{IJ}}{\partial x_I^\alpha}\dot{x}_I^\alpha + \frac{\partial f^{IJ}}{\partial x_J^\alpha}\dot{x}_J^\alpha = J^I - J^J = J^{IJ}, \tag{5}$$

$$df^{IJ} = \frac{\partial f^{IJ}}{\partial x_I^\alpha}dx_I^\alpha + \frac{\partial f^{IJ}}{\partial x_J^\alpha}dx_J^\alpha = \left(\frac{\partial f^{IJ}}{\partial x_I^\alpha}v_I^\alpha + \frac{\partial f^{IJ}}{\partial x_J^\alpha}v_J^\alpha\right)dt, \tag{6}$$

$$J^I = \frac{\partial f^{IJ}}{\partial x_I^\alpha}\dot{x}_I^\alpha, \qquad J^J = \frac{\partial f^{JI}}{\partial x_J^\alpha}\dot{x}_J^\alpha, \tag{7}$$

$$\frac{dp^I(x^I)}{dt} = \frac{\partial p^I}{\partial x_I^\alpha}\dot{x}_I^\alpha = \frac{\partial p^I}{\partial t}\bigg|_X = \frac{\partial p^I}{\partial t}, \quad \frac{dp^I(x^I, t)}{dt} = \frac{\partial p^I}{\partial t} + \frac{\partial p^I}{\partial x_I^\alpha}\dot{x}_I^\alpha = \frac{\partial p^I}{\partial t}\bigg|_X. \tag{8}$$

$$dp^I = \frac{\partial p^I}{\partial x_I^\alpha}dx_I^\alpha = \frac{\partial p^I}{\partial t}\bigg|_X dt = \frac{\partial p^I}{\partial t}dt. \tag{9}$$

Here, J^K, ($K = I, J$), the directional derivative of mass m^K, is defined as a *mass–force–velocity* of m^K.

2.2. Material-Field Variables with Their Time/Space Derivatives

Now, considering our interaction gravity field, the two field functions $f^{IJ}(x^I, x^J)$ and $p^I(x^I, t)$ defined in the Cartesian system $O - x^1 x^2 x^3$ of base-vectors g_α, ($\alpha = 1, 2, 3$), so that their curl vector and divergence can be derived by using the gradient operator ∇ as follows. For the momentum vector, we have the following [15]:

$$\omega^I = \text{curl}\, p^I = \nabla \times p^I, \quad (\text{curl}\, p^I)_\gamma = e_{\gamma\alpha\beta}\frac{\partial p_\beta^I}{\partial x^\alpha}, \tag{10}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\omega}^I = 0. \tag{11}$$

The space derivative of the gravitational force is taken in the following form:

$$\frac{\partial f^{IJ}}{\partial x^\alpha} = \frac{\partial f^{IJ}}{\partial r_{IJ}^\beta}\frac{\partial r_{IJ}^\beta}{\partial x^\alpha}, \qquad \frac{\partial r_{IJ}^\beta}{\partial x^\alpha} = \frac{\partial x_I^\beta}{\partial x^\alpha} - \frac{\partial x_J^\beta}{\partial x^\alpha}. \tag{12}$$

Therefore, we have

$$\operatorname{curl} f^J = \boldsymbol{\nabla}\times f^{IJ}, \quad (\operatorname{curl} f^{IJ})_\gamma = e_{\gamma\alpha\beta}\frac{\partial f_\beta^{IJ}}{\partial x^\alpha} = e_{\gamma\alpha\beta}\frac{\partial f_\beta^{IJ}}{\partial r_{IJ}^\lambda}\frac{\partial r_{IJ}^\lambda}{\partial x^\alpha} = e_{\gamma\alpha\beta}\frac{\partial f_\beta^{IJ}}{\partial r_{IJ}^\lambda}\left(\frac{\partial x_I^\lambda}{\partial x^\alpha} - \frac{\partial x_J^\lambda}{\partial x^\alpha}\right), \tag{13}$$

$$\operatorname{div} f^{IJ} = \boldsymbol{\nabla}\cdot f^{IJ} = \frac{\partial f_\alpha^{IJ}}{\partial x^\alpha} = \frac{\partial f_\alpha^{IJ}}{\partial r_{IJ}^\lambda}\frac{\partial r_{IJ}^\lambda}{\partial x^\alpha} = \frac{\partial f_\alpha^{IJ}}{\partial r_{IJ}^\lambda}\left(\frac{\partial x_I^\lambda}{\partial x^\alpha} - \frac{\partial x_J^\lambda}{\partial x^\alpha}\right). \tag{14}$$

The gravitational flux of mass m^J at x^I must satisfy the Gauss's integration [16],

$$\iiint_V \boldsymbol{\nabla}\cdot f^{IJ}dV = \iint_S f^{IJ}\cdot g_{IJ}dS = \begin{cases} -4\pi\widetilde{G}, & r^{IJ} = 0, \\ 0, & r^{IJ} \neq 0, \end{cases} \qquad g_{IJ} = \frac{r^{IJ}}{|r^{IJ}|}, \tag{15}$$

so that the divergence of Equation (14) becomes

$$\boldsymbol{\nabla}\cdot f^{IJ} = -4\pi\widetilde{G}^{IJ}\Delta(r^{IJ}). \tag{16}$$

Here, Delta function and local gradient operator are as follows:

$$\Delta(r^{IJ}) = \begin{cases} \infty, & r^{IJ} = 0, \\ 0, & r^{IJ} \neq 0, \end{cases} \qquad \boldsymbol{\nabla} = \begin{bmatrix} \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \end{bmatrix}^T, \tag{17}$$

where T denotes matrix transportation.

2.3. Conservation Laws

The gravitational force between two masses is an internal force satisfying the equilibrium equations of the force and its moment,

$$f^{IJ}(x^I, x^J) + f^{JI}(x^J, x^I) = 0, \tag{18}$$

$$x^I\times f^{IJ}(x^I, x^J) + x^J\times f^{JI}(x^J, x^I) = (x^I - x^J)\times f^{IJ}(x^I, x^J) = r^{IJ}\times f^{IJ}(r^{IJ}) = 0, \tag{19}$$

so that there are the following conservation laws for the two masses.

2.3.1. Momentum Conservation

$$\text{Differential form}: \ f^{IJ}dt = dp^I = \frac{\partial p^I}{\partial t}dt, \ f^{JI}dt = dp^J = \frac{\partial p^J}{\partial t}dt, \tag{20}$$

$$\text{Linear momentum}: \ \frac{d}{dt}\left[p^I(x^I, t) + p^J(x^J, t)\right] = f^{IJ} + f^{JI} = 0, \\ p^I(x^I, t) + p^J(x^J, t) = \text{constant}, \tag{21}$$

Angular momentum:

$$\frac{d}{dt}(x^I\times p^I + x^J\times p^J) = x^I\times f^{IJ} + x^J\times f^{JI} = (x^I - x^J)\times f^{IJ}(r^{IJ}) = 0, \\ x^I\times p^I + x^J\times p^J = \text{constant}. \tag{22}$$

2.3.2. Energy Conservation

$$\text{Potential energy}: \quad \Pi(\boldsymbol{r}^{IJ}) = -\frac{1}{(\boldsymbol{r}^{IJ}\cdot\boldsymbol{r}^{IJ})^{1/2}}, \qquad \boldsymbol{f}^{IJ} = -\frac{\partial V}{\partial \boldsymbol{r}^{IJ}}, \tag{23}$$

$$\text{Kinetic energies and complementary ones}: \quad T_I = \frac{1}{2}m^I v_I^2 = \frac{1}{2}\frac{p_I^2}{m^I}, \, T_J = \frac{1}{2}m^J v_J^2 = \frac{1}{2}\frac{p_J^2}{m^J}, \tag{24}$$

$$\Pi + T_I + T_J = \text{constant} = E. \tag{25}$$

2.3.3. Energy-Flow Equation [12,13]

The time change rate of Equation (25) vanishes, so that

$$\begin{aligned}
\frac{dE}{dt} &= \frac{\partial \Pi}{\partial \boldsymbol{r}^{IJ}}\cdot\frac{d\boldsymbol{r}^{IJ}}{dt} + \frac{d(T_I+T_J)}{dt} = -\boldsymbol{f}^{IJ}\cdot\frac{d\boldsymbol{r}^{IJ}}{dt} + \frac{d(T_I+T_J)}{dt} \\
&= \left(\frac{dT_I}{dt} - \boldsymbol{f}^{IJ}\cdot\boldsymbol{v}^I\right) + \left(\frac{dT_J}{dt} - \boldsymbol{f}^{JI}\cdot\boldsymbol{v}^J\right) = 0,
\end{aligned} \tag{26}$$

which, when from Equation (24), noting

$$\frac{dT_I}{dt} = \frac{d\boldsymbol{p}^I}{dt}\cdot\boldsymbol{v}^I, \, \frac{dT_J}{dt} = \frac{d\boldsymbol{p}^J}{dt}\cdot\boldsymbol{v}^J, \tag{27}$$

and Equation (4), gives the energy-flow equation for each mass,

$$\left(\frac{dT_I}{dt} - \boldsymbol{f}^{IJ}\cdot\boldsymbol{v}^I\right) = \left(\frac{d\boldsymbol{p}^I}{dt} - \boldsymbol{f}^{IJ}\right)\cdot\boldsymbol{v}^I = 0. \tag{28}$$

Multiplying both sides of Equation (28) by dt, we obtain the differential forms of energy conservation for each mass, which, generally taking $dx_I^\alpha = dx^\alpha$, is written as follows:

$$dT_I = \boldsymbol{f}^{IJ}\cdot d\boldsymbol{x}^I = \boldsymbol{f}^{IJ}\cdot d\boldsymbol{x} = dW^I, \quad \frac{dp_\alpha^I}{dt}dx_I^\alpha = \frac{dp_\alpha^I}{dt}dx^\alpha = f_\alpha^{IJ}dx_I^\alpha = f_\alpha^{IJ}dx^\alpha, \tag{29}$$

of which the dual equation is an equation of complementary energy conservation, i.e.,

$$d\left(f_\alpha^{IJ}x^\alpha\right) = df_\alpha^{IJ}x^\alpha + \frac{dp_\alpha^I}{dt}dx^\alpha. \tag{30}$$

Understanding of material/directional derivatives, the energy-flow equation, energy/complementary-energy conservations [12,13] in dynamic gravity fields of two masses provides a base to establish the gauge equations for NG field, which, when using the differential equations in Section 2.2 and the exterior derivative rules [2], are derived in the next section.

3. Gravitational Tensor and Gauge Field Equations

3.1. Gravity Tensor and First Set of Gauge Equation

In 4D Cartesian system, a vector $\boldsymbol{x} = x^\alpha \boldsymbol{g}_\alpha$, $\alpha = 0, 1, 2, 3$, $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and the Minkowski metric tensor [4] is $\eta_{\alpha\beta} = \text{diag}\,(-1, 1, 1, 1) = \eta^{\alpha\beta}$. The 2-form [2] of energy conservation Equation (29) is as follows:

$$\begin{aligned}
\frac{\partial f_\alpha^{IJ}}{\partial x^\beta}dx^\beta \wedge dx^\alpha &= \frac{\partial}{\partial x^\beta}\left(\frac{dp_\alpha^I}{dt}\right)dx^\beta \wedge dx^\alpha = \frac{\partial}{\partial x^\beta}\left(\frac{\partial p_\alpha^I}{\partial t} + \frac{\partial p_\alpha^I}{\partial x^\lambda}\frac{dx^\lambda}{dt}\right)dx^\beta \wedge dx^\alpha \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial p_\alpha^I}{\partial x^\beta} - \frac{\partial p_\beta^I}{\partial x^\alpha}\right)\right]dx^\beta \wedge dx^\alpha + \frac{\partial}{\partial x^\lambda}\left\{\left[\frac{1}{2}\left(\frac{\partial p_\alpha^I}{\partial x^\beta} - \frac{\partial p_\beta^I}{\partial x^\alpha}\right)\right]dx^\beta \wedge dx^\alpha\right\}\frac{dx^\lambda}{dt},
\end{aligned} \tag{31}$$

where we have used the Kronecker delta δ_β^λ and the following summation convention [15]:

$$\frac{\partial}{\partial x^\beta}\left(\frac{dx^\lambda}{dt}\right) = \frac{\partial}{\partial x^\beta}\left(\frac{\partial x^\lambda}{\partial t}\right) = \frac{\partial \delta_\beta^\lambda}{\partial t} = 0, \quad \delta_\beta^\lambda = \begin{cases} 1, & \lambda = \beta, \\ 1, & \lambda \neq \beta. \end{cases} \tag{32}$$

The wedge product $dx^\beta \wedge dx^\alpha = -dx^\alpha \wedge dx^\beta$, so that for 3D Cartesian system by cross-product of two vectors, we have

$$dx^\beta \wedge dx^\alpha = e^{\gamma\beta\alpha}dx^\gamma, \quad \frac{\partial f_\alpha^{IJ}}{\partial x^\beta}dx^\beta \wedge dx^\alpha = \frac{\partial f_\alpha^{IJ}}{\partial x^\beta}e^{\gamma\beta\alpha}dx^\gamma = \left(\nabla \times \boldsymbol{f}^{IJ}\right)^\gamma dx^\gamma, \tag{33}$$

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial p_\alpha^I}{\partial x^\beta} - \frac{\partial p_\beta^I}{\partial x^\alpha}\right)\right]dx^\beta \wedge dx^\alpha = \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial p_\alpha^I}{\partial x^\beta} - \frac{\partial p_\beta^I}{\partial x^\alpha}\right)\right]e^{\gamma\beta\alpha}dx^\gamma = \frac{\partial \omega_I^\gamma}{\partial t}dx^\gamma, \tag{34}$$

from which, we obtain

$$\nabla \times \boldsymbol{f}^{IJ} = \frac{\partial \boldsymbol{\omega}^I}{\partial t} + \frac{\partial \boldsymbol{\omega}^I}{\partial x_I^\gamma}\frac{\partial x_I^\gamma}{\partial t} = \frac{\partial \boldsymbol{\omega}^I}{\partial t}(X = x, t). \tag{35}$$

To reveal the physical meaning of Equation (35), we recognize that

$$\frac{\partial f_\alpha^{IJ}}{\partial x^\beta}dx^\beta \wedge dx^\alpha = \partial f_\alpha^{IJ} \wedge dx^\alpha \tag{36}$$

is the cross product of differential force and differential displacement, the force moment caused by the mass motion, which equals the time change rate of the angular velocity of the mass, i.e., the angular acceleration of mass. Therefore, Equation (35) is the result of the angular momentum conservation.

We construct a 2-form $\boldsymbol{F}^I$, a skew-symmetrical *gravity tensor*, as follows:

$$\boldsymbol{F}^I = \boldsymbol{\omega}^I + dt \wedge \boldsymbol{f}^{IJ} = \boldsymbol{\omega}^I + dx^0 \wedge \frac{\boldsymbol{f}^{IJ}}{c}, \tag{37}$$

$$\boldsymbol{f}^{IJ} = f_\alpha^{IJ}dx^\alpha = f_x^{IJ}dx + f_y^{IJ}dy + f_z^{IJ}dz, \tag{38}$$

$$dx^0 \wedge \frac{\boldsymbol{f}^{IJ}}{c} = \frac{f_x^{IJ}}{c}dx^0 \wedge dx + \frac{f_y^{IJ}}{c}dx^0 \wedge dy + \frac{f_z^{IJ}}{c}dx^0 \wedge dz, \tag{39}$$

$$\boldsymbol{\omega}^I = \frac{\partial p_\alpha^I}{\partial x^\beta}dx^\beta \wedge dx^\alpha = \omega_x^I dy \wedge dz + \omega_y^I dz \wedge dx + \omega_z^I dx \wedge dy, \tag{40}$$

$$\boldsymbol{F}^I = \frac{1}{2}F_{\alpha\beta}dx^\alpha \wedge dx^\beta, \quad F_{\alpha\beta} = \begin{bmatrix} 0 & \omega_x^I & \omega_y^I & \omega_z^I \\ -\omega_x^I & 0 & f_z^{IJ}/c & -f_y^{IJ}/c \\ -\omega_y^I & -f_z^{IJ}/c & 0 & f_x^{IJ}/c \\ -\omega_z^I & f_y^{IJ}/c & -f_x^{IJ}/c & 0 \end{bmatrix}, \quad F_{\alpha\beta} = -F_{\beta\alpha}, \tag{41}$$

$$F^{\alpha\beta} = \eta^{\alpha\mu}F_{\mu\gamma}\eta^{\gamma\beta}, \quad F^{\alpha\beta} = \begin{bmatrix} 0 & -\omega_x^I & -\omega_y^I & -\omega_z^I \\ \omega_x^I & 0 & f_z^{IJ}/c & -f_y^{IJ}/c \\ \omega_y^I & -f_z^{IJ}/c & 0 & f_x^{IJ}/c \\ \omega_z^I & f_y^{IJ}/c & -f_x^{IJ}/c & 0 \end{bmatrix}. \tag{42}$$

It is like the electromagnetic tensor [17], from Equation (41), it follows that the Lorentz covariance invariant

$$F_{\alpha\beta}F^{\alpha\beta} = 2\left(\omega^2 - \frac{f^2}{c^2}\right), \tag{43}$$

the Pseudoscalar invariant

$$\frac{1}{2}e_{\alpha\beta\gamma\mu}F^{\alpha\beta}F^{\gamma\mu} = -\frac{4}{c}(\boldsymbol{\omega}\cdot\boldsymbol{f}),\tag{44}$$

where the Levi-Civita symbol with p signum of permutation is

$$e_{\alpha\beta\gamma\mu} = (-1)^{p}e_{1234},\tag{45}$$

and the determinant

$$\det(\boldsymbol{F}) = \frac{1}{c^{2}}(\boldsymbol{\omega}\cdot\boldsymbol{f})^{2}.\tag{46}$$

Vanishing 3-form of Equation (37) and splitting up exterior derivative operator into spacelike and timelike parts gives the following:

$$dF^{I} = d\boldsymbol{\omega}^{I} + dx^{0} \wedge \frac{df^{IJ}}{c} = dx^{0} \wedge \left(\frac{d_{S}f^{IJ}}{c} + \frac{\partial_{x^{0}}f^{IJ}}{c} \wedge dx^{0}\right) + d_{S}\boldsymbol{\omega}^{I} + \partial_{x^{0}}\boldsymbol{\omega}^{I} \wedge dx^{0}$$
$$= (\partial_{t}\boldsymbol{\omega}^{I} - d_{S}\boldsymbol{f}^{IJ}) \wedge dt + d_{S}\boldsymbol{\omega}^{I} = 0, \quad d_{S}\boldsymbol{f}^{IJ} = \partial_{t}\boldsymbol{\omega}^{I}, \quad d_{S}\boldsymbol{\omega}^{I} = 0.\tag{47}$$

Using the Hodge * [2], we obtain the first-set gauge equations,

$$* \, dF^{I} = 0, \ \boldsymbol{\nabla} \times \boldsymbol{f}^{IJ} = \frac{\partial\boldsymbol{\omega}^{I}}{\partial t}, \ \boldsymbol{\nabla}\cdot\boldsymbol{\omega}^{I} = 0,\tag{48}$$

$$\partial_{\beta}F_{\alpha\beta} = \begin{bmatrix} 0 & \omega_{x}^{I} & \omega_{y}^{I} & \omega_{z}^{I} \\ -\omega_{x}^{I} & 0 & f_{z}^{IJ}/c & -f_{y}^{IJ}/c \\ -\omega_{y}^{I} & -f_{z}^{IJ}/c & 0 & f_{x}^{IJ}/c \\ -\omega_{z}^{I} & f_{y}^{IJ}/c & -f_{x}^{IJ}/c & 0 \end{bmatrix} \begin{bmatrix} \partial/c\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} = 0.\tag{49}$$

3.2. Dual Equation

Dual gauge equation is derived from the complementary-energy conservation Equation (30), from which, for the righthand-side terms, we have

$$\hat{\boldsymbol{f}}^{JI} = d\left(df_{\alpha}^{IJ}x_{I}^{\alpha}\right) = \frac{\partial x_{I}^{\alpha}}{\partial x^{\beta}}dx^{\beta} \wedge df_{\alpha}^{IJ} = \delta_{\beta}^{\alpha}dx^{\beta} \wedge df_{\alpha}^{IJ} = dx^{\alpha} \wedge \frac{\partial f_{\alpha}^{IJ}}{\partial x^{\beta}}dx^{\beta}$$
$$= \frac{\partial f_{\alpha}^{IJ}}{\partial x^{\beta}}dx^{\beta} \wedge dx^{\alpha},\tag{50}$$

$$d\left(\frac{\partial p_{\alpha}^{I}}{\partial t}dx_{I}^{\alpha}\right) = \frac{\partial}{\partial t}\left(\frac{\partial p_{\alpha}^{I}}{\partial x^{\beta}}\right)dx^{\beta} \wedge dx^{\alpha},\tag{51}$$

$$\hat{\boldsymbol{\omega}}^{I} = \left(\frac{\partial p_{\alpha}^{I}}{\partial x^{\beta}}\right)dx^{\beta} \wedge dx^{\alpha} = \omega_{\alpha}^{I}dx^{\alpha} = \omega_{x}^{I}dx + \omega_{y}^{I}dy\omega_{z}^{I}dz,\tag{52}$$

and then construct the 2-form of right-hand side of Equation (30), $\hat{\boldsymbol{F}}^{I} = dt \wedge \hat{\boldsymbol{\omega}}^{I} + \hat{\boldsymbol{f}}^{JI}$ with its 3-form,

$$d\hat{\boldsymbol{F}}^{I} = dt \wedge d\hat{\boldsymbol{\omega}}^{I} + d\hat{\boldsymbol{f}}^{JI} = \left(-d_{S}\hat{\boldsymbol{\omega}}^{I} + \partial_{t}\hat{\boldsymbol{\omega}}^{I} \wedge dt\right) \wedge dt + d_{S}\hat{\boldsymbol{f}}^{JI} + \partial_{t}\hat{\boldsymbol{f}}^{JI} \wedge dt$$
$$= \left(-d_{S}\hat{\boldsymbol{\omega}}^{I} + \partial_{t}\hat{\boldsymbol{f}}^{JI}\right) \wedge dt + d_{S}\hat{\boldsymbol{f}}^{JI}.\tag{53}$$

The 3-form of left-hand side of Equation (30) can be constructed as follows. In the first step, we can use the general deferential rule of the two-function product $f_{\alpha}^{IJ}x_{I}^{\alpha}$ to obtain

$$d\left(f_{\alpha}^{IJ}x_{I}^{\alpha}\right) = df_{\alpha}^{IJ}x_{I}^{\alpha} + f_{\alpha}^{IJ}dx^{\alpha} = K, \qquad dK = df_{\alpha}^{IJ}dx^{\alpha} + df_{\alpha}^{IJ}dx^{\alpha},\tag{54}$$

and then use $df_\alpha^{IJ} = \frac{\partial f_\alpha^{IJ}}{\partial x^\beta}dx^\beta = \frac{\partial f_\alpha^{IJ}}{\partial t}dt$ in Equation (54) to yield

$$
\begin{aligned}
dK &= \frac{\partial f_\alpha^{IJ}}{\partial x^\beta}dx^\beta dx^\alpha + \frac{\partial f_\alpha^{IJ}}{\partial t}dt dx^\alpha = \frac{\partial\left(f_\alpha^{IJ}dx^\alpha\right)}{\partial x^\beta}dx^\beta + \frac{\partial\left(f_\alpha^{IJ}dx^\alpha\right)}{\partial t}dt \\
&= \frac{\partial f^{IJ}}{\partial x^\beta}dx^\beta + \frac{\partial f^{IJ}}{\partial t}dt = df^{IJ} + \frac{\partial f^{IJ}}{\partial t}dt,
\end{aligned}
\tag{55}
$$

by which, and noting $df^{IJ} = d(* \ df^{IJ}) = d_S f^{IJ} \wedge dx \wedge dy \wedge dz$ [2], we construct its 3-form

$$
d(K) * 1 = \frac{df^{IJ}}{dt}dt \wedge dx \wedge dy \wedge dz + d_S f^{IJ} \wedge dx \wedge dy \wedge dz.
\tag{56}
$$

Therefore, 3-form of Equation (30) is

$$
\frac{df^{IJ}}{dt}dt \wedge dx \wedge dy \wedge dz + d_S f^{IJ} \wedge dx \wedge dy \wedge dz - d\hat{F}^I = 0,
\tag{57}
$$

$$
d_S \hat{f}^{JI} = d_S f^{JI} \wedge dx \wedge dy \wedge dz, \qquad d_S \hat{\omega}^I - \partial_t \hat{f}^{JI} = \frac{df^{IJ}}{dt}dx \wedge dy \wedge dz.
\tag{58}
$$

Using the Hodge * [2] and the singularity of f^{IJ} at $(x_I^\alpha = x_J^\alpha)$, where m^I locates, we obtain a dual one,

$$
* \ d_S\hat{f}^{JI} = \nabla\cdot f^{JI} = -4\pi\widetilde{G}\delta(x_I^\alpha - x_J^\alpha),\ * \ d_S\hat{\omega}^I - * \ \partial_t \hat{f}^{JI} = \nabla \times \omega^I - \frac{\partial f^{JI}}{\partial t} = \frac{df^{IJ}}{dt},
\tag{59}
$$

$$
\nabla \times \omega^I - \frac{\partial f^{JI}}{\partial t} = J^{IJ}.
\tag{60}
$$

Dual tensor $\hat{F}_{\alpha\beta} = * \ F_{\alpha\beta}$, by replacing the positions of ω_α^I and f_α^{JI} in $F_{\alpha\beta}$, can be obtained. Finally, both gauge equations are

$$
* \ dF^I = 0, \qquad * \ d * \ F^I = \hat{J}^I = \begin{bmatrix} -4\pi\widetilde{G}\delta(x_I^\alpha - x_J^\alpha) \\ J^{IJ} \end{bmatrix},
\tag{61}
$$

$$
\partial_\beta \hat{F}_{\alpha\beta} = \begin{bmatrix} 0 & f_x & f_y & f_z \\ -f_x & 0 & \omega_z/c & -\omega_y/c \\ -f_y & -\omega_z/c & 0 & \omega_x/c \\ -f_x & \omega_y/c & -\omega_x/c & 0 \end{bmatrix}\begin{bmatrix} \partial/c\partial t \\ \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} = \begin{bmatrix} -4\pi\widetilde{G}\delta(x_I^\alpha - x_J^\alpha) \\ J_x^{IJ} \\ J_y^{IJ} \\ J_z^{IJ} \end{bmatrix}.
\tag{62}
$$

3.3. Gravity Vector and Skew-Symmetrical Gravity Tensor

In Sections 3.1 and 3.2, we consider the changes in the gravitational force and the momentum caused by time/position changes that are governed by the conservation laws, which require that the work performed by the force must be the change in kinetic energy of the system. We have known that the change in the gravitational force closely concerns the two essential quantities of Newton's gravity system: (a) the potential Π determined by the two-instant positions of two points in the gravity field, and (b) the dynamic momentum that changes the positions of the masses, so that it affects the gravity force. In mathematical derivations, we have not considered how these two key quantities play their physical roles to obtain the gauge equations but only considered their composited integrating action in the material description to derive the result equations.

Here, based on the field description approach for electromagnetic field [17], we can explore these unrevealed mechanisms of the two key quantities in the dynamic gravity

interactions in the field description as follows. We define the two key field quantities of the gravity field in Appendix C as a field gravity vector,

$$A^\alpha = \left(\frac{\Pi}{c}, \, \boldsymbol{p} \right), \qquad A_\alpha = \left(\frac{\Pi}{c}, \, \boldsymbol{p} \right), \; A^\alpha = \eta^{\alpha\beta} A_\beta = \left(-\frac{\Pi}{c}, \, \boldsymbol{p} \right), \tag{63}$$

from which we can construct a corresponding skew-symmetrical tensor as

$$F_{\alpha\beta} = \frac{\partial A_\alpha}{\partial x^\beta} - \frac{\partial A_\beta}{\partial x^\alpha} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{64}$$

Using the operator,

$$\partial_\alpha = \begin{pmatrix} \frac{\partial}{c\partial t} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{c\partial t} & \boldsymbol{\nabla} \end{pmatrix}, \tag{65}$$

we obtain

$$\partial_\alpha A_\beta = \begin{bmatrix} \frac{\partial}{c\partial t} \\ \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\Pi}{c} & p_x & p_y & p_z \end{bmatrix} = \begin{bmatrix} 0 & p_{x,t}/c & p_{y,t}/c & p_{z,t}/c \\ \Pi_{,x}/c & p_{x,x} & p_{y,x} & p_{z,x} \\ \Pi_{,y}/c & p_{x,y} & p_{y,y} & p_{z,y} \\ \Pi_{,z}/c & p_{x,z} & p_{y,z} & p_{z,z} \end{bmatrix}, \tag{66}$$

$$\partial_\beta A_\alpha = \begin{bmatrix} 0 & p_{x,t}/c & p_{y,t}/c & p_{z,t}/c \\ \Pi_{,x}/c & p_{x,x} & p_{y,x} & p_{z,x} \\ \Pi_{,y}/c & p_{x,y} & p_{y,y} & p_{z,y} \\ \Pi_{,z}/c & p_{x,z} & p_{y,z} & p_{z,z} \end{bmatrix}^T, \tag{67}$$

which, when submitted into Equation (64), yields

$$\begin{aligned} F_{\alpha\beta} &= \begin{bmatrix} 0 & p_{x,t}/c & p_{y,t}/c & p_{z,t}/c \\ \Pi_{,x}/c & p_{x,x} & p_{y,x} & p_{z,x} \\ \Pi_{,y}/c & p_{x,y} & p_{y,y} & p_{z,y} \\ \Pi_{,z}/c & p_{x,z} & p_{y,z} & p_{z,z} \end{bmatrix} - \begin{bmatrix} 0 & p_{x,t}/c & p_{y,t}/c & p_{z,t}/c \\ \Pi_{,x}/c & p_{x,x} & p_{y,x} & p_{z,x} \\ \Pi_{,y}/c & p_{x,y} & p_{y,y} & p_{z,y} \\ \Pi_{,z}/c & p_{x,z} & p_{y,z} & p_{z,z} \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & p_{x,t}/c - \Pi_{,x}/c & p_{y,t}/c - \Pi_{,y}/c & p_{z,t}/c - \Pi_{,z}/c \\ -p_{x,t}/c + \Pi_{,x}/c & 0 & \omega_z & -\omega_y \\ -p_{y,t}/c + \Pi_{,y}/c & -\omega_z & 0 & \omega_x \\ -p_{z,t}/c + \Pi_{,z}/c & \omega_y & -\omega_x & 0 \end{bmatrix}. \end{aligned} \tag{68}$$

Comparing the skew-symmetrical gravitational tensors in Equations (62) and (68), we find that the gravitational force can be formulated as

$$\boldsymbol{f} = \frac{1}{c} \left(-\boldsymbol{\nabla}\Pi + \frac{\partial \boldsymbol{p}}{\partial t} \right), \tag{69}$$

which is the same field expression in Appendix C. Physically, this result implies that in the dynamic interaction gravity field, the dynamic gravity force consists of two parts, one is the gradient of the gravity potential depending on the relative position between the field points of masses, and another is the time change rate of the momentum at the mass point. If there are no two masses in motion, $\partial \boldsymbol{p}/\partial t = 0$, Equation (69) reduces to the case of the classical Newtonian gravity of two fixed masses; the field is conservative and curl-free. However, for the case of dynamic interactions of two movable masses, $\partial \boldsymbol{p}/\partial t \neq 0$, the curl $\boldsymbol{f}$ is governed by Equation (48) developed in this paper.

## 4. Comparison and Example Validation

Comparing the gauge equations developed by kinematical structure in this paper with those of Maxwell equations for electromagnetic field by field structure [2] reveals that both have an exact-same mathematical structure. This implies f^{IJ} and curl $\boldsymbol{p}$ in the NG field, respectively, play the roles like the eclectic $\boldsymbol{E}$ and magnetic $\boldsymbol{B}$ in the electromagnetic field, as compared in detail in Appendix B. Furthermore, they have a similar interaction mechanism. In the electromagnetic theory, a moving electric charge produces a dynamic electric field in which two different points have different electric intensities. Then, this movable electric field produces a dynamic magnetic field interacting with the electric field. In Newton's gravity theory, a moving mass A produces a dynamic gravity field in which two different points have different gravitational intensities, so that another mass B at a field point B will have its local dynamical acceleration, momentum with its local curl vector. Therefore, the curl momentum is produced to interact with the gravitational force of the mass A. Based on the analogy of the mathematical structure of the gauge equations and the interaction mechanisms of these two-gauge fields, we may consider that the mechanism of the dynamic interaction in Newton's gravity field is the "gravitomagnetic effect".

In the derivation of the gauge equation of geometric gravity theory, the geometrical "connection curvature tensor" plays a key link to derive the result. However, here we use the physical "energy conservation" as a link to derive the expected result based on the material description of Newton's gravity. Both aim to produce the corresponding scalar notions. Energy conservation implies that the work performed by the gravity force equals the change in the kinetic energy of the mass. From Figure A1 and the related formulations describing the motion of a single unit mass, we obtain that the variation in the kinetic energy is

$$\delta T = \ddot{\boldsymbol{r}} \cdot \delta \boldsymbol{r} = -\frac{\mathrm{G}}{(\boldsymbol{r} \cdot \boldsymbol{r})^{3/2}} \boldsymbol{r} \cdot \delta \boldsymbol{r} = -\frac{\mathrm{G}}{r^2} \delta r = G\delta\left(\frac{1}{r}\right) = G\delta R,$$

which exactly involves the variation in the curvature R of the unit mass orbit. Therefore, the physical "energy conservation" adopted in the paper plays a similar role as the "geometrical curvature connection" used in geometric field gravity theory for the corresponding gauge equations. But the condition constraining the variation in the curvature R is different; in the geometric gauge theory, this variation is required to vanish (i.e., the kinetic energy is not changed). However, in the gauge equation derivation of this paper, the energy conservation requires the kinetic variation caused by δR to equal the work performed by the gravitational force on the δR. Therefore, in the gauge equation based on the H–E action of geometric gravity theory, only the geometrical variable (the metric tensor) is involved; a cosmological constant Π was introduced by Einstein from his objective consideration. But the gauge equation derived in this paper concerns the gravitational force f and the kinetic variable curl $\boldsymbol{p}$ with their dynamic interactive equilibration.

To validate the developed gauge equations of NG field, we investigate the motion of a unit mass m in the gravity field of another unit mass M in a spherical coordinate system, as shown in Figure 1. We tackle the following two cases.
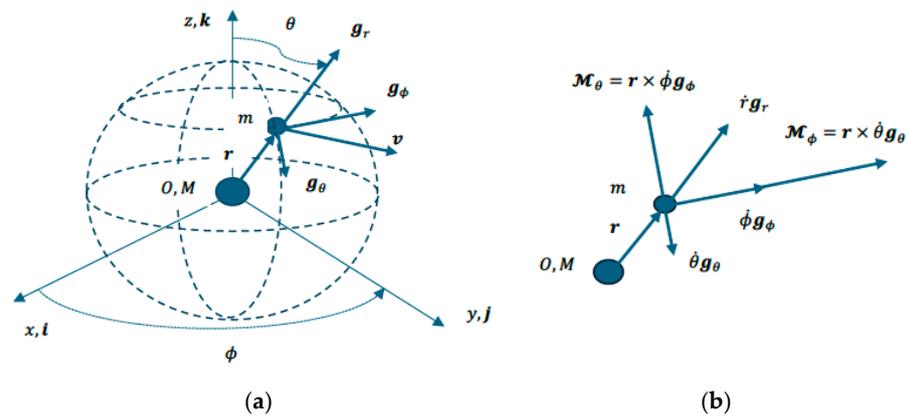
Figure 1. (a) A spherical coordinate system with its local base vectors (g_r, g_θ, g_ϕ) describes the motion of mass m at its instant position r in the gravity field of mass M located at point O; (b) the vector diagram of the components of the velocity/momentum and the angular momentum of the mass m.

4.1. Motion of Mass m Relative to Mass M

In the spherical coordinate system shown in Figure 1, generally, the position vector of the mass m is

$$r(r,\ \theta, \phi) = r\sin\theta\cos\phi i + r\sin\theta\sin\phi j + r\cos\theta k, \tag{70}$$

from which the three base vectors can be obtained as

$$
\begin{aligned}
g_r &= \frac{\partial r}{\partial r} = \sin\theta\cos\phi i + \sin\theta\sin\phi j + \cos\theta k = e_r,\\
g_\theta &= \frac{\partial r}{\partial \theta} = r\cos\theta\cos\phi i + r\cos\theta\sin\phi j - r\sin\theta k = re_\theta,\\
g_\phi &= \frac{\partial r}{\partial \phi} = -r\sin\theta\sin\phi i + r\sin\theta\cos\phi j = r\sin\theta e_\phi,
\end{aligned}
\tag{71}
$$

where e_r, e_θ and e_ϕ are three-unit vectors.

The initial conditions of mass m are defined as

$$
\begin{aligned}
r_0 &= r(r_0,\ \theta_0,\ \phi_0) = r_0 g_r,\ \theta_0 = 0,\quad r_0 = \hat{r},\quad \phi_0 = \pi/2;\\
v_0 &= \dot{r}_0 = \dot{r}_0 g_r + \dot{\theta}_0 g_\theta + \dot{\phi}_0 g_\phi,\quad t = 0.
\end{aligned}
\tag{72}
$$

The gravitational force f, the position vector and velocity of the mass m at an instant in time can be obtained in the following forms:

$$
\begin{aligned}
r &= r(r,\ \theta, \phi_0) = rg_r = re_r, f = -g_r/r^2.\\
v &= \dot{r}g_r + \dot{\theta}g_\theta + \dot{\phi}g_\phi = \dot{r}e_r + r\dot{\theta}e_\theta + \dot{\phi}r\sin\theta e_\phi = p.
\end{aligned}
\tag{73}
$$

The angular momentum vector $\mathcal{M}$ of the mass m about the origin of the coordinate system is a constant vector [3], that is

$$\mathcal{M} = r \times p = re_r \times (\dot{r}e_r + r\dot{\theta}e_\theta + \dot{\phi}r\sin\theta e_\phi) = r^2\dot{\theta}e_\phi - \dot{\phi}r^2\sin\theta e_\theta = \textbf{constant}, \tag{74}$$

$$\mathcal{M}_r = 0,\ \mathcal{M}_\theta = r \times \dot{\phi}g_\phi = -\dot{\phi}r^2\sin\theta e_\theta,\ \mathcal{M}_\phi = r \times \dot{\phi}g_\theta = r^2\dot{\theta}e_\phi. \tag{75}$$

The position vector, the components of the momentum vector, and the ones of angular momentum are shown in Figure 1b.

Using the vector formulation in the spherical coordinate system, we can obtain the curl p and the curl f as

$$
\begin{aligned}
\omega = \nabla \times p &= \left\{\frac{1}{r}\frac{\partial}{\partial r}(r^2\dot{\theta})\right\}e_\phi + \left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\dot{\phi})\right\}e_r - \left\{\frac{1}{r}\frac{\partial}{\partial r}(r^2\dot{\phi}\sin\theta)\right\}e_\theta\\
&= 2\dot{\theta}e_\phi + 2\cos\theta\dot{\phi}e_r - 2\dot{\phi}\sin\theta e_\theta,
\end{aligned}
\tag{76}
$$

$$\boldsymbol{\nabla} \times \boldsymbol{f} = 0. \tag{77}$$

The vector $\boldsymbol{\omega}$ is totally determined by the position of the mass m, so that its partial time derivative vanishes; therefore, we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{f} = 0, \tag{78}$$

which is the same as Equation (35).

As a special case, if the $\dot{\phi} \equiv 0$ in the motion of the mass m, from Equations (72)–(78) above, it follows that

$$\phi = \pi/2, \qquad \boldsymbol{\omega} = 2\dot{\theta} \boldsymbol{e}_\phi, \qquad \boldsymbol{e}_\phi = -\boldsymbol{i}, \tag{79}$$

implying that, *physically*, the mass m moves in the fixed plane $y\mathrm{M}z$, and its instant curl $\boldsymbol{p}$ equals the two times of its instant angular velocity $\dot{\theta} \boldsymbol{e}_\phi$.

4.2. Dynamic NG Gravity Field of Movable Mass m

The initial conditions of m are considered as

$$\dot{r}_0 = 0 = \ddot{r}_0, \qquad r_0 \dot{\theta}_0^2 = 1/r_0^2, \tag{80}$$

implying NG balanced by its centripetal acceleration, and mass m moves in the circle of radius r_0 on $y\mathrm{M}z$ plane with its position and velocity:

$$z = r_0 \cos\theta = r_0 \cos\widetilde{\omega}t, \; y = r_0 \sin\theta = r_0 \sin\widetilde{\omega}t, \; \widetilde{\omega} = \dot{\theta}_0;$$
$$\dot{z} = -r_0\widetilde{\omega}\sin\widetilde{\omega}t, \quad \dot{y} = r_0\widetilde{\omega}\cos\widetilde{\omega}t. \tag{81}$$

Intensity at origin O caused by the gravitational force of m, its curl vector, partial/total time derivatives at O are as follows:

$$f_\mathrm{m}^\mathrm{o} = = (y\mathbf{j} + z\mathbf{k})/r_0^3 = \left(\mathbf{j}\sin\widetilde{\omega}t + \mathbf{k}\cos\widetilde{\omega}t\right)/r_0^2,$$
$$\boldsymbol{\nabla} \times f_\mathrm{m}^\mathrm{o} = (\partial z/\partial y - \partial y/\partial z)/r_0^3 \mathbf{i} = 0, \tag{82}$$
$$\partial f_\mathrm{m}^\mathrm{o}/\partial t = 0, \quad df_\mathrm{m}^\mathrm{o}/dt = \mathbf{j}\widetilde{\omega}\cos\widetilde{\omega}t - \mathbf{k}\widetilde{\omega}\sin\widetilde{\omega}t/r_0^2 = -\boldsymbol{J}^m = -\partial f_\mathrm{o}^\mathrm{m}/\partial t.$$

The relative momentum of mass M to mass m and its curl vector below are the same as given by Equations (35), (59) and (60), i.e.,

$$\widetilde{\boldsymbol{p}} = -r_0\widetilde{\omega}\cos\widetilde{\omega}t\mathbf{j} + r_0\widetilde{\omega}\sin\widetilde{\omega}t\mathbf{k} = -z\widetilde{\omega}\mathbf{j} + y\widetilde{\omega}\mathbf{k},$$
$$\widetilde{\boldsymbol{\omega}} = \boldsymbol{\nabla} \times \widetilde{\boldsymbol{p}} = \left(\frac{\partial(y\widetilde{\omega})}{\partial y} - \frac{\partial(-z\widetilde{\omega})}{\partial z}\right)\mathbf{i} = 2\widetilde{\omega}\mathbf{i}, \tag{83}$$

$$\frac{\partial\widetilde{\boldsymbol{\omega}}}{\partial t} = 0 = \boldsymbol{\nabla} \times f_\mathrm{m}^\mathrm{o}, \quad \frac{df_\mathrm{m}^\mathrm{o}}{dt} = \nabla \times \widetilde{\boldsymbol{\omega}} - \boldsymbol{J}^m = -\boldsymbol{J}^m, \quad \nabla \times \widetilde{\boldsymbol{\omega}} = 0. \tag{84}$$

5. Physical Meanings of Gauge Equations and Extensions

5.1. Physical Meanings of Gauge Equations

The simple example shows the dynamic interactions of two masses in their gravitational field. The movable mass m^J of its instant position $x^J(X^J, t)$ generates its dynamic gravity force at x^I so that the mass m^I at point x^I has the time change rate of its dynamic momentum governed by Newton's second law, which can be written in the form

$$f^{IJ}\left(x^J, x^I\right) = -\widetilde{\mathrm{G}}\frac{\boldsymbol{r}^{IJ}}{r_{IJ}^3} = \widetilde{\mathrm{G}}\frac{(x^{J\alpha} - x^{I\alpha})\boldsymbol{g}_\alpha}{r_{IJ}^3} = \frac{d\boldsymbol{p}^I(x^I, t)}{dt} = \frac{\partial \boldsymbol{p}^I}{\partial t} + \frac{\partial \boldsymbol{p}^I}{\partial x^{I\alpha}}\dot{x}^{I\alpha} = \frac{\partial \boldsymbol{p}^I}{\partial t}\bigg|_{X=x}. \tag{85}$$

Here, the term $\frac{\partial p^I}{\partial x^{I\alpha}} \dot{x}^{I\alpha}$ denotes the gravity force caused by the motion of mass m^I, which causes the change in the distance between the two interacting masses, so that it is covered by the gravitational potential Π, as shown in Equation (69), except for a constant c. Therefore, Equation (85) has its two forms: one considers $p^I(x^I, t)$ as a field function, while another considers it is a material variable, i.e.,

$$f^{IJ}\left(x^I, t\right) = -\boldsymbol{\nabla}\Pi + \frac{\partial \boldsymbol{p}^I}{\partial t}, \qquad f^{IJ}\left(x^J, x^I\right) = \frac{\partial \boldsymbol{p}^I}{\partial t}(x = X, t). \tag{86}$$

The differential changes in this Equation in the space domain and the time domain can derive the gauge equations as follows.

5.1.1. Spacelike Derivative $\partial/\partial x^\alpha$

Taking a curl operation to Equation (86), in which the mass m^J moves with the relative space displacement $(x^{J\alpha} - x^{I\alpha})\boldsymbol{g}_\alpha$ to m^I, and causes the momentum change of m^I at x^I, we obtain the gauge Equation (35), i.e.,

$$\nabla \times f^{IJ} = \nabla \times \frac{\partial \boldsymbol{p}^I}{\partial t} = \frac{\partial(\nabla \times \boldsymbol{p}^I)}{\partial t} = \frac{\partial \boldsymbol{\omega}^I}{\partial t}, \tag{87}$$

5.1.2. Timelike Derivative $\partial/\partial t$

The gravitational force relative to m^J is considered as $f^{IJ}\left(x^I, X^J\right)$ due to the motion of mass m^I, of which the time derivative due to the motion is its directional derivative as given in Equation (5), so that from Equation (86), we have

$$\frac{\partial f^{IJ}}{\partial t} = J^{IJ} + J^J = \frac{\partial}{\partial t}\frac{\partial \boldsymbol{p}^I(x^I, t)}{\partial t}, \tag{88}$$

in which, from the exterior derivative rule, it follows

$$\boldsymbol{p}^I = p^I_\alpha dx^\alpha, \tag{89}$$

with its time derivative equaling its directional derivative, i.e.,

$$\frac{\partial \boldsymbol{p}^I}{\partial t} = \frac{\partial p^I_\alpha}{\partial x^\beta}\frac{dx^\beta}{dt} \wedge dx^\alpha, \quad \frac{\partial \boldsymbol{p}^I}{\partial t}dt = \boldsymbol{\omega}^I = \omega^I_\alpha dx^\alpha, \quad *\, dt = 1. \tag{90}$$

Therefore, we have Equation (60), i.e.,

$$\frac{\partial f^{IJ}}{\partial t} = J^{IJ} + J^J = \frac{\partial}{\partial t}\frac{\partial \boldsymbol{p}^I}{\partial t} = \frac{\partial}{\partial t}\left(\omega^I_\alpha dx^\alpha\right) = \frac{\partial \omega^I_\alpha}{\partial x^\beta}dx^\beta \wedge dx^\alpha = \boldsymbol{\nabla} \times \boldsymbol{\omega}^I, \tag{91}$$

5.2. Extensions to the Gravity Dynamics of N Masses

Now we consider that there are N masses in Newton's gravity field, for which Newton's gravity Equation (86) becomes

$$\sum_{J=1,\neq I}^{N} f^{IJ}\left(x^J, X^J\right) = \sum_{J=1,\neq I}^{N} \widetilde{G}\frac{(x^{J\alpha} - X^{I\alpha})\boldsymbol{g}_\alpha}{r^3_{IJ}} = \frac{D\boldsymbol{p}^I(x^I, t)}{Dt} = \left.\frac{\partial \boldsymbol{p}^I}{\partial t}\right|_X, \quad I = 1, 2, \ldots N, \tag{92}$$

in which the lefthand side represents the resultant gravitational force of $N-1$ masses m^J at positions x^J acting on the mass m^I identified by its instant material coordinate X^I, while the right-hand side is the material derivative of the momentum of mass m^I at position x^I. Considering the differential changes in the space and the time domains, respectively,

as used for Equations (87) and (91), we can derive the following gauge equations for the gravity field of N masses.

5.2.1. Spacelike Derivative $\partial/\partial x^\alpha$

Taking a curl operation to Equation (92), in which the mass m^J moves with the relative space displacement $(x^{J\alpha} - X^{I\alpha})g_\alpha$ to m^I, and causes the momentum change of m^I at x^I, we obtain

$$\sum_{J=1,\neq I}^N \nabla \times \boldsymbol{f}^{IJ} = \nabla \times \frac{\partial \boldsymbol{p}^I}{\partial t} = \frac{\partial(\nabla \times \boldsymbol{p}^I)}{\partial t} = \frac{\partial \boldsymbol{\omega}^I}{\partial t}, \qquad I = 1, 2, \ldots N. \tag{93}$$

5.2.2. Timelike Derivative $\partial/\partial t$

The gravitational force relative to a mass m^J is considered as $\boldsymbol{f}^{IJ}(x^I, X^J)$ due to the motion of m^I, of which the time derivative is due to the motion of mass m^I is its directional derivative as given in Equation (6), i.e.,

$$\frac{\partial \boldsymbol{f}^{IJ}}{\partial t} = \boldsymbol{J}^{IJ} + \boldsymbol{J}^J, \tag{94}$$

from which, when considering all masses as shown in Equation (91), we obtain

$$\sum_{J=1,\neq I}^N \frac{\partial \boldsymbol{f}^{IJ}}{\partial t} = \sum_{J=1,\neq I}^N \left(\boldsymbol{J}^{IJ} + \boldsymbol{J}^J\right) = \nabla \times \boldsymbol{\omega}^I, \qquad I = 1, 2, \ldots N. \tag{95}$$

6. Conclusions and Discussions

6.1. Conclusions

In this paper, the author has developed a gravity tensor and the gauge equations for Newton's gravity field, which address the historic issue that Newton's gravitation is an exception to the Yang–Mills theory. Based on the comparison of the gauge equations of Newton's gravity with the well-known Maxwell equations, it has revealed that both have the same mathematical structure, which implies that the gravitational force and curl–momentum vector in Newton's gravity field, respectively, play the roles as the electric and magnetic intensities in the electromagnetic field. This success is based on understanding the dynamic interactions of two masses in the gravity field, where the material derivative of a field variable is its directional derivative, and energy/complementary-energy conservation equations are the bridges to produce the required scalar notion for dynamic equilibria, with exterior derivatives to develop the gauge equations. A simple example is given to validate the developed equations, which is extended to the case of N masses.

6.2. Discussions

The following points may need to be further investigated.

(a) It is expected that the equations developed in the paper will be used to tackle the interaction motions in the gravity fields, such as the Moon's dynamic interaction with the Earth, the thousands of satellites' motions between their and planets' gravitational force fields. From the author's knowledge, the current Earth–satellite design only considers the gravitational field of the Earth but does not consider the gravitational interactions between the satellites. The number of satellites around the Earth increases very fast. For their safe operations, when two of them are approaching each other, their gravitational interaction may need to be considered, although it is very small compared with the Earth's gravity.

(b) In the derivation of the gauge equations of the paper, energy conservation is a necessary link to generate a scalar energy notion, which is different from geometric gravity

theory, where a curvature connection is used. In the simple example, it has shown that the kinetic energy change is governed by the orbit curvature, generally a curvature tensor in the H–E action. Therefore, the energy conservation link concerning kinetic energy change implies that it also concerns curvature and additional force work. To reveal more details, the mathematical and physical relations of the two links may be further researched.

(c)   From this paper, we have seen that the equations for the electromagnetic field and Newton's gravitational field have the same mathematical structures and similar mechanisms. As we know, Newton's law of gravity indicates the gravitational force between two masses, but we do not know what this physical mechanism is. Based on the similarity compared between the gravity and electromagnetic fields in the paper, we may guess that the gravity force essentially is the macroscopic resultant force of the microscopic electromagnetic force inside the physical bodies. This conjecture needs to be theoretically and experimentally tackled.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.
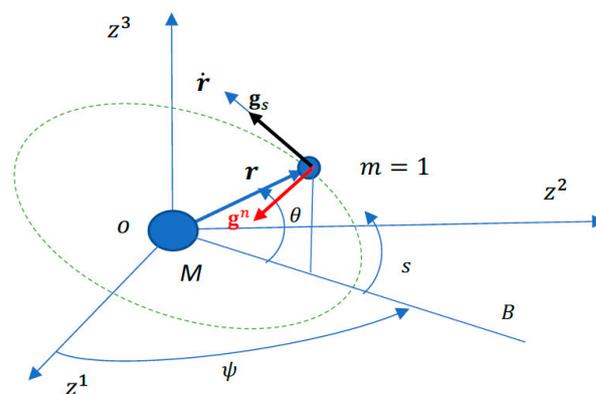
Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

NG   Newton's gravity
GEM   Gravitoelectromagnetism
GFE   Gauge field equations
H–E   Hibert–Einstein

Appendix A. Comparison of Newton's, Newton–Cartan, and Einstein Gravities

The motion of a unit mass in the gravitational field of a unit mass $M = 1$.

Figure A1. Motions of a particle $m = 1$ on the plane Boz^3 in the gravitational field of a mass M of potential $\phi(r)$, identified by arclength S.

As shown in Figure A1, we use the following notations, and the related geometrical/dynamical variables used in the table of Appendix A. The position, velocity, and acceleration of $m = 1$ can be expressed as follows:

$$\boldsymbol{r} = \boldsymbol{r}(r, \theta) = \boldsymbol{r}(s), \quad d\boldsymbol{r} = dr\boldsymbol{g}_r + d\theta\boldsymbol{g}_\theta = ds\boldsymbol{g}_s,$$

$$\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{g}_r + \dot{\theta}\boldsymbol{g}_\theta, \quad \frac{d\boldsymbol{r}}{ds} = \boldsymbol{g}_s, \quad \boldsymbol{g}_\alpha = \partial\boldsymbol{r}/\partial x^\alpha, \quad x^1 = r, \quad x^2 = \theta, \quad R = 1/r,$$

$$\dot{\boldsymbol{r}} = \frac{d\boldsymbol{r}}{dt} = \dot{s}\frac{d\boldsymbol{r}}{ds} = \dot{s}\boldsymbol{g}_s, \quad T = \frac{1}{2}\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} = \frac{1}{2}\dot{s}^2 = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta,$$

$$\ddot{\boldsymbol{r}} = \ddot{s}\boldsymbol{g}_s + \dot{s}^2\frac{d\boldsymbol{g}_s}{ds} = \ddot{s}\boldsymbol{g}_s + R\dot{s}^2\boldsymbol{g}^n, \quad \frac{d\boldsymbol{g}_s}{ds} = R\boldsymbol{g}^n.$$

Therefore, requiring $dT/dt = 0$, shown in the Equation implies the tangent acceleration $\ddot{s} = 0$ with a constant tangent velocity, which physically is that the gravity force does not do work to change the kinetic energy:

$$\frac{dT}{dt} = \ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} = 0, \quad (\ddot{s}\boldsymbol{g}_s + R\dot{s}^2\boldsymbol{g}^n)\cdot\dot{s}\boldsymbol{g}_s = \ddot{s}\dot{s} = \frac{dT}{dt} = 0.$$

$$\delta T = \ddot{\boldsymbol{r}}\cdot\delta\boldsymbol{r} = -\frac{G}{(\boldsymbol{r}\cdot\boldsymbol{r})^{3/2}}\boldsymbol{r}\cdot\delta\boldsymbol{r} = -\frac{G}{r^2}\delta r = G\delta(\frac{1}{r}) = G\delta R,$$

$$\delta\widetilde{H} = \int_0^t \int_{\Omega_z} \delta T\, d\Omega_z dt = \delta\int_0^t \int_{\Omega_z} GR d\,\Omega_z dt = \delta\int_\Omega GR d\,\Omega.$$

Generalized Hilbert–Einstein action for 4D: $\widetilde{H}[g_{\alpha\beta}] = \int_\Omega \frac{1}{2\kappa}R\sqrt{-g}d\Omega$, $\kappa = \frac{8\pi G}{c^4}$. Action of Einstein geometrical gravity: $H[g_{\alpha\beta}] = \int_\Omega \left(\frac{1}{2\kappa}R + \Pi\right)\sqrt{-g}d\Omega$.

Table A1. Comparison of Newton's, Newton–Cartan, and Einstein gravities.

	Newton's Force Gravity	Newton–Cartan Geometrical Gravity	Einstein Geometrical Gravity
Equation	$\ddot{r}(t) = -\partial\phi/\partial r = f$	$\ddot{r}(t) = -\partial\phi/\partial r = f$	$\ddot{r}(t) = -\partial\phi/\partial r = f$
Action	$H[\boldsymbol{r}]$ $= \int_{t_1}^{t_2}\{T(\dot{\boldsymbol{r}}) - \phi(\boldsymbol{r})\}dt$ The variable is the position vector to the particle in the field	$H[\boldsymbol{r}] = \int_{t_1}^{t_2} T(\dot{\boldsymbol{r}})dt$ $\widetilde{H}[R] = \int_0^t \int_{\Omega_z} GR d\Omega_z dt$ $= \int_\Omega GR d\Omega$ R coverture **Generalized 4D case:** **H–E action,** $\kappa = 8\pi G/c^4$ $\widetilde{H}[g_{\alpha\beta}] = \int_\Omega \frac{1}{2\kappa}R\sqrt{-g}d\Omega,$	$H[g_{\alpha\beta}]$ $= \widetilde{H}[g_{\alpha\beta}] + \int_\Omega \Pi\sqrt{-g}d\Omega,$ Adding Π, corresponding to potential ϕ, into H–E action, so that it has a similar form as Newton's one, but the variable is tensor $g_{\alpha\beta}$, a geometrical property of the field.
Constrain	$\delta\boldsymbol{r}(t_1) = 0 = \delta\boldsymbol{r}(t_2)$ $\delta\boldsymbol{r}$ admissible displacements	$\delta\boldsymbol{r}(t_1) = 0 = \delta\boldsymbol{r}(t_2)$ $\delta\boldsymbol{r} = \dot{\boldsymbol{r}}\delta t,$	$\delta g^{\alpha\beta} = 0,$ on the boundary of Ω,
Stational condition	$0 = \delta H$ $= \int_{t_1}^{t_2}\left\{-\ddot{\boldsymbol{r}} - \frac{\partial\phi}{\partial r}\right\}\cdot\delta\boldsymbol{r}dt$ $\ddot{r}(t) = -\partial\phi/\partial r = f$ Newton's equation of motion	$0 = \delta H$ $= -\int_{t_1}^{t_2} \ddot{\boldsymbol{r}}\cdot\delta\boldsymbol{r}dt$ $= -\int_{t_1}^{t_2} \ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}\delta t dt$ $\ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} = dT/dt = 0$ Geometric line, or physical zero-energy-flow line by $\boldsymbol{f}\cdot\dot{\boldsymbol{r}} = 0.$ $0 = \delta\widetilde{H}$ $= \int_{t_1}^{t_2}\delta\widetilde{T}dt$ $= \delta\int_{t_1}^{t_2} GR dt$ $\delta R = 0 = \delta r$ Circle: zero Energy-flow line $\delta R = 0$	$0 = \delta H$ $\int_\Omega\left\{\frac{1}{2\kappa}\left(\overline{R}_{\alpha\beta} - \frac{g_{\alpha\beta}R}{2}\right) - \frac{T_{\alpha\beta}}{2}\right\}$ $\delta g^{\alpha\beta}\sqrt{-g}d\Omega$ $\overline{R}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa T_{\alpha\beta}$ $T_{\alpha\beta} = -\frac{2\partial\Pi}{\partial g^{\alpha\beta}}$

Appendix B. Comparison of Mathematical Structures of Gauge Equations for the Electromagnetic Field and Newton's Gravity Field

Electromagnetic Field		Newton's Gravity Field	
Electric field	E	Gravitational intensity	f^{IJ}
Magnetic field	B	Momentum/curl vector	$p^I,\ \omega^I = \nabla \times p^I$
First-pair equation	$\nabla \times E + \partial B/\partial t = 0$ $\nabla \cdot B = 0$	First pair equation	$\nabla \times f^{IJ} = \partial \omega^I/\partial t,$ $\nabla \cdot \omega^I = 0,$
Second-pair equation	$\nabla \times B - \partial E/\partial t = J$ $\nabla \cdot E = \rho$	Second pair equation	$\nabla \times \omega^I - \partial f^{II}/\partial t = J^{IJ}$ $\nabla \cdot f^{II} = -4\pi \widetilde{G} \delta(x_I^\alpha - x_J^\alpha),$
Electromagnetic tensor	$F_{\alpha\beta} =$ $\begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{bmatrix}$ $F = \tfrac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$	Newton's gravity tensor	$F_{\alpha\beta} =$ $\begin{bmatrix} 0 & \omega_x^I & \omega_y^I & \omega_z^I \\ -\omega_x^I & 0 & f_z^{IJ}/c & -f_y^{IJ}/c \\ -\omega_y^I & -f_z^{IJ}/c & 0 & f_x^{IJ}/c \\ -\omega_z^I & f_y^{IJ}/c & -f_x^{II}/c & 0 \end{bmatrix}$ $F^I = \tfrac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta$
Dual tensor	$*F_{\alpha\beta} =$ $\begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z/c & -B_y/c \\ -E_y & -B_z/c & 0 & B_x/c \\ -E_z & B_y/c & -B_x/c & 0 \end{bmatrix}$ $*F = \tfrac{1}{2} *F_{\alpha\beta} dx^\alpha \wedge dx^\beta$	Dual tensor	$*F_{\alpha\beta} =$ $\begin{bmatrix} 0 & f_x & f_y & f_z \\ -f_x & 0 & \omega_z/c & -\omega_y/c \\ -f_y & -\omega_z/c & 0 & \omega_x/c \\ -f_x & \omega_y/c & -\omega_x/c & 0 \end{bmatrix}$ $*F^I = \tfrac{1}{2} *F_{\alpha\beta} dx^\alpha \wedge dx^\beta$
Gauge equations	$*dF = 0,$ $*d*F = J.$	Gauge equations	$*dF^I = 0,$ $*d*F^I = \hat{J}^I = \begin{bmatrix} -4\pi \widetilde{G} \delta(x_I^\alpha - x_J^\alpha) \\ J^{IJ} \end{bmatrix}.$
Lorentz invariant	$F_{\alpha\beta} F^{\alpha\beta} = 2(B^2 - E^2/c^2)$	Lorentz invariant	$F_{\alpha\beta} F^{\alpha\beta} = 2(\omega^2 - f^2/c^2)$

Appendix C. The Material and Field Descriptions of Newton's Gravity Theory

Variables	Material Description	Field Description
$x^I,\ x^J$	Two updated material coordinates to identify two masses at these two points	Two space points
Gravity force	$f^{IJ}(x^I,\ x^J)$ is the gravity force between two masses. $f^{IJ}(x^I,\ x^J) = -f^{JI}(x^J,\ x^I)$	$f^I(x^I,\ t)$ is the gravity force on mass I at x^I by mass J at x^J. $f^I(x^I,\ t) = -f^J(x^J,\ t)$
Momentum	$p^I(x^I),\ p^J(x^J),$ are totally determined by mass, i.e., material coordinate	$p^I(x^I, t),\ p^J(x^J, t),$ are field momentums changing with time, equaling momentums of masses reaching at the points
Time derivative of variable A	$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial t}(x = (X, t)) = \frac{\partial A}{\partial x^\alpha} \dot{x}^\alpha,$ directional derivative	$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x^\alpha} \dot{x}^\alpha,$ material derivative
Space derivative	respect to material coordinates	respect to field coordinates
Newton's law	$f^{IJ}(x^I,\ x^J) = \left.\frac{\partial p^I(x^I)}{\partial t}\right|_X$ acceleration of mass I equals gravity force on it from mass J.	$f^I(x^I,\ t) = \frac{\partial p^I}{\partial t} + \frac{\partial p^I}{\partial x^\alpha} \dot{x}^\alpha = a_x + a_{\dot{x}}$ $f^I(x^I,\ t) = -\nabla \Pi + \frac{\partial p^I}{\partial t}$ force at a field point equals local a_x + translational $a_{\dot{x}}$. a_x caused by motion of mass J, while $a_{\dot{x}}$ caused by motion mass I that involves potential Π change, so replaced by $-\nabla \Pi$.

References

1. Yang, C.N.; Mills, R.L. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.* **1954**, *96*, 191–195. [CrossRef]
2. Baez, J.; Muniain, J.P. *Gauge Fields, Knots and Gravity*; World Scientific: New Jersey, NJ, USA, 1961.
3. Landau, L.D.; Lifshitz, E.M. *Mechanics*, 3rd ed.; Pergamon Press plc: Oxford, UK, 1976.
4. Wald, R. *General Relativity*; University of Chicago Press: Chicago, IL, USA, 1984.
5. Christian, J. Exactly soluble sector of quantum gravity. *Phys. Rev. D* **1997**, *56*, 4844–4877. [CrossRef]
6. Knox, E. Newton-Cartan theory and teleparallel gravity: The force of a formulation. *St. His Phil Mod. Phys.* **2011**, *42*, 264–275. [CrossRef]
7. Blagojevic, M.; Hekl, F.W. *Commentaries from the Edited Collection of Reprints, Gauge Theories of Gravitation, A Reader with Commentaries*; Imperial College Press: London, UK, 2013. [CrossRef]
8. Duel, C.; Burdet, G.; Künzle, H.P.; Perrin, M. Bargmann structures and Newton-Cartan theory. *Phys. Rev. D* **1985**, *31*, 1841. [CrossRef] [PubMed]
9. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; W. H. Freeman: San Francisco, CA, USA, 1973.
10. Brandenburg, J. The GEM Theory of the Unification of Gravitation and Electro-Magnetism. *Astro. Rev.* **2012**, *7*, 48–67. [CrossRef]
11. Rabounski, D. The theory of vortical gravitational fields. *Prog. Phys.* **2007**, *2*, 3–10.
12. Xing, J.T. *Energy Flow Theory of Nonlinear Dynamical Systems with Applications*; Springer: Heidelberg, Germany, 2015.
13. Xing, J.T. Generalised energy conservation law of chaotic phenomena. *Acta Mech. Sin.* **2019**, *35*, 1257–1268. [CrossRef]
14. Xing, J.T. A gravity tensor and gauge equations for newton's nonlinear dynamical gravity field. In Proceedings of the ICTAM2024, Daegu, Republic of Korea, 25–30 August 2024. Available online: https://www.Ictam2024.org (accessed on 28 December 2025).
15. Xing, J.T. *Fluid-Solid Interaction Dynamics: Theory, Variational Principles, Numerical Methods, and Applications*; Elsevier: New York, NY, USA, 2019.
16. Griffiths, D.J. *Introduction to Electrodynamics*, 4th ed.; Pearson Education: New York, NY, USA, 1942.
17. Wikipedia. Electromagnetic Tensor. Available online: https://en.wikipedia.org (accessed on 28 December 2025).